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JORGE L. RAMÍREZ ALFONSÍN[†] AND IVÁN RASSKIN[‡]

Abstract. We present a generalization of Descartes' theorem for the family of polytopal sphere packings arising from uniform polytopes. The corresponding quadratic equation is expressed in terms of geometric invariants of uniform polytopes which are closely connected to canonical realizations of edge-scribable polytopes. We use our generalization to construct integral Apollonian packings based on the Platonic solids. Additionally, we also introduce and discuss a new spectral invariant for edge-scribable polytopes.

Contents

1. Introduction

A classical *Apollonian packing* of circles arises from an initial configuration of four pairwise tangent disks on the plane. By adding the inscribed disk to the interstice between each triple of disks, and then repeating this process *ad finitum*, we obtain an Apollonian packing, as shown in Figure 1.

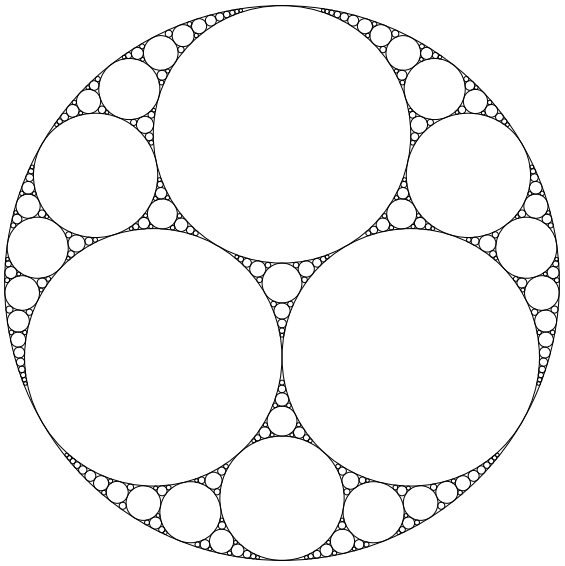

Figure 1. An Apollonian packing.

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Apollonian packings and their generalizations have proven to be effective tools for studying various structures in natural science (see [She20] for a nice survey). They also appear in many different branches in mathematics such us geometric group theory [Zha23], fractal geometry [MEG04], discrete geometry [Che16a], knot theory [RR23] or number theory [Gra+03]. The underlying integral structure of Apollonian packings was first noticed by Soddy in [Sod36]. He observed that if the four *curvatures*[1] (reciprocal of radii) of the initial disks are integers, then the corresponding Apollonian packing is *integral*, i.e. the set of curvatures is a subset of $\mathbb{Z}$ (see Figure 2). Understanding the behaviour of the integers arising from these configurations is currently an active area of research that is connected to several central problems in number theory [Kon13].

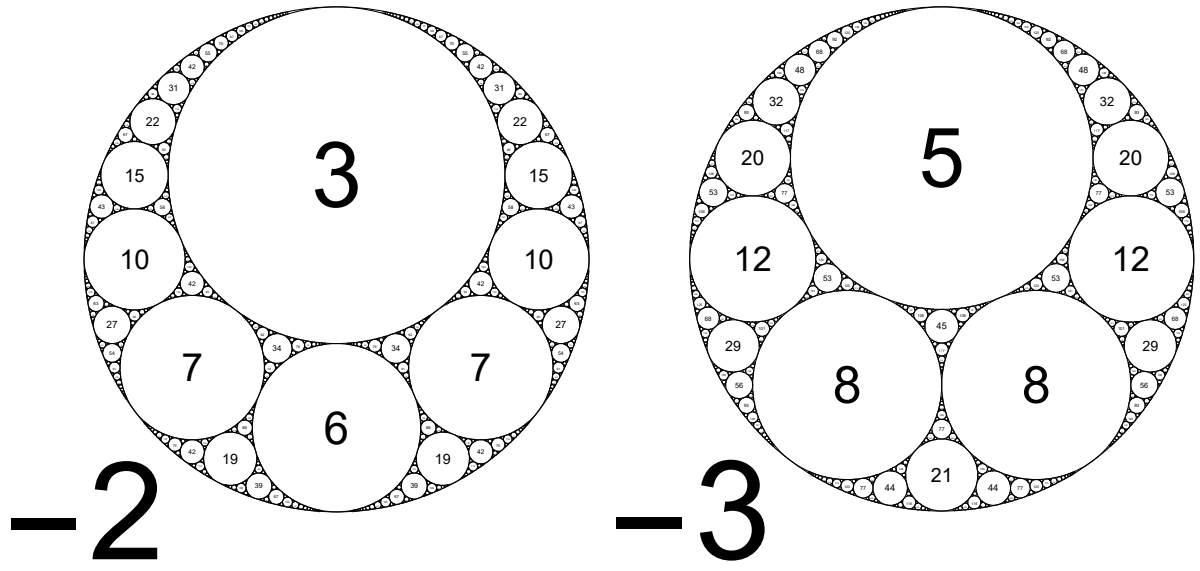

FIGURE 2. Two integral Apollonian packings. The labels are the curvatures.

Soddy's observation follows from an old algebraic relation on the curvatures of four pairwise tangent circles on the plane. This relation, known as *Descartes' theorem*, was written in the correspondence between Descartes and the Princess Elizabeth of Bohemia around 1643 [Bos10].

Theorem 1.1 (Descartes)**.** *The curvatures of four pairwise tangent disks on the plane satisfy*

$$(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)^2 = 2(\kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2). \tag{1}$$

Both Apollonian packings and Descartes' theorem have been widely generalized. For instance, the higher-dimensional analogues of Descartes' theorem for configurations of $d+2$ pairwise tangent spheres on the d-dimensional Euclidean space were given by Soddy for $d = 3$ [Sod36] and by Gosset for every $d \geq 2$ [Gos37]. Another type of generalizations in the plane arise by modifying the initial set of disks according to a polyhedron whose edges are tangent to the unit sphere. The classic Apollonian packing arises from the tetrahedron, but this approach can also be used to obtain the octahedral *octahedral* [Boy74; GM08], *cubic* [Sta15], or *icosahedral* [BBH18] analogues of Apollonian packings. Each of these works admits a generalization of Descartes' theorem, and all of them can be derived from the most general version mentioned by Boyd in [Boy74]. However, none of these Descartes' equations takes into account the interaction with the geometry of the corresponding polyhedron.

The Apollonian-like packings that we have mentioned belong to the family of *polyhedral packings* described in [KN19]. In this paper, we explore the higher dimensional analogues, which we call *polytopal sphere packings*. The connection between polytopes and sphere packings has been previously explored in the works of Boyd [Boy74], Maxwell [Max82], Eppstein, Kuperberg and Ziegler [EKZ03], Chen [Che16a; Che16b] and Chen and Labbé [CL15]. Our main contribution is the following generalization of Descartes' theorem for the class of polytopal sphere packings arising from uniform polytopes in every dimension.

[1]Also called *bends*.

Theorem 1.2 (Polytopal Descartes' theorem)**.** *Let $\mathcal{B}_\mathcal{P}$ a polytopal sphere packing where $\mathcal{P}$ is a uniform $(d+1)$-polytope with $d \geq 2$. The polytopal curvatures of $\mathcal{B}_\mathcal{P}$ with respect to the faces in any flag $(f_0, \ldots, f_d, f_{d+1} = \mathcal{P})$ satisfy*

$$(\kappa_{f_0} - \kappa_{f_1})^2 + \ell_{f_2}^2(\kappa_{f_1} - \kappa_{f_2})^2 + \sum_{i=2}^{d} \frac{1}{\ell_{f_{i+1}}^{-2} - \ell_{f_i}^{-2}}(\kappa_{f_i} - \kappa_{f_{i+1}})^2 = \ell_\mathcal{P}^2 \kappa_\mathcal{P}^2 \tag{2}$$

One of the special features of this generalization is that the quadratic equation (15), which links the notions of *canonical lengths* ℓ_{f_i} with *polytopal curvatures* κ_{f_i}, is given in terms of the geometry of the underlying polytope. The latter provides a better understanding of the equations involving previous generalizations, which turns out to be useful in computing the integrality conditions necessary to construct integral Apollonian packings for other polytopes, like the packings in Figure 3.

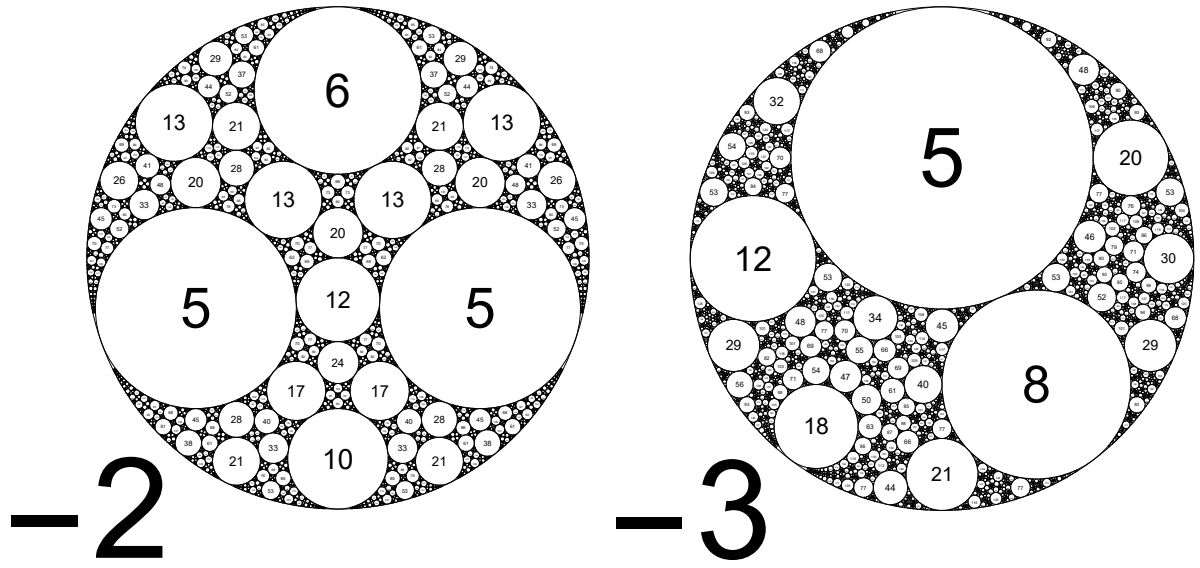

FIGURE 3. Two integral Apollonian packings based on the cube (left) and the truncated tetrahedron (right).

1.1. **Organization of the paper.** In Section 2, we give the theoretical background on the Lorentzian and projective models of the space of spheres and polytopes, necessary for understanding the notion of polytopal sphere packings. Then, we discuss several aspects concerning the duality, Apollonian groups and packings, and uniqueness under Möbius transformations. We also introduce a novel spectral invariant of edge-scribable polytopes that we call *Möbius spectrum*.

In Section 3, after defining the notions of *canonical length* of uniform polytopes and *polytopal curvatures* of polytopal sphere packings, and presenting some lemmas, we prove the polytopal Descartes' theorem (Theorem 3.1).

In Section 4, we apply the main theorem to give formulas of a Descartes' theorem (Proposition 4.1), integral matrix representations of the Apollonian groups (Proposition 4.2), integrality conditions (Corollary 4.2), and parametrizations of the integral Apollonian packings (Corollary 4.4), in terms of the Schläfli symbol of each Platonic solid. We also show the existence of the sequence of perfect squares among the curvatures of certain Platonic packings (Corollary 4.4). At the end of the section, we present a catalogue of the Platonic packings describing all the properties discussed in this paper.

We conclude with some final remarks in Section 5.

2. Polytopal sphere packings: background

For any $d, m, n \in \mathbb{N}$ with $d = m + n$, let $\mathbb{R}^{m,n}$ be the real vector space of dimension d endowed with the inner product $\langle \cdot, \cdot \rangle$ of signature (m, n). The *unit sphere* $\mathbb{S}(\mathbb{R}^{m,n})$ is the space $\{\mathbf{x} \in \mathbb{R}^{m,n} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\}$. As usually, we shall write $\mathbb{R}^d$ and $\mathbb{S}^d$ instead of $\mathbb{R}^{d,0}$ and $\mathbb{S}(\mathbb{R}^{d+1})$, respectively. An *oriented hypersphere* (in short, *sphere*) of $\widehat{\mathbb{R}^d} := \mathbb{R}^d \cup \{\infty\}$, is the image of a spherical cap of $\mathbb{S}^d$ under the stereographic projection. According to the position of the North Pole of $\mathbb{S}^d$ with respect to the corresponding spherical cap, there are three types of spheres, namely, *solid sphere* (positive radius), *hollow sphere* (negative radius) or *half-space* (infinite radius). Any sphere is determined by its center $c \in \widehat{\mathbb{R}^d}$ and its curvature $\kappa \in \mathbb{R}$ when is not a half-space, and by its normal vector $\widehat{n} \in \mathbb{S}^d$ pointing to the interior and signed distance to the origin $\delta \in \mathbb{R}$ otherwise. We denote by $\mathscr{B}(\widehat{\mathbb{R}^d})$ the space of spheres of $\widehat{\mathbb{R}^d}$.

2.1. **The Lorentzian model of $\mathscr{B}(\widehat{\mathbb{R}^d})$.** The space $\mathbb{R}^{d,1}$ and its corresponding inner product are called the *Lorentzian space* and the *Lorentzian product* in dimension $d+1$, respectively. A vector of $\mathbf{x} \in \mathbb{R}^{d,1}$ is *future-directed* (resp. *past-directed)* if $x_{d+2} > 0$ (resp. $x_{d+2} < 0$). There is a well-known bijection between $\mathscr{B}(\widehat{\mathbb{R}^d})$ and the Lorentzian unit sphere $\mathbb{S}(\mathbb{R}^{d+1,1})$ (see [Wil81] as well as [RR21, Section 2] for further details on this bijection). For any sphere $s \in \mathscr{B}(\widehat{\mathbb{R}^d})$, we denote by $\mathbf{x}_s \in \mathbb{S}(\mathbb{R}^{d+1,1})$ the *Lorentzian vector* corresponding to s. The *inversive coordinates* $\mathbf{i}(s)$ given by Wilker [Wil81] corresponds to the Cartesian coordinates of $\mathbf{x}_s$ in the canonical basis of $\mathbb{R}^{d+1,1}$, which can be computed by

$$\mathbf{i}(s) = \begin{cases} \frac{\kappa}{2}(2c, \|c\|^2 - \kappa^{-2} - 1, \|c\|^2 - \kappa^{-2} + 1)^T & \text{if } \kappa \neq 0, \\ (\widehat{n}, \delta, \delta)^T & \text{if } \kappa = 0 \end{cases} \tag{3}$$

where $\| \cdot \|$ denotes the Euclidean norm of $\widehat{\mathbb{R}^d}$. The *inversive product* of two spheres is defined as $\langle s, s' \rangle := \langle \mathbf{x}_s, \mathbf{x}_{s'} \rangle$. If $\mathbf{x}_s, \mathbf{x}_{s'}$ are not both past-directed then

$$\langle s, s' \rangle \begin{cases} < -1 & \text{if } s \text{ and } s' \text{ are disjoint} \\ = -1 & \text{if } s \text{ and } s' \text{ are externally tangent (with disjoint interiors)} \\ = 0 & \text{if } s \text{ and } s' \text{ are orthogonal} \\ = 1 & \text{if } s \text{ and } s' \text{ are internally tangent (with overlapping interiors)} \\ > 1 & \text{if } s \text{ and } s' \text{ are nested} \end{cases} \tag{4}$$

An arrangement of spheres $\mathcal{B}$ is a *packing* if any two spheres are externally tangent or disjoint. For any arrangement $\mathcal{B}$, we denoted by $\mathrm{Gram}(\mathcal{B})$ the Gramian of the corresponding Lorentzian vectors in $\mathbb{R}^{d+1,1}$. The *Möbius group* $\mathsf{M\ddot{o}b}(\widehat{\mathbb{R}^d})$ is the group generated by inversions through spheres. It is isomorphic to the *Orthochronous group*, made by the linear automorphisms of $\mathbb{R}^{d+1,1}$ preserving the Lorentz product and the time direction. Therefore, $\mathsf{M\ddot{o}b}(\widehat{\mathbb{R}^d})$ acts transitively on $\mathscr{B}(\widehat{\mathbb{R}^d})$ preserving the inversive product.

2.2. **The projective model of $\mathscr{B}(\widehat{\mathbb{R}^d})$.** The Lorentzian unit sphere $\mathbb{S}(\mathbb{R}^{d+1,1})$ can be regarded in the *oriented projective space* $\mathbb{P}_+\mathbb{R}^{d+1,1} = \{\mathbf{x} \in \mathbb{R}^{d+1,1} \setminus 0\}/_\sim$ where $\mathbf{x} \sim \mathbf{y}$ if there is a real number $\lambda > 0$ such that $\mathbf{x} = \lambda \mathbf{y}$. Then $\mathbb{P}_+\mathbb{R}^{d+1,1}$ is equivalent to the Euclidean unit sphere $\mathbb{S}^{d+1} \subset \mathbb{R}^{d+1,1}$ which, under the gnomonic projection, becomes the union of two affine hyperplanes $\Pi_{\pm 1} = \{x_{d+2} = \pm 1\}$, which can be both identified with $\mathbb{R}^{d+1}$, together with $\Sigma_0 = \{(\mathbf{x}, 0) \mid \mathbf{x} \in \mathbb{S}^{d+1}\}$. The composition of the isomorphism $\mathscr{B}(\widehat{\mathbb{R}^d}) \to \mathbb{S}(\mathbb{R}^{d+1,1})$ with the projection

$$\begin{array}{ccc} \mathbb{S}(\mathbb{R}^{d+1,1}) & \longrightarrow & \Pi_1 \cup \Sigma_0 \cup \Pi_{-1} \\ \mathbf{x} & \longmapsto & \begin{cases} \mathbf{x} & \text{if } x_{d+2} = 0 \\ \frac{1}{|x_{d+2}|}\mathbf{x} & \text{otherwise} \end{cases} \end{array}$$

gives an isomorphism between $\mathscr{B}(\widehat{\mathbb{R}^d})$ and $\Pi_1^\circ \cup \Sigma_0 \cup \Pi_{-1}^\circ$, where $\Pi_{\pm 1}^\circ$ is the set of points in $\Pi_{\pm 1}$ whose Euclidean norm is greater than 1. Such a point will be called an *outer point* of $\mathbb{R}^{d+1}$. We call $\Pi_1^\circ \cup \Sigma_0 \cup \Pi_{-1}^\circ$ the *projective model of the space of spheres*. In this way, we can construct a bijection between the subset of spheres whose Lorentzian vector is future-directed and the set of outer points of $\mathbb{R}^{d+1}$. The reciprocal bijection between an outer point $v \in \mathbb{R}^{d+1}$ and a sphere $s_v \in \mathscr{B}(\widehat{\mathbb{R}^d})$ can be obtained geometrically by taking a *light source* which illuminates $\mathbb{S}^d$ from v. The illuminated region on $\mathbb{S}^d$ is a spherical cap is stereographically projected to the sphere s_v. We say that v is the *light source* of s_v. In Figure 4, we illustrate a sphere and its light source within the projective model of the space of spheres.

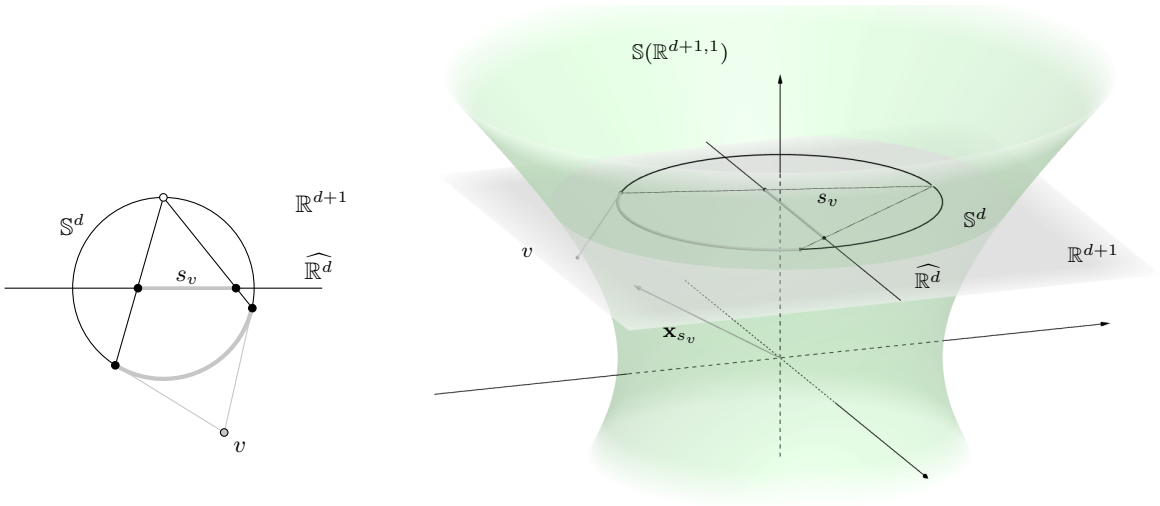

FIGURE 4. (Left) A sphere of $\widehat{\mathbb{R}^d}$ and its light source; (right) same setting in the projective model, together with the corresponding lorentzian vector.

The inversive coordinates of a sphere whose Lorentzian vector is future-directed, can be computed from the coordinates of its light source by

$$\mathbf{i}(s_v) \quad = \quad \frac{1}{\sqrt{\|v\|^2 - 1}} \begin{pmatrix} v \\ 1 \end{pmatrix} \tag{5}$$

The above equality implies that for any two spheres whose Lorentzian vectors are future-directed, their inversive product is related to the Euclidean inner product of their corresponding light sources by the following equation

$$\langle s_u, s_v \rangle = \frac{1}{\sqrt{(\|u\|^2 - 1)(\|v\|^2 - 1)}} (u \cdot v - 1) \tag{6}$$

where $\cdot$ denotes the Euclidean inner product. The Möbius group acts on the outer points of $\mathbb{R}^{d+1}$ as the group of projective transformations preserving $\mathbb{S}^d$ [Spr05].

2.3. **Polytopes.** We recall some basic notions of polytopes needed for the rest of the paper. We refer the reader to [Sch04] for further details. A $(d+1)$-*polytope* $\mathcal{P}$ is the convex hull of a finite collection of points in $\mathbb{R}^{d+1}$. A 2-polytope and a 3-polytope are usually called (*convex*) *polygon* and *polyhedron*, respectively. For every $0 \leq k \leq d+1$, we denote by $F_k(\mathcal{P})$ the set of k-faces of $\mathcal{P}$ and by $\mathcal{F}(\mathcal{P}) = \{\emptyset\} \cup \bigcup_{k=0}^{d+1} F_k(\mathcal{P})$. The elements of $V(\mathcal{P}) := F_0(\mathcal{P})$, $E(\mathcal{P}) := F_1(\mathcal{P})$, $F_{d-1}(\mathcal{P})$ and $F_d(\mathcal{P})$ are called *vertices*, *edges*, *ridges* and *facets* of $\mathcal{P}$, respectively. The *graph of* $\mathcal{P}$ is the graph induced by the vertices and the edges of $\mathcal{P}$. The *face lattice* $(\mathcal{F}(\mathcal{P}), \subset)$ encodes all the combinatorial information about $\mathcal{P}$. A *flag* of $\mathcal{P}$ is a sequence of faces $\Phi = (f_0, f_1, \ldots, f_d, \mathcal{P})$ where for each $k = 0, \ldots, d$, $f_k \subset f_{k+1}$. Two polytopes $\mathcal{P}$ and $\mathcal{P}'$ are combinatorially equivalent if there exists an isomorphism between their face lattices. If they are combinatorially equivalent, we say they have the same *combinatorial type* and $\mathcal{P}'$ is said to be a *realization* of $\mathcal{P}$.

The *polar* of a set $X \subset \mathbb{R}^{d+1}$ is defined as the set of points $X^* = \{u \in \mathbb{R}^{d+1} \mid u \cdot v \leq 1 \text{ for all } v \in X\}$ where $\cdot$ denotes the Euclidean inner product. If $\mathcal{P}$ is a $(d+1)$-polytope containing the origin in its interior then $\mathcal{P}^*$ is also a $(d+1)$-polytope containing the origin in its interior and holding the dual relation $(\mathcal{P}^*)^* = \mathcal{P}$. There is a bijection between $\mathcal{F}(\mathcal{P})$ and $\mathcal{F}(\mathcal{P}^*)$ which reverses incidences and maps every facet f of $\mathcal{P}$ to a vertex v_f of $\mathcal{P}^*$. For every vertex $u \in f$, one has

$$u \cdot v_f = 1 \tag{7}$$

The *symmetry group* of $\mathcal{P}$ is defined as the group of Euclidean isometries of $\mathbb{R}^{d+1}$ preserving $\mathcal{P}$. $\mathcal{P}$ is *regular* if its symmetry group acts transitively on the set of flags $\mathcal{P}$. For any $d \geq 2$, a d-polytope is called *uniform* if is regular or, recursively, if $d \geq 2$, its facets are uniform, and the symmetry group acts transitively on its vertices. It is well-known that the uniform 3-polytopes are the 5 Platonic solids, the 13 Archimedean solids, and the infinite families of prisms and antiprisms [Grü+03].

For every $0 \leq k \leq d$, a $(d+1)$-polytope $\mathcal{P}$ is said to be k-scribed if all its k-faces are tangent to the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$. A k-scribed polytope is called *inscribed*, *edge-scribed*, *ridge-scribed* and *circunscribed* if $k = 0, 1, d-1, d$ respectively. A polytope is said to be *k-scribable* if it admits a realization which is k-scribed. If $\mathcal{P}$ is a $(d+1)$-polytope containing the origin in its interior then $\mathcal{P}$ is k-scribed if and only if $\mathcal{P}^*$ is $(d-k)$-scribed [CP17]. Concerning edge-scribability, in dimension 2 and 3 all polytopes are edge-scribable. In dimension $d \geq 4$, there are non-edge-scribable polytopes [Sch87]. An edge-scribed $(d+1)$-polytope is said to be *canonical* [Zie94] if the barycenter of all its tangency points with $\mathbb{S}^d$ is the origin. It follows from the work of Springborn in [Spr05], that for any edge-scribable polytope $\mathcal{P}$, there is a unique canonical realization $\mathcal{P}_0$, up to Euclidean isometries. For 3-polytopes, canonical realizations were also called *Springborn realizations* in [BJP22].

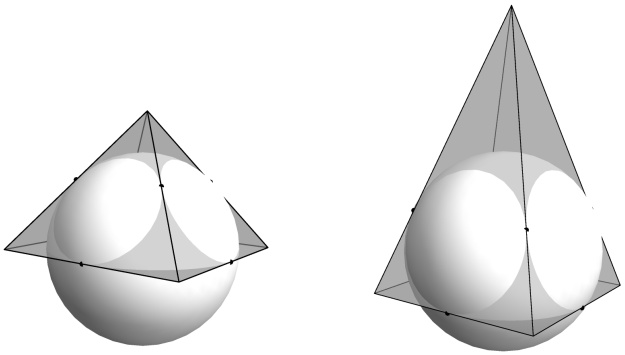

FIGURE 5. An edge-scribed realization (left) and a canonical realization (right) of a 4-pyramid.

2.4. **Polytopal sphere packings.** Let $\mathcal{P}$ be an *outer* $(d+1)$-polytope, i.e. whose vertices are outside the unit sphere $\mathbb{S}^d$. We define the *arrangement projection* of $\mathcal{P}$ as the arrangement of spheres of $\widehat{\mathbb{R}^d}$ given by $\beta(\mathcal{P}) := \{s_v \,|\, v \in V(\mathcal{P})\}$. For any edge uv of $\mathcal{P}$, the spheres s_u and s_v are disjoint, externally tangent or with intersecting interiors, if and only if uv cuts transversely, is tangent or avoids strictly $\mathbb{S}^d$, respectively. Therefore, if $\mathcal{P}$ is an edge-scribed $(d+1)$-polytope then $\beta(\mathcal{P})$ is a packing.

Definition 2.1. For every $d \geq 2$, a sphere packing $\mathcal{B}_\mathcal{P}$ in $\widehat{\mathbb{R}^d}$ is *polytopal* if there is an edge-scribable $(d+1)$-polytope $\mathcal{P}$ such that $\mathcal{B}_\mathcal{P} = \beta(\mathcal{P}_0)$, up to Möbius transformations.

In the previous definition $\mathcal{P}_0$ denotes a canonical realization of $\mathcal{P}$. Clearly, there are sphere packings which are not polytopal (see Figure 6). As Chen noticed in [Che16a], if $\beta(\mathcal{P})$ is a packing, then the tangency graph of $\beta(\mathcal{P})$ is a spanning subgraph of the graph of $\mathcal{P}$. If in addition, $\mathcal{P}$ is edge-scribed, then the two graphs are isomorphic. Therefore, the tangency relations of any polytopal sphere packing $\mathcal{B}_\mathcal{P}$ are encoded by the edges of the polytope $\mathcal{P}$.

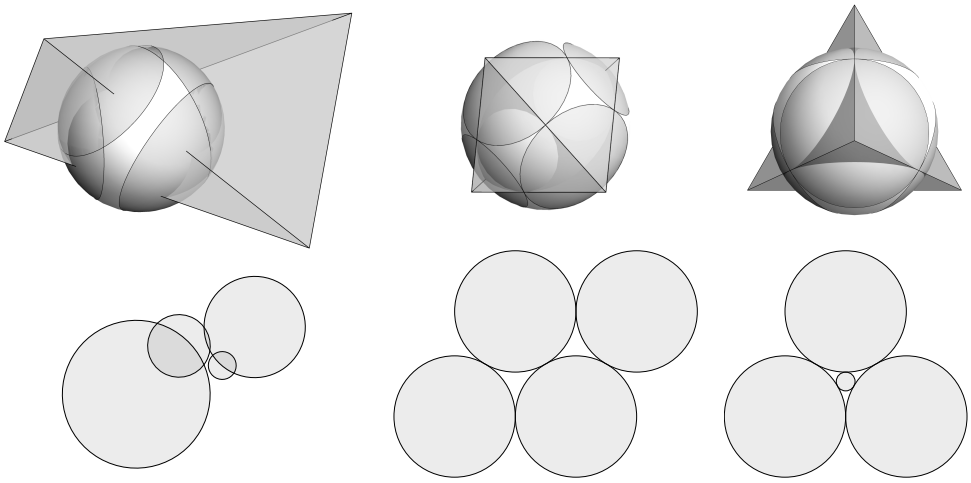

FIGURE 6. (Top) Three outer polyhedra with the spherical caps illuminated by their vertices. (Below) The arrangement projection of the three polyhedra. The last two are packings but only the third one is polytopal.

2.5. **Duality.** Let $\mathcal{B}_\mathcal{P}$ any polytopal sphere packing where $\mathcal{P}$ is the edge-scribable $(d+1)$-polytope with $d \geq 2$. By definition, there is a Möbius transformation μ such that $\mathcal{B}_\mathcal{P} = \mu\beta(\mathcal{P}_0)$. We define the *dual arrangement* of $\mathcal{B}_\mathcal{P}$ as the arrangement $\mathcal{B}^*_\mathcal{P} := \mu\beta(\mathcal{P}^*_0)$. Notice that $\mathcal{B}^*_\mathcal{P}$ is well-defined since $\mathcal{P}_0$ contains the origin in its interior. By combining equations (4) (6) 7, we have that for any two vertices $(v, v_f) \in V(\mathcal{P}) \times V(\mathcal{P}^*)$, where v_f is the vertex corresponding to a facet f of $\mathcal{P}$ containing v, the spheres s_v and s_{v_f} are orthogonal. Since $\mathcal{P}_0$ is edge-scribed, $\mathcal{P}^*_0$ is ridge-scribed. For $d = 2$, $\mathcal{P}^*_0$ is also edge-scribed and therefore $\mathcal{B}^*_\mathcal{P}$ is also a packing. The union $\mathcal{B}_\mathcal{P} \cup \mathcal{B}^*_\mathcal{P}$ has been called a *primal-dual circle representation* of $\mathcal{P}$ [FR18]. Brightwell and Scheinerman proved in [BS93] the existence and the uniqueness up to Möbius transformations of primal-dual circle representations for every polyhedron. It can be seen as a stronger version of the Koebe-Andreev-Thurston Circle packing theorem [BS04].

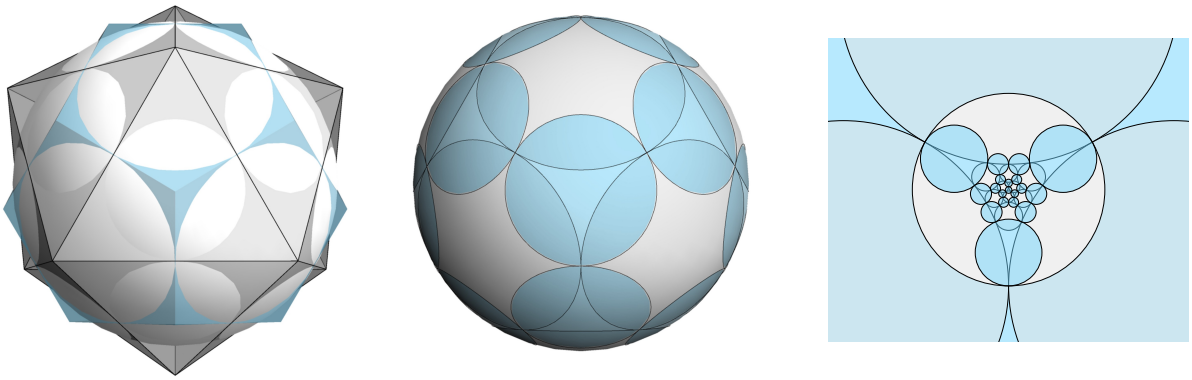

FIGURE 7. (Left) An edge-scribed icosahedron $\mathcal{I}^3$ and its polar in blue; (center) the spherical illuminated regions of $\mathcal{I}^3$ and its polar; (right) a primal-dual circle representation of $\mathcal{I}^3$.

2.6. **Polytopal Apollonian packings.** We define the *symmetry group* of $\mathcal{B}_\mathcal{P}$ as the stabilizer of $\mathsf{M\ddot{o}b}(\widehat{\mathbb{R}^d})$ for $\mathcal{B}_\mathcal{P}$, which is isomorphic to the symmetry group of $\mathcal{P}$. The *Apollonian group* of $\mathcal{B}_\mathcal{P}$ is the subgroup $\mathsf{A}(\mathcal{B}_\mathcal{P}) < \mathsf{M\ddot{o}b}(\widehat{\mathbb{R}^d})$ generated by the inversions through the spheres of $\mathcal{B}^*_\mathcal{P}$. Since inversions of $\mathsf{M\ddot{o}b}(\widehat{\mathbb{R}^d})$ correspond to hyperbolic reflections of the $(d+1)$-dimensional hyperbolic space, Apollonian groups are hyperbolic Coxeter groups. The *Apollonian arrangement* of $\mathcal{B}_\mathcal{P}$, denoted by $\mathscr{P}(\mathcal{B}_\mathcal{P})$, is the union of the orbits of the action of $\mathsf{A}(\mathcal{B}_\mathcal{P})$ on $\mathcal{B}_\mathcal{P}$ (see Figure 8). If $\mathscr{P}(\mathcal{B}_\mathcal{P})$ is a *packing*, then we shall call it a *polytopal Apollonian packing*. For $d = 2$, $\mathscr{P}(\mathcal{B}_\mathcal{P})$ is always a packing, but this is not true in higher dimensions. As instance, the Apollonian arrangement of the simplicial packing $\mathscr{P}(\mathcal{B}_{\mathcal{T}^{d+1}})$ is not a packing for $d \geq 4$. The property of being a packing depends on the intersection angles of the dual spheres of $\mathcal{B}^*_\mathcal{P}$, which must satisfy the *crystallographic restriction* [Boy74], or equivalently, the polar $\mathcal{P}^*$ must be a Coxeter polytope [Che16b].

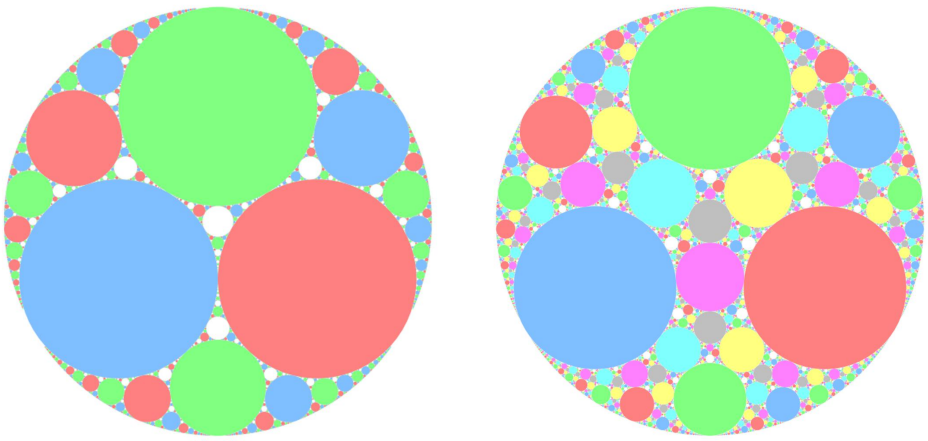

FIGURE 8. A tetrahedral (left) and cubic (right) Apollonian packing. Each colour represents an orbit under the action of the Apollonian group.

We define the *symmetrized Apollonian group* of $\mathcal{B}_\mathcal{P}$, denoted by $\mathsf{SA}(\mathcal{B}_\mathcal{P})$, as the stabilizer of $\mathsf{M\ddot{o}b}(\widehat{\mathbb{R}^d})$ for $\mathscr{P}(\mathcal{B}_\mathcal{P})$. It can be generated by union of a set of generators of the symmetry and the Apollonian group of $\mathcal{B}_\mathcal{P}$. This group was used by Baragar in [Bar18; Bar22] for studying the simplicial cases $\mathcal{P} = \mathcal{T}^{d+1}$ for $d = 2, \dots, 8$.

Polytopal Apollonian packings belong to the family of *Kleinian sphere packings* introduced by Kapovich and Kontorovich [KK23] or *Boyd-Maxwell packings* introduced by Chen and Labbé in [CL15]. In dimension 2, polytopal Apollonian packings coincide with the family of *polyhedral packings* introduced by Kontorovich and Nakamura in [KN19] as a particular case of *crystallographic sphere packings*, also studied in more detail by Chait, Cuit and Stier in [CCS20] (see Section 5 for further details).

2.7. **Möbius uniqueness and the Möbius spectrum.** Two sphere packings are *Möbius equivalent* (also called *conformally equivalent*) if one can be sent to the other by a Möbius transformation. Let $\mathcal{B}_\mathfrak{C}$ be a sphere packing whose combinatorial structure is given by a combinatorial object $\mathfrak{C}$. We say that $\mathcal{B}_\mathfrak{C}$ is *Möbius unique* with respect to $\mathfrak{C}$ if any other sphere packing $\mathcal{B}'_\mathfrak{C}$ is Möbius equivalent to $\mathcal{B}_\mathfrak{C}$. If $\mathfrak{C}$ is the contact graph, then $\mathcal{B}_\mathfrak{C}$ is in general not Möbius unique (see [RR21]). As it is mentioned in the previous section, in dimension 2, polytopal sphere packings are Möbius unique due to Brightwell-Schneirmann Theorem [BS93]. Analogously to the case of crystallographic sphere packings [KN19], Mostow Rigidity Theorem implies that polytopal sphere packings are also Möbius unique in higher dimensions. To see this, let us consider two sphere packings $\mathcal{B}_{\mathcal{P}_1}$ to $\mathcal{B}_{\mathcal{P}_2}$ in $\widehat{\mathbb{R}^d}$ with $d \geq 2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are two edge-scribed $(d+1)$-polytopes with same combinatorial type. The polytopes of $\mathcal{Q}_1 = \mathcal{P}_1 \cap \mathcal{P}_1^*$ and $\mathcal{Q}_2 = \mathcal{P}_2 \cap \mathcal{P}_2^*$ are inscribed to $\mathbb{S}^d$, so they correspond to ideal hyperbolic polytopes of $\mathbb{H}^{d+1}$ with finite volume. Moreover, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are the fundamental domains of the hyperbolic reflection groups generated by the reflections on their facets. Due to polarity, all the dihedral angles of the facets are right angles so these groups are isomorphic. Thus, by Mostow Rigidity Theorem, there is a hyperbolic isometry of $\mathbb{H}^{d+1}$ carrying $\mathcal{Q}_1$ to $\mathcal{Q}_2$ which can be extended to Möbius transformation of $\widehat{\mathbb{R}^d}$ carrying $\mathcal{B}_{\mathcal{P}_1}$ to $\mathcal{B}_{\mathcal{P}_2}$.

Möbius uniqueness of polytopal sphere packings implies several things. Firstly, by definition 2.1, a polytopal sphere packing $\mathcal{B}_\mathcal{P}$ does not depend on the realization but on the combinatorial type of $\mathcal{P}$. Secondly, the (symmetrized) Apollonian groups of two polytopal sphere packings induced by the same edge-scribable $(d+1)$-polytope for $d \geq 2$, are congruent in $\mathsf{M\ddot{o}b}(\widehat{\mathbb{R}^d})$. Therefore, the corresponding Apollonian arrangements are also Möbius unique. From the polytopal perspective, Möbius uniqueness is equivalent to say that the space of edge-scribable realizations of any edge-scribable polytope, under projective transformations preserving the sphere, is a point. We can also use the Möbius uniqueness to define an spectral invariant of edge-scribable d-polytopes for $d \geq 3$.

2.7.1. *The Möbius spectrum of an edge-scribable polytope.* For every $d \geq 3$, we define the *Möbius spectrum* $\mathfrak{M}(\mathcal{P})$ of an edge-scribable d-polytope $\mathcal{P}$ as the multiset of the eigenvalues of the Gramian of $\mathcal{B}_\mathcal{P}$. By combining the Möbius uniqueness of edge-scribable polytopes with the invariance of the inversive product under Möbius transformations, we have that $\mathfrak{M}(\mathcal{P})$ does not depend on the packing and therefore it is well-defined. In Section 3, we give the Möbius spectrum of each Platonic solid.

3. The polytopal Descartes' theorem

We present first some notions and lemmas needed for the proof of the main theorem.

3.1. **The canonical length of uniform polytopes.** It is clear that any regular $(d+1)$-polytope is *k-scribable* for every $0 \leq k \leq d$. With a little extra effort, it can be proved that uniform polytopes are inscribable, edge-scribable and, in general, non-circunscribable. Moreover, we have the following.

Lemma 3.1. *Every edge-scribed uniform polytope is canonical and its barycenter is the origin.*

Proof. Let $\mathcal{P}$ be a uniform $(d+1)$-polytope with $d \geq 2$. Let us first clarify that $\mathcal{P}$ admits a midsphere. The vertex-transitivity implies that every symmetry of $\mathcal{P}$ fixes the barycenter $\mathrm{bar}(\mathcal{P})$. It follows that the distance from each vertex to $\mathrm{bar}(\mathcal{P})$ is constant. Since every 2-face of $\mathcal{P}$ is regular, every edge of $\mathcal{P}$ has equal length. By combining these two facts, we get that for every $e \in E(\mathcal{P})$, $\|bar(e) - \mathrm{bar}(\mathcal{P})\|$ is constant and e is orthogonal to the line passing through $\mathrm{bar}(e)$ and $\mathrm{bar}(\mathcal{P})$. Thus, the sphere S centered at $\mathrm{bar}(\mathcal{P})$ and radius $\|\mathrm{bar}(e) - \mathrm{bar}(\mathcal{P})\|$ is tangent to every edge of $\mathcal{P}$ at the barycenters of the edges.

Let us now suppose that $\mathcal{P}$ is also edge-scribed, i.e. $S = \mathbb{S}^d$ and hence, $\mathrm{bar}(\mathcal{P})$ is the origin $\overrightarrow{0} \in \mathbb{R}^{d+1}$. On the other hand, the vertex-transitivity implies also that every vertex has same degree $\delta \geq d+1$, and

therefore $2|E(\mathcal{P})| = \delta|V(\mathcal{P})|$. We conclude by showing that $\mathcal{P}$ is canonical.

$$\sum_{x\in\mathcal{P}\cap\mathbb{S}^d} x = \sum_{e\in E(\mathcal{P})} \mathrm{bar}(e) = \frac{1}{|E(\mathcal{P})|} \sum_{u,v\in e\in E(\mathcal{P})} \frac{1}{2}(u+v)$$
$$= \frac{1}{2|E(\mathcal{P})|} \sum_{v\in V(\mathcal{P})} \deg(v)v = \frac{\delta}{\delta|V(\mathcal{P})|} \sum_{v\in V(\mathcal{P})} v = \mathrm{bar}(\mathcal{P}) = \overrightarrow{0}$$

□

As it is described in the above proof, uniform polytopes have equal edge length. We define the *canonical length* $\ell_{\mathcal{P}}$ of every uniform polytope $\mathcal{P}$ as the half edge length of its canonical realization.

3.2. **The polytopal curvatures of a polytopal sphere packing.** Let $d \geq 1$ and let $\mathcal{P}$ be an edge-scribed $(d+1)$-polytope. We define the *Lorentzian barycenter* of a face $f \in \mathcal{F}(\mathcal{P})$ as the vector $\mathbf{x}_f \in \mathbb{R}^{d+1,1}$ given by

$$\mathbf{x}_f := \frac{1}{|V(f)|} \sum_{v\in V(f)} \mathbf{x}_{s_v} \tag{8}$$

where s_v is the sphere corresponding to v.

Lemma 3.2. *Let $\mathcal{P}$ be a uniform edge-scribed $(d+1)$-polytope with $d \geq 2$. Then, for every $f \in \mathcal{F}(\mathcal{P})$ we have that $\langle \mathbf{x}_f, \mathbf{x}_{\mathcal{P}} \rangle = -\ell_{\mathcal{P}}^{-2}$.*

Proof. By Lemma 3.1, $\mathcal{P}$ is canonical and the barycenter of $\mathcal{P}$ is the origin of $\mathbb{R}^{d+1}$. Thus, the Euclidean norm of any vertex v of $\mathcal{P}$ and the canonical length of $\mathcal{P}$ are related by $\|v\|^2 = \ell_{\mathcal{P}}^2 + 1$. Therefore, by Equation (5), the Lorentzian vectors of the vertices of $\mathcal{P}$ are contained in the affine hyperplane $\Pi_{\ell_{\mathcal{P}}^{-1}} = \{x_{d+2} = \ell_{\mathcal{P}}^{-1}\} \subset \mathbb{R}^{d+1,1}$. Therefore, for every face f of $\mathcal{P}$, its Lorentzian barycenter $\mathbf{x}_f \in \Pi_{\ell_{\mathcal{P}}^{-1}}$. Since $\mathrm{bar}(\mathcal{P})$ is the origin of $\mathbb{R}^{d+1}$, then $\mathbf{x}_{\mathcal{P}} = \ell_{\mathcal{P}}^{-1} e_{d+2} \in \Pi^{\perp}_{\ell_{\mathcal{P}}^{-1}}$ and thus,

$$\langle \mathbf{x}_f - \mathbf{x}_{\mathcal{P}}, \mathbf{x}_{\mathcal{P}} \rangle = 0 \Leftrightarrow \langle \mathbf{x}_f, \mathbf{x}_{\mathcal{P}} \rangle = \langle \mathbf{x}_{\mathcal{P}}, \mathbf{x}_{\mathcal{P}} \rangle = -\ell_{\mathcal{P}}^{-2}.$$

□

Boyd's generalization of Descartes' theorem given in [Boy74] states that for any arrangement $\mathcal{B} = \{s_1, \ldots, s_{d+2}\} \subset \mathscr{B}(\widehat{\mathbb{R}^d})$ with full-rank Gramian, the vector of curvatures $\mathbf{K} = (\kappa_1, \ldots, \kappa_{d+2})^T$ satisfy

$$\mathbf{K}^T \,\mathrm{Gram}(\mathcal{B})^{-1} \mathbf{K} = 0 \tag{9}$$

We may consider (9) from the Lorentzian perspective in order to obtain a stronger statement. Let $\mathbf{x}_N = e_{d+1} + e_{d+2}$ be the vector given by the North Pole in the projective model of the space of spheres, where e_i denotes the i-th canonical vector of $\mathbb{R}^{d+1,1}$ (see Figure 4). Notice that $\mathbf{x}_N$ lies on the *light-cone* $\mathbb{L}(\mathbb{R}^{d+1,1})$, i.e.

$$\langle \mathbf{x}_N, \mathbf{x}_N \rangle = 0 \tag{10}$$

Let $s \in \mathscr{B}(\widehat{\mathbb{R}^d})$ of curvature κ_s and let $\mathbf{x}_s$ be its Lorentzian vector. From the definition of inversive coordinates given in (3), we have

$$\kappa_s = -\langle \mathbf{x}_N, \mathbf{x}_s \rangle \tag{11}$$

It follows from this equation that the linear transformation which maps $(\mathbf{x}_{s_1}, \ldots, \mathbf{x}_{s_{d+2}})$ to $(\kappa_{s_1}, \ldots, \kappa_{s_{d+2}})$ translates the equation (10) into (9). In other words, the statement of the generalized Descartes' theorem is equivalent to say that the vector of curvatures can be obtained from the North Pole by basis exchange, and the Lorentzian norm of this vector is 0. In order to extend the relation (9) to any basis of $\mathbb{R}^{d+1,1}$, we need to define a notion of *curvature* to any vector $\mathbf{x} \in \mathbb{R}^{d+1,1}$ by

$$\kappa_{\mathbf{x}} := -\langle \mathbf{x}_N, \mathbf{x} \rangle \tag{12}$$

We thus clearly have that equation (9) holds not only for basis of vectors of $\mathbb{R}^{d+1,1}$ corresponding to spheres, but for any basis.

Lemma 3.3. *For any basis $\Delta = (\mathbf{x}_1, \ldots, \mathbf{x}_{d+2})$ of $\mathbb{R}^{d+1,1}$ the vector of curvatures $\mathbf{K} = (\kappa_{\mathbf{x}_1}, \ldots, \kappa_{\mathbf{x}_{d+2}})^T$ satisfies*

$$\mathbf{K}^T \,\mathrm{Gram}(\Delta)^{-1} \mathbf{K} = 0 \tag{13}$$

For every polytopal sphere packing $\mathcal{B}_\mathcal{P}$, we define the *polytopal curvature* of $\mathcal{B}_\mathcal{P}$ with respect to a face f of $\mathcal{P}$ as $\kappa_f := \kappa_{\mathbf{x}_f}$ where $\mathbf{x}_f$ denotes the Lorentzian barycenter of f. Notice that if f is a vertex $v \in \mathcal{P}$, then $\kappa_v = \kappa_{s_v}$. By linearity, we have that κ_f corresponds to the following arithmetic mean.

$$\kappa_f := \frac{1}{|V(f)|} \sum_{v \in V(f)} \kappa_{s_v} \tag{14}$$

3.3. **Proof of the main theorem.** We have now all the ingredients to prove the polytopal Descartes' theorem that we restate below.

Theorem 3.1. *Let $\mathcal{B}_\mathcal{P}$ a polytopal sphere packing where $\mathcal{P}$ is a uniform $(d+1)$-polytope with $d \geq 2$. The polytopal curvatures of $\mathcal{B}_\mathcal{P}$ with respect to the faces in any flag $(f_0, \dots, f_d, f_{d+1} = \mathcal{P})$ satisfy*

$$(\kappa_{f_0} - \kappa_{f_1})^2 + \ell_{f_2}^2(\kappa_{f_1} - \kappa_{f_2})^2 + \sum_{i=2}^{d} \frac{1}{\ell_{f_{i+1}}^{-2} - \ell_{f_i}^{-2}}(\kappa_{f_i} - \kappa_{f_{i+1}})^2 = \ell_\mathcal{P}^2 \kappa_\mathcal{P}^2 \tag{15}$$

Proof. Let $\mathcal{P}$ be an edge-scribed uniform $(d+1)$-polytope with $d \geq 2$ and let $\Phi = (f_0, f_1, \dots, f_d, f_{d+1} = \mathcal{P})$ be a flag of $\mathcal{P}$. Let $\Delta = (\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_d, \mathbf{y}_{d+1})$, where $\mathbf{y}_i := \mathbf{x}_{f_i} - \mathbf{x}_{f_{i+1}}$ for every $i = 1, \dots, d$, and $\mathbf{y}_{d+1} := \mathbf{x}_\mathcal{P}$. Let us compute the Gramian of Δ. Let $v = f_0$, $e = f_1$ and let v' be the other vertex of e. Then,

$$\begin{aligned}
\langle \mathbf{y}_0, \mathbf{y}_0 \rangle &= \langle \mathbf{x}_v - \mathbf{x}_e, \mathbf{x}_v - \mathbf{x}_e \rangle \\
&= \langle \mathbf{x}_v, \mathbf{x}_v \rangle - 2\langle \mathbf{x}_v, \mathbf{x}_e \rangle + \langle \mathbf{x}_e, \mathbf{x}_e \rangle \\
&= 1 - 2\langle \mathbf{x}_v, \frac{1}{2}(\mathbf{x}_v + \mathbf{x}_{v'}) \rangle + \langle \frac{1}{2}(\mathbf{x}_v + \mathbf{x}_{v'}), \frac{1}{2}(\mathbf{x}_v + \mathbf{x}_{v'}) \rangle = 1 - (1 - 1) + 0 = 1 \\
\langle \mathbf{y}_0, \mathbf{y}_1 \rangle &= \langle \mathbf{x}_v, \mathbf{x}_v - \mathbf{x}_e \rangle \\
&= \langle \mathbf{x}_v, \frac{1}{2}(\mathbf{x}_{v'} - \mathbf{x}_v) \rangle \\
&= \frac{1}{2}(\langle \mathbf{x}_v, \mathbf{x}_{v'} \rangle - \langle \mathbf{x}_v, \mathbf{x}_v \rangle) = \frac{1}{2}(-1 + 1) = 0 \\
\langle \mathbf{y}_1, \mathbf{y}_1 \rangle &= \langle \mathbf{x}_e - \mathbf{x}_{f_2}, \mathbf{x}_e - \mathbf{x}_{f_2} \rangle \\
&= \langle \mathbf{x}_e, \mathbf{x}_e \rangle - 2\langle \mathbf{x}_e, \mathbf{x}_{f_2} \rangle + \langle \mathbf{x}_{f_2}, \mathbf{x}_{f_2} \rangle \\
&= 0 + 2\ell_{f_2}^{-2} - \ell_{f_2}^{-2} = \ell_{f_2}^{-2}
\end{aligned}$$

By definition, f_i is uniform for every $2 \leq i \leq d+1$. Moreover, the intersection of $\mathbb{S}^d$ with the affine subspace spanned by f_i induces an edge-scribed realization of f_i. Therefore, by Lemma 3.2, $\langle \mathbf{x}_{f_i}, \mathbf{x}_{f_j} \rangle = -\ell_{f_j}^{-2}$ for every $i \leq j \leq d+1$. Then, for the diagonal entries with $2 \leq i \leq d$, we have

$$\begin{aligned}
\langle \mathbf{y}_i, \mathbf{y}_i \rangle &= \langle \mathbf{x}_{f_i} - \mathbf{x}_{f_{i+1}}, \mathbf{x}_{f_i} - \mathbf{x}_{f_{i+1}} \rangle \\
&= \langle \mathbf{x}_{f_i}, \mathbf{x}_{f_i} \rangle - 2\langle \mathbf{x}_{f_i}, \mathbf{x}_{f_{i+1}} \rangle + \langle \mathbf{x}_{f_{i+1}}, \mathbf{x}_{f_{i+1}} \rangle \\
&= -\ell_{f_i}^{-2} + 2\ell_{f_{i+1}}^{-2} + -\ell_{f_{i+1}}^{-2} \\
&= \ell_{f_{i+1}}^{-2} - \ell_{f_i}^{-2} \\
\langle \mathbf{y}_{d+1}, \mathbf{y}_{d+1} \rangle &= \langle \mathbf{x}_\mathcal{P}, \mathbf{x}_\mathcal{P} \rangle = -\ell_\mathcal{P}^{-2}
\end{aligned}$$

and for the non-diagonal entries with $1 \leq i < j \leq d+1$, we have

$$\begin{aligned}
\langle \mathbf{y}_i, \mathbf{y}_j \rangle &= \langle \mathbf{x}_{f_i} - \mathbf{x}_{f_{i+1}}, \mathbf{x}_{f_j} - \mathbf{x}_{f_{j+1}} \rangle \\
&= \langle \mathbf{x}_{f_i}, \mathbf{x}_{f_j} \rangle - \langle \mathbf{x}_{f_i}, \mathbf{x}_{f_{j+1}} \rangle - \langle \mathbf{x}_{f_{i+1}}, \mathbf{x}_{f_j} \rangle + \langle \mathbf{x}_{f_{i+1}}, \mathbf{x}_{f_{j+1}} \rangle \\
&= -\ell_{f_j}^{-2} + \ell_{f_{j+1}}^{-2} + \ell_{f_j}^{-2} - \ell_{f_{j+1}}^{-2} = 0
\end{aligned}$$

Consequently, $\mathrm{Gram}(\Delta) = \mathrm{diag}(1, \ell_{f_2}^{-2}, \dots, \ell_{f_{i+1}}^{-2} - \ell_{f_i}^{-2}, \dots, -\ell_\mathcal{P}^{-2})$. On the other hand, by linearity, we have that the vector of curvatures of Δ is equal to $\mathbf{K} = (\kappa_{f_0} - \kappa_{f_1}, \dots, \kappa_{f_d} - \kappa_\mathcal{P}, \kappa_\mathcal{P})^T$. Since Δ is a basis of $\mathbb{R}^{d+1,1}$, we can apply Lemma 3.3 which gives us

$$\mathbf{K}^T \,\mathrm{diag}(1, \ell_{f_2}^2, \dots, \frac{1}{\ell_{f_{i+1}}^{-2} - \ell_{f_i}^{-2}}, \dots, -\ell_\mathcal{P}^2)\mathbf{K} = 0$$

which is equivalent to (15). □

4. The Platonic packings

In this section, we study the packings based on the simplest family of polytopes covered by the polytopal Descartes' theorem: the regular 3-polytopes, also known as the Platonic solids. By applying Theorem 3.1, we shall present a generalization of Descartes' theorem and the Apollonian group in terms of the *Schläfli symbol*, which will give us the recipe to construct analogues of integral Apollonian packings for the Platonic solids. At the end of the section, we present a catalogue summarizing all the properties discussed in this paper for each Platonic solid.

4.1. **Characteristic sequences.** Let $\mathcal{B}_\mathcal{P}$ a polytopal sphere packing in $\widehat{\mathbb{R}^d}$ where $\mathcal{P}$ regular $(d+1)$-polytope. For any flag $\Phi = (f_0, \dots, f_d, \mathcal{P})$, the set of curvatures of $\mathcal{B}_\mathcal{P}$ can be deduced from the polytopal curvatures of the faces of Φ. To do so, we shall define a canonical set of curvatures which will serve us as a basis. The *characteristic simplex* of $\mathcal{P}$ with respect to the flag Φ is the simplex

$$\Delta_\Phi := \operatorname{conv}(\operatorname{bar}(f_0), \dots, \operatorname{bar}(f_d), \operatorname{bar}(\mathcal{P}))$$

where $\operatorname{bar}(f)$ denotes the barycenter of the vertices of f (see Figure 9). When $\mathcal{P}$ is regular, the symmetry group of $\mathcal{P}$ coincide to the symmetry group of the characteristic simplex Δ_Φ. This group is generated by the reflections $\mathfrak{r}_0, \dots, \mathfrak{r}_d$ on the walls of Δ_Φ containing the barycenter of $\mathcal{P}$, where $\mathfrak{r}_i$ denotes the reflection on the wall opposite to the vertex of Δ_Φ corresponding to the face $f_i \in \Phi$. We define the *characteristic vertices* of $\mathcal{P}$ with respect to Φ as the subset of vertices $V_\Phi = (v_1, \dots, v_{d+2}) \subset V(\mathcal{P})$ given by

$$v_1 := f_0, \qquad \text{and} \qquad v_{i+1} := \mathfrak{r}_0 \cdots \mathfrak{r}_{i-1}(v_i) \tag{16}$$

subindices of $\mathfrak{r}_i$ $k = 1, ..., d+1$ corrected for every $i = 1, \dots, d+1$ (see Figure 9). A *characteristic bend vector* of $\mathcal{S}_\mathcal{P}$ is a vector $\mathbf{B} = (b_1, \dots, b_{d+2})$ with the bends of the spheres corresponding to a set of characteristic vertices of $\mathcal{P}$.

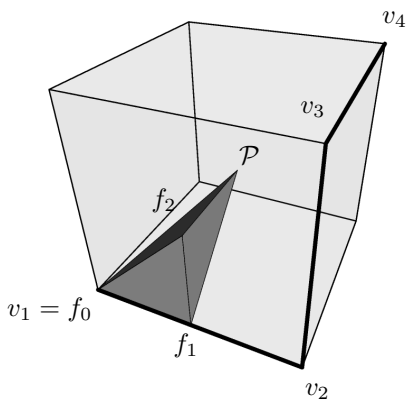

FIGURE 9. The characteristic simplex Δ_Φ (in dark gray) and the characteristic vertices $V_\Phi = (v_1, v_2, v_3, v_4)$ with respect to the flag $\Phi = (f_0, f_1, f_2, \mathcal{P})$ of a cube.

4.2. **The Platonic Descartes Theorem.** The classic Descartes' theorem is usually stated in terms of the *Descartes quadratic form* [Fuc13] whose corresponding matrix is

$$\mathbf{Q}_D := \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \tag{17}$$

The matrix $\mathbf{Q}_D$ derives from Boyd's generalization (9). For every Schläfli symbol $\{p, q\}$ of a Platonic solid, we define the *Platonic quadratic form* with corresponding matrix $\mathbf{Q}_{\{p,q\}} := \mathbf{Q}_D + \mathbf{P}_{\{p,q\}}$ where

$$\mathbf{P}_{\{p,q\}} := \begin{pmatrix} 0 & -\omega_p - \omega_q & \omega_p - \omega_q & 0 \\ -\omega_p - \omega_q & (\omega_p + \omega_q)(\omega_p + \omega_q + 2) & \omega_q^2 - \omega_p^2 & \omega_p - \omega_q \\ \omega_p - \omega_q & \omega_q^2 - \omega_p^2 & (\omega_p + \omega_q)(\omega_p + \omega_q + 2) & -\omega_p - \omega_q \\ 0 & \omega_p - \omega_q & -\omega_p - \omega_q & 0 \end{pmatrix}$$

and $\omega_n := 1 + 2\cos\frac{2\pi}{n}$. We notice that $\omega_3 = 0$, $\omega_4 = 1$ and $\omega_5 = \varphi$ is the Golden ratio.

Proposition 4.1. *Let* $\mathbf{K} = (\kappa_1,\ \kappa_2,\ \kappa_3,\ \kappa_4)^T$ *be a vector of four characteristic curvatures of a Platonic circle packing* $\mathcal{B}_\mathcal{P}$ *with* $\mathcal{P} = \{p, q\}$. *Then*

$$\mathbf{K}^T \mathbf{Q}_{\{p,q\}} \mathbf{K} = 0. \tag{18}$$

Proof. Let $\Phi = (v, e, f, \mathcal{P})$ be a flag of $\mathcal{P}$. Theorem 3.1 states that

$$(\kappa_v - \kappa_e)^2 + \ell_f^2(\kappa_e - \kappa_f)^2 + \frac{1}{\ell_\mathcal{P}^{-2} - \ell_f^{-2}}(\kappa_f - \kappa_\mathcal{P})^2 = \ell_\mathcal{P}^2 \kappa_\mathcal{P}^2 \tag{19}$$

where ℓ_f and $\ell_\mathcal{P}$ are the canonical length of $f = \{p\}$ and $\mathcal{P} = \{p, q\}$. These values can be easily computed in terms of ω_p and ω_q by

$$\ell_f = \sqrt{\frac{3 - \omega_p}{1 + \omega_p}} \qquad\qquad \ell_\mathcal{P} = \sqrt{\frac{2 - \omega_p - \omega_q}{1 + \omega_p}} \tag{20}$$

Let $V_\Phi = (v_1, v_2, v_3, v_4)$ and let $\Phi' = (v', e', f', \mathcal{P})$ be the flag of $\mathcal{P}$ where $V_{\Phi'} = (v_0, v_1, v_2, v_3)$ ($v_0 = v_4$ when $\mathcal{P}$ is the tetrahedron). Then, e and e' share the vertex v, and f and f' share the edge e. By solving (19) in each of the polytopal curvatures κ_e, κ_f, after replacing the values in (20), and then adding both solutions, we obtain the following relations.

$$\kappa_e + \kappa_{e'} = \frac{(1 + \omega_p)\kappa_v + (3 - \omega_p)\kappa_f}{2} \tag{21}$$

$$\kappa_f + \kappa_{f'} = 2\frac{(1 + \omega_q)\kappa_e + (2 - \omega_p - \omega_q)\kappa_\mathcal{P}}{3 - \omega_p} \tag{22}$$

Let $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ be the characteristic curvatures with respect to V_Φ. By combining the equations (21) and (22) with the definition of polytopal curvatures of the faces in $\Phi \cup \Phi'$, we obtain the following equations

$$\kappa_v = \kappa_1 \tag{23}$$

$$\kappa_e = \frac{\kappa_1 + \kappa_2}{2} \tag{24}$$

$$\kappa_f = \frac{\kappa_1 + \kappa_2 + \kappa_3 - \omega_p \kappa_2}{3 - \omega_p} \tag{25}$$

$$\kappa_\mathcal{P} = \frac{\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 - (\omega_p + \omega_q)(\kappa_2 + \kappa_3)}{4 - 2(\omega_p + \omega_q)} \tag{26}$$

The above relations define a transition matrix $\mathbf{T}$ satisfying

$$\mathbf{K}_\Phi = \mathbf{T}\mathbf{K} \tag{27}$$

where $\mathbf{K}_\Phi = (\kappa_v, \kappa_e, \kappa_f, \kappa_\mathcal{P})^T$ and $\mathbf{K} = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)^T$. Let $\mathbf{Q}_\Phi$ the matrix of the quadratic form induced by (19) after combining with (20). Then, equation (19) becomes

$$\mathbf{K}_\Phi^T \mathbf{Q}_\Phi \mathbf{K}_\Phi = 0 \Leftrightarrow \mathbf{K}^T \mathbf{T}^T \mathbf{Q}_\Phi \mathbf{T}\mathbf{K} = 0. \tag{28}$$

It can be checked by direct computations that $\mathbf{Q}_{\{p,q\}} = 4(1 + \omega_p + \omega_q + \omega_p\omega_q)\mathbf{T}^T \mathbf{Q}_\Phi \mathbf{T}$. □

4.3. **Matrix representations and integrality conditions.** Polytopal Apollonian groups are discrete subgroups of the Möbius group. The classic linear representation of tetrehadral Apollonian group introduced by Hirst in [Hir67], and studied for the first time from the number theoretical point of view by Graham et al. in [Gra+05], is defined as the discrete group $\langle \mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_4 \rangle < O_D(\mathbb{Z})$ where $O_D(\mathbb{Z})$ is the group of orthogonal matrices with entries in $\mathbb{Z}$ preserving the Descartes quadratic form. The generating matrices, called *bend matrices* in [CCS20], do not depend on the packing and gives the linear relations on the curvatures under the action of the Apollonian group. We shall give a similar representation of for each Platonic solid $\mathcal{P}$. To do so, we first give a linear representation of the symmetrized Apollonian group $\langle \mathbf{R}_0, \mathbf{R}_1, \mathbf{R}_2, \mathbf{S} \rangle < O_{\{p,q\}}(\mathbb{Z}[\omega_p, \omega_q])$ where $O_{\{p,q\}}(\mathbb{Z}[\omega_p, \omega_q])$ is the group of orthogonal matrices which preserve the Platonic quadratic form with entries in $\mathbb{Z}[\omega_p, \omega_q]$. The action by conjugation of $\langle \mathbf{R}_0, \mathbf{R}_1, \mathbf{R}_2 \rangle$ on $\mathbf{S}$ gives the finite set of bend matrices $\{\mathbf{S}_f \mid f \in F_2(\mathcal{P})\}$ which generate the Apollonian group of $\mathcal{P}$. We denote by $\mathscr{K}(\mathcal{B}_\mathcal{P})$ the set of curvatures of the Apollonian packing $\mathscr{P}(\mathcal{B}_\mathcal{P})$.

Proposition 4.2. *Let $\mathcal{B}_\mathcal{P}$ a Platonic circle packing where $\mathcal{P} = \{p, q\}$. The symmetrized Apollonian group of $\mathcal{B}_\mathcal{P}$ acts transitively on $\mathscr{K}(\mathcal{B}_\mathcal{P})$ as the discrete subgroup $\langle \mathbf{R}_0, \mathbf{R}_1, \mathbf{R}_2, \mathbf{S}\rangle < O_{\{p,q\}}(\mathbb{Z}[\omega_p, \omega_q])$ where*

$$\mathbf{R}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \omega_p & -\omega_p & 1 & 0 \\ \omega_p(\omega_p+\omega_q)+\omega_q & -\omega_p(\omega_p+\omega_q)-\omega_q & 0 & 1 \end{pmatrix} \tag{29}$$

$$\mathbf{R}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \omega_p & -\omega_p & 1 & 0 \\ \omega_p(-1+\omega_p) & 1-\omega_p^2 & \omega_p & 0 \\ \omega_p^2(-1+\omega_p+\omega_q) & -\omega_p(1+\omega_p)(-1+\omega_p+\omega_q) & \omega_p(-1+\omega_p+\omega_q) & 1 \end{pmatrix} \tag{30}$$

$$\mathbf{R}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \omega_q & \omega_p & -(\omega_p+\omega_q) & 1 \\ \omega_q(-1+\omega_p+\omega_q) & \omega_p(-1+\omega_p+\omega_q) & 1-(\omega_p+\omega_q)^2 & \omega_p+\omega_q \end{pmatrix} \tag{31}$$

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2(1-\omega_p+\omega_q) & 2(1+\omega_p+\omega_q) & -1 \end{pmatrix} \tag{32}$$

Proof. Let $\Phi = (v, e, f, \mathcal{P})$ be a flag of $\mathcal{P}$ and let $\{\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2\}$ the fundamental generators of the symmetry group of $\mathcal{P}$ with respect to Φ. Let $v' := \mathfrak{r}_0(v)$, $e' := \mathfrak{r}_1(e)$ and $f' := \mathfrak{r}_2(f)$. Then,

$$\mathfrak{r}_0(v,e,f,\mathcal{P}) = (v',e,f,\mathcal{P}), \qquad \mathfrak{r}_1(v,e,f,\mathcal{P}) = (v,e',f,\mathcal{P}), \qquad \mathfrak{r}_2(v,e,f,\mathcal{P}) = (v,e,f',\mathcal{P}).$$

From equations (21) and (22), we have that $\mathfrak{r}_0$, $\mathfrak{r}_1$ and $\mathfrak{r}_2$ correspond to the matrices

$$\mathbf{R}_0' \begin{pmatrix} \kappa_v \\ \kappa_e \\ \kappa_f \\ \kappa_\mathcal{P} \end{pmatrix} = \begin{pmatrix} 2\kappa_v - \kappa_e \\ \kappa_e \\ \kappa_f \\ \kappa_\mathcal{P} \end{pmatrix} \tag{33}$$

$$\mathbf{R}_1' \begin{pmatrix} \kappa_v \\ \kappa_e \\ \kappa_f \\ \kappa_\mathcal{P} \end{pmatrix} = \begin{pmatrix} \kappa_v \\ \frac{1+\omega_p}{2}\kappa_v - \kappa_e + \frac{3-\omega_p}{2}\kappa_f \\ \kappa_f \\ \kappa_\mathcal{P} \end{pmatrix} \tag{34}$$

$$\mathbf{R}_2' \begin{pmatrix} \kappa_v \\ \kappa_e \\ \kappa_f \\ \kappa_\mathcal{P} \end{pmatrix} = \begin{pmatrix} \kappa_v \\ \kappa_e \\ \frac{2(1+\omega_q)}{3-\omega_p}\kappa_e - \kappa_f + \frac{2(2-\omega_p-\omega_q)}{3-\omega_p}\kappa_\mathcal{P} \\ \kappa_\mathcal{P} \end{pmatrix} \tag{35}$$

By conjugating with the transition matrix $\mathbf{T}$ described in (27), we obtain the matrices $\mathbf{R}_0, \mathbf{R}_1, \mathbf{R}_2$. By resolving (18) on κ_4, we obtain

$$\kappa_4, \kappa_4' = \kappa_1 + (1-\omega_p+\omega_q)\kappa_2 + (1+\omega_p+\omega_q)\kappa_3 \pm 2\sqrt{(1+\omega_q)(\kappa_1\kappa_2+\kappa_1\kappa_3+\kappa_2\kappa_3-\omega_p\kappa_2^2)} \tag{36}$$

where κ_4' is the curvature of the sphere $\mathfrak{s}(s_{v_4})$ and $\mathfrak{s}$ is the inversion through the sphere corresponding to the face f. Therefore, since $\mathfrak{s}$ fixes the spheres corresponding to the vertices of f, then $\mathfrak{s}$ acts in $\mathscr{K}(\mathcal{B}_\mathcal{P})$ as the matrix

$$\mathbf{S} \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \\ \kappa_4 \end{pmatrix} = \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \\ 2\kappa_1 + 2(1-\omega_p+\omega_q)\kappa_2 + 2(1+\omega_p+\omega_q)\kappa_3 - \kappa_4 \end{pmatrix} \tag{37}$$

The elements $\{\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{s}\}$ generate the symmetrized Apollonian group of $\mathcal{B}_\mathcal{P}$, which acts transitively on $\mathscr{K}(\mathcal{B}_\mathcal{P})$ since $\mathcal{P}$ is regular. It can be easily checked that the four matrices belong to $O_{\{p,q\}}(\mathbb{Z}[\omega_p, \omega_q])$. □

Corollary 4.1. *The Apollonian groups of the Platonic solids are discrete subgroups of $O_{\{p,q\}}(\mathbb{Z}[\omega_p, \omega_q])$.*

For determining all the curvatures in a Platonic Apollonian packing, only three characteristic curvatures are necessary. We denote by $\mathscr{K}_{\{p,q\}}(\kappa_1,\kappa_2,\kappa_3) := \mathscr{K}(\mathcal{B}_\mathcal{P})$ where $\mathcal{P}$ is the Platonic solid with Schläfli symbol $\{p,q\}$ and $\kappa_1,\kappa_2,\kappa_3$ are three characteristic curvatures of $\mathcal{B}_\mathcal{P}$.

Corollary 4.2 (Integrality condition)**.** *Let $\mathcal{B}_\mathcal{P}$ be a Platonic packing where $\mathcal{P}=\{p,q\}$. If $\mathcal{B}_\mathcal{P}$ has three curvatures $(\kappa_1,\kappa_2,\kappa_3)$ in a characteristic sequence satisfying*

$$\kappa_1,\kappa_2,\kappa_3,\sqrt{(1+\omega_q)(\kappa_1\kappa_2+\kappa_1\kappa_3+\kappa_2\kappa_3-\omega_p\kappa_2^2)} \tag{38}$$

then all the curvatures of the Apollonian packing $\mathscr{P}(\mathcal{B}_\mathcal{P})$ are in $\mathbb{Z}[\omega_p,\omega_q]$.

Proof. If κ_1, κ_2,κ_3 satisfy (38) then, by equation (36), $\kappa_4\in\mathbb{Z}[\omega_p,\omega_q]$. By Proposition 4.2, we have that $\mathscr{K}(\mathcal{B}_\mathcal{P}) = \langle \mathbf{R}_0,\mathbf{R}_1,\mathbf{R}_2,\mathbf{S}\rangle\cdot\{\kappa_1,\kappa_2,\kappa_3,\kappa_4\}\subset\mathbb{Z}[\omega_p,\omega_q]$. □

Corollary 4.3. *For every Platonic solid $\{p,q\}$, $\mathscr{K}_{\{p,q\}}(0,0,1)\subset\mathbb{Z}[\omega_p,\omega_q]$. Moreover, if $q=3$, then $\mathscr{K}_{p,3}(0,0,1)$ contains the sequence of perfect squares.*

Proof. The integrality follows from Corollary 4.2. From (36), if $\kappa_1=0,\kappa_2=0,\kappa_3=1$, then $\kappa_4=1+\omega_p+\omega_q$. Let $\mathbf{K}_0 := (0,0,1,1+\omega_p+\omega_q)^T$ and let, for every $n\geq 0$, $\mathbf{M}_n := \mathbf{R}_1(\mathbf{R}_2\mathbf{S})^n\mathbf{R}_1$ and $\mathbf{K}_n = (\kappa_1^{(n)},\kappa_2^{(n)},\kappa_3^{(n)},\kappa_4^{(n)})^T := (\mathbf{M}_n\mathbf{K}_0)^T$. By Proposition 4.2, $\mathbf{K}_n\subset\mathscr{K}(\mathcal{B}_\mathcal{P})$. It can be proved by induction on n that $\kappa_2^{(n)}=n^2(1+\omega_q)$. Therefore, for $q=3$, $\kappa_2^{(n)}=n^2$. □

The packings giving $\mathscr{K}_{\{p,q\}}(0,0,1)$ are illustred in Figures 12, 15, 18, 21 and 24. We can use the previous two corollaries to parametrize the triples satisfying the integrality condition. To do so, we must solve the Diophantine equation derived from (38).

$$(1+\omega_q)(xy+yz+xz-\omega_p y^2)=n^2 \tag{39}$$

for $n,x,y,z\in\mathbb{Z}[\omega_p,\omega_q]$. Since this equation is a homogeneous of degree 2, we can find all the solutions from the initial solution $n=0$, $x=0$, $y=0$, $z=1$ by classical methods.

Corollary 4.4. *For every Platonic solid $\{p,q\}$, the triples $(\kappa_1,\kappa_2,\kappa_3)\in\mathbb{Z}[\omega_p,\omega_q]^3$ satisfying (38) are given by*

$$\kappa_1 = k\frac{(1+\omega_q)(t_2+t_3)t_2}{d},\qquad \kappa_2 = k\frac{(1+\omega_q)(t_2+t_3)t_3}{d}\qquad \kappa_3 = k\frac{t_1^2-(1+\omega_q)(t_2-\omega_p t_3)t_3}{d} \tag{40}$$

where $t_1,t_2,t_3\in\mathbb{Z}[\omega_p,\omega_q]$ coprimes, $d=\gcd((1+\omega_q)(t_2+t_3)t_2,(1+\omega_q)(t_2+t_3)t_3,t_1^2-(1+\omega_q)(t_2-\omega_p t_3)t_3)$ and $k\in\mathbb{Z}[\omega_p,\omega_q]$ coprime with d.

We use the previous parametrization to find triples for generate the primitive Platonic Apollonian packings, i.e. the gcd of all the curvatures is 1, which are depicted in Figures 12, 15, 18, 21 and 24. This is given by setting $k=1$ in the previous parametrization. We notice that different triples can give the same packing, so we cannot use it for counting the different primitive packings. For enumerating methods, see [Gra+03].

4.4. **Linear relations on the curvatures.** The set of curvatures of a Platonic packing $\mathcal{B}_\mathcal{P}$ needs four characteristic curvatures $\mathbf{K}=(\kappa_1,\kappa_2,\kappa_3,\kappa_4)$ to be fully determined. Indeed, by Proposition 4.2, the set of curvatures of $\mathcal{B}_\mathcal{P}$ is obtained by multiplication of the matrices $\mathbf{R}_0$, $\mathbf{R}_1$, $\mathbf{R}_2$ on $\mathbf{K}$. If we also multiply by $\mathbf{S}$, then we generate all the curvatures in the Apollonian packing $\mathscr{P}(\mathcal{B}_\mathcal{P})$. Equivalently, by applying recursively the following linear relations, which derive from equations (25), (26), (36), we can obtain the curvatures of $\mathscr{P}(\mathcal{B}_\mathcal{P})$ from $\mathbf{K}$ without using the matrices.

- (*Face relation*) If $(\kappa_0,\kappa_1,\kappa_2,\kappa_3)$ correspond to four consecutive vertices in a face, then

$$\kappa_0 = \omega_p(\kappa_1-\kappa_2)+\kappa_3 \tag{41}$$

- (*Characteristic sequences*)If $(\kappa_1,\kappa_2,\kappa_3,\kappa_4)$ and $(\kappa_2,\kappa_3,\kappa_4,\kappa_5)$ are two characteristic sequences of curvatures of $\mathcal{B}_\mathcal{P}$ then

$$\kappa_1-\kappa_5 = (\omega_p+\omega_q)(\kappa_2-\kappa_4) \tag{42}$$

- (*Dual inversion*) If $(\kappa_1,\kappa_2,\kappa_3,\kappa_4)$ and $(\kappa_1,\kappa_2,\kappa_3,\kappa_4')$ are two characteristic sequences of $\mathcal{B}_\mathcal{P}$ and $\mathcal{B}'_\mathcal{P}$, respectively, where $\mathcal{B}'_\mathcal{P}$ is obtained from $\mathcal{B}_\mathcal{P}$ by the inversion through the dual circle orthogonal to the circles corresponding to $\kappa_1,\kappa_2,\kappa_3$, then

$$\kappa_4' = 2\kappa_1+2(1-\omega_p+\omega_q)\kappa_2+2(1+\omega_p+\omega_q)\kappa_3-\kappa_4 \tag{43}$$

4.5. **Tetrahedron** $(p = 3, q = 3)$**.** This is the classical case which has been extensively studied. We write it for the sake of completeness.

4.5.1. *Canonical realization and centered arrangement projections.* The canonical tetrahedron with the spherical illuminated regions of its vertices is represented in Figure 10. Its canonical length is $\ell_{\mathcal{T}^3} = \sqrt{2}$.

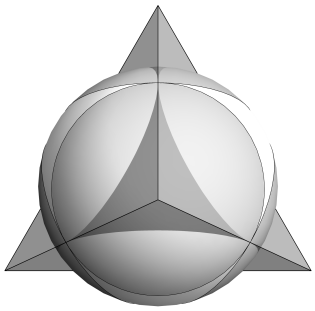

FIGURE 10. A canonical tetrahedron with the spherical illuminated regions.

In Figure 11, we show three tetrahedral packings obtained by the arrangement projections of the canonical tetrahedron with a vertex, edge and face, centered at the Pole North, respectively.

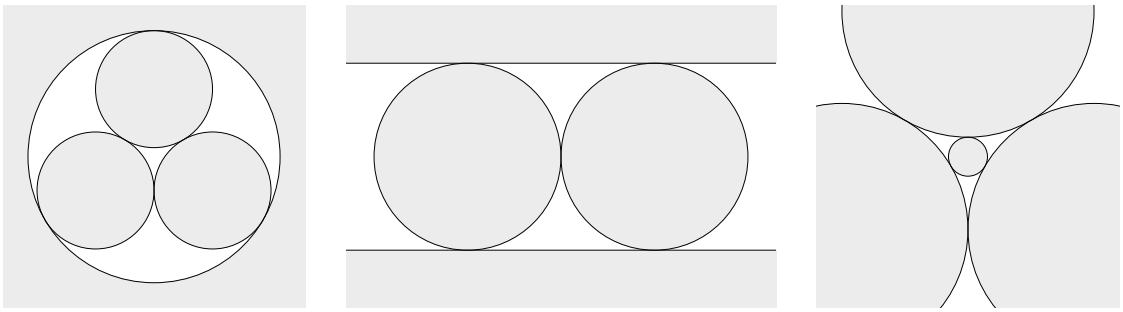

FIGURE 11. (From left to right) Vertex-centered, edge-centered and face-centered arrangement projection of the canonical tetrahedron.

4.5.2. *Möbius spectrum.* The Gramian is equal to the Descartes quadratic form, so the Möbius spectrum of the tetrahedron is is $\mathfrak{M}(\mathcal{T}^3) = (\lambda_0^{(1)}, \lambda_1^{(3)})$ where $\lambda_0 = -2$ and $\lambda_1 = 2$.

4.5.3. *Descartes' theorem.* The polytopal curvatures with respect to any flag $(v, e, f, \mathcal{T}^3)$ satisfy

$$(\kappa_v - \kappa_e)^2 + 3(\kappa_e - \kappa_f)^2 + 6(\kappa_f - \kappa_{\mathcal{T}^3})^2 = 2\kappa_{\mathcal{T}^3}^2 \tag{44}$$

The Platonic quadratic form for the tetrahedron is the same as the Descartes quadratic form

$$\mathbf{Q}_{\{3,3\}} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \tag{45}$$

which implies the classic Descartes' theorem, i.e. the curvatures $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ of any tetrahedral packing satisfies the quadratic equation

$$(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)^2 = 2(\kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2) \tag{46}$$

4.5.4. *Linear relation on the curvatures.* The curvatures of any tetrahedral packing $\mathcal{B}_{\mathcal{T}^3}$ satisfy the following *dual inversion* relation.

- If $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ and $(\kappa_1, \kappa_2, \kappa_3, \kappa_4')$ are the curvatures of $\mathcal{B}_{\mathcal{T}^3}$ and $\mathcal{B}'_{\mathcal{T}^3}$, respectively, where $\mathcal{B}'_{\mathcal{T}^3}$ is obtained from $\mathcal{B}_{\mathcal{T}^3}$ by the inversion through the dual circle orthogonal to the circles corresponding to $\kappa_1, \kappa_2, \kappa_3$, then

$$\kappa_4 + \kappa_4' = 2(\kappa_1 + \kappa_2 + \kappa_3) \tag{47}$$

4.5.5. *Apollonian group.* The following matrices generate the symmetrized Apollonian group of the tetrahedron.

$$\mathbf{R}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{R}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{R}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix}$$

The action by conjugation of $\langle \mathbf{R}_0, \mathbf{R}_1, \mathbf{R}_2 \rangle$ on $\mathbf{S}$ gives the following set of 4 matrices

$$\mathbf{S}_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{S}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{S}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{S}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix}$$

The discrete subgroup $\langle \mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_4 \rangle < O_{\{3,3\}}(\mathbb{Z}) = O_D(\mathbb{Z})$ is the classic linear representation of the tetrahedral Apollonian group.

4.5.6. *Primitive Apollonian packings.* If a triple of curvatures of a tetrahedral packing $\mathcal{B}_{\mathcal{T}^3}$ verify

$$\kappa_1, \kappa_2, \kappa_3, \sqrt{\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3} \in \mathbb{Z} \tag{48}$$

then the Apollonian packing $\mathscr{P}(\mathcal{B}_{\mathcal{T}^3})$ is integral. The primitive triples of curvatures satisfying the previous condition are parametrized by

$$\kappa_1 = \frac{t_2^2 + t_2 t_3}{d}, \qquad \kappa_2 = \frac{t_2 t_3 + t_3^2}{d}, \qquad \kappa_3 = \frac{t_1^2 - t_2 t_3}{d}, \tag{49}$$

where t_1, t_2, t_3 are three coprime integers and $d = \gcd\left(t_2^2 + t_2 t_3, t_2 t_3 + t_3^2, t_1^2 - t_2 t_3\right)$, In Figure 12, we show two examples given by this parametrization.

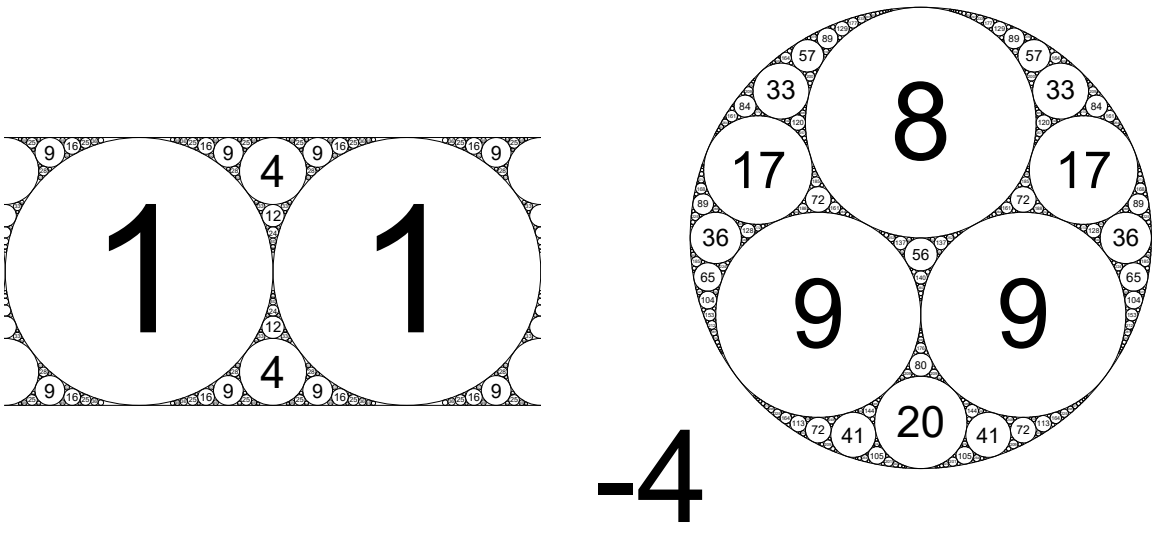

FIGURE 12. The tetrahedral Apollonian gaskets generated by $t_1 = 1$, $t_2 = 0$, $t_3 = 0 \Rightarrow (\kappa_1, \kappa_2, \kappa_3) = (0, 0, 1)$, and $t_1 = 1$, $t_2 = -2$, $t_3 = 4 \Rightarrow (\kappa_1, \kappa_2, \kappa_3) = (-4, 8, 9)$.

4.6. **Octahedron** $(p = 3, q = 4)$**.** Octahedral packings appear in the works of Boyd in [Boy74], Guettler and Mallows [GM08] and Zhang [Zha18].

4.6.1. *Canonical realization and centered arrangement projections.* A canonical octahedron with the spherical illuminated regions of its vertices is represented in Figure 13. Its canonical length is $\ell_{\mathcal{O}^3} = 1$.

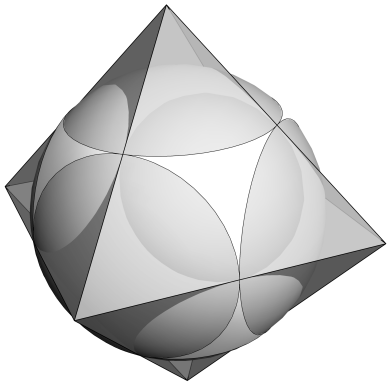

FIGURE 13. A canonical octahedron with the spherical illuminated regions.

In Figure 14, we show three octahedral packings obtained by centered arrangement projections of a canonical octahedron.

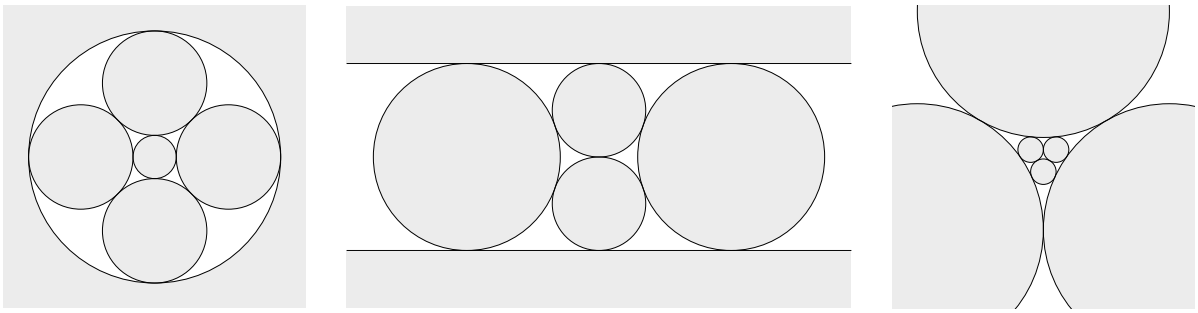

FIGURE 14. (From left to right) Vertex-centered, edge-centered and face-centered arrangement projection of the canonical octahedron.

4.6.2. *Möbius spectrum.* The Möbius spectrum of the octahedron is $\mathfrak{M}(\mathcal{O}^3) = (\lambda_0^{(1)}, \lambda_1^{(2)}, \lambda_2^{(3)})$ where $\lambda_0 = -4$, $\lambda_1 = 0$ and $\lambda_2 = 6$. It can be computed with the values of Table 1.

Graph distance	0	1	2
Inversive product	1	-1	-3

TABLE 1. The inversive product given by the distance in the graph for octahedral packings.

4.6.3. *Descartes' theorem.* The polytopal curvatures of an octahedral packing with respect to any flag $(v, e, f, \mathcal{O}^3)$ satisfy

$$(\kappa_v - \kappa_e)^2 + 3(\kappa_e - \kappa_f)^2 + \frac{3}{2}(\kappa_f - \kappa_{\mathcal{O}^3})^2 = \kappa_{\mathcal{O}^3}^2 \tag{50}$$

The Platonic quadratic form for the octahedron is given by the matrix

$$\mathbf{Q}_{\{3,4\}} = \begin{pmatrix} 1 & -2 & -2 & -1 \\ -2 & 4 & 0 & -2 \\ -2 & 0 & 4 & -2 \\ -1 & -2 & -2 & 1 \end{pmatrix} \tag{51}$$

which implies that any sequence of characteristic curvatures $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ of an octahedral packing satisfies the quadratic equation

$$(\kappa_1 - \kappa_4)^2 + (\kappa_1 - 2\kappa_2 + \kappa_4)^2 + (\kappa_1 - 2\kappa_3 + \kappa_4)^2 = 2(\kappa_1 + \kappa_4)^2 \tag{52}$$

which is equivalent to the quadratic equation given in [GM08].

4.6.4. *Linear relations on the curvatures.* The set of curvatures of any octahedral packing $\mathcal{B}_{\mathcal{O}^3}$ satisfy the following linear relations.

- (*Characteristic sequences*) If $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ and $(\kappa_2, \kappa_3, \kappa_4, \kappa_5)$ are two characteristic sequences of curvatures of $\mathcal{B}_{\mathcal{O}^3}$, then

$$\kappa_1 + \kappa_4 = \kappa_2 + \kappa_5 \tag{53}$$

  which is equivalent to the linear equation given in [GM08].
- (*Dual inversion*) If $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ and $(\kappa_1, \kappa_2, \kappa_3, \kappa_4')$ are two characteristic sequences of $\mathcal{B}_{\mathcal{O}^3}$ and $\mathcal{B}'_{\mathcal{O}^3}$, respectively, where $\mathcal{B}'_{\mathcal{O}^3}$ is obtained from $\mathcal{B}_{\mathcal{O}^3}$ by the inversion through the dual circle orthogonal to the circles corresponding to $\kappa_1, \kappa_2, \kappa_3$, then

$$\kappa_4 + \kappa_4' = 2(\kappa_1 + 2\kappa_2 + 2\kappa_3) \tag{54}$$

4.6.5. *Apollonian group.* The following matrices generate the symmetrized Apollonian group of the octahedron.

$$\mathbf{R}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \quad \mathbf{R}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{R}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 4 & 4 & -1 \end{pmatrix}$$

The action by conjugation of $\langle \mathbf{R}_0, \mathbf{R}_1, \mathbf{R}_2 \rangle$ on $\mathbf{S}$ gives the set of 8 matrices which generates the linear representation of the octahedral Apollonian group as a discrete subgroup of $O_{\{3,4\}}(\mathbb{Z})$.

4.6.6. *Primitive Apollonian packings.* If an octahedral packing $\mathcal{B}_{\mathcal{O}^3}$ has a characteristic sequence $(\kappa_1, \kappa_2, \kappa_3)$ satisfying

$$\kappa_1, \kappa_2, \kappa_3, \sqrt{2(\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3)} \in \mathbb{Z} \tag{55}$$

then the Apollonian packing $\mathscr{P}(\mathcal{B}_{\mathcal{O}^3})$ is integral. The primitive triples of curvatures satisfying the previous condition are parametrized by

$$\kappa_1 = \frac{2t_2^2 + 2t_2t_3}{d}, \qquad \kappa_2 = \frac{2t_2t_3 + 2t_3^2}{d}, \qquad \kappa_3 = \frac{t_1^2 - 2t_2t_3}{d}, \tag{56}$$

where t_1, t_2, t_3 are three coprime integers and $d = \gcd\left(2t_2^2 + 2t_2t_3, 2t_2t_3 + 2t_3^2, t_1^2 - 2t_2t_3\right)$. In Figure 15, we show two primitive octahedral Apollonian packings given by this parametrization.

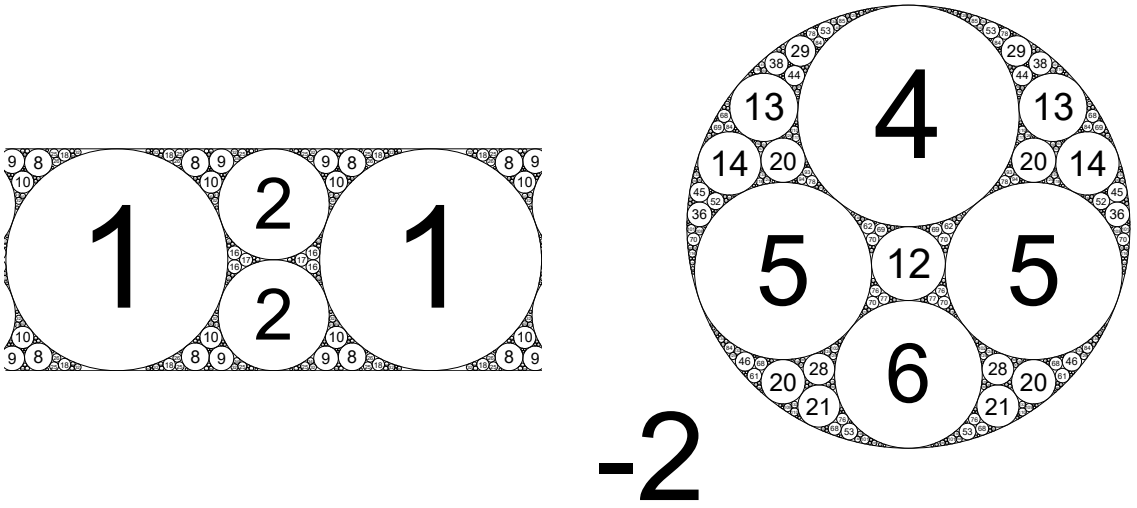

FIGURE 15. The octahedral Apollonian gaskets generated by $t_1 = 1$, $t_2 = 0$, $t_3 = 0 \Rightarrow (\kappa_1, \kappa_2, \kappa_3) = (0, 0, 1)$, and $t_1 = 1$, $t_2 = 1$, $t_3 = -2 \Rightarrow (\kappa_1, \kappa_2, \kappa_3) = (-2, 4, 5)$.

4.7. **Cube** $(p = 4, q = 3)$**.** Cubic packings were studied by Stange in [Sta15] as a particular case of Schmidt arrangements.

4.7.1. *Canonical realization and centered arrangement projections.* A canonical cube with the spherical illuminated regions of its vertices is represented in Figure 16. Its canonical length is $\ell_{\mathcal{C}^3} = \frac{1}{\sqrt{2}}$.

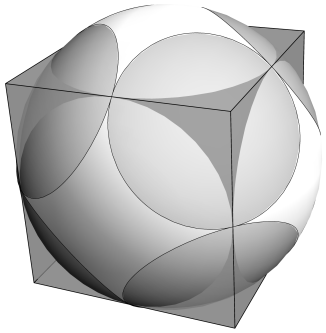

FIGURE 16. The canonical cube with the spherical illuminated regions.

In Figure 17, we show three cubic packings obtained by centered arrangement projections of a canonical cube.

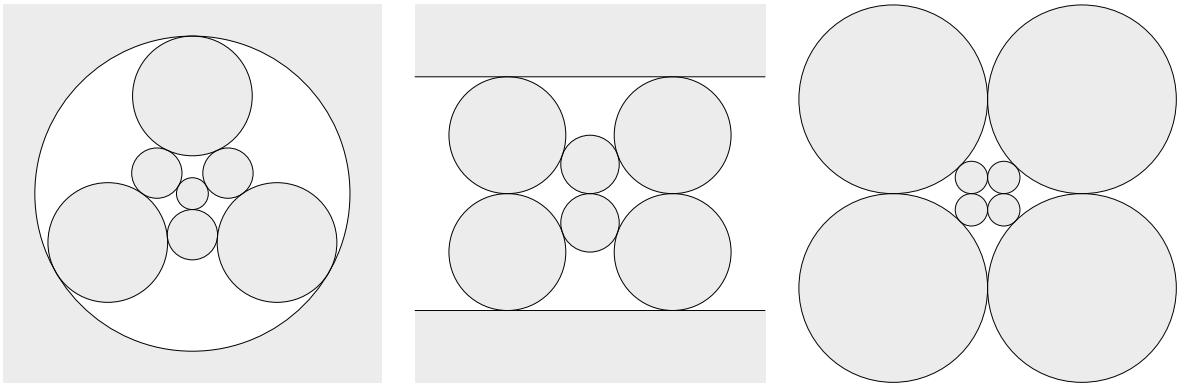

FIGURE 17. (From left to right) Vertex-centered, edge-centered and face-centered arrangement projection of the canonical cube.

4.7.2. *Möbius spectrum.* The Möbius spectrum of the cube is $\mathfrak{M}(\mathcal{C}^3) = (\lambda_0^{(1)}, \lambda_1^{(4)}, \lambda_2^{(3)})$ where $\lambda_0 = -16$, $\lambda_1 = 0$ and $\lambda_2 = 8$. It can be computed with the values of Table 2.

Graph distance	0	1	2	3
Inversive product	1	-1	-3	-5

TABLE 2. The inversive product given by the distance in the graph for cubic packings.

4.7.3. *Descartes' theorem.* The polytopal curvatures of a cubic packing with respect to any flag $(v, e, f, \mathcal{C}^3)$ satisfy

$$2(\kappa_v - \kappa_e)^2 + 2(\kappa_e - \kappa_f)^2 + 2(\kappa_f - \kappa_{\mathcal{C}^3})^2 = \kappa_{\mathcal{C}^3}^2 \tag{57}$$

The Platonic quadratic form for the cube is given by the matrix

$$\mathbf{Q}_{\{4,3\}} = \begin{pmatrix} 1 & -2 & 0 & -1 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ -1 & 0 & -2 & 1 \end{pmatrix} \tag{58}$$

which implies that any sequence of characteristic curvatures $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ of a cubic packing satisfy

$$2(\kappa_1 - \kappa_2)^2 + 2(\kappa_2 - \kappa_3)^2 + 2(\kappa_3 - \kappa_4)^2 = (\kappa_1 + \kappa_4)^2 \tag{59}$$

4.7.4. *Linear relations on the curvatures.* The set of curvatures of any cubic packing $\mathcal{B}_{\mathcal{C}^3}$ satisfy the following linear relations.

- (*Face relation*) If $(\kappa_0, \kappa_1, \kappa_2, \kappa_3)$ correspond to four consecutive vertices in a square face, then

$$\kappa_0 + \kappa_2 = \kappa_1 + \kappa_3 \tag{60}$$

- (*Characteristic sequences*) If $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ and $(\kappa_2, \kappa_3, \kappa_4, \kappa_5)$ are two characteristic sequences of curvatures of $\mathcal{B}_{\mathcal{C}^3}$, then

$$\kappa_1 + \kappa_4 = \kappa_2 + \kappa_5 \tag{61}$$

- (*Dual inversion*) If $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ and $(\kappa_1, \kappa_2, \kappa_3, \kappa_4')$ are two characteristic sequences of $\mathcal{B}_{\mathcal{C}^3}$ and $\mathcal{B}'_{\mathcal{C}^3}$, respectively, where $\mathcal{B}'_{\mathcal{C}^3}$ is obtained from $\mathcal{B}_{\mathcal{C}^3}$ by the inversion through the dual circle orthogonal to the circles corresponding to $\kappa_1, \kappa_2, \kappa_3$, then

$$\kappa_4 + \kappa_4' = 2(\kappa_1 + 2\kappa_3) \tag{62}$$

4.7.5. *Apollonian group.* The symmetrized Apollonian group of the cube is generated by

$$\mathbf{R}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \quad \mathbf{R}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{R}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 4 & -1 \end{pmatrix}$$

The action by conjugation of $\langle \mathbf{R}_0, \mathbf{R}_1, \mathbf{R}_2 \rangle$ on $\mathbf{S}$ gives a set of 6 matrices which generates the linear representation of the cubic Apollonian group as a discrete subgroup of $O_{\{4,3\}}(\mathbb{Z})$.

4.7.6. *Primitive Apollonian packings.* If a cubic packing $\mathcal{B}_{\mathcal{C}^3}$ has a characteristic sequence of curvatures $(\kappa_1, \kappa_2, \kappa_3)$ satisfying

$$\kappa_1, \kappa_2, \kappa_3, \sqrt{\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3 - \kappa_2^2} \in \mathbb{Z} \tag{63}$$

then the Apollonian packing $\mathscr{P}(\mathcal{B}_{\mathcal{C}^3})$ is integral. The primitive triples of curvatures satisfying the previous condition are parametrized by

$$\kappa_1 = \frac{t_2^2 + t_2 t_3}{d}, \qquad \kappa_2 = \frac{t_2 t_3 + t_3^2}{d}, \qquad \kappa_3 = \frac{t_1^2 - t_2 t_3 + t_3^2}{d}, \tag{64}$$

where t_1, t_2, t_3 are three coprime integers and $d = \gcd\left(t_2^2 + t_2 t_3, t_2 t_3 + t_3^2, t_1^2 - t_2 t_3 + t_3^2\right)$. In Figure 18, we show two primitive cubic Apollonian gaskets given by this parametrization.

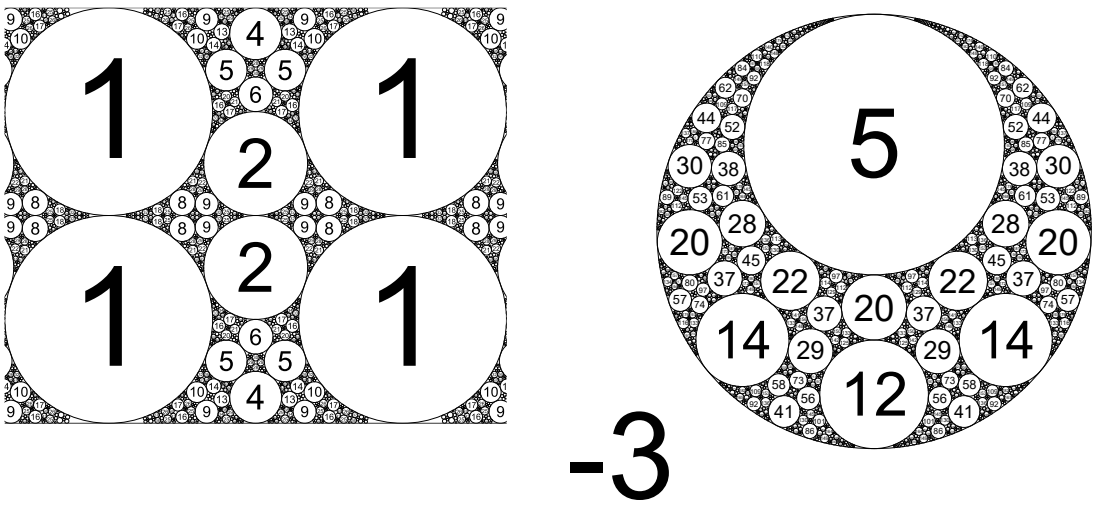

FIGURE 18. The cubic Apollonian packings generated by $t_1 = 1$, $t_2 = 0$, $t_3 = 0 \Rightarrow (\kappa_1, \kappa_2, \kappa_3) = (0, 0, 1)$, and $t_1 = 0$, $t_2 = 5$, $t_3 = -3 \Rightarrow (\kappa_1, \kappa_2, \kappa_3) = (5, -3, 12)$.

4.8. **Icosahedron** $(p = 3, q = 5)$**.** Icosahedral packings were studied by Bolt et al. as a particular case of *Apollonian ring packings* [BBH18]

4.8.1. *Canonical realization and centered arrangement projections.* A canonical icosahedron is represented in Figure 19. The canonical length of the icosahedron $\ell_{\mathcal{I}^3} = \varphi^{-1}$.

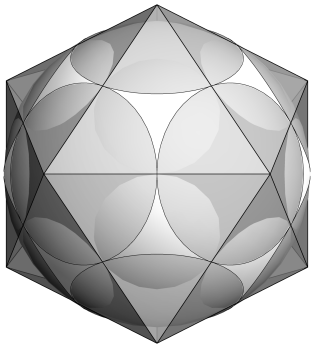

FIGURE 19. A canonical icosahedron with the spherical illuminated regions.

In Figure 20, we show three icosahedral packings obtained by centered arrangement projections of a canonical icosahedron.

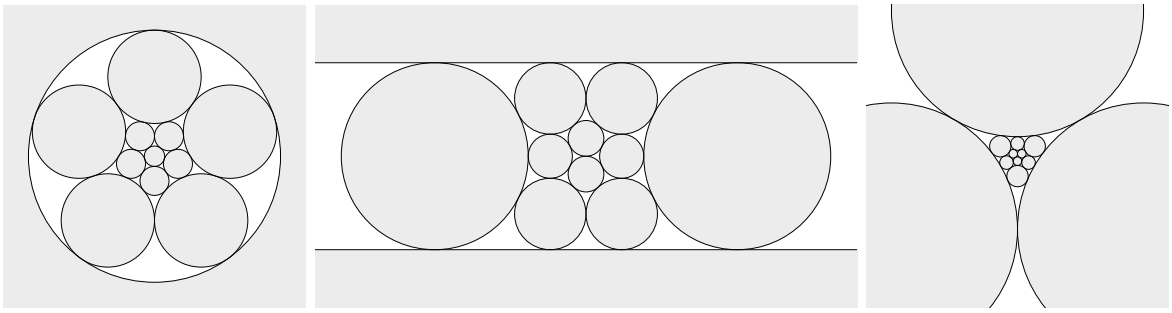

FIGURE 20. (From left to right) Vertex-centered, edge-centered and face-centered arrangement projection of the canonical icosahedron.

4.8.2. *Möbius spectrum.* The Möbius spectrum of the icosahedron is $\mathfrak{M}(\mathcal{I}^3) = (\lambda_0^{(1)}, \lambda_1^{(8)}, \lambda_2^{(3)})$ where $\lambda_0 = -12\varphi^2$, $\lambda_1 = 0$ and $\lambda_2 = 4(1 + \varphi^2)$. It can be computed from the values in Table 3.

Graph distance	0	1	2	3
Inversive product	1	-1	$1 - 2\varphi^2$	$-1 - 2\varphi^2$

TABLE 3. The inversive product given by the distance in the graph for icosahedral packings.

4.8.3. *Descartes' theorem.* The polytopal curvatures of an icosahedral packing with respect to any flag $(v, e, f, \mathcal{I}^3)$ satisfy

$$(\kappa_v - \kappa_e)^2 + 3(\kappa_e - \kappa_f)^2 + 3\varphi^{-4}(\kappa_f - \kappa_{\mathcal{I}^3})^2 = \varphi^{-2}\kappa_{\mathcal{I}^3}^2 \tag{65}$$

The Platonic quadratic form for the icosahedron is given by the matrix

$$\mathbf{Q}_{\{3,5\}} = \begin{pmatrix} 1 & -\varphi^2 & -\varphi^2 & -1 \\ -\varphi^2 & \varphi^4 & \varphi & -\varphi^2 \\ -\varphi^2 & \varphi & \varphi^4 & -\varphi^2 \\ -1 & -\varphi^2 & -\varphi^2 & 1 \end{pmatrix} \tag{66}$$

which implies that any sequence of characteristic curvatures $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ of an icosahedral packing satisfy

$$(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)^2 + \varphi^{-1}(\kappa_1 - \kappa_4)^2 = 2(\kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2) + \varphi(\kappa_2 + \kappa_3)^2 \tag{67}$$

4.8.4. *Linear relations on the curvatures.* The set of curvatures of any icosahedral packing $\mathcal{B}_{\mathcal{I}^3}$ satisfy the following linear relations.

- (*Characteristic sequences*) If $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ and $(\kappa_2, \kappa_3, \kappa_4, \kappa_5)$ are two characteristic sequences of curvatures of $\mathcal{B}_{\mathcal{I}^3}$, then

$$\kappa_1 - \kappa_5 = \varphi(\kappa_2 - \kappa_4) \tag{68}$$

- (*Dual inversion*) If $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ and $(\kappa_1, \kappa_2, \kappa_3, \kappa_4')$ are two characteristic sequences of $\mathcal{B}_{\mathcal{I}^3}$ and $\mathcal{B}'_{\mathcal{I}^3}$, respectively, where $\mathcal{B}'_{\mathcal{I}^3}$ is obtained from $\mathcal{B}_{\mathcal{I}^3}$ by the inversion through the dual circle orthogonal to the circles corresponding to $\kappa_1, \kappa_2, \kappa_3$, then

$$\kappa_4 + \kappa_4' = 2(\kappa_1 + \varphi^2\kappa_2 + \varphi^2\kappa_3) \tag{69}$$

4.8.5. *Apollonian group.* The symmetrized Apollonian group of the icosahedron is generated by

$$\mathbf{R}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \varphi & -\varphi & 0 & 1 \end{pmatrix} \quad \mathbf{R}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{R}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \varphi & 0 & -\varphi & 1 \\ 1 & 0 & -\varphi & \varphi \end{pmatrix} \quad \mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2\varphi^2 & 2\varphi^2 & -1 \end{pmatrix}$$

The action by conjugation of $\langle \mathbf{R}_0, \mathbf{R}_1, \mathbf{R}_2 \rangle$ on $\mathbf{S}$ gives a set of 20 matrices which generates the linear representation of the icosahedral Apollonian group as a discrete subgroup of $O_{\{3,5\}}(\mathbb{Z}[\varphi])$.

4.8.6. *Primitive Apollonian packings.* If an icosahedral packing $\mathcal{B}_{\mathcal{I}^3}$ has a characteristic sequence of curvatures $(\kappa_1, \kappa_2, \kappa_3)$ satisfying

$$\kappa_1, \kappa_2, \kappa_3, \sqrt{\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3} \in \mathbb{Z}[\varphi] \tag{70}$$

Error in the previous version then the curvatures of icosahedral Apollonian packing $\mathscr{P}(\mathcal{B}_{\mathcal{I}^3})$ are in $\mathbb{Z}[\varphi]$. The primitive triples of curvatures satisfying the previous condition are parametrized by

$$\kappa_1 = \frac{\varphi^2(t_2+t_3)t_2}{d}, \qquad \kappa_2 = \frac{\varphi^2(t_2+t_3)t_3}{d}, \qquad \kappa_3 = \frac{t_1^2 - \varphi^2 t_2 t_3}{d}, \tag{71}$$

where $t_1, t_2, t_3 \in \mathbb{Z}[\varphi]$ coprimes and $d = \gcd\left(\varphi^2(t_2+t_3)t_2, \varphi^2(t_2+t_3)t_3, t_1^2 - \varphi^2 t_2 t_3\right)$. In Figure 21, we show two primitive icosahedral Apollonian gasket given by this parametrization.

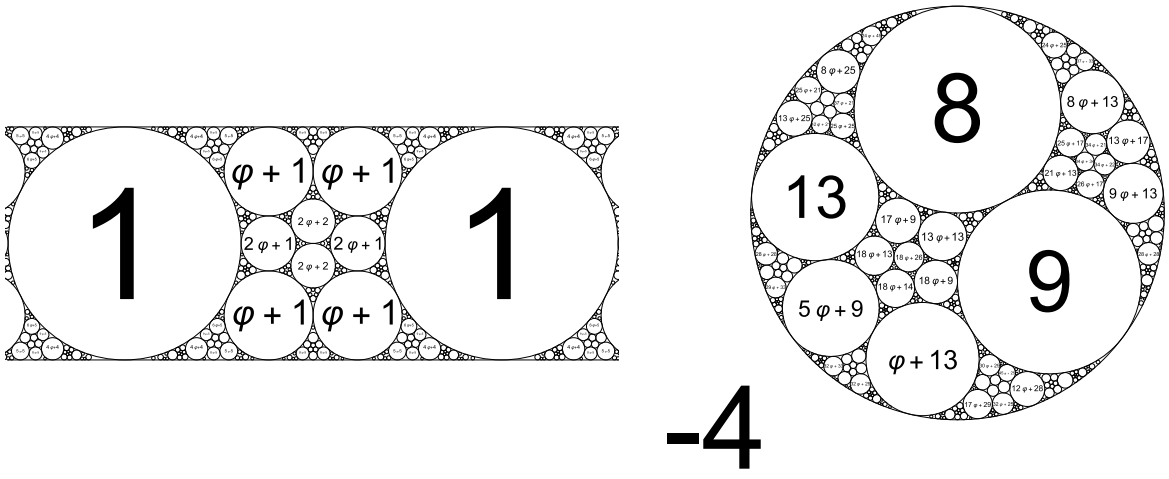

FIGURE 21. The icosahedral Apollonian packings generated by $t_1 = 1$, $t_2 = 0$, $t_3 = 0 \Rightarrow (\kappa_1, \kappa_2, \kappa_3) = (0, 0, 1)$, and $t_1 = 1$, $t_2 = -2 + 2\varphi$, $t_3 = 4 - 4\varphi \Rightarrow (\kappa_1, \kappa_2, \kappa_3) = (-4, 8, 9)$.

It follows from the integrality conditions of the tetrahedron (49) and the icosahedron (70), that any primitive triple of characteristic curvatures generates both a tetrahedral and an icosahedral Apollonian gaskets (see Figures 12 and 21).

4.9. **Dodecahedron** $(p = 5, q = 3)$**.**

4.9.1. *Canonical realization and centered arrangement projections.* A canonical dodecahedron is represented in Figure 22. Its canonical length is $\ell_{\mathcal{D}^3} = \varphi^{-2}$.

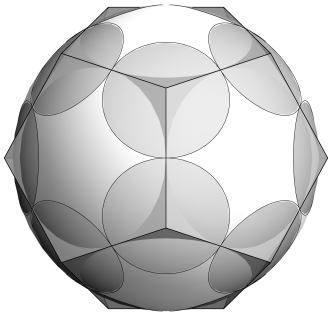

FIGURE 22. A canonical dodecahedron with the spherical illuminated regions.

In Figure 23, we show three dodecahedral packings obtained by centered arrangement projections of a canonical dodecahedron.

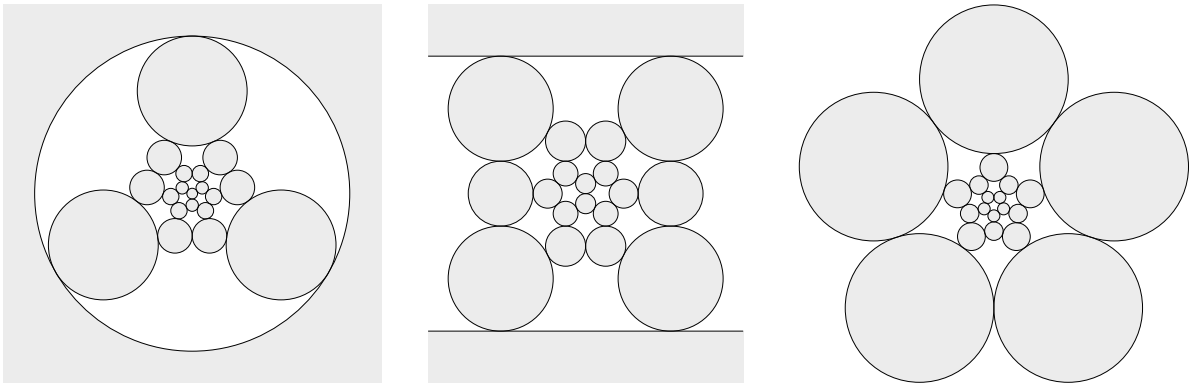

FIGURE 23. (From left to right) Vertex-centered, edge-centered and face-centered arrangement projection of the canonical dodecahedron.

4.9.2. *Möbius spectrum.* The Möbius spectrum of the dodecahedron is $\mathfrak{M}(\mathcal{D}^3) = (\lambda_0^{(1)}, \lambda_1^{(16)}, \lambda_2^{(3)})$ where $\lambda_0 = -20\varphi^4$, $\lambda_1 = 0$ and $\lambda_2 = 20\varphi^2$. It can be computed with the values in Table 4.

Graph distance	0	1	2	3	4	5
Inversive product	1	-1	$1-2\varphi^2$	$1-4\varphi^2$	$1-2\varphi^4$	$1-6\varphi^2$

TABLE 4. The inversive product compared to the graph-distance for dodecahedral packings.

4.9.3. *Descartes' theorem.* The polytopal curvatures of a dodecahedral circle packing with respect to any flag $(v, e, f, \mathcal{D}^3)$ satisfy

$$(\kappa_v - \kappa_e)^2 + (7-4\varphi)(\kappa_e - \kappa_f)^2 + (18-11\varphi)(\kappa_f - \kappa_{\mathcal{D}^3})^2 = (5-3\varphi)\kappa_{\mathcal{D}^3}^2 \tag{72}$$

The Platonic quadratic form for the dodecahedron is given by the matrix

$$\mathbf{Q}_{\{5,3\}} = \begin{pmatrix} 1 & -\varphi^2 & \varphi^{-1} & -1 \\ -\varphi^2 & \varphi^4 & -1-\varphi^2 & \varphi^{-1} \\ \varphi^{-1} & -1-\varphi^2 & \varphi^4 & -\varphi^2 \\ -1 & \varphi^{-1} & -\varphi^2 & 1 \end{pmatrix} \tag{73}$$

which implies that any characteristic curvatures $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ of a dodecahedral packing satisfy

$$\begin{aligned} \varphi^{-1}((\kappa_1+\kappa_3)^2 + (\kappa_2+\kappa_4)^2) + (\kappa_1-\kappa_4)^2 + (\kappa_2-\kappa_3)^2 \\ +\varphi^2\left((\kappa_1-\kappa_2)^2 + (\kappa_2-\kappa_3)^2 + (\kappa_3-\kappa_4)^2\right) = 2\varphi(\kappa_1+\kappa_4)^2 \end{aligned} \tag{74}$$

4.9.4. *Linear relations on the curvatures.* The set of curvatures of any dodecahedral packing $\mathcal{B}_{\mathcal{D}^3}$ satisfy the following linear relations.

- (*Face relation*) If $(\kappa_0, \kappa_1, \kappa_2, \kappa_3)$ correspond to four consecutive vertices in a pentagonal face, then

$$\kappa_0 - \kappa_3 = \varphi(\kappa_1 - \kappa_2) \tag{75}$$

- (*Characteristic sequences*) If $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ and $(\kappa_2, \kappa_3, \kappa_4, \kappa_5)$ are two characteristic sequences of curvatures of $\mathcal{B}_{\mathcal{D}^3}$, then

$$\kappa_1 - \kappa_5 = \varphi(\kappa_2 - \kappa_4) \tag{76}$$

- (*Dual inversion*) If $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ and $(\kappa_1, \kappa_2, \kappa_3, \kappa_4')$ are two characteristic sequences of $\mathcal{B}_{\mathcal{D}^3}$ and $\mathcal{B}'_{\mathcal{D}^3}$, respectively, where $\mathcal{B}'_{\mathcal{D}^3}$ is obtained from $\mathcal{B}_{\mathcal{D}^3}$ by the inversion through the dual circle orthogonal to the circles corresponding to $\kappa_1, \kappa_2, \kappa_3$, then

$$\kappa_4 + \kappa_4' = 2(\kappa_1 - \varphi^{-1}\kappa_2 + \varphi^2\kappa_3) \tag{77}$$

4.9.5. *Apollonian group.* The symmetrized Apollonian group of the dodecahedron is generated by

$$\mathbf{R}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \varphi & -\varphi & 1 & 0 \\ \varphi^2 & -\varphi^2 & 0 & 1 \end{pmatrix} \quad \mathbf{R}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \varphi & -\varphi & 1 & 0 \\ 1 & -\varphi & \varphi & 0 \\ \varphi & -\varphi^2 & 1 & 1 \end{pmatrix} \quad \mathbf{R}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \varphi & -\varphi & 1 \\ 0 & 1 & -\varphi & \varphi \end{pmatrix} \quad \mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2\varphi^{-1} & 2\varphi^2 & -1 \end{pmatrix}$$

The action by conjugation of $\langle \mathbf{R}_0, \mathbf{R}_1, \mathbf{R}_2 \rangle$ on $\mathbf{S}$ gives the set of 12 matrices which generates a linear representation of the dodecahedral Apollonian group as a discrete subgroup of $O_{\{5,3\}}(\mathbb{Z}[\varphi])$.

4.9.6. *Primitive Apollonian packings.* If a dodecahedral packing $\mathcal{B}_{\mathcal{D}^3}$ has a characteristic sequence of curvatures $(\kappa_1, \kappa_2, \kappa_3)$ satisfying

$$\kappa_1, \kappa_2, \kappa_3, \sqrt{\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3 - \varphi\kappa_2^2} \in \mathbb{Z}[\varphi] \tag{78}$$

then the curvatures of Apollonian packing $\mathscr{P}(\mathcal{B}_{\mathcal{D}^3})$ are in $\mathbb{Z}[\varphi]$. The primitive triples of curvatures satisfying the previous condition are parametrized by

$$\kappa_1 = \frac{t_2^2 + t_2 t_3}{d}, \qquad \kappa_2 = \frac{t_2 t_3 + t_3^2}{d}, \qquad \kappa_3 = \frac{t_1^2 - t_2 t_3 + \varphi t_3^2}{d}, \tag{79}$$

where $t_1, t_2, t_3 \in \mathbb{Z}[\varphi]$ coprimes and $d = \gcd\left(t_2^2 + t_2 t_3, t_2 t_3 + t_3^2, t_1^2 - t_2 t_3 + \varphi t_3^2\right)$. In Figure 21, we show two primitive dodecahedral Apollonian gaskets given by this parametrization.

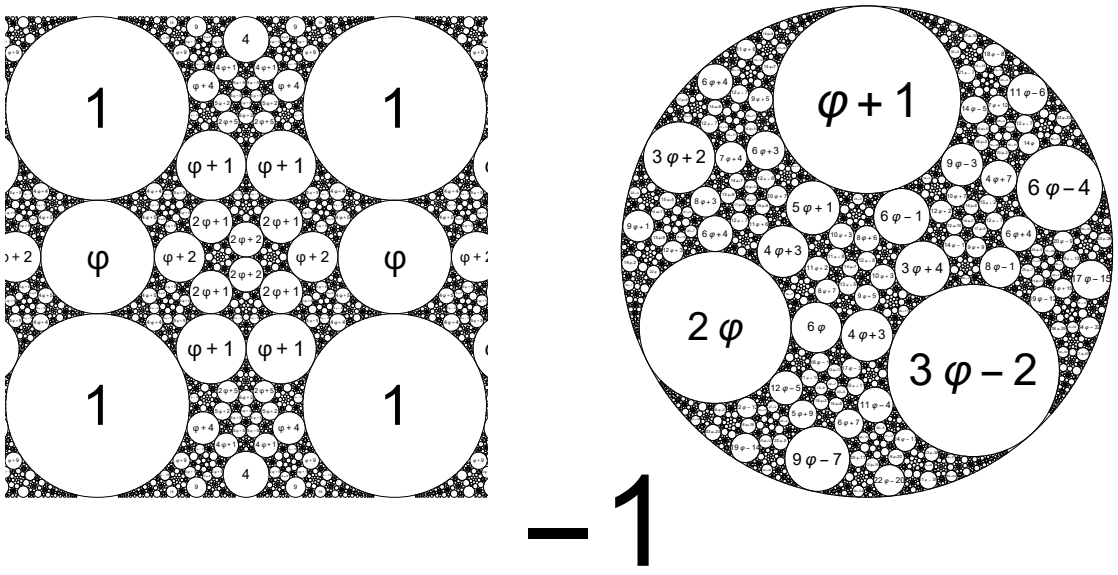

FIGURE 24. The dodecahedral Apollonian packings generated by $t_1 = 1$, $t_2 = 0$, $t_3 = 0 \Rightarrow (\kappa_1, \kappa_2, \kappa_3) = (0, 0, 1)$, and $t_1 = 1$, $t_2 = 1 + \varphi$, $t_3 = -1 \Rightarrow (\kappa_1, \kappa_2, \kappa_3) = (\varphi + 1, -1, 2\varphi)$.

5. Conclusions

5.1. **Integral polytopes.** We have explored the family of *polytopal sphere packings*, which are packings in $\widehat{\mathbb{R}^d}$ carrying a rich structure induced by an edge-scribable $(d+1)$-polytope, with $d \geq 2$. Specifically, the vertices, edges and facets of the polytope correspond to spheres, tangency relations, and dual mirrors of the packing, respectively. The action of the group generated by the dual mirrors on the packing gives an infinite arrangement of spheres which, in certain cases, results in an integral Apollonian packing. While it remains unknown which polytopes are *integral*[2], in the sense that they admit an integral Apollonian packing, significant progress has been made for polyhedra. Based on the work on *crystallographic sphere packings* due to Kontorovich and Nakamura [KN19], Chair, Cui and Stier gaved in [CCS20] the list of the only 10 integral uniform polyhedra, and defined the *glueing operations* on polyhedra preserving integrality. We wonder if a similar list and operations can be defined in higher dimensions.

5.2. **Polytopal Descartes' theorem.** Descartes's theorem is a central algebraic tool used to study integral Apollonian packings. In this paper, we present a Descartes' theorem for polytopal sphere packings that arise from uniform polytopes. The equation we obtain encompasses several generalizations of Descartes' theorem, and we can express it in terms of the geometry of the polytope. This enables us to obtain integrality conditions from the combinatorial information of the polytope without the coordinates of an initial packing, as we have done for the truncated tetrahedron depicted in Figure 3.

In the first version of this paper, the polytopal Descartes' theorem was restricted to regular polytopes exclusively. However, we later discovered that the same formula is also valid for uniform polytopes, and we are confident that it could be extended to other families, such as the *quasi-uniform polytopes*. These polytopes are obtained by replacing the vertex-transitive condition with vertex-congruence, which means that all vertex-figures are Euclidean congruent. In dimension 3, there is only one quasi-uniform polyhedron that is not uniform: the 37th Johnson solid [Grü+03]. We have checked that for this polyhedron the polytopal Descartes' theorem holds.

5.3. **Regular polytopes and beyond.** We have studied the polytopal Apollonian packings arising from Platonic solids as the simplest family of polytopes to which the polytopal Descartes' theorem applies. Analogue constructions for the regular polytopes in higher dimensions are investigated in [Ras21]. One of the advantages of the regular case, is that the polytopal Descartes' theorem yields a unique quadratic form for the polytopal curvatures with respect to the faces of any flag. In the case of non-regular uniform polytopes, there is a different quadratic form for each class of flags up to symmetry. We believe that it would be interesting to study the nature of the quadratic forms arising from uniform polytopes in every dimension.

5.4. **Möbius spectrum of polyhedral graphs.** Spectral techniques, which rely on the eigenvalues and eigenvectors of the adjacency or Laplace matrices of graphs, have proven to be powerful and effective tools for studying various graph properties. In this vein, we have defined an spectral invariant for edge-scribable d-polytopes, which we call the *Möbius spectrum*, for every $d \leq 3$. Steinitz's well-known theorem [Ste28] states that the graph of a polyhedron is a 3-connected simple planar graph, which is also known as a polyhedral graph. Given that all polyhedra are edge-scribable, the Möbius spectrum can be defined for any polyhedral graph. We are curious whether the Möbius spectrum serves as a complete invariant for edge-scribable polytopes, and particularly for polyhedral graphs.

Question 1. *For any $d \geq 3$, are there two combinatorially different edge-scribable d-polytopes with same Möbius spectrum? In particular, are there two non isomorphic polyhedral graphs with same Möbius spectrum?*

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