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## ► To cite this version:

Walter Boscheri, Raphaël Loubère, Jean-Philippe Braeunig, Pierre-Henri Maire. A geometrically and thermodynamically compatible finite volume scheme for continuum mechanics on unstructured polygonal meshes. *Journal of Computational Physics*, 2024, 507, pp.112957. 10.1016/j.jcp.2024.112957 . hal-04526320

**HAL Id: hal-04526320**

**<https://hal.science/hal-04526320>**

Submitted on 29 Mar 2024

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# A geometrically and thermodynamically compatible finite volume scheme for continuum mechanics on unstructured polygonal meshes

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## Abstract

We present a novel Finite Volume (FV) scheme on unstructured polygonal meshes that is provably compliant with the Second Law of Thermodynamics and the Geometric Conservation Law (GCL) at the same time. The governing equations are provided by a subset of the class of symmetric and hyperbolic thermodynamically compatible (SHTC) models introduced by Godunov in 1961. Specifically, our numerical method discretizes the equations for the conservation of momentum, total energy, distortion tensor and thermal impulse vector, hence accounting in one single unified mathematical formalism for a wide range of physical phenomena in continuum mechanics, spanning from ideal and viscous fluids to hyperelastic solids. By means of two conservative corrections directly embedded in the definition of the numerical fluxes, the new schemes are proven to satisfy two extra conservation laws, namely an entropy balance law and a geometric equation that links the distortion tensor to the density evolution. As such, the classical mass conservation equation can be discarded. Firstly, the GCL is derived at the continuous level, and subsequently it is satisfied by introducing the new concepts of general potential and generalized Gibbs relation. The new potential is nothing but the determinant of the distortion tensor, and the associated Gibbs relation is derived by introducing a set of dual or thermodynamic variables such that the GCL is retrieved by dot multiplying the original system with the new dual variables. Once compatibility of the GCL is ensured, thermodynamic compatibility is tackled in the same manner, thus achieving the satisfaction of a local cell entropy inequality. The two corrections are orthogonal, meaning that they can coexist simultaneously without interfering with each other. The compatibility of the new FV schemes holds true at the semi-discrete level, and time integration of the governing PDE is carried out relying on Runge-Kutta schemes. A large suite of test cases demonstrates the structure preserving properties of the schemes at the discrete level as well.

**Keywords:** exact preservation of determinant constraint, thermodynamically compatible finite volume schemes, entropy preserving, entropy stability, unstructured mesh, continuum mechanics

## 1. Introduction

Born in 1929, Godunov embarked on a remarkable career that spanned several decades until his death in July 2023. He made profound contributions to the development of numerical techniques for solving Partial Differential Equations (PDEs), particularly in the field of fluid dynamics. The celebrated Godunov theorem and Godunov scheme revolutionized Computational Fluid Dynamics (CFD), enabling scientists and engineers to simulate complex fluid flows. However, Godunov research developed beyond fluid dynamics, and in [31] he found a connection between symmetric hyperbolicity in the sense of Friedrichs [23] and thermodynamic compatibility. In this work of Godunov, one learns that for hyperbolic systems having an underlying variational formulation, the total energy conservation law

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19 can be derived as the dot product of the other equations with the so-called *thermodynamic dual variables* that are given  
20 by the partial derivative of the total energy potential with respect to the conservative variables of the system. These  
21 variables are also known as *main field*, or even *Godunov variables*, see for instance [22]. Later on, Godunov and  
22 Romenski [54, 33] extended the theory of symmetric hyperbolic and thermodynamically compatible (SHTC) systems  
23 to a wide class of mathematical models: magnetohydrodynamics [32], nonlinear hyperelasticity [28], compressible  
24 multi-phase flows [53, 51] as well as relativistic fluid and solid mechanics [27, 52]. Within this theory the total  
25 energy potential plays a crucial role which is coming from the variational principle from which the system is derived.  
26 Moreover, the entropy density equation is part of the master system, while the total energy conservation law is an extra  
27 conservation law, since it is obtained by a linear combination of the other conservation equations. Due to the very  
28 general formalism, SHTC systems have been derived for modeling a wide range of physical phenomena that cover  
29 magnetohydrodynamics [29], nonlinear hyperelasticity [34], compressible multi-phase flows [55] and even relativistic  
30 fluid and solid mechanics [30, 56].

31 The SHTC models are therefore compliant with the Second Law of Thermodynamics by construction, and they are  
32 derived as first order hyperbolic systems, where the stress tensor is a function of the inverse deformation gradient  $\mathbf{A}$   
33 rather than velocity gradients, even for fluids. Indeed, irreversible dissipative processes are accounted by the presence  
34 of source terms with one or more characteristic strain relaxation times  $\tau$ . The hyperbolicity of SHTC models implies  
35 finite wave speeds for all involved physical processes, even dissipative ones, thus making their mathematical structure  
36 substantially different from those PDE systems which admit parabolic dissipation and diffusion terms. Indeed, in [49],  
37 heat conduction is derived in first order hyperbolic form proving consistency with the Fourier law in the asymptotic  
38 regime [19]. Likewise, the stress tensor in the SHTC model proposed in [49] is asymptotically consistent with the  
39 Navier-Stokes model. The distortion tensor is defined as the inverse of the deformation gradient, hence it is defined by  
40 construction as the inverse of the Jacobian matrix associated to the Lagrange-Euler mapping between the Lagrangian  
41 (or material) to the Eulerian (or updated) configuration. Consequently, the distortion tensor accounts for the deforma-  
42 tion and rotation of the matter subject to mechanical and thermal loads. A direct link exists between the scalar density  
43 and the distortion tensor at the continuous level, which is also known as Geometric Conservation Law (GCL) in the  
44 Lagrangian formulation of the governing equations [18, 44]. As a consequence, the density equation in the original  
45 model is redundant at the continuous level [47]. This can be viewed as an internal consistency constraint. However,  
46 ensuring this compatibility at the discrete level is not obvious, and this is one goal of this work.

47 As already mentioned, in the SHTC formalism the total energy equation plays the role of an extra conservation  
48 law that can be deduced from the other equations of the system at the continuous level. This means that the entropy  
49 balance law is part of the master system, and it becomes an equality in the absence of shock waves. Nevertheless, at  
50 the discrete level, the energy equation is typically solved, hence ensuring energy conservation and numerical stability  
51 in the energy norm, and a lot of research has been conducted in order to achieve thermodynamic compatibility, i.e.  
52 obtaining an entropy balance law as a consequence of the chosen discretization. This research line started from  
53 the pioneering work presented in [57], with the aim of devising provably entropy preserving and entropy stable  
54 numerical schemes that has been further investigated in [35, 26, 40, 43, 42, 6, 39, 41]. Other important contributions  
55 to the design and implementation of entropy preserving and stable schemes can be found for instance in [20, 16,  
56 36, 17, 25, 21] and references therein. The numerical strategy proposed in [1, 3] has been recently employed to  
57 construct a new family of thermodynamically compatible schemes in which the entropy inequality is solved instead  
58 of the energy [12, 13, 2, 14, 58], hence strictly mimicking the SHTC framework at the discrete level. The numerical  
59 methods are provably energy preserving at the semi-discrete level thanks to a scalar correction factor that is directly  
60 embedded in the definition of the numerical fluxes, hence ensuring conservation. In Lagrangian hydrodynamics,  
61 thermodynamically compatible schemes have been developed in order to obtain the total energy conservation and the  
62 satisfaction of an entropy inequality as a consequence of a compatible discretization of the equations of continuity,  
63 momentum and internal energy, see for instance [15, 4, 45]. A recent attempt in directly solving the entropy inequality  
64 and obtaining conservation and stability in the energy has been forwarded in [11].

65 In this work, we make use of the general framework introduced in [1] for the construction of thermodynamically  
66 compatible schemes. We choose to discretize the total energy conservation law and deduce the entropy equation as a  
67 consequence, hence implying that the entropy inequality is one extra **equation satisfied by the mathematical model,**  
68 **which** must also be fulfilled at discrete level. This choice is the classical one because it is simpler to monitor energy or  
69 temperature in experimental devices compared to measuring entropy variations. From the continuous point of view,  
70 choosing the total energy equation or the entropy one is totally equivalent. Furthermore, the numerical scheme must

71 feature a discrete compatibility with the entropy inequality, and a discrete internal consistency between the determinant  
72 of the inverse of the deformation gradient and the discrete mass equation. We propose to resort to the approach  
73 originally forwarded in [1], and subsequently used in [2] for achieving thermodynamic compatibility for the SHTC  
74 model presented in [49]. Our novel idea is to define a new geometrical potential that plays the role of total energy in  
75 SHTC schemes, and consequently to derive the associated dual variables. In this way, another extra conservation law  
76 can be obtained which accounts for the geometric consistency, that is nothing but the Geometric Conservation Law  
77 written in the Eulerian framework. Up to the knowledge of the authors, no geometrically compatible schemes on fixed  
78 unstructured meshes are part of the state-of-the-art numerical schemes for continuum mechanics. In this work we will  
79 design a first order Finite Volume (FV) **scheme** on unstructured two-dimensional polygonal grids that is compatible  
80 with the Second Law of Thermodynamics and with the GCL, meaning that two extra conservation laws are satisfied  
81 by the scheme at the same time. This will ultimately allow to discard the classical mass conservation equation since  
82 the density can be deduced by the geometric compatibility achieved by the numerical method.

83 The paper is organized in three main sections. In Section 2 we introduce the governing equations, the extra  
84 conservation laws and the final reduced compatible continuous model that is derived. Section 3 is devoted to the  
85 design of the numerical scheme, including two theorems that demonstrate the compatibility of the new methods at the  
86 semi-discrete level. Appendix A contains all the details related to the compatibility of the reduced model with the  
87 geometric constraint in the framework of SHTC systems. The numerical experiments are gathered in the dedicated  
88 Section 4, where we numerically **verify** that the structural properties of the continuous model are preserved at the  
89 discrete level. Finally, we draw some conclusions and an outlook to future developments in Section 5.

## 90 2. Mathematical model

### 91 2.1. Governing equations

92 The governing equations are given by the unified first order hyperbolic model of continuum mechanics proposed  
93 in [49] that belongs to the class of hyperbolic thermodynamically compatible (HTC) systems [31, 28, 54, 33]. Let  
94 us assume Einstein summation convention over repeated indices, and let us adopt bold symbols to label vectors and  
95 matrices. Following [2], the mathematical model is written in three space dimensions with indices  $1 \leq i, k, m \leq 3$  as  
96 follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_k)}{\partial x_k} - \frac{\partial}{\partial x_m} \left( \epsilon \frac{\partial \rho}{\partial x_m} \right) = 0, \quad (1a)$$

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial(\rho v_i v_k + p \delta_{ik} + \sigma_{ik} + \phi_{ik})}{\partial x_k} - \frac{\partial}{\partial x_m} \left( \epsilon \frac{\partial \rho v_i}{\partial x_m} \right) = 0, \quad (1b)$$

$$\frac{\partial \mathcal{S}}{\partial t} + \frac{\partial(\mathcal{S} v_k + \beta_k)}{\partial x_k} - \frac{\partial}{\partial x_m} \left( \epsilon \frac{\partial \mathcal{S}}{\partial x_m} \right) = \Pi + \frac{\alpha_{ik} \alpha_{ik}}{\theta_1(\tau_1) T} + \frac{\beta_i \beta_i}{\theta_2(\tau_2) T} \geq 0, \quad (1c)$$

$$\frac{\partial A_{ik}}{\partial t} + \frac{\partial(A_{im} v_m)}{\partial x_k} + v_m \left( \frac{\partial A_{ik}}{\partial x_m} - \frac{\partial A_{im}}{\partial x_k} \right) - \frac{\partial}{\partial x_m} \left( \epsilon \frac{\partial A_{ik}}{\partial x_m} \right) = -\frac{\alpha_{ik}}{\theta_1(\tau_1)}, \quad (1d)$$

$$\frac{\partial J_k}{\partial t} + \frac{\partial(J_m v_m + T)}{\partial x_k} + v_m \left( \frac{\partial J_k}{\partial x_m} - \frac{\partial J_m}{\partial x_k} \right) - \frac{\partial}{\partial x_m} \left( \epsilon \frac{\partial J_k}{\partial x_m} \right) = -\frac{\beta_k}{\theta_2(\tau_2)}, \quad (1e)$$

97 with  $t \in \mathbb{R}_+^0$  being the time and  $\mathbf{x} = \{x_k\}$  denoting the spatial position vector. The state vector  $\mathbf{q} = \{q_j\} =$   
98  $(\rho, \rho v_i, \mathcal{S}, A_{ik}, J_k)^\top$  is composed of mass density  $\rho > 0$ , momentum  $\rho \mathbf{v} = \{\rho v_i\}$ , total entropy  $\mathcal{S}$ , distortion tensor  
99  $\mathbf{A} = \{A_{ik}\}$  and thermal impulse  $\mathbf{J} = \{J_k\}$ . Furthermore,  $p > 0$  and  $T > 0$  denote the fluid pressure and temperature,  
100 respectively. The fluid is also characterized by a polytropic index  $\gamma = c_p/c_v$ , given as the ratio of specific heats at  
101 constant pressure and volume, namely  $c_p$  and  $c_v$ , respectively. The above system also accounts for parabolic van-  
102 ishing viscosity terms with the parameter  $\epsilon > 0$ , that yield a production contribution  $\Pi$  in the entropy equation (1c)  
103 **which, according to [13] writes**

$$\Pi = \frac{\epsilon}{T} (\partial_{x_m} q_i) (\partial_{q_i q_j}^2 \mathcal{E}) (\partial_{x_m} q_j) \geq 0. \quad (2)$$

104 The positivity of the production term is ensured by assuming a **convex total energy potential implying that** the Hessian  
105 of the total energy potential is at least positive semi-definite, i.e.  $\partial_{q_i q_j}^2 \mathcal{E} \geq 0$ , therefore the physical entropy is

106 increasing, in accordance with the Second Law of Thermodynamics. Here,  $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4$  is the total energy  
 107 of the system which is obtained as the sum of four terms [19]:

$$\mathcal{E}_1 = \frac{\rho^\gamma}{\gamma - 1} e^{S/c_v}, \quad \mathcal{E}_2 = \frac{1}{2} \rho v_i v_i, \quad \mathcal{E}_3 = \frac{1}{4} \rho c_s^2 \mathring{G}_{ij} \mathring{G}_{ij}, \quad \mathcal{E}_4 = \frac{1}{2} c_h^2 \rho J_i J_i, \quad (3)$$

108 where  $\mathbf{G} = \{G_{ik}\} := \{A_{ij} A_{kj}\}$  represents the metric tensor and  $\mathring{\mathbf{G}} = \{\mathring{G}_{ik}\} = \{G_{ik} - \frac{1}{3} G_{mm} \delta_{ik}\}$  denotes its trace-free  
 109 part with  $\delta_{ik}$  being the Kronecker delta. The first term  $\mathcal{E}_1$  corresponds to the internal energy, for which we assume the  
 110 ideal gas equation of state, then the kinetic energy is considered by  $\mathcal{E}_2$ , whereas  $\mathcal{E}_3$  is the shear energy with the shear  
 111 sound speed  $c_s$ , and the last term  $\mathcal{E}_4$  takes into account the thermal energy with  $c_h$  being the heat wave speed. Let us  
 112 now introduce the set of thermodynamic dual variables  $\mathbf{p} := \partial_{\mathbf{q}} \mathcal{E} = \{p_j\} = (r, v_i, T, \alpha_{ik}, \beta_k)^T$  which are explicitly  
 113 given by the derivative of the energy potential (3) with respect to the state vector  $\mathbf{q}$ , that is

$$r = \partial_\rho \mathcal{E}, \quad v_i = \partial_{\rho v_i} \mathcal{E}, \quad T = \partial_S \mathcal{E}, \quad \alpha_{ik} = \partial_{A_{ik}} \mathcal{E}, \quad \beta_k = \partial_{J_k} \mathcal{E}. \quad (4)$$

114 In the momentum equation (1b), the shear stress tensor  $\boldsymbol{\sigma} = \{\sigma_{ik}\}$  and the thermal stress tensor  $\boldsymbol{\phi} = \{\phi_{ik}\}$  are defined  
 115 in terms of the dual variables  $\alpha_{ik}$  and  $\beta_k$  as

$$\sigma_{ik} = A_{ji} \partial_{A_{jk}} \mathcal{E} = A_{ji} \alpha_{jk} = \rho c_s^2 G_{ij} \mathring{G}_{jk}, \quad \phi_{ik} = J_i \partial_{J_k} \mathcal{E} = J_i \beta_k = \rho c_h^2 J_i J_k. \quad (5)$$

116 The work of the shear and thermal stress tensors  $\chi_k$  as well as the heat flux  $h_k$  are given by

$$\chi_k = \partial_{\rho v_i} \mathcal{E} \left( A_{ji} \partial_{A_{jk}} \mathcal{E} + J_i \partial_{J_k} \mathcal{E} \right) = v_i (\sigma_{ik} + \phi_{ik}), \quad h_k = \partial_S \mathcal{E} \partial_{J_k} \mathcal{E} = T \beta_k = \rho c_h^2 T J_k, \quad (6)$$

117 Finally, the mathematical model (1) is also endowed with algebraic source terms which contain two positive  
 118 functions  $\theta_1(\tau_1) > 0$  and  $\theta_2(\tau_2) > 0$  that depend on  $\mathbf{q}$  and on the relaxation times  $\tau_1 > 0$  and  $\tau_2 > 0$  as follows:

$$\theta_1 = \frac{1}{3} \rho z_1 \tau_1 c_s^2 |\mathbf{A}|^{-\frac{5}{3}}, \quad \theta_2 = \rho z_2 \tau_2 c_h^2, \quad z_1 = \frac{\rho_0}{\rho}, \quad z_2 = \frac{\rho_0 T_0}{\rho T}, \quad (7)$$

119 with  $\rho_0$  and  $T_0$  being a reference density and a reference temperature, respectively, and  $|\mathbf{A}|$  denoting the determinant  
 120 of  $\mathbf{A}$ . The asymptotic limit of the model (1) has been analyzed in [19] at the continuous level and in [7] in the fully  
 121 discrete setting, showing that for small relaxation times, i.e. when  $\tau_1 \rightarrow 0$  and  $\tau_2 \rightarrow 0$ , the Navier-Stokes-Fourier  
 122 limit is obtained. Indeed, the stress tensor  $\sigma_{ik}$  and the heat flux  $h_k$  tend to

$$\sigma_{ik} = -\frac{1}{6} \rho_0 c_s^2 \tau_1 \left( \partial_k v_i + \partial_i v_k - \frac{2}{3} (\partial_m v_m) \delta_{ik} \right), \quad h_k = -\rho_0 T_0 c_h^2 \tau_2 \partial_k T, \quad (8)$$

123 with  $\sigma_{ik}$  fulfilling Stokes hypothesis. In the asymptotic regime, the relaxation time  $\tau_1$  is directly related to the viscosity  
 124 of the fluid by  $\mu = \frac{1}{6} \rho_0 c_s^2 \tau_1$ . Analogously, there is a direct link between the relaxation time  $\tau_2$  and the thermal  
 125 conductivity coefficient which is explicitly given by  $\kappa = \rho_0 T_0 c_h^2 \tau_2$ .

126 The eigenstructure of the system (1) has been studied in [38]. Here, we are only interested in an estimate of the  
 127 maximum eigenvalues that can be chosen according to [7] as

$$\lambda = \sqrt{\frac{\gamma p}{\rho} + \frac{4}{3} c_s^2 + c_h^2}. \quad (9)$$

## 128 2.2. Overdetermined systems: extra conservation laws

129 By construction, see [49], the model (1) is an overdetermined hyperbolic system, thus implying the satisfaction of  
 130 additional (or extra) conservation laws. Firstly, we obtain total energy conservation from the HTC framework, then  
 131 we focus on the derivation of the Geometric Conservation Law that imposes a geometric constraint on the determinant  
 132 of the distortion tensor  $|\mathbf{A}|$ .

133 2.2.1. Total energy conservation law

134 By dot multiplying equations (1a)-(1e) with the associated thermodynamic variables  $\mathbf{p}$ , one obtains the total energy  
 135 equation

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial (\mathcal{E} v_k + v_i p \delta_{ik} + \chi_k + h_k)}{\partial x_k} - \frac{\partial}{\partial x_m} \left( \epsilon \frac{\mathcal{E}}{\partial x_m} \right) = 0, \quad (10)$$

136 meaning that the following Gibbs relation is satisfied:

$$\begin{aligned} 1 \cdot d\mathcal{E} &= r \cdot d\rho + v_i \cdot d(\rho v_i) + T \cdot d\mathcal{S} + \alpha_{ik} \cdot dA_{ik} + \beta_k \cdot dJ_k := \mathbf{p} \cdot d\mathbf{q} \\ 1 \cdot (10) &= r \cdot (1a) + v_i \cdot (1b) + T \cdot (1c) + \alpha_{ik} \cdot (1d) + \beta_k \cdot (1e) \end{aligned} \quad (11)$$

137 This also implies that the entropy production term  $\Pi$  in (1c) must be compatible with the parabolic dissipation terms

$$\partial_S \mathcal{E} \cdot \Pi + \mathbf{p} \cdot \partial_m (\epsilon \partial_m \mathbf{q}) = \partial_m (\epsilon \partial_m \mathcal{E}), \quad (12)$$

138 and that the dot product of  $\mathbf{p}$  with the algebraic relaxation source terms must vanish

$$\mathbf{p} \cdot \mathbf{S}(\mathbf{q}) = 0. \quad (13)$$

139 Although the rigorous formalism and derivation of HTC systems implies the use of the entropy as state variable,  
 140 let us remark that the energy equation (10) could be solved instead, and the associated entropy balance can be retrieved  
 141 again from the Gibbs relation (11) as

$$T d\mathcal{S} = -r \cdot d\rho - v_i \cdot d(\rho v_i) + 1 \cdot d\mathcal{E} - \alpha_{ik} \cdot dA_{ik} - \beta_k \cdot dJ_k, \quad (14)$$

142 with a set of dual variables

$$\mathbf{r} = \{r_j\} = \frac{1}{T} (-r, -v_i, 1, -\alpha_{ik}, -\beta_k)^\top. \quad (15)$$

143 This implies the assumption of a physical entropy potential  $\mathcal{S}$  such that  $\mathbf{r} = \partial_{\mathbf{q}} \mathcal{S}$  with an associated positive semi-  
 144 negative Hessian matrix  $\partial_{q_i q_j}^2 \mathcal{S} \leq 0$ .

145 2.2.2. Geometric Conservation Law (GCL)

146 The governing equations (1) involve a geometric constraint on the determinant of the distortion tensor  $\mathbf{A}$ ,  
 147 which corresponds to the inverse deformation gradient for reversible processes in the material. To properly derive this  
 148 geometric constraint, let us consider the Lagrange-Euler mapping between the Lagrangian domain  $\Omega \subset \mathbb{R}^3$  and the  
 Eulerian domain  $\omega(t) \subset \mathbb{R}^3$  at time  $t > 0$ , that deforms in time through the movement of the material. At the aid

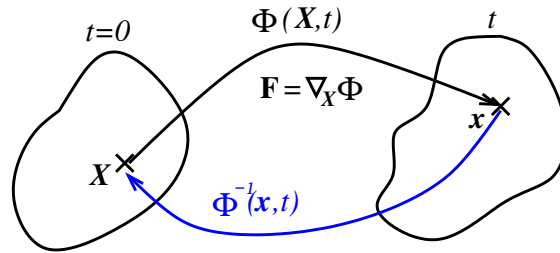


Figure 1: Sketch of the Lagrangian-Eulerian transformation.

149 of Figure 1, let  $\mathbf{X} = \{X_k\}$  and  $\mathbf{x} = \{x_k\}$  represent the coordinate of any Lagrangian point in  $\Omega$  and Eulerian point in  
 150  $\omega(t)$ , respectively. Then, the Lagrange-Euler mapping  $\Phi = \{\Phi_i\}$  is such that  $\mathbf{x} = \Phi(\mathbf{X}, t) \in \omega(t)$ , and the kinematic  
 151 velocity of the material in the Eulerian frame is given by  
 152

$$v_i(\mathbf{X}, t) = \frac{\partial \Phi_i(\mathbf{X}, t)}{\partial t}. \quad (16)$$

153 The deformation gradient tensor  $\mathbf{F} = \{F_{ik}\}$  is nothing but the Jacobian matrix associated to the flow map  $\Phi$  and  
 154 verifies

$$F_{ik} = \frac{\partial \Phi_i}{\partial X_k}. \quad (17)$$

155 We assume that for all  $t > 0$ , the determinant of  $\mathbf{F}$ , called the Jacobian of the transformation, satisfies  $\det(\mathbf{F}(\mathbf{X}, t)) :=$   
 156  $|\mathbf{F}(\mathbf{X}, t)| > 0$ , so that the flow map is always invertible. The inverse of the transformation links the Eulerian coordinate  
 157 to the Lagrangian one, i.e.  $\mathbf{X} = \Phi^{-1}(\mathbf{x}, t)$ , and the distortion tensor in the mathematical model (1) is geometrically  
 158 defined as  $\mathbf{A} = \mathbf{F}^{-1}$  for reversible processes. The determinant of the deformation gradient represents the ratio of the  
 159 Eulerian volume element to the Lagrangian volume element, that is

$$dv = |\mathbf{F}|dV. \quad (18)$$

160 Following [24], the mass conservation law with respect to the Lagrangian configuration is expressed for  $t \geq 0$  by

$$\frac{d}{dt} \int_{\Omega} \rho(\mathbf{X}, t) |\mathbf{F}(\mathbf{X}, t)| dV = 0, \quad (19)$$

161 with the Lagrangian or material derivative given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_k \frac{\partial}{\partial x_k}. \quad (20)$$

162 Since relation (19) must hold for an arbitrary domain  $\Omega$ , it implies

$$\rho(\mathbf{X}, t) |\mathbf{F}(\mathbf{X}, t)| = \rho(\mathbf{X}, 0) \implies |\mathbf{F}| = \frac{\rho_0}{\rho}, \quad (21)$$

163 where we recall that  $\rho_0 = \rho(\mathbf{X}, 0)$  is the initial density of the material. Consequently, thanks to the relationship  
 164  $\mathbf{A} = \mathbf{F}^{-1}$ , the determinant of the distortion matrix must obey the following constraint:

$$|\mathbf{A}| = \frac{\rho}{\rho_0}. \quad (22)$$

165 This geometric constraint is extremely difficult to be respected at the discrete level, especially for Eulerian  
 166 schemes. To the best knowledge of the authors, this has never been achieved so far on fixed grids. Therefore, our aim  
 167 is to satisfy the constraint (22) by proposing a new approach, that requires the satisfaction of an extra conservation  
 168 law for the quantity  $|\mathbf{A}|$ . Using the Lagrangian derivative (20) and neglecting viscous and source terms, the evolution  
 169 equation (1d) writes

$$\frac{dA_{ki}}{dt} + A_{kj} \frac{\partial v_j}{\partial x_i} = 0. \quad (23)$$

170 Employing the Jacobi formula and the above relation, the time derivative of the determinant of the distortion tensor  
 171 leads to

$$\begin{aligned} \frac{d|\mathbf{A}|}{dt} &= \text{tr} \left( |\mathbf{A}| \mathbf{W} \frac{d\mathbf{A}}{dt} \right), \quad \mathbf{W} = \mathbf{A}^{-1}, \\ &= -|\mathbf{A}| W_{ik} A_{kj} \frac{\partial v_j}{\partial x_i} \\ &= -|\mathbf{A}| \delta_{ij} \frac{\partial v_j}{\partial x_i}. \end{aligned} \quad (24)$$

172 By replacing the material derivative on the left hand side of (24) with its Eulerian counterpart according to (20), the  
 173 Geometric Conservation Law is obtained as an extra conservation law satisfied by the governing equations (1), which  
 174 is explicitly given by

$$\frac{\partial |\mathbf{A}|}{\partial t} + \frac{\partial}{\partial x_k} (|\mathbf{A}| v_k) = 0. \quad (25)$$

175 Therefore, satisfying the GCL (25) implies that the constraint (22) is also respected. To mimic the HTC approach,  
 176 let us introduce a new set of pseudo-*thermodynamic variables*  $\mathbf{w} = \{w_{ik}\}$  that are dual with respect to a pseudo-  
 177 *potential* given by  $|\mathbf{A}|$ , thus obtaining  $\mathbf{w} := \partial_{\mathbf{A}}|\mathbf{A}|$ . Then, by construction, one can verify that a pseudo-*Gibbs relation*  
 178 is satisfied, that is

$$\begin{aligned} d(|\mathbf{A}|) &= w_{ik} \cdot d(A_{ik}) := \mathbf{w} \cdot d\mathbf{A} \\ (25) &= w_{ik} \cdot (1d) \end{aligned} \quad (26)$$

179 More precisely, the source term of the distortion tensor equation (1d), referred to as  $\mathbf{S}_{\mathbf{A}}$ , has been designed in [49] not  
 180 to affect the mass conservation equation. Indeed, it is proportional to  $\partial_{\mathbf{A}}\mathcal{E}$ , namely

$$\mathbf{S}_{\mathbf{A}} = -\frac{\partial_{\mathbf{A}}\mathcal{E}}{\theta_1(\tau_1)} = -\frac{\boldsymbol{\alpha}}{\theta_1(\tau_1)} = -\frac{3}{\tau_1} |\mathbf{A}|^{\frac{5}{3}} \mathbf{A} \mathring{\mathbf{G}}. \quad (27)$$

181 On the other hand, the dual variables  $\mathbf{w}$  are given by

$$\mathbf{w} = |\mathbf{A}| \mathbf{A}^{-\top}. \quad (28)$$

182 Therefore, the contraction  $\mathbf{w} : \mathbf{S}_{\mathbf{A}} = \text{tr}(\mathbf{w}^{\top} \mathbf{S}_{\mathbf{A}})$  yields

$$\mathbf{w} : \mathbf{S}_{\mathbf{A}} = -\frac{3}{\tau_1} |\mathbf{A}|^{\frac{8}{3}} \text{tr}(\mathbf{A}^{-1} \mathbf{A} \mathring{\mathbf{G}}) = -\frac{3}{\tau_1} |\mathbf{A}|^{\frac{8}{3}} \text{tr}(\mathring{\mathbf{G}}) = 0, \quad (29)$$

183 since  $\mathring{\mathbf{G}}$  is the trace-free part of the metric tensor  $\mathbf{G} = \mathbf{A}^{\top} \mathbf{A}$ . The details concerning the derivation of the GCL in  
 184 terms of the dual variables  $\mathbf{w}$  can be found in Appendix A.

### 185 2.3. Reduced compatible model

186 The previous considerations incline us to consider a reduced model consisting of the following equations:

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial (\rho v_i v_k + p \delta_{ik} + \sigma_{ik} + \phi_{ik})}{\partial x_k} - \frac{\partial}{\partial x_m} \left( \epsilon \frac{\partial \rho v_i}{\partial x_m} \right) = 0, \quad (30a)$$

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial (\mathcal{E} v_k + v_i p \delta_{ik} + \chi_k + h_k)}{\partial x_k} - \frac{\partial}{\partial x_m} \left( \epsilon \frac{\partial \mathcal{E}}{\partial x_m} \right) = 0, \quad (30b)$$

$$\frac{\partial A_{ik}}{\partial t} + \frac{\partial (A_{im} v_m)}{\partial x_k} + v_m \left( \frac{\partial A_{ik}}{\partial x_m} - \frac{\partial A_{im}}{\partial x_k} \right) - \frac{\partial}{\partial x_m} \left( \epsilon \frac{\partial A_{ik}}{\partial x_m} \right) = -\frac{\alpha_{ik}}{\theta_1(\tau_1)}, \quad (30c)$$

$$\frac{\partial J_k}{\partial t} + \frac{\partial (J_m v_m + T)}{\partial x_k} + v_m \left( \frac{\partial J_k}{\partial x_m} - \frac{\partial J_m}{\partial x_k} \right) - \frac{\partial}{\partial x_m} \left( \epsilon \frac{\partial J_k}{\partial x_m} \right) = -\frac{\beta_k}{\theta_2(\tau_2)}. \quad (30d)$$

187 This system satisfies the entropy inequality (1c) and the Geometric Conservation Law (25). We underline that no  
 188 evolution equation for the mass density is embedded in the model, since the material density can be easily computed  
 189 from the determinant constraint (22) thanks to the GCL compatibility, that is  $\rho = \rho_0 |\mathbf{A}|$ . Likewise, the entropy balance  
 190 is also satisfied by the reduced model (30) which is compliant with the Gibbs relation (14).

191 Here we consider the state variables  $\mathbf{u} = (\rho v_i, \mathcal{E}, A_{ik}, J_k)$ , and the governing equations can be written in a compact  
 192 matrix-vector formulation as

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}_k(\mathbf{u})}{\partial x_k} + \mathbf{B}_k(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_k} - \frac{\partial}{\partial x_m} \left( \epsilon \frac{\partial \mathbf{u}}{\partial x_m} \right) = \mathbf{S}(\mathbf{u}), \quad (31)$$

193 where  $\mathbf{f}_k(\mathbf{u})$  is the nonlinear flux tensor and  $\mathbf{B}_k(\mathbf{u}) \partial_k \mathbf{u}$  contains the non-conservative part of the system in block-  
 194 matrix notation for  $\mathbf{A}$  and  $\mathbf{J}$ . The algebraic sources are gathered in the term  $\mathbf{S}(\mathbf{u})$ , while the regularizing viscous terms  
 195 are given by  $\partial_m (\epsilon \partial_m \mathbf{u})$ .

196 The model (30) is solved with a finite volume method on general unstructured meshes that is proven to preserve  
 197 both the geometric and the thermodynamic compatibility. Indeed, the novel numerical method only solves the reduced  
 198 model (30) because it is compliant with (1c) and (25). All the details of the numerical scheme are provided in the next  
 199 section.



### 200 3. Numerical scheme

#### 201 3.1. Semi-discrete finite volume scheme on unstructured meshes

202 To ease the notation and the readability, subscripts are used for tensor indices while superscripts denote the spatial  
 203 discretization index. The two-dimensional computational domain  $\Omega \in \mathbb{R}^2$  is discretized with a total number  $N^\ell$  of  
 204 non-overlapping unstructured polygonal control volumes  $\omega^\ell$  with border  $\partial\omega^\ell$  and barycenter coordinates  $\mathbf{x}^\ell$ . We  
 205 underline that Voronoi meshes can also be employed as well as any other general polygonal elements. The surface of  
 206 the element is denoted with  $|\omega^\ell|$ , whereas  $|\partial\omega^\ell|$  refers to the cell perimeter. The set of neighbors of cell  $\omega^\ell$  is labeled  
 207 with  $\mathcal{N}^\ell$ , and  $\partial\omega^{\ell r}$  is the common edge shared by two adjacent elements  $\omega^\ell$  and  $\omega^r$  with outward pointing unit normal  
 vector  $\mathbf{n}^{\ell r}$ . Figure 2 shows an example of an unstructured polygonal mesh and a sketch of the adopted notation.

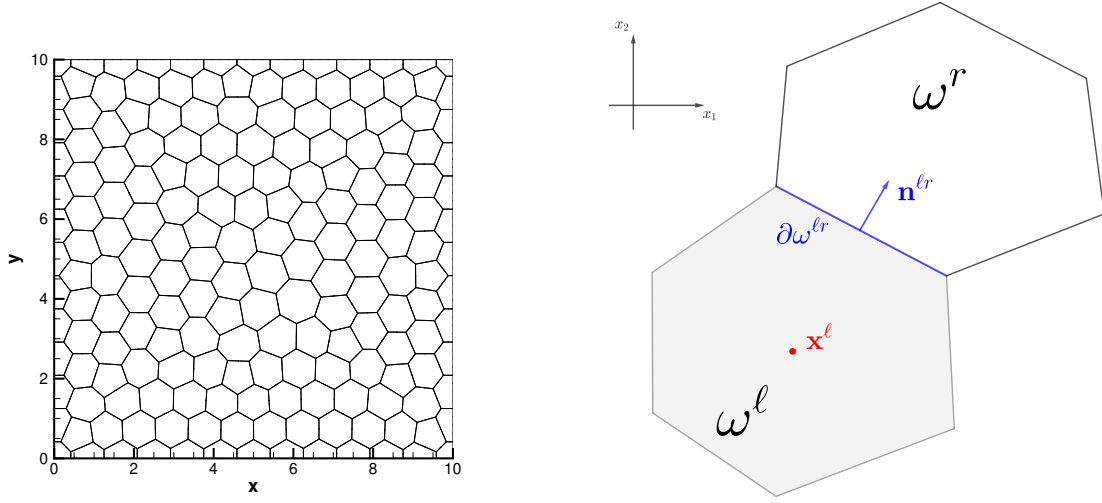


Figure 2: Left: example of unstructured Voronoi mesh. Right: notation used for cell  $\omega^\ell$  and one direct neighbor cell  $\omega^r$ .

208 We start the derivation of a finite volume method for discretizing the reduced model (31). For finite volume  
 209 schemes, data are stored and evolved in time as piecewise constant cell averages which are defined as  
 210

$$\mathbf{u}^\ell := \frac{1}{|\omega^\ell|} \int_{\omega^\ell} \mathbf{u} \, d\mathbf{x}. \quad (32)$$

211 To obtain a semi-discrete finite volume scheme, let us integrate in space the governing equations over the control  
 212 volume  $\omega^\ell$  by keeping time continuous:

$$|\omega^\ell| \frac{\partial \mathbf{u}^\ell}{\partial t} + \int_{\omega^\ell} \left( \frac{\partial \mathbf{f}_k(\mathbf{u})}{\partial x_k} + \mathbf{B}_k(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_k} - \frac{\partial}{\partial x_m} \left( \epsilon \frac{\partial \mathbf{u}}{\partial x_m} \right) \right) d\mathbf{x} = \int_{\omega^\ell} \mathbf{S}(\mathbf{u}) \, d\mathbf{x}. \quad (33)$$

213 Application of the theorem of Gauss on the nonlinear flux and viscosity terms yields

$$|\omega^\ell| \frac{\partial \mathbf{u}^\ell}{\partial t} + \int_{\partial\omega^\ell} \left( \mathbf{f}_k(\mathbf{u}) - \left( \epsilon \frac{\partial \mathbf{u}}{\partial x_m} \right) \right) \cdot \mathbf{n} \, dS + \int_{\omega^\ell} \mathbf{B}_k(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_k} \, d\mathbf{x} = \int_{\omega^\ell} \mathbf{S}(\mathbf{u}) \, d\mathbf{x}. \quad (34)$$

214 The non-conservative terms are integrated using a path-conservative approach, see [48, 10] and references therein,  
 215 hence obtaining

$$\int_{\omega^\ell} \mathbf{B}_k(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_k} \, d\mathbf{x} = \int_{\partial\omega^\ell} \mathbf{D}_k \cdot \mathbf{n} \, dS + \int_{\omega^\ell \setminus \partial\omega^\ell} \mathbf{B}_k(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_k} \, d\mathbf{x}, \quad (35)$$

216 where the new term  $\mathbf{D}_k \cdot \mathbf{n}$  takes into account potential jumps of the solution across the element boundaries and it is  
 217 defined for the interface between  $\omega^\ell$  and  $\omega^i$  as

$$\mathbf{D}_k \cdot \mathbf{n} = \frac{1}{2} \int_0^1 \mathbf{B}_k(\boldsymbol{\psi}(\mathbf{u}^\ell, \mathbf{u}^i, s)) \cdot \mathbf{n}^{\ell i} \frac{\partial \boldsymbol{\psi}}{\partial s} ds. \quad (36)$$

218 The integration path  $\boldsymbol{\psi}$  is chosen to be a simple straight-line segment according to [48, 10], thus it is given by  $\boldsymbol{\psi} =$   
 219  $\boldsymbol{\psi}(\mathbf{u}^\ell, \mathbf{u}^i, s) = \mathbf{u}^\ell + s(\mathbf{u}^i - \mathbf{u}^\ell)$ , and the jump term (36) reduces to

$$\mathbf{D}_k \cdot \mathbf{n} = \frac{1}{2} \left( \int_0^1 \mathbf{B}_k(\boldsymbol{\psi}(\mathbf{u}^\ell, \mathbf{u}^i, s)) \cdot \mathbf{n}^{\ell i} ds \right) (\mathbf{u}^i - \mathbf{u}^\ell). \quad (37)$$

220 The first order semi-discrete finite volume scheme for the governing equations (34) writes

$$\frac{\partial \mathbf{u}^\ell}{\partial t} = - \sum_{i \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell i}|}{|\omega^\ell|} \left( \mathcal{F}(\mathbf{u}^\ell, \mathbf{u}^i) + \mathcal{D}(\mathbf{u}^\ell, \mathbf{u}^i) + \mathcal{G}(\mathbf{u}^\ell, \mathbf{u}^i) \right) \cdot \mathbf{n}^{\ell i} + \mathbf{S}(\mathbf{u}^\ell), \quad (38)$$

221 where the volume term in (35) vanishes since at first order any gradient of  $\mathbf{u}$  inside the cell is zero, and the source  
 222 term integral simply corresponds to the evaluation of  $\mathbf{S}$  with  $\mathbf{u}^\ell$  thanks to the definition of the finite volume solution  
 223 (32). The numerical flux is given by a central approximation, that is

$$\mathcal{F}(\mathbf{u}^\ell, \mathbf{u}^i) \cdot \mathbf{n}^{\ell i} = \frac{1}{2} (\mathbf{f}_k^\ell + \mathbf{f}_k^i) n_k^{\ell i}, \quad (39)$$

224 and the following discretization is chosen for the non-conservative terms (37)

$$\mathcal{D}(\mathbf{u}^\ell, \mathbf{u}^i) \cdot \mathbf{n}^{\ell i} = \frac{1}{2} \mathbf{B}_k(\bar{\mathbf{u}}^{\ell i}) n_k^{\ell i} (\mathbf{u}^i - \mathbf{u}^\ell), \quad \bar{\mathbf{u}}^{\ell i} = \frac{1}{2} (\mathbf{u}^i + \mathbf{u}^\ell), \quad (40)$$

225 which corresponds to a midpoint rule for the evaluation of the path-integral (37). The dissipative numerical flux is  
 226 computed by a Rusanov-type scheme with

$$\mathcal{G}(\mathbf{u}^\ell, \mathbf{u}^i) \cdot \mathbf{n}^{\ell i} = -\epsilon^{\ell i} \|\mathbf{x}^i - \mathbf{x}^\ell\| \frac{\mathbf{u}^i - \mathbf{u}^\ell}{\|\mathbf{x}^i - \mathbf{x}^\ell\|} = -\epsilon^{\ell i} (\mathbf{u}^i - \mathbf{u}^\ell), \quad \epsilon^{\ell i} = \frac{1}{2} \max(|\lambda(\mathbf{u}^\ell)|, |\lambda(\mathbf{u}^i)|), \quad (41)$$

227 where we use the estimate of the maximum eigenvalue of the system according to (9) for the definition of the positive  
 228 coefficient  $\epsilon^{\ell i}$ . To ease the notation, let us introduce the abbreviations

$$\tilde{\mathcal{F}}^{\ell i} := \tilde{\mathcal{F}}(\mathbf{u}^\ell, \mathbf{u}^i), \quad \mathcal{D}^{\ell i} := \mathcal{D}(\mathbf{u}^\ell, \mathbf{u}^i), \quad \mathcal{G}^{\ell i} := \mathcal{G}(\mathbf{u}^\ell, \mathbf{u}^i). \quad (42)$$

229  
 230 As it stands, the finite volume scheme (38) is not structure preserving: it is neither compliant with the extra  
 231 conservation laws (1c) nor with (25). Therefore some *ad hoc* modifications have to be designed in order to make the  
 232 scheme *structure preserving*.

### 233 3.2. Geometrically compatible finite volume scheme

234 We start by designing a modification of the finite volume scheme (38) such that geometric compatibility is ensured  
 235 by satisfying the GCL equation (25) as an extra conservation law of the mathematical model. For the moment, we  
 236 neglect the source terms and the dissipation fluxes, i.e we assume  $\mathcal{G}^{\ell i} \cdot \mathbf{n}^{\ell i} = \mathbf{0}$  and  $\mathbf{S}(\mathbf{u}) = \mathbf{0}$ . We rely on a  
 237 very general method firstly proposed for achieving thermodynamic compatibility in [1], and more recently extended  
 238 to hyperbolic systems of the type (1) to recover energy conservation from the direct discretization of the entropy  
 239 inequality [13, 2, 14]. Here, we apply this strategy for the first time to preserve a different structural property rather  
 240 than thermodynamics.

241 Let us interpret the determinant of the distortion matrix  $|\mathbf{A}|$  as a *thermodynamic potential*, and the associated  
 242 dual variables  $\mathbf{w} = \{w_{ik}\} = \partial_{\mathbf{A}}|\mathbf{A}|$  as a set of *thermodynamic variables*. Moreover, let  $\mathcal{F}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau}$  denote the central  
 243 fluxes related to the distortion tensor equation (30c) according to (39). Likewise,  $\mathbf{f}_{\mathbf{A},m}$  represents the physical flux  
 244 of equation (30c) and  $\mathcal{D}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau}$  are the corresponding fluctuations of the non-conservative terms restricted to (30c).  
 245 According to [1], these fluxes are modified by a correction factor  $\alpha_{\mathbf{A}}^{\ell\tau}$ , which is defined at the cell interface, hence  
 246 obtaining the modified fluxes

$$\tilde{\mathcal{F}}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} = \mathcal{F}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} - \alpha_{\mathbf{A}}^{\ell\tau} (\mathbf{w}^{\tau} - \mathbf{w}^{\ell}) \cdot \mathbf{n}^{\ell\tau} = \frac{1}{2} \left( \mathbf{f}_{\mathbf{A},k}^{\ell} + \mathbf{f}_{\mathbf{A},k}^{\tau} \right) n_k^{\ell\tau} - \alpha_{\mathbf{A}}^{\ell\tau} (\mathbf{w}^{\tau} - \mathbf{w}^{\ell}) \cdot \mathbf{n}^{\ell\tau}. \quad (43)$$

247 The scalar correction factor  $\alpha_{\mathbf{A}}^{\ell\tau}$  has no sign, and it can add or subtract the total amount of the jump in the dual variables  
 248 which is needed to reach geometric compatibility with the GCL (25). To determine  $\alpha_{\mathbf{A}}^{\ell\tau}$ , the conservation principle is  
 249 invoked. Indeed, across each cell boundary  $\partial\omega^{\ell\tau}$ , a consistent condition implies that the sum of the fluctuations must  
 250 balance the sum of the fluxes which have to be preserved. This sum, namely  $|\mathbf{A}|_{v_k}$  in (25), must be recovered as the  
 251 dot product of equation (30c) with the dual variables  $\mathbf{w}^{\ell}$ , that is

$$\begin{aligned} \mathbf{w}^{\ell} \cdot \left( \tilde{\mathcal{F}}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} - \mathbf{f}_{\mathbf{A},k}^{\ell} \cdot n_k^{\ell\tau} \right) + \mathbf{w}^{\tau} \cdot \left( \mathbf{f}_{\mathbf{A},k}^{\tau} \cdot n_k^{\ell\tau} - \tilde{\mathcal{F}}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} \right) + \\ \mathbf{w}^{\ell} \cdot \mathcal{D}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} + \mathbf{w}^{\tau} \cdot \mathcal{D}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} = \left( (|\mathbf{A}|_{v_k})^{\tau} - (|\mathbf{A}|_{v_k})^{\ell} \right) n_k^{\ell\tau}. \end{aligned} \quad (44)$$

252 By inserting the definition of the modified fluxes (43) in the condition (44), we obtain the sought correction factor  
 253 defined at the cell interface  $\partial\omega^{\ell\tau}$ :

$$\begin{aligned} \alpha_{\mathbf{A}}^{\ell\tau} = \frac{\left( (|\mathbf{A}|_{v_k})^{\tau} - (|\mathbf{A}|_{v_k})^{\ell} \right) n_k^{\ell\tau} + \left( \mathcal{F}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} \right) \cdot (\mathbf{w}^{\tau} - \mathbf{w}^{\ell}) - \left( \mathbf{w}^{\tau} \cdot \mathbf{f}_{\mathbf{A},k}^{\tau} - \mathbf{w}^{\ell} \cdot \mathbf{f}_{\mathbf{A},k}^{\ell} \right) n_k^{\ell\tau}}{(\mathbf{w}^{\tau} - \mathbf{w}^{\ell})^2} \\ - \frac{(\mathbf{w}^{\tau} + \mathbf{w}^{\ell}) \cdot \mathcal{D}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau}}{(\mathbf{w}^{\tau} - \mathbf{w}^{\ell})^2}. \end{aligned} \quad (45)$$

254 Obviously, if  $\mathbf{w}^{\ell} = \mathbf{w}^{\tau}$  then no correction occurs and we simply set  $\alpha_{\mathbf{A}}^{\ell\tau} = 0$ . We underline that the correction factor  
 255  $\alpha_{\mathbf{A}}$  has no sign and, in principle, it is unbounded. From the numerical viewpoint, we only take care about avoiding  
 256 division by zero.

257 *Geometric compatibility with dissipation fluxes and source terms.* Even with smooth initial data, the solution of  
 258 the PDE system (30) can exhibit shocks and other discontinuities, which require a stabilization of the numerical  
 259 scheme that is carried out relying on parabolic vanishing viscosity terms. Also in this case, the compatibility with the  
 260 geometric extra conservation law (25) must be respected. To that aim, let us add to the compatible fluxes (43) also the  
 261 dissipative fluxes  $\mathcal{G}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau}$  as well as the source terms  $\mathbf{S}_{\mathbf{A}}(\mathbf{u}^{\ell}) = -\frac{\alpha^{\ell}}{\theta(\tau_1)}$ , so that the semi-discrete evolution equation  
 262 (30c) for  $\mathbf{A}$  now becomes

$$\frac{\partial \mathbf{A}^{\ell}}{\partial t} + \sum_{\tau \in \mathcal{N}^{\ell}} \frac{|\partial\omega^{\ell\tau}|}{|\omega^{\ell}|} \left( \tilde{\mathcal{F}}_{\mathbf{A}}^{\ell\tau} + \mathcal{D}_{\mathbf{A}}^{\ell\tau} \right) \cdot \mathbf{n}^{\ell\tau} = - \sum_{\tau \in \mathcal{N}^{\ell}} \mathcal{G}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} + \mathbf{S}_{\mathbf{A}}(\mathbf{u}^{\ell}). \quad (46)$$

263 The part on the left hand side of the above equation is already compatible with the GCL thanks to the modified fluxes  
 264 (43) with the scalar correction factor given by (45). Therefore, we focus on the compatibility of the right hand side  
 265 of (46). Recalling the definition (41) and following [13], after multiplication by the dual variables  $\mathbf{w}^{\ell}$ , for the viscous  
 266 terms we obtain

$$\begin{aligned} \mathbf{w}^{\ell} \cdot \mathcal{G}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} &= \frac{1}{2} \left( \mathbf{w}^{\ell} \cdot \mathcal{G}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} + \mathbf{w}^{\tau} \cdot \mathcal{G}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} + \mathbf{w}^{\ell} \cdot \mathcal{G}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} - \mathbf{w}^{\tau} \cdot \mathcal{G}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} \right) \\ &= \frac{1}{2} \underbrace{(\mathbf{w}^{\tau} - \mathbf{w}^{\ell}) \cdot \epsilon^{\ell\tau} (\mathbf{A}^{\tau} - \mathbf{A}^{\ell})}_{\mathcal{G}_1} - \frac{1}{2} \underbrace{(\mathbf{w}^{\tau} + \mathbf{w}^{\ell}) \cdot \epsilon^{\ell\tau} (\mathbf{A}^{\tau} - \mathbf{A}^{\ell})}_{\mathcal{G}_2}. \end{aligned} \quad (47)$$

267 The second term  $\mathcal{G}_2$  is the approximation of the jump term related to the numerical dissipation in the GCL (25), that  
 268 is

$$-\frac{1}{2}(\mathbf{w}^\tau + \mathbf{w}^\ell) \cdot \epsilon^{\ell\tau} (\mathbf{A}^\tau - \mathbf{A}^\ell) \approx -\epsilon^{\ell\tau} (|\mathbf{A}|^\tau - |\mathbf{A}|^\ell). \quad (48)$$

269 Indeed, applying path integration in the state variables  $\mathbf{A}$ , the following relation holds true by construction:

$$\int_{\mathbf{A}^\ell}^{\mathbf{A}^\tau} \mathbf{w} \cdot d\mathbf{A} = \int_{\mathbf{A}^\ell}^{\mathbf{A}^\tau} \partial_{\mathbf{A}} |\mathbf{A}| \cdot d\mathbf{A} = |\mathbf{A}|^\tau - |\mathbf{A}|^\ell, \quad (49)$$

270 and the term  $\frac{1}{2}(\mathbf{w}^\tau + \mathbf{w}^\ell)(\mathbf{A}^\tau - \mathbf{A}^\ell)$  in (48) can be seen as a numerical approximation of the path integral in (49) using  
 271 a trapezoidal rule. Therefore, we still remain with an additional contribution given by the first term  $\mathcal{G}_1$  in (47). To  
 272 control its production of numerical dissipation, we reformulate the jump in the dual variables  $\mathbf{w}$  as a jump in the state  
 273 variables  $\mathbf{A}$  through the Hessian matrix  $\partial_{\mathbf{A}\mathbf{A}}^2 |\mathbf{A}|^{\ell\tau}$  which verifies the Roe property

$$\partial_{\mathbf{A}\mathbf{A}}^2 |\mathbf{A}|^{\ell\tau} \cdot (\mathbf{A}^\tau - \mathbf{A}^\ell) = \mathbf{w}^\tau - \mathbf{w}^\ell. \quad (50)$$

274 The Hessian matrix at the cell interface is computed as

$$\partial_{\mathbf{A}\mathbf{A}}^2 |\mathbf{A}|^{\ell\tau} = \int_0^1 \partial_{\mathbf{A}\mathbf{A}}^2 |\mathbf{A}|(\xi(s)) ds, \quad \xi(s) = \mathbf{A}^\ell + s(\mathbf{A}^\tau - \mathbf{A}^\ell), \quad 0 \leq s \leq 1, \quad (51)$$

275 where  $\partial_{\mathbf{A}\mathbf{A}}^2 |\mathbf{A}|$  explicitly writes

$$\partial_{\mathbf{A}\mathbf{A}}^2 |\mathbf{A}| = \begin{bmatrix} 0 & 0 & 0 & 0 & A_{33} & -A_{32} & 0 & -A_{23} & A_{22} \\ 0 & 0 & 0 & -A_{33} & 0 & A_{31} & A_{23} & 0 & -A_{21} \\ 0 & 0 & 0 & A_{32} & -A_{31} & 0 & -A_{22} & A_{21} & 0 \\ 0 & -A_{33} & A_{32} & 0 & 0 & 0 & 0 & A_{13} & -A_{12} \\ A_{33} & 0 & -A_{31} & 0 & 0 & 0 & -A_{13} & 0 & A_{11} \\ -A_{32} & A_{31} & 0 & 0 & 0 & 0 & A_{12} & -A_{11} & 0 \\ 0 & A_{23} & -A_{22} & 0 & -A_{13} & A_{12} & 0 & 0 & 0 \\ -A_{23} & 0 & A_{21} & A_{13} & 0 & -A_{11} & 0 & 0 & 0 \\ A_{22} & -A_{21} & 0 & -A_{12} & A_{11} & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (52)$$

276 These contributions, which come from all the faces  $\mathcal{N}^\ell$  of the cell, must vanish in order to obtain compatibility with  
 277 the GCL equation (25). Consequently, a production term  $\Pi_{\mathbf{A}}^\ell$  is introduced to balance these terms with opposite sign,  
 278 that is given by

$$\Pi_{\mathbf{A}}^\ell = \sum_{\tau \in \mathcal{N}^\ell} \frac{|\partial\omega^{\ell\tau}|}{|\omega^\ell|} \frac{1}{2} \epsilon^{\ell\tau} (\mathbf{A}^\tau - \mathbf{A}^\ell) \cdot \partial_{\mathbf{A}\mathbf{A}}^2 |\mathbf{A}|^{\ell\tau} (\mathbf{A}^\tau - \mathbf{A}^\ell), \quad (53)$$

279 where only jumps in the state variables appear because of the use of the Roe-type Hessian matrix (51). The production  
 280 term  $\Pi_{\mathbf{A}}^\ell$  is a scalar, that now needs to be distributed among all the components  $A_{ik}^\ell$  of the distortion tensor, hence  
 281 obtaining new contributions  $P_{ik}^\ell$ . Here, we adopt a rescaling with respect to the trace of the distortion tensor, as  
 282 proposed in [12] for the redistribution of a production term associated to the Reynolds stress tensor, thus we define  
 283  $P_{ik}^\ell$  as

$$P_{ik}^\ell = \Pi_{\mathbf{A}}^\ell \frac{w_{ik}^\ell}{\text{tr}(\mathbf{w}^\ell \mathbf{w}^{\ell,\top})}, \quad (54)$$

284 with the positive trace  $\text{tr}(\mathbf{w}^\ell \mathbf{w}^{\ell,\top}) = w_{ik}^\ell w_{ik}^\ell \geq 0$ .

285 At last, it remains to verify the compatibility with the source terms. Multiplication of  $\mathbf{S}_{\mathbf{A}}(\mathbf{u}^\ell)$  by the dual variables  
 286 yields

$$-\mathbf{w}^\ell \cdot \frac{\boldsymbol{\alpha}^\ell}{\theta(\tau_1)} = -w_{ik}^\ell \cdot \frac{\alpha_{ik}^\ell}{\theta(\tau_1)} = 0, \quad (55)$$

287 therefore the compatibility is proven by construction of the dual variables  $\mathbf{w}$ , as demonstrated at the continuous level  
 288 by (29). All the related details are reported in Appendix A.

289 **Theorem 1** (Geometric compatibility). *The semi-discrete finite volume scheme for the equation of the distortion*  
 290 *tensor  $\mathbf{A}$  given by*

$$\frac{\partial \mathbf{A}^\ell}{\partial t} + \sum_{\tau \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell\tau}|}{|\omega^\ell|} \left( \tilde{\mathcal{F}}_{\mathbf{A}}^{\ell\tau} + \mathcal{D}_{\mathbf{A}}^{\ell\tau} + \mathcal{G}_{\mathbf{A}}^{\ell\tau} \right) \cdot \mathbf{n}^{\ell\tau} = -\frac{\boldsymbol{\alpha}^\ell}{\theta(\tau_1)} + \Pi_{\mathbf{A}}^\ell \frac{\mathbf{w}^\ell}{\text{tr}(\mathbf{w}^\ell \mathbf{w}^{\ell,\top})}, \quad (56)$$

291 *with the geometrically compatible fluxes (43), the non-conservative products (40), the dissipation terms (41) and the*  
 292 *production term (53), satisfies the extra conservation law (25) with the following conservative semi-discrete scheme:*

$$\frac{\partial |\mathbf{A}|^\ell}{\partial t} + \sum_{\tau \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell\tau}|}{|\omega^\ell|} \frac{1}{2} \left( F_{|\mathbf{A}|}^\ell + F_{|\mathbf{A}|}^\tau \right) \cdot \mathbf{n}^{\ell\tau} = 0. \quad (57)$$

293 *Proof.* Let us recall that the discrete Gauss theorem over a closed surface yields the relation

$$\sum_{\tau \in \mathcal{N}^\ell} |\partial \omega^{\ell\tau}| \mathbf{n}^{\ell\tau} = \mathbf{0}. \quad (58)$$

294 **Dot multiplying** the distortion equation (56) by the dual variables  $\mathbf{w}^\ell$ , we obtain

$$\mathbf{w}^\ell \cdot \frac{\partial \mathbf{A}^\ell}{\partial t} + \sum_{\tau \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell\tau}|}{|\omega^\ell|} \mathbf{w}^\ell \cdot \left( \tilde{\mathcal{F}}_{\mathbf{A}}^{\ell\tau} + \mathcal{D}_{\mathbf{A}}^{\ell\tau} + \mathcal{G}_{\mathbf{A}}^{\ell\tau} \right) \cdot \mathbf{n}^{\ell\tau} = \mathbf{w}^\ell \cdot \left( -\frac{\boldsymbol{\alpha}^\ell}{\theta(\tau_1)} + \Pi_{\mathbf{A}}^\ell \frac{\mathbf{w}^\ell}{\text{tr}(\mathbf{w}^\ell \mathbf{w}^{\ell,\top})} \right). \quad (59)$$

295 On the right hand side, we have

$$\mathbf{w}^\ell \cdot \left( -\frac{\boldsymbol{\alpha}^\ell}{\theta(\tau_1)} + \Pi_{\mathbf{A}}^\ell \frac{\mathbf{w}^\ell}{\text{tr}(\mathbf{w}^\ell \mathbf{w}^{\ell,\top})} \right) = 0 + \Pi_{\mathbf{A}}^\ell, \quad (60)$$

296 where the first term vanishes thanks to the compatibility condition (55) (see Appendix A) and the second term verifies  
 297 by construction the relation

$$\mathbf{w}^\ell \cdot \Pi_{\mathbf{A}}^\ell \frac{\mathbf{w}^\ell}{\text{tr}(\mathbf{w}^\ell \mathbf{w}^{\ell,\top})} = w_{ik}^\ell \Pi_{\mathbf{A}}^\ell \frac{w_{ik}^\ell}{w_{ik}^\ell w_{ik}^\ell} = \Pi_{\mathbf{A}}^\ell. \quad (61)$$

298 On the left hand side of (59), we add and subtract the terms  $\frac{1}{2} \mathbf{w}^\tau \cdot \tilde{\mathcal{F}}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau}$ ,  $\frac{1}{2} \mathbf{w}^\tau \cdot \mathcal{D}_{\mathbf{A}}^{\tau\ell} \cdot \mathbf{n}^{\tau\ell}$  and  $\frac{1}{2} \mathbf{w}^\tau \cdot \mathcal{G}_{\mathbf{A}}^{\tau\ell} \cdot \mathbf{n}^{\tau\ell}$ , hence  
 299 obtaining

$$\begin{aligned} & \frac{\partial |\mathbf{A}|^\ell}{\partial t} + \frac{1}{2} \sum_{\tau \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell\tau}|}{|\omega^\ell|} \left( (\mathbf{w}^\ell + \mathbf{w}^\tau) \cdot \tilde{\mathcal{F}}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} + (\mathbf{w}^\ell - \mathbf{w}^\tau) \cdot \tilde{\mathcal{F}}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} \right) \\ & + \frac{1}{2} \sum_{\tau \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell\tau}|}{|\omega^\ell|} \left( \mathbf{w}^\ell \cdot \mathcal{D}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} + \mathbf{w}^\tau \cdot \mathcal{D}_{\mathbf{A}}^{\tau\ell} \cdot \mathbf{n}^{\tau\ell} + \mathbf{w}^\ell \cdot \mathcal{D}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} - \mathbf{w}^\tau \cdot \mathcal{D}_{\mathbf{A}}^{\tau\ell} \cdot \mathbf{n}^{\tau\ell} \right) \\ & + \frac{1}{2} \sum_{\tau \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell\tau}|}{|\omega^\ell|} \left( (\mathbf{w}^\ell + \mathbf{w}^\tau) \cdot \mathcal{G}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} + (\mathbf{w}^\ell - \mathbf{w}^\tau) \cdot \mathcal{G}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} \right) \\ & = \Pi_{\mathbf{A}}^\ell. \end{aligned} \quad (62)$$

300 Due to the continuity of the computational mesh, it holds that  $\mathbf{n}^{\ell\tau} = -\mathbf{n}^{\tau\ell}$ . Furthermore, the term  $(\mathbf{w}^\ell - \mathbf{w}^\tau) \cdot \tilde{\mathcal{F}}_{\mathbf{A}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau}$

301 can be rewritten by means of the compatibility condition (44), which leads to

$$\begin{aligned}
& \frac{\partial |\mathbf{A}|^\ell}{\partial t} + \frac{1}{2} \sum_{z \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell z}|}{|\omega^\ell|} (\mathbf{w}^\ell + \mathbf{w}^z) \cdot \tilde{\mathcal{F}}_{\mathbf{A}}^{\ell z} \cdot \mathbf{n}^{\ell z} \\
& + \frac{1}{2} \sum_{z \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell z}|}{|\omega^\ell|} \left( (|\mathbf{A}|v_k)^z - (|\mathbf{A}|v_k)^\ell \right) + \left( \mathbf{w}^\ell \cdot \mathbf{f}_{\mathbf{A},k}^\ell - \mathbf{w}^z \cdot \mathbf{f}_{\mathbf{A},k}^z \right) n_k^{\ell z} \\
& + \frac{1}{2} \sum_{z \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell z}|}{|\omega^\ell|} \left( \mathbf{w}^\ell \cdot \mathcal{D}_{\mathbf{A}}^{\ell z} + \mathbf{w}^z \cdot \mathcal{D}_{\mathbf{A}}^{z\ell} \right) \cdot \mathbf{n}^{\ell z} \\
& + \frac{1}{2} \sum_{z \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell z}|}{|\omega^\ell|} \left( (\mathbf{w}^\ell + \mathbf{w}^z) \cdot \mathcal{G}_{\mathbf{A}}^{\ell z} \cdot \mathbf{n}^{\ell z} + (\mathbf{w}^\ell - \mathbf{w}^z) \cdot \mathcal{G}_{\mathbf{A}}^{z\ell} \cdot \mathbf{n}^{\ell z} \right) \\
& = \Pi_{\mathbf{A}}^\ell.
\end{aligned} \tag{63}$$

By virtue of the discrete Gauss theorem (58), we can add to the left hand side of the above equation a zero term given by

$$\sum_{z \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell z}|}{|\omega^\ell|} \left( (|\mathbf{A}|v_k)^\ell - \mathbf{w}^\ell \cdot \mathbf{f}_{\mathbf{A},k}^\ell \right) n_k^{\ell z} = 0,$$

302 and we reformulate the numerical dissipation according to (47)-(48) with the Roe-type property (49), hence obtaining

$$\begin{aligned}
& \frac{\partial |\mathbf{A}|^\ell}{\partial t} + \frac{1}{2} \sum_{z \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell z}|}{|\omega^\ell|} (\mathbf{w}^\ell + \mathbf{w}^z) \cdot \tilde{\mathcal{F}}_{\mathbf{A}}^{\ell z} \cdot \mathbf{n}^{\ell z} \\
& + \frac{1}{2} \sum_{z \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell z}|}{|\omega^\ell|} \left( (|\mathbf{A}|v_k)^z + (|\mathbf{A}|v_k)^\ell \right) n_k^{\ell z} \\
& - \frac{1}{2} \sum_{z \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell z}|}{|\omega^\ell|} \left( \mathbf{w}^\ell \cdot \mathbf{f}_{\mathbf{A},k}^\ell + \mathbf{w}^z \cdot \mathbf{f}_{\mathbf{A},k}^z \right) n_k^{\ell z} \\
& + \frac{1}{2} \sum_{z \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell z}|}{|\omega^\ell|} \left( \mathbf{w}^\ell \cdot \mathcal{D}_{\mathbf{A}}^{\ell z} + \mathbf{w}^z \cdot \mathcal{D}_{\mathbf{A}}^{z\ell} \right) \cdot \mathbf{n}^{\ell z} \\
& + \sum_{z \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell z}|}{|\omega^\ell|} \frac{1}{2} \epsilon^{\ell z} (\mathbf{A}^z - \mathbf{A}^\ell) \cdot \partial_{\mathbf{A}\mathbf{A}}^2 |\mathbf{A}|^{\ell z} (\mathbf{A}^z - \mathbf{A}^\ell) - \epsilon^{\ell z} (|\mathbf{A}|^z - |\mathbf{A}|^\ell) \\
& = \Pi_{\mathbf{A}}^\ell.
\end{aligned} \tag{64}$$

303 The last term on the left hand side partially cancels with the production term  $\Pi_{\mathbf{A}}^\ell$ , that follows by the definition (53).  
304 Therefore, the extra conservation law (25) is satisfied by defining the following fluxes in the semi-discrete finite  
305 volume scheme (57):

$$\begin{aligned}
F_{|\mathbf{A}|}^\ell \cdot \mathbf{n}^{\ell z} &= \left( (|\mathbf{A}|v_k)^\ell - \mathbf{w}^\ell \cdot \mathbf{f}_{\mathbf{A},k}^\ell \right) n_k^{\ell z} + \mathbf{w}^\ell \cdot \left( \tilde{\mathcal{F}}_{\mathbf{A}}^{\ell z} + \mathcal{D}_{\mathbf{A}}^{\ell z} \right) \cdot \mathbf{n}^{\ell z} + 2 \epsilon^{\ell z} |\mathbf{A}|^\ell, \\
F_{|\mathbf{A}|}^z \cdot \mathbf{n}^{\ell z} &= \left( (|\mathbf{A}|v_k)^z - \mathbf{w}^z \cdot \mathbf{f}_{\mathbf{A},k}^z \right) n_k^{\ell z} + \mathbf{w}^z \cdot \left( \tilde{\mathcal{F}}_{\mathbf{A}}^{\ell z} + \mathcal{D}_{\mathbf{A}}^{z\ell} \right) \cdot \mathbf{n}^{\ell z} - 2 \epsilon^{\ell z} |\mathbf{A}|^z.
\end{aligned} \tag{65}$$

306 □

### 307 3.3. Thermodynamically compatible finite volume scheme

308 After achieving compatibility with the extra conservation law (25), the semi-discrete finite volume scheme must be  
309 modified again to be compliant with the Second Law of Thermodynamics, meaning that it must fulfill also the entropy  
310 balance (1c). This is equivalent to satisfy the Gibbs relation (14), implying that we need to work with all the state  
311 variables  $\mathbf{u}$  plus the density. However, thanks to the geometrically compatible discretization achieved for the distortion

312 tensor  $\mathbf{A}$ , we can deduce the density directly from the determinant of  $\mathbf{A}$  as  $\rho = \rho_0|\mathbf{A}|$ , therefore the full vector of state  
 313 variables is simply given by  $\tilde{\mathbf{u}} = (\rho_0|\mathbf{A}|, \mathbf{u})^\top = (\rho_0|\mathbf{A}|, \rho v_i, \varepsilon, A_{ik}, J_k)^\top$ . The thermodynamic correction is carried  
 314 out in analogy with the one employed for the geometric compatibility, hence we introduce a modified set of numerical  
 315 fluxes of the form

$$\hat{\mathcal{F}}(\tilde{\mathbf{u}}^\ell, \tilde{\mathbf{u}}^\tau) \cdot \mathbf{n}^{\ell\tau} = \tilde{\mathcal{F}}(\tilde{\mathbf{u}}^\ell, \tilde{\mathbf{u}}^\tau) - \alpha_S^{\ell\tau} (\tilde{\mathbf{r}}^\tau - \tilde{\mathbf{r}}^\ell), \quad (66)$$

316 where  $\alpha_S^{\ell\tau}$  is a scalar correction factor that must be determined to obtain thermodynamic compatibility. The fluxes  
 317  $\tilde{\mathcal{F}}(\tilde{\mathbf{u}}^\ell, \tilde{\mathbf{u}}^\tau) \cdot \mathbf{n}^{\ell\tau}$  coincide with the central fluxes for all equations of (30) except for the distortion tensor equation, for  
 318 which they are given by (43). Furthermore, the flux in the continuity equation (1a) is computed from the compatible  
 319 fluxes (65) of the semi-discrete equation for  $|\mathbf{A}|$  given by (57) upon multiplication by  $\rho_0^\ell$ . Consequently, we have that

$$\tilde{\mathcal{F}}(\tilde{\mathbf{u}}^\ell, \tilde{\mathbf{u}}^\tau) \cdot \mathbf{n}^{\ell\tau} = \begin{cases} \frac{1}{2} \rho_0^\ell \left( F_{|\mathbf{A}|}^\ell + F_{|\mathbf{A}|}^\tau \right) \cdot \mathbf{n}^{\ell\tau} & \text{for (1a) with fluxes (65)} \\ \mathcal{F}(\mathbf{u}^\ell, \mathbf{u}^\tau) \cdot \mathbf{n}^{\ell\tau} & \text{for (30a)-(30b)-(30d)} \\ \tilde{\mathcal{F}}_{\mathbf{A}}(\mathbf{u}^\ell, \mathbf{u}^\tau) \cdot \mathbf{n}^{\ell\tau} & \text{for (30c)} \end{cases}. \quad (67)$$

320 Since the compliance with the GCL must not be destroyed by this new modification, we deliberately choose to add the  
 321 correction factor  $\alpha_S^{\ell\tau}$  to only a subset of dual variables among  $\mathbf{r}$  in (15), which is referred to as  $\tilde{\mathbf{r}}$  in (66). Specifically,  
 322 we allow the momentum and the thermal impulse equations to account for the thermodynamic compatibility, thus  
 323 defining

$$\tilde{\mathbf{r}} = \frac{1}{T} (0, \partial_{\rho v} \varepsilon, 0, \mathbf{0}, \partial_{\mathbf{J}} \varepsilon)^\top = \{\tilde{r}_j\} = \frac{1}{T} (0, -v_i, 0, 0_{ik}, -\beta_k)^\top. \quad (68)$$

324 We remark that the equations for density and distortion tensor are not affected by the thermodynamic corrections  
 325 since they already carry the geometric compatibility correction. Let us also note that the total energy equation is not  
 326 modified in order to maintain stationary solutions of the governing PDE, as explained at the end of this section. To  
 327 compute the correction factor  $\alpha_S^{\ell\tau}$  we can now proceed along the lines of [1, 3], hence requiring that the sum of all the  
 328 fluctuations across an element interface is equal to the flux difference of the entropy equation (1c), thus leading to

$$\begin{aligned} \mathbf{r}^\ell \cdot \left( \hat{\mathcal{F}}(\tilde{\mathbf{u}}^\ell, \tilde{\mathbf{u}}^\tau) \cdot \mathbf{n}^{\ell\tau} - \mathbf{f}_k^\ell \cdot n_k^{\ell\tau} \right) + \mathbf{r}^\tau \cdot \left( \mathbf{f}_k^\tau \cdot n_k^{\ell\tau} - \hat{\mathcal{F}}(\tilde{\mathbf{u}}^\ell, \tilde{\mathbf{u}}^\tau) \cdot \mathbf{n}^{\ell\tau} \right) + \\ \mathbf{r}^\ell \cdot \mathcal{D}(\tilde{\mathbf{u}}^\ell, \tilde{\mathbf{u}}^\tau) \cdot \mathbf{n}^{\ell\tau} + \mathbf{r}^\tau \cdot \mathcal{D}(\tilde{\mathbf{u}}^\tau, \tilde{\mathbf{u}}^\ell) \cdot \mathbf{n}^{\ell\tau} = \left( (\mathcal{S}v_k + \beta_k)^\tau - (\mathcal{S}v_k + \beta_k)^\ell \right) n_k^{\ell\tau}. \end{aligned} \quad (69)$$

329 By employing the flux definition (66) in the conservation condition (69), the thermodynamic correction scalar  $\alpha_S^{\ell\tau}$  is  
 330 found to be given by

$$\begin{aligned} \alpha_S^{\ell\tau} = \frac{\left( (\mathcal{S}v_k + \beta_k)^\tau - (\mathcal{S}v_k + \beta_k)^\ell \right) n_k^{\ell\tau} + \left( \tilde{\mathcal{F}}(\tilde{\mathbf{u}}^\ell, \tilde{\mathbf{u}}^\tau) \cdot \mathbf{n}^{\ell\tau} \right) \cdot (\mathbf{r}^\tau - \mathbf{r}^\ell) - (\mathbf{r}^\tau \cdot \mathbf{f}_k^\tau - \mathbf{r}^\ell \cdot \mathbf{f}_k^\ell) n_k^{\ell\tau}}{(\tilde{\mathbf{r}}^\tau - \tilde{\mathbf{r}}^\ell)^2} \\ - \frac{(\mathbf{r}^\tau + \mathbf{r}^\ell) \cdot \mathcal{D}(\tilde{\mathbf{u}}^\ell, \tilde{\mathbf{u}}^\tau) \cdot \mathbf{n}^{\ell\tau}}{(\tilde{\mathbf{r}}^\tau - \tilde{\mathbf{r}}^\ell)^2}. \end{aligned} \quad (70)$$

331 Obviously, we set  $\alpha_S^{\ell\tau} = 0$  if  $\tilde{\mathbf{r}}^\ell = \tilde{\mathbf{r}}^\tau$ . **Even in this case, the correction factor  $\alpha_S$  may be unbounded, and no special**  
 332 **treatment is numerically applied apart from avoiding division by zero.** The source terms in equations (30c) and (30d)  
 333 are compatible by construction also for thermodynamic compatibility. Indeed, multiplication of the sources by the  
 334 dual variables  $\partial_{A_{ik}^\ell} \varepsilon = \alpha_{ik}^\ell$  and  $\partial_{J_k^\ell} \varepsilon = \beta_k^\ell$ , with negative sign and divided by the temperature according to (14), yields

$$-\frac{\alpha_{ik}^\ell}{T^\ell} \left( -\frac{\alpha_{ik}^\ell}{\theta_1(\tau_1)} \right) - \frac{\beta_k^\ell}{T^\ell} \left( -\frac{\beta_k^\ell}{\theta_2(\tau_2)} \right) = \frac{\alpha_{ik}^\ell \alpha_{ik}^\ell}{\theta_1(\tau_1) T^\ell} + \frac{\beta_k^\ell \beta_k^\ell}{\theta_2(\tau_2) T^\ell} \geq 0, \quad (71)$$

335 which is exactly the source term in the entropy equation (1c).

336 The geometrically and thermodynamically compatible semi-discrete finite volume scheme without numerical dis-  
 337 sipation is then given by

$$\frac{\partial \mathbf{u}^\ell}{\partial t} + \sum_{\tau \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell\tau}|}{|\omega^\ell|} \left( \hat{\mathcal{F}}(\mathbf{u}^\ell, \mathbf{u}^\tau)^{\ell\tau} + \mathcal{D}(\mathbf{u}^\ell, \mathbf{u}^\tau)^{\ell\tau} \right) \cdot \mathbf{n}^{\ell\tau} = \mathbf{S}(\mathbf{u}^\ell), \quad (72)$$

338 with the definition of the compatible fluxes (66)-(67) and the thermodynamic correction factor (70).

339 **Remark** (On the energy equation in the thermodynamic correction). *The subset of dual variables  $\tilde{\mathbf{r}}$  in the flux cor-*  
 340 *rection (66) does not take into account the dual variable in the energy equation, which is equal to  $1/T$ . Without loss*  
 341 *of generality, let us consider a computational domain  $\Omega = [-L; L]^2$  with periodic boundaries and  $L \in \mathbb{R}$ , and the*  
 342 *following initial condition with only a discontinuity in the density field located at  $\|\mathbf{x}\| = R_0 \subset \Omega$ :*

$$\rho(\mathbf{x}, t = 0) = \begin{cases} \rho_0^L & \text{for } \|\mathbf{x}\| \leq R_0 \\ \rho_0^R & \text{for } \|\mathbf{x}\| > R_0 \end{cases}, \quad \mathbf{v}(\mathbf{x}, t = 0) = \mathbf{0}, \quad p(\mathbf{x}, t = 0) = p_0, \quad \mathbf{A}(\mathbf{x}, t = 0) = \mathbf{I}, \quad \mathbf{J}(\mathbf{x}, t = 0) = \mathbf{0}. \quad (73)$$

343 In this case, the total energy is only given by the internal energy contribution which is constant, namely

$$\mathcal{E}(\mathbf{x}, t = 0) = \frac{p_0}{\gamma - 1}, \quad (74)$$

344 therefore the semi-discrete finite volume scheme (72) yields

$$\frac{\partial \mathbf{u}^\ell}{\partial t} = \mathbf{0}, \quad (75)$$

345 and the initial condition (73) also represents the exact solution. The correction factor is a priori nonzero, i.e.  $\alpha_S^{\ell\tau} \neq 0$ ,  
 346 because of the pressure and temperature terms in the momentum and thermal impulse equations, respectively. If we  
 347 add the thermodynamic correction in the energy equation, the associated flux  $\hat{\mathcal{F}}_\mathcal{E}(\mathbf{u}^\ell, \mathbf{u}^\tau) \cdot \mathbf{n}^{\ell\tau}$  should be corrected  
 348 with the jump term in the dual variable  $1/T$ , leading to

$$\hat{\mathcal{F}}_\mathcal{E}(\mathbf{u}^\ell, \mathbf{u}^\tau) \cdot \mathbf{n}^{\ell\tau} = \mathcal{F}_\mathcal{E}(\mathbf{u}^\ell, \mathbf{u}^\tau) \cdot \mathbf{n}^{\ell\tau} - \alpha_S^{\ell\tau} \left( \frac{1}{T^\tau} - \frac{1}{T^\ell} \right). \quad (76)$$

349 This would no longer preserve the constant energy density (74) because the discontinuity in the density profile causes  
 350 a jump in the temperature which is defined as

$$T(\mathbf{x}, t = 0) = \frac{p_0}{\rho(\mathbf{x}, t = 0) c_v (\gamma - 1)}. \quad (77)$$

351 As a consequence, the artificial flux (76) is not physical and thus the thermodynamic dual variable  $1/T$  is not included  
 352 in the dual vector  $\tilde{\mathbf{r}}$  for computing the scalar factor  $\alpha_S^{\ell\tau}$  in (70), so that the geometrically and thermodynamically  
 353 compatible scheme still maintains this physical equilibrium, i.e. we obtain again the correct stationary solution given  
 354 by (75).

355 *Thermodynamic compatibility with dissipation terms.* As done for the geometric compatibility, to ensure the stability  
 356 of the scheme in case of discontinuous solutions we also take into account the parabolic dissipation terms (41),  
 357 thus we supplement the geometrically and thermodynamically compatible scheme (72) with the dissipative fluxes  
 358  $\mathcal{G}(\mathbf{u}^\ell, \mathbf{u}^\tau) \cdot \mathbf{n}^{\ell\tau}$ , hence giving rise to the non-negative production term  $\Pi^\ell$  in the entropy inequality (1c).

359 **Theorem 2** (Thermodynamic compatibility). *The semi-discrete finite volume scheme for the reduced model (30)*

$$\frac{\partial \mathbf{u}^\ell}{\partial t} + \sum_{\tau \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell\tau}|}{|\omega^\ell|} \left( \hat{\mathcal{F}}(\mathbf{u}^\ell, \mathbf{u}^\tau)^{\ell\tau} + \mathcal{D}(\mathbf{u}^\ell, \mathbf{u}^\tau)^{\ell\tau} + \mathcal{G}(\mathbf{A}^\ell, \mathbf{A}^\tau) \right) \cdot \mathbf{n}^{\ell\tau} = \mathbf{S}(\mathbf{u}^\ell), \quad (78)$$

360 with the geometrically and thermodynamically compatible fluxes (66)-(67), the non-conservative products (40) and  
 361 the dissipation terms (41), satisfies the extra conservation law (1c) with the following conservative semi-discrete  
 362 scheme:

$$\frac{\partial \mathcal{S}^\ell}{\partial t} + \sum_{\tau \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell\tau}|}{|\omega^\ell|} \frac{1}{2} \left( F_S^\ell + F_S^\tau \right) \cdot \mathbf{n}^{\ell\tau} = \Pi^\ell + \frac{\alpha_{ik}^\ell \alpha_{ik}^\tau}{\theta_1(\tau_1) T^\ell} + \frac{\beta_k^\ell \beta_k^\tau}{\theta_2(\tau_2) T^\ell}. \quad (79)$$



363 Furthermore, assuming  $T^\ell > 0$  and  $\partial_{\mathbf{u}\mathbf{u}}\mathcal{S}^{\ell\tau} \leq 0$ , the right hand side of the entropy balance is non-negative:

$$\Pi^\ell + \frac{\alpha_{ik}^\ell \alpha_{ik}^\ell}{\theta_1(\tau_1)T^\ell} + \frac{\beta_k^\ell \beta_k^\ell}{\theta_2(\tau_2)T^\ell} \geq 0, \quad (80)$$

364 therefore the scheme (78) also satisfies a cell entropy inequality.

365 *Proof.* The proof is similar to the one already carried out for Theorem 1. We introduce the abbreviations

$$\hat{\mathcal{F}}^{\ell\tau} := \hat{\mathcal{F}}^{\ell\tau}(\mathbf{u}^\ell, \mathbf{u}^\tau), \quad \mathcal{D}^{\ell\tau} := \mathcal{D}(\mathbf{u}^\ell, \mathbf{u}^\tau), \quad \mathcal{G}^{\ell\tau} := \mathcal{G}(\mathbf{u}^\ell, \mathbf{u}^\tau). \quad (81)$$

366 We consider the dual variables  $\tilde{\mathbf{u}} = (\rho_0|\mathbf{A}|, \mathbf{u})^\top$  where the density is directly evaluated from the determinant of the  
 367 distortion tensor, and the associated fluxes are computed using (65) multiplied by  $\rho_0^\ell$  in the GCL equation (57). With  
 368 a little abuse of notation, let us omit the tilde symbol and assume that the additional fluxes related to the density  
 369 equation are embedded in the flux tensor, thus we will simply use  $\mathbf{u}^\ell$  (and thus the dual variables  $\mathbf{r}^\ell$ ) and  $\hat{\mathcal{F}}^{\ell\tau}$ .  
 370 After dot multiplying the semi-discrete system (78) by the dual variables  $\mathbf{r}^\ell$ , and adding and subtracting the terms  
 371  $\frac{1}{2} \mathbf{r}^\tau \cdot \hat{\mathcal{F}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau}$ ,  $\frac{1}{2} \mathbf{r}^\tau \cdot \mathcal{D}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau}$  and  $\frac{1}{2} \mathbf{r}^\tau \cdot \mathcal{G}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau}$ , we obtain

$$\begin{aligned} & \frac{\partial \mathcal{S}^\ell}{\partial t} + \frac{1}{2} \sum_{\tau \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell\tau}|}{|\omega^\ell|} \left( (\mathbf{r}^\ell + \mathbf{r}^\tau) \cdot \hat{\mathcal{F}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} + (\mathbf{r}^\ell - \mathbf{r}^\tau) \cdot \hat{\mathcal{F}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} \right) \\ & + \frac{1}{2} \sum_{\tau \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell\tau}|}{|\omega^\ell|} \left( \mathbf{r}^\ell \cdot \mathcal{D}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} + \mathbf{r}^\tau \cdot \mathcal{D}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} + \mathbf{r}^\ell \cdot \mathcal{D}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} - \mathbf{r}^\tau \cdot \mathcal{D}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} \right) \\ & + \frac{1}{2} \sum_{\tau \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell\tau}|}{|\omega^\ell|} \left( (\mathbf{r}^\ell + \mathbf{r}^\tau) \cdot \mathcal{G}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} + (\mathbf{r}^\ell - \mathbf{r}^\tau) \cdot \mathcal{G}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} \right) \\ & = \mathbf{r}^\ell \cdot \mathbf{S}(\mathbf{u}^\ell). \end{aligned} \quad (82)$$

372 We analyze the compatibility of the source terms, which explicitly write

$$\mathbf{r}^\ell \cdot \mathbf{S}(\mathbf{u}^\ell) = -\frac{\alpha_{ik}^\ell}{T^\ell} \left( -\frac{\alpha_{ik}^\ell}{\theta_1(\tau_1)} \right) - \frac{\beta_k^\ell}{T^\ell} \left( -\frac{\beta_k^\ell}{\theta_2(\tau_2)} \right) - \underbrace{\alpha_{ik}^\ell \Pi_{\mathbf{A}}^\ell \frac{w_{ik}^\ell}{w_{ik}^\ell w_{ik}^\ell}}_{=0} = \frac{\alpha_{ik}^\ell \alpha_{ik}^\ell}{\theta_1(\tau_1)T^\ell} + \frac{\beta_k^\ell \beta_k^\ell}{\theta_2(\tau_2)T^\ell}, \quad (83)$$

373 where the production term vanishes thanks to the relation  $\alpha_{ik}^\ell w_{ik}^\ell = 0$  as proven for the geometric compatibility (see  
 374 Appendix A). Thus we retrieve the source terms of the entropy balance law (79). Back to equation (82), we use the  
 375 compatibility condition (69) to rewrite the term  $(\mathbf{r}^\ell - \mathbf{r}^\tau) \cdot \hat{\mathcal{F}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau}$ , and we get

$$\begin{aligned} & \frac{\partial \mathcal{S}^\ell}{\partial t} + \frac{1}{2} \sum_{\tau \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell\tau}|}{|\omega^\ell|} (\mathbf{r}^\ell + \mathbf{r}^\tau) \cdot \hat{\mathcal{F}}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} \\ & + \frac{1}{2} \sum_{\tau \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell\tau}|}{|\omega^\ell|} \left( ((Sv_k + \beta_k)^\tau - (Sv_k + \beta_k)^\ell) + (\mathbf{r}^\ell \cdot \mathbf{f}_k^\ell - \mathbf{r}^\tau \cdot \mathbf{f}_k^\tau) \right) n_k^{\ell\tau} \\ & + \frac{1}{2} \sum_{\tau \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell\tau}|}{|\omega^\ell|} (\mathbf{r}^\ell \cdot \mathcal{D}^{\ell\tau} + \mathbf{r}^\tau \cdot \mathcal{D}^{\ell\tau}) \cdot \mathbf{n}^{\ell\tau} \\ & + \frac{1}{2} \sum_{\tau \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell\tau}|}{|\omega^\ell|} \left( (\mathbf{r}^\ell + \mathbf{r}^\tau) \cdot \mathcal{G}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} + (\mathbf{r}^\ell - \mathbf{r}^\tau) \cdot \mathcal{G}^{\ell\tau} \cdot \mathbf{n}^{\ell\tau} \right) \\ & = \frac{\alpha_{ik}^\ell \alpha_{ik}^\ell}{\theta_1(\tau_1)T^\ell} + \frac{\beta_k^\ell \beta_k^\ell}{\theta_2(\tau_2)T^\ell}. \end{aligned} \quad (84)$$

Adding on the left hand side term

$$\frac{1}{2} \sum_{\tau \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell\tau}|}{|\omega^\ell|} \left( (Sv_k + \beta_k)^\ell - \mathbf{r}^\ell \cdot \mathbf{f}_k^\ell \right) n_k^{\ell\tau} = 0,$$

376 which corresponds to a zero contribution thanks to the property (58), leads to

$$\begin{aligned}
& \frac{\partial \mathcal{S}^\ell}{\partial t} + \frac{1}{2} \sum_{z \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell z}|}{|\omega^\ell|} (\mathbf{r}^\ell + \mathbf{r}^z) \cdot \hat{\mathcal{F}}^{\ell z} \cdot \mathbf{n}^{\ell z} \\
& + \frac{1}{2} \sum_{z \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell z}|}{|\omega^\ell|} \left( (\mathcal{S} v_k + \beta_k)^z + (\mathcal{S} v_k + \beta_k)^\ell \right) n_k^{\ell z} \\
& - \frac{1}{2} \sum_{z \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell z}|}{|\omega^\ell|} \left( \mathbf{r}^\ell \cdot \mathbf{f}_k^\ell + \mathbf{r}^z \cdot \mathbf{f}_k^z \right) n_k^{\ell z} \\
& + \frac{1}{2} \sum_{z \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell z}|}{|\omega^\ell|} \left( \mathbf{r}^\ell \cdot \mathcal{D}^{\ell z} + \mathbf{r}^z \cdot \mathcal{D}^{z\ell} \right) \cdot \mathbf{n}^{\ell z} \\
& + \frac{1}{2} \sum_{z \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell z}|}{|\omega^\ell|} \left( (\mathbf{r}^\ell + \mathbf{r}^z) \cdot \mathcal{G}^{\ell z} \cdot \mathbf{n}^{\ell z} + (\mathbf{r}^\ell - \mathbf{r}^z) \cdot \mathcal{G}^{\ell z} \cdot \mathbf{n}^{\ell z} \right) \\
& = \frac{\alpha_{ik}^\ell \alpha_{ik}^\ell}{\theta_1(\tau_1) T^\ell} + \frac{\beta_k^\ell \beta_k^\ell}{\theta_2(\tau_2) T^\ell}.
\end{aligned} \tag{85}$$

377 Relying on the same reasoning applied for the geometric compatibility, the dissipation terms  $\mathbf{r}^\ell \cdot \mathcal{G}^{\ell z} \cdot \mathbf{n}^{\ell z}$  can be  
378 rearranged as in (47), that is

$$\mathbf{r}^\ell \cdot \mathcal{G}^{\ell z} \cdot \mathbf{n}^{\ell z} = \frac{1}{2} (\mathbf{r}^z - \mathbf{r}^\ell) \cdot \epsilon^{\ell z} (\mathbf{u}^z - \mathbf{u}^\ell) - \frac{1}{2} (\mathbf{r}^z + \mathbf{r}^\ell) \cdot \epsilon^{\ell z} (\mathbf{u}^z - \mathbf{u}^\ell). \tag{86}$$

379 Likewise in (48), due to the path integral

$$\int_{\mathbf{u}^\ell}^{\mathbf{u}^z} \mathbf{r} \cdot d\mathbf{u} = \int_{\mathbf{u}^\ell}^{\mathbf{u}^z} \partial_{\mathbf{u}} \mathcal{S} \cdot d\mathbf{u} = \mathcal{S}^z - \mathcal{S}^\ell, \tag{87}$$

380 we interpret the second term on the right hand side as an approximation of the jump term in the entropy variables,  
381 thus

$$-\frac{1}{2} (\mathbf{r}^z + \mathbf{r}^\ell) \cdot \epsilon^{\ell z} (\mathbf{u}^z - \mathbf{u}^\ell) \approx -\epsilon^{\ell z} (\mathcal{S}^z - \mathcal{S}^\ell). \tag{88}$$

382 The jump in the dual variables present in the first term in (86) is converted into a jump in the state variables by  
383 introducing the Hessian matrix  $\partial_{\mathbf{u}\mathbf{u}}^2 \mathcal{S}^{\ell z}$  which verifies the Roe property

$$\partial_{\mathbf{u}\mathbf{u}}^2 \mathcal{S}^{\ell z} \cdot (\mathbf{u}^z - \mathbf{u}^\ell) = \mathbf{r}^z - \mathbf{r}^\ell. \tag{89}$$

384 Therefore, using (88) and (89) in (85), we arrive at

$$\begin{aligned}
& \frac{\partial \mathcal{S}^\ell}{\partial t} + \frac{1}{2} \sum_{z \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell z}|}{|\omega^\ell|} (\mathbf{r}^\ell + \mathbf{r}^z) \cdot \hat{\mathcal{F}}^{\ell z} \cdot \mathbf{n}^{\ell z} \\
& + \frac{1}{2} \sum_{z \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell z}|}{|\omega^\ell|} \left( (\mathcal{S} v_k + \beta_k)^z + (\mathcal{S} v_k + \beta_k)^\ell \right) n_k^{\ell z} \\
& - \frac{1}{2} \sum_{z \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell z}|}{|\omega^\ell|} \left( \mathbf{r}^\ell \cdot \mathbf{f}_k^\ell + \mathbf{r}^z \cdot \mathbf{f}_k^z \right) n_k^{\ell z} \\
& + \frac{1}{2} \sum_{z \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell z}|}{|\omega^\ell|} \left( \mathbf{r}^\ell \cdot \mathcal{D}^{\ell z} + \mathbf{r}^z \cdot \mathcal{D}^{z\ell} \right) \cdot \mathbf{n}^{\ell z} \\
& - \sum_{z \in \mathcal{N}^\ell} \frac{|\partial \omega^{\ell z}|}{|\omega^\ell|} \epsilon^{\ell z} (\mathcal{S}^z - \mathcal{S}^\ell) \\
& = \Pi^\ell + \frac{\alpha_{ik}^\ell \alpha_{ik}^\ell}{\theta_1(\tau_1) T^\ell} + \frac{\beta_k^\ell \beta_k^\ell}{\theta_2(\tau_2) T^\ell},
\end{aligned} \tag{90}$$

385 with the production term given by

$$\Pi^\ell = - \sum_{\tau \in \mathcal{N}^\ell} \frac{|\partial\omega^{\ell\tau}|}{|\omega^\ell|} \frac{1}{2} \epsilon^{\ell\tau} (\mathbf{u}^\tau - \mathbf{u}^\ell) \cdot \partial_{\mathbf{uu}}^2 \mathcal{S}^{\ell\tau} (\mathbf{u}^\tau - \mathbf{u}^\ell). \quad (91)$$

386 The compatibility with the extra conservation law (1c) is then achieved by defining the following fluxes in the semi-  
387 discrete finite volume scheme (79):

$$\begin{aligned} F_S^\ell \cdot \mathbf{n}^{\ell\tau} &= ((\mathcal{S}v_k + \beta_k)^\ell - \mathbf{r}^\ell \cdot \mathbf{f}_k^\ell) n_k^{\ell\tau} + \mathbf{r}^\ell \cdot (\hat{\mathcal{F}}^{\ell\tau} + \mathcal{D}^{\ell\tau}) \cdot \mathbf{n}^{\ell\tau} + 2 \epsilon^{\ell\tau} \mathcal{S}^\ell, \\ F_S^\tau \cdot \mathbf{n}^{\ell\tau} &= ((\mathcal{S}v_k + \beta_k)^\tau - \mathbf{r}^\tau \cdot \mathbf{f}_k^\tau) n_k^{\ell\tau} + \mathbf{r}^\tau \cdot (\hat{\mathcal{F}}^{\ell\tau} + \mathcal{D}^{\ell\tau}) \cdot \mathbf{n}^{\ell\tau} - 2 \epsilon^{\ell\tau} \mathcal{S}^\tau. \end{aligned} \quad (92)$$

388 Finally, in the presence of numerical viscosity, i.e. when  $\epsilon^{\ell\tau} > 0$ , the entropy inequality is retrieved since the resulting  
389 term on the right hand side of (90) is non-negative, meaning that the positivity condition (80) is fulfilled due to the  
390 assumptions  $\theta_1 > 0$ ,  $\theta_2 > 0$ ,  $T^\ell > 0$  and  $\partial_{\mathbf{uu}}^2 \mathcal{S}^{\ell\tau} \leq 0$ . The cell entropy inequality is thus satisfied at the semi-discrete  
391 level by the finite volume scheme (78).  $\square$

### 392 3.4. Time discretization

393 The explicit time marching algorithm is given by Runge-Kutta schemes that are listed in Appendix B for order  
394 one, two and four. The associated time step is computed according to a classical CFL-type stability condition based  
395 on the maximum hyperbolic eigenvalue estimate given by (9) and the maximum viscous eigenvalue related to the  
396 parabolic dissipative terms:

$$\Delta t \leq \text{CFL} \frac{\min_{\ell \in \mathcal{N}^\ell} h^\ell}{\max_{\ell \in \mathcal{N}^\ell} (|\lambda^\ell| + 2 \frac{\epsilon^\ell}{h^\ell})}, \quad (93)$$

397 where CFL is the Courant-Friedrichs-Lewy number and  $h^\ell = \sqrt{|\omega^\ell|}$  is the characteristic cell size. **In the case of stiff**  
398 **source terms, i.e. when  $\tau_1 \rightarrow 0$  or  $\tau_2 \rightarrow 0$ , the time step must be reduced according to the time scale imposed by the**  
399 **sources because of the explicit time discretization.**

## 400 4. Numerical results

401 In this section, we propose a suite of test cases aiming at validating the accuracy and the robustness of the novel  
402 Hyperbolic Geometrically and Thermodynamically Compatible finite volume schemes (78), which will be labeled  
403 as HGTC. We demonstrate that the compatibility is preserved at the semi-discrete level up to the order of the time  
404 integrator, and we systematically measure the errors of mass conservation ( $\varepsilon_A$ ) and total entropy balance ( $\varepsilon_S$ ). More  
405 precisely, we monitor over time the following quantities in  $L_\infty$  norm over the entire computational domain  $\Omega$ :

$$\delta_{\mathbf{A}} = \| |\mathbf{A}| - \rho/\rho_0 \|_\infty, \quad \delta_S = \| \mathcal{S} - \mathcal{S}(\rho, p) \|_\infty. \quad (94)$$

406 For  $\delta_A$ , the quantity  $|\mathbf{A}|$  is computed by evaluating the determinant of the distortion tensor  $\mathbf{A}$  by using the components  
407  $A_{ik}$  that are evolved according to the semi-discrete scheme (56), whereas the quantity  $\rho/\rho_0$  is obtained with  $|\mathbf{A}|$  taken  
408 from the solution of the extra conservation law (25) discretized by the scheme (57) with fluxes (65). For  $\delta_S$ ,  $\mathcal{S}$  is  
409 the total entropy computed from the entropy equation (1c) solved as an extra conservation law with the semi-discrete  
410 scheme (79), while  $\mathcal{S}(\rho, p)$  is evaluated from the equation of state given by  $\varepsilon_1$  in (3), namely

$$\mathcal{S}(\rho, p) = \rho \log \left( \frac{p}{\rho^\gamma} c_v \right), \quad \rho = \rho_0 |\mathbf{A}|. \quad (95)$$

411 As such, the structure-preserving properties of the scheme are numerically investigated. If not stated otherwise, we  
412 set the CFL number to  $\text{CFL} = 0.5$  in (93) and the polytropic index of the gas is assumed to be  $\gamma = 7/5$ , whereas  
413 the specific heat at constant volume is always chosen to be  $c_v = 2.5$ . Whenever a viscosity coefficient  $\mu$  is specified,

414 the relaxation time  $\tau_1$  is computed according to  $\mu = \frac{1}{6}\rho_0 c_s^2 \tau_1$ . Likewise, if a heat conduction coefficient  $\kappa$  is set,  
415 the corresponding relaxation time  $\tau_2$  is evaluated from the asymptotic relation  $\kappa = \rho_0 T_0 c_h^2 \tau_2$ . In the other cases, no  
416 source terms are considered, thus we set  $\tau_1 = \tau_2 = 10^{20}$  hence retrieving the behavior of elastic solids without heat  
417 conduction. The distortion matrix is always initialized as  $\mathbf{A} = \mathbf{I}$ , and the thermal impulse is initially given  $\mathbf{J} = \mathbf{0}$ . The  
418 reference density and temperature are set to  $\rho_0 = \rho(\mathbf{x}, t = 0)$  and  $T_0 = 1$ , if not specified. We depict the absolute  
419 values of the correction factors  $\alpha_A$  and  $\alpha_S$  in (45) and (70), respectively, in order to better appreciate the order of  
420 magnitude and the location of the structure-preserving corrections. **If not specified otherwise, we use the fourth order**  
421 **Runge-Kutta scheme for time integration.**

#### 4.1. Numerical convergence studies

422 The accuracy of the new HGTC schemes is verified on the isentropic vortex problem forwarded in [37]. The  
423 computational domain is the square  $\Omega = [0; 10]^2$  with periodic boundaries, and the generic radial position is  $r =$   
424  $\sqrt{(x_1 - 5)^2 + (x_2 - 5)^2}$ . The parameters of the model are such that an ideal inviscid fluid is retrieved, hence we  
425 set  $c_s = c_h = 0$ , and the initial condition is prescribed in terms of some perturbations that are superimposed on a  
426 background constant state:  
427

$$\rho(t = 0, \mathbf{x}) = (1 + \delta T)^{\frac{1}{\gamma-1}}, \quad \mathbf{v}(t = 0, \mathbf{x}) = \mathbf{0}, \quad p(t = 0, \mathbf{x}) = (1 + \delta T)^{\frac{\gamma}{\gamma-1}}, \quad (96)$$

428 with the perturbations for temperature  $\delta T$  given by

$$\delta T = -\frac{(\gamma - 1)\epsilon^2}{8\gamma\pi^2} e^{1-r^2}. \quad (97)$$

429 The simulation is carried out until the final time  $t_f = 0.25$  on a sequence of successively refined Voronoi meshes, and  
430 the errors are measured in  $L_2$  norms and reported in Table 1, showing that the formal order of accuracy is retrieved.  
431 No numerical dissipation is added to the scheme because the flow does not exhibit any discontinuity, thus  $\epsilon^{\ell x} = 0$  in  
432 (41).

Table 1: Numerical convergence results for the isentropic vortex problem using the HGTC scheme. The errors are measured in the  $L_2$  norm and refer to the variables  $\rho = \rho_0 |\mathbf{A}|$  (density),  $v_1$  (horizontal velocity) and pressure  $p$  at time  $t_f = 0.25$ .

$h$	$\ \rho_0  \mathbf{A} \ _2$	$O(\rho_0  \mathbf{A} )$	$\ v_1\ _2$	$\mathcal{O}(v_1)$	$\ p\ _2$	$\mathcal{O}(p)$
3.20E-01	6.2483E-02	-	1.5675E-01	-	7.9011E-02	-
1.65E-01	3.1941E-02	1.01	7.9131E-02	1.03	4.0536E-02	1.01
1.09E-01	2.1427E-02	0.97	5.3292E-02	0.96	2.7139E-02	0.98
8.57E-02	1.6231E-02	1.14	3.9863E-02	1.19	2.0620E-02	1.13

433 We also use this test case to analyze the time convergence which ultimately affects the preservation of the deter-  
434 minant and the entropy compatibility. Therefore we measure the errors of the total mass and entropy conservation  
435 according to (94) while running this simulation until the time  $t_f = 1$  on one single unstructured mesh with character-  
436 istic size of  $h = 1/3$ . Three different Runge-Kutta time integrators are used of order  $N = \{1, 2, 4\}$  (see Appendix B),  
437 and the results are collected in Table 2. We observe that the convergence rates for the entropy conservation exhibits  
438 order of accuracy  $\mathcal{O}(N + 1)$ , and convergence of order  $\mathcal{O}(N + 2)$  is achieved for the total mass conservation. The time  
439 evolution of the mass and entropy conservation errors are plot in Figure 3, where we also show the map of the scalar  
440 correction factors  $\alpha_A$  and  $\alpha_S$  at the final time.

#### 4.2. Riemann problems

441 The novel HGTC scheme is here validated against three one-dimensional Riemann problems taken from [59, 2].  
442 The computational domain is the rectangular box  $\Omega = [-0.5; 0.5] \times [-0.05; 0.05]$  with periodic boundaries in the  
443  $y$ -direction and transmissive boundaries along the  $x$ -direction. The computational mesh is unstructured made of  
444

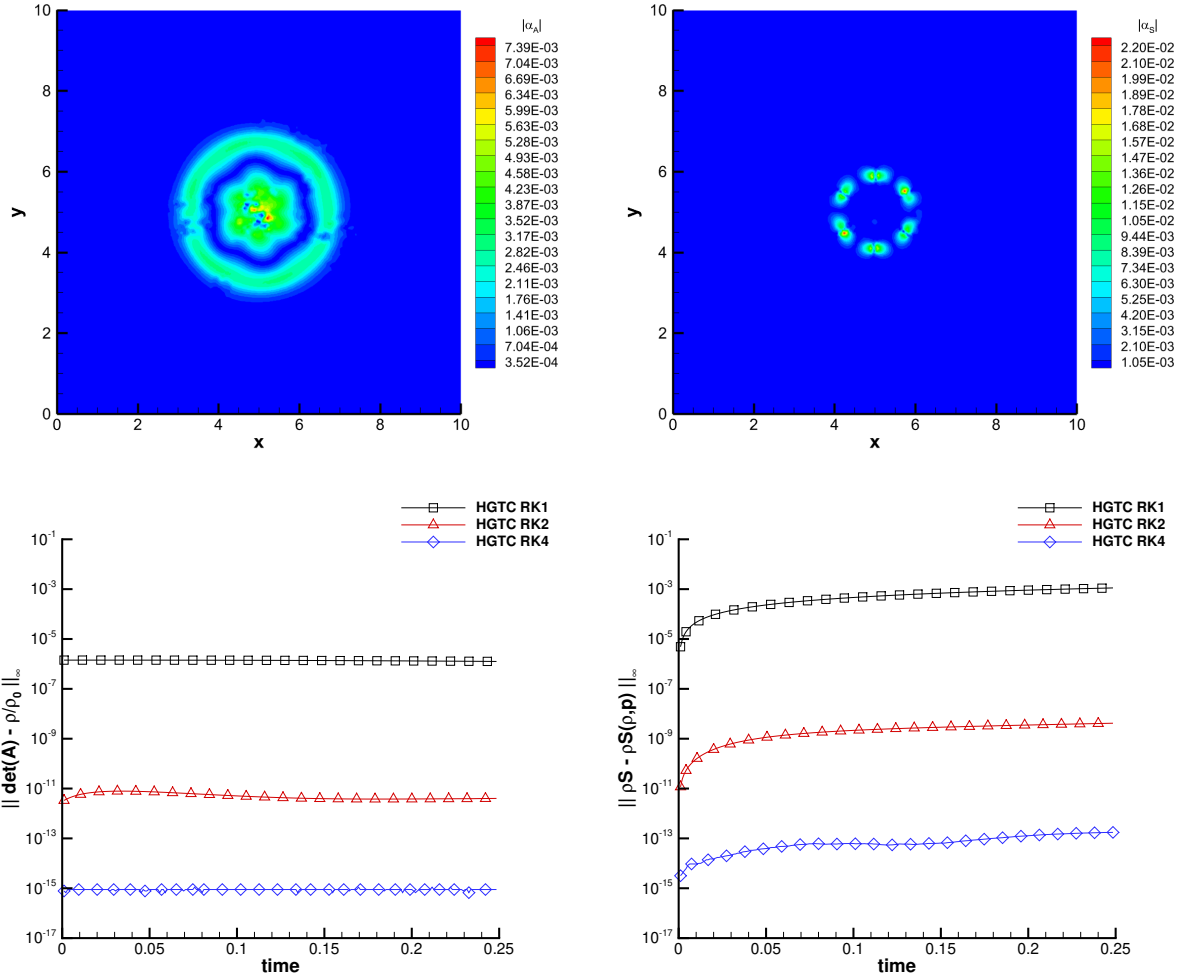


Figure 3: Isentropic vortex problem at time  $t_f = 0.25$ . Top: map of the geometric correction factor  $|\alpha_A|$  (left) and of the thermodynamic correction factor  $|\alpha_S|$  (right) with mesh size  $h = 1/6$ . Bottom: time evolution of the mass (left) and entropy (right) conservation errors for Runge-Kutta time integration schemes of order 1 (black line), 2 (red line) and 4 (blue line).

Table 2: Time convergence study related to total mass and entropy conservation for the isentropic vortex problem at time  $t_f = 1$  with three different Runge-Kutta time integration schemes on a mesh with size  $h = 1/3$ . The errors are measured in the  $L_\infty$  norm and refer to the geometric thermodynamic errors given by (94).

Runge-Kutta $\mathcal{O}(1)$				
$\Delta t$	$\delta_A$		$\delta_S$	
8.00E-03	5.1415E-05	-	2.2516E-02	-
4.00E-03	1.2874E-05	2.00	1.1238E-02	1.00
2.00E-03	3.2211E-06	2.00	5.6138E-03	1.00

Runge-Kutta $\mathcal{O}(2)$				
$\Delta t$	$\delta_A$		$\delta_S$	
8.00E-03	3.3043E-09	-	1.3195E-06	-
4.00E-03	3.3385E-10	3.31	2.5736E-07	2.36
2.00E-03	3.6793E-11	3.18	5.5271E-08	2.22

Runge-Kutta $\mathcal{O}(4)$				
$\Delta t$	$\delta_A$		$\delta_S$	
8.00E-03	2.9510E-13	-	6.6099E-11	-
4.00E-03	9.1038E-15	5.02	4.1208E-12	4.00
2.00E-03	2.8818E-16	4.98	2.5709E-13	4.00

445 polygons and it has a characteristic size of  $h = 1/4096$  and all the simulations are run in 2D, thus the properties of  
 446 symmetry preservation of the numerical solution are verified as well. Indeed, despite the one-dimensional setting of  
 447 the Riemann problems, these test cases become fully multidimensional in the case of unstructured Voronoi meshes,  
 448 where no mesh edges are in principle aligned with the flow. The initial condition is given in terms of a left and a right  
 449 state separated at position  $x = x_d$ . Table 3 summarizes the setup of the three Riemann problems considered here.

Table 3: Initialization of Riemann problems. Initial states left (L) and right (R) are reported as well as the final time of the simulation  $t_f$  and the position of the initial discontinuity  $x_d$ .

Name	$t_f$	$x_d$	$\rho_L$	$v_{1,L}$	$v_{2,L}$	$p_L$	$\rho_R$	$v_{1,R}$	$v_{2,R}$	$p_R$
RP1	0.035	-0.2	5.99924	19.5975	0.0	460.894	5.99924	-6.19633	0.0	46.095
RP2	0.15	0.0	1.0	-2.0	0.0	0.4	1.0	2.0	0.0	0.4
RP3	0.20	0.0	1.0	0.0	-0.2	1.0	0.5	0.0	0.2	0.5

450 The first two Riemann problems RP1 and RP2 involve the Euler equations for compressible gas dynamics (i.e.  
 451  $c_s = c_h = 0$ ), and the reference solution is computed with the exact Riemann solver detailed in [59]. The last Riemann  
 452 problem is concerned with the full model (1) and we set  $\mu = \kappa = 10^{-5}$ , so that the stiff relaxation limit of the model is  
 453 retrieved and numerically assessed. The reference solution is obtained numerically using a second order TVD finite  
 454 volume method on a very fine mesh of 100000 control volumes. The results are collected in Figures 4-6, showing  
 455 a good agreement with the reference solution in all cases. To appreciate that the one-dimensional symmetry of  
 456 the solution is well preserved, we show in Figure 7 a three-dimensional view of the solution for the three Riemann  
 457 problems considered here.

### 458 4.3. Circular explosion problem

459 We consider a cylindrical explosion problem to test the HGTC schemes with numerical dissipation, which is  
 460 here activated since the solution exhibits an outward traveling shock wave. The computational domain is given by

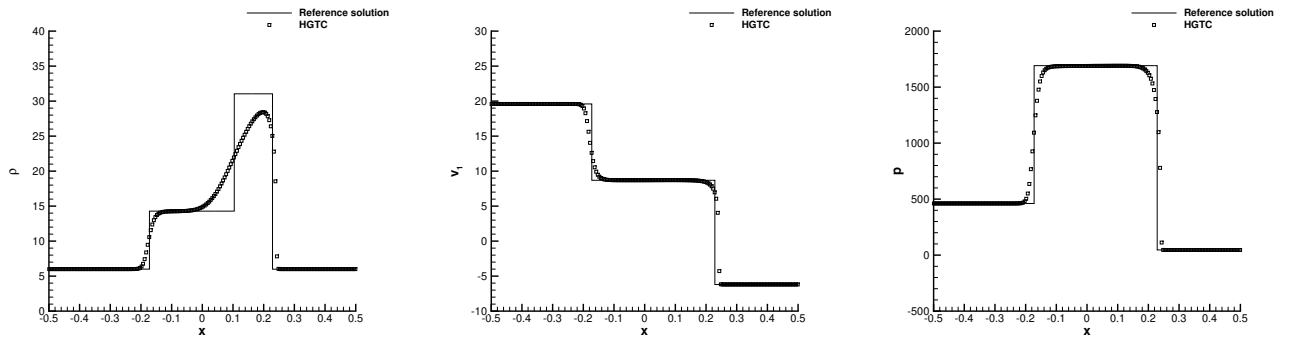


Figure 4: Riemann problem RP1 at final time  $t_f = 0.035$ . Comparison of density, horizontal velocity and pressure against the reference solution extracted with a one-dimensional cut of 200 equidistant points along the  $x$ -direction at  $y = 0$ .

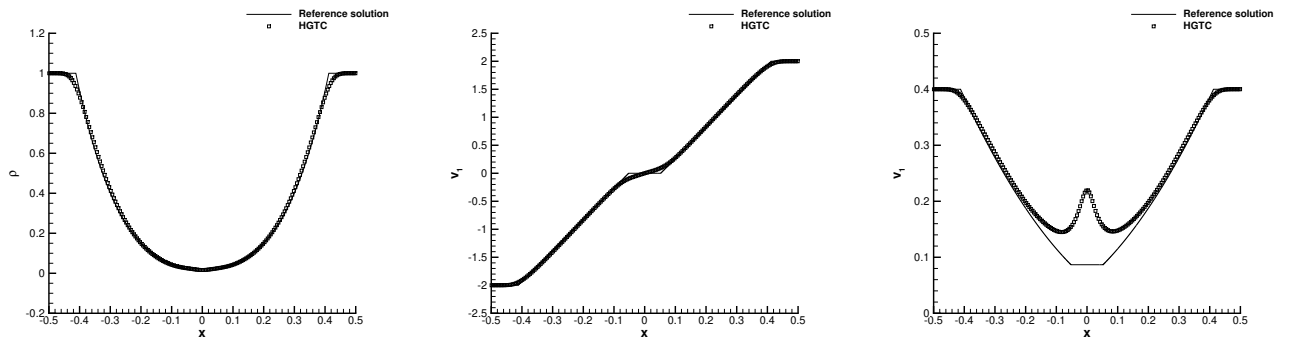


Figure 5: Riemann problem RP2 at final time  $t_f = 0.15$ . Comparison of density, horizontal velocity and temperature against the reference solution extracted with a one-dimensional cut of 200 equidistant points along the  $x$ -direction at  $y = 0$ .

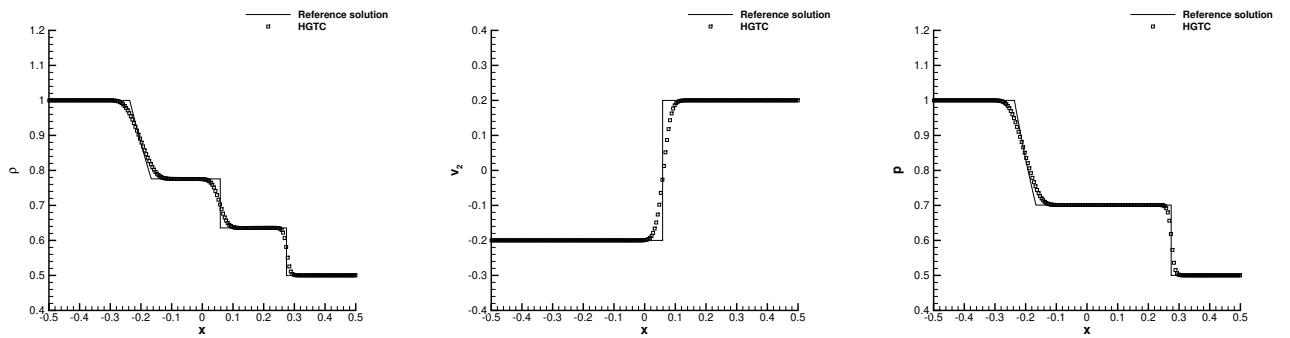


Figure 6: Riemann problem RP3 at final time  $t_f = 0.2$ . Comparison of density, vertical velocity and pressure against the reference solution for the compressible Euler equations extracted with a one-dimensional cut of 200 equidistant points along the  $x$ -direction at  $y = 0$ .

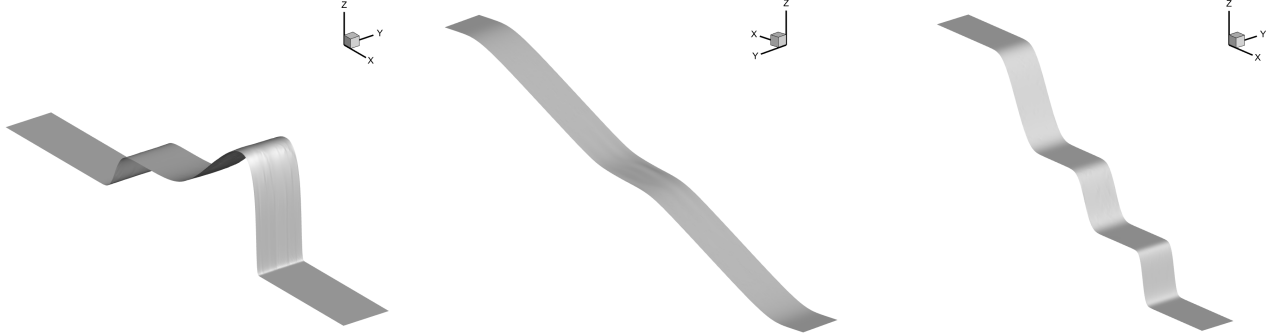


Figure 7: Three-dimensional view of density for RP1 (left), horizontal velocity for RP2 (middle) and density for RP3 (right) at their corresponding final times.

461  $\Omega = [-1; 1]^2$  with transmissive boundaries, and the fluid is initially assigned as follows:

$$(\rho, v_1, v_2, v_3, p) = \begin{cases} (1, 0, 0, 0, 1) & r < R \\ (0.125, 0, 0, 0, 0.1) & r \geq R \end{cases}, \quad t = 0, \quad \mathbf{x} \in \Omega, \quad (98)$$

462 where  $R = 0.5$  denotes the radius of the initial discontinuity and  $r = \sqrt{x_1^2 + x_2^2}$  represents the generic radial coordinate.  
 463 An inviscid fluid is considered by setting  $c_s = c_h = 0$  and the final time of the simulation is chosen to be  $t_f = 0.25$ . We  
 464 run this test on three different Voronoi meshes with characteristic mesh size of  $h = 1/256$ ,  $h = 1/128$  and  $h = 1/64$ .  
 465 The numerical results are compared against the reference solution that has been computed by solving the compressible  
 466 Euler equations with geometric sources [59] employing a classical second order TVD finite volume scheme on a very  
 467 fine mesh composed of 20000 cells. An overall very good agreement can be observed in Figure 8, that numerically  
 468 confirms the convergence of the new HGTC schemes as the mesh resolution gets finer. Figure 9 depicts a map of  
 469 the correction factor  $\alpha_A$  as well as the time evolution of the total mass conservation errors for all the simulations.  
 470 The solution preserves an excellent cylindrical symmetry despite the unstructured nature of the mesh, as shown by  
 471 the three-dimensional density distribution plot in Figure 9. Furthermore, we notice that the highest correction for the  
 472 preservation of the determinant constraint occurs across the contact wave.

#### 473 4.4. Viscous shock profile

474 Next, we model compressible heat-conducting viscous flows by setting  $c_s = c_h = 10$ ,  $\mu = 2 \cdot 10^{-2}$  and  
 475  $\kappa = 9.3333 \cdot 10^{-2}$ . The computational domain is the channel  $\Omega = [0; 1] \times [0; 0.2]$  that is paved with Voronoi poly-  
 476 gons of characteristic size of  $h = 1/1024$ . Periodic boundaries are imposed in  $y$ -direction, while a constant inflow  
 477 velocity is prescribed for  $x = 0$  and outflow boundary conditions are set at  $x = 1$ . In [5], an exact solution of the one-  
 478 dimensional compressible Navier-Stokes equations is derived for Prandtl number  $\text{Pr} = 0.75$  and constant viscosity that  
 479 involves a stationary viscous shock wave at a shock mach number  $M_s$ . The Reynolds number is  $\text{Re}_s = \rho_0 c_0 M_s L \mu^{-1}$ ,  
 480 with the reference length that is assumed to be  $L = 1$ . This is an interesting test case since all the terms characteristics  
 481 of the one-dimensional compressible Navier-Stokes equations can be verified, including viscous stress and heat con-  
 482 duction. According to [5], the exact solution is given in terms of dimensionless density, pressure and velocity. The  
 483 dimensionless velocity  $\bar{v} = \frac{v}{M_s c_0}$  is related to the stationary shock wave, which can be determined as the root of the  
 484 following equation:

$$\frac{|\bar{v} - 1|}{|\bar{v} - \lambda^2| \lambda^2} = \left| \frac{1 - \lambda^2}{2} \right|^{(1-\lambda^2)} \exp\left(\frac{3}{4} \text{Re}_s \frac{M_s^2 - 1}{\gamma M_s^2} x\right), \quad (99)$$

485 with

$$\lambda^2 = \frac{1 + \frac{\gamma-1}{2} M_s^2}{\frac{\gamma+1}{2} M_s^2}. \quad (100)$$



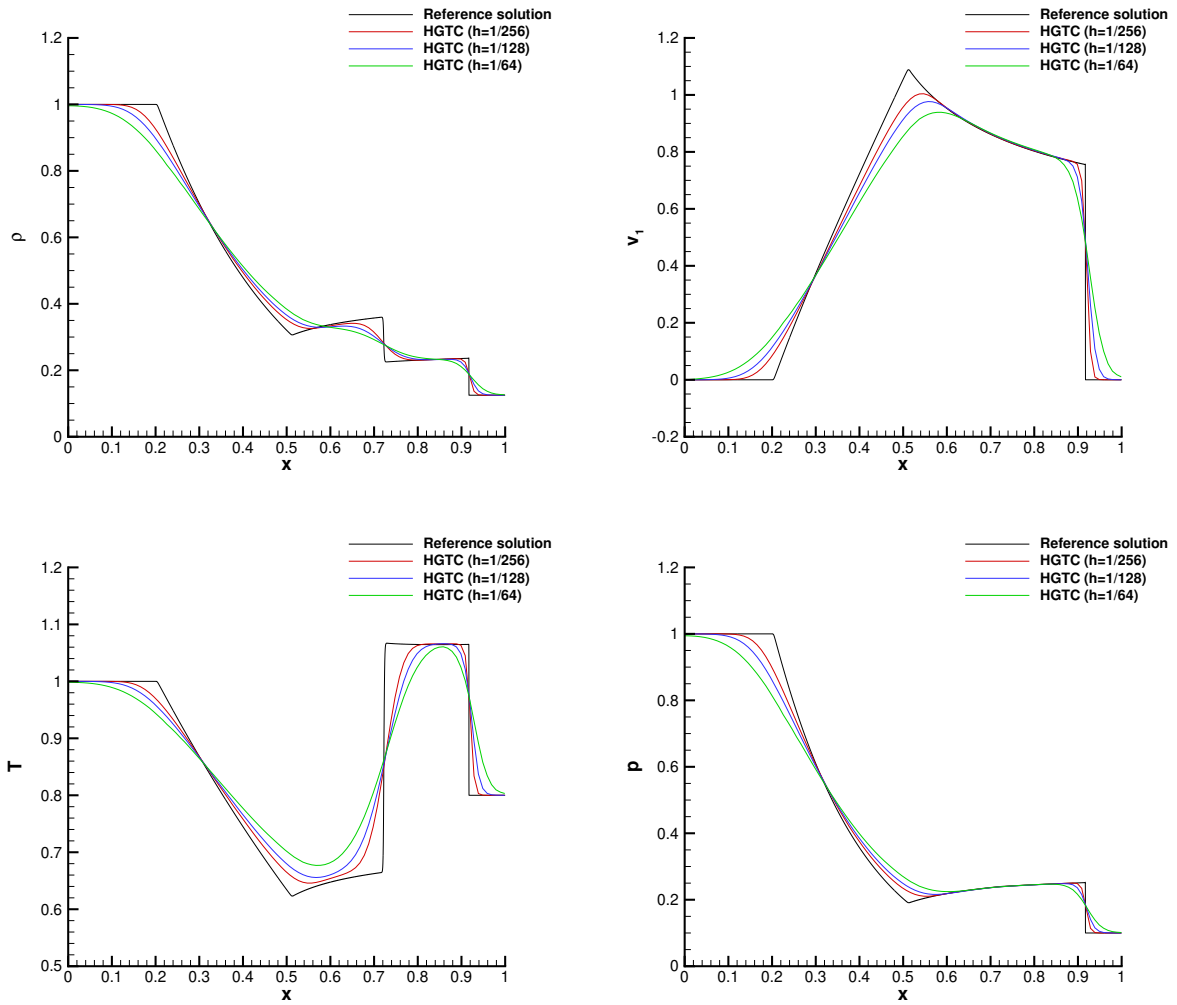


Figure 8: Explosion problem at time  $t_f = 0.25$ . Numerical results for density, horizontal velocity, pressure and temperature (from top left to bottom right) compared against the reference solution extracted with a one-dimensional cut of 200 equidistant points along the  $x$ -direction at  $y = 0$ . Mesh convergence analysis with characteristic mesh size  $h = 1/256$  (red line),  $h = 1/128$  (red line) and  $h = 1/64$  (green line).

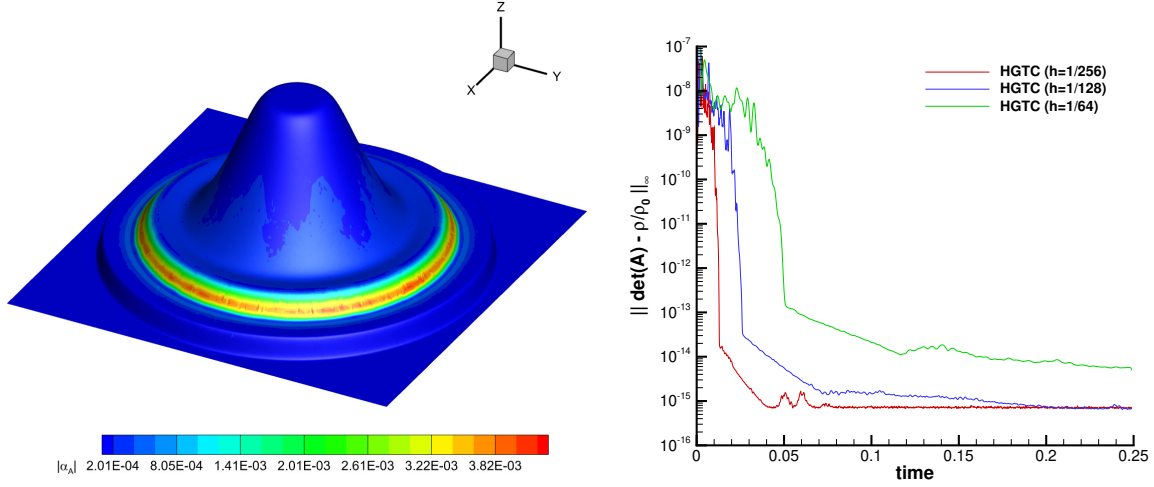


Figure 9: Explosion problem at time  $t_f = 0.25$ . Left: three-dimensional view of the density distribution with the map of the correction factor  $\alpha_A$  for the results obtained with  $h = 1/256$ . Right: time evolution of the mass conservation errors for  $h = 1/256$  (red line),  $h = 1/128$  (blue line) and  $h = 1/64$  (green line).

486 Once the solution of equation (99) is computed, the dimensionless velocity  $\bar{v}$  is expressed as a function of  $x$ . The form  
 487 of the viscous profile of the dimensionless pressure  $\bar{p} = \frac{p - p_0}{\rho_0 c_0^2 M_s^2}$  is given by the relation

$$\bar{p} = 1 - \bar{v} + \frac{1}{2\gamma} \frac{\gamma + 1}{\gamma - 1} \frac{(\bar{v} - 1)}{\bar{v}} (\bar{v} - \lambda^2). \quad (101)$$

488 Finally, the profile of the dimensionless density  $\bar{\rho} = \frac{\rho}{\rho_0}$  is derived from the integrated continuity equation:  $\bar{\rho}\bar{v} = 1$ .  
 489 Here, we make the simulation unsteady by adding a constant velocity background field  $v = M_s c_0$ . The initial condition  
 490 is given by a shock wave centered at  $x = 0.25$  which is propagating at Mach  $M_s = 2$  with  $\text{Re}_s = 100$ . The upstream  
 491 shock state is defined by

$$\rho(t = 0, \mathbf{x}) = \rho_0, \quad \mathbf{v}(t = 0, \mathbf{x}) = \mathbf{0}, \quad p_0(t = 0, \mathbf{x}) = 1/\gamma, \quad (102)$$

492 with  $c_0 = 1$ . The numerical solution obtained with the HGTC schemes without numerical dissipation at the final time  
 493  $t_f = 0.2$  is compared against the reference solution of the one-dimensional compressible Navier-Stokes equations.  
 494 The numerical solutions for the main primitive variables are plot in Figure 10. An excellent agreement is obtained,  
 495 demonstrating the capability of the HGTC schemes of retrieving the correct physical solution for heat-conducting  
 496 viscous fluids.

497 We also plot the correction factor  $\alpha_A$  in Figure 11 as well as the time evolution of the total mass conservation  
 498 errors  $\delta_A$  for the simulations run with and without numerical dissipation. In both cases, the  $L_\infty$  determinant error  
 499 remains at machine accuracy.

#### 500 4.5. 2D Taylor-Green vortex

501 The two-dimensional Taylor-Green vortex problem is a well-known test case for the incompressible Navier-Stokes  
 502 equations. The exact solution writes

$$\begin{aligned} u(t, \mathbf{x}) &= \sin(x_1) \cos(x_2) e^{-2\nu t}, \\ v(t, \mathbf{x}) &= -\cos(x_1) \sin(x_2) e^{-2\nu t}, \\ p(t, \mathbf{x}) &= C + \frac{1}{4} (\cos(2x_1) + \cos(2x_2)) e^{-4\nu t}, \end{aligned} \quad (103)$$

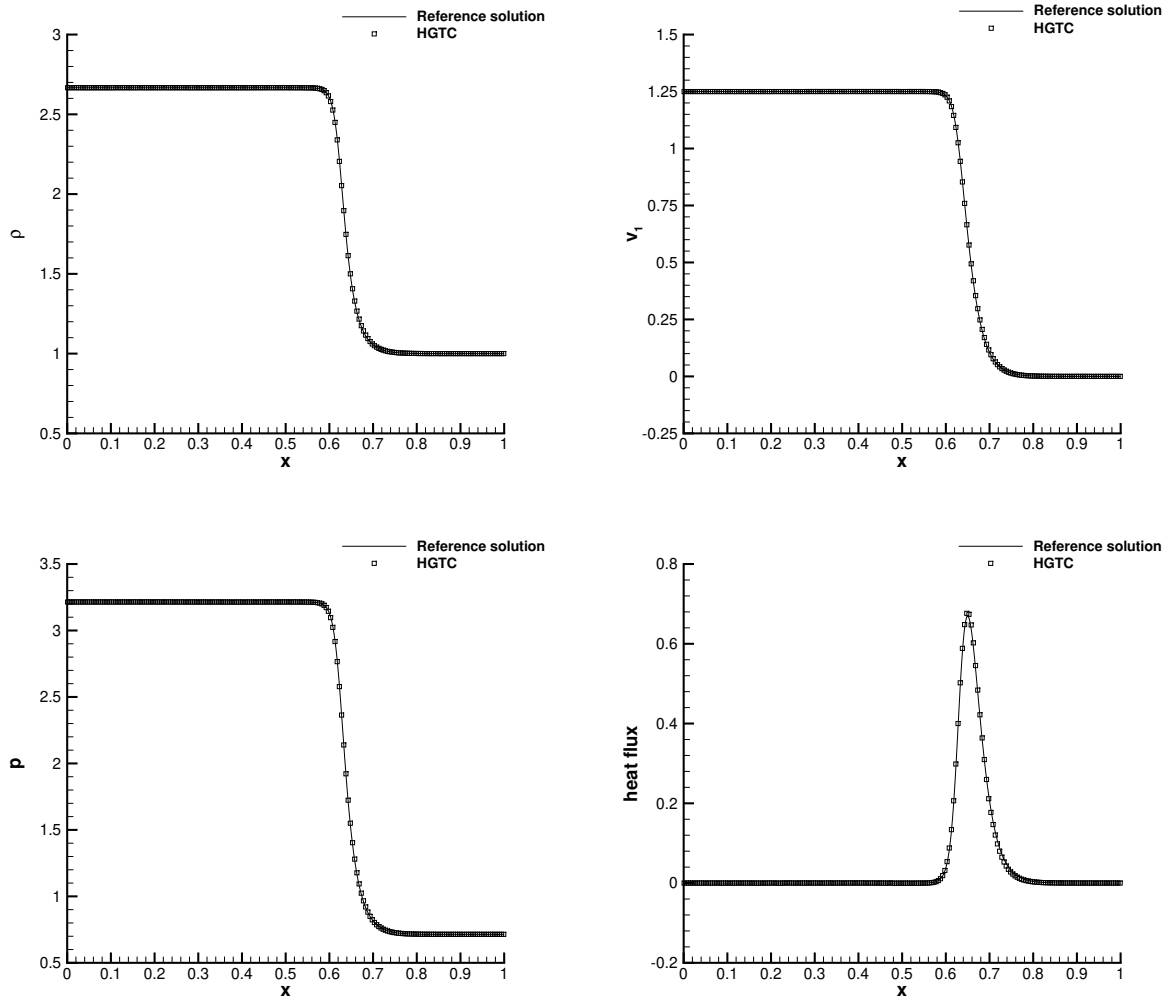


Figure 10: Viscous shock problem at time  $t_f = 0.2$ . Numerical results for density, horizontal velocity, pressure and heat flux extracted with a one-dimensional cut of 200 equidistant points along the  $x$ -direction at  $y = 0.1$  compared against the reference solution.

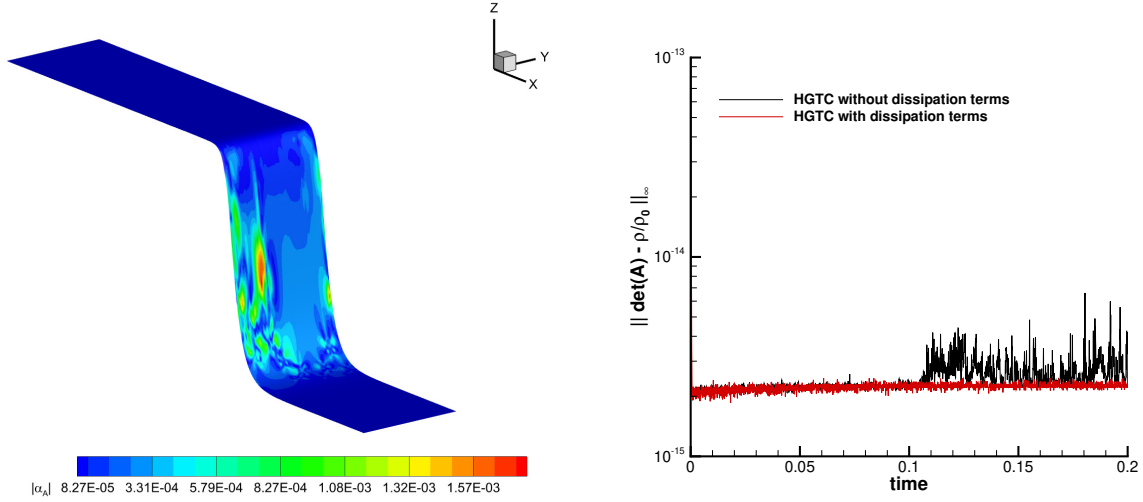


Figure 11: Viscous shock problem at time  $t_f = 0.2$ . Map of the thermodynamic correction factor  $|\alpha_A|$  with a three-dimensional view of the density distribution (left) and time evolution of the mass conservation errors for the HGTC with and without viscous terms (right).

503 where  $\nu = \mu/\rho$  denotes the kinematic viscosity of the fluid and the density is  $\rho(t, \mathbf{x}) = 1$ . To model an incompressible  
 504 viscous fluid, we set  $c_s = 10$  and  $\mu = 10^{-2}$ , and the additive constant to the pressure is chosen to be  $C = 100/\gamma$  so that  
 505 a maximum Mach number of 0.1 is retrieved. Heat conduction is neglected, thus we set  $c_h = 0$ . The initial condition is  
 506 provided by the exact solution (103) at time  $t = 0$ . The computational domain is given by  $\Omega = [0; 2\pi]^2$  with periodic  
 507 boundary conditions everywhere, and it is paved with a Voronoi grid of characteristic mesh size  $h = 2\pi/200$ . Figure  
 508 12 depicts the numerical results at the final time  $t_f = 0.2$  that are compared against the reference solution, obtaining  
 509 an excellent matching. Furthermore, we also show the time evolution of the mass and entropy conservation errors,  
 510 that remain bounded and preserved thanks to the compatibility corrections of our novel HGTC schemes.

#### 511 4.6. Solid rotor problem

512 Finally, a test case for solid mechanics is solved, namely the solid rotor problem introduced in [50, 8]. The  
 513 relaxation times of the mathematical model are set to  $\tau_1 = \tau_2 = 10^{20}$ , hence nonlinear hyperelastic solids are genuinely  
 514 modeled by the governing PDE presented in [49]. We fix  $c_s = c_h = 1$  and the final time of the simulation is  $t_f = 0.3$ .  
 515 The computational domain is  $\Omega = [-1; 1]^2$  with periodic boundaries, and the initial condition of the material writes

$$(\rho, v_1, v_2, v_3, p) = \begin{cases} (1, -x_2/R, x_1/R, 0, 1) & r < R \\ (1, 0, 0, 0, 1) & r \geq R \end{cases}, \quad t = 0, \quad \mathbf{x} \in \Omega, \quad (104)$$

516 with the initial discontinuity located at  $R = 0.2$  and  $r = \sqrt{x_1^2 + x_2^2}$ . To show mesh convergence, we run the solid rotor  
 517 problem on two different meshes with characteristic size of  $h = 1/256$  and  $h = 1/128$ . The results are compared with  
 518 each other in Figure 13, where the horizontal velocity distribution is plot. The maps of the scalar correction factors  $\alpha_A$   
 519 and  $\alpha_S$  at the final time level are also depicted. We observe that both corrections may act at the same spatial locations  
 520 without negatively interfering between each other.

## 521 5. Conclusions

522 In this paper we have presented a novel finite volume scheme on unstructured Voronoi meshes for the solution  
 523 of a reduced unified model for continuum mechanics, where the mass conservation equation is discarded. This has  
 524 been achieved enforcing the compatibility of the new schemes with the Geometric Conservation Law that links the  
 525 distortion tensor to the density within each control volume. The geometric compatibility is obtained by introducing

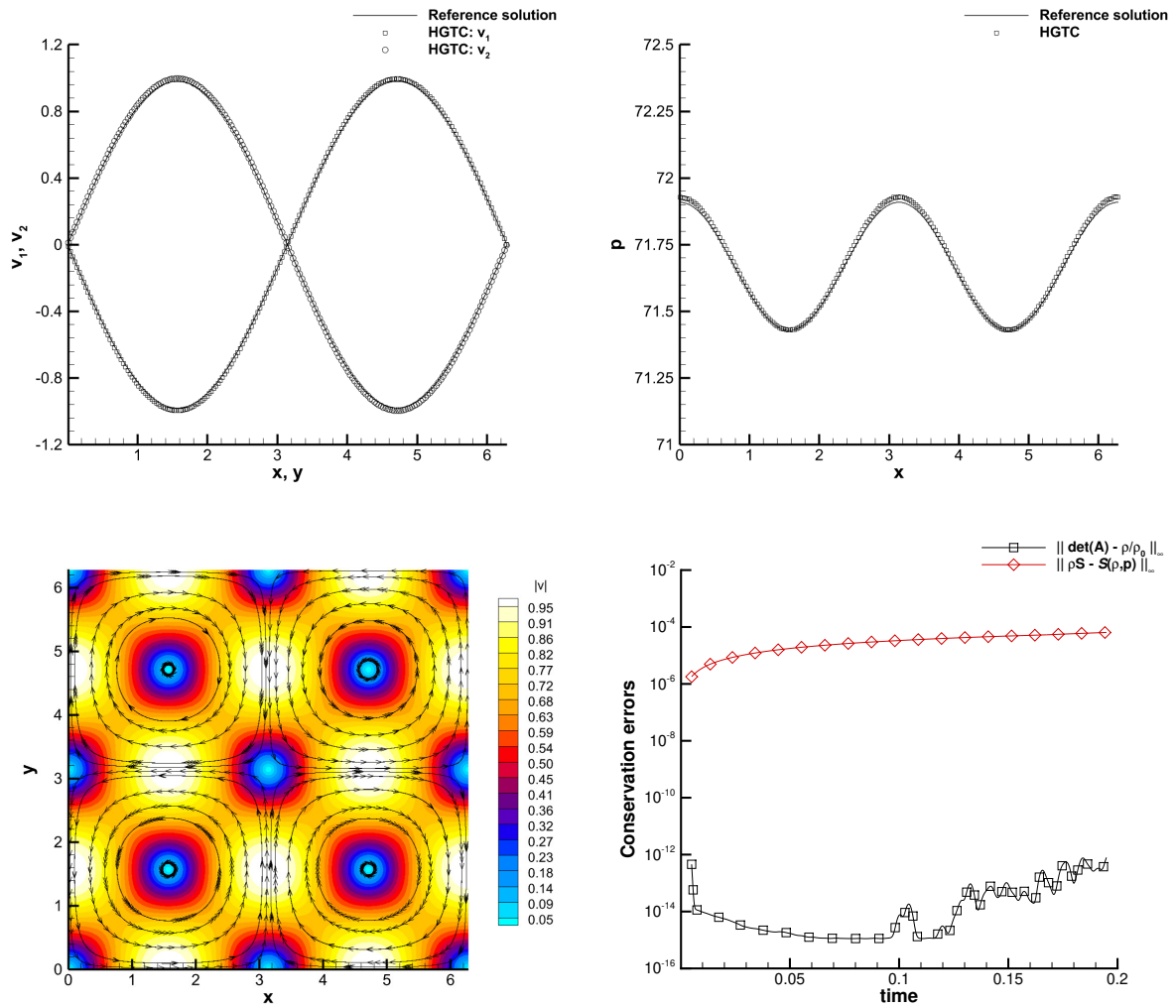


Figure 12: 2D Taylor-Green vortex at time  $t_f = 0.2$  with viscosity  $\mu = 10^{-2}$ . Top: one-dimensional cut of 200 equidistant points along the  $x$ -axis and the  $y$ -axis for the velocity components  $v_1$  and  $v_2$  (left) and for the pressure  $p$  (right). Bottom: distribution of the velocity magnitude with stream-traces (left) and time evolution of the mass and entropy conservation errors for the HGTC scheme.

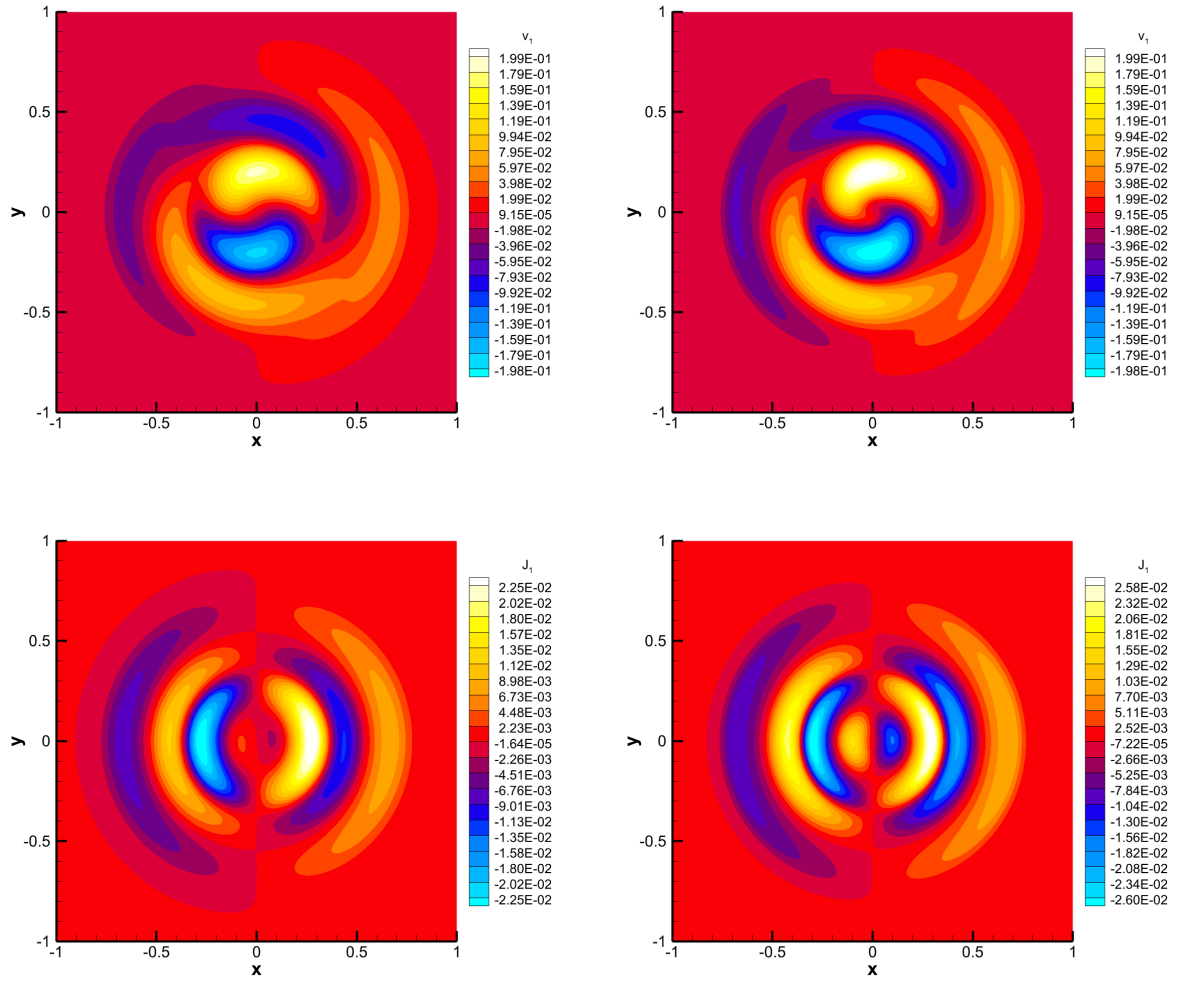


Figure 13: Solid rotor problem at time  $t_f = 0.3$ . Top: numerical results for the horizontal velocity  $v_1$  with  $h = 1/128$  (left) and  $h = 1/256$  (right). Bottom: numerical results for the thermal impulse component  $J_1$  with  $h = 1/128$  (left) and  $h = 1/256$  (right).

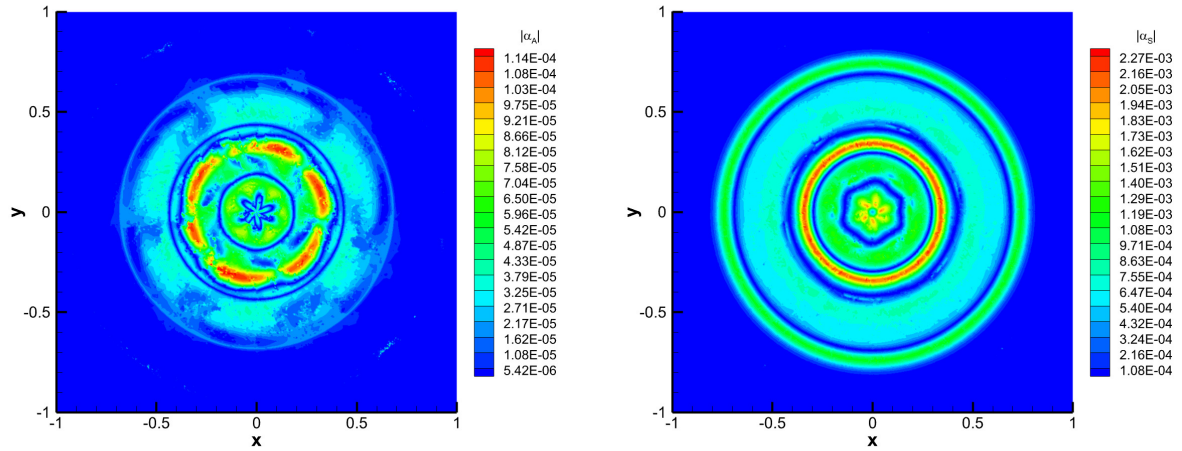


Figure 14: Solid rotor problem at time  $t_f = 0.3$ . Map of the geometric correction factor  $|\alpha_A|$  (left) and of the thermodynamic correction factor  $|\alpha_S|$  (right) obtained with a characteristic mesh size of  $h = 1/512$ .

526 a new generalized concept of potential, that is assumed to be the determinant of the distortion tensor. Consequently,  
 527 a set of associated pseudo-dual variables is retrieved, which play the role of the thermodynamic variables for the  
 528 total energy potential. By means of a conservative correction directly embedded in the numerical fluxes, the novel  
 529 schemes are proven to be compliant with the GCL at the semi-discrete level. Once the geometric compatibility is  
 530 achieved, thermodynamic compatibility is also guaranteed using the same strategy that derives from the formalism of  
 531 symmetric and hyperbolic thermodynamically compatible (SHTC) systems introduced by Godunov in 1961. These  
 532 two corrections can coexist at the discrete level and they do not interfere with each other, hence making it possible  
 533 for the first time on unstructured fixed grids to ensure geometric and thermodynamic compatibility at the same time.  
 534 Two theorems demonstrate that these properties are respected at the semi-discrete level. A two-dimensional first order  
 535 finite volume scheme with up to fourth order Runge-Kutta time integrators has been implemented and tested. A large  
 536 suite of test cases is shown to numerically **assess** the structure preserving properties of the new schemes.

537 In the future we plan to exploit this strategy to tackle other types of constraints, namely involution-constraints  
 538 like the solenoidal property of the magnetic field in magnetohydrodynamics or the irrotational behavior of the de-  
 539 formation gradient and the thermal impulse vector in reversible processes in solid mechanics. Finally, the extension  
 540 of the proposed approach to high order discontinuous Galerkin schemes is also foreseen as well as the development  
 541 of implicit-explicit [9, 8] asymptotic preserving discretizations to make the numerical schemes consistent with the  
 542 Navier-Stokes-Fourier limit exhibited by the governing equations.

### 543 Acknowledgments

544 WB received financial support by Fondazione Cariplo and Fondazione CDP (Italy) under the grant No. 2022-1895  
 545 and by the Italian Ministry of University and Research (MUR) with the PRIN Project 2022 No. 2022N9BM3N. WB  
 546 and RL acknowledge the support of LRC Anabase and the CEA-Cesta.

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## 658 Appendix A. Geometric Conservation Law derived from the determinant potential

659 In this appendix we show that the pseudo-*Gibbs relation* (26) holds true. The starting point is the evolution  
660 equation of the distortion tensor  $\mathbf{A} = \{A_{ik}\}$ , in which the viscous terms are neglected. Let us explicitly compute the  
661 dual variables  $\mathbf{w}$  of the potential  $|\mathbf{A}|$ :

$$\mathbf{w} := \{w_{ik}\} = \partial_{A_{ik}} |\mathbf{A}| = \begin{pmatrix} A_{22}A_{33} - A_{23}A_{32} & -A_{21}A_{33} + A_{23}A_{31} & A_{21}A_{32} - A_{22}A_{31} \\ -A_{12}A_{33} + A_{13}A_{32} & A_{11}A_{33} - A_{13}A_{31} & -A_{11}A_{32} + A_{12}A_{31} \\ A_{12}A_{23} - A_{13}A_{22} & -A_{11}A_{23} + A_{13}A_{21} & A_{11}A_{22} - A_{12}A_{21} \end{pmatrix}. \quad (\text{A.1})$$

662 The determinant of the distortion tensor  $\mathbf{A}$  is explicitly given by

$$|\mathbf{A}| = A_{11}A_{22}A_{33} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31}. \quad (\text{A.2})$$

663 Firstly, we investigate the compatibility with the source terms which are present in the equations (30c). The source  
664 terms  $\mathbf{S}_{\mathbf{A}} = \{S_{\mathbf{A},ik}\}$  are given by

$$S_{\mathbf{A},ik} = -\frac{3}{\tau_1} |\mathbf{A}|^{\frac{5}{3}} A_{im} \dot{G}_{mk}, \quad \dot{G}_{mk} = G_{mk} - \frac{1}{3} G_{jj} \delta_{mk}, \quad G_{mk} = A_{jm} A_{jk}. \quad (\text{A.3})$$

665 Multiplication of the source terms with the dual variables yields the contributions  $\{\tilde{S}_{\mathbf{A},ik}\} = \{w_{ik} S_{\mathbf{A},ik}\}$ , which write

$$\begin{aligned}
\tilde{S}_{\mathbf{A},11} &= -\frac{(-2A_{11}^3 + (-2A_{12}^2 - 2A_{13}^2 - 2A_{21}^2 + A_{22}^2 + A_{23}^2 - 2A_{31}^2 + A_{32}^2 + A_{33}^2)A_{11} + (-3A_{21}A_{22} - 3A_{31}A_{32})A_{12} - 3A_{13}(A_{21}A_{23} + A_{31}A_{33}))(-A_{22}A_{33} + A_{23}A_{32})|\mathbf{A}|^{\frac{5}{3}}}{\tau_1}, \\
\tilde{S}_{\mathbf{A},12} &= \frac{2(A_{12}^3 + (A_{32}^2 - \frac{A_{33}^2}{2} + A_{11}^2 + A_{13}^2 - \frac{A_{21}^2}{2} + A_{22}^2 - \frac{A_{23}^2}{2} - \frac{A_{31}^2}{2})A_{12} + (\frac{3A_{31}A_{32}}{2} + \frac{3A_{21}A_{22}}{2})A_{11} + \frac{3A_{13}(A_{22}A_{23} + A_{32}A_{33})}{2})(A_{21}A_{33} - A_{23}A_{31})|\mathbf{A}|^{\frac{5}{3}}}{\tau_1}, \\
\tilde{S}_{\mathbf{A},13} &= \frac{(-2A_{13}^3 + (-2A_{11}^2 - 2A_{12}^2 + A_{21}^2 + A_{22}^2 - 2A_{23}^2 + A_{31}^2 + A_{32}^2 - 2A_{33}^2)A_{13} + (-3A_{21}A_{23} - 3A_{31}A_{33})A_{11} - 3A_{12}(A_{22}A_{23} + A_{32}A_{33}))(A_{21}A_{32} - A_{22}A_{31})|\mathbf{A}|^{\frac{5}{3}}}{\tau_1}, \\
\tilde{S}_{\mathbf{A},21} &= \frac{(-2A_{21}^3 + (-2A_{11}^2 + A_{12}^2 + A_{13}^2 - 2A_{22}^2 - 2A_{23}^2 - 2A_{31}^2 + A_{32}^2 + A_{33}^2)A_{21} + (-3A_{12}A_{22} - 3A_{13}A_{23})A_{11} - 3A_{31}(A_{22}A_{32} + A_{23}A_{33}))(-A_{12}A_{33} + A_{13}A_{32})|\mathbf{A}|^{\frac{5}{3}}}{\tau_1}, \\
\tilde{S}_{\mathbf{A},22} &= -\frac{2(A_{11}A_{33} - A_{13}A_{31})(A_{22}^3 + (A_{32}^2 - \frac{A_{33}^2}{2} - \frac{A_{11}^2}{2} + A_{12}^2 - \frac{A_{13}^2}{2} + A_{21}^2 + A_{23}^2 - \frac{A_{22}^2}{2} - \frac{A_{23}^2}{2})A_{22} + \frac{3A_{11}A_{12}A_{21}}{2} + \frac{3A_{12}A_{13}A_{23}}{2} + \frac{3A_{32}(A_{21}A_{31} + A_{23}A_{33})}{2})|\mathbf{A}|^{\frac{5}{3}}}{\tau_1}, \\
\tilde{S}_{\mathbf{A},23} &= -\frac{(A_{11}A_{32} - A_{12}A_{31})(-2A_{23}^3 + (A_{11}^2 + A_{12}^2 - 2A_{13}^2 - 2A_{21}^2 - 2A_{22}^2 + A_{21}^2 + A_{32}^2 - 2A_{33}^2)A_{23} - 3A_{11}A_{13}A_{21} - 3A_{12}A_{13}A_{22} - 3A_{33}(A_{21}A_{31} + A_{22}A_{32}))|\mathbf{A}|^{\frac{5}{3}}}{\tau_1}, \\
\tilde{S}_{\mathbf{A},31} &= -\frac{2(A_{12}A_{23} - A_{13}A_{22})(A_{31}^3 + (A_{32}^2 + A_{33}^2 + A_{11}^2 - \frac{A_{12}^2}{2} - \frac{A_{13}^2}{2} + A_{21}^2 - \frac{A_{22}^2}{2} - \frac{A_{23}^2}{2})A_{31} + (\frac{3A_{32}A_{12}}{2} + \frac{3A_{33}A_{13}}{2})A_{11} + \frac{3A_{21}(A_{22}A_{32} + A_{23}A_{33})}{2})|\mathbf{A}|^{\frac{5}{3}}}{\tau_1}, \\
\tilde{S}_{\mathbf{A},32} &= \frac{2(A_{32}^3 + (A_{33}^2 - \frac{A_{11}^2}{2} + A_{12}^2 - \frac{A_{13}^2}{2} - \frac{A_{21}^2}{2} + A_{22}^2 - \frac{A_{23}^2}{2} + A_{31}^2)A_{32} + \frac{3A_{11}A_{12}A_{31}}{2} + \frac{3A_{12}A_{13}A_{33}}{2} + \frac{3A_{22}(A_{21}A_{31} + A_{23}A_{33})}{2})(A_{11}A_{23} - A_{13}A_{21})|\mathbf{A}|^{\frac{5}{3}}}{\tau_1}, \\
\tilde{S}_{\mathbf{A},33} &= -\frac{2(A_{11}A_{22} - A_{12}A_{21})(A_{33}^3 + (A_{32}^2 - \frac{A_{11}^2}{2} - \frac{A_{12}^2}{2} + A_{13}^2 - \frac{A_{21}^2}{2} - \frac{A_{22}^2}{2} + A_{23}^2 + A_{31}^2)A_{33} + \frac{3A_{11}A_{13}A_{31}}{2} + \frac{3A_{12}A_{13}A_{32}}{2} + \frac{3A_{23}(A_{21}A_{31} + A_{22}A_{32})}{2})|\mathbf{A}|^{\frac{5}{3}}}{\tau_1}.
\end{aligned} \tag{A.4}$$

666 At the aid of a linear algebra software [46], by summing up all the above terms, i.e. dot multiplying the source terms  
667 with the dual variables, one obtains

$$\tilde{S}_{\mathbf{A},ik} = -w_{ik} \cdot \frac{\alpha_{ik}}{\theta(\tau_1)} = 0. \tag{A.5}$$

668 Thus, we retrieve no source on the right hand side of the GCL (25) as expected.

669 Next, the compatibility with the flux and non-conservative terms on the left hand side of (1d) has to be verified.  
670 To that aim, the equation of the distortion tensor as well as the GCL are written in fully non-conservative form as  
671 follows:

$$\frac{\partial A_{ik}}{\partial t} + A_{im} \frac{\partial v_m}{\partial x_k} + v_m \frac{\partial A_{ik}}{\partial x_m} = -\frac{\alpha_{ik}}{\theta_1(\tau_1)}, \tag{A.6}$$

$$\frac{\partial |\mathbf{A}|}{\partial t} + |\mathbf{A}| \frac{\partial v_k}{\partial x_k} + v_k \cdot \frac{\partial |\mathbf{A}|}{\partial x_k} = 0. \tag{A.7}$$

672 The non-conservative terms in (A.6) for each component of the distortion tensor  $\mathbf{A}$ , i.e.  $D_{ik} = A_{im} \frac{\partial v_m}{\partial x_k} + v_m \frac{\partial A_{ik}}{\partial x_m}$ , are



675 On the other side, the non-conservative products in (A.7) explicitly write

$$|\mathbf{A}| \frac{\partial v_k}{\partial x_k} = (A_{11}A_{22}A_{33} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31}) \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) \quad (\text{A.10})$$

$$\begin{aligned} v_1 \frac{\partial |\mathbf{A}|}{\partial x_1} = & v_1 \left( A_{22}A_{33} \left( \frac{\partial A_{11}}{\partial x_1} \right) + A_{11}A_{22} \left( \frac{\partial A_{33}}{\partial x_1} \right) + A_{11}A_{33} \left( \frac{\partial A_{22}}{\partial x_1} \right) - A_{23}A_{32} \left( \frac{\partial A_{11}}{\partial x_1} \right) - A_{11}A_{23} \left( \frac{\partial A_{32}}{\partial x_1} \right) - A_{11}A_{32} \left( \frac{\partial A_{23}}{\partial x_1} \right) \right. \\ & - A_{21}A_{33} \left( \frac{\partial A_{12}}{\partial x_1} \right) - A_{12}A_{21} \left( \frac{\partial A_{33}}{\partial x_1} \right) - A_{12}A_{33} \left( \frac{\partial A_{21}}{\partial x_1} \right) + A_{23}A_{31} \left( \frac{\partial A_{12}}{\partial x_1} \right) + A_{12}A_{23} \left( \frac{\partial A_{31}}{\partial x_1} \right) + A_{12}A_{31} \left( \frac{\partial A_{23}}{\partial x_1} \right) \\ & \left. + A_{21}A_{32} \left( \frac{\partial A_{13}}{\partial x_1} \right) + A_{13}A_{21} \left( \frac{\partial A_{32}}{\partial x_1} \right) + A_{13}A_{32} \left( \frac{\partial A_{21}}{\partial x_1} \right) - A_{22}A_{31} \left( \frac{\partial A_{13}}{\partial x_1} \right) - A_{13}A_{22} \left( \frac{\partial A_{31}}{\partial x_1} \right) - A_{13}A_{31} \left( \frac{\partial A_{22}}{\partial x_1} \right) \right) \end{aligned} \quad (\text{A.11})$$

$$\begin{aligned} v_2 \frac{\partial |\mathbf{A}|}{\partial x_2} = & v_2 \left( A_{22}A_{33} \left( \frac{\partial A_{11}}{\partial x_2} \right) + A_{11}A_{22} \left( \frac{\partial A_{33}}{\partial x_2} \right) + A_{11}A_{33} \left( \frac{\partial A_{22}}{\partial x_2} \right) - A_{23}A_{32} \left( \frac{\partial A_{11}}{\partial x_2} \right) - A_{11}A_{23} \left( \frac{\partial A_{32}}{\partial x_2} \right) - A_{11}A_{32} \left( \frac{\partial A_{23}}{\partial x_2} \right) \right. \\ & - A_{21}A_{33} \left( \frac{\partial A_{12}}{\partial x_2} \right) - A_{12}A_{21} \left( \frac{\partial A_{33}}{\partial x_2} \right) - A_{12}A_{33} \left( \frac{\partial A_{21}}{\partial x_2} \right) + A_{23}A_{31} \left( \frac{\partial A_{12}}{\partial x_2} \right) + A_{12}A_{23} \left( \frac{\partial A_{31}}{\partial x_2} \right) + A_{12}A_{31} \left( \frac{\partial A_{23}}{\partial x_2} \right) \\ & \left. + A_{21}A_{32} \left( \frac{\partial A_{13}}{\partial x_2} \right) + A_{13}A_{21} \left( \frac{\partial A_{32}}{\partial x_2} \right) + A_{13}A_{32} \left( \frac{\partial A_{21}}{\partial x_2} \right) - A_{22}A_{31} \left( \frac{\partial A_{13}}{\partial x_2} \right) - A_{13}A_{22} \left( \frac{\partial A_{31}}{\partial x_2} \right) - A_{13}A_{31} \left( \frac{\partial A_{22}}{\partial x_2} \right) \right) \end{aligned} \quad (\text{A.12})$$

$$\begin{aligned} v_3 \frac{\partial |\mathbf{A}|}{\partial x_3} = & v_3 \left( A_{22}A_{33} \left( \frac{\partial A_{11}}{\partial x_3} \right) + A_{11}A_{22} \left( \frac{\partial A_{33}}{\partial x_3} \right) + A_{11}A_{33} \left( \frac{\partial A_{22}}{\partial x_3} \right) - A_{23}A_{32} \left( \frac{\partial A_{11}}{\partial x_3} \right) - A_{11}A_{23} \left( \frac{\partial A_{32}}{\partial x_3} \right) - A_{11}A_{32} \left( \frac{\partial A_{23}}{\partial x_3} \right) \right. \\ & - A_{21}A_{33} \left( \frac{\partial A_{12}}{\partial x_3} \right) - A_{12}A_{21} \left( \frac{\partial A_{33}}{\partial x_3} \right) - A_{12}A_{33} \left( \frac{\partial A_{21}}{\partial x_3} \right) + A_{23}A_{31} \left( \frac{\partial A_{12}}{\partial x_3} \right) + A_{12}A_{23} \left( \frac{\partial A_{31}}{\partial x_3} \right) + A_{12}A_{31} \left( \frac{\partial A_{23}}{\partial x_3} \right) \\ & \left. + A_{21}A_{32} \left( \frac{\partial A_{13}}{\partial x_3} \right) + A_{13}A_{21} \left( \frac{\partial A_{32}}{\partial x_3} \right) + A_{13}A_{32} \left( \frac{\partial A_{21}}{\partial x_3} \right) - A_{22}A_{31} \left( \frac{\partial A_{13}}{\partial x_3} \right) - A_{13}A_{22} \left( \frac{\partial A_{31}}{\partial x_3} \right) - A_{13}A_{31} \left( \frac{\partial A_{22}}{\partial x_3} \right) \right) \end{aligned} \quad (\text{A.13})$$

676 After some tedious algebraic manipulations, we arrive at the result

$$w_{ik} \cdot D_{ik} = |\mathbf{A}| \frac{\partial v_k}{\partial x_k} + v_k \frac{\partial |\mathbf{A}|}{\partial x_k}, \quad (\text{A.14})$$

$$(\text{A.9}) = (\text{A.10}) + (\text{A.11}) + (\text{A.12}) + (\text{A.13}),$$

677 therefore the GCL (25) is retrieved as the dot product of the dual variables  $\mathbf{w}$  with the evolution equations of the  
678 distortion tensor  $\mathbf{A}$  given by (1d).

## 679 Appendix B. Runge-Kutta schemes

680 Runge-Kutta methods represent a quite popular technique to carry out time integration and they are based on the  
681 method of lines (MOL) approach. The governing equations can be written in semi-discrete form as

$$\frac{d\mathbf{U}}{dt} = \mathcal{L}_h(\mathbf{U}), \quad (\text{B.1})$$

682 where  $\mathcal{L}_h(\mathbf{U})$  contains the spatial discretization of the numerical fluxes, non-conservative products and source terms.  
683 A generic Runge-Kutta scheme with a total number of  $s$  sub-stages is described by a Butcher tableau of the form  
684 shown in Table B.4. The numerical solution is determined at the next time step as

$$\mathbf{U}^{n+1} = \mathbf{U}^n + \Delta t \sum_{i=1}^s c_i \kappa_i. \quad (\text{B.2})$$

685 The generic Runge-Kutta stage  $\kappa_i$  is evaluated at the intermediate time level  $t^{(i)} = t^n + \alpha_i \Delta t$  by

$$\kappa_i = \mathcal{L}_h \left( \mathbf{U}_h^n + \Delta t \sum_{j=1}^i \beta_{ij} \kappa_j \right) \quad (\text{B.3})$$

Table B.4: Butcher tableau for Runge-Kutta explicit methods.

$0$					
$\alpha_2$	$\beta_{21}$				
$\alpha_3$	$\beta_{31}$	$\beta_{32}$			
$\vdots$	$\vdots$	$\vdots$	$\ddots$		
$\alpha_s$	$\beta_{s1}$	$\beta_{s2}$	$\dots$	$\beta_{s(s-1)}$	
	$c_1$	$c_2$	$\dots$	$c_{s-1}$	$c_s$

686 with  $\mathbf{U}_h^n$  denoting the numerical solution at the current time level  $t^n$ . In this work we consider three different Runge-  
 687 Kutta schemes:

- 688 • Euler method with accuracy  $\mathcal{O}(1)$

$0$	$0$
	$1$

- 690 • Heun method with accuracy  $\mathcal{O}(2)$

$0$	$0$	$0$
$1$	$1$	$0$
	$1/2$	$1/2$

- 692 • RK4 method with accuracy  $\mathcal{O}(4)$

$0$	$0$	$0$	$0$	$0$
$1/2$	$1/2$	$0$	$0$	$0$
$1/2$	$0$	$1/2$	$0$	$0$
$1$	$0$	$0$	$1$	$0$
	$1/6$	$1/3$	$1/3$	$1/6$