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Mikhail Hlushchanka, Nikolai Prochorov. Critically fixed Thurston maps: classification, recognition, and twisting. 2023. hal-04525890

**HAL Id: hal-04525890**

**<https://hal.science/hal-04525890>**

Preprint submitted on 29 Mar 2024

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# CRITICALLY FIXED THURSTON MAPS: CLASSIFICATION, RECOGNITION, AND TWISTING

MIKHAIL HLUSHCHANKA AND NIKOLAI PROCHOROV

ABSTRACT. An orientation-preserving branched covering map  $f: S^2 \rightarrow S^2$  is called a *critically fixed Thurston map* if  $f$  fixes each of its critical points. It was recently shown that there is an explicit one-to-one correspondence between Möbius conjugacy classes of critically fixed rational maps and isomorphism classes of planar embedded connected graphs. In the paper, we generalize this result to the whole family of critically fixed Thurston maps. Namely, we show that each critically fixed Thurston map  $f$  is obtained by applying the *blow-up operation*, introduced by Kevin Pilgrim and Tan Lei, to a pair  $(G, \varphi)$ , where  $G$  is a planar embedded graph in  $S^2$  without isolated vertices and  $\varphi$  is an orientation-preserving homeomorphism of  $S^2$  that fixes each vertex of  $G$ . This result allows us to provide a classification of combinatorial equivalence classes of critically fixed Thurston maps. We also develop an algorithm that reconstructs (up to isotopy) the pair  $(G, \varphi)$  associated with a critically fixed Thurston map  $f$ . Finally, we solve some special instances of the *twisting problem* for the family of critically fixed Thurston maps obtained by blowing up pairs  $(G, \text{id}_{S^2})$ .

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*Date:* January 2, 2023.

*2010 Mathematics Subject Classification.* Primary 37F20, 37F10.

*Key words and phrases.* Thurston maps, critically fixed maps, blow-up operation, decomposition theory, classification problem, twisting problem.

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## 1. INTRODUCTION

One-dimensional holomorphic dynamics studies the iteration of analytic functions on the Riemann sphere  $\widehat{\mathbb{C}}$  or on the complex plane  $\mathbb{C}$ . One of the most influential ideas in the subject, and in dynamics of complex rational maps in particular, has been to abstract from the rigid underlying complex structure and consider the more general setup of *branched self-coverings* of a topological 2-sphere  $S^2$  or of a topological plane. Such a branched self-covering is called *postcritically-finite*, or *pcf* for short, if all its critical points are periodic or preperiodic. Nowadays, orientation-preserving pcf branched covers  $f: S^2 \rightarrow S^2$  of topological degree  $\deg(f) \geq 2$  are called *Thurston maps*, in honor of William Thurston, who introduced them in order to better understand dynamics of pcf rational maps on  $\widehat{\mathbb{C}}$ .

In this paper, we study the following special subclass of Thurston maps.

**Definition 1.1.** A branched covering map  $f: S^2 \rightarrow S^2$  is called a *critically fixed Thurston map* if each of its critical points is fixed under  $f$  and  $\deg(f) \geq 2$ .

Pcf rational maps may seem to form a rather specific family, as there are only countably many pcf rational maps of a fixed degree up to Möbius conjugation. However, they play a crucial role in the study of dynamics of rational maps. In particular, the entire complicated structure of the *Mandelbrot set* can be described in terms of pcf polynomials [DH84]. Furthermore, the existence of a strong dynamical similarity between a dense subset of the space of all rational maps (of any fixed degree) and pcf rational maps was conjectured in [McM94, Conjecture 1.1]; see also the discussion in [McM94, Chapter 4].

We denote the set of critical points of a Thurston map  $f$  by  $\text{Crit}(f)$ . The set  $\text{Post}(f) := \bigcup_{n=1}^{\infty} f^n(\text{Crit}(f))$  of the forward iterates of the critical points is called the *postcritical set* of  $f$ . For critically fixed Thurston maps, we obviously have  $\text{Post}(f) = \text{Crit}(f)$ . Two Thurston maps are called *combinatorially* (or *Thurston*) *equivalent* if they are conjugate up to isotopy relative to their postcritical sets; see Definition 2.8.

One of the key features of Thurston maps is that they often admit a description in purely combinatorial terms. A fundamental question in this context is whether a given Thurston map  $f$  can be *realized* by a rational map with the same combinatorics, that is, if  $f$  is combinatorially equivalent to a rational map. William Thurston answered this question in his celebrated *characterization of rational maps*: If a Thurston map  $f$  has a *hyperbolic orbifold* (this is always true, except for some well-understood special maps), then  $f$  is realized by a rational map  $F$  if and only if  $f$  has no *Thurston obstruction* [DH93]. Such an obstruction is given by a finite collection of disjoint Jordan curves in  $S^2 \setminus \text{Post}(f)$  with certain invariance properties under the map  $f$ . Furthermore, the rational map  $F$  is unique up to Möbius conjugation (see Section 2.5 for more discussion).

Thurston’s characterization of rational maps allows to address the *classification problem*, which asks to describe all possible *combinatorial models* that are realized by rational maps within a given family. More precisely, we want to assign some finite combinatorial certificate to each map from the family so that different certificates correspond to different maps and vice versa. This question is well-understood for pcf complex polynomials. Specifically, they can be classified by the so-called *Hubbard trees* [Poi10] or *external angles* [Poi93]. Recently, the classification problem was also solved for pcf *Newton maps* [DMRS19, LMS22] and *critically fixed rational maps* [Hlu19]. In both cases, the classification is given in terms of *planar embedded graphs* on the 2-sphere. Nevertheless, the classification problem for the family of *all* pcf rational maps is extremely challenging and remains wide open.

Once we have an answer to the classification problem within some class of rational maps, we can assign canonical combinatorial data to any map from the considered family. Thus, we can ask the *recognition question* of computing the canonical combinatorial model for a given map. For example, this question was recently solved for pcf polynomials [BLMW22, BD17]. A more detailed survey of known results can be found in [BLMW22, Section 1.4].

A particular instance of the recognition question above is the *twisting problem*. Suppose  $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is a pcf rational map and  $\varphi$  is an element of  $\text{Homeo}^+(S^2, \text{Post}(f))$ , that is,  $\varphi$  is an orientation-preserving homeomorphism of  $S^2$  that fixes each postcritical point of  $f$ . The twisting problem asks to determine if the *twisted map*  $g := \varphi \circ f$  is realized by a rational map and, if yes, find the canonical combinatorial model for  $g$ .

The twisting problem was initially investigated for the case when  $f$  is a complex polynomial using algebraic machinery provided by *iterated monodromy groups* [BN06]. An alternative approach in the polynomial case, using *mapping class groups* methods, was recently suggested in [BLMW22]. Nevertheless, in the case of non-polynomial rational maps, not many results are known. For instance, in [Lod13, KL19] the twisting problem was studied for rational maps of low degree with four postcritical points.

**1.1. Main results.** In this paper, we construct and study combinatorial models for the class of critically fixed Thurston maps. Below, we provide an overview of our main results.

1.1.1. *Classification of critically fixed Thurston maps.* The classification problem for the family of critically fixed rational maps was studied in several works [Tis89, CGN<sup>+</sup>15], and in [Hlu19] it was completely solved as stated in the following result.

**Theorem 1.2.** [Hlu19, Theorem 2] *There is a canonical bijection between the combinatorial equivalence classes (or, equivalently, Möbius conjugacy classes) of critically fixed rational maps and the isomorphism classes of planar embedded connected graphs.*

Here and in the following, a planar embedded graph is allowed to have multiple edges but no loops. To associate a critically fixed rational map to a planar embedded connected graph, one uses the so-called “*blow-up operation*”, introduced by Kevin Pilgrim and Tan Lei in [PL98]. It is a special surgery on Thurston maps, which we roughly describe now in the context relevant to us (see a detailed discussion in Section 3.1).

Let  $G$  be a planar embedded graph in  $S^2$  and  $\varphi$  be an element of  $\text{Homeo}^+(S^2, V(G))$ . First, we cut the sphere  $S^2$  open along each edge  $e \in E(G)$  and glue in a closed Jordan region  $D_e$  in each slit along the boundary. Then we define a branched covering map  $\widehat{f}: \widehat{S}^2 \rightarrow S^2$  on the resulting 2-sphere  $\widehat{S}^2$  as follows:  $\widehat{f}$  maps the complement of  $\bigcup_{e \in E(G)} \text{int}(D_e)$  in the same way as the homeomorphism  $\varphi$ , and it maps each  $\text{int}(D_e)$ ,  $e \in E(G)$ , to the complement of  $\varphi(e)$

in  $S^2$  by a homeomorphism whose extension to  $\partial D_e$  matches the map  $\varphi|_e$ . After natural identification of the domain sphere  $\widehat{S^2}$  with the target sphere  $S^2$ , we obtain a critically fixed Thurston map  $f: S^2 \rightarrow S^2$  of degree  $d = |E(G)| + 1$  and with critical points at the non-isolated vertices of  $G$ . We say that the map  $f$  is obtained by *blowing up the pair*  $(G, \varphi)$ .

It is easy to check that the critically fixed Thurston map obtained by blowing up a pair  $(G, \text{id}_{S^2})$  is realized by a rational map if  $G$  is connected [CGN<sup>+</sup>15, Theorem 9]. It is shown in [Hlu19, Proposition 7] that the converse is also true: every critically fixed rational map  $F$  is obtained by blowing up the pair  $(G_F, \text{id}_{\widehat{\mathbb{C}}})$  for some planar embedded connected graph  $G_F$  in  $\widehat{\mathbb{C}}$ . The graph  $G_F$  is called the *charge graph* of  $F$ . The existence of such a graph is one of the crucial ingredients in the proof of Theorem 1.2.

In this paper, we provide an extension of the classification result above to the class of *all* critically fixed Thurston maps (including the obstructed ones). In particular, we prove that every critically fixed Thurston map  $f: S^2 \rightarrow S^2$  can be obtained by blowing a pair  $(G_f, \varphi_f)$ , where  $G_f$  is a planar embedded graph in  $S^2$  with the vertex set  $V(G_f) = \text{Crit}(f)$  and exactly  $\deg(f) - 1$  edges, and  $\varphi_f$  is a homeomorphism in  $\text{Homeo}^+(S^2, V(G_f))$  satisfying some extra assumptions. Namely, the image  $\varphi_f(e)$  of each edge  $e \in E(G_f)$  is isotopic to  $e$  rel.  $V(G_f)$ .

Similar to the rational case, we call the graph  $G_f$  the *charge graph* of  $f$  and denote it by  $\text{Charge}(f)$ . We note that it is *invariant* under  $f$ . More precisely,  $G_f \subset f^{-1}(G_f)$  up to isotopy rel.  $\text{Crit}(f)$ .

We show that the pair  $(G_f, \varphi_f)$  as above is uniquely defined, up to a natural equivalence relation, within the combinatorial equivalence class of  $f$ . This allows us to completely classify all critically fixed Thurston maps. Before we can provide the precise statement of this classification, we need the following definitions.

**Definition 1.3.** Let  $G$  be a planar embedded graph in  $S^2$  and  $\varphi$  be a homeomorphism in  $\text{Homeo}^+(S^2, V(G))$ . We say that  $(G, \varphi)$  is an *admissible pair* (in  $S^2$ ) if  $G$  does not have isolated vertices and  $\varphi(e)$  is isotopic to  $e$  rel.  $V(G)$  for each edge  $e \in E(G)$ .

Two admissible pairs  $(G_1, \varphi_1)$  and  $(G_2, \varphi_2)$  in two topological 2-spheres  $S^2$  and  $\widehat{S^2}$ , respectively, are called *equivalent* if there exists an orientation-preserving homeomorphism  $\psi: S^2 \rightarrow \widehat{S^2}$  such that  $\psi(G_1) = G_2$  and  $\psi \circ \varphi_1 \circ \psi^{-1}$  is isotopic to  $\varphi_2$  rel.  $V(G_2)$ .

The following theorem provides a complete combinatorial classification of critically fixed Thurston maps.

**Main Theorem A.** *There is a canonical bijection between the combinatorial equivalence classes of critically fixed Thurston maps and the equivalence classes of admissible pairs.*

The proof of Main Theorem A is based on *Pilgrim's decomposition theory* of Thurston maps [Pil01]. In particular, we show that each critically fixed Thurston map  $f$  can be canonically decomposed into homeomorphisms and critically fixed Thurston maps that are realized (Theorem 3.22). The charge graph of  $f$  is then defined as a union of the charge graphs of the ‘‘rational pieces’’ of this decomposition; see Section 3.6 for details.

We also point out the following simple criterion for realizability of critically fixed Thurston maps, which also follows from a decomposition result.

**Proposition 1.4.** *Let  $f$  be a critically fixed Thurston map. Then  $f$  is obstructed if and only if  $f$  has a Levy fixed curve, that is, there is an essential Jordan curve  $\gamma \subset S^2 \setminus \text{Crit}(f)$  and a component  $\gamma'$  of  $f^{-1}(\gamma)$  such that  $\gamma'$  is isotopic to  $\gamma$  rel.  $\text{Crit}(f)$  and  $f|_{\gamma'}: \gamma' \rightarrow \gamma$  is a homeomorphism.*

Here, a Jordan curve  $\gamma \subset S^2 \setminus \text{Crit}(f)$  is *essential* if each connected component of  $S^2 \setminus \gamma$  contains at least two critical points of  $f$ .

1.1.2. *Recognition of combinatorial models of critically fixed Thurston maps.* We study the question of determining the combinatorial equivalence class of a given critically fixed Thurston map  $f$ . To answer this question, we develop an algorithm that finds an admissible pair  $(G_f, \varphi_f)$  associated with  $f$ . The algorithm is based on iteration of a *pullback operation* on planar embedded trees, in the spirit of the *lifting algorithm* (for pcf polynomial maps) from [BLMW22] and the *ivy iteration* (for pcf quadratic rational maps) from [ST19]. We note that, in contrast with our algorithm, the lifting algorithm does not have any complexity estimates, and the ivy iteration lacks convergence results.

Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map and  $T$  be a planar embedded tree in  $S^2$  with  $\text{Crit}(f) \subset V(T)$ . It is easy to check that the preimage  $f^{-1}(T)$  is a connected planar embedded graph in  $S^2$  containing  $\text{Crit}(f)$ . Hence, we can take a *spanning subtree* of the critical points in the graph  $f^{-1}(T)$ . We “simplify” the obtained tree by removing all non-critical vertices of degree 2 (if there any). The result of this *pullback operation* is again some planar embedded tree  $T'$  with  $\text{Crit}(f) \subset V(T')$ . We call the tree  $T'$  a *pullback* of the tree  $T$  under the map  $f$  and denote by  $\Pi_f(T)$  the set of all such pullbacks.

Note that a pullback of  $T$  is not uniquely defined, as we may choose different spanning trees of  $\text{Crit}(f)$  in  $f^{-1}(T)$ . This contrasts with the lifting operation for the polynomial case from [BLMW22]. Nevertheless, we can still iterate our pullback operation and consider the sequence  $T_{n+1} \in \Pi_f(T_n)$ ,  $n \geq 0$ , of planar embedded trees. We show that this sequence eventually lands in a special set, dependent only on the function  $f$  and not on the starting tree  $T_0$ .

To be more precise, suppose that  $[T]$  denotes the equivalence class of a planar embedded tree  $T$  modulo isotopy rel.  $\text{Crit}(f)$ . Let us consider the set  $\mathcal{N}_f$  of the equivalence classes  $[T]$  of planar embedded trees  $T$  that satisfy the following three conditions:

- (i)  $\text{Crit}(f) \subset V(T)$ ;
- (ii)  $T$  does not have non-critical vertices of degree 2;
- (iii)  $T \cap \text{Charge}(f) = \text{Crit}(f)$ .

We note that the set  $\mathcal{N}_f$  is finite if and only if  $f$  is realized by a rational map. This easily follows from the connectivity of the charge graph in the rational case. Moreover, if  $f$  is realized, then every tree  $T$  as above is invariant under  $f$  (up to isotopy).

Now the following statement is true.

**Main Theorem B.** *Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map and  $T_0$  be a planar embedded tree with  $\text{Crit}(f) \subset V(T_0)$ . Consider the sequence  $(T_n)_{n \geq 0}$  of planar embedded trees in  $S^2$  such that  $T_{n+1} \in \Pi_f(T_n)$  for each  $n \geq 0$ . Then there exists  $N \in \mathbb{N}$  depending on  $f$  and  $T_0$  such that  $[T_n] \in \mathcal{N}_f$  for every  $n \geq N$ .*

In other words, for every sufficiently large  $n$ , the tree  $T_n$  intersects the charge graph of  $f$  only in critical points (up to isotopy). To prove this result, we establish a certain “topological contraction” property of the pullback operation, see Proposition 4.6.

Main Theorem B allows us to develop an algorithm that answers the raised question of recognizing the combinatorial equivalence class of critically fixed Thurston maps; see Algorithm 1. The key idea here is that, once we get a tree  $T$  with  $[T] \in \mathcal{N}_f$ , we can reconstruct  $\text{Charge}(f)$  and compute an admissible pair for  $f$  using a simple combinatorial operation; see Section 4.3 for more details. We provide an upper bound for the speed of

convergence of Algorithm 1 in Theorem 4.7. Note that, once we know  $\text{Charge}(f)$ , we can also determine if  $f$  is realized and, if not, find its canonical Thurston obstruction.

1.1.3. *Twisting problem.* The machinery we develop may be used to study the twisting problem for critically fixed Thurston maps.

Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map and  $\varphi \in \text{Homeo}^+(S^2, \text{Crit}(f))$ . The *twisting problem* asks to determine the combinatorial equivalence class of the twisted map  $g := \varphi \circ f$ . In our case, the map  $g$  is again a critically fixed Thurston map with  $\text{Crit}(g) = \text{Crit}(f)$ . Thus, we may use Algorithm 1 for solving the twisting problem; see Example 5.1. At the same time, in some special cases, we may solve the twisting problem explicitly using a simple combinatorial operation applied to the charge graph of the original map  $f$ .

To be exact, we consider critically fixed Thurston maps  $f$  obtained by blowing up the pair  $(G, \text{id}_{S^2})$ , where  $G$  is a planar embedded graph in  $S^2$ . Note that this family includes all critically fixed rational maps (i.e., maps for which the graph  $G$  is connected). Further, let  $\gamma$  be an essential Jordan curve in  $S^2 \setminus \text{Crit}(f)$  such that  $i_f(G, \gamma) = |G \cap \gamma|$  and  $|\gamma \cap e| \leq 1$  for each edge  $e \in E(G)$ . Here,  $i_f(\cdot, \cdot)$  denotes the (unsigned) *intersection number* rel.  $\text{Crit}(f)$ ; see Section 2.3 for the precise definition. Finally, we consider the twisted maps of the form  $T_\gamma^n \circ f$ , where  $n \in \mathbb{Z}$ ,  $f$  and  $\gamma$  are as described above, and  $T_\gamma$  is the *Dehn twist* about the curve  $\gamma$ . We prove that the combinatorial equivalence class of such maps can be determined by applying a combinatorial operation, which we call the *n-rotation about the curve  $\gamma$* , to the original graph  $G = \text{Charge}(f)$ . Roughly speaking, this operation acts like a “fractional Dehn twist” on  $G$ ; see the precise definition in Section 5.1.

**Main Theorem C.** *Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map obtained by blowing up a pair  $(G, \text{id}_{S^2})$ . Suppose  $n \in \mathbb{Z}$  is arbitrary and  $\gamma$  is an essential Jordan curve in  $S^2 \setminus \text{Crit}(f)$  such that  $i_f(G, \gamma) = |G \cap \gamma|$  and  $|\gamma \cap e| \leq 1$  for each  $e \in E(G)$ .*

- (i) *If  $i_f(G, \gamma) \geq 1$ , then the twisted map  $T_\gamma^n \circ f$  is isotopic to the map obtained by blowing up the pair  $(H, \text{id}_{S^2})$ , where  $H$  is the result of the  $(-n)$ -rotation about the curve  $\gamma$  applied to  $G$ .*
- (ii) *If  $i_f(G, \gamma) = 0$ , then the twisted map  $T_\gamma^n \circ f$  is isotopic to the map obtained by blowing up the (admissible) pair  $(G, T_\gamma^n)$ .*

This theorem allows to conclude the following statement.

**Corollary 1.5.** *Suppose we are in the setup of Main Theorem C with  $i_f(G, \gamma) \geq 1$ . Then the sequence of combinatorial equivalence classes of  $(T_\gamma^n \circ f)_{n \in \mathbb{Z}}$  is strictly periodic with the period dividing  $i_f(G, \gamma)$ . In other words, if  $n_1 \equiv n_2 \pmod{i_f(G, \gamma)}$ , then the twisted maps  $T_\gamma^{n_1} \circ f$  and  $T_\gamma^{n_2} \circ f$  are combinatorially equivalent.*

1.2. **Organization of the paper.** Our paper is organized as follows. In Section 2, we review some general background. In Section 2.1, we fix notation and state some basic definitions. We discuss planar embedded graphs and intersection numbers in Sections 2.2 and 2.3, respectively. In Sections 2.4 and 2.5, we provide the necessary background in Thurston theory and formulate Thurston’s characterization of rational maps. In Section 2.6, we discuss notions and results from Pilgrim’s decomposition theory. We also formulate some auxiliary results about branched covers and planar graphs in Section 2.7.

In Section 3, we construct canonical combinatorial models for critically fixed Thurston maps. In Sections 3.1–3.3, we introduce the blow-up operation and discuss its properties. We review the classification of critically fixed rational maps in Section 3.4. In Section 3.5,

we study invariant multicurves and decompositions of critically fixed Thurston maps. In Section 3.6, we prove Main Theorem A, i.e., provide a complete classification of critically fixed Thurston maps.

Further, in Section 4, we work on the algorithmic recognition of the combinatorial equivalence classes for critically fixed Thurston maps. In Section 4.1, we introduce the pullback operation on planar embedded trees. Next, we explore contraction properties of the pullback operation and prove Main Theorem B in Section 4.2. In Section 4.3, we discuss how to recover a canonical combinatorial model for critically fixed Thurston maps. In particular, we present the Lifting Algorithm (Algorithm 1) that recovers the charge graph and discuss its complexity.

In Section 5, we study the twisting problem for the family of critically fixed Thurston maps. We start by briefly reviewing the previous work on the problem and discussing a few specific examples. In Section 5.1, we define the combinatorial operation of the  $n$ -rotation of a planar embedded graph about a Jordan curve. We introduce the concept of *simple transversals* with respect to a planar embedded graph in Section 5.2. Finally, we prove Main Theorem C and discuss its implications in Section 5.3.

**Acknowledgments.** This material is based upon work supported by the National Science Foundation under Grant No. 1440140, while the authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring semester of 2022. The authors were also partially supported by the ERC advanced grant ‘‘HOLOGRAM’’. We would like to thank Laurent Bartholdi, Mario Bonk, Kostya Drach, Dima Dudko, Lukas Geyer, Russell Lodge, Yusheng Luo, Daniel Meyer, Sabyasachi Mukherjee, Kevin Pilgrim, Palina Salanevich, Dierk Schleicher, Vladlen Timorin, and Rebecca Winarski for useful comments, remarks, and discussions. We are also grateful to Laurent Bartholdi for help with his GAP-package IMG.

## 2. BACKGROUND

**2.1. Notation and basic concepts.** The sets of positive integers, integers, and complex numbers are denoted by  $\mathbb{N}$ ,  $\mathbb{Z}$ , and  $\mathbb{C}$ , respectively. We use the notation  $i$  for the imaginary unit in  $\mathbb{C}$ ,  $\mathbb{I} := [0, 1]$  for the closed unit interval on the real line,  $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$  for the open unit disk in  $\mathbb{C}$ , and  $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$  for the Riemann sphere.

The cardinality of a set  $X$  is denoted by  $|X|$  and the identity map on  $X$  by  $\text{id}_X$ . If  $f: X \rightarrow X$  is a map and  $n \in \mathbb{N}$ , we denote the  $n$ -th iterate of  $f$  by  $f^n$ . If  $f: X \rightarrow Y$  is map and  $U \subset X$ , then  $f|_U$  stands for the restriction of  $f$  to  $U$ .

If  $X$  is a topological space and  $U \subset X$ , then  $\overline{U}$  denotes the closure,  $\text{int}(U)$  the interior, and  $\partial U$  the boundary of  $U$  in  $X$ .

Let  $S$  be a connected and oriented topological 2-manifold. We denote its *Euler characteristic* by  $\chi(S)$ . We use the notation  $S^2$  for an (oriented) *topological 2-sphere*, that is, a 2-manifold homeomorphic to the Riemann sphere  $\widehat{\mathbb{C}}$ . Let  $V \subset S^2$  be an open and connected subset of  $S^2$ . Then  $\chi(V)$  is given by  $2 - k_V$ , where  $k_V$  is the number of complementary components of  $V$ .

A *Jordan curve* in  $S^2$  is the image of an injective continuous map of  $\partial\mathbb{D}$  into the sphere  $S^2$ . A *Jordan arc*  $e$  in  $S^2$  is the image  $e = \iota(\mathbb{I})$  of an injective continuous map  $\iota: \mathbb{I} \rightarrow S^2$ . We will use the notation  $\partial e := \{\iota(0), \iota(1)\}$  for the *endpoints* and  $\text{int}(e) := e \setminus \partial e$  for the *interior*



of the Jordan arc  $e$  (these, of course, differ from the boundary and interior of  $e$  as a subset of  $S^2$ ).

A subset  $U \subset S^2$  is called an *open* or *closed Jordan region* (in  $S^2$ ) if there exists an injective continuous map  $\eta: \mathbb{D} \rightarrow S^2$  such that  $U = \eta(\mathbb{D})$  or  $U = \eta(\overline{\mathbb{D}})$ , respectively. In both cases,  $\partial U = \eta(\partial \mathbb{D})$  is a Jordan curve in  $S^2$ . A *crosscut* in an open or closed Jordan region  $U$  is a Jordan arc  $e$  such that  $\text{int}(e) \subset \text{int}(U)$  and  $\partial e \subset \partial U$ .

A *closed annulus* in  $S^2$  is the image  $A = \phi(\partial \mathbb{D} \times \mathbb{I})$  of an injective continuous map  $\phi: \partial \mathbb{D} \times \mathbb{I} \rightarrow S^2$ . A *core curve* of  $A$  is a Jordan curve  $\gamma \subset \text{int}(A)$  such that the two boundary curves of  $A$  are in distinct components  $S^2 \setminus \gamma$ .

Usually, we work with a *finitely marked sphere*, that is, a pair  $(S^2, Z)$ , where  $Z$  is a finite set of *marked points* in  $S^2$ . We say that  $e \subset S^2$  is a Jordan arc in a marked sphere  $(S^2, Z)$  if  $e$  is a Jordan arc in  $S^2$  with  $\partial e \subset Z$  and  $\text{int}(e) \subset S^2 \setminus Z$ . A Jordan curve in  $(S^2, Z)$  is a Jordan curve  $\gamma$  in  $S^2$  with  $\gamma \subset S^2 \setminus Z$ . Such a Jordan curve  $\gamma$  is called *essential* if each of the two connected components of  $S^2 \setminus \gamma$  contains at least two points from  $Z$ ; otherwise, we say that  $\gamma$  is *non-essential*. A non-essential Jordan curve  $\gamma$  in  $(S^2, Z)$  is called *null-homotopic* if one of the components of  $S^2 \setminus \gamma$  contains no points from  $Z$ ; otherwise, we say that  $\gamma$  is *peripheral*.

Let  $X$  and  $Y$  be topological spaces. A continuous map  $H: X \times \mathbb{I} \rightarrow Y$  is called a *homotopy* from  $X$  to  $Y$ . The homotopy  $H$  is called an *isotopy* if the *time- $t$  map*  $H_t := H(t, \cdot): X \rightarrow Y$  is a homeomorphism for each  $t \in \mathbb{I}$ .

Suppose  $Z \subset X$ . A homotopy  $H: X \times \mathbb{I} \rightarrow Y$  is said to be a homotopy *relative to*  $Z$  (abbreviated “ $H$  is a homotopy rel.  $Z$ ”) if  $H_t(p) = H_0(p)$  for all  $p \in Z$  and  $t \in \mathbb{I}$ . Similarly, we define an isotopy rel.  $Z$ .

Two homeomorphisms  $h_0, h_1: X \rightarrow Y$  are called *isotopic (rel.  $Z \subset X$ )* if there exists an isotopy  $H: X \times \mathbb{I} \rightarrow Y$  (rel.  $Z$ ) with  $H_0 = h_0$  and  $H_1 = h_1$ .

Given  $M, N \subset X$ , we say that  $M$  is *homotopic to*  $N$  (rel.  $Z \subset X$ ) if there exists a homotopy  $H: X \times \mathbb{I} \rightarrow X$  (rel.  $Z$ ) with  $H_0 = \text{id}_X$  and  $H_1(M) = N$ . If  $H$  is an isotopy rel.  $Z$  we say that  $M$  is *isotopic to*  $N$  rel.  $Z$  (or  $M$  can be isotoped into  $N$  rel.  $Z$ ) and denote this by  $M \sim N$  rel.  $Z$ .

Let  $(S^2, Z)$  be a finitely marked sphere. We denote by  $\text{Homeo}^+(S^2, Z)$  the group of all orientation-preserving homeomorphisms of  $S^2$  that fix the set  $Z$  element-wise. We will use the notation  $\text{Homeo}_0^+(S^2, Z)$  for the subgroup of  $\text{Homeo}^+(S^2, Z)$  consisting of homeomorphisms isotopic to the identity rel.  $Z$ . The *pure mapping class group* of the marked sphere  $(S^2, Z)$  is then defined as the quotient

$$\text{PMCG}(S^2, Z) := \text{Homeo}^+(S^2, Z) / \text{Homeo}_0^+(S^2, Z).$$

Let  $\gamma$  be a Jordan curve in  $(S^2, Z)$ . We will use the notation  $T_\gamma$  to denote a *Dehn twist* about  $\gamma$  in  $(S^2, Z)$ . To define it, consider first the (left) *twist map*  $T: \partial \mathbb{D} \times \mathbb{I} \rightarrow \partial \mathbb{D} \times \mathbb{I}$  given by the formula  $T(e^{i\theta}, t) = (e^{i(\theta+2\pi t)}, t)$ . We assume that the cylinder  $\partial \mathbb{D} \times \mathbb{I}$  is oriented so that its embedding into the complex plane  $\mathbb{C}$  via the map  $(e^{i\theta}, t) \mapsto e^{i\theta}(t+1)$  is orientation-preserving. Let now  $A \subset S^2 \setminus Z$  be a closed annulus in  $S^2$  with core curve  $\gamma$  and  $\phi: \partial \mathbb{D} \times \mathbb{I} \rightarrow A$  be an orientation-preserving homeomorphism. Then a Dehn twist  $T_\gamma$  about the curve  $\gamma$  is defined by

$$T_\gamma(p) = \begin{cases} (\phi \circ T \circ \phi^{-1})(p) & \text{if } p \in A \\ p & \text{if } p \in S^2 \setminus A. \end{cases}$$

By construction, the map  $T_\gamma \in \text{Homeo}^+(S^2, Z)$ . It is uniquely defined up to isotopy rel.  $Z$  independently of the choice of  $A$  and  $\phi$ . Furthermore, the isotopy class of  $T_\gamma$  does not depend on the choice of the Jordan curve  $\gamma$  within its isotopy class rel.  $Z$ .

**2.2. Planar embedded graphs.** We refer the reader to [Die05] for general background in graph theory. Below we conduct a discussion in the context of planar embedded graphs, though many of the concepts are also relevant for abstract graphs.

A *planar embedded graph* in a sphere  $S^2$  is a pair  $G = (V, E)$ , where  $V$  is a finite set of points in  $S^2$  and  $E$  is a finite set of Jordan arcs in  $(S^2, V)$  with pairwise disjoint interiors. The sets  $V$  and  $E$  are called the *vertex* and *edge sets* of  $G$ , respectively. Note that our notion of a planar embedded graph allows *multiple edges*, that is, distinct edges that connect the same pair of vertices. However, it does not allow *loops*, that is, edges that connect a vertex to itself.

In the following, suppose  $G = (V, E)$  is a planar embedded graph in  $S^2$ . The *degree* of a vertex  $v$  in  $G$ , denoted by  $\deg_G(v)$ , is the number of edges in  $G$  incident to  $v$ . If  $\deg_G(v) = 0$ , we say that the vertex  $v$  is *isolated*, and if  $\deg_G(v) = 1$ , we say that the vertex  $v$  is a *leaf*.

The subset  $\mathcal{G} := V \cup \bigcup_{e \in E} e$  of  $S^2$  is called the *realization* of  $G$ . A *face* of the graph  $G$  is a connected component of  $S^2 \setminus \mathcal{G}$ . Given a planar embedded graph  $G$ , we will denote by  $V(G)$ ,  $E(G)$ , and  $F(G)$  the sets of vertices, edges, and faces of  $G$ , respectively.

It will be convenient to conflate a planar embedded graph  $G$  with its realization  $\mathcal{G}$ . In this case, we will specify a finite set  $V(\mathcal{G}) \subset \mathcal{G}$  of distinguished points that serve as the vertices of the graph. Then the edge set  $E(\mathcal{G})$  consists of the closures of the components of  $\mathcal{G} \setminus V(\mathcal{G})$ .

A *walk*  $P$  of length  $n$  between vertices  $v$  and  $v'$  in  $G$  is a sequence  $(v_0 := v, e_0, v_1, e_1, \dots, v_{n-1}, e_{n-1}, v_n := v')$ , where  $e_j$  is an edge in  $G$  incident to the vertices  $v_j$  and  $v_{j+1}$  for each  $j = 0, \dots, n-1$ . If it does not create ambiguity, we may describe the walk  $P$  by the sequence  $(v_0, v_1, \dots, v_n)$  of its consecutive vertices. The walk  $P$  is called a *path* if all its edges  $e_j$ ,  $j = 0, \dots, n-1$ , are distinct, and it is called a *simple path* if all its vertices  $v_j$ ,  $j = 0, \dots, n$ , are distinct.

A path  $(v_0, e_0, v_1, e_1, \dots, v_{n-1}, e_{n-1}, v_n)$  in  $G$  with  $v_0 = v_n$  and  $n \geq 2$  is called a *cycle of length*  $n$ . Such a cycle is called *simple* if all vertices  $v_j$ ,  $j = 0, \dots, n-1$ , are distinct.

The graph  $G$  is called *connected* if there is a path in  $G$  between every two vertices  $v, v' \in V$ . In other words,  $G$  is connected if its realization is a connected subset of  $S^2$ . It follows that the graph  $G$  is connected if and only if each face of  $G$  is simply connected. We say that the graph  $G$  is a *tree* if  $G$  is connected and there are no cycles in  $G$ .

A *subgraph* of  $G$  is a planar embedded graph  $G' = (V', E')$  such that  $V' \subset V$  and  $E' \subset E$ . A *connected component* of  $G$  is a maximal connected subgraph of  $G$ . The number  $k_G$  of connected components of  $G$  is given by the *Euler formula*

$$k_G = |F(G)| - |E(G)| + |V(G)| - 1.$$

Let  $A$  be a non-empty subset of  $V$ . A *spanning subtree of  $A$  in  $G$*  is a minimal subtree  $T$  of  $G$  with  $A \subset V(T)$ . Such a subtree  $T$  exists if and only if all vertices in  $A$  belong to the same connected component of  $G$ . Note that each leaf of  $T$  must be in  $A$  due to minimality.

Suppose  $S^2$  and  $\widehat{S^2}$  are two topological 2-spheres. Let  $G = (V, E)$  and  $\widehat{G} = (\widehat{V}, \widehat{E})$  be two planar embedded graphs in  $S^2$  and  $\widehat{S^2}$ , respectively. We say that  $G$  is *isomorphic* to  $\widehat{G}$  if there exists an orientation-preserving homeomorphism  $\psi: S^2 \rightarrow \widehat{S^2}$  that maps vertices and edges of  $G$  into vertices and edges of  $\widehat{G}$ , that is,  $\psi(v) \in \widehat{V}$  and  $\psi(e) \in \widehat{E}$  for all  $v \in V$  and

$e \in E$ . In this case, we call  $\psi$  an *isomorphism* between  $G$  and  $\widehat{G}$ . Clearly, isomorphisms induce an equivalence relation on the set of all planar embedded graphs. An equivalence class of this relation is called an *isomorphism class* of planar embedded graphs.

**2.3. Isotopies and intersection numbers.** In various constructions of isotopies throughout this paper, we use the following fact without explicit reference (the proof follows from [Bus10, Theorem A.6(ii)]).

**Lemma 2.1.** *Let  $W$  be an open Jordan region in  $S^2$ . Suppose  $\alpha$  and  $\beta$  are Jordan arcs in  $S^2$  with  $\text{int}(\alpha), \text{int}(\beta) \subset W$  and  $\partial\alpha = \partial\beta$ . Then  $\alpha$  and  $\beta$  are isotopic rel.  $\partial\alpha \cup (S^2 \setminus W)$ .*

We will frequently consider planar embedded graphs up to isotopy rel. finite number of marked points in  $S^2$ .

**Definition 2.2.** Let  $G$  and  $G'$  be two planar embedded graphs in  $S^2$  and  $Z \subset S^2$  be a finite set of points. We say that  $G$  and  $G'$  are *isotopic rel.  $Z$* , denoted by  $G \sim G'$  rel.  $Z$ , if there exists an isotopy  $H: S^2 \times \mathbb{I} \rightarrow S^2$  rel.  $Z$  such that the following conditions are satisfied:

- (i)  $H_0 = \text{id}_{S^2}$ ;
- (ii)  $H_1(V(G)) = V(G')$ ;
- (iii)  $H_1(G) = G'$ .

Note that (ii) and (iii) imply that  $H_1$  provides a one-to-one correspondence between the edges of  $G$  and  $G'$ .

The next statement guarantees that we do not run into topological difficulties while studying planar embedded graphs (the proof follows from [Bus10, Lemma A.8]).

**Proposition 2.3.** *Let  $G$  be a planar embedded graph in  $S^2$  and  $Z \subset S^2$  be a finite set of points. Then there exists a planar embedded graph  $G'$  such that  $G \sim G'$  rel.  $V(G) \cup Z$  and such that each edge of  $G'$  is a piecewise geodesic arc in  $S^2$  (with respect to some fixed Riemannian metric on  $S^2$ ).*

Using [Bus10, Theorem A.5] and Lemma 2.1, we obtain the following criterion for two planar embedded graphs to be isotopic.

**Proposition 2.4.** *Let  $G$  and  $G'$  be two planar embedded graphs with a common vertex set  $V$ . Then  $G$  and  $G'$  are isotopic rel.  $V$  if and only if for each edge  $e \in E(G)$  there is an edge  $e' \in E(G')$  such that  $e$  and  $e'$  are isotopic rel.  $V$  and  $m_G(e) = m_{G'}(e')$ .*

Here,  $m_G(e)$  denotes the *multiplicity* of an edge  $e$  in a planar embedded graph  $G$ , that is, the total number of edges of  $G$  that are isotopic to  $e$  rel.  $V(G)$ .

In the following, let  $(S^2, Z)$  be a finitely marked sphere. The (unsigned) *intersection number* between two Jordan arcs or curves  $\alpha$  and  $\beta$  in  $(S^2, Z)$  is defined as

$$i_Z(\alpha, \beta) := \inf_{\substack{\alpha' \sim \alpha \text{ rel. } Z, \\ \beta' \sim \beta \text{ rel. } Z}} |(\alpha' \cap \beta') \setminus Z|,$$

where the infimum is taken over all Jordan arcs or curves  $\alpha'$  and  $\beta'$  in  $(S^2, Z)$  that are isotopic to  $\alpha$  and  $\beta$  rel.  $Z$ , respectively. Note that the intersection number is finite, because we can always reduce to the case when  $\alpha$  and  $\beta$  are piecewise geodesic with respect to some Riemannian metric on  $S^2$ . We say that  $\alpha$  and  $\beta$  are in *minimal position* rel.  $Z$  if  $|(\alpha \cap \beta) \setminus Z| = i_Z(\alpha, \beta)$ .

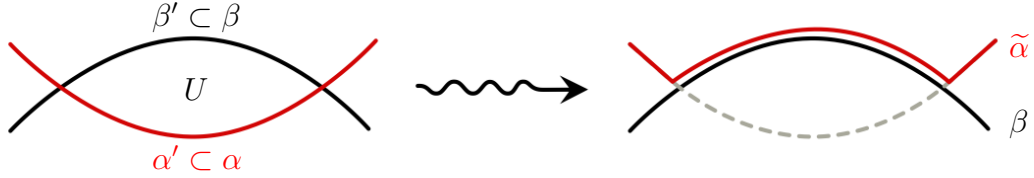


FIGURE 1. Removing a bigon between two Jordan arcs or curves  $\alpha$  and  $\beta$ .

Let  $\alpha$  and  $\beta$  be two Jordan arcs or curves in  $(S^2, Z)$ . We say that subarcs  $\alpha' \subset \alpha$  and  $\beta' \subset \beta$  form a *bigon*  $U$  in  $(S^2, Z)$  if  $\partial\alpha' = \partial\beta'$ ,  $\text{int}(\alpha') \cap \text{int}(\beta') = \emptyset$ , and  $U$  is a connected component of  $S^2 \setminus (\alpha' \cup \beta')$  with  $U \cap Z = \emptyset$ ; see the left part of Figure 1 for an illustration. It is easy to see that in this situation  $\alpha$  and  $\beta$  are not in minimal position rel.  $Z$ . Indeed, one of the curves, say  $\alpha$ , may be isotoped into a new curve  $\tilde{\alpha}$  rel.  $Z$  with  $|(\tilde{\alpha} \cap \beta) \setminus Z| < |(\alpha \cap \beta) \setminus Z|$ ; see the right part of Figure 1 for an illustration. We will call this procedure “*removing a bigon*” between  $\alpha$  and  $\beta$ . In fact, the converse is also true. If two Jordan arcs or curves  $\alpha$  and  $\beta$  in  $(S^2, Z)$  with transverse intersections are not in minimal position, then there are subarcs  $\alpha' \subset \alpha$  and  $\beta' \subset \beta$  forming a bigon (see [FM12, Proposition 1.7 and Section 1.2.7]).

Let  $G$  be a planar embedded graph in  $S^2$  without isolated vertices and  $\alpha$  be a Jordan arc (or curve) in  $(S^2, Z)$ . The *intersection number*  $i_Z(G, \alpha)$  between  $G$  and  $\alpha$  rel.  $Z$  is defined as

$$i_Z(G, \alpha) := \inf_{\substack{G' \sim G \text{ rel. } Z, \\ \alpha' \sim \alpha \text{ rel. } Z}} |(G' \cap \alpha') \setminus Z|,$$

where the infimum is taken over all planar embedded graphs  $G'$  in  $S^2$  and Jordan arcs (curves)  $\alpha'$  in  $(S^2, Z)$  that are isotopic to  $G$  and  $\alpha$  rel.  $Z$ , respectively. We say that  $G$  and  $\alpha$  are in *minimal position* rel.  $Z$  if they satisfy

$$i_Z(G, \alpha) = |(G \cap \alpha) \setminus Z|$$

The following lemma follows easily from the definitions and Proposition 2.3 above.

**Lemma 2.5.** *Let  $G$  be a planar embedded graph in  $S^2$  without isolated vertices and  $\alpha$  be a Jordan arc (or curve) in  $(S^2, Z)$ , where  $Z \subset S^2$  is a finite set of points. Then the following statements are true:*

- (i) *The intersection number  $i_Z(G, \alpha)$  is finite.*
- (ii) *There exists a Jordan arc (curve)  $\alpha'$  in  $(S^2, Z)$  that is isotopic to  $\alpha$  rel.  $Z$  such that  $G$  and  $\alpha'$  are in minimal position rel.  $Z$ .*
- (iii) *There exists a planar embedded graph  $G'$  that is isotopic to  $G$  rel.  $Z$  such that  $G'$  and  $\alpha$  are in minimal position rel.  $Z$ .*

Lemma 2.5(ii) implies that if  $i_Z(G, \alpha) = 0$ , then we may isotope  $\alpha$  so that  $G \cap \alpha \subset Z$ . We record the following extension of this fact, where we replace a single arc  $\alpha$  in  $(S^2, Z)$  by a planar embedded graph  $H$  with vertices in  $Z$ .

**Proposition 2.6.** *Let  $(S^2, Z)$  be a finitely marked sphere and  $G$  be a planar embedded graph in  $S^2$  without isolated vertices. Suppose  $H$  is a planar embedded graph in  $S^2$  with  $V(H) \subset Z$  and such that  $i_Z(G, e) = 0$  for each  $e \in E(H)$ . Then there exists a planar embedded graph  $H'$  isotopic to  $H$  rel.  $Z$  such that  $H' \cap G \subset Z$ .*

*Proof.* We will only give an outline of the proof, leaving some straightforward details to the reader.

First, it is sufficient to consider the case when  $V(H) = Z$ . We prove the statement by induction on  $|E(H)|$ . If  $|E(H)| = 0$ , then there is nothing to prove. Also, if  $|E(H)| = 1$ , then the statement follows from part (ii) of Lemma 2.5.

Assume the statement is true if  $|E(H)| \leq n$ , where  $n \in \mathbb{N}$ . Suppose now that  $|E(H)| = n + 1$ , and consider the graph  $H_\alpha := (V(H), E(H) \setminus \{\alpha\})$ , which is obtained from  $H$  by removing some edge  $\alpha \in E(H)$ . By induction hypothesis, there is a planar embedded graph  $H'_\alpha$  isotopic to  $H_\alpha$  rel.  $Z$  such that  $H'_\alpha \cap G \subset Z$ .

*Claim.* There exists a Jordan arc  $\alpha' \sim \alpha$  rel.  $Z$  such that  $\alpha' \cap (G \cup H'_\alpha) \subset Z$ .

Let  $\mathcal{A}$  be the set of all Jordan arcs  $\alpha'$  in  $(S^2, Z)$  that are isotopic to  $\alpha$  rel.  $Z$  and satisfy  $\alpha' \cap G \subset Z$ . Since  $i_Z(G, \alpha) = 0$ , the set  $\mathcal{A}$  is non-empty by Lemma 2.5(ii). Furthermore,  $i_Z(e', \alpha') = 0$  for every  $e' \in E(H'_\alpha)$  and  $\alpha' \in \mathcal{A}$ . Consider now the following intersection number

$$(2.1) \quad N := \inf_{\alpha' \in \mathcal{A}} |(H'_\alpha \cap \alpha') \setminus Z|.$$

Proposition 2.3 implies that  $N$  is finite and there exists a Jordan arc  $\alpha' \in \mathcal{A}$  that realizes the infimum in (2.1). We claim that  $N$  must be equal to 0. Otherwise, there is an edge  $e' \in E(H'_\alpha)$  such that  $|(e' \cap \alpha') \setminus Z| > 0$ , which means that  $e'$  and  $\alpha'$  are not in minimal position rel.  $Z$ . But then some subarcs of  $e'$  and  $\alpha'$  must form a bigon in  $(S^2, Z)$  or have a non-transverse intersection. We may remove this bigon or non-transverse intersection between  $e'$  and  $\alpha'$  and get a Jordan arc  $\tilde{\alpha} \in \mathcal{A}$  that will satisfy

$$|(H'_\alpha \cap \tilde{\alpha}) \setminus Z| < |(H'_\alpha \cap \alpha') \setminus Z|.$$

But this contradicts the choice of  $\alpha'$ . Thus,  $N = 0$  and the claim follows.

Let  $\alpha'$  be a Jordan arc as in the claim. Then  $H' := H'_\alpha \cup \alpha'$  is a planar embedded graph with  $V(H') = V(H) = Z$  and  $H' \cap G \subset Z$ . By construction,  $H'$  and  $H$  satisfy the conditions of Proposition 2.4, and thus they are isotopic rel.  $Z$ . This finishes the proof.  $\square$

**2.4. Thurston maps.** A continuous surjective map  $f: S^2 \rightarrow S^2$  is called a *branched covering map* if it locally acts as the power map  $z \mapsto z^d$  for some  $d \in \mathbb{N}$  in orientation-preserving coordinate charts in domain and target. More precisely, for each  $p \in S^2$  we require that there are two open Jordan regions  $U$  and  $V$  containing  $p$  and  $f(p)$ , respectively, two orientation-preserving homeomorphisms  $\varphi: \mathbb{D} \rightarrow U$  and  $\psi: \mathbb{D} \rightarrow V$ , and a number  $d \in \mathbb{N}$  such that

- (i)  $\varphi(0) = p$  and  $\psi(0) = f(p)$ ;
- (ii)  $(\psi^{-1} \circ f \circ \varphi)(z) = z^d$  for all  $z \in \mathbb{D}$ .

The integer  $d$  as in (ii) is uniquely determined by  $f$  and  $p$ . It is called the *local degree* of the map  $f$  at the point  $p$  and denoted by  $\deg(f, p)$ . We also denote the *topological degree* of  $f$  by  $\deg(f)$ .

A point  $p \in S^2$  is called a *critical point* of  $f$  if  $\deg(f, p) > 1$ , that is, if  $f$  is not locally injective at  $p$ . We denote the set of all critical points of  $f$  by  $\text{Crit}(f)$ .

Let  $f: S^2 \rightarrow S^2$  be a branched covering map and  $V \subset S^2$  be an open and connected set. Suppose  $U$  is a connected component of  $f^{-1}(V)$ . Then  $f(U) = V$  and each point  $q \in V$  has the same number  $d \in \mathbb{N}$  of preimages under  $f|_U$  counting multiplicities (given by the local degrees of  $f$  at the preimage points). This number  $d$  is called the *degree of  $f$  on  $U$*  and

denoted by  $\deg(f|U)$ . If the Euler characteristic  $\chi(V)$  is finite, then  $\chi(U)$  is also finite and we have the *Riemann-Hurwitz formula*

$$\chi(U) + \sum_{c \in U \cap \text{Crit}(f)} (\deg(f, c) - 1) = \deg(f|U) \cdot \chi(V);$$

see the discussion in [BM17, Section 13.2].

The set

$$\text{Post}(f) := \bigcup_{n \in \mathbb{N}} f^n(\text{Crit}(f))$$

is called the *postcritical set* of a branched covering map  $f: S^2 \rightarrow S^2$ . We say that the map  $f$  is *postcritically-finite* if  $\text{Post}(f)$  is finite.

**Definition 2.7.** A *Thurston map* is a postcritically-finite branched covering map  $f: S^2 \rightarrow S^2$  with  $\deg(f) \geq 2$ .

In other words, a branched covering map  $f$  on  $S^2$  is called a Thurston map if it is not a homeomorphism and each critical point of  $f$  has a finite orbit under iteration. Natural examples of Thurston maps are provided by *rational Thurston maps*, that is, postcritically-finite rational maps on the Riemann sphere  $\widehat{\mathbb{C}}$ .

**Definition 2.8.** Suppose  $S^2$  and  $\widehat{S^2}$  are two topological 2-spheres. Two Thurston maps  $f: S^2 \rightarrow S^2$  and  $g: \widehat{S^2} \rightarrow \widehat{S^2}$  are called *combinatorially* (or *Thurston*) *equivalent* if there are orientation-preserving homeomorphisms  $\psi_0, \psi_1: S^2 \rightarrow \widehat{S^2}$  that are isotopic rel.  $\text{Post}(f)$  and satisfy  $\psi_0 \circ f = g \circ \psi_1$ .

We say that a Thurston map is *realized* (by a rational map) if it is combinatorially equivalent to a rational map. Otherwise, we say that it is *obstructed*.

Thurston maps have the following isotopy lifting property (see, for example, [BM17, Proposition 11.3]).

**Proposition 2.9.** *Suppose  $f: S^2 \rightarrow S^2$  and  $g: \widehat{S^2} \rightarrow \widehat{S^2}$  are two Thurston maps, and  $h_0, \tilde{h}_0: S^2 \rightarrow \widehat{S^2}$  are homeomorphisms such that  $h_0|_{\text{Post}(f)} = \tilde{h}_0|_{\text{Post}(f)}$  and  $h_0 \circ f = g \circ \tilde{h}_0$ . Let  $H: S^2 \times \mathbb{I} \rightarrow \widehat{S^2}$  be an isotopy rel.  $\text{Post}(f)$  with  $H_0 = h_0$ .*

*Then the isotopy  $H$  uniquely lifts to an isotopy  $\tilde{H}: S^2 \times \mathbb{I} \rightarrow \widehat{S^2}$  rel.  $f^{-1}(\text{Post}(f))$  such that  $\tilde{H}_0 = \tilde{h}_0$  and  $g \circ \tilde{H}_t = H_t \circ f$  for all  $t \in \mathbb{I}$ .*

We will frequently work with Thurston maps defined by combinatorial constructions and the following definition appears to be useful.

**Definition 2.10.** Two Thurston maps  $f: S^2 \rightarrow S^2$  and  $g: S^2 \rightarrow S^2$  are called *isotopic* if  $\text{Post}(f) = \text{Post}(g)$  and there exist  $\psi_0, \psi_1 \in \text{Homeo}_0^+(S^2, \text{Post}(f))$  such that  $\psi_0 \circ f = g \circ \psi_1$ .

Note that Proposition 2.9 implies that two Thurston maps  $f, g: S^2 \rightarrow S^2$  are isotopic if and only if  $f = g \circ \psi$  for some  $\psi \in \text{Homeo}_0^+(S^2, \text{Post}(f))$ .

The *ramification function* of a Thurston map  $f: S^2 \rightarrow S^2$  is a function  $\nu_f: S^2 \rightarrow \mathbb{N} \cup \{\infty\}$  defined as follows:  $\nu_f(p)$  equals the lowest common multiple of all local degrees  $\deg(f^n, q)$ , where  $q \in f^{-n}(p)$  and  $n \in \mathbb{N}$  are arbitrary. It easily follows that  $\nu_f(p) \geq 2$  if and only if  $p \in \text{Post}(f)$ .

The pair  $\mathcal{O}_f := (S^2, \nu_f)$  is called the *orbifold* associated with  $f$ . The *Euler characteristic* of  $\mathcal{O}_f$  is given by

$$\chi(\mathcal{O}_f) := 2 - \sum_{p \in \text{Post}(f)} \left(1 - \frac{1}{\nu_f(p)}\right).$$

One can check that  $\chi(\mathcal{O}_f) \leq 0$  for every Thurston map  $f$ ; see [BM17, Proposition 2.12]. We say that  $\mathcal{O}_f$  is *hyperbolic* if  $\chi(\mathcal{O}_f) < 0$ , and *parabolic* if  $\chi(\mathcal{O}_f) = 0$ . Thurston maps with a parabolic orbifold are rather special and may be completely classified; see, for example, [BM17, Chapters 3 and 7]. We note that if  $\text{Post}(f) \geq 5$ , then  $\mathcal{O}_f$  is always hyperbolic.

**2.5. Thurston's characterization of rational maps.** In the following, let  $f: S^2 \rightarrow S^2$  be a Thurston map. A natural question to ask is when  $f$  is combinatorially equivalent to a rational map. William Thurston provided a topological criterion that answers this question in his celebrated *characterization of rational maps* [DH93, Theorem 1]. To formulate this result we need to introduce several concepts.

A *multicurve* is a finite collection  $\Gamma$  of essential Jordan curves in  $(S^2, \text{Post}(f))$  that are pairwise disjoint and pairwise non-isotopic rel.  $\text{Post}(f)$ . We say that a multicurve  $\Gamma$  is *invariant* if, for every curve  $\gamma \in \Gamma$ , each essential component of  $f^{-1}(\gamma)$  is isotopic rel.  $\text{Post}(f)$  to a curve in  $\Gamma$ .

Let  $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ ,  $n \in \mathbb{N}$ , be an invariant multicurve for  $f$ . We will now associate an  $(n \times n)$ -matrix  $M(f, \Gamma) = (m_{ij})$  with  $\Gamma$  as follows. Fix  $i, j \in \{1, \dots, n\}$ , and let  $\delta_1, \dots, \delta_K$ , where  $K = K(i, j) \geq 0$ , be all the components of  $f^{-1}(\gamma_j)$  that are isotopic to  $\gamma_i$  rel.  $\text{Post}(f)$ . We denote by  $\deg(f|_{\delta_k})$  the (unsigned) mapping degree of the covering map  $f|_{\delta_k}: \delta_k \rightarrow \gamma_j$ . Then the  $(i, j)$ -entry  $m_{ij}$  of the matrix  $M(f, \Gamma)$  is given by

$$m_{ij} := \sum_{k=1}^{K(i,j)} \frac{1}{\deg(f|_{\delta_k})}.$$

If  $K(i, j) = 0$ , then the sum is empty and  $m_{ij} = 0$ .

Note that  $M(f, \Gamma)$  depends only on the isotopy classes of curves in  $\Gamma$  (this easily follows from Proposition 2.9). The Perron-Frobenius theorem implies that the spectral radius of  $M(f, \Gamma)$  is given by the largest non-negative (real) eigenvalue  $\lambda(f, \Gamma)$  of this matrix. The invariant multicurve  $\Gamma$  is called a (*Thurston*) *obstruction* for  $f$  if  $\lambda(f, \Gamma) \geq 1$ .

With these definitions, we are finally in a position to state Thurston's theorem; the proof can be found in [DH93], see also [Hub16, Theorem 10.1.14].

**Theorem 2.11.** *A Thurston map  $f: S^2 \rightarrow S^2$  with a hyperbolic orbifold is combinatorially equivalent to a rational map  $F: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  if and only if  $f$  does not have a Thurston obstruction. Moreover, the rational map  $F$  is unique up to conjugation by a Möbius transformation.*

The easiest examples of obstructions are provided by *Levy fixed curves*.

**Definition 2.12.** Let  $f: S^2 \rightarrow S^2$  be a Thurston map and  $\gamma$  be an essential Jordan curve in  $(S^2, \text{Post}(f))$ . We call  $\gamma$  a *Levy fixed curve* if there is a connected component  $\gamma'$  of  $f^{-1}(\gamma)$  such that  $\gamma$  and  $\gamma'$  are isotopic rel.  $\text{Post}(f)$  and  $f|_{\gamma'}: \gamma' \rightarrow \gamma$  is a homeomorphism.

Note that if a Thurston map  $f$  has a Levy fixed curve  $\gamma$ , then it must be obstructed (since postcritically-finite rational maps are expanding with respect to the orbifold metric; see [Mil06, Theorem 19.6]). A priori the multicurve  $\{\gamma\}$  does not need to be invariant, but

we can always find an invariant multicurve  $\Gamma \supset \{\gamma\}$  by taking iterative preimages of  $\gamma$ ; see [Lei92, Lemma 2.2] for details. Then this multicurve  $\Gamma$  is a Thurston obstruction for  $f$ .

A Thurston obstruction may contain curves that are “extraneous” in some natural sense. The simplest instance of this is when one combines two invariant multicurves  $\Gamma_1$  and  $\Gamma_2$  having pairwise disjoint and pairwise non-isotopic curves, where  $\Gamma_1$  is an obstruction and  $\Gamma_2$  is not. Then  $\Gamma_1 \cup \Gamma_2$  is a Thurston obstruction as well, even though the curves from  $\Gamma_2$  are obviously redundant in there. These considerations motivate the following definition.

**Definition 2.13.** Let  $f$  be a Thurston map with an obstruction  $\Gamma$ . We say that  $\Gamma$  is *simple* if there is no permutation of the curves in  $\Gamma$  that puts the matrix  $M(f, \Gamma)$  in the block form

$$M(f, \Gamma) = \begin{bmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{bmatrix},$$

where the spectral radius of the square matrix  $M_{11}$  is less than 1.

One can easily check from the definition that every Thurston obstruction contains a simple one.

*Remark.* We note that Thurston’s characterization theorem remains true in a more general setup of marked Thurston maps. A *marked Thurston map* on  $S^2$  is a pair  $(f, Z)$  where  $f: S^2 \rightarrow S^2$  is a Thurston map and  $Z \subset S^2$  is a finite set of marked points that satisfies  $\text{Post}(f) \subset Z$  and  $f(Z) \subset Z$ . The notions of combinatorial equivalence, isotopy, and Thurston obstructions naturally extend to these maps by considering isotopies rel.  $Z$  and (multi)curves in  $(S^2, Z)$ . Then Theorem 2.11 holds in the same form for marked Thurston maps; see, for example, [BCL14, Theorem 2.1].

**2.6. Decomposition theory.** We outline a procedure due to Kevin Pilgrim that allows to naturally decompose a branched covering map into “simpler” pieces. We refer the reader to [Pil03] for details.

Let  $(S^2, Z)$  be a marked sphere and  $\Gamma$  be a collection of pairwise disjoint Jordan curves in  $(S^2, Z)$ . We denote by  $\mathcal{S}_\Gamma$  the set of all components of  $S^2 \setminus \bigcup_{\gamma \in \Gamma} \gamma$ . Each such component  $S$  may be viewed as a punctured sphere and we call the corresponding closure, denoted by  $\widehat{S}$ , a *small sphere* with respect to  $\Gamma$ . The points in  $Z$  and the curves in  $\Gamma$  induce a marking on small spheres. Namely, the small sphere  $\widehat{S}$  is marked by points corresponding to  $S \cap Z$  and  $\partial S \cap \Gamma$ . The set of all (marked) small spheres with respect to  $\Gamma$  is denoted by  $\widehat{\mathcal{S}}_\Gamma$ ; see Figure 2 (bottom) and Figure 3 for an illustration.

We now strengthen the definition of an invariant multicurve for a Thurston map  $f$ . We say that a multicurve  $\Gamma$  is *completely invariant* if the following two conditions are satisfied:

- (i) each essential component of  $f^{-1}(\bigcup_{\gamma \in \Gamma} \gamma)$  is isotopic rel.  $\text{Post}(f)$  to a curve in  $\Gamma$ ;
- (ii) each curve in  $\Gamma$  is isotopic rel.  $\text{Post}(f)$  to a component of  $f^{-1}(\bigcup_{\gamma \in \Gamma} \gamma)$ .

We note that every simple Thurston obstruction must be completely invariant.

In the following, suppose that  $f: S^2 \rightarrow S^2$  is a Thurston map and  $\Gamma$  is a completely invariant multicurve. For convenience, we denote by  $f^{-1}(\Gamma)$  the set of all components of the set  $f^{-1}(\bigcup_{\gamma \in \Gamma} \gamma) \subset S^2$ .

Let  $\widehat{\mathcal{S}}_\Gamma = \{\widehat{S}_j\}_{j \in J}$  be the set of all small spheres with respect to  $\Gamma$ . Since  $\Gamma$  is completely invariant, we may identify each small sphere  $\widehat{S}_j$ ,  $j \in J$ , with a unique small sphere  $\widehat{S}'_j$  with respect to  $f^{-1}(\Gamma)$  as follows. Let  $S_j \in \mathcal{S}_\Gamma$  be the component corresponding to the small



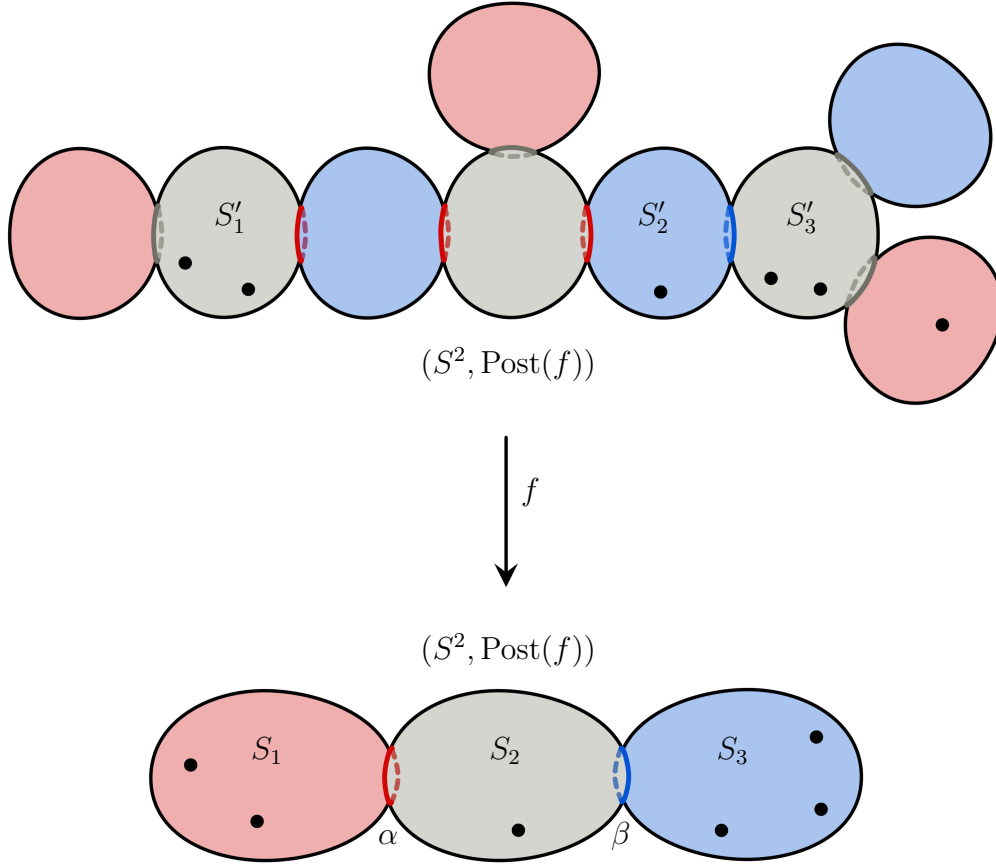


FIGURE 2. Decomposing a Thurston map  $f: S^2 \rightarrow S^2$  along a completely invariant multicurve  $\Gamma := \{\alpha, \beta\}$ . The bottom indicates the multicurve  $\Gamma$  and  $\mathcal{S}_\Gamma = \{S_1, S_2, S_3\}$ . The top illustrates  $f^{-1}(\Gamma)$  and  $\mathcal{S}_{f^{-1}(\Gamma)}$ . The black dots correspond to the postcritical points of  $f$ . The map  $f$  sends each component in  $\mathcal{S}_{f^{-1}(\Gamma)}$  onto the component in  $\mathcal{S}_\Gamma$  of the same color. At the top, the red curves are isotopic to  $\alpha$ , the blue curves are isotopic to  $\beta$ , and the gray curves are non-essential in  $(S^2, \text{Post}(f))$ .

sphere  $\widehat{S}_j$ . Then there exists a unique component  $S'_j \in \mathcal{S}_{f^{-1}(\Gamma)}$  such that  $S'_j$  and  $S_j$  are homotopic in  $S^2 \setminus \text{Post}(f)$ ; see the top of Figure 2 for an illustration. Furthermore, each component  $U$  of the complement  $S^2 \setminus \bigcup_{j \in J} S'_j$  is either

- (a) a closed Jordan region with  $|U \cap \text{Post}(f)| \leq 1$ , so that  $\partial U$  is a non-essential Jordan curve;
- (b) a closed annulus whose boundary components are isotopic rel.  $\text{Post}(f)$  to a curve  $\gamma_U \in \Gamma$ ;
- (c) or a Jordan curve from  $f^{-1}(\Gamma)$  that is isotopic rel.  $\text{Post}(f)$  to a curve  $\gamma_U \in \Gamma$ .

We now pick a homotopy that sends each component  $S'_j$  onto  $S_j$  and collapses each component  $U$  of  $S^2 \setminus \bigcup_{j \in J} S'_j$  to a point in case (a) or to the curve  $\gamma_U \in \Gamma$  in cases (b) and (c). More precisely, we choose a homotopy  $H: S^2 \times \mathbb{I} \rightarrow S^2$  rel.  $\text{Post}(f)$  with the following properties:

- (A)  $H_t := H(\cdot, t)$  is a homeomorphism for every  $t \in [0, 1)$ ;

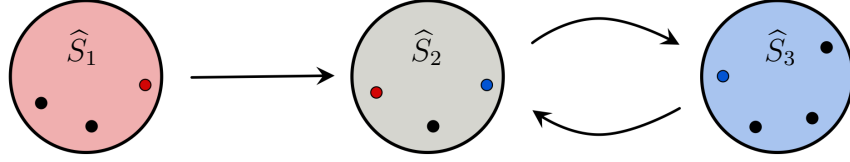


FIGURE 3. Dynamics of  $\widehat{f}: \widehat{\mathcal{S}}_\Gamma \rightarrow \widehat{\mathcal{S}}_\Gamma$  on the small spheres for the example from Figure 2. The black, red, and blue dots correspond to the postcritical points of  $f$ , the curve  $\alpha \in \Gamma$ , and the curve  $\beta \in \Gamma$ , respectively.

- (B)  $H_0 = \text{id}_{S^2}$ ;
- (C)  $H_1(\overline{S'_j}) = \overline{S_j}$  for each  $j \in J$ ;
- (D)  $H_1|_{S'_j}$  is a homeomorphism of  $S'_j$  onto the image  $H_1(S'_j) \subset S_j$  for each  $j \in J$ ;
- (E) for each  $j \in J$  the homotopy  $H_1$  sends each essential component  $\gamma'$  of  $\partial S'_j$  homeomorphically onto the component  $\gamma$  of  $\partial S_j$  that is isotopic to  $\gamma'$ ;
- (F)  $H_1(\gamma')$  is a single point for each non-essential component  $\gamma'$  of  $\partial S'_j$  and each  $j \in J$ .

Then for every  $j \in J$  the inverse of  $H_1|_{S'_j}: \overline{S'_j} \rightarrow \overline{S_j}$  defines an identification  $i^*: \widehat{S}_j \rightarrow \widehat{S}'_j$  between the small spheres (sending marked points to marked points).

By construction, for each  $j \in J$  the image  $f(S'_j)$  is a component in  $\mathcal{S}_\Gamma$ , which we will denote by  $S_{f(j)}$ . Then, by filling in the punctures, we get a branched covering map  $f_*: \widehat{S}'_j \rightarrow \widehat{S}_{f(j)}$  between the corresponding small spheres (and respecting the marked points).

The composition  $\widehat{S}_j \rightarrow \widehat{S}'_j \rightarrow \widehat{S}_{f(j)}$  defines a branched covering map  $\widehat{f} := f_* \circ i^*: \widehat{S}_j \rightarrow \widehat{S}_{f(j)}$ , which we call a *small sphere map*. It is uniquely defined up to isotopy (relative to marked points) for each  $j \in J$ .

The considerations above imply that  $f$  induces a map

$$\widehat{f}: \bigsqcup_{j \in J} \widehat{S}_j \rightarrow \bigsqcup_{j \in J} \widehat{S}_j$$

on (the disjoint union of) small spheres; see Figure 3. With a slight abuse of notation, we will simply denote this map by  $\widehat{f}: \widehat{\mathcal{S}}_\Gamma \rightarrow \widehat{\mathcal{S}}_\Gamma$ . Since  $\widehat{\mathcal{S}}_\Gamma$  consists of only finitely many spheres, each small sphere is eventually periodic under  $\widehat{f}$ . Suppose  $\widehat{S}_j$  is a periodic small sphere. Then the first return map  $\widehat{f}^{k(j)}: \widehat{S}_j \rightarrow \widehat{S}_j$  is a postcritically-finite branched covering map. Hence this first return map is either a (marked) Thurston map or a homeomorphism.

To summarize the discussion above, a completely invariant multicurve  $\Gamma$  allows to *decompose* the dynamics of a Thurston map  $f$  on  $S^2$  into dynamics of the induced map  $\widehat{f}$  on the (periodic) small spheres with respect to  $\Gamma$ .

In [Pil01], Kevin Pilgrim introduces the notion of a *canonical Thurston obstruction* for a Thurston map  $f$ . It is a special multicurve  $\Gamma_{\text{Th}}$ , defined up to isotopy rel.  $\text{Post}(f)$ , which has the following property: in the case  $f$  has a hyperbolic orbifold, the map  $f$  is realized by a rational map if and only if  $\Gamma_{\text{Th}}$  is empty. If  $\Gamma_{\text{Th}} \neq \emptyset$ , the multicurve  $\Gamma_{\text{Th}}$  is a simple Thurston obstruction and provides the *canonical decomposition* of the given Thurston map  $f$ . Nikita Selinger gave the following topological characterization of the canonical Thurston obstruction in terms of the pieces of this decomposition; see [Sel13, Theorem 5.6] for a precise statement.

**Theorem 2.14.** *Let  $f: S^2 \rightarrow S^2$  be a Thurston map. Then the canonical Thurston obstruction of  $f$  is a unique minimal (with respect to inclusion) completely invariant Thurston*

obstruction  $\Gamma$  such that for each periodic small sphere  $\widehat{S} \in \widehat{\mathcal{S}}_\Gamma$  the first return map  $\widehat{f}^k: \widehat{S} \rightarrow \widehat{S}$  is either

- (i) a homeomorphism;
- (ii) a Thurston map with a parabolic orbifold of special type (see [Sel13, Theorem 5.6] for the details);
- (iii) or realized by a rational map.

**2.7. Branched covers and graphs.** Let  $f: S^2 \rightarrow S^2$  be a branched covering map and  $\alpha$  be a Jordan arc in  $S^2$ . We say that a Jordan arc  $\widetilde{\alpha}$  is a *lift* of  $\alpha$  under  $f$  if  $f|_{\widetilde{\alpha}}$  is a homeomorphism of  $\widetilde{\alpha}$  onto  $\alpha$ . It easily follows from the existence and uniqueness statements for lifts of paths under covering maps (see, for example, [BM17, Lemma A.6]) that if  $\alpha$  is a Jordan arc in  $S^2$  with  $\text{int}(\alpha) \subset S^2 \setminus f(\text{Crit}(f))$ ,  $p \in \text{int}(\alpha)$ , and  $q \in f^{-1}(p)$ , then there exists a unique lift  $\widetilde{\alpha}$  of  $\alpha$  under  $f$  with  $q \in \text{int}(\widetilde{\alpha})$ .

Suppose now that  $f: S^2 \rightarrow S^2$  is a Thurston map and  $G$  is a planar embedded graph with  $V(G) \supset \text{Post}(f)$ . Then the preimage  $f^{-1}(G)$  may be viewed as a planar embedded graph with the vertex set  $V(f^{-1}(G)) := f^{-1}(V(G))$  and the edge set  $E(f^{-1}(G))$  consisting of all lifts of the edges of  $G$  under  $f$ . The graph  $f^{-1}(G)$  is then called the *complete preimage* of  $G$  under the map  $f$ . We note that

$$V(f^{-1}(G)) = f^{-1}(V(G)) \supset f^{-1}(\text{Post}(f)) \supset \text{Post}(f) \cup \text{Crit}(f).$$

Furthermore, each face  $\widetilde{W}$  of  $f^{-1}(G)$  is a component of  $f^{-1}(W)$  for some face  $W$  of  $G$  and  $f|_{\widetilde{W}}: \widetilde{W} \rightarrow W$  is a covering map.

**Lemma 2.15.** *Let  $f$  be a Thurston map and  $G$  be a connected planar embedded graph with  $V(G) \supset \text{Post}(f)$ . Then the complete preimage  $f^{-1}(G)$  is a connected planar embedded graph with  $\text{Post}(f) \subset V(f^{-1}(G))$ .*

*Proof.* Indeed, since  $G$  is connected and  $\text{Post}(f) \subset V(G)$ , each face  $W$  of  $G$  is simply connected and  $W \cap \text{Post}(f) = \emptyset$ . Thus, by the Riemann-Hurwitz formula, each component of  $f^{-1}(W)$  is simply connected as well. Hence,  $f^{-1}(G)$  is connected and the statement follows.  $\square$

Finally, we discuss extensions of maps between planar embedded graphs to maps between the underlying spheres.

Suppose  $S^2$  and  $\widehat{S}^2$  are two topological 2-spheres. Let  $G = (V, E)$  and  $\widehat{G} = (\widehat{V}, \widehat{E})$  be two planar embedded graphs in  $S^2$  and  $\widehat{S}^2$ , respectively. A continuous map  $f: G \rightarrow \widehat{G}$  is called a *graph map* if forward and inverse images of vertices are vertices (i.e.,  $f(V) \subset \widehat{V}$  and  $f^{-1}(\widehat{V}) \subset V$ ), and  $f$  is injective on each edge of  $G$ . An (orientation-preserving) branched covering map  $\bar{f}: S^2 \rightarrow \widehat{S}^2$  is called a *regular extension* of a graph map  $f: G \rightarrow \widehat{G}$  if  $\bar{f}|_G = f$  and  $\bar{f}$  is injective on each face of  $G$ .

A criterion for the existence of regular extensions is provided in [BFH92, Proposition 6.4]. Here, we only record the following uniqueness result from the same paper, which we use in the sequel; see [BFH92, Corollary 6.3].

**Proposition 2.16.** *Let  $G$  and  $\widehat{G}$  be two planar embedded connected graphs in  $S^2$  and  $\widehat{S}^2$ , respectively. Suppose that  $f, g: G \rightarrow \widehat{G}$  are two graph maps such that  $f(v) = g(v)$  and  $f(e) = g(e)$  for each  $v \in V(G)$  and  $e \in E(G)$ . If  $f$  and  $g$  have regular extensions  $\bar{f}$  and  $\bar{g}$ , respectively, then there exists  $\psi \in \text{Homeo}_0^+(S^2, V(G))$  such that  $\bar{f} = \bar{g} \circ \psi$ .*

The following corollary follows easily from the proposition above and Proposition 2.4.

**Corollary 2.17.** *Let  $G$  be of planar embedded connected graph in  $S^2$  and  $\varphi_1, \varphi_2 \in \text{Homeo}^+(S^2)$ . Suppose that  $\varphi_1(e) \sim \varphi_2(e)$  rel.  $V(G)$  for every  $e \in E(G)$ . Then  $\varphi_1, \varphi_2$  are isotopic rel.  $V(G)$ .*

### 3. CRITICALLY FIXED THURSTON MAPS

The main goal of this section is to provide a classification of *critically fixed Thurston maps*, that is, Thurston maps that fix (pointwise) each of its critical points.

First, we define the “*blow-up operation*”, introduced by Kevin Pilgrim and Tan Lei in [PL98, Section 2.5], which provides a surgery for constructing and modifying Thurston maps and plays a crucial role in our classification result. We do not define this operation in complete generality, but in some rather particular case. Namely, we will be blowing up only pairs  $(G, \varphi)$ , where  $G$  is a planar embedded graph in  $S^2$  and  $\varphi \in \text{Homeo}^+(S^2, V(G))$ . The result of this operation is a critically fixed Thurston map. (Recall that  $G$  may have multiple edges but no loops and that  $\text{Homeo}^+(S^2, Z)$  denotes the group of all orientation-preserving homeomorphisms of  $S^2$  that fix each point in a finite set  $Z \subset S^2$ .)

Conversely, to each critically fixed Thurston map  $f$  we associate a pair  $(G_f, \varphi_f)$ , where  $G_f$  is a planar embedded graph with vertices in  $\text{Crit}(f)$  and  $\varphi_f$  is a homeomorphism in  $\text{Homeo}^+(S^2, \text{Crit}(f))$ . We show that  $f$  is isotopic to the map obtained by blowing up the pair  $(G_f, \varphi_f)$ .

**3.1. The blow-up operation.** In the following, let  $G$  be a planar embedded graph in  $S^2$  and  $\varphi$  be an element of  $\text{Homeo}^+(S^2, V(G))$ . We will now describe a construction that associates a critically fixed Thurston map  $f: S^2 \rightarrow S^2$  to every such pair  $(G, \varphi)$  by “*blowing up*” each edge of  $G$ . Mostly, we will follow the discussion from [BHI21, Section 4.1], but will slightly simplify the definition based on the specific features of the considered case.

First, for each edge  $e \in E(G)$  we choose an open Jordan region  $W_e \subset S^2$  such that the following conditions hold:

- (A1) Each  $e \in E(G)$  is a crosscut in  $W_e$ , that is,  $\text{int}(e) \subset W_e$  and  $\partial e \subset \partial W_e$ ;
- (A2)  $\overline{W_e} \cap V(G) = \partial e$  for each  $e \in E(G)$ ;
- (A3) For distinct edges  $e_1, e_2 \in E(G)$ , we have  $\overline{W_{e_1}} \cap \overline{W_{e_2}} = \partial e_1 \cap \partial e_2$ . In particular, the open Jordan regions  $W_e$  are pairwise disjoint.

Next, we choose closed Jordan regions  $D_e, e \in E(G)$ , so that  $e$  is a crosscut in  $D_e$  and  $D_e \setminus \partial e \subset W_e$ . The two endpoints of  $e$  partition  $\partial D_e$  into two Jordan arcs, which we denote by  $\partial D_e^+$  and  $\partial D_e^-$ . One can think of  $D_e$  as the resulting region if we cut the sphere  $S^2$  along the edge  $e$  and “open up” the slit.

Now we define a map that collapses each  $D_e$  back to  $e$ . More precisely, we choose a homotopy  $h: S^2 \times \mathbb{I} \rightarrow S^2$  with the following properties:

- (B1)  $h_t := h(\cdot, t)$  is a homeomorphism on  $S^2$  for each  $t \in [0, 1)$ ;
- (B2)  $h_0 = \text{id}_{S^2}$ ;
- (B3)  $h_t$  is the identity map on  $S^2 \setminus \bigcup_{e \in E(G)} W_e$  for each  $t \in \mathbb{I}$ ;
- (B4)  $h_1$  is a homeomorphism of  $S^2 \setminus \bigcup_{e \in E(G)} D_e$  onto  $S^2 \setminus \bigcup_{e \in E(G)} e$ ;
- (B5)  $h_1$  maps  $\partial D_e^+$  and  $\partial D_e^-$  homeomorphically onto  $e$  for each  $e \in E(G)$ .

It easily follows that  $h_1(D_e) = e$ , so the homotopy  $h$  collapses each closed Jordan region  $D_e$  onto  $e$  at time 1, while keeping each point in  $S^2 \setminus \bigcup_{e \in E(G)} W_e$  fixed at all times.

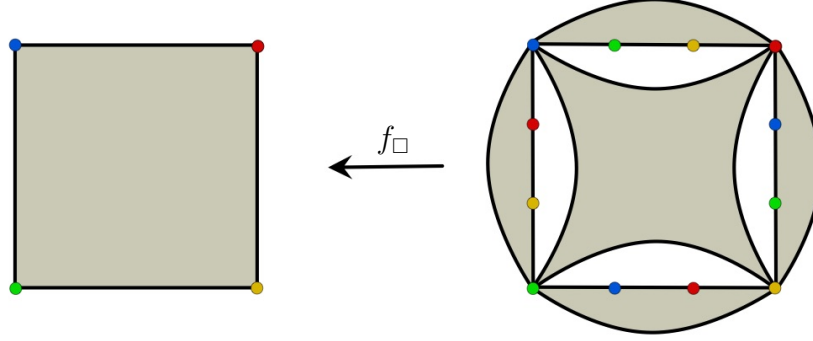


FIGURE 4. A critically fixed Thurston map  $f_{\square}$  obtained by blowing up the pair  $(G_{\square}, \text{id}_{S^2})$ , where  $G_{\square}$  is the planar embedded graph on the left.

Finally, for each  $e \in E(G)$  we choose a continuous map  $f_e: D_e \rightarrow S^2$  with the following properties:

- (C1)  $f_e|_{\text{int}(D_e)}: \text{int}(D_e) \rightarrow S^2 \setminus \varphi(e)$  is an orientation-preserving homeomorphism;
- (C2)  $f_e|\partial D_e^+ = \varphi \circ h_1|\partial D_e^+$  and  $f_e|\partial D_e^- = \varphi \circ h_1|\partial D_e^-$ .

Now we may define a map  $f: S^2 \rightarrow S^2$  as follows:

$$(3.1) \quad f(p) = \begin{cases} (\varphi \circ h_1)(p) & \text{if } p \in S^2 \setminus \bigcup_{e \in E(G)} D_e \\ f_e(p) & \text{if } p \in D_e. \end{cases}$$

**Definition 3.1.** We say that the map  $f: S^2 \rightarrow S^2$  as described above is obtained by *blowing up* the pair  $(G, \varphi)$ . The operation of constructing this map is called the *blow-up operation*.

One can check that the constructed map  $f: S^2 \rightarrow S^2$  is in fact a critically fixed Thurston map with properties summarized in the following proposition. (For the proof, see [BHI21, Lemma 4.3].)

**Proposition 3.2.** *Suppose a map  $f: S^2 \rightarrow S^2$  is obtained by blowing up a pair  $(G, \varphi)$ , where  $G$  is a planar embedded graph in  $S^2$  and  $\varphi \in \text{Homeo}^+(S^2, V(G))$ . Then the following statements are true:*

- (i)  $f$  is a Thurston map with  $\deg(f) = |E(G)| + 1$ ;
- (ii)  $\text{Crit}(f) = \{v \in V(G) : \deg_G(v) > 0\}$ ;
- (iii)  $\deg(f, v) = \deg_G(v) + 1$  for each  $v \in V(G)$ ;
- (iv) each  $v \in V(G)$  is fixed under  $f$ , in particular,  $f$  is a critically fixed Thurston map.

*Example 3.3.* Consider the “square graph” shown on the left in Figure 4, which we will denote by  $G_{\square}$ . Here and in the following, all the graphs in figures are assumed to be embedded in an underlying 2-sphere. In particular, the graph  $G_{\square}$  has two simply connected faces, which we denote by  $U_g$  and  $U_w$ . The face  $U_g$  corresponds to the interior of the square, which is colored gray, and the face  $U_w$  corresponds to the exterior of the square, which is colored white. Figure 4 illustrates a critically fixed Thurston map  $f_{\square}$  obtained by blowing up the pair  $(G_{\square}, \text{id}_{S^2})$ . We will use this map as a prototypical example in our paper. In the figure, the closure of each gray region  $U$  on the right is mapped by  $f_{\square}$  homeomorphically onto the closure of the gray face  $U_g$  on the left so that the marked vertices on  $\partial U$  are sent to the vertices of the same color on  $\partial U_g$ . An analogous statement is true for each white region on

the right. Note that  $\deg(f_\square) = 5$ , and each vertex of  $G_\square$  is a fixed critical point of  $f_\square$  with local degree 3.

It follows from [PL98, Proposition 2] (and the remark after it) that the map  $f$  is uniquely defined up to isotopy (rel.  $V(G)$ ) independently of the choices in the construction above. Moreover, up to isotopy  $f$  depends only on the isotopy classes of  $G$  and  $\varphi$ .

**Proposition 3.4.** *For  $j = 1, 2$ , suppose  $G_j$  is a planar embedded graph in  $S^2$  and  $\varphi_j$  is a homeomorphism in  $\text{Homeo}^+(S^2, V(G_j))$ . Let  $f_1$  and  $f_2$  be Thurston maps obtained by blowing up the pairs  $(G_1, \varphi_1)$  and  $(G_2, \varphi_2)$ , respectively.*

*Suppose that  $G_1$  and  $G_2$  are isotopic rel.  $V(G_1)$ , and  $\varphi_1$  and  $\varphi_2$  are isotopic rel.  $V(G_1)$ . Then the marked Thurston maps  $(f_1, V(G_1))$  and  $(f_2, V(G_2))$  are isotopic.*

The proposition above easily implies the following statement.

**Proposition 3.5.** *Suppose  $f: S^2 \rightarrow S^2$  and  $g: S^2 \rightarrow S^2$  are critically fixed Thurston maps obtained by blowing up pairs  $(G, \varphi)$  and  $(G, \text{id}_{S^2})$ , where  $G$  is a planar embedded graph in  $S^2$  and  $\varphi \in \text{Homeo}^+(S^2, V(G))$ . Then the marked Thurston maps  $(f, V(G))$  and  $(\varphi \circ g, V(G))$  are isotopic.*

Since we frequently work with Thurston maps defined up to isotopy (or modulo combinatorial equivalence), the following easy statement will be useful.

**Proposition 3.6.** *Let  $f: S^2 \rightarrow S^2$  be a Thurston map obtained by blowing up a pair  $(G, \varphi)$ , where  $G$  is a planar embedded graph in  $S^2$  and  $\varphi \in \text{Homeo}^+(S^2, V(G))$ . Suppose that a marked Thurston map  $(\hat{f}, Q_{\hat{f}})$  on a topological 2-sphere  $\widehat{S}^2$  is combinatorially equivalent to  $(f, V(G))$ . Then  $\hat{f}$  is obtained by blowing a pair  $(\widehat{G}, \widehat{\varphi})$ , where  $\widehat{G}$  is a planar embedded graph in  $\widehat{S}^2$  and  $\widehat{\varphi} \in \text{Homeo}^+(\widehat{S}^2, V(\widehat{G}))$ .*

*Proof.* Suppose that  $f: S^2 \rightarrow S^2$  is a Thurston map obtained by blowing up a pair  $(G, \varphi)$  as in the statement, that is, we fix a choice of  $W_e, D_e, f_e$ , and  $h$  as in the construction above. Moreover, we assume that  $\psi_0, \psi_1: S^2 \rightarrow \widehat{S}^2$  are orientation-preserving homeomorphisms that are isotopic rel.  $V(G)$  and satisfy  $\psi_0 \circ f = \hat{f} \circ \psi_1$ .

$$\begin{array}{ccccc}
 S^2 \setminus \bigcup_{e \in E(G)} D_e & \xrightarrow{\psi_1} & \widehat{S}^2 \setminus \bigcup_{\widehat{e} \in E(\widehat{G})} \widehat{D}_{\widehat{e}} & & \\
 \downarrow f & \searrow h_1 & \swarrow \widehat{h}_1 & & \downarrow \widehat{f} \\
 & S^2 \setminus G & \xrightarrow{\psi_1} & \widehat{S}^2 \setminus \widehat{G} & \\
 & \swarrow \varphi & & \searrow \widehat{\varphi} & \\
 S^2 \setminus \varphi(G) & \xrightarrow{\psi_0} & \widehat{S}^2 \setminus \widehat{\varphi}(\widehat{G}) & & 
 \end{array}$$

Let  $\widehat{G}$  be the planar embedded graph  $\psi_1(G)$  in  $\widehat{S}^2$  with the vertex set  $V(\widehat{G}) = \psi_1(V(G)) = Q_{\widehat{f}}$ . Then the edges of  $\widehat{G}$  are given by the images  $\widehat{e} := \psi_1(e)$ ,  $e \in E(G)$ . Furthermore, let  $\widehat{\varphi} := \psi_0 \circ \varphi \circ \psi_1^{-1}$ . Then  $\widehat{\varphi} \in \text{Homeo}^+(\widehat{S}^2, V(\widehat{G}))$ . We claim that the map  $\widehat{f}$  is obtained by blowing up the pair  $(\widehat{G}, \widehat{\varphi})$  with the following choices:

(i) for each edge  $\widehat{e} = \psi_1(e)$ ,  $e \in E(G)$ , we set

$$\widehat{W}_{\widehat{e}} := \psi_1(W_e), \quad \widehat{D}_{\widehat{e}} := \psi_1(D_e), \quad \text{and} \quad \widehat{f}_{\widehat{e}} := \psi_0 \circ f_e \circ \psi_1^{-1};$$

(ii) for each  $t \in \mathbb{I}$  we set  $\widehat{h}_t := \psi_1 \circ h_t \circ \psi_1^{-1}$ .

Indeed, this can be easily verified from the identity  $\widehat{f}_e = \widehat{f}|_{\widehat{D}_e}$  and the commutative diagram above, and we leave the straightforward details to the reader.  $\square$

*Remark 3.7.* Let  $(\widehat{G}, \widehat{\varphi})$  be as constructed in the proof of Proposition 3.6. Then the following is true:

- (i) If the pair  $(G, \varphi)$  is admissible in  $S^2$  then the pair  $(\widehat{G}, \widehat{\varphi})$  is admissible in  $\widehat{S}^2$ . Furthermore, the pairs  $(G, \varphi)$  and  $(\widehat{G}, \widehat{\varphi})$  are equivalent (see Definition 1.3).
- (ii) If  $S^2 = \widehat{S}^2$  and the marked Thurston maps  $(f, V(G))$  and  $(\widehat{f}, Q_{\widehat{f}})$  are isotopic then  $\widehat{G}$  and  $\widehat{\varphi}$  are isotopic to  $G$  and  $\varphi$  rel.  $V(G)$ , respectively.

**3.2. Admissible pairs and a blow-up criterion.** Let  $G$  be a planar embedded graph in  $S^2$  and  $\varphi \in \text{Homeo}^+(S^2, V(G))$  be a homeomorphism. Recall from the introduction that the pair  $(G, \varphi)$  is called *admissible* (in  $S^2$ ) if  $G$  has no isolated vertices and  $\varphi(e)$  is isotopic to  $e$  for each  $e \in E(G)$ . In this section, we provide a criterion for checking if a given critically fixed Thurston map arises by blowing up some admissible pair  $(G, \varphi)$ .

First, we summarize mapping properties of (critically fixed) Thurston maps obtained by blowing up admissible pairs.

**Proposition 3.8.** *Let  $f: S^2 \rightarrow S^2$  be a Thurston map obtained by blowing up an admissible pair  $(G, \varphi)$  in  $S^2$ . Suppose that  $K$  is a planar embedded graph in  $S^2$  that is isotopic to  $G$  rel.  $V(G)$ . Then for each  $\alpha \in E(K)$  there is a triple  $(\alpha^+, \alpha^-, U_\alpha)$  satisfying the following conditions:*

- (D1)  $\alpha^+$  and  $\alpha^-$  are distinct lifts of  $\alpha$  under  $f$  that are isotopic to  $\alpha$  rel.  $\text{Crit}(f)$ ;
- (D2)  $U_\alpha$  is a connected component of  $S^2 \setminus (\alpha^+ \cup \alpha^-)$ ;
- (D3)  $\overline{U_{\alpha_1}} \cap \overline{U_{\alpha_2}} = \alpha_1 \cap \alpha_2$  for distinct  $\alpha_1, \alpha_2 \in E(K)$ ;

Furthermore, let us consider the planar embedded graph  $K^\pm := \bigcup_{\alpha \in E(K)} (\alpha^+ \cup \alpha^-)$  with the vertex set  $V(K^\pm) = V(K) = V(G)$ . Then  $f$  sends each face  $\widetilde{W}$  of  $K^\pm$  homeomorphically onto its image. More precisely, the following statements are true:

- (E1) If  $\widetilde{W} = U_\alpha$  for some  $\alpha \in E(K)$ , then  $f$  sends  $\widetilde{W}$  homeomorphically onto  $S^2 \setminus \alpha$ .
- (E2) If  $\widetilde{W} \neq U_\alpha$  for every  $\alpha \in E(K)$ , then  $f(\widetilde{W})$  is a face of  $K$  with  $\partial f(\widetilde{W}) = f(\partial \widetilde{W})$  and  $f|_{\widetilde{W}}: \widetilde{W} \rightarrow f(\widetilde{W})$  is a homeomorphism.
- (E3)  $f$  sends  $S^2 \setminus \bigcup_{\alpha \in E(K)} \overline{U_\alpha}$  homeomorphically onto  $S^2 \setminus K$ . Moreover, the restrictions  $f|_{S^2 \setminus \bigcup_{\alpha \in E(K)} (\overline{U_\alpha} \setminus \partial \alpha)}$  and  $\varphi|_{S^2 \setminus \bigcup_{\alpha \in E(K)} (\overline{U_\alpha} \setminus \partial \alpha)}$  are isotopic rel.  $V(G)$ .

The graph  $K^\pm$  as above is called the *blow-up* of  $K$  under  $f$ .

*Proof.* First, suppose that  $K$  is the planar embedded graph  $\varphi(G)$  with the vertex set  $V(K) = V(G)$ . Note that since  $(G, \varphi)$  is admissible, Proposition 2.4 implies that  $K$  is isotopic to  $G$  rel.  $V(G)$ . Let  $\alpha$  be an edge of  $K = \varphi(G)$ . Then  $\alpha = \varphi(e)$  for some  $e \in E(G)$ . We may now set  $\alpha^+ := \partial D_e^+$ ,  $\alpha^- := \partial D_e^-$ , and  $U_\alpha := \text{int}(D_e)$ , where  $D_e$ ,  $\partial D_e^+$ , and  $\partial D_e^-$  are as chosen in the construction of  $f$  by blowing up the pair  $(G, \varphi)$ ; see Section 3.1. One can easily check that all the conditions (D1)-(D3) and (E1)-(E3) are satisfied. The general case, when  $K$  is an arbitrary planar embedded graph isotopic to  $G$ , then follows from this by the isotopy lifting property for Thurston maps (see Proposition 2.9). We leave it to the reader to fill in the details.  $\square$

We are ready now to state the following “blow-up criterion” for critically fixed Thurston maps.

**Proposition 3.9.** *Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map and  $K$  be a planar embedded graph in  $S^2$  with  $V(K) = \text{Crit}(f)$  and  $|E(K)| = \deg(f) - 1$ . Suppose that for each  $\alpha \in E(K)$  there is a triple  $(\alpha^+, \alpha^-, U_\alpha)$  satisfying conditions (D1)-(D3).*

*Then  $f$  is obtained by blowing an admissible pair  $(G, \varphi)$ , where  $G$  is a planar embedded graph isotopic to  $K$  rel.  $\text{Crit}(f)$ . Furthermore,  $f$  is isotopic to a critically fixed Thurston map obtained by blowing up the (admissible) pair  $(K, \varphi)$ .*

*Proof.* Suppose  $f$  is a critically fixed Thurston map and  $K$  is a planar embedded graph satisfying conditions in the statement. We will view the set  $K^\pm := \bigcup_{\alpha \in E(K)} (\alpha^+ \cup \alpha^-)$  as a planar embedded graph with  $V(K^\pm) = \text{Crit}(f)$ . Then  $K^\pm$  is a subgraph of the complete preimage  $f^{-1}(K)$ .

*Claim 1.*  $f(U_\alpha) \supset S^2 \setminus \alpha$  for each  $\alpha \in E(K)$ .

Indeed, fix a point  $q \in U_\alpha \setminus f^{-1}(\alpha)$  and suppose  $p \in S^2 \setminus \alpha$  is arbitrary. We may connect  $p$  and  $f(q)$  by a Jordan arc  $\beta$  inside  $S^2 \setminus \alpha$ . Then there is a lift  $\tilde{\beta}$  of  $\beta$  under  $f$  connecting the point  $q$  and a point  $\tilde{p} \in f^{-1}(p)$  (see, for example, [BM17, Lemma A.18]). Note that  $\tilde{\beta}$  must stay inside  $U_\alpha$ , and thus  $\tilde{p} \in U_\alpha$ . Claim 1 follows.

Note that Claim 1 implies that  $\deg(f, q) \geq \deg_K(q) + 1$  for each  $q \in V(K) = \text{Crit}(f)$ .

*Claim 2.* Suppose  $H$  is a component of  $f^{-1}(K) \setminus K^\pm$ . Then  $H \subset U_\alpha$  for some  $\alpha \in E(K)$ .

This is an easy counting argument. Let  $p \in H$  and  $q := f(p) \in K$ . Then either  $q \in \text{int}(\alpha_q)$  for some  $\alpha_q \in E(K)$  or  $q \in \text{Crit}(f)$ . In the first case, Claim 1 implies that  $q$  has at least one preimage in each  $U_\alpha$  for  $\alpha \neq \alpha_q$ . At the same time,  $q$  has two preimages in  $\partial U_{\alpha_q}$ . Since  $|E(K)| = \deg(f) - 1$ , the point  $p$  must be in one of the regions  $U_\alpha$  with  $\alpha \neq \alpha_q$ . In the latter case,  $q$  is a fixed critical point with  $\deg(f, q) \geq \deg_K(q) + 1$ . At the same time, Claim 1 implies that  $q$  has at least one preimage in every  $U_\alpha$  for which  $\alpha$  is not incident to  $q$ . Again, since  $|E(K)| = \deg(f) - 1$ , the point  $p$  must be in one of these regions  $U_\alpha$ . Claim 2 follows.

The proof of Claim 2 implies that  $\deg(f, q) = \deg_K(q) + 1$  for each  $q \in \text{Crit}(f)$ . In particular, the graph  $K$  has no isolated vertices. Furthermore, for each  $\alpha \in E(K)$  we have  $f(U_\alpha) \subset S^2 \setminus \alpha$ , and thus  $f(U_\alpha) = S^2 \setminus \alpha$  by Claim 1.

*Claim 3.* The map  $f$  sends each face  $\tilde{W}$  of  $K^\pm$  homeomorphically onto its image. More precisely,  $f$  satisfies conditions (E1)-(E3).

The proof is again based on a counting argument. Suppose first that  $\tilde{W}$  is a face of  $K^\pm$  different from each  $U_\alpha$ ,  $\alpha \in E(K)$ . It easily follows from the Euler formula that there are exactly  $|F(K)|$  such faces. Claim 2 implies that  $\tilde{W}$  is also a face of  $f^{-1}(K)$ . Thus,  $f(\tilde{W})$  is a face of  $K$  (with  $\partial f(\tilde{W}) = f(\partial \tilde{W})$ ) and  $f|_{\tilde{W}}: \tilde{W} \rightarrow f(\tilde{W})$  is a covering map. Since  $|E(K)| = \deg(f) - 1$ , Claim 1 implies that  $f$  maps  $S^2 \setminus \bigcup_{\alpha \in E(K)} \overline{U_\alpha}$  injectively into  $S^2 \setminus K$ . So  $f$  has to satisfy (E2) and (E3).

Suppose now that  $\tilde{W} = U_\alpha$  for some  $\alpha \in E(K)$ . Since  $f(U_\alpha) = S^2 \setminus \alpha$  and  $f(\partial U_\alpha) = \alpha$ , we conclude that  $U_\alpha$  is a component of  $f^{-1}(S^2 \setminus \alpha)$ . Note that  $U_\alpha \cap \text{Crit}(f) = \emptyset$ . Thus, by the Riemann-Hurwitz formula,  $\deg(f|_{U_\alpha}) = 1$  and so  $f$  satisfies (E1). Claim 3 follows.



For each  $\alpha \in E(K)$  let us now choose a Jordan arc  $e_\alpha$  with  $\text{int}(e_\alpha) \subset U_\alpha$  and  $\partial e_\alpha = \partial\alpha$ . Let  $G$  be the planar embedded graph with the vertex set  $V(G) = \text{Crit}(f)$  and the edge set  $E(G) = \{e_\alpha : \alpha \in E(K)\}$ . Lemma 2.1 and Proposition 2.4 imply that  $G$  is isotopic to  $K$  rel.  $\text{Crit}(f)$ . We will prove that  $f$  is obtained by blowing up a pair  $(G, \varphi)$ , where  $\varphi \in \text{Homeo}^+(S^2, \text{Crit}(f))$ .

For each edge  $e = e_\alpha \in E(G)$ , we set  $D_e := \overline{U_\alpha}$ ,  $\partial D_e^+ := \alpha^+$ ,  $\partial D_e^- := \alpha^-$ , and  $f_e := f|_{\overline{U_\alpha}}$ . We also choose open Jordan regions  $W_e \supset D_e \setminus \partial e$  satisfying conditions (A1)-(A3). Next, we consider a homotopy  $h: S^2 \times \mathbb{I} \rightarrow S^2$  that satisfies properties (B1)-(B5) together with the following extra condition:

(\*) for each  $e \in E(G)$  and arbitrary  $x \in \partial D_e^+$  and  $y \in \partial D_e^-$ ,  $h_1(x) = h_1(y)$  if and only if  $f(x) = f(y)$ .

Let  $\varphi: S^2 \setminus G \rightarrow S^2 \setminus K$  be the map defined by  $\varphi(p) := f \circ h_1^{-1}(p)$  for each  $p \in S^2 \setminus G$ . Condition (B4) and Claim 3 imply that  $\varphi$  is a homeomorphism.

*Claim 4.* The map  $\varphi: S^2 \setminus G \rightarrow S^2 \setminus K$  extends to a homeomorphism  $\varphi \in \text{Homeo}^+(S^2, \text{Crit}(f))$  so that  $\varphi(e_\alpha) = \alpha$  for each  $\alpha \in E(K)$ .

We will only give an outline of the argument and leave some details to the reader. We claim that the inverse map  $\varphi^{-1}: S^2 \setminus K \rightarrow S^2 \setminus G$  may be extended to a continuous bijection  $\varphi^{-1}: S^2 \rightarrow S^2$ . Indeed, let  $p \in K$  be arbitrary, then  $p \in \alpha$  for some  $\alpha \in E(K)$ . We set  $\varphi^{-1}(p) := h_1(q)$  for  $q \in f^{-1}(p) \cap (\alpha^+ \cup \alpha^-)$ . Condition (\*) ensures that  $\varphi^{-1}(p)$  is well-defined and condition (B5) insures that  $\varphi^{-1}|_\alpha: \alpha \rightarrow e_\alpha$  is a homeomorphism for each  $\alpha \in E(K)$ . Thus  $\varphi^{-1}: S^2 \rightarrow S^2$  is a bijection. The continuity of  $\varphi^{-1}$  can be easily deduced from the properties of  $h_1$  and the fact that  $f$  acts locally as a power map. This implies that  $\varphi^{-1}: S^2 \rightarrow S^2$  is a homeomorphism. The rest follows from the construction of  $\varphi$ .

It is now straightforward to check that  $f$  is obtained by blowing up the pair  $(G, \varphi)$  with the choices above. It also follows from Proposition 3.4 that  $f$  is isotopic to the map obtained by blowing up the pair  $(K, \varphi)$ . Finally, since  $\alpha \sim e_\alpha$  rel.  $\text{Crit}(f)$  for each  $\alpha \in E(K)$ , we have  $\varphi(\alpha) \sim \varphi(e_\alpha) = \alpha$  rel.  $\text{Crit}(f)$ . Hence the pairs  $(G, \varphi)$  and  $(K, \varphi)$  are admissible. This completes the proof of Proposition 3.9.  $\square$

**3.3. Admissible pairs and arc lifting.** In the following, let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map obtained by blowing up an admissible pair  $(G, \varphi)$  in  $S^2$ . We also suppose that the triples  $(e^+, e^-, U_e)$ ,  $e \in E(G)$ , are as provided by Proposition 3.8 for  $K = G$ . In this section, we prove several facts about lifts of Jordan arcs in  $(S^2, \text{Crit}(f))$  under the map  $f$ .

First, we introduce the following notion.

**Definition 3.10.** Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map and  $\alpha$  be a Jordan arc in  $(S^2, \text{Crit}(f))$ . Suppose  $\alpha$  has exactly  $k$  distinct lifts under  $f$  that are isotopic to  $\alpha$  rel.  $\text{Crit}(f)$ . Then  $k$  is called the *blow-up degree of  $\alpha$  under  $f$*  (or simply the *blow-up degree of  $\alpha$*  if  $f$  is understood) and denoted by  $\text{deg}(f, \alpha)$ . If  $\text{deg}(f, \alpha) \geq 2$ , we say that the Jordan arc  $\alpha$  *blows up* under  $f$ .

One can easily see that if two Jordan arcs  $\alpha$  and  $\alpha'$  in  $(S^2, \text{Crit}(f))$  are isotopic rel.  $\text{Crit}(f)$ , then their blow-up degrees under  $f$  coincide, i.e.,  $\text{deg}(f, \alpha) = \text{deg}(f, \alpha')$ .

**Lemma 3.11.** *Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map obtained by blowing up an admissible pair  $(G, \varphi)$ . Suppose  $\alpha$  is a Jordan arc in  $(S^2, \text{Crit}(f))$  such that  $\text{deg}(f, \alpha) > 0$ . Then  $i_f(\alpha, e) = 0$  for each edge  $e \in E(G)$ .*

To simplify the notation, here and in the following we write  $i_f(\cdot, \cdot)$  instead of  $i_{\text{Crit}(f)}(\cdot, \cdot)$ .

*Proof.* Without loss of generality, we may assume that  $\alpha$  and  $e$  are in minimal position rel.  $\text{Crit}(f)$ . Suppose that  $\tilde{\alpha}$  is a lift of  $\alpha$  under  $f$  that is isotopic to  $\alpha$  rel.  $\text{Crit}(f)$ . Then by (D1) we have

$$\begin{aligned} i_f(\alpha, e) &= |\alpha \cap \text{int}(e)| = |\tilde{\alpha} \cap f^{-1}(\text{int}(e))| \\ &\geq |\tilde{\alpha} \cap \text{int}(e^+)| + |\tilde{\alpha} \cap \text{int}(e^-)| \\ &\geq i_f(\tilde{\alpha}, e^+) + i_f(\tilde{\alpha}, e^-) = 2i_f(\alpha, e). \end{aligned}$$

It follows that  $i_f(\alpha, e) = 0$  for each edge  $e$  of  $G$ , as desired.  $\square$

**Lemma 3.12.** *Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map obtained by blowing up an admissible pair  $(G, \varphi)$ . Then a Jordan arc  $\alpha$  in  $(S^2, \text{Crit}(f))$  blows up under  $f$  if and only if  $\alpha$  is isotopic to an edge of  $G$  rel.  $\text{Crit}(f)$ .*

*Proof.* Each edge  $e$  of  $G$  blows up under  $f$ , because  $e^+$  and  $e^-$  are isotopic to  $e$  rel.  $\text{Crit}(f)$ . Thus, if  $\alpha$  is a Jordan arc isotopic to  $e$  rel.  $\text{Crit}(f)$ , then it blows up as well.

Suppose now that  $\alpha$  is a Jordan arc in  $(S^2, \text{Crit}(f))$  that blows up under  $f$ . It follows from Lemma 3.11 that we may assume that  $\alpha$  intersects the graph  $G$  only in its vertices. Then by (E1) each of the  $\deg(f) - 1$  regions  $\overline{U}_e$ ,  $e \in E(G)$ , contains exactly one lift  $\tilde{\alpha}_e$  of  $\alpha$  under  $f$ . Since  $\alpha$  blows up, it follows that one of these lifts  $\tilde{\alpha}_e$  has to be isotopic to  $\alpha$  rel.  $\text{Crit}(f)$ . This is possible only if  $\partial\tilde{\alpha}_e = \partial e$ , which implies that  $e^+ \sim \tilde{\alpha}_e$  by Lemma 2.1. Hence  $e \sim e^+ \sim \tilde{\alpha}_e \sim \alpha$  rel.  $\text{Crit}(f)$ . This finishes the proof of the lemma.  $\square$

**Lemma 3.13.** *Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map obtained by blowing up an admissible pair  $(G, \varphi)$  and  $e$  be an edge of  $G$ . Suppose  $\beta$  is a Jordan arc in  $S^2$  with  $\partial\beta = \partial e$  (and possibly with critical points of  $f$  in its interior) that satisfies  $i_f(\beta, e) = 0$ . Then  $\beta$  has a lift  $\tilde{\beta}$  under  $f$  that is isotopic to  $e$  rel.  $\text{Crit}(f)$ .*

*Proof.* Without loss of generality, we may assume that  $\beta \cap e = \partial e$ . Property (E1) implies that there is a lift  $\tilde{\beta}$  of  $\beta$  under  $f$  with  $\text{int}(\tilde{\beta}) \subset U_e$  and  $\partial\tilde{\beta} = \partial e$ . Again we have  $\tilde{\beta} \sim e^+ \sim e$  rel.  $\text{Crit}(f)$ . This completes the proof.  $\square$

The following lemma provides a quantitative version of Lemma 3.12.

**Lemma 3.14.** *Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map obtained by blowing up an admissible pair  $(G, \varphi)$ . Then for each  $\alpha \in E(G)$  we have  $\deg(f, \alpha) = m_G(\alpha) + 1$ . More generally, for every Jordan arc  $\alpha$  in  $(S^2, \text{Crit}(f))$  there are exactly  $\max(0, \deg(f, \alpha) - 1)$  edges in  $G$  that are isotopic to  $\alpha$  rel.  $\text{Crit}(f)$ .*

*Proof.* Properties (D1), (E1), and (E3) imply that each  $\alpha \in E(G)$  has the following lifts under  $f$ :  $\alpha^+$ ,  $\alpha^-$ , and a unique lift  $\tilde{\alpha}_e$  with  $\text{int}(\tilde{\alpha}_e) \subset U_e$  for every  $e \in E(G) \setminus \{\alpha\}$ . Note that  $\partial\tilde{\alpha}_e = \partial e$  if and only if  $\partial\alpha = \partial e$ . It follows that  $\tilde{\alpha}_e$  is isotopic to  $\alpha$  rel.  $\text{Crit}(f)$  if and only if  $e$  is isotopic to  $\alpha$  rel.  $\text{Crit}(f)$ , and therefore  $\deg(\alpha, f) = m_G(\alpha) + 1$ . The second part of the lemma now follows from these considerations and Lemma 3.12.  $\square$

**3.4. Rational case.** Let  $f$  be a critically fixed rational map with  $\deg(f) \geq 2$  and  $c \in \text{Crit}(f)$  be a critical point of  $f$ . Note that  $c$  is a superattracting fixed point of  $f$ , and thus all points in a neighborhood of  $c$  converge to  $c$  under iteration. The *basin of attraction* of  $c$  is defined to be the set

$$B_c := \{z \in \mathbb{C}: \lim_{n \rightarrow \infty} f^n(z) = c\}.$$

The connected component of  $B_c$  containing the point  $c$  is called the *immediate basin* of  $c$  and denoted by  $\Omega_c$ . It follows from [Mil06, Theorem 9.3] that  $\Omega_c$  is a simply connected open set and there exists a biholomorphic map  $\tau_c: \mathbb{D} \rightarrow \Omega_c$  such that

$$(\tau_c^{-1} \circ f \circ \tau_c)(z) = z^{d_c},$$

where  $d_c := \deg(f, c)$ . Furthermore, the map  $\tau_c$  extends to a continuous and surjective map  $\tau_c: \overline{\mathbb{D}} \rightarrow \overline{\Omega_c}$ , which provides a semi-conjugacy between the map  $f$  on  $\overline{\Omega_c}$  and the power map  $z \mapsto z^{d_c}$  on  $\overline{\mathbb{D}}$ .

An *internal ray of angle*  $\theta \in [0, 2\pi)$  in the immediate basin  $\Omega_c$  is the image of the radial arc  $r(\theta) := \{te^{i\theta} : t \in \mathbb{I}\}$  under the map  $\tau_c$ . The point  $\tau_c(e^{i\theta}) \in \partial\Omega_c$  is called the *landing point* of the internal ray of angle  $\theta$ . Note that the internal ray of angle  $\theta$  is fixed under  $f$  if and only if  $\theta = 2\pi \frac{j}{d_c-1}$  for some  $j \in \{0, \dots, d_c-2\}$ . The landing points of such rays are repelling fixed points of the map  $f$ .

The *Tischler graph* of a critically fixed rational map  $f$  is the planar embedded graph  $\text{Tisch}(f)$  whose edge set consists of the fixed internal rays taken in the immediate basins  $\Omega_c$  for all  $c \in \text{Crit}(f)$  and vertex set consists of the endpoints of all these rays. That is, as a subset of  $\widehat{\mathbb{C}}$ ,  $\text{Tisch}(f)$  is the union of all fixed internal rays described in the previous paragraph.

Let us denote by  $\text{Fix}(f)$  the set of all fixed points of a (critically fixed) rational map  $f$ . Recall that the holomorphic fixed point formula implies that  $|\text{Fix}(f)| = \deg(f) + 1$ , if counted with multiplicity.

Due to [Hlu19, Theorem 1 and Corollary 6] and [CGN<sup>+</sup>15, Lemma 3] we have the following properties of the Tischler graph.

**Proposition 3.15.** *Let  $f$  be a critically fixed rational map with  $\deg(f) \geq 2$ . Then the following statements are true:*

- (i) *The vertex set of  $\text{Tisch}(f)$  consists of all fixed points of  $f$ . In particular,  $|V(\text{Tisch}(f))| = \deg(f) + 1$ .*
- (ii)  *$\text{Tisch}(f)$  is a bipartite graph: each edge of  $\text{Tisch}(f)$  connects a superattracting fixed point and a repelling fixed point of  $f$ .*
- (iii)  *$\text{Tisch}(f)$  is connected.*
- (iv) *The boundary  $\partial Q$  of each face  $Q$  of  $\text{Tisch}(f)$  is either a quadrilateral or a bigon with a sticker inside (see Figure 5).*

The proposition above justifies the following definition.

**Definition 3.16.** Let  $f$  be a critically fixed rational map with  $\deg(f) \geq 2$ . For each face  $Q$  of the Tischler graph  $\text{Tisch}(f)$ , choose a Jordan arc  $e(Q)$  joining the (only) two critical points of  $f$  on  $\partial Q$  so that  $\text{int}(e(Q)) \subset Q$  (see Figure 5 for an illustration). Let  $G$  be the planar embedded graph with the vertex set  $\text{Crit}(f)$  and the edge set  $\{e(Q) : Q \in F(\text{Tisch}(f))\}$ . Then any planar embedded graph isotopic to  $G$  rel.  $\text{Crit}(f)$  is called the *charge graph* of  $f$  and denoted by  $\text{Charge}(f)$ .

Note that the Tischler graph  $\text{Tisch}(f)$  of a critically fixed rational map  $f$  with  $\deg(f) \geq 2$  is uniquely defined, while the charge graph  $\text{Charge}(f)$  is defined only up to isotopy rel.  $\text{Crit}(f)$ .

The next statement relates critically fixed rational maps and their charge graphs via the blow-up operation. (It immediately follows from Proposition 3.4 and [Hlu19, Proposition 7].)

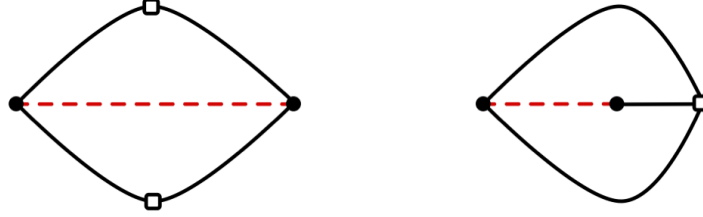


FIGURE 5. Constructing an edge  $e(Q)$  of  $\text{Charge}(f)$  inside a face  $Q$  of  $\text{Tisch}(f)$  if  $\partial Q$  is a quadrilateral (left) and a bigon with a sticker inside (right). The boundary of  $Q$  consists of black edges; the critical points of  $f$  on  $\partial Q$  are represented by black dots; the repelling fixed points of  $f$  on  $\partial Q$  are represented by white squares; and the edge  $e(Q)$  is represented by a dashed red line.

**Proposition 3.17.** *Let  $f$  be a critically fixed rational map with  $\deg(f) \geq 2$  and  $g$  be a critically fixed Thurston map obtained by blowing up the pair  $(\text{Charge}(f), \text{id}_{\widehat{\mathbb{C}}})$ . Then the maps  $f$  and  $g$  are isotopic.*

*Remark 3.18.* In fact, Proposition 3.4 and [Hlu19, Proposition 7] imply a slightly stronger result. Suppose  $f$  is a critically fixed rational map and  $G$  is the planar embedded graph in  $\widehat{\mathbb{C}}$  constructed in Definition 3.16 from the Tischler graph of  $f$ . Consider the planar embedded graph  $G' := G \cup \text{Fix}(f)$  with the vertex set  $\text{Fix}(f)$ . We note that each face of  $G'$  contains exactly one point from  $\text{Fix}(f) \setminus \text{Crit}(f)$ ; see [Hlu19, Lemma 8]. Let  $g'$  be the critically fixed Thurston map obtained by blowing up the pair  $(G', \text{id}_{\widehat{\mathbb{C}}})$ . Then the marked Thurston maps  $(f, \text{Fix}(f))$  and  $(g', \text{Fix}(f))$  are isotopic (rel.  $\text{Fix}(f)$ ).

The following converse statement to Proposition 3.17 easily follows from [PL98, Corollary 3].

**Proposition 3.19.** [CGN<sup>+</sup>15, Theorem 9] *Let  $f$  be a critically fixed Thurston map obtained by blowing up a pair  $(G, \text{id}_{S^2})$ , where  $G$  is a planar embedded graph in  $S^2$  without isolated vertices. Then  $f$  is combinatorially equivalent to a rational map if and only if  $G$  is connected.*

*Remark.* In fact, the following stronger statement easily follows from the discussion in Remark 3.18. Let  $f$  be a critically fixed Thurston map obtained by blowing up a pair  $(G, \text{id}_{S^2})$ . Then the marked Thurston map  $(f, V(G))$  is realized if and only if  $G$  has exactly one non-trivial connected component  $H$  and each face of  $H$  contains at most one isolated vertex of  $G$ . Here, a connected graph  $H$  is called *non-trivial* if  $H$  has at least one edge.

The family of critically fixed rational maps may be completely classified using their charge graphs. Namely, Theorem 1.2 is an immediate corollary of the following result.

**Proposition 3.20** ([Hlu19, Section 5]). *Two critically fixed rational maps  $f$  and  $g$  are combinatorially equivalent if and only if their charge graphs  $\text{Charge}(f)$  and  $\text{Charge}(g)$  are isomorphic.*

*Example 3.21.* Let us consider the following rational map

$$F_{\square}(z) = \frac{3z^5 - 20z}{5z^4 - 12}.$$

One can easily check that  $\text{Crit}(F_{\square}) = \{1+i, -1+i, -1-i, 1-i\}$  and that  $F_{\square}(c) = c$  for each  $c \in \text{Crit}(F_{\square})$ , that is,  $F_{\square}$  is a critically fixed rational map. The extra two fixed points of  $F_{\square}$

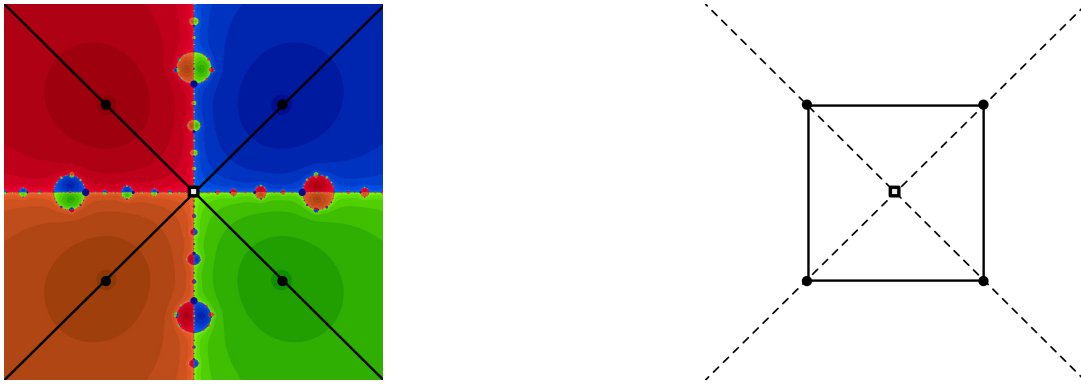


FIGURE 6. The Tischler graph of  $F_{\square}$  (left) and the construction of the charge graph of  $F_{\square}$  from  $\text{Tisch}(F_{\square})$  (right).

are 0 and  $\infty$ . Furthermore, we have  $\deg(F_{\square}) = 5$  and  $\deg(F_{\square}, c) = 3$  for each  $c \in \text{Crit}(F_{\square})$ . The attracting basins and the Tischler graph of  $F_{\square}$  are shown on the left part of Figure 6. The right part of Figure 6 illustrates the construction of the charge graph  $\text{Charge}(F_{\square})$ .

Note that  $\text{Charge}(F_{\square})$  is isomorphic to  $G_{\square}$ . Proposition 3.19 implies that the critically fixed Thurston map  $f_{\square}$  introduced in Example 3.3 is realized. It now follows from Proposition 3.20 that the rational map  $F_{\square}$  is combinatorially equivalent to  $f_{\square}$ . We will use the map  $f_{\square}$  in our further examples, but the same observations will hold for any Thurston map equivalent to  $f_{\square}$ , in particular, for  $F_{\square}$ .

**3.5. Decomposition.** Throughout this subsection we follow the notation and terminology developed in Section 2.6. Our goal is to prove the following result, which shows that completely invariant multicurves (and the induced decompositions) for critically fixed Thurston maps satisfy very restrictive conditions.

**Theorem 3.22.** *Let  $f$  be a critically fixed Thurston map and  $\Gamma$  be a non-empty completely invariant multicurve. Suppose  $\hat{f}: \widehat{\mathcal{S}}_{\Gamma} \rightarrow \widehat{\mathcal{S}}_{\Gamma}$  is the corresponding map on the small spheres with respect to  $\Gamma$ . Then the following statements are true.*

- (i) *For each curve  $\gamma \in \Gamma$  there is exactly one component  $\gamma'$  of  $f^{-1}(\gamma)$  that is isotopic to  $\gamma$  rel.  $\text{Crit}(f)$ . All other components  $\delta'$  of  $f^{-1}(\gamma)$  are null-homotopic with  $\deg(f|\delta') = 1$ .*
- (ii) *Each small sphere  $\hat{S} \in \widehat{\mathcal{S}}_{\Gamma}$  is fixed under  $\hat{f}$ .*

Before we proceed with the proof of this theorem, we first provide some auxiliary definitions and results.

Let us assume that  $f: S^2 \rightarrow S^2$  is an arbitrary Thurston map and  $\Gamma$  is a completely invariant multicurve. Consider an (abstract) graph  $T_{\Gamma}$  with the vertex set  $\mathcal{S}_{\Gamma}$  and the edge set  $\Gamma$ , where we connect two distinct components  $S_1, S_2 \in \mathcal{S}_{\Gamma}$  by an edge  $\gamma \in \Gamma$  if and only if  $\gamma$  is a boundary curve in each of them. It easily follows that  $T_{\Gamma}$  is connected. In fact,  $T_{\Gamma}$  must be a tree. Indeed, the removal of any edge disconnects  $T_{\Gamma}$ , because each curve  $\gamma \in \Gamma$  disconnects the sphere  $S^2$ . We will denote by  $T_{f^{-1}(\Gamma)}$  the corresponding tree for  $f^{-1}(\Gamma)$ .

If  $S$  is a component in  $\mathcal{S}_{\Gamma}$ , we will denote by  $\hat{S}$  the corresponding small sphere in  $\widehat{\mathcal{S}}_{\Gamma}$ , and similarly for the components in  $\mathcal{S}_{f^{-1}(\Gamma)}$ . Recall that  $f$  maps each component  $S' \in \mathcal{S}_{f^{-1}(\Gamma)}$  onto a component  $f(S') \in \mathcal{S}_{\Gamma}$ , which induces a branched covering map  $f_*: \hat{S}' \rightarrow \hat{f}(S')$  between the associated small spheres (that respects the marked points). Note that  $f$  sends

adjacent vertices in  $T_{f^{-1}(\Gamma)}$  to adjacent vertices in  $T_\Gamma$ . Recall also that, since  $\Gamma$  is completely invariant, for each component  $S \in \mathcal{S}_\Gamma$  there is a unique component  $i(S) \in \mathcal{S}_{f^{-1}(\Gamma)}$  that is homotopic to  $S$  in  $S^2 \setminus \text{Post}(f)$ . This allows to identify the corresponding small spheres  $\widehat{S} \in \widehat{\mathcal{S}}_\Gamma$  and  $i(\widehat{S}) \in \widehat{\mathcal{S}}_{f^{-1}(\Gamma)}$  via a homeomorphism  $i^*: \widehat{S} \rightarrow i(\widehat{S})$ . Then  $\widehat{f}: \widehat{\mathcal{S}}_\Gamma \rightarrow \widehat{\mathcal{S}}_{f^{-1}(\Gamma)}$  is defined as the composition  $f_* \circ i^*$ .

We will now introduce two special subtrees of  $T_{f^{-1}(\Gamma)}$ . The first one, which we denote by  $T_{f^{-1}(\Gamma)}^{ess}$ , is the (unique) spanning subtree of the vertex set  $\{i(S): S \in \mathcal{S}_\Gamma\}$  in  $T_{f^{-1}(\Gamma)}$ . It is easy to see that the edges of  $T_{f^{-1}(\Gamma)}^{ess}$  are exactly all the essential curves in  $f^{-1}(\Gamma)$ . In fact, the following claim is true (the proof is straightforward from the definitions).

**Lemma 3.23.** *The tree  $T_{f^{-1}(\Gamma)}^{ess}$  is obtained from  $T_\Gamma$  by edge subdivision: if two components  $S_1, S_2 \in \mathcal{S}_\Gamma$  are connected in  $T_\Gamma$  by an edge  $\gamma \in \Gamma$  then the components  $i(S_1), i(S_2)$  are connected in  $T_{f^{-1}(\Gamma)}^{ess}$  by a simple path consisting of all edges  $\delta' \in f^{-1}(\Gamma)$  that are isotopic to  $\gamma$  rel.  $\text{Post}(f)$ .*

To define the second subtree of  $T_{f^{-1}(\Gamma)}$ , let us consider the set

$$\mathcal{S}_{f^{-1}(\Gamma)}^\bullet := \{S' \in \mathcal{S}_{f^{-1}(\Gamma)}: S' \cap \text{Post}(f) \neq \emptyset\},$$

which represent the vertices of  $T_{f^{-1}(\Gamma)}$  that contain the postcritical points of  $f$ . We denote by  $T_{f^{-1}(\Gamma)}^\bullet$  the (unique) spanning subtree of  $\mathcal{S}_{f^{-1}(\Gamma)}^\bullet$  in  $T_{f^{-1}(\Gamma)}$ . The next lemma describes the structure of  $T_{f^{-1}(\Gamma)}^\bullet$ . Again, the proof follows easily from the definitions and is left to the reader.

**Lemma 3.24.** *The following statements are true:*

- (i) *The edges of  $T_{f^{-1}(\Gamma)}^\bullet$  are all the curves in  $f^{-1}(\Gamma)$  that are not null-homotopic. In particular,  $T_{f^{-1}(\Gamma)}^{ess}$  is a subtree of  $T_{f^{-1}(\Gamma)}^\bullet$ .*
- (ii) *Let  $S' \in \mathcal{S}_{f^{-1}(\Gamma)}^\bullet$ . Suppose  $p$  is a postcritical point in  $S'$  and  $S \in \mathcal{S}_\Gamma$  is the component containing  $p$ . Then either  $i(S) = S'$  and  $S'$  is a vertex of  $T_{f^{-1}(\Gamma)}^{ess}$ , or  $i(S) \neq S'$  and  $S' \in V(T_{f^{-1}(\Gamma)}^\bullet) \setminus V(T_{f^{-1}(\Gamma)}^{ess})$ . In the latter case,  $S'$  is a leaf of  $T_{f^{-1}(\Gamma)}^\bullet$  with  $S' \cap \text{Post}(f) = \{p\}$ . Furthermore,  $S'$  and  $i(S)$  are connected in  $T_{f^{-1}(\Gamma)}^\bullet$  by a simple path consisting of all  $p$ -peripheral curves  $\delta' \in f^{-1}(\Gamma)$ .*

Here a curve  $\delta' \in f^{-1}(\Gamma)$  is called  $p$ -peripheral if for a component  $U$  of  $S^2 \setminus \delta'$  we have  $U \cap \text{Post}(f) = \{p\}$ . Note that any two  $p$ -peripheral curves are isotopic rel.  $\text{Post}(f)$ .

We now turn to the proof of Theorem 3.22.

*Proof of Theorem 3.22.* Suppose  $f$  and  $\Gamma$  are as in the statement. Then  $\text{Post}(f) = \text{Crit}(f)$ . We will say that a vertex of  $T_\Gamma$  or  $T_{f^{-1}(\Gamma)}^\bullet$  is *critical* if it contains a critical point.

Let

$$P' := (v'_0, \delta'_1, v'_1, \dots, \delta'_n, v'_n)$$

be a simple path in  $T_{f^{-1}(\Gamma)}^\bullet$ . Here, each  $v'_j \in V(T_{f^{-1}(\Gamma)}^\bullet)$  and each  $\delta'_j \in E(T_{f^{-1}(\Gamma)}^\bullet)$  is a (non null-homotopic) curve in  $f^{-1}(\Gamma)$ . Then

$$f(P') := (f(v'_0), f(\delta'_1), f(v'_1), \dots, f(\delta'_n), f(v'_n))$$

is a walk in  $T_\Gamma$ .

Suppose now that  $P'$  is a maximal (i.e., non-extendable) simple path in  $T_{f^{-1}(\Gamma)}^\bullet$  such that  $f(P')$  is a simple path in  $T_\Gamma$ . Clearly,  $P'$  has positive length. We will refer to  $P'$  as a *maximal injective path* in  $T_{f^{-1}(\Gamma)}^\bullet$ .

*Claim 1.* The start and end vertices of  $P'$  are critical.

It is sufficient to show that  $P'$  starts at a critical vertex. If  $v'_0$  is a leaf of  $T_{f^{-1}(\Gamma)}^\bullet$  then  $v'_0$  immediately has to be critical, because  $T_{f^{-1}(\Gamma)}^\bullet$  is the spanning subtree of  $\mathcal{S}_{f^{-1}(\Gamma)}^\bullet$  in  $T_{f^{-1}(\Gamma)}$ .

In the following we assume that  $v'_0$  is not a leaf of  $T_{f^{-1}(\Gamma)}^\bullet$ . We will argue by contradiction and suppose that  $v'_0 \cap \text{Crit}(f) = \emptyset$ . Set  $\gamma := f(\delta'_1)$  and let  $E(v'_0)$  be the set of all edges that are incident to  $v'_0$  in  $T_{f^{-1}(\Gamma)}^\bullet$ . By maximality of  $P'$ , each edge  $\delta' \in E(v'_0)$  satisfies  $f(\delta') = \gamma$ . Furthermore, each critical point of  $f$  is separated from  $v'_0$  by one of these edges  $\delta'$ . Let us denote by  $\tilde{S}'$  the unique component in  $\mathcal{S}_{f^{-1}(\{\gamma\})}$  that contains  $v'_0$ . Then  $f(\tilde{S}') \in \mathcal{S}_{\{\gamma\}}$  is an open Jordan region. Note that each edge  $\delta' \in E(v'_0)$  is a boundary curve of  $\tilde{S}'$ . It follows that  $\chi(\tilde{S}') \leq 0$  and  $\tilde{S}' \cap \text{Crit}(f) = \emptyset$ . Hence  $f|_{\tilde{S}'}: \tilde{S}' \rightarrow f(\tilde{S}')$  is a covering, which is impossible. This contradiction implies that  $v'_0 \cap \text{Crit}(f) \neq \emptyset$ , which completes the proof of the claim.

Claim 1 implies that  $v'_0$  contains a critical point  $c_0$  and  $v'_n$  contains a critical point  $c_n$  of  $f$ . Let  $v_0$  and  $v_n$  be the vertices of  $T_\Gamma$  that contain  $c_0$  and  $c_n$ , respectively. By Lemma 3.24(ii), either  $v'_0 = i(v_0)$  and  $v'_0$  is a vertex of  $T_{f^{-1}(\Gamma)}^{ess}$ , or  $v'_0 \neq i(v_0)$  and  $v'_0$  is a leaf of  $T_{f^{-1}(\Gamma)}^\bullet$  that is connected to  $i(v_0)$  by a simple path consisting of all  $c_0$ -peripheral curves in  $f^{-1}(\Gamma)$ . In either case, we obtain that  $i(v_0)$  is the first vertex of  $T_{f^{-1}(\Gamma)}^{ess}$  on the path  $P'$ . Similarly,  $i(v_n)$  is the last vertex of  $T_{f^{-1}(\Gamma)}^{ess}$  on the path  $P'$ .

Since  $f$  is critically fixed,  $f(v'_0) = v_0$  and  $f(v'_n) = v_n$ . Thus,  $f(P')$  is a simple path that connects  $v_0$  and  $v_n$  in  $T_\Gamma$ . At the same time, it follows from Lemma 3.23 that the sub-path of  $P'$  that connects  $i(v_0)$  and  $i(v_n)$  is not shorter than  $f(P')$ . In fact, this subpath has to pass through all the vertices  $i(f(v'_0)) = i(v_0), i(f(v'_1)), \dots, i(f(v'_{n-1})), i(f(v'_n)) = i(v_n)$  and in this order. These facts together imply the following two claims.

*Claim 2.* For each  $j = 0, \dots, n$  we have  $i(f(v'_j)) = v'_j$ . In particular, all vertices of  $P'$  are in  $T_{f^{-1}(\Gamma)}^{ess}$ .

*Claim 3.* For each  $j = 1, \dots, n$  the curve  $\delta'_j$  is essential and isotopic to  $f(\delta'_j)$  rel.  $\text{Crit}(f)$ . Furthermore,  $\delta'_j$  is the only curve in  $f^{-1}(\Gamma)$  that is isotopic to  $f(\delta'_j)$  rel.  $\text{Crit}(f)$ .

Let us now fix an arbitrary curve  $\gamma \in \Gamma$ . Since  $\Gamma$  is completely invariant, there must be a component  $\gamma'$  of  $f^{-1}(\Gamma)$  that is isotopic to  $\gamma$  rel.  $\text{Crit}(f)$ . The curve  $\gamma'$  corresponds to an edge of  $T_{f^{-1}(\Gamma)}^\bullet$ , and thus it is contained in some maximal injective path in  $T_{f^{-1}(\Gamma)}^\bullet$ . It follows from Claim 3 that  $\gamma'$  satisfies  $f(\gamma') = \gamma$ , that is,  $\gamma'$  is a component of  $f^{-1}(\gamma)$ . Furthermore,  $\gamma'$  is the only curve in  $f^{-1}(\Gamma)$  that is isotopic to  $\gamma$  rel.  $\text{Crit}(f)$ .

Suppose now that  $\delta'$  is an arbitrary component of  $f^{-1}(\gamma)$ . If  $\delta'$  is not null-homotopic, it is contained in some maximal injective path in  $T_{f^{-1}(\Gamma)}^\bullet$ . Claim 3 implies that  $\delta'$  is isotopic to  $f(\delta') = \gamma$  rel.  $\text{Crit}(f)$ , and thus it must be the curve  $\gamma'$ . If  $\delta'$  is null-homotopic, then there is a component  $U$  of  $S^2 \setminus \delta'$  such that  $U \cap \text{Crit}(f) = \emptyset$ . It follows from the Riemann-Hurwitz formula that  $\deg(f|_U) = 1$ , and hence  $\deg(f|\delta') = 1$ . This proves part (i) of the theorem.

To prove part (ii), consider an arbitrary small sphere  $\widehat{S} \in \widehat{\mathcal{S}}_\Gamma$ . Let  $S$  be the corresponding component in  $\mathcal{S}_\Gamma$ . Then  $i(S)$  is a vertex of  $T_{f^{-1}(\Gamma)}^\bullet$ , and thus it is contained in some maximal injective path in  $T_{f^{-1}(\Gamma)}^\bullet$ . By Claim 2 we have  $i(f(i(S))) = i(S)$ . It follows that  $f(i(S)) = S$ , which means that  $\widehat{f}(\widehat{S}) = \widehat{S}$  as required. This finishes the proof of the theorem.  $\square$

We record the following immediate corollary of Theorem 3.22.

**Corollary 3.25.** *Let  $f$  be a critically fixed Thurston map and  $\Gamma$  be a non-empty completely invariant multicurve. Then  $T_{f^{-1}(\Gamma)}^\bullet = T_{f^{-1}(\Gamma)}^{\text{ess}}$  and  $f$  provides an isomorphism between  $T_{f^{-1}(\Gamma)}^\bullet$  and  $T_\Gamma$ .*

The next corollary describes properties of canonical Thurston obstructions for critically fixed Thurston maps. In particular, it shows that every critically fixed Thurston map can be canonically decomposed into homeomorphisms and critically fixed rational maps.

**Corollary 3.26.** *Let  $f$  be a critically fixed Thurston map and  $\Gamma_{\text{Th}}$  be the canonical Thurston obstruction of  $f$ . Then the following statements are true.*

- (i) *For every curve  $\gamma \in \Gamma_{\text{Th}}$  there is exactly one component  $\gamma'$  of  $f^{-1}(\gamma)$  that is isotopic to  $\gamma$  and satisfies  $\deg(f|\gamma') = 1$ . All other components  $\delta'$  of  $f^{-1}(\gamma)$  are null-homotopic and also satisfy  $\deg(f|\delta') = 1$ . In particular, each curve  $\gamma \in \Gamma_{\text{Th}}$  is a fixed Levy curve.*
- (ii) *Each small sphere  $\widehat{S} \in \widehat{\mathcal{S}}_\Gamma$  is fixed under  $\widehat{f}$ .*
- (iii) *If  $S \in \mathcal{S}_\Gamma$  satisfies  $S \cap \text{Crit}(f) = \emptyset$ , then the small sphere map  $\widehat{f}: \widehat{S} \rightarrow \widehat{S}$  is a homeomorphism.*
- (iv) *If  $S \in \mathcal{S}_\Gamma$  satisfies  $S \cap \text{Crit}(f) \neq \emptyset$ , then the small sphere map  $\widehat{f}: \widehat{S} \rightarrow \widehat{S}$  is realized by a critically fixed rational map of degree  $d(\widehat{S}) = 1 + \frac{1}{2} \sum_{c \in S \cap \text{Crit}(f)} (\deg(f, c) - 1)$ .*

*Proof.* The proof is immediate from Theorem 3.22 since the canonical Thurston obstruction  $\Gamma_{\text{Th}}$  is simple. The formula for  $d(\widehat{S})$  follows from the Riemann-Hurwitz formula.  $\square$

The next corollary shows that it is sufficient to check absence of fixed Levy curves to conclude that a given critically fixed rational map is realized.

**Corollary 3.27.** *Let  $f$  be a critically fixed Thurston map. Then every Thurston obstruction of  $f$  contains a fixed Levy curve. In particular,  $f$  is realized if and only if  $f$  does not have a fixed Levy curve.*

*Proof.* The proof is immediate from Theorem 3.22 since every Thurston obstruction contains a simple one.  $\square$

**3.6. The charge graph and classification.** The goal of this subsection is to extend the notion of the charge graph of a critically fixed rational map to the more general setup of critically fixed Thurston maps. Moreover, we will provide a classification of critically fixed Thurston maps in terms of admissible pairs.

Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map and  $\Gamma := \Gamma_{\text{Th}}$  be the canonical Thurston obstruction for  $f$ . Let us decompose  $\widehat{\mathcal{S}}_\Gamma$  as the union  $\widehat{\mathcal{S}}_{\Gamma, \text{Rat}} \sqcup \widehat{\mathcal{S}}_{\Gamma, \text{Homeo}}$ , where  $\widehat{\mathcal{S}}_{\Gamma, \text{Rat}}$  consists of the small spheres containing critical points of  $f$ , and  $\widehat{\mathcal{S}}_{\Gamma, \text{Homeo}}$  consists of the small spheres without critical points. By Corollary 3.26, each small sphere  $\widehat{S} \in \widehat{\mathcal{S}}_\Gamma$  is fixed under  $\widehat{f}$ . Moreover, each small sphere map  $\widehat{f}|_{\widehat{S}}$ ,  $\widehat{S} \in \widehat{\mathcal{S}}_{\Gamma, \text{Homeo}}$ , is a homeomorphism, and each small sphere map  $\widehat{f}|_{\widehat{S}}$ ,  $\widehat{S} \in \widehat{\mathcal{S}}_{\Gamma, \text{Rat}}$ , is combinatorially equivalent to a critically fixed rational map  $F_{\widehat{S}}$ . In the latter case, Proposition 3.6 implies that  $\widehat{f}|_{\widehat{S}}$  is obtained by blowing up a pair  $(G_{\widehat{S}}, \varphi_{\widehat{S}})$ , where  $G_{\widehat{S}}$  is a (connected) graph isomorphic to  $\text{Charge}(F_{\widehat{S}})$ . Note that we may naturally view  $G_{\widehat{S}}$  as a planar embedded graph in  $S^2$ . Then the union

$$(3.2) \quad \text{Charge}(f) := \bigsqcup_{\widehat{S} \in \widehat{\mathcal{S}}_{\Gamma, \text{Rat}}} G_{\widehat{S}},$$



of all such graphs is a planar embedded graph in  $S^2$  with the vertex set  $\text{Crit}(f)$ . This observation leads us to the following definition.

**Definition 3.28.** Let  $f$  be a critically fixed Thurston map. We define the *charge graph* of  $f$  to be the planar embedded graph  $\text{Charge}(f)$  with the vertex set  $\text{Crit}(f)$  constructed as above.

Note that the charge graph is defined only up to isotopy rel.  $\text{Crit}(f)$ . In fact, its isotopy class is uniquely characterized by the following result.

**Theorem 3.29.** *Let  $f$  be a critically fixed Thurston map. Then the following statements are true.*

- (i) *The map  $f$  is isotopic to a critically fixed Thurston map obtained by blowing up an admissible pair  $(\text{Charge}(f), \varphi_f)$ .*
- (ii) *If  $f$  is isotopic to a critically fixed Thurston map  $f'$  obtained by blowing up an admissible pair  $(G', \varphi')$ , then  $G'$  and  $\varphi'$  are isotopic to  $\text{Charge}(f)$  and  $\varphi_f$  rel.  $\text{Crit}(f)$ , respectively.*

*Proof.* Suppose  $f$  is a critically fixed Thurston map,  $G := \text{Charge}(f)$  is the charge graph of  $f$ , and  $\Gamma := \Gamma_{\text{Th}}$  is the canonical Thurston obstruction of  $f$ .

Consider a small sphere  $\widehat{S} \in \widehat{\mathcal{S}}_{\Gamma, \text{Rat}}$ , and let  $S$  be the respective component in  $\mathcal{S}_{\Gamma}$ . Note that  $\widehat{S}$  is marked by a finite set  $Q(\widehat{S})$  corresponding to  $\text{Crit}(f) \cap S$  and  $\Gamma \cap \partial S$ . In fact, each of these points is a fixed point of  $\widehat{f}|_{\widehat{S}}$  (see Corollary 3.26(i)). By construction,  $S \subset S^2$  contains exactly one component  $G_S$  of the charge graph  $G$ . Combining Proposition 3.2, Corollary 3.26(iv), and the Riemann-Hurwitz formula, we obtain

$$\begin{aligned} |E(G)| &= \sum_{\widehat{S} \in \widehat{\mathcal{S}}_{\Gamma, \text{Rat}}} |E(G_S)| = \sum_{\widehat{S} \in \widehat{\mathcal{S}}_{\Gamma, \text{Rat}}} (\deg(\widehat{f}|_{\widehat{S}}) - 1) = \\ &= \sum_{\widehat{S} \in \widehat{\mathcal{S}}_{\Gamma, \text{Rat}}} \frac{1}{2} \sum_{c \in S \cap \text{Crit}(f)} (\deg(f, c) - 1) = \frac{1}{2} \sum_{c \in \text{Crit}(f)} (\deg(f, c) - 1) = \deg(f) - 1. \end{aligned}$$

We claim that each edge  $\alpha$  of  $G$  admits a triple  $(\alpha^+, \alpha^-, U_\alpha)$  that satisfies the conditions of Proposition 3.9. Indeed, if  $\alpha \in E(G_S)$  then the map  $\widehat{f}|_{\widehat{S}}$  induces such a triple in  $S^2$  by Proposition 3.8. Here, we use the fact that the marked small sphere map  $(\widehat{f}|_{\widehat{S}}, Q(\widehat{S}))$  is combinatorially equivalent to a marked critically fixed rational map  $(F_{\widehat{S}}, \text{Fix}(F_{\widehat{S}}))$ ; see Proposition 3.6 and Remark 3.18. Part (i) of the theorem now follows from Proposition 3.9.

Suppose now that  $f$  is isotopic to a map  $f'$  obtained by blowing up an admissible pair  $(G', \varphi')$ . Then each edge  $\alpha' \in E(G')$  blows up under  $f$  by Proposition 2.9. Lemma 3.14 implies that  $\alpha'$  is isotopic to an edge  $\alpha$  of  $G$  and  $m_G(\alpha) = m_{G'}(\alpha')$ . It then follows from Proposition 2.4 that  $G$  and  $G'$  are isotopic rel.  $\text{Crit}(f) = \text{Crit}(f')$ .

It is now left to prove that  $\varphi_f$  and  $\varphi'$  are isotopic rel.  $\text{Crit}(f)$ . Let  $g$  be the map obtained by blowing up the pair  $(G, \varphi')$ . Since  $f$  and  $f'$  are isotopic, Propositions 3.4 implies that  $f$  and  $g$  are isotopic as well. Therefore, we may write  $f = g \circ \psi$  for some homeomorphism  $\psi \in \text{Homeo}_0^+(S^2, \text{Crit}(f))$ . Then  $G_f^\pm = \psi^{-1}(G_g^\pm)$ , where  $G_f^\pm = \bigcup_{\alpha \in E(G)} (\alpha^+ \cup \alpha^-)$  and  $G_g^\pm$  are the blow-ups of  $G$  under  $f$  and  $g$ , respectively. Choose a connected graph  $H \subset S^2 \setminus \bigcup_{\alpha \in E(G)} (\overline{U}_\alpha \setminus \partial \alpha)$  with  $V(H) = \text{Crit}(f)$ . Then, (E3) implies that  $(g \circ \psi)(e) = f(e) \sim \varphi_f(e)$  rel.  $\text{Crit}(f)$  for every  $e \in E(H)$ . At the same time, since  $\text{int}(\psi(e)) \subset S^2 \setminus \bigcup_{\alpha \in E(G)} \psi(\overline{U}_\alpha)$ ,

we get  $(g \circ \psi)(e) = g(\psi(e)) \sim \varphi'(\psi(e)) \sim \varphi'(e)$  rel.  $V(G)$ . Corollary 2.17 now implies that  $\varphi_f$  and  $\varphi'$  are isotopic rel.  $\text{Crit}(f)$ . This finishes the proof of the theorem.  $\square$

Based on the previous results, we can now complete a solution to the classification problem for the entire class of critically fixed Thurston maps.

**Definition 3.30.** Let  $(G_1, \varphi_1)$  and  $(G_2, \varphi_2)$  be two admissible pairs in  $S^2$  and  $\widehat{S^2}$ , respectively. We say that  $(G_1, \varphi_1)$  and  $(G_2, \varphi_2)$  are equivalent if there is an orientation-preserving homeomorphism  $\psi: S^2 \rightarrow \widehat{S^2}$  such that  $\psi(G_1) = G_2$  and  $\psi \circ \varphi_1 \circ \psi^{-1}$  is isotopic to  $\varphi_2$  rel.  $V(G_2)$ .

The following result establishes Main Theorem A from the introduction.

**Theorem 3.31.** *There is a canonical bijection between the combinatorial equivalence classes of critically fixed Thurston maps and the equivalence classes of admissible pairs.*

*Proof.* Let  $\text{CritFixMaps}$  be the set of all combinatorial equivalence classes of critically fixed Thurston maps and  $\text{AdmPairs}$  be the set of all equivalence classes of admissible pairs. As common, we will denote the equivalence class of an element by  $[\cdot]$ .

Let  $f$  be a critically fixed Thurston map. Theorem 3.29(i) implies that  $f$  is isotopic to a map obtained by blowing up an admissible pair  $(G_f, \varphi_f)$ . By Theorem 3.29(ii) and Remark 3.7, this induces a well-defined map  $\Phi: \text{CritFixMaps} \rightarrow \text{AdmPairs}$  given by

$$\Phi([f]) = [(G_f, \varphi_f)].$$

The map  $\Phi$  is clearly surjective since blowing up of any admissible pair gives raise to some critically fixed Thurston map according to Proposition 3.2. To check if the map  $\Phi$  is injective, let  $f: S^2 \rightarrow S^2$  and  $g: \widehat{S^2} \rightarrow \widehat{S^2}$  be two critically fixed Thurston maps defined by blowing up some admissible pairs  $(G_f, \varphi_f)$  and  $(G_g, \varphi_g)$  in  $S^2$  and  $\widehat{S^2}$ , respectively. Suppose that these admissible pairs are equivalent, that is, there exists an orientation-preserving homeomorphism  $\psi: S^2 \rightarrow \widehat{S^2}$  such that  $\psi(G_f) = G_g$  and the homeomorphisms  $\psi \circ \varphi_f \circ \psi^{-1}$  and  $\varphi_g$  are isotopic rel.  $\text{Crit}(g)$ . The proof of Proposition 3.6 implies that the map  $\psi \circ f \circ \psi^{-1}$  is obtained by blowing up the pair  $(\widehat{G}, \widehat{\varphi})$ , where  $\widehat{G}$  is the planar embedded graph  $\psi(G)$  in  $\widehat{S^2}$  with the vertex set  $V(\widehat{G}) = \psi(V(G)) = \text{Crit}(g)$  and  $\widehat{\varphi} = \psi \circ \varphi_f \circ \psi^{-1}$ . Proposition 3.4 then implies that the maps  $\psi \circ f \circ \psi^{-1}$  and  $g$  are isotopic. Thus, the maps  $f$  and  $g$  are combinatorially equivalent, which means that  $\Phi$  is injective. This finishes the proof of the theorem.  $\square$

We close this section by noting that we can easily determine the canonical obstruction of a critically fixed Thurston map obtained by blowing up an admissible pair. To do so, we need to introduce some terminology.

Let  $G$  be a planar embedded graph in  $S^2$ . Suppose  $H$  is a connected component of  $G$  and  $U$  is one of its faces. Since  $U$  is simply connected, we may fix a homeomorphism  $\eta: \mathbb{D} \rightarrow U$ . Let  $0 < \varepsilon < 1$ . We say that a Jordan curve  $\gamma \subset U$  is an  $\varepsilon$ -boundary of  $G$  (with respect to  $U$ ) if  $\eta^{-1}(\gamma) \subset \{z: 1 - \varepsilon < |z| < 1\}$  and  $\eta^{-1}(\gamma)$  separates 0 from  $\partial\mathbb{D}$ . Note that the isotopy class of  $\gamma$  rel.  $V(G)$  is fixed for all sufficiently small  $\varepsilon$  and is independent of the choice of  $\eta$ . In the following, whenever we talk about  $\varepsilon$ -boundaries we assume that  $\varepsilon$  is sufficiently small.

**Theorem 3.32.** *Let  $f$  be a critically fixed Thurston map obtained by blowing an admissible pair  $(G, \varphi)$ . Set  $\Gamma$  to be the multicurve obtained by taking all essential  $\varepsilon$ -boundaries of  $G$  and identifying the isotopic ones. Then  $\Gamma$  is the canonical Thurston obstruction for  $f$ .*

*Proof.* We will only give an outline of the argument and leave some straightforward details to the reader.

Let  $f$  be a critically fixed Thurston map obtained by blowing an admissible pair  $(G, \varphi)$ . Without loss of generality, we may assume that  $\varphi(\alpha) = \alpha$  for each edge  $\alpha \in E(G)$ . Suppose  $(\alpha^+, \alpha^-, U_\alpha)$ ,  $\alpha \in E(G)$ , are the triples provided by Proposition 3.8 for  $K = G$ . We also denote by  $G^\pm = \bigcup_{\alpha \in E(G)} (\alpha^+ \cup \alpha^-)$  the corresponding blow-up of  $G$  under the map  $f$ .

*Claim.* Let  $\gamma$  be an essential  $\varepsilon$ -boundary of  $f$ . Then there is exactly one component  $\gamma'$  of  $f^{-1}(\gamma)$  that is isotopic to  $\gamma$  rel.  $\text{Crit}(f)$  and satisfies  $\deg(f|_{\gamma'}) = 1$ . All other components  $\delta'$  of  $f^{-1}(\gamma)$  are null-homotopic and satisfy  $\deg(f|_{\delta'}) = 1$ . In particular,  $\gamma$  is a fixed Levy curve of  $f$ .

Suppose  $\gamma$  is an essential  $\varepsilon$ -boundary of  $G$  with respect to  $U$ . Then  $\gamma \subset W \subset U$  where  $W$  is a multiply connected face of  $G$ . Let  $\widetilde{W}$  be the face of  $G^\pm$  such that  $f|_{\widetilde{W}}: \widetilde{W} \rightarrow W$  is a homeomorphism (see Proposition 3.8). Then  $\widetilde{W}$  contains a unique component  $\gamma'$  of  $f^{-1}(\gamma)$ . Clearly,  $\deg(f|_{\gamma'}) = 1$ . By (E1) and (E3), any other component  $\delta'$  of  $f^{-1}(\gamma)$  belongs to  $U_\alpha$  for some  $\alpha \in E(G)$ . Therefore,  $\delta'$  is null-homotopic and satisfies  $\deg(f|_{\delta'}) = 1$ .

To prove the claim, it remains to show that  $\gamma'$  is isotopic to  $\gamma$  rel.  $\text{Crit}(f)$ . We may naturally view  $H := \partial U$  a subgraph of  $G$ . Similarly,  $H^\pm := \bigcup_{\alpha \in E(H)} (\alpha^+ \cup \alpha^-) \supset \partial \widetilde{W}$  is a subgraph of  $G^\pm$ . By the uniform continuity of  $f$ , the curve  $\gamma'$  is an  $\varepsilon$ -boundary of  $G^\pm$  with respect to a unique face  $\widetilde{U}$  of  $H^\pm$  that contains  $\widetilde{W}$ . Let  $Q$  be the component of  $S^2 \setminus \gamma$  that contains  $H$ . Changing  $f$  by an isotopy rel.  $\text{Crit}(f)$ , we may assume that  $Q$  contains each  $\overline{U_\alpha}$  with  $\alpha \in E(H)$ , while all  $\overline{U_\alpha}$  with  $\alpha \in E(G) \setminus E(H)$  remain the same. Then  $\gamma$  is also an  $\varepsilon$ -boundary of  $G^\pm$  with respect to  $\widetilde{U}$ . Therefore, the curves  $\gamma'$  and  $\gamma$  are isotopic rel.  $\text{Crit}(f)$ .

The claim implies that the multicurve  $\Gamma$  is completely invariant. Suppose  $\widehat{S} \in \widehat{\mathcal{S}}_\Gamma$  is a small sphere with respect to  $\Gamma$ , and  $S$  is the respective component in  $\mathcal{S}_\Gamma$ . Recall that  $\widehat{S}$  is marked by a finite set  $Q(\widehat{S})$  corresponding to  $\text{Crit}(f) \cap S$  and  $\Gamma \cap \partial S$ . By Theorem 3.22(ii), the small sphere  $\widehat{S}$  is fixed under  $\widehat{f}$ . If  $S \cap \text{Crit}(f) = \emptyset$ , then the small sphere map  $\widehat{f}|_{\widehat{S}}$  is a homeomorphism. Otherwise,  $S$  contains a unique connected component  $G_S$  of  $G$ . Furthermore, each face  $W$  of  $G_S$  contains at most one marked point in  $Q(\widehat{S})$ . It now easily follows from Remark 3.18 that the marked small sphere map  $(\widehat{f}|_{\widehat{S}}, Q(\widehat{S}))$  is realized. We claim that the multicurve  $\Gamma$  satisfies the criterion in Theorem 2.14, and thus  $\Gamma$  is a canonical obstruction of  $f$ . Indeed, to check minimality, assume that  $\Gamma' \subsetneq \Gamma$  and  $\gamma \in \Gamma \setminus \Gamma'$ . Let  $S' \in \mathcal{S}_{\Gamma'}$  be the component containing  $\gamma$ . Note that  $\gamma$  is an essential curve in the respective marked small sphere  $\widehat{S}'$ . Moreover,  $\widehat{S}' \cap \text{Crit}(f) \neq \emptyset$ . The claim now implies that  $\gamma$  is a Levy fixed curve in the (marked) small sphere map associated with  $\widehat{S}'$ . Therefore,  $\Gamma'$  does not meet the requirements from Theorem 2.14, and  $\Gamma$  must be the canonical obstruction for  $f$ .  $\square$

#### 4. LIFTING ALGORITHM

In this section, we develop an algorithm that for a given critically fixed Thurston map  $f$  finds its charge graph  $G_f := \text{Charge}(f)$ . This also allows us to reconstruct an admissible pair  $(G_f, \varphi_f)$  that is associated with  $f$  by Theorem 3.29(i). Namely, we provide a combinatorial description of the homeomorphism  $\varphi_f$  using the charge graph  $G_f$  and the original map  $f$ . By the discussion in Section 3.4, knowing the graph  $G_f$  we can decide whether  $f$  is obstructed or not, depending on the connectivity of  $G_f$ . Furthermore, if  $f$  is realized, the charge graph

$G_f$  uniquely determines the combinatorial equivalence class of  $f$ . Otherwise, we find the canonical obstruction of  $f$  by looking at the connectivity components of  $G_f$ .

**4.1. The pullback operation.** In the following, we suppose that  $f: S^2 \rightarrow S^2$  is a critically fixed Thurston map and  $T$  is a planar embedded tree in  $S^2$  with  $\text{Crit}(f) \subset V(T)$ . By Lemma 2.15, the complete preimage  $f^{-1}(T)$  is a connected graph with  $\text{Crit}(f) \subset V(f^{-1}(T))$ . Hence, we can choose a spanning subtree of the critical set  $\text{Crit}(f)$  in the graph  $f^{-1}(T)$ , that is, a minimal under inclusion subtree  $\tilde{T}$  of  $f^{-1}(T)$  such that  $\text{Crit}(f) \subset V(\tilde{T})$ . Note that each non-critical vertex of  $\tilde{T}$  must then have degree at least 2. Next, we further “simplify” the tree  $\tilde{T}$  by removing all non-critical vertices of degree 2 (if there are any). More formally, we consider the tree  $T'$  embedded in  $S^2$  with the same realization as  $\tilde{T}$  but with the vertex set  $V(T')$  given by

$$V(T') = \text{Crit}(f) \cup \{v \in V(\tilde{T}) : \deg_{\tilde{T}}(v) \geq 3\}.$$

**Definition 4.1.** The operation of deriving a planar embedded tree  $T'$  from the tree  $T$  as described above is called the *pullback operation*. We also say that  $T'$  is a *pullback* of the tree  $T$  under the map  $f$  and denote by  $\Pi_f(T)$  the set of all such pullbacks.

We emphasize that the result of the pullback operation is not uniquely determined, since the choice of the spanning subtree  $\tilde{T}$  may not be unique. Nevertheless, independently of this choice, each pullback  $T'$  is a planar embedded tree with  $\text{Crit}(f) \subset V(T')$ . This allows us to iterate the pullback operation and get a sequence  $(T_n)_{n \geq 0}$  of planar embedded trees, where

$$T_0 = T \text{ and } T_{n+1} \in \Pi_f(T_n) \text{ for } n \geq 0.$$

The following proposition implies that the “combinatorial complexity” of the pullbacks  $T_n, n \in \mathbb{N}$ , is uniformly bounded. In the next subsection, we will show that the “topological complexity” of these pullbacks eventually stabilizes.

**Proposition 4.2.** *Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map and  $T$  be a planar embedded tree in  $S^2$  with  $\text{Crit}(f) \subset V(T)$ . Suppose that  $T'$  is a pullback of  $T$  under  $f$ . Then  $|V(T')| \leq 2|\text{Crit}(f)| - 2$ .*

*Proof.* By definition of the pullback operation, each non-critical vertex of  $T'$  has degree greater than 2. Hence the following inequality holds:

$$\sum_{v \in V(T')} \deg_{T'}(v) \geq |\text{Crit}(f)| + 3(|V(T')| - |\text{Crit}(f)|).$$

At the same time, since  $T'$  is a tree, we have

$$\sum_{v \in V(T')} \deg_{T'}(v) = 2|V(T')| - 2.$$

It follows that

$$2|V(T')| - 2 \geq |\text{Crit}(f)| + 3(|V(T')| - |\text{Crit}(f)|),$$

and thus  $|V(T')| \leq 2|\text{Crit}(f)| - 2$ , as desired.  $\square$

*Example 4.3.* Let us consider the critically fixed Thurston map  $f_{\square}$  from Example 3.3 and a planar embedded tree  $T$  with  $V(T) = \text{Crit}(f_{\square})$  shown on the left in Figure 7. The tree  $T$  has three edges colored red, green and blue. The complete preimage  $f_{\square}^{-1}(T)$  is illustrated on the right in Figure 7. The map  $f_{\square}$  sends each edge and vertex of  $f_{\square}^{-1}(T)$  to the edge and

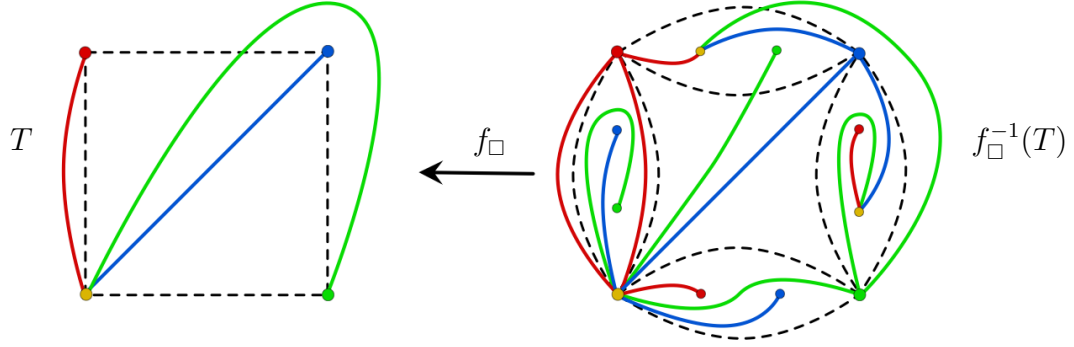


FIGURE 7. Taking the complete preimage of a tree  $T$  with  $V(T) = \text{Crit}(f_\square)$  under the critically fixed Thurston map  $f_\square$ .

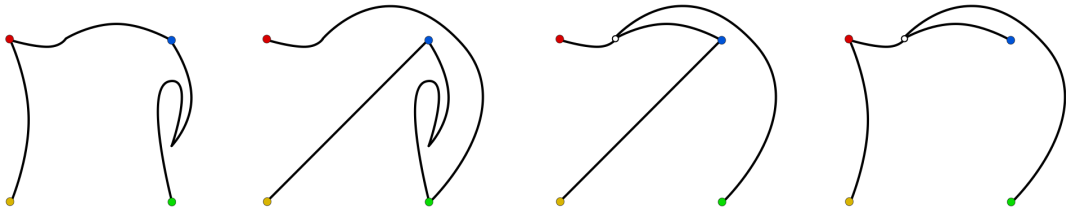


FIGURE 8. Examples of pullbacks of the tree  $T$  from Figure 7 under the map  $f_\square$ .

vertex of the same color. The graphs in dashed lines indicate the charge graph of  $f_\square$  (on the left) and its blow-up (on the right).

Figure 8 illustrates some of the possible pullbacks of the tree  $T$  under the map  $f_\square$ . Note that the first two examples are trees with vertices only in the critical points of  $f_\square$ , while the second two examples are trees with an extra non-critical vertex of degree 3 indicated in white.

**4.2. Topological contraction of the pullback operation.** To control the “topological complexity” of iterated pullbacks of a tree under a critically fixed Thurston map  $f$ , we will use intersection numbers rel.  $\text{Crit}(f)$ . As before, we simply write  $i_f(\cdot, \cdot)$  instead of  $i_{\text{Crit}(f)}(\cdot, \cdot)$ .

**Definition 4.4.** Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map and  $G$  be a planar embedded graph in  $S^2$  with  $\text{Crit}(f) \subset V(G)$ . We define the *norm*  $\|G\|_f$  of  $G$  with respect to the map  $f$  as

$$\|G\|_f := \max_{\alpha \in E(\text{Charge}(f))} i_f(G, \alpha).$$

Our goal is to show the following result.

**Theorem 4.5.** *Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map and  $T_0$  be a planar embedded tree in  $S^2$  with  $V(T_0) \supset \text{Crit}(f)$ . Consider the sequence  $(T_n)_{n \geq 0}$  of planar embedded trees in  $S^2$  obtained by the iterative application of the pullback operation starting with the tree  $T_0$ , that is,  $T_n \in \Pi_f(T_{n-1})$  for each  $n \in \mathbb{N}$ . Then  $\|T_n\|_f = 0$  for  $n \geq \|T_0\|_f$ .*

In other words, after at most  $\|T_0\|_f$  iterations of the pullback operation we obtain a tree that, up to isotopy rel.  $\text{Crit}(f)$ , intersects the charge graph of  $f$  only in critical points (to conclude this from the theorem, use Proposition 2.6). This establishes Main Theorem B.

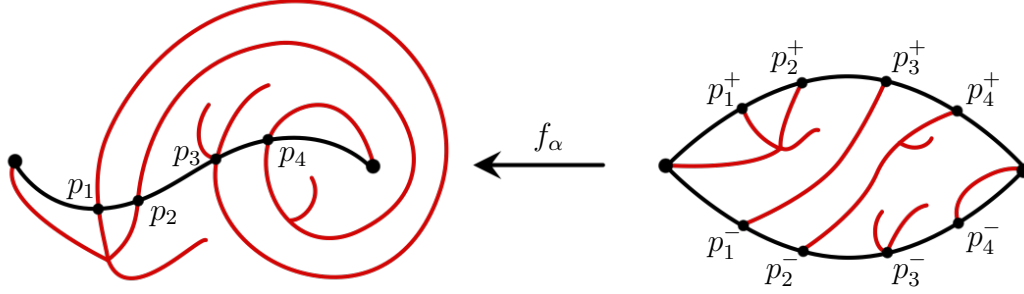


FIGURE 9. Left: a planar embedded tree  $T$  (in red) intersecting an edge  $\alpha$  (in black). Right: the preimage  $f_\alpha^{-1}(T)$  (in red) and  $\partial U_\alpha = \alpha^+ \cup \alpha^-$  (in black).

*Example.* Let us consider the map  $f_\square$  and the tree  $T$  from Example 4.3. One can check that  $\|T\|_f = 1$ , but for every pullback  $T'$  of  $T$  under  $f$  we have  $\|T'\|_f = 0$ .

Theorem 4.5 will easily follow from the next statement.

**Proposition 4.6.** *Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map and  $T$  be a planar embedded tree in  $S^2$ . Then for each edge  $\alpha \in E(\text{Charge}(f))$  we have:*

- (i)  $i_f(f^{-1}(T), \alpha) \leq i_f(T, \alpha)$ ;
- (ii)  $i_f(f^{-1}(T), \alpha) < i_f(T, \alpha)$ , if  $i_f(T, \alpha) > 0$ .

*Proof.* Let  $f$ ,  $T$ , and  $\alpha$  be as in the statement. Since the charge graph of  $f$  is defined up to isotopy, we may assume without loss of generality that the tree  $T$  and the edge  $\alpha \in E(\text{Charge}(f))$  are in minimal position rel.  $\text{Crit}(f)$ . By Proposition 3.8, there are distinct lifts  $\alpha^+, \alpha^-$  of  $\alpha$  under  $f$  and a component  $U_\alpha$  of  $S^2 \setminus (\alpha^+ \cup \alpha^-)$  such that  $\alpha^+, \alpha^-$  are isotopic to  $\alpha$  rel.  $\text{Crit}(f)$  and  $f|_{U_\alpha}: U_\alpha \rightarrow S^2 \setminus \alpha$  is a homeomorphism.

We first show that inequality (i) holds. Since  $f|_{\alpha^+}: \alpha^+ \rightarrow \alpha$  is a homeomorphism, we have

$$|(f^{-1}(T) \cap \alpha^+) \setminus \text{Crit}(f)| = |(T \cap \alpha) \setminus \text{Crit}(f)| = i_f(T, \alpha).$$

At the same time, since  $\alpha^+ \sim \alpha$  rel.  $\text{Crit}(f)$ , we have

$$i_f(f^{-1}(T), \alpha) = i_f(f^{-1}(T), \alpha^+) \leq |(f^{-1}(T) \cap \alpha^+) \setminus \text{Crit}(f)|,$$

which implies the desired inequality.

Suppose now that  $i_f(T, \alpha) > 0$ . By the argument above, to prove (ii) it is sufficient to show that

$$(4.1) \quad i_f(f^{-1}(T), \alpha^+) < |(f^{-1}(T) \cap \alpha^+) \setminus \text{Crit}(f)|$$

or

$$(4.2) \quad i_f(f^{-1}(T), \alpha^-) < |(f^{-1}(T) \cap \alpha^-) \setminus \text{Crit}(f)|$$

In other words, our goal is to show that some intersection between  $f^{-1}(T)$  and either  $\alpha^+$  or  $\alpha^-$  can be removed by an isotopy rel.  $\text{Crit}(f)$ .

Let  $P = \{p_1, p_2, \dots, p_m\}$ ,  $m > 0$ , be the set of intersections between  $\text{int}(\alpha)$  and  $T$  listed from one endpoint of  $\alpha$  to another. If we set  $P^+ := \text{int}(\alpha^+) \cap f^{-1}(T)$  and  $P^- := \text{int}(\alpha^-) \cap f^{-1}(T)$ , we may write  $P^+ = \{p_1^+, p_2^+, \dots, p_m^+\}$  and  $P^- = \{p_1^-, p_2^-, \dots, p_m^-\}$  where  $f(p_i^+) = f(p_i^-) = p_i$  for each  $i = 1, \dots, m$ .

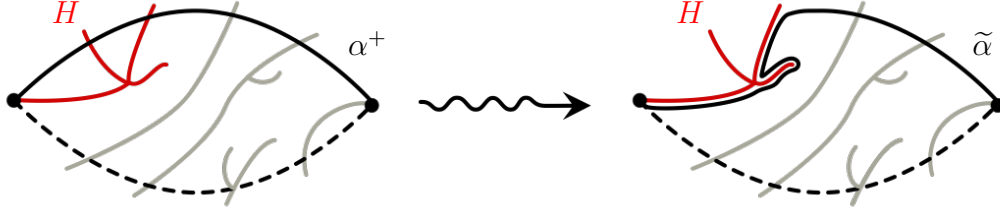


FIGURE 10. Modifying the Jordan arc  $\alpha^+$  into a Jordan arc  $\tilde{\alpha}$  by an isotopy rel.  $\text{Crit}(f)$  to reduce intersections with  $f^{-1}(T)$ .

Consider the preimage  $f_\alpha^{-1}(T)$  of the tree  $T$  under the map  $f_\alpha := f|_{\overline{U_\alpha}: \overline{U_\alpha} \rightarrow S^2}$ . Note that  $f_\alpha^{-1}(T)$  is a planar embedded graph with the vertex set  $f_\alpha^{-1}(V(T)) \cup P^+ \cup P^-$ . Furthermore, each connected component of  $f_\alpha^{-1}(T)$  is a planar embedded tree with at least one vertex on  $\partial U_\alpha = \alpha^+ \cup \alpha^-$ ; see Figure 9 for an illustration.

*Claim.* There exists a connected component  $H$  of  $f_\alpha^{-1}(T)$  such that  $H \cap \text{int}(\alpha^+) = \emptyset$  or  $H \cap \text{int}(\alpha^-) = \emptyset$ .

Indeed, let  $H^+$  and  $H^-$  be the components of  $f_\alpha^{-1}(T)$  such that  $p_1^+ \in H^+$  and  $p_1^- \in H^-$ . Note that  $H^+ \cap H^- = \emptyset$ , because otherwise  $H^+ = H^-$  and  $f_\alpha(H^+) \subset T$  contains a cycle, which leads to a contradiction. If either  $H^+ \cap \text{int}(\alpha^-) = \emptyset$  or  $H^- \cap \text{int}(\alpha^+) = \emptyset$ , then we are done. Otherwise,  $H^+ \cap \{p_2^-, \dots, p_m^-\} \neq \emptyset$  and  $H^- \cap \{p_2^+, \dots, p_m^+\} \neq \emptyset$ ; but then  $H^+ \cap H^- \neq \emptyset$ , which is a contradiction. The claim follows.

Without loss of generality we may assume that there is a connected component  $H$  of  $f_\alpha^{-1}(T)$  with  $H \cap \text{int}(\alpha^-) = \emptyset$ . But since  $\overline{U_\alpha} \cap \text{Crit}(f) = \partial \alpha$ , the Jordan arc  $\alpha^+$  may be isotoped into a Jordan arc  $\tilde{\alpha}$  rel.  $\text{Crit}(f)$  so that

$$|(f^{-1}(T) \cap \tilde{\alpha}) \setminus \text{Crit}(f)| < |(f^{-1}(T) \cap \alpha^+) \setminus \text{Crit}(f)|;$$

see Figure 10 for an illustration. Thus (4.1) holds, which completes the proof of part (ii).  $\square$

The first part of Proposition 4.6 remains true if  $T$  is an arbitrary planar embedded graph in  $S^2$  (in fact, the same proof applies). However, as the next example shows, the assumption that  $T$  is a tree is crucial for the second part of the statement. This explains why using the pullback operation, which extracts a subtree out of a complete preimage, instead of taking the complete preimage itself is essential.

*Example.* Consider the left part of Figure 11. Let  $G$  be the planar embedded graph shown in black dashed lines and  $H$  be the planar embedded graph shown in colored solid lines. Suppose  $f$  is a critically fixed Thurston map obtained by blowing up the pair  $(G, \text{id}_{S^2})$ . The colored lines on the right in Figure 11 indicate a subgraph  $\tilde{H}$  of  $f^{-1}(H)$ . Let  $H'$  be the planar embedded graph in  $S^2$  with the same realization as  $\tilde{H}$  but with vertex set  $V(H') = V(H) = \text{Crit}(f)$ . Note that  $H'$  is isotopic to  $H$  rel.  $\text{Crit}(f)$ , and thus  $i_f(f^{-1}(H), \alpha) = i_f(H, \alpha)$  for each edge  $\alpha \in E(G)$ . This obviously implies that  $\|f^{-n}(H)\|_f = \|H\|_f = 2$  for each  $n \geq 0$ .

*Proof of Theorem 4.5.* Let  $f$  and  $(T_n)_{n \geq 0}$  be as in the statement. Proposition 4.6 implies that for each  $n \geq 0$  we have the inequality  $\|f^{-1}(T_n)\|_f \leq \|T_n\|_f$ , which is strict unless  $\|T_n\|_f = 0$ . At the same time, since  $T_{n+1}$  is a subset of  $f^{-1}(T_n)$ , we have  $\|T_{n+1}\|_f \leq \|f^{-1}(T_n)\|_f$  for each  $n \geq 0$ . Thus the sequence  $\|T_0\|_f, \|T_1\|_f, \dots$  strictly decreases until we reach 0. This finishes the proof of the theorem.  $\square$

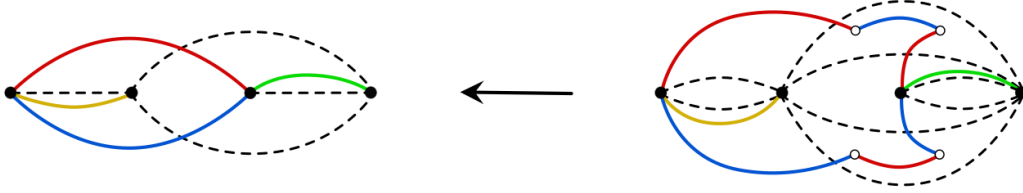


FIGURE 11. Left: a graph  $G$  in dashed black lines and a graph  $H$  in colored solid lines. Right: the blow-up of  $G$  in dashed black lines, and a subgraph  $\tilde{H}$  of  $f^{-1}(H)$  in colored solid lines. The colors indicate onto which edges of  $H$  the edges of  $\tilde{H}$  are mapped under  $f$ .

**4.3. Lifting algorithm.** We are finally ready to describe an algorithm that finds the charge graph of a given critically fixed Thurston map  $f: S^2 \rightarrow S^2$ .

Let  $T_0$  be a planar embedded tree in  $S^2$  with  $\text{Crit}(f) = V(T_0)$ , and suppose that  $T_{n+1} \in \Pi_f(T_n)$  is a pullback of  $T_n$  for every  $n \geq 0$ . By construction, each  $T_n$  is a planar embedded tree in  $S^2$  with  $V(T_n) \supset \text{Crit}(f)$ . Let  $\beta$  be a simple path in  $T_n$  with endpoints in  $\text{Crit}(f)$ , which we view as a Jordan arc in  $S^2$  joining the corresponding endpoints. Note that  $\beta$  may have critical points in its interior. A lift  $\tilde{\beta}$  of  $\beta$  under  $f$  is called *critical* if  $\tilde{\beta}$  is a Jordan arc in  $(S^2, \text{Crit}(f))$ , that is,  $\partial\tilde{\beta} \subset \text{Crit}(f)$  and  $\text{int}(\tilde{\beta}) \cap \text{Crit}(f) = \emptyset$ . By Lemma 3.14, there are exactly  $\max(0, \deg(f, \tilde{\beta}) - 1)$  edges in  $\text{Charge}(f)$  that are isotopic to a critical lift  $\tilde{\beta}$ . At the same time, if  $i_f(\beta, \alpha) = 0$  for some edge  $\alpha \in E(\text{Charge}(f))$  with  $\partial\alpha = \partial\beta$ , then there is a critical lift  $\tilde{\beta}$  of  $\beta$  under  $f$  such that  $\tilde{\beta} \sim \alpha$  rel.  $\text{Crit}(f)$  (see Lemma 3.13). Combining these facts with Theorem 4.5, we propose an algorithm for reconstruction of the charge graph of  $f$ ; see Algorithm 1.

---

**Algorithm 1** Lifting Algorithm

---

**Input:** a critically fixed Thurston map  $f: S^2 \rightarrow S^2$

**Output:** the charge graph  $\text{Charge}(f)$  of the map  $f$

- 1: set  $\mathcal{E} := \emptyset$  and  $n := 0$
  - 2: choose any planar embedded tree  $T_0$  in  $S^2$  with  $V(T_0) = \text{Crit}(f)$
  - 3: **while**  $|\mathcal{E}| \leq \deg(f) - 1$  **do**
  - 4:     **for** every simple path  $\beta$  in  $T_n$  with endpoints in  $\text{Crit}(f)$  **do**
  - 5:         **for** every critical lift  $\tilde{\beta}$  of  $\beta$  under  $f$  **do**
  - 6:             **if** there is no arc in  $\mathcal{E}$  isotopic to  $\tilde{\beta}$  rel.  $\text{Crit}(f)$  **then**
  - 7:                 add  $\max(0, \deg(f, \tilde{\beta}) - 1)$  edges isotopic to  $\tilde{\beta}$  rel.  $\text{Crit}(f)$  to  $\mathcal{E}$  so that
  - 8:                 all edges in  $\mathcal{E}$  have pairwise disjoint interiors
  - 9:             **end if**
  - 10:         **end for**
  - 11:     **end for**
  - 12:     take  $T_{n+1} \in \Pi_f(T_n)$
  - 13:      $n \leftarrow n + 1$
  - 14: **end while**
  - 15: **return**  $(\text{Crit}(f), \mathcal{E})$
-



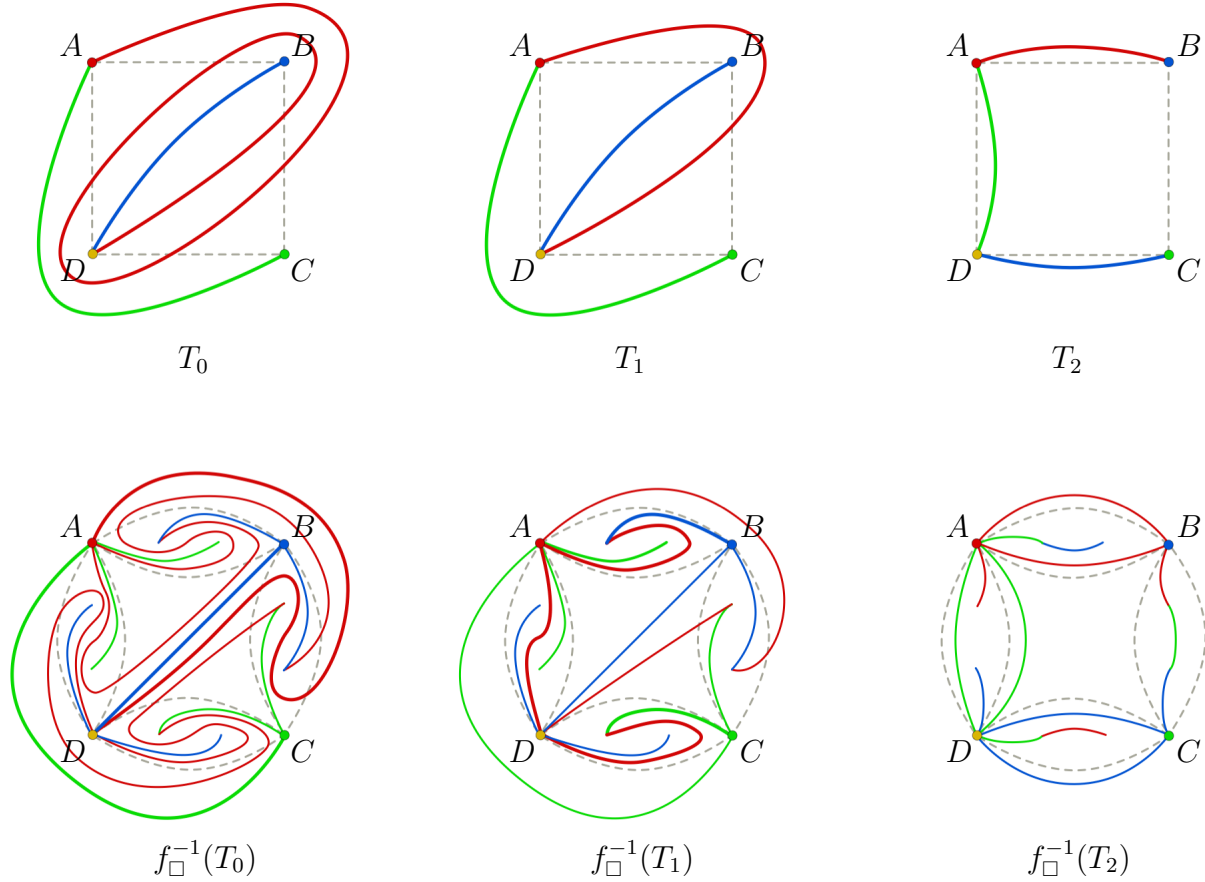


FIGURE 12. Iterations of the Lifting Algorithm applied to the critically fixed Thurston map  $f_{\square}$ .

**Theorem 4.7.** *Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map and  $T_0$  be an initial planar embedded tree in  $S^2$  with  $V(T_0) = \text{Crit}(f)$ . Then Algorithm 1 stops after taking at most  $\|T_0\|_f$  pullback operations and returns the charge graph of  $f$ .*

*Proof.* Theorem 4.5 implies that after taking at most  $\|T_0\|_f$  iterations of the pullback operation, we obtain a planar embedded tree  $T'$  with  $\text{Crit}(f) \subset V(T')$  and  $\|T'\|_f = 0$ . The statement now follows from the discussion above.  $\square$

*Remark.* Note that the running of Algorithm 1 significantly depends on the choice of the starting tree  $T_0$  and the choices of the pullbacks  $T_{n+1} \in \Pi_f(T_n)$  for  $n \geq 0$ . Nevertheless, the trees  $T_n$  have uniformly bounded combinatorial complexity. More precisely, Proposition 4.2 implies that  $|V(T_n)| \leq 2|\text{Crit}(f)| - 2$ , so  $|E(T_n)| \leq 2|\text{Crit}(f)| - 3$  for each  $n \geq 0$ .

*Example 4.8.* To illustrate Algorithm 1, we apply it to the critically fixed Thurston map  $f_{\square}$  from Example 3.3.

We start with the tree  $T_0$  shown in color on the top left in Figure 12. The top part of the figure illustrates (up to isotopy rel.  $\text{Crit}(f_{\square})$ ) the trees  $T_1 \in \Pi_f(T_0)$  and  $T_2 \in \Pi_f(T_1)$  constructed during the algorithm. The bottom part of the figure shows the corresponding complete preimages. Here, as usual, the map  $f_{\square}$  sends each edge of  $f_{\square}^{-1}(T_n)$  to the edge of  $T_n$

of the same color. Slightly thicker colored edges in  $f_{\square}^{-1}(T_0)$  and  $f_{\square}^{-1}(T_1)$  indicate the choice of the pullbacks  $T_1$  and  $T_2$ , respectively. The graphs in dashed lines illustrate the charge graph of  $f_{\square}$  (on the top pictures) and its blow-up (on the bottom pictures).

We now describe how the set  $\mathcal{E}$  changes during each iteration of the algorithm; see Figure 12 as a reference.

- (1) On the first iteration, we discover the edges  $AB$  and  $CD$  of the charge graph  $G_{\square}$ .
- (2) On the second iteration, we discover the edge  $AD$  of  $G_{\square}$ . Note that the simple paths  $(A, D, B)$  and  $(C, A, D)$  in  $T_1$  also have critical lifts that blow up under  $f_{\square}$ , but they provide us only with the edges  $AB$  and  $CD$  of  $G_{\square}$  obtained on the previous iteration.
- (3) On the third iteration, we discover the last edge  $BC$  of  $G_{\square}$  and the algorithm stops.

Note that  $\|T_0\|_f = 2$ , thus the bound in Theorem 4.7 is sharp.

We close this section by noting that we may combinatorially encode a homeomorphism from the admissible pair by knowing only the charge graph of the map.

Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map and  $G$  be the charge graph of  $f$ . By Theorem 3.29(i),  $f$  is isotopic to a map  $g: S^2 \rightarrow S^2$  obtained by blowing up some admissible pair  $(G, \varphi_g)$ . Proposition 3.8 now implies that we may find triples  $(\alpha^+, \alpha^-, U_{\alpha})$ ,  $\alpha \in E(G)$ , that satisfy conditions (D1)-(D3) and (E1)-(E3) with respect to  $f$ . Note that these triples are uniquely determined by the map  $f$  and the graph  $G$  when  $|\text{Crit}(f)| > 2$ : for every edge  $\alpha \in E(G)$  there is a unique pair  $\alpha^+, \alpha^-$  of its lifts that satisfies conditions (D1)-(D3).

Let  $H$  be a connected graph with  $V(H) = V(G) = \text{Crit}(f)$  and  $H \cap \overline{U_{\alpha}} \in \{\alpha^+, \alpha^-\}$  for each  $\alpha \in E(G)$ . Conditions (D1) and (E3) imply that  $f$  sends  $H$  homeomorphically onto its image. The map  $f|_H$  may be continuously extended to a homeomorphism  $\varphi_f \in \text{Homeo}^+(S^2, V(H))$ . This easily follows from [Hlu17, Proposition 3.4.3] since  $f$  preserves the cyclic order of edges incident to every vertex of  $H$ . Alternatively, since  $f|_H: H \rightarrow f(H)$  is a graph map, we may use the criterion [BFH92, Proposition 6.4] to check that  $f|_H$  admits a regular extension, which then must be a homeomorphism. Note also that  $\varphi_f$  is unique up to isotopy rel.  $V(H)$  (see Proposition 2.16).

The following statement implies that the pair  $(G, \varphi_f)$  corresponds to the map  $f$ .

**Proposition 4.9.** *The pair  $(G, \varphi_f)$  is admissible. Moreover, the homeomorphisms  $\varphi_f$  and  $\varphi_g$  are isotopic rel.  $V(G)$ .*

*Proof.* It is sufficient to show that  $\varphi_f$  and  $\varphi_g$  are isotopic rel.  $V(G)$ . Suppose  $f = g \circ \psi$  for some  $\psi \in \text{Homeo}_0^+(S^2, \text{Crit}(f))$ . Note that  $G_f^{\pm} = \psi^{-1}(G_g^{\pm})$ , where  $G_f^{\pm} = \bigcup_{\alpha \in E(G)} (\alpha^+ \cup \alpha^-)$  and  $G_g^{\pm}$  are the blow-ups of  $G$  with respect to  $f$  and  $g$ , respectively (see Proposition 3.8).

We claim that  $\varphi_f(e)$  and  $\varphi_g(e)$  are isotopic rel.  $V(G)$  for every  $e \in E(H)$ .

*Case 1.* Suppose  $e \in \{\alpha^+, \alpha^-\}$  for some  $\alpha \in E(G)$ . Then  $\varphi_f(e) = f(e) = \alpha$ . At the same time,  $\varphi_g(e) \sim \varphi_g(\alpha)$  rel.  $V(G)$ , because  $e \sim \alpha$  rel.  $V(G)$ , and  $\varphi_g(\alpha) \sim \alpha$  rel.  $V(G)$ , because the pair  $(G, \varphi_g)$  is admissible. Thus  $\varphi_f(e) \sim \varphi_g(e)$  rel.  $V(G)$ .

*Case 2.* Suppose  $e \notin \{\alpha^+, \alpha^-\}$  for every  $\alpha \in E(G)$ . By (E3),  $(g \circ \psi)(e) = f(e) \sim \varphi_f(e)$  rel.  $V(G)$ . By construction,  $\text{int}(\psi(e)) \subset S^2 \setminus \bigcup_{\alpha \in E(G)} \psi(\overline{U_{\alpha}})$ . We get  $(g \circ \psi)(e) \sim \varphi_g(\psi(e)) \sim \varphi_g(e)$  rel.  $V(G)$ , where the first equivalence follows from (E3) and the second one from  $\psi \in \text{Homeo}_0^+(S^2, \text{Crit}(f))$ . Therefore,  $\varphi_f(e) \sim \varphi_g(e)$  rel.  $V(G)$  as claimed.

The statement now follows from Corollary 2.17. □

## 5. TWISTING PROBLEM

Let  $f: S^2 \rightarrow S^2$  be a Thurston map and  $\varphi \in \text{Homeo}^+(S^2, \text{Post}(f))$  be a homeomorphism. Consider the map  $g := \varphi \circ f: S^2 \rightarrow S^2$ , called the *twist of  $f$  by  $\varphi$*  (or simply a *twisted map*). Note that  $g$  is a branched covering map on  $S^2$  with  $\deg(g) = \deg(f)$  and  $\text{Crit}(g) = \text{Crit}(f)$ . Moreover,  $f$  and  $g$  have the same dynamics on the critical set, so  $g$  is a Thurston map. In particular, if  $f$  is critically fixed, then the twisted map  $g$  is critically fixed as well. By the *twisting problem* we mean the problem of determining the combinatorial equivalence class of the twisted map  $g = \varphi \circ f$ , knowing the maps  $f$  and  $\varphi$ .

Since we are interested in  $g$  only up to combinatorial equivalence, we may consider  $f$  and  $\varphi$  up to isotopy rel.  $\text{Post}(f)$ . In particular, we may treat  $\varphi$  as an element of  $\text{PMCG}(S^2, \text{Post}(f))$ . Recall that  $\text{PMCG}(S^2, \text{Post}(f))$  is generated by finitely many Dehn twists; see, for example, [FM12, Theorems 4.9 and 4.11]. Thus, understanding the case when  $\varphi = T_\gamma^n$ , where  $n \in \mathbb{Z}$  and  $T_\gamma$  is the Dehn twist about an essential Jordan curve  $\gamma$  on  $(S^2, \text{Post}(f))$ , has been of particular interest.

The twisting problem for polynomial maps has been sufficiently well studied in the last decade [BN06, KL19, BLMW22, MW22, LW22]. However, in the non-polynomial case, the problem has been previously considered only for rational maps of low degree with four postcritical points [Lod13, KL19].

In this section, we address the twisting problem for the family of critically fixed Thurston maps. We start though by briefly discussing the principal previous work in the polynomial case; see [BLMW22] for a more thorough overview.

The first instance of the twisted problem was posed by John Hubbard in the 1980s. Namely, let us consider the quadratic polynomials of the form  $z^2 + c$  for which the critical point 0 is 3-periodic. There are exactly three such polynomials, called the *rabbit polynomial*  $p_R$  (with  $c \approx -0.1225 + 0.7448i$ ), the *co-rabbit polynomial*  $p_C$  (with  $c \approx -0.1225 - 0.7448i$ ), and the *airplane polynomial*  $p_A$  (with  $c \approx -1.7548$ ). Take the rabbit polynomial  $p_R$  and postcompose it with (an iterate of) the Dehn twist  $T_\gamma$  about a Jordan curve  $\gamma$  surrounding the postcritical points  $p_R(0)$  and  $p_R^2(0)$  and staying in the upper half-plane. The Levy-Bernstein theorem [Hub16, Theorem 10.3.9] implies that the twisted map  $T_\gamma^n \circ p_R$  is realized for every  $n \in \mathbb{Z}$ , and thus it must be combinatorially equivalent to precisely one of the polynomials  $p_R$ ,  $p_C$ , and  $p_A$ . The problem of understanding the combinatorial class of  $T_\gamma^n \circ p_R$  (in terms of  $n \in \mathbb{Z}$ ) is called *Hubbard's twisted rabbit problem*.

Laurent Bartholdi and Volodymyr Nekrashevych solved the twisted rabbit problem in [BN06] using a novel algebraic machinery provided by *iterated monodromy groups*. Quite unexpectedly, the answer depends on the 4-adic expansion of the power  $n$  of the twist. A similar algebraic approach can be applied to the twisting problem for some other polynomial and non-polynomial maps. For example, twisting questions for  $z^2 + i$  and  $\frac{3z^2}{2z^3 + 1}$  are considered in [BN06] and [Lod13], respectively. We remark that in these cases the twisted maps may be obstructed.

Recently, an alternative solution to the twisting problem for postcritically-finite polynomials was proposed by James Belk, Justin Lanier, Dan Margalit, and Rebecca R. Winarski [BLMW22]. Similarly to the work of Bartholdi-Nekrashevych, the solution is algorithmic, but it is based on ideas from combinatorial topology to a greater extent; see [BLMW22] for a comparison of the two methods.

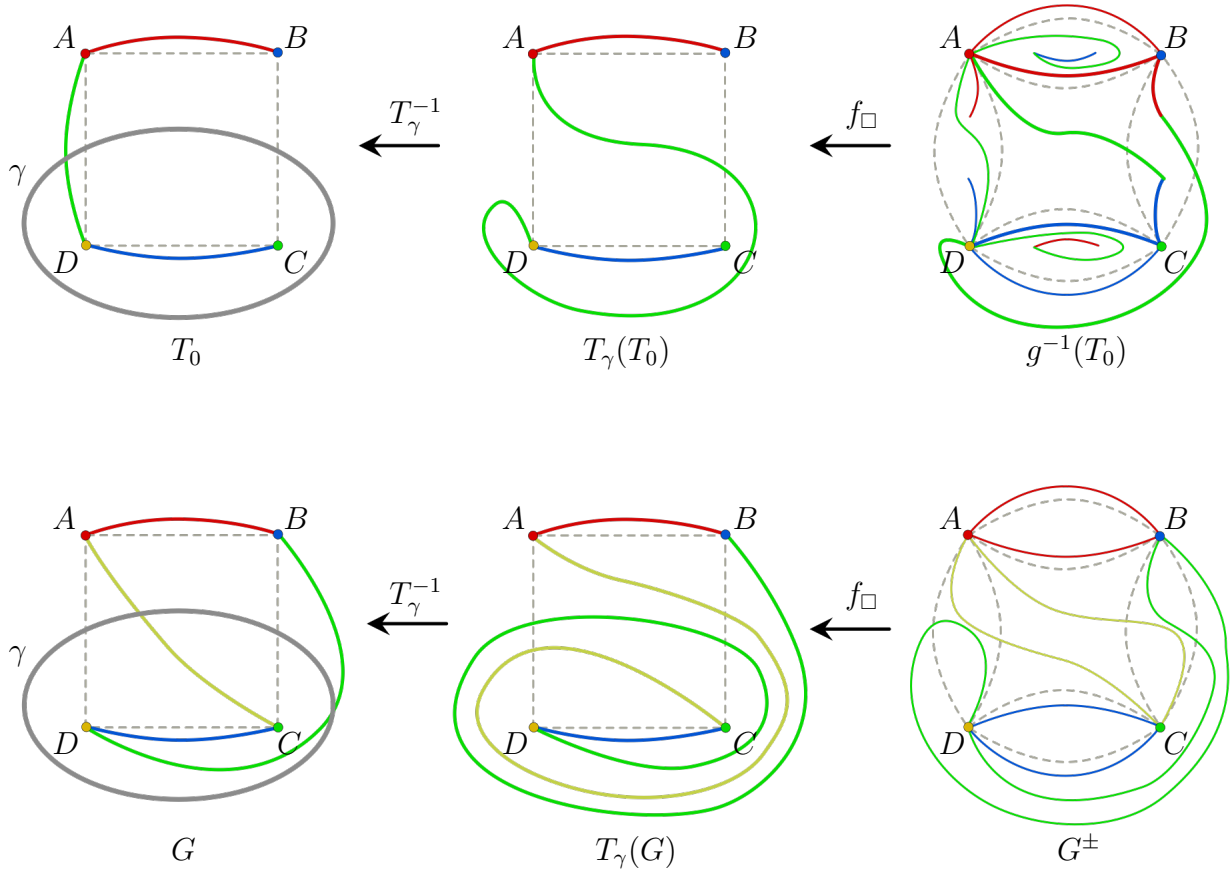


FIGURE 13. Finding the combinatorial class of the map  $g = T_\gamma^{-1} \circ f_\square$ .

The classification results and the lifting algorithm we developed in the preceding sections naturally suggest an algorithmic solution to the twisting problem for the family of critically fixed Thurston maps.

*Example 5.1.* Let us show how using Algorithm 1 one can find the combinatorial class of the twisted map  $g := T_\gamma^{-1} \circ f_\square$ , where  $\gamma$  is the curve in gray on the top left picture in Figure 13. Here, the graph in dashed lines represents the charge graph  $G_\square$ . We run Algorithm 1 for the map  $g$  starting with the tree  $T_0$  shown in color on the top left picture in the figure.

During the first iteration of the algorithm, it will find that there are critical lifts of the paths  $(A, B)$ ,  $(C, D)$ ,  $(A, D, C)$ , and  $(B, A, D)$  in the tree  $T_0$  providing four different edges of the charge graph of the map  $g$ . Examples of such lifts are shown in slightly thicker colored lines on the top right picture in Figure 13, which illustrates the full preimage  $g^{-1}(T_0)$ . Here, the graph in dashed lines represents the blow-up of  $G_\square$ . Furthermore,  $f_\square$  sends each edge of  $g^{-1}(T_0)$  to the edge of the same color in the tree  $T_\gamma(T_0)$  on the top middle picture. Similarly,  $T_\gamma^{-1}$  sends each edge of  $T_\gamma(T_0)$  to the edge of the tree  $T_0$  of the same color.

The bottom part of Figure 13 verifies that the found lifts indeed blow up under the twisted map  $g$ . Namely, the picture on the left shows a graph  $G$  composed of these lifts (up to isotopy rel.  $\text{Crit}(f_\square)$ ); the middle picture shows  $T_\gamma(G)$ ; and the picture on the right illustrates the blow-up  $G^\pm$  of  $G$  with respect to the map  $g$ .

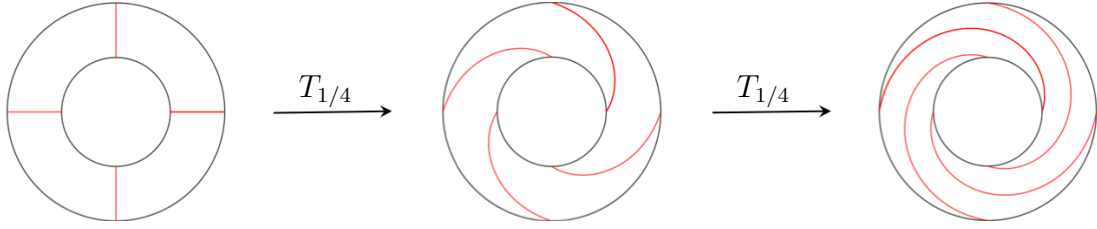


FIGURE 14. The action of the 1/4-twist on radial arcs.

It follows that Algorithm 1 stops after the very first iteration. Note that the charge graph  $G_{\square}$  of  $f$  and the charge graph  $G$  of  $g = T_{\gamma}^{-1} \circ f$  are connected and isomorphic to each other. Therefore, the maps  $f_{\square}$  and  $T_{\alpha}^{-1} \circ f_{\square}$  are combinatorially equivalent by Proposition 3.20, even though they are not isotopic.

The main goal of this section is to address some special instances of the twisting problem for the family of critically fixed Thurston maps  $f$  obtained by blowing up a pair  $(G, \text{id}_{S^2})$ . Note that up to isotopy this family includes all critically fixed rational maps (see Proposition 3.17). We will show that for some special Jordan curves  $\gamma$  in  $(S^2, \text{Crit}(f))$  the combinatorial class of the twisted map  $T_{\gamma}^n \circ f$ ,  $n \in \mathbb{Z}$ , can be obtained by applying a simple combinatorial operation to the charge graph of the initial map  $f$ .

**5.1. Graph rotation.** Our goal now is to introduce a simple combinatorial operation on planar embedded graphs that will allow us to describe the action of (special) Dehn twists on critically fixed rational maps.

Consider the map  $T_{n/m}: \partial\mathbb{D} \times \mathbb{I} \rightarrow \partial\mathbb{D} \times \mathbb{I}$ , where  $n \in \mathbb{Z}$  and  $m \in \mathbb{N}$ , defined as

$$T_{n/m}(e^{i\theta}, x) = \left( e^{i(\theta + 2\pi x \frac{n}{m})}, x \right).$$

The map  $T_{n/m}$  is called the  $n/m$ -twist on the cylinder  $\partial\mathbb{D} \times \mathbb{I}$ . Note that  $T_{n/m}$  fixes pointwise the boundary circle  $\partial\mathbb{D} \times \{0\}$ , and “rotates” the boundary circle  $\partial\mathbb{D} \times \{1\}$  counterclockwise by angle  $2\pi \frac{n}{m}$ . Furthermore,  $T_{n/m} = (T_{1/m})^n$  for all  $n \in \mathbb{Z}$  and  $m \in \mathbb{N}$ . Figure 14 illustrates the action of the 1/4-twist (here, the cylinder  $\partial\mathbb{D} \times \mathbb{I}$  is viewed as an annulus in  $\mathbb{C}$  under the embedding  $(e^{i\theta}, t) \mapsto e^{i\theta}(t + 1)$ ).

Let  $(S^2, Z)$  be a finitely marked sphere and  $G$  be a planar embedded graph in  $S^2$  with  $G \cap Z = V(G)$ . Suppose  $\gamma$  is a Jordan curve in  $(S^2, Z)$  such that  $|\gamma \cap e| \leq 1$  for each edge  $e \in E(G)$ . We also assume that  $i_Z(G, \gamma) = |G \cap \gamma| \geq 1$ , and denote by  $\beta_1, \dots, \beta_m$  all the edges of  $G$  that  $\gamma$  intersects. We label these edges in the order they are met by  $\gamma$  (for some chosen basepoint and orientation on  $\gamma$ ).

Let us thicken the curve  $\gamma$  to a (small) closed annulus  $A \subset S^2 \setminus Z$  so that  $A \setminus G$  has exactly  $m$  components. Choose an orientation-preserving homeomorphism  $\phi: \partial\mathbb{D} \times \mathbb{I} \rightarrow A$  so that  $\phi(r_{j,m}) = \beta_j \cap A$  for each  $j = 1, \dots, m$ , where  $r_{j,m} := \left\{ \left( e^{2\pi i \frac{j-1}{m}}, x \right) : x \in \mathbb{I} \right\}$  are radial arcs in the cylinder  $\partial\mathbb{D} \times \mathbb{I}$  subdividing it into  $m$  congruent pieces.

Consider now the map  $T_{n/m, \phi}: S^2 \rightarrow S^2$  defined as

$$T_{n/m, \phi}(p) = \begin{cases} (\phi \circ T_{n/m} \circ \phi^{-1})(p) & \text{if } p \in A \\ p & \text{if } p \in S^2 \setminus A. \end{cases}$$

We call  $T_{n/m, \phi}$  the  $n/m$ -twist of  $A$  with respect to  $\phi$ .

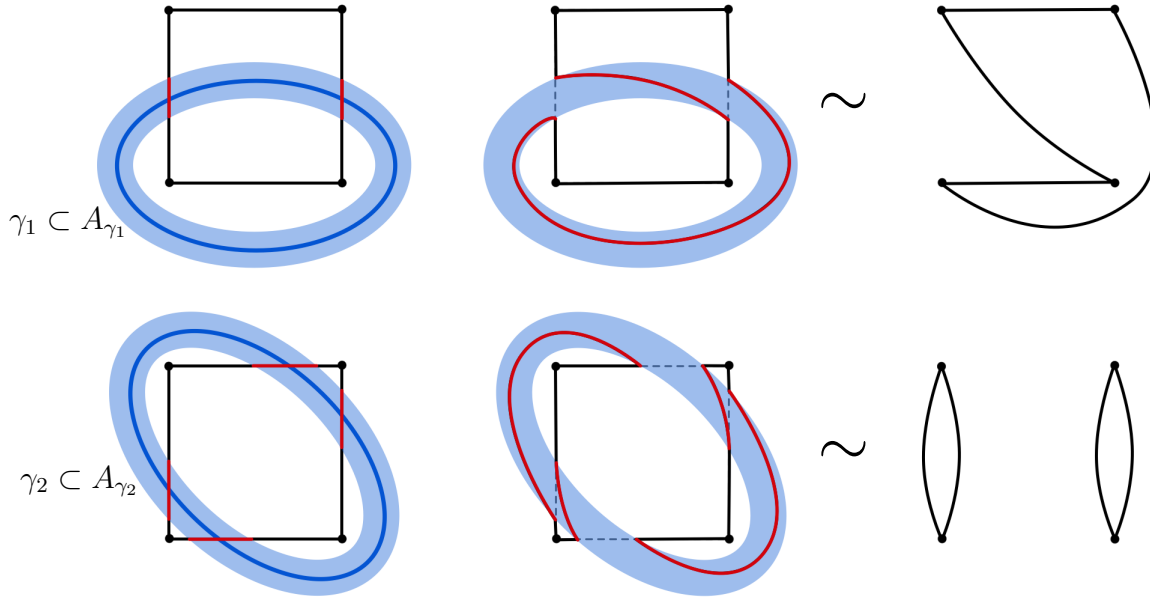


FIGURE 15. The 1-rotations of  $G_{\square}$  about the curves  $\gamma_1$  (top) and  $\gamma_2$  (bottom).

Note that  $T_{n/m,\phi}$  is a homeomorphism of  $S^2$  if and only if  $n/m \in \mathbb{Z}$ . In fact, the map  $T_{n/m,\phi}$  should be thought of as a “fractional” Dehn twist: when  $q := n/m \in \mathbb{Z}$ ,  $T_{n/m,\phi}$  coincides (up to isotopy rel.  $Z$ ) with the  $q$ th iterate  $T_{\gamma}^q$  of the Dehn twist  $T_{\gamma}$  about the curve  $\gamma$ . For  $n/m \notin \mathbb{Z}$ , the map  $T_{n/m,\phi}$  fixes one of the boundary curves of  $A$  and “rotates” the other one to the left when viewed from the inside of  $A$ .

Consider the image  $G' := T_{n/m,\phi}(G)$  of the graph  $G$  under the  $n/m$ -twist of  $A$  with respect to  $\phi$ . It is easy to see that  $G'$  may be viewed as a planar embedded graph in  $S^2$  with the vertex set  $V(G)$ . Note that then  $T_{n/m,\phi}$  modifies only the edges  $\beta_1, \dots, \beta_m$  of  $G$  and keeps all other edges fixed. One can check that, up to isotopy rel.  $Z$ , the graph  $G'$  is uniquely defined independently of the choice of  $A$  and  $\phi$ .

**Definition 5.2.** The graph  $G'$  constructed as above is called the  $n$ -rotation of the graph  $G$  about the curve  $\gamma$ .

*Example 5.3.* Figure 15 illustrates the 1-rotations of the graph  $G_{\square}$  about two Jordan curves  $\gamma_1$  and  $\gamma_2$ . Here, the pictures in the left column indicate the chosen annuli  $A_{\gamma_1}$  and  $A_{\gamma_2}$  around  $\gamma_1$  and  $\gamma_2$ , respectively. The red arcs correspond to the intersections of the annuli with the graph. In the middle column, we see the images of  $G_{\square}$  under the the  $1/2$ -twist of  $A_{\gamma_1}$  (top) and the  $1/4$ -twist of  $A_{\gamma_2}$  (bottom), that is, the 1-rotations of  $G_{\square}$  about  $\gamma_1$  and  $\gamma_2$ , respectively. The red arcs indicate the modifications of the edges. Finally, the right column shows the same graphs after simplification by isotopy rel.  $V(G_{\square})$ .

In analogy with the usual Dehn twists, we have the following statement.

**Proposition 5.4.** *Let  $(S^2, Z)$  be a finitely marked sphere. For  $j = 1, 2$ , suppose that  $G_j$  is a planar embedded graph in  $S^2$  with  $G_j \cap Z = V(G_j)$ , and  $\gamma_j$  is a Jordan curve in  $(S^2, Z)$  with  $i_Z(G_j, \gamma_j) = |G_j \cap \gamma_j| \geq 1$  and  $|e_j \cap \gamma_j| \leq 1$  for each  $e_j \in E(G_j)$ .*

*If  $G_1$  is isotopic to  $G_2$  and  $\gamma_1$  is isotopic to  $\gamma_2$  rel.  $Z$ , then the  $n$ -rotations of  $G_1$  about  $\gamma_1$  and of  $G_2$  about  $\gamma_2$  are isotopic rel.  $Z$  for all  $n \in \mathbb{Z}$ .*

**5.2. Simple transversals and their properties.** Let  $G$  be a planar embedded graph in  $S^2$ . We say that an essential Jordan curve  $\gamma$  in  $(S^2, V(G))$  is a *simple transversal with respect to  $G$*  if it satisfies the following conditions:

- (i)  $i_{V(G)}(G, \gamma) = |G \cap \gamma|$ , that is,  $G$  and  $\gamma$  are in minimal position;
- (ii)  $|e \cap \gamma| \leq 1$  for each edge  $e \in E(G)$ .

**Lemma 5.5.** *Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map obtained by blowing up a pair  $(G, \text{id}_{S^2})$  and  $\gamma$  be a simple transversal with respect to  $G$ . Then the following are true:*

- (i) *There exists a unique component  $\gamma'$  of  $f^{-1}(\gamma)$  that is isotopic to  $\gamma$  rel.  $V(G)$ . Moreover,  $\deg(f|\gamma') = i_{V(G)}(G, \gamma) + 1$ .*
- (ii) *All other components  $\delta'$  of  $f^{-1}(\gamma)$  are null-homotopic in  $(S^2, V(G))$  and satisfy  $\deg(f|\delta') = 1$ .*

*Proof.* The lemma easily follows from the definition of the blow-up operation; see (3.1). Indeed, let  $f$  and  $\gamma$  be as in the statement. In particular, we fix a choice of  $W_e$ ,  $D_e$ ,  $f_e$ , and  $h$  as in Section 3.1.

We assume below that  $m := i_{V(G)}(G, \gamma) \geq 2$ ; the proof can be easily adapted for the remaining two cases. Let  $\beta_1, \dots, \beta_m$  be all the edges of  $G$  that  $\gamma$  intersects. We label these edges in the order they are met by  $\gamma$  (for some chosen basepoint and orientation on  $\gamma$ ). Then the Jordan curve  $\gamma$  is broken into  $m$  consecutive Jordan arcs  $\gamma_1, \gamma_2, \dots, \gamma_m$  having endpoints  $x_j \in \beta_j$  and  $x_{j+1} \in \beta_{j+1}$  for each  $j = 1, 2, \dots, m$ . Here and further all indices are considered modulo  $m$ .

By (3.1),  $f^{-1}(\gamma) \cap D_{\beta_j}$  is a Jordan arc  $\gamma_j^\pm$  connecting two preimages  $x_j^- \in \partial D_{\beta_j}^-$  and  $x_j^+ \in \partial D_{\beta_j}^+$  of  $x_j$  under  $f$ . Moreover, up to relabeling, we may assume that  $x_j^+$  and  $x_{j+1}^-$  are connected by a lift  $\gamma'_j \subset S^2 \setminus \bigcup_{j=1}^m \text{int}(D_j)$  of  $\gamma_j$  under  $f$ . The concatenation of the arcs  $\gamma_1^\pm, \gamma'_1, \gamma_2^\pm, \gamma'_2, \dots, \gamma_m^\pm, \gamma'_m$  is a Jordan curve  $\gamma'$ . Moreover,  $f|\gamma': \gamma' \rightarrow \gamma$  is a covering of degree  $m + 1$ . We may now modify  $h$  within  $\bigcup_{j=1}^m \text{int}(D_j)$  so that  $h$  realizes a homotopy between  $\gamma'$  and  $\gamma$  in  $S^2 \setminus V(G)$ . Hence  $\gamma'$  and  $\gamma$  are isotopic rel.  $V(G)$ ; see, for example, [FM12, Proposition 1.10].

Finally, if  $\delta'$  is a component of  $f^{-1}(\gamma)$  different from  $\gamma'$ , then  $\delta' \subset \text{int}(D_e)$  for some  $e \in E(G) \setminus \{\beta_1, \beta_2, \dots, \beta_m\}$ . Hence,  $\delta'$  is null-homotopic in  $(S^2, V(G))$  and  $\deg(f|\delta') = 1$ . This completes the proof of the lemma.  $\square$

*Remark.* We note that simple transversals with respect to the charge graph of a critically fixed rational map  $f$  correspond exactly to the essential curves in the *global curve attractor* of  $f$ ; see [Hlu19, Proposition 10] and Lemma 5.5. In particular, for every Jordan curve  $\gamma$  in  $(S^2, \text{Crit}(f))$  there exists  $n \in \mathbb{N}$  such that each component of  $f^{-n}(\gamma)$  is either non-essential or isotopic rel.  $\text{Crit}(f)$  to a simple transversal with respect to  $\text{Charge}(f)$ . In fact, the same statement is true for all critically fixed Thurston maps obtained by blowing up a pair  $(G, \text{id}_{S^2})$ .

**5.3. Twists about simple transversals.** We now look at a special instance of the twisting problem. Namely, we consider a critically fixed Thurston map  $f$  obtained by blowing up a pair  $(G, \text{id}_{S^2})$  and (iterates of) Dehn twists  $T_\gamma$  about simple transversals  $\gamma$  with respect to  $G$ . We are going to describe the combinatorial classes of the twisted maps using the graph rotation operation introduced in Section 5.1.

**Proposition 5.6.** *Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map obtained by blowing up a pair  $(G, \text{id}_{S^2})$  and  $\gamma$  be a simple transversal with respect to  $G$  with  $|G \cap \gamma| \geq 1$ . Then*

$T_\gamma^{-1} \circ f$  is isotopic to the map obtained by blowing up the pair  $(H, \text{id}_{S^2})$ , where  $H$  is the 1-rotation of  $G$  about the curve  $\gamma$ . In particular,  $H$  is the charge graph of  $g$ .

*Proof.* Suppose  $f$  and  $\gamma$  are as in the statement. In particular, we fix a choice of  $W_e, D_e, f_e$ , and  $h$  as in Section 3.1. Then  $V(G) = \text{Crit}(f)$  and  $\deg(f) = |E(G)| + 1$ . Set  $m := i_{V(G)}(G, \gamma)$ , and let  $A$  be a (small) closed annulus in  $S^2 \setminus V(G)$  obtained by thickening the curve  $\gamma$  so that  $A \setminus G$  has exactly  $m$  components.

Let us denote by  $H$  the 1-rotation of  $G$  about the curve  $\gamma$  realized by a  $1/m$ -twist  $T_{1/m, \phi}$  of  $A$  with respect to some orientation-preserving homeomorphism  $\phi: \partial\mathbb{D} \times \mathbb{I} \rightarrow A$  as in Section 5.1. Without loss of generality, we may also assume that the Dehn twist  $T_\gamma$  is defined with respect to this homeomorphism  $\phi$ , so that  $T_\gamma$  is the identity on  $S^2 \setminus \text{int}(A)$ . We are going to show that each edge  $\alpha \in E(H)$  of  $H$  has a triple  $(\alpha^+, \alpha^-, U_\alpha)$  satisfying conditions (D1)-(D3) with respect to the map  $g := T_\gamma^{-1} \circ f$ . It would then follow from Proposition 3.9 and Theorem 3.29(ii) that  $H$  is the charge graph of  $g$ .

By adjusting  $D_e, f_e, h$  (which does not change the isotopy class of  $f$ ), we may assume the following:

- $D_e \cap A = \emptyset$  if  $e \cap A = \emptyset$  (i.e., if  $e \cap \gamma = \emptyset$ );
- $A$  is a component of  $f^{-1}(A)$ , so that  $f|_A: A \rightarrow A$  is a self-cover of degree  $m + 1$  (cf. Lemma 5.5).

We are going to set up now some notation. Suppose  $X$  and  $Y$  are the two components of  $S^2 \setminus \text{int}(A)$ , where  $X$  is the component with  $\partial X = \phi(\partial\mathbb{D} \times \{0\})$  and  $Y$  is the component with  $\partial Y = \phi(\partial\mathbb{D} \times \{1\})$ . Then  $T_{1/m, \phi}$  fixes  $\partial X$  pointwise, and it rotates  $\partial Y$  to the left when viewed from the inside of  $A$ .

Let  $E^A(G) := \{\beta_1, \dots, \beta_m\}$  be the set of all edges of  $G$  that intersect  $A$ . These edges subdivide the annulus  $A$  into  $m$  closed components  $A_1, \dots, A_m$ . We label these edges and components so that they are met in the order

$$\beta_1, A_1, \beta_2, A_2, \dots, \beta_m, A_m$$

when we walk around the Jordan domain  $X$  in the counter-clockwise direction. For a reference, see the top middle picture in Figure 16. Here, the graph  $G$  and the annulus  $A$  are drawn in black and blue, respectively.

The arcs  $\beta_j^+ := \partial D_{\beta_j}^+$  and  $\beta_j^- := \partial D_{\beta_j}^-$ ,  $j = 1, \dots, m$ , subdivide the annulus  $A$  into  $2m$  closed components  $A_1^\pm, A'_1, \dots, A_m^\pm, A'_m$  so that  $A_j^\pm = A \cap D_{\beta_j}^\pm$  and  $A'_j \subset A_j$  for each  $j$ . Up to relabeling, we may assume that these arcs and components are met in the order

$$\beta_1^-, A_1^\pm, \beta_1^+, A'_1, \beta_2^-, A_2^\pm, \beta_2^+, A'_2, \dots, \beta_m^-, A_m^\pm, \beta_m^+, A'_m$$

when we walk around  $X$  in the counter-clockwise direction; see the top right picture in Figure 16 for a reference.

Let us decompose each edge  $\beta_j$ ,  $j = 1, \dots, m$ , as the union

$$\beta_j = \beta_{j,X} \cup \beta_{j,A} \cup \beta_{j,Y},$$

where  $\beta_{j,X} := \beta_j \cap X$ ,  $\beta_{j,A} := \beta_j \cap A$ , and  $\beta_{j,Y} := \beta_j \cap Y$ . We decompose the edges  $\beta_j^+$  and  $\beta_j^-$  in the same manner.

Set

$$\alpha_j := \beta_{j,X} \cup T_{1/m, \phi}(\beta_{j,A}) \cup \beta_{j+1,Y};$$



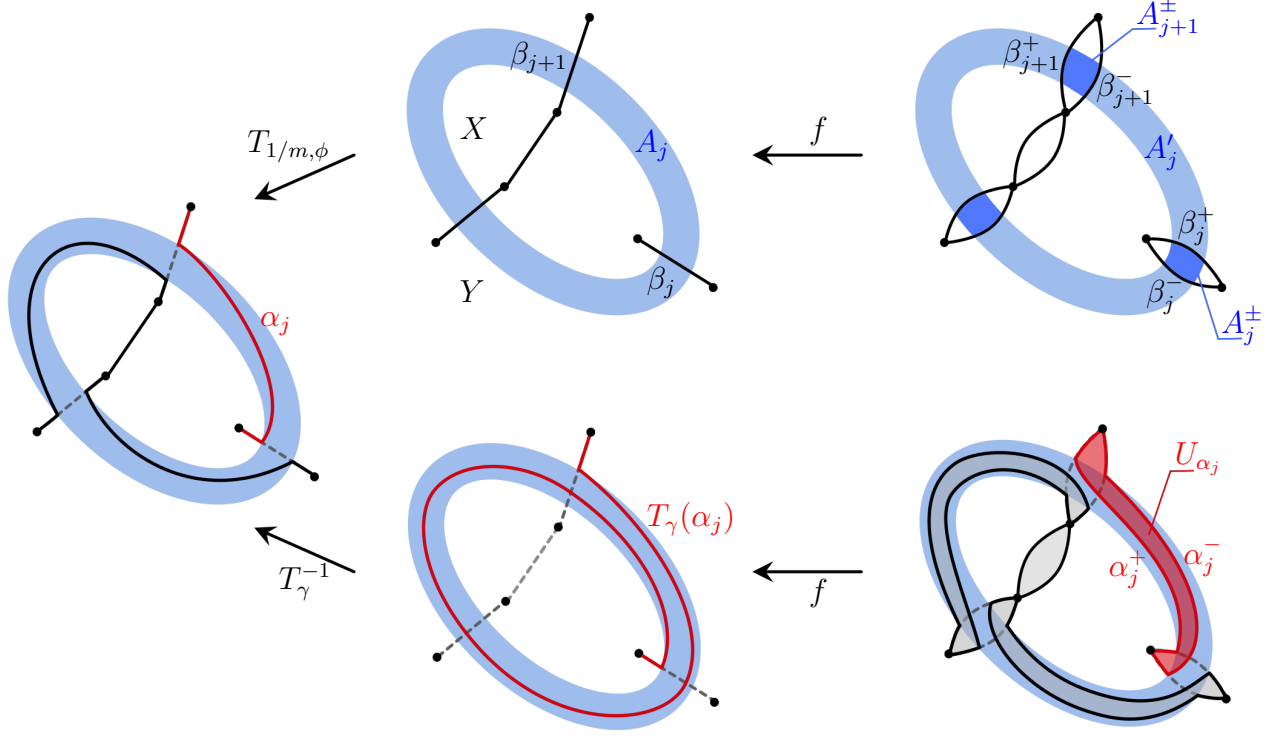


FIGURE 16. Proof of Proposition 5.6.

here and in the following, all indices are considered modulo  $m$ . By definition of the graph rotation,  $\alpha_1, \dots, \alpha_m$  are all the edges of  $H$  that intersect  $A$ . Therefore,

$$E(H) = \{\alpha_1, \dots, \alpha_m\} \cup (E(G) \setminus E^A(G));$$

see the left picture in Figure 16, where  $H$  is shown in solid black and red lines. We set  $\alpha_{j,A} := T_{1/m,\phi}(\beta_{j,A}) = \alpha_j \cap A$ . Note that  $\alpha_{j,A} \subset A_j$  for each  $j = 1, \dots, m$ .

Let  $\alpha \in E(H)$  be arbitrary. We will now define triples  $(\alpha^+, \alpha^-, U_\alpha)$  that will satisfy conditions (D1)-(D3) with respect to the map  $g = T_\gamma^{-1} \circ f$ .

First, if  $\alpha \in E(G) \setminus E^A(G)$ , then we set  $\alpha^+ := \partial D_\alpha^+$ ,  $\alpha^- := \partial D_\alpha^-$ , and  $U_\alpha := \text{int}(D_\alpha)$ .

Otherwise,  $\alpha = \alpha_j$  for some  $j = 1, \dots, m$ . Recall that the Dehn twist  $T_\gamma$  about  $\gamma$  is defined with respect to the same homeomorphism  $\phi$  as the  $1/m$ -twist  $T_{1/m,\phi}$ . Thus

$$T_\gamma(\alpha_j) = \beta_{j,X} \cup T_\gamma(\alpha_{j,A}) \cup \beta_{j+1,Y},$$

where  $\text{int}(T_\gamma(\alpha_{j,A}))$  intersects (transversely) each arc  $\beta_{j,A}$  and  $\beta_{j+1,A}$  exactly once; see the red arc in the bottom middle picture in Figure 16. Since  $f|_{\text{int}(A_j^\pm)}: \text{int}(A_j^\pm) \rightarrow \text{int}(A) \setminus \beta_j$  and  $f|_{A'_j}: A'_j \rightarrow A_j$  are homeomorphisms, there are two lifts  $\alpha_j^+$  and  $\alpha_j^-$  of  $\alpha_j$  under  $g = T_\gamma^{-1} \circ f$  such that

$$\alpha_j^- = \beta_{j,X}^- \cup \alpha_{j,A}^- \cup \beta_{j+1,Y}^-$$

and

$$\alpha_j^+ = \beta_{j,X}^+ \cup \alpha_{j,A}^+ \cup \beta_{j+1,Y}^+$$

where  $\alpha_{j,A}^-$  and  $\alpha_{j,A}^+$  are lifts of  $T_\gamma(\alpha_{j,A})$  under  $f$  that satisfy

$$\alpha_{j,A}^- \subset A_j^\pm \cup A'_j \quad \text{and} \quad \alpha_{j,A}^+ \subset A'_j \cup A_{j+1}^\pm.$$

Finally, let us set  $U_{\alpha_j}$  to be the connected component of  $S^2 \setminus (\alpha_j^+ \cup \alpha_j^-)$  that contains  $\text{int}(D_{\beta_j}) \cap X$  (and  $\text{int}(D_{\beta_{j+1}}) \cap Y$ ); see the bottom right picture in Figure 16 for a reference.

*Claim.* The constructed triples  $(\alpha^+, \alpha^-, U_\alpha)$ ,  $\alpha \in E(H)$ , satisfy conditions (D1)-(D3) for  $g = T_\gamma^{-1} \circ f$ .

This follows easily from the construction. Indeed, condition (D2) is immediate, as well as condition (D1) for  $\alpha \in E(G) \setminus E^A(G)$ . Now, if  $\alpha = \alpha_j$  for some  $j = 1, \dots, m$ , then  $\alpha_j$ ,  $\alpha_j^+$  and  $\alpha_j^-$  satisfy  $\partial\alpha_j = \partial\alpha_j^+ = \partial\alpha_j^-$ . Moreover, these arcs are inside the closed Jordan region

$$W' := (D_{\beta_j} \cap X) \cup A_j^\pm \cap A'_j \cup A_{j+1}^\pm \cup (D_{\beta_{j+1}} \cap Y).$$

Lemma 2.1 now implies that  $\alpha_j$ ,  $\alpha_j^+$  and  $\alpha_j^-$  are all isotopic rel.  $\text{Crit}(g) = \text{Crit}(f)$  (take  $W$  to be an open Jordan region with  $W' \subset W$  and  $W \cap \text{Crit}(f) = W' \cap \text{Crit}(f) = \partial\alpha_j$ ). Lastly, condition (D3) for the triples  $(\alpha^+, \alpha^-, U_\alpha)$ ,  $\alpha \in E(H)$ , follows from the fact that the triples  $(\partial D_e^+, \partial D_e^-, \text{int}(D_e))$ ,  $e \in E(G)$ , satisfy it by Proposition 3.8. We leave this straightforward verification to the reader.

The claim above and Proposition 3.9 imply that  $g = T_\gamma^{-1} \circ f$  is isotopic to a critically fixed Thurston map obtained by blowing up an admissible pair  $(H, \varphi)$ . Hence,  $H$  is isotopic to the charge graph of  $g$  by Theorem 3.29(ii).

To complete the proof we need to check that  $\varphi$  is isotopic to  $\text{id}_{S^2}$  rel.  $\text{Crit}(f)$ . Let  $H'$  be a connected graph with  $V(H') = \text{Crit}(f)$  and

$$H' \cap \left( A \cup \bigcup_{\alpha \in E(H)} \bar{U}_\alpha \right) = \alpha_1^+ \cup \dots \cup \alpha_m^+.$$

Note that  $f(\alpha') \sim \alpha'$  for every  $\alpha' \in E(H')$ . It follows now from Proposition 4.9 and Corollary 2.17 that  $\varphi \sim \text{id}_{S^2}$  rel.  $\text{Crit}(f)$ . This finishes the proof of the proposition.  $\square$

The following result establishes Main Theorem C.

**Theorem 5.7.** *Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map obtained by blowing up a pair  $(G, \text{id}_{S^2})$ . Suppose  $\gamma$  is a simple transversal with respect to  $G$  and  $n \in \mathbb{Z}$  is arbitrary.*

- (i) *If  $i_f(G, \gamma) \geq 1$ , then the twisted map  $T_\gamma^n \circ f$  is isotopic to the map obtained by blowing up the pair  $(H, \text{id}_{S^2})$ , where  $H$  is the  $(-n)$ -rotation of  $G$  about the curve  $\gamma$ .*
- (ii) *If  $i_f(G, \gamma) = 0$ , then the twisted map  $T_\gamma^n \circ f$  is isotopic to the map obtained by blowing up the (admissible) pair  $(G, T_\gamma^n)$ .*

*Proof.* (i) Proposition 5.6 implies the statement for  $n = -1$ . By a similar argument, we can also show the statement for  $n = 1$ . The rest of the statement follows from these two cases and Proposition 5.4 by induction on  $n$ .

(ii) The statement immediately follows from Proposition 3.5 and Theorem 3.29(ii), because the pair  $(G, T_\gamma^n)$  is admissible.  $\square$

We can now easily deduce Corollary 1.5 from the introduction.

**Corollary 5.8.** *Let  $f: S^2 \rightarrow S^2$  be a critically fixed Thurston map obtained by blowing up a pair  $(G, \text{id}_{S^2})$  and  $\gamma$  be a simple transversal with respect to  $G$  with  $|G \cap \gamma| \geq 1$ .*

*Then the sequence of combinatorial equivalence classes of  $(T_\gamma^n \circ f)_{n \in \mathbb{Z}}$  is strictly periodic with the period dividing  $i_f(G, \gamma)$ . In other words, if  $n_1 \equiv n_2 \pmod{i_f(G, \gamma)}$ , then the twisted maps  $T_\gamma^{n_1} \circ f$  and  $T_\gamma^{n_2} \circ f$  are combinatorially equivalent.*

*Proof.* Let  $n \in \mathbb{Z}$  be arbitrary and suppose that  $n = mk + r$ , where  $m := i_f(\gamma, G)$ ,  $k \in \mathbb{Z}$ , and  $0 \leq r < m$ . By Theorem 5.7(i), the twisted map  $T_\gamma^n \circ f$  is isotopic to the map obtained by blowing up the pair  $(H, \text{id}_{S^2})$ , where  $H$  is the  $(-n)$ -rotation of  $G$  about the curve  $\gamma$ .

Let  $G'$  be the  $(-r)$ -rotation of  $G$  about the curve  $\gamma$ . By definition of the graph rotation, the graph  $H$  is isotopic to the graph  $T_\gamma^k(G')$  rel.  $\text{Crit}(f)$ . Thus the map  $T_\gamma^n \circ f$  is isotopic to the map obtained by blowing up the pair  $(T_\gamma^k(G'), \text{id}_{S^2})$ , while  $T_\gamma^r \circ f$  is isotopic to the map obtained by blowing up the pair  $(G', \text{id}_{S^2})$ . Since the pairs  $(T_\gamma^k(G'), \text{id}_{S^2})$  and  $(G', \text{id}_{S^2})$  are equivalent, Theorem 3.31 implies that the maps  $T_\gamma^n \circ f$  and  $T_\gamma^r \circ f$  are combinatorially equivalent. This finishes the proof of the corollary.  $\square$

*Example 5.9.* Let  $\gamma_1, \gamma_2$  be the simple transversals with respect to  $G_\square$  as in Figure 15. By Proposition 5.6, the charge graphs of  $T_{\gamma_1}^{-1} \circ f_\square$  and  $T_{\gamma_2}^{-1} \circ f_\square$  are as shown on the top right and bottom right in the same figure. In particular, we see that  $T_{\gamma_1}^{-1} \circ f_\square$  is combinatorially equivalent to  $f_\square$  and  $T_{\gamma_2}^{-1} \circ f_\square$  is obstructed. (Note that the former agrees with what we have obtained in Example 5.1.) Furthermore, using Theorem 5.7(i) and Corollary 5.8, one can verify that  $T_{\gamma_1}^n \circ f_\square$  is combinatorially equivalent to  $f_\square$  for all  $n \in \mathbb{Z}$ . We also obtain that the twisted map  $T_{\gamma_2}^n \circ f_\square$  is equivalent to  $T_{\gamma_2} \circ f_\square$ , and thus obstructed, for odd  $n \in \mathbb{Z}$ ; and it is equivalent to  $f_\square$ , and thus realized, for even  $n \in \mathbb{Z}$ .

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