

HOLOMORPHICALLY CONJUGATE POLYNOMIAL AUTOMORPHISMS OF \mathbb{C}^2 ARE POLYNOMIALLY CONJUGATE

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ABSTRACT. We confirm a conjecture of Friedland and Milnor: if two polynomial automorphisms f and $g \in \text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ with dynamical degree > 1 are conjugate by some holomorphic diffeomorphism $\varphi: \mathbb{C}^2 \rightarrow \mathbb{C}^2$, then φ is a polynomial automorphism; thus, f and g are conjugate inside $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$. We also discuss a number of variations on this result.

1. INTRODUCTION

1.1. **Conjugacy classes in $\text{Aut}(\mathbb{A}_{\mathbf{K}}^2)$.** In their very influential paper [10], Friedland and Milnor study the dynamical and group-theoretic properties of polynomial automorphisms of the affine plane defined over the field $\mathbf{K} = \mathbf{R}$ or \mathbf{C} . In particular, they provide a dichotomy for conjugacy classes in $\text{Aut}(\mathbb{A}_{\mathbf{C}}^2)$, which we describe below.

For any field \mathbf{K} , Jung's theorem says that the group $\text{Aut}(\mathbb{A}_{\mathbf{K}}^2)$ is the amalgamated product of the group of *affine automorphisms* $\text{GL}_2(\mathbf{K}) \ltimes \mathbf{K}^2$ and the group of *elementary automorphisms*

$$(x, y) \mapsto (\alpha x + p(y), \beta y + \gamma),$$

with amalgamation along their intersection. If f is an element of $\text{Aut}(\mathbb{A}_{\mathbf{K}}^2)$, we denote by $\deg(f)$ the degree of the formulas defining f , and set

$$\lambda_1(f) = \lim_{n \rightarrow +\infty} \deg(f^n)^{1/n}.$$

Friedland and Milnor prove the following:

- either $\lambda_1(f) = 1$, and then f is conjugate to an elementary or an affine automorphism; moreover, if \mathbf{K} is algebraically closed, every affine automorphism is conjugate to an elementary automorphism by a linear change of variable;
- or $\lambda_1(f)$ is an integer larger than 1, f is conjugate to a *generalized Hénon map*, that is a finite composition of Hénon mappings $(x, y) \mapsto (ay + p_i(x), x)$ with $\deg(p_i) \geq 2$. In that case, $\lambda_1(f)$ is the product of the degrees of the p_i .

As explained in [10], it follows that the dynamics of f is interesting only when f is conjugate to a generalized Hénon map; for instance, for $f \in \text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$, the topological entropy of the map $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is equal to $\log(\lambda_1(f))$ (see [13]).

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The Bass-Serre theory shows that $\text{Aut}(\mathbb{A}_{\mathbf{K}}^2)$ acts on a simplicial tree, with stabilizers of vertices corresponding to conjugates of the affine group and the elementary group. Then, the classification of Friedland and Milnor corresponds to the following dichotomy: $\lambda_1(f) = 1$ if and only if the induced action of f on the Bass-Serre tree fixes some vertex, while $\lambda(f) > 1$ when f acts as a non-trivial translation on some invariant geodesic axis. In accordance with this description, we will refer to these two cases as *elliptic* and *loxodromic*, respectively.

1.2. Conjugation under biholomorphisms and rigidity. Let us now suppose that $\mathbf{K} = \mathbf{C}$. Friedland and Milnor also study the following question: *if two polynomial automorphisms are conjugate by some holomorphic diffeomorphism of \mathbb{C}^2 , are they also conjugate by an element of $\text{Aut}(\mathbb{A}_{\mathbf{C}}^2)$?* Automorphisms with distinct dynamical degrees cannot be conjugate by a homeomorphism $\mathbb{C}^2 \rightarrow \mathbb{C}^2$, because their topological entropies are distinct. In particular, elliptic and loxodromic classes should be treated separately. A complete answer is given in [10] for elementary automorphisms: the answer is *yes* if such an automorphism admits a periodic point, and *no* otherwise [10, Theorem 6.10 and Lemma 6.12]. For generalized Hénon maps, Friedland and Milnor also prove that the answer is *yes* for maps of degree 2 and 3, and conjecture that the same result holds in the general case [10, Theorem 7.1]. In this note we confirm this conjecture.

Theorem A. *Let f and g be loxodromic polynomial automorphisms of the affine plane, defined over some subfield \mathbf{K} of \mathbf{C} . If $\varphi: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a biholomorphism that conjugates f to g , i.e. $\varphi \circ f \circ \varphi^{-1} = g$, then φ is a polynomial automorphism, and it is defined over a finite extension of \mathbf{K} . Moreover, one can find $\psi \in \text{Aut}(\mathbb{A}_{\mathbf{K}}^2)$ such that $\psi \circ f = g \circ \psi$ and $\deg(\psi) \leq 2^{57}(\deg(f) \deg(g))^{29}$.*

A natural approach to this problem is to use eigenvalues at periodic orbits and to show that loxodromic automorphisms with the same eigenvalues are conjugate in $\text{Aut}(\mathbb{A}_{\mathbf{C}}^2)$. This is precisely what is done in [10] in degree ≤ 3 . However, for arbitrary degree, such a multiplier rigidity result is not available, so we use a different method. In a nutshell, the main ingredient will be the existence and rigidity of the canonical invariant currents of f and g ; these currents T_f^{\pm} and T_g^{\pm} must be respected by the conjugacy, i.e. $\varphi_* T_f^{\pm} = c T_g^{\pm}$ for some $c > 0$, and their rigidity properties will be the key to our proof.

When $f = g$, the corresponding result was proven in [6, Proposition 8.1] and [5]; we shall use this particular case below. Related results were also obtained by Bera, Pal and Verma in [4], with an approach similar to ours.

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2. PROOF OF THEOREM A

2.1. Preliminaries on Hénon maps (see [10] and [12]). Let $f \in \text{Aut}(\mathbb{A}_{\mathbf{C}}^2)$ be a generalized Hénon map of degree d (so $d = \lambda_1(f)$). Its birational extension to the projective plane $\mathbb{P}^2(\mathbf{C})$ has an indeterminacy point at $I_f^+ := [0 : 1 : 0]$, and it contracts the line at infinity $\{z = 0\}$ onto the indeterminacy point $I_f^- := [1 : 0 : 0]$ of f^{-1} . Let K_f^+ (resp. K_f^-) be the subset of \mathbb{C}^2 of

points with a bounded forward (resp. backward) orbit; its closure in $\mathbb{P}^2(\mathbf{C})$ intersects the line at infinity at I_f^+ (resp. at I_f^-). Let $\|\cdot\|$ denote the usual euclidean norm on \mathbf{C}^2 . Then, the function $G_f^+ : \mathbf{C}^2 \rightarrow \mathbf{R}$ defined by

$$G_f^+(p) = \lim_{n \rightarrow +\infty} \frac{1}{d^n} \log^+ \|f^n(p)\|$$

is continuous, plurisubharmonic, and non-negative; it vanishes exactly along K_f^+ and satisfies $G_f^+ \circ f = dG_f^+$. The difference $p \mapsto G_f^+(p) - \log^+ \|p\|$ extends to a continuous function on $\mathbb{P}^2(\mathbf{C}) \setminus \{I_f^+\}$, as follows for instance from [12, Thm 1.7.1]. The closed, positive current

$$T_f^+ = \frac{i}{\pi} \partial \bar{\partial} G_f^+$$

satisfies $f^* T_f^+ = dT_f^+$; it is supported on the boundary of K_f^+ and up to a positive multiplicative constant, *it is the unique closed positive current supported by K_f^+* (see [9, Theorem 7.12]). Changing f into f^{-1} , one constructs similar objects K_f^- , G_f^- and T_f^- .

The intersection $K_f = K_f^+ \cap K_f^-$ is a compact f -invariant subset of \mathbf{C}^2 , it is maximal for this property, and the topological entropy of f on K_f is equal to $\log(d)$. The automorphism f has infinitely many saddle periodic points, all of them contained in K_f (see [1]). If $q \in \mathbf{C}^2$ is such a saddle periodic point, its stable manifold $W^s(q)$ is the image of a holomorphic, injective, immersion $\mathbf{C} \rightarrow \mathbf{C}^2$, *the image of which is dense in ∂K_f^+* ; the current T_f^+ can be recovered asymptotically as a current of integration on $W^s(p)$ (see [3]). The function $G_f := \max(G_f^+, G_f^-)$ is continuous and plurisubharmonic and vanishes exactly on K_f . Moreover

$$G_f^+(p) \leq \log^+ \|p\| + O(1) \quad \text{and} \quad G_f(p) = \log^+ \|p\| + O(1)$$

as $\|p\|$ goes to $+\infty$; see Corollary 2.6 and Proposition 3.8 in [2]. The product $T_f^+ \wedge T_f^-$ is an invariant probability measure, supported in K_f ; it coincides with $(\frac{i}{\pi} \partial \bar{\partial} G_f)^2$.

2.2. The proof. Conjugating f and g by elements of $\text{Aut}(\mathbb{A}_{\mathbf{C}}^2)$, we may assume that both of them are generalized Hénon maps. Let $\varphi : \mathbf{C}^2 \rightarrow \mathbf{C}^2$ be a biholomorphism such that $\varphi \circ f \circ \varphi^{-1} = g$. Then $\varphi \circ f^n = g^n \circ \varphi$ for all $n \in \mathbf{Z}$, $\varphi(K_f^\pm) = K_g^\pm$, $\varphi(K_f) = K_g$, and f and g have the same topological entropy, hence the same degree d .

Lemma. *There exists a positive real number c such that $\varphi^* T_g^+ = c T_f^+$ and $G_g^+ \circ \varphi = c G_f^+$.*

Proof. The first statement follows from the fact, recalled above, that the only positive closed $(1, 1)$ -currents supported on K_g^+ are of the form $c T_g^+$, $c > 0$.

To prove the second fact, we observe that $H = c G_f^+ - G_g^+ \circ \varphi$ is a pluriharmonic function on \mathbf{C}^2 with $K_f^+ \subset \{H = 0\}$. Assume that $H \not\equiv 0$ and fix a holomorphic function h on \mathbf{C}^2 such that H is the real part $\Re(h)$. Then $\{H = 0\}$ admits a unique singular foliation by Riemann surfaces: its leaves are the level sets $\{h = i\alpha\}$, $\alpha \in \mathbf{R}$. On the other hand, for any saddle periodic point q of f , the stable manifold $W^s(q) \subset \mathbf{C}^2$ is a connected, immersed, Riemann surface contained in K_f^+ ; thus $W^s(q)$ is an irreducible component of $\{h = i\alpha\}$ for some α . This is a contradiction since $W^s(q)$ is dense in the boundary of K_f^+ , so it is not embedded at any point. \square

Similarly, there exists a constant $c' > 0$ such that $G_g^- \circ \varphi = c'G_f^-$. In fact, from

$$\int \varphi^*(T_g^+ \wedge T_g^-) = \int T_f^+ \wedge T_f^- = 1$$

we deduce that $c' = 1/c$, but this will not be used in what follows. From this we infer that

$$0 \leq G_g \circ \varphi = \max(G_g^+ \circ \varphi, G_g^- \circ \varphi) = \max(cG_f^+, c'G_f^-) \leq \max(c, c')G_f.$$

On the other hand, $G_f(p) = \log \|p\| + O(1)$, and $G_g(p) = \log \|p\| + O(1)$ as $\|p\| \rightarrow \infty$. Since φ is proper, we deduce that

$$\log \|\varphi(p)\| + O(1) = G_g(\varphi(p)) \leq \max(c, c')G_f(p) \leq \max(c, c') \log \|p\| + O(1),$$

that is $\|\varphi(p)\| \leq C\|p\|^e$ for some constant $C > 0$ and for $e = \max(c, c')$. This implies that φ is defined by polynomial formulas. Similarly, φ^{-1} is also defined by polynomial formulas, and φ is a polynomial automorphism.

If φ and ψ are two automorphisms conjugating f to g , then $\varphi \circ \psi^{-1}$ commutes with f . In [11, 5], it is proven that the centralizer of f in $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ contains $f^{\mathbb{Z}}$ as a finite index subgroup. Thus, given any integer $D \geq 1$, the set $\{\psi \in \text{Aut}(\mathbb{A}_{\mathbb{C}}^2); \psi \circ f \circ \psi^{-1} = g, \deg(\psi) \leq D\}$ is finite, and it is non-empty for $D \geq \deg(\varphi)$. Write $\psi = (\psi_1, \psi_2)$, with ψ_1 and ψ_2 in $\mathbb{C}[x, y]$ of degree $\leq D$. The constraints $\psi \circ f \circ \psi^{-1} = g$ and $\psi \in \text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ correspond to polynomial equations in the coefficients of ψ_1 and ψ_2 ; the coefficients of these equations are in the field of definition \mathbf{K} of f and g . Thus, the solutions lie in a finite extension of \mathbf{K} .

It remains to show that one can find an automorphism ψ that conjugates f to g and has degree at most $2^{57}(\deg(f) \deg(g))^{29}$. From [5, Theorem 4.10], one can find such a conjugacy in the group $\text{Bir}(\mathbb{P}_{\mathbf{K}}^2)$. Thus, to conclude one only needs to apply the following lemma of independent interest. \square

Lemma. *Let f and $g \in \text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ be two Hénon automorphisms. If ψ is a birational transformation of the affine plane that f to g , then ψ is an automorphism.*

Proof. Let us consider f , g , and ψ as birational transformations of the projective plane $\mathbb{P}_{\mathbb{C}}^2$. Let E be the union of the irreducible curves $E_i \subset \mathbb{A}_{\mathbb{C}}^2$ which are contracted by ψ (there strict transform is a point of $\mathbb{P}_{\mathbb{C}}^2$). From $\psi \circ f = g \circ \psi$ we deduce that E is f -invariant, and since a Hénon map does not preserve any algebraic curve, we deduce that E is empty. Similarly, ψ^{-1} does not contract any curve $F \subset \mathbb{A}_{\mathbb{C}}^2$.

It remains to show that ψ does not have any indeterminacy point in $\mathbb{A}^2(\mathbb{C})$. To see this, we resolve the indeterminacies of ψ by a finite sequence of blow-ups: this provides a smooth projective surface X together with two birational morphisms $\varepsilon, \eta: X \rightarrow \mathbb{P}_{\mathbb{C}}^2$ such that $\psi = \eta \circ \varepsilon^{-1}$; we assume that X is minimal for this property. Suppose that $q \in \mathbb{A}^2(\mathbb{C})$ is an indeterminacy point of ψ and denote by $D \subset X$ the tree of rational curves which is mapped to q by ε . The curve D must intersect an irreducible curve $C \subset X$ which is contracted by η and is not contained in D ; indeed, otherwise, X would not be a minimal resolution of the indeterminacies. But then, C is the strict transform by ε of an irreducible plane curve C' that contains q . This curve C' is contracted by ψ and contains q , in contradiction with $E = \emptyset$. \square

Example. Fix an integer $m \geq 2$, and a positive integer D which is not the m -th power of an integer. One easily sees that the Hénon maps $f(x, y) = (y, x + y^{m+1})$ and $g(x, y) = (y, x + Dy^{m+1})$ are conjugate over $\mathbb{Q}(D^{1/m})$ but not over \mathbb{Q} . Indeed, $h(x, y) = (\alpha x, \alpha y)$ conjugates f to g if $\alpha^m = 1/D$, and from [11] we know that the centralizer of f is the semi-direct product of $f^{\mathbb{Z}}$ and the cyclic group of order m generated by $(x, y) \mapsto (e^{2i\pi/m}x, e^{2i\pi/m}y)$.

3. DISCUSSION AND COMPLEMENTS

(a) – Theorem A is still true when φ is a proper holomorphic semiconjugacy:

Theorem B. *If f and g are loxodromic automorphisms of \mathbb{C}^2 and $\varphi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a proper holomorphic map such that $\varphi \circ f = g \circ \varphi$, then φ is a polynomial automorphism.*

Proof. Note that we also have $\varphi \circ f^{-1} = g^{-1} \circ \varphi$. The properness of φ implies successively that

- (1) $\varphi(\mathbb{C}^2) = \mathbb{C}^2$ by Remmert's proper mapping theorem;
- (2) $\varphi^{-1}(K_g^\pm) = K_f^\pm$;
- (3) $\varphi^*(T_g^+)$ (resp. $\varphi^*(T_g^-)$) is a positive closed current supported on K_f^+ (resp. K_f^-).

Indeed, $\varphi^*(T_g^+)$ (resp. $\varphi^*(T_g^-)$) is non-zero by the first item. Hence $\varphi^*T_g^\pm = c^\pm T_f^\pm$, with $c^\pm > 0$, and exactly as in Theorem A we conclude that φ is polynomial. Finally, the Jacobian of φ does not vanish, because otherwise, by the relation $\varphi \circ f = g \circ \varphi$, $\{\text{Jac}(\varphi) = 0\}$ would be an f -invariant algebraic curve, and such a curve does not exist by [2, Proposition 4.2]. Thus, since φ is proper, it is a covering, hence an automorphism, and we are done. \square

(b) – Take $\mathbf{K} = \mathbf{R}$. If $f \in \text{Aut}(\mathbb{A}_{\mathbf{R}}^2)$ has no fixed point in \mathbf{R}^2 and f preserves the orientation of \mathbf{R}^2 , then f is conjugate to a translation by some real analytic diffeomorphism of \mathbf{R}^2 (see [7]). Thus, *Theorem A fails for polynomial automorphisms of \mathbf{R}^2* . On the other hand one may expect some rigidity properties when $h_{\text{top}}(f|_{\mathbf{R}^2}) > 0$, for instance when $K_f \subset \mathbf{R}^2$.

(c) – If U is a small ball in the complement of $K_f^+ \cup K_f^-$, then its orbit under the action of f is wandering. So, we can find a C^∞ diffeomorphism of \mathbb{C}^2 that commutes to f , is the identity on the complement of the f -orbit of U , and is not the identity on U . This shows that, as it is stated, *Theorem A fails in the C^∞ setting*.

On the other hand, the following lemma shows that the canonical currents are preserved under C^1 conjugacy. Thus, the rigidity of positive closed currents with support in K^+ is not really needed in the proof of Theorem A, we could replace it by this lemma.

Lemma. *Let f and g be Hénon automorphisms of the plane $\mathbb{A}_{\mathbb{C}}^2$. If φ is a C^1 diffeomorphism of \mathbb{C}^2 conjugating f to g , then $\varphi^*T_g^+ = cT_f^+$ for some real number $c \neq 0$.*

Sketch of proof. For both f and g , T^+ is a laminar current whose laminar structure is entirely defined in terms of the smooth dynamics [1]. More precisely, the structure theory of strongly laminar currents [8, §5] shows that these currents can be viewed in a precise way as foliation cycles on a weak lamination, whose leaves are Pesin stable manifolds and whose transverse measure is induced by the unstable conditionals of the unique measure of maximal entropy. It

follows that if φ is a C^1 diffeomorphism conjugating f and g , $\varphi^*T_g^+$ is a closed laminar current subordinated to the same weak lamination as T_f^+ ; more precisely, $\varphi^*T_g^+$ is

- a closed current;
- a laminar current with respect a transverse signed measures (the sign takes care of the orientation of the local leaves, since φ may reverse this orientation);
- absolutely continuous with respect to T_f^+ ; in particular it is supported by K_f^+ .

On the other hand, the fact that T_f^+ is an extremal point in the cone of positive closed currents (see e.g. [12, §2.2]) entails that the transverse invariant measure of T_f^+ is ergodic (see [8, Cor. 5.8]) and this implies that $\varphi^*T_g^+$ is a constant multiple of T_f^+ . \square

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