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► **To cite this version:**

Juan Sepúlveda, Luce Brotcorne, Hélène Le Cadre. A Reverse Stackelberg Model for Demand Response in Local Energy Markets. 2024. hal-04525404v1

**HAL Id: hal-04525404**

**<https://hal.science/hal-04525404v1>**

Preprint submitted on 28 Mar 2024 (v1), last revised 4 Nov 2024 (v2)

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# A Reverse Stackelberg Model for Demand Response in Local Energy Markets <sup>★</sup>

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## ARTICLE INFO

### Keywords:

Time-of-use tariff  
Pricing  
Reverse Stackelberg Game  
Bilevel optimization  
Demand response  
Local electricity markets  
Reactive power

## Abstract

In an era where renewable energy resources are increasingly integrated into our power systems, and consumer-centric approaches gain traction, local energy markets emerge as a pivotal mechanism for empowering prosumers. This paper presents a novel bilevel optimization model that uniquely blends the dynamics of peer-to-peer energy markets with the physical realities of power distribution networks. The innovation stems from introducing a tariff design approach based on affine functions to shape prosumer behavior towards operationally efficient and secure energy exchanges. This is critical as previous market designs often overlooked the physical constraints of power flows, leading to potential risks in voltage regulation and economic efficiency. The lower level of the model encapsulates the interactions among prosumers in a generalized Nash equilibrium problem (GNEP), modeling active and reactive power injections of prosumers. The upper level, representing the role of the distribution system operator, strategically computes tariffs to steer the market to an operationally efficient equilibrium. The paper relies on the classical Nikaido–Isoda (NI) reformulation to characterize the GNEP, a key aspect in leveraging a proof of strong stability of the lower-level solution. Computational experiments on various IEEE test feeder instances reveal the model’s capacity to efficiently align prosumer behavior with operational objectives, utilizing only the tariff information, thereby simplifying the decision-making process in complex distribution systems.

## 1. Introduction

The energy transition represents a path toward reshaping the global energy sector from its fossil-fuel foundations to a carbon-neutral state by the latter half of this century. This journey demands transformative shifts within power systems, particularly at the distribution level. One central aspect involves the widespread adoption of distributed energy resources such as renewable generators, storage systems, adaptable loads, and electric vehicles. Simultaneously, future regulations are expected to pivot towards a consumer-centric model, leveraging local electricity markets and empowering the prosumers to actively manage their energy assets.

While the potential benefits of this transition are substantial, it also presents notable challenges. The uncertain and fast-changing nature of renewable resources, coupled with the autonomy of prosumers, may compromise the efficient and secure operation of the distribution network. To address this concern, the distribution system operators (DSOs) could potentially implement strategies to steer the behavior of prosumers. We believe these strategies must go beyond the control of energy usage patterns and extend to reactive power injections as well as participation in local electricity markets, therefore extending the notion of demand-side management.

The objective of the work is to design an incentive policy in the form of a tariff function to promote prosumer behavior that is both operationally safe and economically efficient. The tariff applied to each prosumer is a function of its net active and

reactive power output, thereby being a price signal that acts on local variables of the system. The tariff design problem considers two classes of decision-makers: (i) the DSO whose interest is to achieve efficient and secure operation of the distribution network, and (ii) the prosumers, selfishly participating in a peer-to-peer market, agnostic of the global system operation. Naturally, we adopt a Stackelberg game setting in which the DSO is the *leader* (upper level) and the prosumers are the *followers* (lower level). The main research questions revolve around evaluating the performance of the proposed tariff design specially when compared against conventional constant tariffs.

Recent studies have increasingly explored tariff design as a method for demand-side management, capitalizing on the responsiveness of prosumers to price signals. These approaches are often based on Stackelberg games and bilevel optimization to reflect the complex interplay between the different classes of decision makers involved. Notably, Schittekatte et al. (2018) pioneered the use of a best-response algorithm to achieve a Stackelberg equilibrium between the DSO and the prosumers, a methodology that was expanded by Hoarau and Perez (2019) to encompass electromobility considerations. Pricing techniques based on bilevel optimization have been proposed in the works Askeland et al. (2020, 2021) and Anjos et al. (2021). In an innovative approach, Aussel et al. (2020) developed a customized solution concept for a single-leader multi-follower trilevel problem, considering energy price and quantity as prosumers variables. Similarly, the authors in Tushar et al. (2014) examined a single-leader multi-follower Stackelberg game where the prosumers were modeled by a generalized Nash equilibrium problem. While Maharjan et al. (2016) applied an iterative approach for a multi-leader multi-follower Stackelberg game.

The works mentioned above reveal gaps in incorporating the impact of prosumers’ actions on physical variables such

<sup>★</sup> This work emanates from research supported by the ERA-NET Cofund grant under the CHIST-ERA IV Joint Call on Novel Computational Approaches for Environmental Sustainability (CES) project “Supporting Energy Communities-Operational Research and Energy Analytics” (SEC-OREA) and ANR.

ORCID(s):

as voltage levels, congestion, and reactive power injections. The integration of distributed energy resources (DERs) and the decentralization of power grids have brought to the forefront the critical role of reactive power support Dall’Anese et al. (2015). DERs are increasingly recognized for their potential to provide reactive power, a capability that is essential for the efficient operation of networks leveraging these resources. Recent literature, including studies like Potter et al. (2023), Bozionic et al. (2022), and Wolgast et al. (2022) underscores the growing emphasis on creating reactive power markets and tariffs designed to address reactive power contributions. We highlight works like Padiaditis et al. (2021), which aim to incorporate physical aspects such as flow constraints and reactive power injection as part of the tariff design problem while considering a peer-to-peer market of energy, illustrating a move towards more comprehensive solutions.

Current literature in tariff design remains in the scope of classical approaches such as Labbé et al. (1998), where the tariff is synthesized as a constant coefficient of a linear expression in the objective function of the prosumers, thereby introducing a linear cost expression. While recognizing state-of-the-art information systems and local energy management technologies Jin et al. (2017). These advancements allow for a reevaluation of conventional assumptions about the intelligibility of tariffs, as the ones in Labbé et al. (1998) and Padiaditis et al. (2021).

As a result, we move beyond constant tariffs based on discrete levels, opting for tariff coefficients that are seen by the prosumers as affine functions of their decision variables. Despite this increased complexity, the focus on simplicity is concretized by ensuring that the decision-making model addressed by prosumers is convex and computationally tractable. This approach finds its origins in works like Zheng and Basar (1982) and is based on the control-theoretic view of incentives presented in Ho et al. (1980). There, under a Stackelberg game setting, the leader calculates optimal cost terms on the objective function of the follower by means of announcing its own reaction function in advance—in our case, a tariff function—thereby called reverse Stackelberg game Groot et al. (2016). Although the finer distinctions within the definitions of the game-theoretic framework are discussed in greater detail in Ho et al. (1981), in the end, the computation of solutions boils down to solving a mathematical program with equilibrium constraints (MPECs).

The MPECs are a class of problems, of which bilevel optimization is a special case, that allows for modeling two layers of decision-making, in which an equilibrium problem (the lower-level problem) is nested into an optimization problem (the upper-level problem); see Luo et al. (1996). They have been applied in tariff design and taxation through modeling single-leader multiple-follower games Hart et al. (2021). The formulation of these programs involves the integration of complementarity conditions into optimization models, enabling the detailed analysis of single-leader multi-follower interactions in competitive markets Raghunathan and Biegler (2003); Flegel and Kanzow (2005); Ye (2005).

Different notions of equilibrium might be chosen to represent the lower-level problem of a MPEC. We focus on the generalized Nash equilibrium problem (GNEP) for its ability to represent a peer-to-peer energy market equilibrium, as done in Le Cadre et al. (2020). This solution concept extends that of the classical Nash equilibrium problem, representing the equilibrium of multiple optimization problems coupled through their objective functions and feasibility sets; detailed reviews can be found in Facchinei and Kanzow (2007) and Fischer et al. (2014). Jointly-convex GNEPs appear as an important family of problems for which there are well-known and useful reformulations: (i) the KKT reformulation, in the case standard constraint qualifications are met, and (ii) the Nikaido–Isoda reformulation; both reformulations are detailed in Facchinei and Kanzow (2007). The Nikaido–Isoda reformulation links jointly convex GNEPs and traditional optimization problems, allowing the analysis of GNEPs at the lower-level of MPECs as parametric optimization problems. Consequently, this allows the application of the notion of stability in parametric optimization, as developed in Dempe (2002), whose importance in an application context has been highlighted in Caruso et al. (2020).

Our paper pioneers the application of reverse Stackelberg games to demand response in power distribution systems, exploring the dynamics of peer-to-peer markets alongside voltage and reactive power management. We address this through a bilevel programming framework with second-order conic constraints that uniquely overcomes the common issue of non-uniqueness in lower-level solutions. Notably, our research is the first within the domain of electricity market demand response to offer a proof of stability for the solutions. Our main contributions are:

1. The development of an inclusive demand-response model that integrates a peer-to-peer market, battery operations, and the AC power flow within the distribution network to ensure both security and efficiency.
2. A strong stability proof of the solution, based on the Nikaido–Isoda reformulation of the lower-level generalized Nash equilibrium problem (GNEP).

Together, these contributions significantly advance demand response strategies, enhancing the functionality and efficiency of local energy markets.

The rest of the paper is organized as follows: Section 2 details the decision-making models for the DSO and prosumers, using a GNEP to model the peer-to-peer market. Moving forward, Section 3 is dedicated to establishing the stability of our proposed solution, profiting from the Nikaido–Isoda reformulation of the lower-level GNEP. Section 4 showcases our numerical findings on instances built on top of standard IEEE distribution grids, illustrating the practicality and effectiveness of our approach. Lastly, Section 5 concludes the paper, summarizing our contributions and suggesting directions for future research.

## Notation

Boldface letters are used for matrices and vectors,  $(\cdot)^T$  for transposition, and  $|\cdot|$  denotes the absolute value of a number or the cardinality of a set. Let  $\mathbf{A}$  be a matrix,  $\|\mathbf{A}\|_F$  and  $\|\mathbf{A}\|_2$  denote the Frobenius norm and the 2–norm, respectively.

Let  $\mathcal{N}$  be the set of players of a game; we use classical game theoretic notation to define  $\mathbf{z}_{-i} := (\mathbf{z}_j)_{j \in \mathcal{N} \setminus \{i\}}$ , i.e., a vector stacking the entries of  $\mathbf{z}$  that correspond to all players except  $i$ .

## Nomenclature

### Sets and set-valued maps

$\mathcal{N}^0$	Set of prosumers including the slack prosumer ( $\mathcal{N}^0 = \mathcal{N} \cup \{0\}$ ).
$\mathcal{N}$	Set of prosumers.
$\mathcal{K}$	Set of time steps.
$\mathcal{G}_i$	Feasible generator dispatch set of prosumer $i$ .
$\Omega_i(\cdot)$	Feasibility set of prosumer $i$ . $\Omega_i : \mathbb{R}^{ \mathcal{N}  - n_i} \rightrightarrows \mathbb{R}^{n_i}$ .

### Parameters

$\mathbf{B}$	Incidence matrix associating prosumers to corresponding buses.
$\mathbf{H}$	Matrix of linear coefficients associating bus injections to voltage magnitudes.
$\tilde{\mathbf{H}}$	Matrix of linear coefficients associating prosumer injections to voltage magnitudes.
$\mathbf{h}$	Power flow affine model intercept.
$\mathbf{R}$	Matrix of resistive linear coefficients of the <i>LinDis-<i>t</i>Flow</i> model.
$\mathbf{X}$	Matrix of reactive linear coefficients of the <i>LinDis-<i>t</i>Flow</i> model.
$E_i$	Connection capacity of prosumer $i$ .
$c_{ij}^k$	Bilateral energy trade price from prosumer $i$ to $j$ at time step $k \in \mathcal{K}$ , for $(i, j) \in \mathcal{E}$ .
$\hat{v}_i, \check{v}_i$	Upper and lower limits of the squared voltage magnitude at bus $i \in \mathcal{B}$ , respectively.
$\hat{s}_{G,i}$	Distributed generator capacity of prosumer $i \in \mathcal{N}$ .
$\hat{p}_{G,i}^k$	Maximum available active power for the distributed generator of prosumer $i$ at time step $k$ .
$\check{\theta}_{G,i}$	Minimum power factor of prosumer $i$ .
$\eta_{B,i}$	Square root of the battery's round trip efficiency of prosumer $i$ .
$\eta_{G,i}$	Loss coefficient of the distributed generator of prosumer $i$ .
$\hat{s}_{B,i}$	Capacity of the battery of prosumer $i$ .
$\hat{p}_{B,i}$	Nominal power of the battery of prosumer $i$ .
$n_i$	Number of decision variables of prosumer $i$ .

## Leader variables

$\mathbf{x}$	Vector of leader variables stacking the tariff function coefficients $\mathbf{x} := (\Phi_i^k, \phi_i^k)_{i \in \mathcal{N}, k \in \mathcal{K}}$ .
$\Phi_i^k$	Matrix in $\mathbb{R}^{2 \times 2}$ representing the affine coefficients of the tariff function assigned to prosumer $i$ at time step $k$ .
$\phi_i^k$	Vector in $\mathbb{R}^2$ representing the constant coefficients of the tariff function assigned to prosumer $i$ at time step $k$ .
$\gamma_i^k$	Affine function of tariff coefficients such that $\gamma_i^k(\mathbf{u}_i^k) = \Phi_i^k \mathbf{u}_i^k + \phi_i^k$ .

## Followers variables

$\mathbf{u}_i^k$	Vector of variables of prosumer $i$ at time step $k$ .
$p_i^k$	Net active power injection of prosumer $i$ at time step $k$ .
$q_i^k$	Net reactive power injection of prosumer $i$ at time step $k$ .
$p_{G,i}^k$	Active power generated by prosumer $i$ at time step $k$ .
$q_{G,i}^k$	Reactive power generated by prosumer $i$ at time step $k$ .
$p_{E,i,j}^k$	Energy offered by prosumer $i$ to prosumer $j$ at time step $k$ .

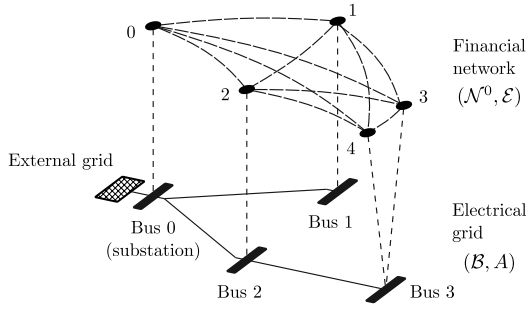
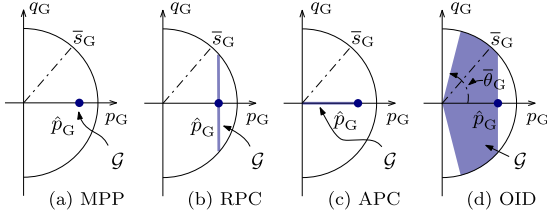
## 2. Problem formulation

The proposed Stackelberg model involves two distinct types of agents: (i) prosumers and (ii) the DSO. Prosumers participate in a peer-to-peer market, where their interactions are driven by self-interest, potentially causing adverse effects on the physical integrity of the underlying electrical grid. To address this challenge, we consider a DSO with the capability to enforce economic incentives aimed at enhancing the grid's operational performance.

Let  $(\mathcal{N}^0, \mathcal{E})$  be a symmetric simple directed graph modeling the financial network, where  $\mathcal{N}^0 := \mathcal{N} \cup \{0\}$ ,  $\mathcal{N}$  is the set of prosumers, and  $\mathcal{E}$  is the set of edges representing possible peer-to-peer interactions. We define prosumer 0 as a slack prosumer representing the external grid at the substation. As usual, the distribution network is a directed rooted tree with  $\mathcal{B}^0$  being the set of nodes (buses). The substation node (root) is labeled as 0, and  $\mathcal{B} := \mathcal{B}^0 \setminus \{0\}$ , where  $\mathcal{B}$  is the set of buses that may host prosumers. We define a map  $i \mapsto A_i$ , from  $\mathcal{B}$  to  $\mathcal{B}^0$  that returns the parent of each node.

The link between the market network and the electrical grid is illustrated in Figure 1. Each prosumer is connected to exactly one bus. We define a mapping  $B : \mathcal{N}^0 \rightarrow \mathcal{B}^0$  that assigns the hosting bus to each prosumer, thereby  $B(2) = 1$  in the example shown in Figure 1. The slack prosumer (external grid) is always connected to the substation bus; therefore,  $B(0) = 0$  is always true.

In the following sections, we will outline the models that capture the decision-making processes of the two classes of players: the prosumer and the DSO.


**Figure 1:** The financial network and the supporting electrical grid.

**Figure 2:** Distributed generator capacity set (depicted in purple) for different operating modes in the active/reactive power plane.

## 2.1. The prosumer model

Each prosumer owns and manages a single photovoltaic generator and a single battery, and may buy or sell energy in a peer-to-peer fashion with its neighbors in  $(\mathcal{N}, \mathcal{E})$ . The prosumer schedules its decisions in order to maximize their utility function. To do that, each prosumer solves a dynamic optimization problem over a finite horizon with time step indices  $k \in \mathcal{K} := \{1, \dots, K\}$ .

To begin, we establish a model for the production capacity of individual prosumers. In this context, we assume that each prosumer possesses a photovoltaic distributed generator, with  $p_{G,i}^k$  and  $q_{G,i}^k$  representing the active and reactive power output of prosumer  $i \in \mathcal{N}$  at time step  $k \in \mathcal{K}$ . The feasible dispatch set in the active and reactive power space is modeled as follows

$$\mathcal{G}_i^k := \{(p_{G,i}, q_{G,i}) \in \mathbb{R}^2 : \quad (1a)$$

$$\hat{p}_{G,i}^k \leq p_{G,i} \leq \hat{p}_{G,i}^k \quad (1b)$$

$$p_{G,i}^2 + q_{G,i}^2 \leq s_{G,i}^2 \quad (1c)$$

$$q_{G,i} \leq \check{\theta}_i p_{G,i} \quad (1d)$$

The definition provided in (1) comprehensively encompasses various standard operating modes, as detailed in Dall'Anese et al. (2015). These modes are illustrated in Figure 2, each corresponding to distinct selections of the parameters  $\hat{p}_{G,i}^k$  and  $\check{\theta}_i$ . Specifically, the modes include the Maximum Power Point (MPP), Reactive Power Control (RPC), Active Power Control (APC), and Optimal Inverter Dispatch (OID).

Apart from the distributed generator, each prosumer owns a battery, modeled as a 1-dimensional dynamic system such that the state of charge  $s_{B,i}^k$  satisfies

$$s_{B,i}^k = s_{B,i}^{k-1} + \eta_{B,i} p_{B,i}^{+,k} - \frac{1}{\eta_{B,i}} p_{B,i}^{-,k}, \quad \forall k \in \mathcal{K} \quad (2)$$

where  $p_{B,i}^k = p_{B,i}^{+,k} - p_{B,i}^{-,k}$  is the energy absorbed by the battery of prosumer  $i$ , over time step  $k$ , and  $p_{B,i}^{+,k} \geq 0$  and  $p_{B,i}^{-,k} \geq 0$  are auxiliary variables that help modeling the battery energy losses in the charging and discharging cycle. The positive parameter  $\eta_{B,i}$  is the square root of the round cycle battery efficiency. In addition, we impose the following capacity bounds

$$0 \leq s_{B,i}^k \leq \hat{s}_{B,i}, \quad \forall k \in \mathcal{K} \quad (3)$$

$$-\hat{p}_{B,i} \leq p_{B,i}^k \leq \hat{p}_{B,i}, \quad \forall k \in \mathcal{K} \quad (4)$$

Having defined the models for the distributed generator and battery, we now delve into the elements that shape the preferences of prosumers. We assume there are three components: (i) the tariff/incentive cost terms, (ii) the income obtained via peer-to-peer exchanges, and (iii) the distributed generator losses. Combining the three components results in the following prosumer disutility function

$$J_{F,i}(T(\cdot), \mathbf{u}_i, \mathbf{u}_{-i}) = \sum_{k \in \mathcal{K}} \left( T_i^k(p_i^k, q_i^k) - U_i^k \left( \mathbf{p}_{E,i}^k, (p_{E,ji}^k)_{j \in \mathcal{N}_i^0} \right) + l_i^k(p_{G,i}^k, q_{G,i}^k) \right) \quad (5)$$

where  $\mathbf{u}_i^k := (p_i^k, q_i^k, p_{G,i}^k, q_{G,i}^k, p_{B,i}^{+,k}, p_{B,i}^{-,k}, s_{B,i}^k, \mathbf{p}_{E,i}^k)$  comprises the decision variables of prosumer  $i \in \mathcal{N}$  at time step  $k \in \mathcal{K}$ , with  $\mathbf{p}_{E,i}^k := (p_{E,0i}^k, (p_{E,ij}^k)_{j \in \mathcal{N}_i^0})$ . The function  $T_i^k(\cdot)$  is the tariff/incentive function,  $U_i^k(\cdot)$  characterizes the gain obtained in the peer-to-peer market, and  $l_i^k(\cdot)$  is the distributed generator losses. Note that minimizing the disutility function (5) is the same as maximizing the utility.

The profit from peer-to-peer energy transactions can be expressed as follows:

$$U_i^k \left( \mathbf{p}_{E,i}^k, (p_{E,ji}^k)_{j \in \mathcal{N}_i^0} \right) := \sum_{j \in \mathcal{N}_i^0} c_{ij}^k p_{E,ij}^k - c_{ji}^k p_{E,ji}^k \quad (6)$$

where  $c_{ij}^k$  is the price of energy sold from prosumer  $i$  to prosumer  $j$  at time step  $k$ ,  $p_{E,ij}^k \geq 0$  represents the amount of energy that prosumer  $i$  is willing to offer to prosumer  $j$ . Hence, the actual energy transfer from  $i$  to  $j$ , is given by

$$\Delta p_{E,ij}^k = p_{E,ij}^k - p_{E,ji}^k, \quad \forall j \in \mathcal{N}_i^0 \quad (7)$$

which is a joint decision between prosumer  $i$  and prosumer  $j$ . Note that the sign constraint forcing  $p_{E,ij}^k$  to be non-negative allows modeling the difference between buying and selling price associated with the transfer  $\Delta p_{E,ij}^k$ .

We introduce a 1-norm regularization term that accounts for the losses of the photovoltaic generator. It is defined as

$$l_i^k(p_{G,i}^k, q_{G,i}^k) := \eta_{G,i} \left( p_{G,i}^k + |q_{G,i}^k| \right), \quad \forall (i, k) \in \mathcal{N} \times \mathcal{K} \quad (8)$$

This captures the preference of prosumers in favor of not dispatching active or reactive power unless it benefits them, whether through incentives, market conditions, or self-consumption.



In the following, we present the optimization problem to be solved by each prosumer

$$\min_{\mathbf{u}_i} J_{F,i}(T(\cdot), \mathbf{u}_i, \mathbf{u}_{-i}) \quad (9a)$$

$$\text{s.t. } p_i^k = \sum_{j \in \mathcal{N}_i^0} (p_{E,ij}^k - p_{E,ji}^k), \quad \forall k \in \mathcal{K} \quad (\mu_i^k) \quad (9b)$$

$$p_i^k = p_{G,i}^k - p_{B,i}^k - p_{D,i}^k, \quad \forall k \in \mathcal{K} \quad (v_{P,i}^k) \quad (9c)$$

$$q_i^k = q_{G,i}^k - q_{D,i}^k, \quad \forall k \in \mathcal{K} \quad (v_{Q,i}^k) \quad (9d)$$

$$s_{B,i}^k = s_{B,i}^{k-1} + \eta_{B,i} p_{B,i}^{+,k} - \frac{p_{B,i}^{-,k}}{\eta_{B,i}}, \quad \forall k \in \mathcal{K} \quad (v_{B,i}^{-,k}) \quad (9e)$$

$$(p_{G,i}^k, q_{G,i}^k) \in \mathcal{G}_i^k, \quad \forall k \in \mathcal{K} \quad (\lambda_{G,i}^k) \quad (9f)$$

$$0 \leq s_{B,i}^k \leq \hat{s}_{B,i}, \quad \forall k \in \mathcal{K} \quad (\lambda_{SL,i}^k, \lambda_{SU,i}^k) \quad (9g)$$

$$0 \leq p_{B,i}^{+,k} \leq \hat{p}_{B,i}, \quad \forall k \in \mathcal{K} \quad (\lambda_{BL,i}^{+,k}, \lambda_{BU,i}^{+,k}) \quad (9h)$$

$$0 \leq p_{B,i}^{-,k} \leq \hat{p}_{B,i}, \quad \forall k \in \mathcal{K} \quad (\lambda_{BL,i}^{-,k}, \lambda_{BU,i}^{-,k}) \quad (9i)$$

$$0 \leq p_{E,ij}^k, \quad \forall (j, k) \in \mathcal{N}_i^0 \times \mathcal{K} \quad (\lambda_{EL,ij}^k) \quad (9j)$$

where (9b) represents the balance between the net active power injection and the peer-to-peer market trade, constraints (9c) and (9d) are the active and reactive power balance with parameters  $p_{D,i}^k$  and  $q_{D,i}^k$  representing the active and reactive power demand, respectively. The dynamics of the battery are modeled via (9e), with capacity constraints (9g), (9h), and (9i). Finally, the sign constraints imposed on the bilateral offers are enforced through (9j).

The computational complexity of (9) drastically depends on the selected  $T(\cdot)$ . If  $T(\cdot)$  is a convex quadratic function, problem (9) forms a convex quadratic program with second-order cone constraints.

**Remark 1.**  $p_{E,0i}^k$  is a decision variable of prosumer  $i$  representing the energy they are willing to buy from the DSO (slack prosumer indexed by 0) at time step  $k$ . Therefore the energy transfer  $\Delta p_{E,i0}^k = p_{E,i0}^k - p_{E,0i}^k$  is a decision of prosumer  $i$ . In fact, we could replace  $(p_{E,i0}^k - p_{E,0i}^k)$  by  $\Delta p_{E,i0}^k$  whenever it appears in problem (9). As a consequence, we have that if  $T$  is convex, then (9) is always feasible because prosumer can always satisfy their power consumption needs as  $\Delta p_{E,i0}^k$  is assumed unbounded.

## 2.2. The peer-to-peer market model

The objective of this section is to leverage a model for the peer-to-peer market equilibrium. We consider two relevant classes of market designs: with and without a local market operator (LMO). The corresponding solution concepts representing the output of the market in each of the cases mentioned above are: (i) generalized Nash equilibria (GNEs) Facchinei and Kanzow (2007) and (ii) welfare optima. In the following paragraphs, we characterize the two solution concepts. Then, in Appendix A, we establish explicit connections between them in terms of dual variables and parameters of the prosumer model.

### 2.2.1. The set of GNEs

We assume that if there is no LMO in place and prosumers are free to trade with each other, then the set of market equilibria is characterized by the GNEs. In the context of this work, a GNE is a solution where no prosumer can improve their objective function by unilaterally changing their decision, given the decisions of all other prosumers. Prosumers behave selfishly and independently, minimizing their objective function under local and coupling constraints, based on (10). It extends the classical Nash equilibrium by also considering coupling constraints between the players.

The optimization problem (9) is coupled to the decision making of the other prosumers by the objective function (9a) and constraints (9b), since  $\mathbf{p}_{E,ji}$ , with  $j \in \mathcal{N}_i$ , is a vector of parameters in prosumer  $i$ 's decision problem, and simultaneously a vector of variables for prosumer  $j$ . We explicitly denote the coupling between the optimization problems by writing (9) in compact form as

$$\min_{\mathbf{u}_{F,i} \in \Omega_{F,i}(\mathbf{u}_{F,-i})} J_{F,i}(T_i(\cdot), \mathbf{u}_{F,i}, \mathbf{u}_{F,-i}) \quad (10)$$

where  $\mathbf{u}_{F,i} := (\mathbf{u}_{F,i}^k)_{k \in \mathcal{K}}$  is the action profile of prosumer  $i$  over the whole planning horizon, and  $\Omega_{F,i}(\cdot)$  is a point-to-set mapping—also referred to as *set-valued map* (see (Rockafellar and Wets, 1998, Chapter 5)),—representing the feasible set of prosumer  $i$  as a function of the action profiles of the other prosumers.

More precisely, consider a game with a set  $\mathcal{N}$  of players, and the tuple  $(\Omega_{F,i}, J_{F,i}, T_i)_{i \in \mathcal{N}}$  as the data.

**Definition 1.** The set of GNEs, denoted by  $S(T)$ , is the set of elements  $\bar{\mathbf{u}} \in \prod_{i \in \mathcal{N}} \mathbb{R}^{n_i}$  such that

$$\bar{\mathbf{u}}_i \in \arg \min_{\mathbf{u}_i \in \Omega_i(\bar{\mathbf{u}}_{-i})} J_{F,i}(T_i(\cdot), \mathbf{u}_i, \bar{\mathbf{u}}_{-i}), \quad \forall i \in \mathcal{N} \quad (11)$$

The concept of GNEs is also referred to as pseudo-Nash equilibria; see Harker (1991). In our case, the set  $S(T)$  can be interpreted as modeling the behavior of prosumers uncoordinately making bilateral contracts of energy, subject to tariff  $T$ . The fundamental task of this work is to design  $T$  such that all elements in  $S(T)$  represent operationally secure and efficient action profiles. A representation of  $S(T)$  in terms of a system of equations is studied in Appendix A for convex quadratic  $T$ .

### 2.2.2. The set of welfare optima

On the other hand, when there is a market clearing mechanism driving the system to an economically efficient equilibrium, i.e., one that maximizes the social welfare of all prosumers or, equivalently, minimizes the total disutility. The following definition captures the corresponding solution concept

**Definition 2.** The set of economically efficient market outputs, denoted by  $\mathcal{W}(T)$ , is the set of elements  $\bar{\mathbf{u}} \in \prod_{i \in \mathcal{N}} \mathbb{R}^{n_i}$  such that

$$\bar{\mathbf{u}} \in \arg \min_{\mathbf{u} \in \Omega} \sum_{i \in \mathcal{N}} J_{F,i}(T_i(\cdot), \mathbf{u}_i, \bar{\mathbf{u}}_{-i}) \quad (12)$$

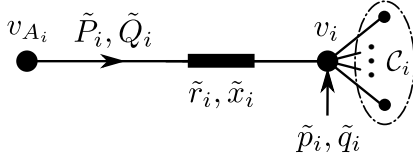


Figure 3: Electric branch associated with bus  $i \in \mathcal{B}$ .

We can interpret (12) as representing the decision-making of all prosumers jointly minimizing the total disutility.

**Remark 2.** *It is natural to ask if there exist conditions under which individual prosumers, if acting selfishly, might want to deviate from  $\mathcal{W}(T)$ . This is the same as asking if there exists an instance of the problem for which  $\bar{u}_F \in \mathcal{W}(T) \setminus \mathcal{S}(T)$ . Similarly, we may pose questions regarding the reciprocal difference  $\mathcal{S}(T) \setminus \mathcal{W}(T)$ , i.e., if there exists an output of a market without LMO that is not economically efficient. Following a similar framework as in Le Cadre et al. (2020), in Appendix A, we derive explicit expressions characterizing the relationship between  $\mathcal{S}$ ,  $\mathcal{W}$  in terms of dual variables and parameters of (9).*

### 2.3. The effect of prosumers on the electrical grid

This section focuses on modeling the impact of prosumers' behavior on the operation of the distribution system. Specifically, we aim to establish the relationship between the net active and reactive power injected by prosumers and the resulting changes in voltage levels and system losses. We drop the time-step indices unless said otherwise because all relations presented here are static.

The optimal power flow (OPF) model is a non-convex quadratically constrained quadratic problem (QCQP) which is NP-Hard in general. Because of the difficulty of solving the OPF, several approximations have been made. Between them, convex relaxations are of the families: SDP, SOCP, and linear; see Low (2014a). SDP relaxations present the difficulty of extracting solutions if the rank constraint is not met. SOCP relaxations are not exact when voltage upper bound constraints are active Low (2014b). Therefore, we believe linear approximations are better suited for situations with high penetration of renewable energy resources since the system is exposed to overvoltage conditions. Among the linear OPF models, we highlight the linear OPF model presented in Coffrin and Van Hentenryck (2014), the two models for unbalanced networks presented in Bernstein et al. (2018), and the widely referenced *LinDistFlow* model proposed in Baran and Wu (1989). In the proposed work, we adopt the *LinDistFlow* model which is detailed in the following.

For each bus  $i \in \mathcal{B}$ , the following equations hold

$$\tilde{P}_i = \sum_{j \in \mathcal{C}_i} \tilde{P}_j - \tilde{P}_i \quad (13a)$$

$$\tilde{Q}_i = \sum_{j \in \mathcal{C}_i} \tilde{Q}_j - \tilde{Q}_i \quad (13b)$$

$$v_{A_i} = v_i - 2\tilde{r}_i\tilde{P}_i - 2\tilde{x}_i\tilde{Q}_i \quad (13c)$$

Here,  $\tilde{P}_i$  and  $\tilde{Q}_i$  represent the active and reactive power flows, respectively, on the upstream branch of bus  $i \in \mathcal{B}$ , measured in the downstream direction. The parameters  $\tilde{r}_i$  and  $\tilde{x}_i$  denote the resistance and reactance of that branch, while  $v_i$  corresponds to the square of the voltage magnitude at bus  $i \in \mathcal{B}$ . The set  $\mathcal{C}_i$  represents the set of successor buses connected to  $i$ . A summary of the notation is provided in Figure 3. The *LinDistFlow* is limited to radial (tree graph) networks with balanced voltage and currents. If the reader requires a linear model capturing a more general setting, such as unbalanced and/or non-radial networks, they may refer to the first-order Taylor linearization in Bernstein et al. (2018).

To calculate the system losses, define matrices  $\tilde{\mathbf{R}} \in \mathbb{R}^{|\mathcal{B}| \times |\mathcal{B}|}$  and  $\tilde{\mathbf{X}} \in \mathbb{R}^{|\mathcal{B}| \times |\mathcal{B}|}$  with coefficients

$$\tilde{\mathbf{R}}_{ij} := \sum_{k \in \mathcal{L}(i) \cap \mathcal{L}(j)} \tilde{r}_k, \quad \tilde{\mathbf{X}}_{ji} := \sum_{k \in \mathcal{L}(i) \cap \mathcal{L}(j)} \tilde{x}_k \quad (14)$$

where  $\mathcal{L}(i)$  is the set of lines between bus  $i$  and the slack bus 0, including  $i$ . Both  $\tilde{\mathbf{R}}$  and  $\tilde{\mathbf{X}}$  are positive definite matrices, and the losses of the system can be expressed as a convex quadratic function

$$L(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) = \tilde{\mathbf{p}}^T \tilde{\mathbf{R}} \tilde{\mathbf{p}}^k + \tilde{\mathbf{q}}^T \tilde{\mathbf{R}} \tilde{\mathbf{q}} \quad (15)$$

The operational security constraints are voltage magnitude bounds at each bus. Recall that in (13), our proxy for the voltage magnitudes are the voltage magnitudes squared. Therefore, we express the security constraints limits as the following linear constraints

$$\check{v}_i^2 \leq v_i \leq \hat{v}_i^2, \quad \forall i \in \mathcal{B} \quad (16)$$

Now, it is necessary to translate the previous expressions into prosumer related-quantities. The correspondence between prosumers and buses can be described by a relation  $\mathcal{B} \times \mathcal{N} \rightarrow \{0, 1\}$ , represented by the incidence matrix  $\mathbf{B} \in \{0, 1\}^{|\mathcal{B}| \times |\mathcal{N}|}$ , such that

$$B_{ij} := \begin{cases} 1, & \text{if prosumer } j \text{ is connected to bus } i \\ 0, & \text{otherwise} \end{cases} \quad (17)$$

Let us consider again the time step indices and obtain the voltage magnitude and losses in terms of the prosumers' injections. Recall that  $\mathbf{p}^k := (p_i^k)_{i \in \mathcal{N}}$  and  $\mathbf{q}^k := (q_i^k)_{i \in \mathcal{N}}$ , for all  $k \in \mathcal{K}$ , and aggregate the effect of prosumers by bus by applying the following transformation based on (17)

$$\tilde{\mathbf{p}}^k = \mathbf{B} \mathbf{p}^k \quad (18a)$$

$$\tilde{\mathbf{q}}^k = \mathbf{B} \mathbf{q}^k \quad (18b)$$

Such that

$$\mathbf{R} := \mathbf{B}^T \tilde{\mathbf{R}} \mathbf{B} \quad (19a)$$

$$\mathbf{X} := \mathbf{B}^T \tilde{\mathbf{X}} \mathbf{B} \quad (19b)$$

The linear transformation  $\mathbf{B}$  preserves positive definiteness; therefore, again, we obtain a convex quadratic function for the

system losses in terms of the prosumers' active and reactive power injections

$$L(\mathbf{p}^k, \mathbf{q}^k) = \mathbf{p}^k \mathbf{R} \mathbf{p}^k + \mathbf{q}^k \mathbf{R} \mathbf{q}^k \quad (20)$$

We further compactify notation by defining  $\mathbf{H} = (\mathbf{R}^\top, \mathbf{X}^\top)$  to obtain a linear relation between squared voltage magnitude and the net active and reactive power injections made by prosumers

$$\mathbf{v}^k = \mathbf{H} \begin{pmatrix} \mathbf{p}^k \\ \mathbf{q}^k \end{pmatrix} + \mathbf{1} \quad (21)$$

Finally, recall that at the substation (bus 0), the distribution network is connected to an external grid (transmission system). We may express the net systemic power output export as the concave function

$$P(\mathbf{p}^k, \mathbf{q}^k) = \mathbf{1}^\top \mathbf{p}^k - L(\mathbf{p}^k, \mathbf{q}^k), \quad (22)$$

interpreted as the difference between the total amount of energy produced within the distribution network and the losses.

#### 2.4. The incentive design problem

In this section, we will leverage a reverse Stackelberg model to determine the incentive functions  $T_i^k(\cdot)$ , for each pair  $(i, k) \in \mathcal{N} \times \mathcal{K}$ . We refer to it as the incentive design problem, which is to be solved assuming global knowledge of the parameters that characterize the electrical grid and the prosumers. The objective is to drive the behavior of prosumers towards maximizing a predefined operational performance metric considering security constraints. We interpret the incentive terms  $T_i^k(p_i^k, q_i^k)$  as a local perturbation on the utility function of each prosumer. We refer to them as "local" because the domain corresponds to the decisions of only prosumer  $i$ . Ultimately, we expect optimal global operation to emerge by locally perturbing the utility functions of the players.

The objective function of the regulatory entity (the leader) is an operational performance metric. For simplicity of exposition, we define it as the total exported active power

$$J_L(\mathbf{p}, \mathbf{q}) := \sum_{k \in \mathcal{K}} P(\mathbf{p}^k, \mathbf{q}^k) \quad (23)$$

which is a strictly concave quadratic function to be maximized.

It is worth noting that the leader's objective function solely relies on the variables  $\mathbf{p}$  and  $\mathbf{q}$ . Therefore, the applied incentive is specifically targeted toward these variables. For computational tractability reasons, we restrict ourselves to a parametric family of functions on  $(\mathbf{p}, \mathbf{q})$ , such that

$$T_i^k(p, q) := \gamma_i^k(p, q)^\top \begin{pmatrix} p \\ q \end{pmatrix} \quad (24)$$

where  $\gamma_i^k : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a semidefinite positive affine transformation. To be more precise, we state that  $\gamma_i^k \in \mathcal{A}$  for all  $(i, k) \in \mathcal{N} \times \mathcal{K}$  iff. there exist coefficients  $\Phi_i^k \in \mathbb{R}^{2 \times 2}$  and  $\phi_i^k$ , such that  $\Phi_i^k = \Phi_i^{k\top} \geq 0$  and

$$\gamma_i^k(p, q) = \frac{1}{2} \Phi_i^k \begin{pmatrix} p \\ q \end{pmatrix} + \phi_i^k \quad (25)$$

where

$$\Phi_i^k := \begin{pmatrix} \Phi_{pp,i}^k & \Phi_{pq,i}^k \\ \Phi_{pq,i}^k & \Phi_{qq,i}^k \end{pmatrix}, \quad \phi_i^k := \begin{pmatrix} \phi_{p,i}^k \\ \phi_{q,i}^k \end{pmatrix}$$

are the incentive function coefficients meant to be determined by the DSO. Hence, a tuple  $(\Phi, \phi)$  uniquely determines the incentive function  $\gamma$ .

We write the vector of tariff costs as a function of the vector of control actions for each time step  $k$  by stacking the entries corresponding to each prosumer

$$T^k(\mathbf{u}^k) = (\gamma^k(\mathbf{u}^k))^\top \mathbf{u}^k \quad (26)$$

Accordingly, we define the matrix  $\tilde{\Phi}^k$  and vector  $\tilde{\phi}^k$  such that

$$\gamma^k(\mathbf{u}^k) = \tilde{\Phi}^k \mathbf{u}^k + \tilde{\phi}^k \quad (27)$$

Prosumers are assumed to behave rationally, solving problem (9) to minimize their disutility (maximize their profit). This is the main driver for the proposed approach. To that end, some design choices over the shape of functions  $T_i$  should be put in place so that (9) is simple enough for prosumers to solve. First, the incentive cost term,  $T_i^k(p_i^k, q_i^k)$ , is known in advance to each prosumer, and it is a function of their own local active and reactive power output. Second,  $T_i^k$  is a quadratic convex function.

Operation performance metrics of distribution networks ultimately capture the cost of operation. It might consider Ohmic losses on electrical branches<sup>1</sup>, total exported energy, voltage violations, line congestion, etc.

##### 2.4.1. The reverse Stackelberg game model

Determining the incentive functions requires characterizing how they influence the behavior of prosumers and, ultimately, the system's operation. There is a need for a different kind of equilibrium concept than the one utilized to model the peer-to-peer market since the roles of the DSO and the prosumers are not symmetric.

In the peer-to-peer market equilibrium model described in Section 2.2, prosumers announced their strategies simultaneously. Here, the DSO has to announce the incentive functions first and then let the prosumers react accordingly. To capture this asymmetry, we rely on a particular case of Stackelberg equilibrium described in the following paragraphs.

As in Ho et al. (1981), we acknowledge a difference between *announcing a strategy* and *performing an action*. It is the order of announcing strategies rather than the order of actions that distinguishes the leader from the follower. The reverse Stackelberg equilibrium solution concept considers the following timeline:

1. the leader announces their strategy
2. knowing the leader's strategy, the followers choose their action

<sup>1</sup>Lines and transformers.



3. the action of the leader is determined based on their previously announced strategy as a function of the actions of the followers

In the proposed model, the strategy of the leader is the tariff function  $\gamma(\cdot)$ , the action of the follower is the vector  $\mathbf{u}$  whose elements are the active and reactive power injections, and the action of the leader is  $\gamma(\mathbf{u})$ . The reverse Stackelberg optimization problem is given as

$$\text{“max”}_{\gamma(\cdot)} J_L(\gamma(\cdot), \mathbf{u}) \quad (28a)$$

$$\text{s.t. } \gamma(\cdot) \in \mathcal{A} \quad (28b)$$

$$\gamma(\cdot) \in \mathcal{D} \quad (28c)$$

$$\check{\mathbf{v}} \leq \mathbf{H}\mathbf{u}^k + \mathbf{1} \leq \hat{\mathbf{v}} \quad \forall k \in \mathcal{K} \quad (28d)$$

$$\mathbf{u} \in \mathcal{S}(\gamma(\cdot)) \quad (28e)$$

where (28b) forces  $\gamma(\cdot)$  to belong to the family of semidefinite positive affine functions from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ , denoted by  $\mathcal{A}$ , constraint (28c) imposes the budget condition limiting the action of the leader, constraint (28d) enforces the voltage levels, and (28e) is the lower-level equilibrium problem. The quoted  $\max\{\cdot\}$  operator is utilized to emphasize that (28) might be ill-posed as the lower level problem might have a multiplicity of solutions (28e), see Dempe and Dutta (2012). Uniqueness of solutions in the space of active and reactive net power injection is studied in Section 3.

Depending on the focus, the lower-level (28e) might be replaced by

$$\mathbf{u} \in \mathcal{W}(\gamma(\cdot)) \quad (29)$$

to solve the problem as if the market clearing mechanism were efficient in the economic sense, i.e., if the equilibrium maximizes the welfare or, equivalently, minimizes the total disutility.

#### 2.4.2. The incentive budget constraints

For each  $(i, k) \in \mathcal{N} \times \mathcal{K}$ , we assume that there is a physical bound on  $\mathbf{w}_i^k := (p_i^k, q_i^k)$  given by the prosumer-to-grid connection capacity, which takes the form

$$\|\mathbf{w}_i^k\|_2 \leq E_i \quad (30)$$

where  $E_i$  is the connection capacity measured in the same physical units of apparent power, i.e. per unit or volt-ampere (VA) units. Therefore, we restrict  $(\Phi_i^k, \phi_i^k)$  so that we ensure that for any  $\mathbf{w}_i^k$  satisfying (30), the incentive cost terms are bounded. More precisely, we impose the following budget condition

$$(\Phi_i^k, \phi_i^k) \in \mathcal{D}_i, \quad \forall (i, k) \in \mathcal{N} \times \mathcal{K} \quad (31)$$

such that

$$\mathcal{D}_i^k := \left\{ (\Phi_i^k, \phi_i^k) \in \mathbb{S}_2 \times \mathbb{R}^2 \mid \forall \mathbf{w}_i^k \in \mathbb{R}^2 : \|\mathbf{w}_i^k\|_2 \leq E_i, \right. \\ \left. -M_i^k \leq (\mathbf{w}_i^k)^\top \Phi_i^k \mathbf{w}_i^k + \phi_i^k \mathbf{w}_i^k \leq M_i^k \right\} \quad (32)$$

where  $\mathbb{S}_2$  is the set of symmetric positive semidefinite matrices of size 2.

**Proposition 1.** For each  $(i, k) \in \mathcal{N} \times \mathcal{K}$ , if  $(\Phi_i^k, \phi_i^k)$  satisfies

$$E_i \|\Phi_i^k\|_F + \|\phi_i^k\|_2 \leq \frac{M_i}{E_i}, \quad (33)$$

then  $(\Phi_i^k, \phi_i^k) \in \mathcal{D}_i$ .

PROOF. For simplicity of notation, ignore the indices  $i$  and  $k$ . We write (33) as

$$E \|\Phi\|_F + \|\phi\|_2 \leq \frac{M}{E}$$

Recall that the Frobenius norm is an upper bound on the matrix 2-norm. Then,

$$E^2 \|\Phi\|_2 + E \|\phi\|_2 \leq M$$

The connection capacity condition (30) states that  $E$  is an upper bound on  $\|\mathbf{w}\|_2$ , therefore the following is true

$$\|\mathbf{w}\|_2^2 \|\Phi\|_2 + \|\phi\|_2 \|\mathbf{w}\|_2 \leq M$$

Then, we apply the the Cauchy-Schwarz inequality to obtain

$$\|\mathbf{w}\|_2 \|\Phi \mathbf{w}\|_2 + \|\phi\|_2 \|\mathbf{w}\|_2 \leq M$$

followed by

$$\|\mathbf{w}^\top \Phi \mathbf{w} + \phi^\top \mathbf{w}\|_2 \leq M$$

Consequently,

$$-M \leq \mathbf{w}^\top \Phi \mathbf{w} + \phi^\top \mathbf{w} \leq M$$

□

Problem (28) can be more explicitly written such that  $\gamma(\cdot)$  is parameterized in terms of the coefficients  $(\Phi, \phi)$ .

Problem (28) can be more explicitly written such that  $\gamma(\cdot)$  is parameterized in terms of the coefficients  $(\Phi, \phi)$ .

The formulation of Problem (28) can be further refined by explicitly parameterizing  $\gamma(\cdot)$  in terms of the coefficients  $(\Phi, \phi)$ .

$$\max_{\Phi, \phi} J_L(\Phi, \phi, \mathbf{u}) \quad (34a)$$

$$\text{s.t. } \Phi_{pp,i}^k + \Phi_{qq,i}^k \geq 0 \quad \forall (i, k) \in \mathcal{N} \times \mathcal{K} \quad (34b)$$

$$(\Phi_{pq,i}^k)^2 \leq \Phi_{pp,i}^k \Phi_{qq,i}^k \quad \forall (i, k) \in \mathcal{N} \times \mathcal{K} \quad (34c)$$

$$E_i \|\Phi_i^k\|_F + \|\phi_i^k\|_2 \leq \frac{M_i}{E_i} \quad \forall (i, k) \in \mathcal{N} \times \mathcal{K} \quad (34d)$$

$$\check{\mathbf{v}} \leq \mathbf{H}^k \mathbf{u}^k + \mathbf{h}^k \leq \hat{\mathbf{v}} \quad \forall k \in \mathcal{K} \quad (34e)$$

$$\mathbf{u} \in \mathcal{S}(\Phi, \phi) \quad (34f)$$

where the objective function (34a) is concave quadratic, constraints (34b) and (34b) ensure that each matrix  $\Phi_i^k \in \mathbb{R}^{2 \times 2}$

is positive semidefinite, equation (34d) represents the budget conditions, (34e) are the voltage limits, and (34f) models the lower-level. Note that the complexities of problem (28) arise primarily from (34f), as it would otherwise be a simpler second-order conic problem.

The problem can be further compactly written as

$$\gamma(\cdot) \in \arg \min_{\gamma(\cdot) \in \Gamma(\mathbf{u})} J_L(\gamma(\cdot), \mathbf{u}) \quad (35a)$$

$$\mathbf{u}_i \in \arg \min_{\tilde{\mathbf{u}}_i \in \Omega_{F,i}(\mathbf{u}_{-i})} J_F(\gamma_i(\cdot), \tilde{\mathbf{u}}_i, \mathbf{u}_{-i}), \quad \forall i \in \mathcal{N} \quad (35b)$$

### 3. Properties

In this section, our concern is to study how the lower-level equilibrium might respond to perturbations in the parameters defining the objective function of prosumers. Specifically, our concern is on studying stability in the sense of Dempe (2002) for the lower level as a parametric optimization problem. First, we show that the GNEP can be reformulated as an optimization problem, classically known as the Nikaido–Isoda (NI) reformulation, see Facchinei and Kanzow (2007). Then we show that the set of solutions to the GNEP,  $S(T)$ , is strongly stable towards perturbations in the parameters that define it by proving that the NI reformulation is strongly stable. As corollaries, we leverage guarantees of uniqueness of solutions in the net active and reactive power output space.

**Definition 3.** (Jointly convex GNEP) We say that (11) is a jointly convex GNEP if, for every prosumer  $i \in \mathcal{N}$  and every  $\mathbf{u}_{-i}$ , the objective function  $J_F(T(\cdot), \cdot, \mathbf{u}_{-i})$  is convex, the set  $\Omega_i(\mathbf{u}_{-i})$  is closed and convex, and for some closed convex  $\Omega \subset \mathbb{R}^n$  and all  $i \in \mathcal{N}$  we have

$$\Omega_i(\mathbf{u}_{-i}) = \{\mathbf{u}_i \in \mathbb{R}^{n_i} \mid (\mathbf{u}_i, \mathbf{u}_{-i}) \in \Omega\}$$

As remarked in Facchinei and Kanzow (2007), if the feasible set of each player of a GNEP is described by inequalities and if all the shared constraints, i.e., the ones that involve variables from more than one player are the same for every player involved in the constraint, then the GNEP in question is a jointly convex GNEP. By this remark, it is clear that the GNEP in (11) is jointly convex. By the discussion above we derive the following result

**Lemma 1.** The game defined in (11) is a jointly convex GNEP.

First, we establish a classical result that bridges jointly convex GNEPs and optimization problems

**Lemma 2.** If  $T_i$  is convex for each  $i \in \mathcal{N}$ , a vector  $\bar{\mathbf{u}}$  is a solution to (11) if and only if  $\bar{\mathbf{u}}$  is a global minimum of the optimization problem

$$\min_{\mathbf{u}} \hat{V}(T, \mathbf{u}) \quad (36a)$$

$$\text{s.t. } \mathbf{u} \in \Omega, \quad (36b)$$

with zero objective function value, where

$$\hat{V}(T, \mathbf{u}) := \sup_{\mathbf{y} \in \Omega} \psi(T, \mathbf{u}, \mathbf{y}) \quad (37)$$

and

$$\psi(T, \mathbf{u}, \mathbf{y}) := \sum_{i \in \mathcal{N}} (J_{F,i}(T_i, \mathbf{u}_i, \mathbf{u}_{-i}) - J_{F,i}(T_i, \mathbf{y}_i, \mathbf{u}_{-i})) \quad (38)$$

Lemma 2 corresponds to the classical Nikaido–Isoda (NI) reformulation in the case of jointly convex GNEPs, hence the proof is omitted, see Facchinei and Kanzow (2007). The NI reformulation is an important analytical tool that allows to transform the equilibrium in (11) into an optimization problem.

**Lemma 3.** If  $T_i(\cdot)$  is convex for each  $i \in \mathcal{N}$ , problem (36) is a convex optimization problem with polyhedral feasible set

PROOF. Note that the objective function of each follower can be expressed as the sum

$$J_F(T_i(\cdot), \mathbf{u}_i, \mathbf{u}_{-i}) = h_i(\mathbf{u}_i, T_i(\cdot)) + g_i(\mathbf{u}_{-i}), \quad \forall i \in \mathcal{N} \quad (39)$$

with

$$h_i(\mathbf{u}_i, T_i(\cdot)) := \sum_{k \in \mathcal{K}} (T_i^k(p_i^k, q_i^k) + l_i(p_{G,i}^k, q_{G,i}^k))$$

and

$$g_i(\mathbf{u}_{-i}) := \sum_{k \in \mathcal{K}} U_i^k (p_{E,ij}, (p_{E,ij})_{j \in \mathcal{N}_i^0})$$

Then the NI-function becomes

$$\psi(T(\cdot), \mathbf{u}, \mathbf{y}) = \sum_{i \in \mathcal{N}} (h_i(\mathbf{u}_i, T_i(\cdot)) - h_i(\mathbf{y}_i, T_i(\cdot))) \quad (40)$$

For fixed  $T(\cdot)$ , function  $\psi(T(\cdot), \mathbf{u}, \mathbf{y})$  is convex in  $\mathbf{u}$ , for all  $\mathbf{y} \in \Omega$ , therefore  $\hat{V}(T(\cdot), \mathbf{u})$  is convex in  $\mathbf{u}$  as the supremum operation in (37) preserves convexity.  $\square$

**Theorem 1.** If  $T(\cdot)$  is convex, the set of GNEs  $S(T)$  of (11) is convex. Moreover, if  $T(\cdot)$  is strictly convex, the set of GNEs of (11),  $S(T)$ , is uniquely defined in  $(\mathbf{p}, \mathbf{q})$ .

PROOF. By Lemma 3, problem (36) is a convex optimization problem. Therefore, the set of optimal solutions is a convex set. By Lemma 2, the solution set of (36) corresponds to  $S(T)$  if the objective value of (36) is zero at optimality. Therefore, to prove Theorem 1, it suffices to prove that the optimal value of (36) is zero for any given  $T(\cdot)$ . For that, we express the NI-function as in (40), and considering  $\Omega$  is closed, the optimal value of (36) can be written as

$$\hat{V}^* = \min_{\mathbf{u} \in \Omega} \left\{ \max_{\mathbf{y} \in \Omega} \sum_{i \in \mathcal{N}} (h_i(\mathbf{u}_i, T_i(\cdot)) - h_i(\mathbf{y}_i, T_i(\cdot))) \right\} \quad (41)$$

which is zero. Therefore  $S(T)$  is convex.

If  $T(\cdot)$  is strictly convex, (40) is strictly convex with respect to  $(\mathbf{p}, \mathbf{q})$ , therefore the NI reformulation (36) is unique in  $(\mathbf{p}, \mathbf{q})$ , thus, by Lemma 2, so does the set  $S(T)$ .  $\square$

The DSO exclusively considers the net active and reactive power output, denoted as  $(\mathbf{p}, \mathbf{q})$ , as the operational performance metrics hinge on these variables. Hence, our focus is

on ensuring the robustness of the solution, specifically within the  $(\mathbf{p}, \mathbf{q})$ -space, against variations in lower-level model parameters. Thus, providing a robust stability guarantee in this projected space suffices.

Expanding on this concept, it worth noting that the objective function (34a) and the coupling constraints (34e) in the tariff design problem (34) solely depend on  $(\mathbf{p}, \mathbf{q})$ . Consequently, establishing a uniqueness guarantee for  $\mathcal{S}(T(\cdot))$  in  $(\mathbf{p}, \mathbf{q})$  circumvents challenges associated with non-uniqueness of solutions in mathematical programs with equilibrium constraints. This guarantee is easily attained using the tools we have leveraged thus far.

To ensure robustness of the solution against parameter variations in the lower-level problem, we extend our approach beyond merely ensuring uniqueness to focus the notion of stability. Our approach involves drawing upon the concept of strong stability for local solutions, as defined in Dempe (2002), which we paraphrase as follows

**Definition 4 (Dempe (2002)).** *A local solution  $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$  of a parametric optimization problem*

$$\min_{\mathbf{x}} f(\mathbf{x}, \mathbf{y}) \quad (42a)$$

$$\text{s.t. } g(\mathbf{x}, \mathbf{y}) \leq 0 \quad (42b)$$

is said to be strongly stable if there exist neighborhoods  $\mathcal{U}_\delta(\bar{\mathbf{y}})$  with  $\delta > 0$  of  $\bar{\mathbf{y}}$  and  $\mathcal{V}_\epsilon(\bar{\mathbf{x}})$  with  $\epsilon > 0$  of  $\bar{\mathbf{x}}$ , and a uniquely determined continuous vector-valued function  $x : \mathcal{U}_\delta(\bar{\mathbf{y}}) \rightarrow \mathcal{V}_\epsilon(\bar{\mathbf{x}})$  such that  $x(y)$  is the unique local optimal solution of problem (42) in  $\mathcal{V}_\epsilon(\bar{\mathbf{x}})$  for all  $y \in \mathcal{U}_\delta(\bar{\mathbf{y}})$ .

First, we introduce a projection operator. Let  $\bar{\mathbf{u}} \in \Omega$  represent a solution at the lower level, with  $\bar{\mathbf{u}}_i^k := \left( \bar{p}_i^k, \bar{q}_i^k, \bar{p}_{G,i}^k, \bar{q}_{G,i}^k, \bar{p}_{B,i}^k, \bar{q}_{B,i}^k, \bar{s}_{B,i}^k, \bar{p}_{E,i}^k \right)$ . We define  $\Pi_{pq}(\bar{\mathbf{u}}_i^k) := (\bar{p}_i^k, \bar{q}_i^k)$  and extend the notation to aggregate variables, such that  $\Pi_{pq}(\bar{\mathbf{u}}) := (\bar{\mathbf{p}}, \bar{\mathbf{q}})$ .

**Proposition 2.** *If  $(\bar{\Phi}, \bar{\phi}, \bar{\mathbf{u}})$  is an optimal solution to (34) and  $\bar{\mathbf{w}} = \Pi_{pq}(\bar{\mathbf{u}})$ , then the projected solution  $(\bar{\Phi}, \bar{\phi}, \bar{\mathbf{w}})$  is strongly stable.*

PROOF. From (41),  $V(T(\cdot), \mathbf{u})$  can be written as

$$V(T(\cdot), \mathbf{u}) = \sum_{i \in \mathcal{N}} h_i(T(\cdot), \mathbf{u}) + C \quad (43)$$

where  $C$  is constant with respect to  $\mathbf{u}$ , given by

$$C := \max_{\mathbf{y} \in \Omega} \left\{ - \sum_{i \in \mathcal{N}} h_i(T(\cdot), \mathbf{y}) \right\}$$

thus,  $V(T(\cdot), \mathbf{u})$  is differentiable in  $\mathbf{u}$ . In fact, the partial hessian matrix of the lagrangian function of the NI reformulation (36) with respect to  $\mathbf{w}$  is a block-diagonal matrix

$$\frac{\partial^2 L}{\partial \mathbf{w}^2} = \text{diag} \left( \left( \bar{\Phi}_i^k \right)_{i \in \mathcal{N}, k \in \mathcal{K}} \right)$$

Therefore  $\frac{\partial^2 L}{\partial \mathbf{w}^2}$  is positive definite because each block diagonal element is positive definite as  $\epsilon > 0$ ; thus  $(\bar{\Phi}, \bar{\phi}, \bar{\mathbf{w}})$  satisfies the strong sufficient optimality condition of second order Dempe (2002). In addition, (36) satisfies the Mangasarian-Fromowitz constraint qualification (MFCQ). Then, Proposition 2 follows directly from Theorem 4.4 in Dempe (2002).  $\square$

Proposition 2 stands as a robustness guarantee, ensuring that the tariff solution remains reliable in the face of parameter variations of the lower-level. Note that stability defined in this sense implies uniqueness of the market output with respect to  $(\mathbf{p}, \mathbf{q})$ .

Furthermore, the stability property not only applies for the GNEP-based market setup. It remains true for both when the market is modeled as a GNEP and when it is modeled as an optimization problem that aims to minimize total disutility, as presented in Definition 2. To see this, note that if  $T(\cdot)$  is strictly convex, then (12) is a strictly convex problem with respect to  $(\mathbf{p}, \mathbf{q})$ , and the same line of analysis developed in this section can be applied. This implies that the stability condition we have established is equally applicable to both managed and unmanaged market designs, solidifying the versatility of the proposed model.

We highlight that the NI reformulation is a valuable tool for characterizing the stability of jointly convex GNEPs. Note that by characterizing the stability of the solution of the NI reformulation, we bypass the notions of stability in variational systems such as Rockafellar (2023) which are commonly used in more general variational systems when there is no optimization problem reformulation at hand.

## 4. Numerical results

An instance of the problem comprises a distribution network, the prosumers with their devices, and the financial grid. For each instance, the computational experiments follow a scheme in which the tariff is calculated by the DSO and passed to the prosumers in a rolling horizon fashion<sup>2</sup>. In this case, tariff functions for all prosumers are calculated considering a planning horizon of 4 hours and applied only for 1 hour. We run the experiments utilizing a 10-minute time step setting; therefore, each instance of (34) to be solved has  $|\mathcal{K}| = 6 \times 4 = 24$  time steps.

### 4.1. A toy problem

This section is dedicated to leveraging a toy problem, with the objective of gaining deeper insights into the proposed tariff design and offering a tangible proof of concept. First, we show how the DSO is able to induce safe operation by prompting the voltage profiles to remain within nominal operation bounds. Then, we establish the main differences between the proposed approach and the conventional constant tariff, which involves only bilinear terms between leader and follower variables.

The experimental setup for this toy problem involves a set  $\mathcal{N} = \{1, 2, 3\}$  of prosumers connected to a 4 bus electrical

<sup>2</sup>Also called receding horizon.

grid where the set of buses is  $\mathcal{B}^0 = \{0, 1, 2, 3\}$ , with parent buses  $A_1 = 0, A_2 = 1$ , and  $A_3 = 0$ . The set of prosumers is  $\mathcal{N} = \{1, 2, 3\}$  such that  $\mathbf{B} = \text{diag}(1, 1, 1)$  as depicted in Figure 1. The distribution system corresponds to the IEEE 4-bus test feeder IEEE (1991) which has been adapted to this setting by balancing the line segments using Carson's equation and Kron reduction. Distributed loads have been converted into balanced three-phase loads. The resistance of each electrical branch has been increased so that the resistance to reactance ratio is exactly 2.3—a normal ratio in distribution systems.—In this case, each prosumer model represents a group of actual prosumers. This is because the IEEE 4-bus system is a simplified electrical equivalent of a whole distribution system. Therefore, to analyze the effect of the prosumers in the operation of the grid, it is necessary to aggregate the effect of a large group of them.

The time series data for the problem is depicted in Figure 4. On top is the maximum available solar power, and at the bottom is the active power load. The signals have been obtained from one-minute time series data from a set of individual homes from the U.S. Austin region Pecan Street (2021). The time-series have been upsampled, averaging the values over periods of 10 minutes to accommodate them to the current time setting. A constant power factor of 0.85 is considered; therefore, the reactive power load is proportional to the active power load by a constant factor of  $\tan(\cos^{-1}(0.85)) \approx 0.61$ .

Naturally, the high solar irradiance period occurs around midday. During this period, there are high chances of violating the upper voltage limit because the net active power injections increase and  $\mathbf{R}$  in (19a) is a positive definite matrix that maps the increment in power injection into an increment of voltage magnitudes. Without the tariff incentive, the voltage magnitude of electrical Node 2 would have been out of bounds. Figure 5 depicts the voltage profiles considering the tariff. Note that the voltage profile gets flattened to remain within bounds during the high irradiance interval. This is achieved by inducing the active and reactive net power profiles shown in figures 6 and 8. A similar process occurs during the high-demand periods in the evening, where voltages tend to lower, and they get flattened as a result of the tariff to avoid a lower limit voltage violation.

In Figure 6, we observe that in time periods near midday, the induced net power injection depicted in orange has less noise than the blue line representing the maximum available net active power. Recall that the tariff gets updated and passed to the prosumers hourly. The red vertical lines show the moments in which the tariff is recomputed and passed to the prosumers. Notably, within these intervals, the net active power injection maintains a near-constant level despite the underlying potential for variation. This observation suggests a denoising effect associated with the tariff updates, which likely contributes to enhancing the predictability of the system's operation—a beneficial characteristic for operating the distribution network.

## 4.2. Experiments on larger instances

In evaluating our proposed method, we look at various metrics and extend our tests to larger scenarios based on the IEEE 4-bus and 34-bus systems. We adopt the same rolling horizon approach as we did in the earlier section solving 168

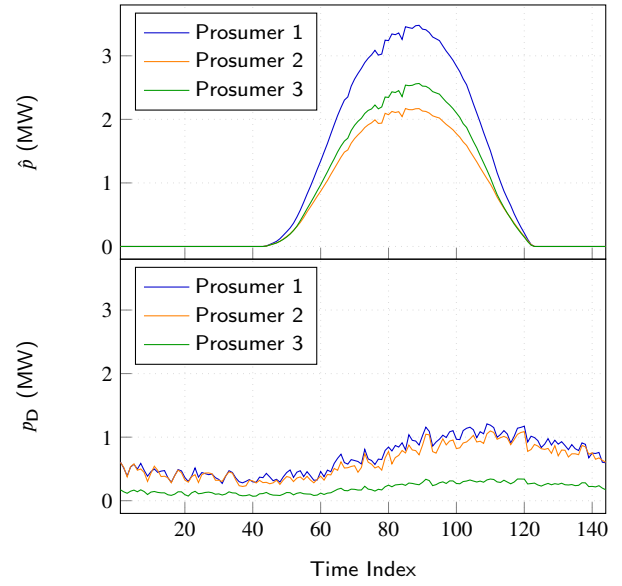


Figure 4: Time series data of maximum available power (top), and active power load (bottom).

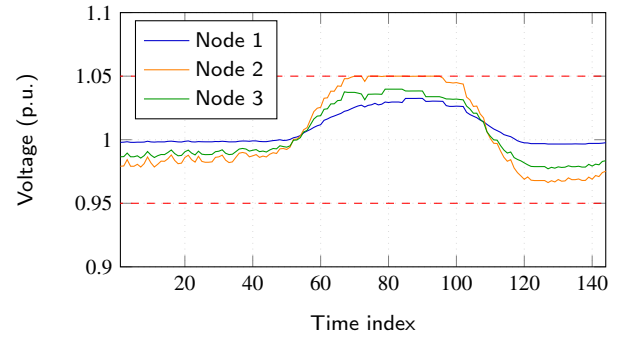
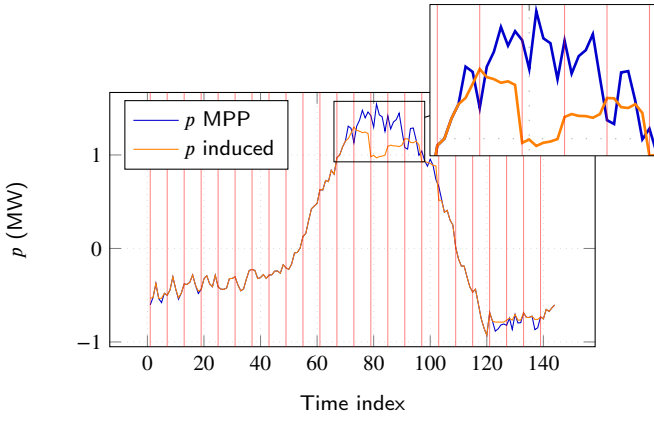


Figure 5: Voltage magnitude per bus.

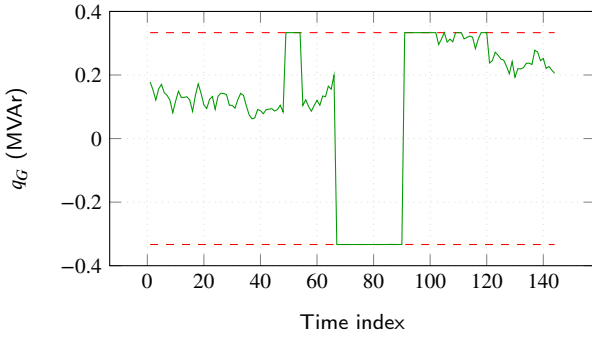
instances of the tariff design problem, one each hour over a week. We considered the following performance metrics:

- **Hourly Average Number of Voltage Violations per Bus (NVV):** This is the count of average voltage magnitude deviations from the acceptable range per bus, calculated over 10-minute intervals.
- **Hourly Average Net Energy Export per Prosumer (NEE) in kW:** We measure the total energy exported to the grid, divided by the evaluation period and the number of prosumers.
- **Hourly Average Distribution Grid Losses (DGL) in kW:** This represents the total resistive losses in the grid, averaged over the evaluation period.
- **Hourly Average Disutility per Prosumer (DPP) in \$:** This is the sum of each prosumer's disutility, averaged over the time of the evaluation.
- **Hourly Average Tariff Cost per Prosumer (TC) in \$:** We accumulate the tariff costs from the prosumers'

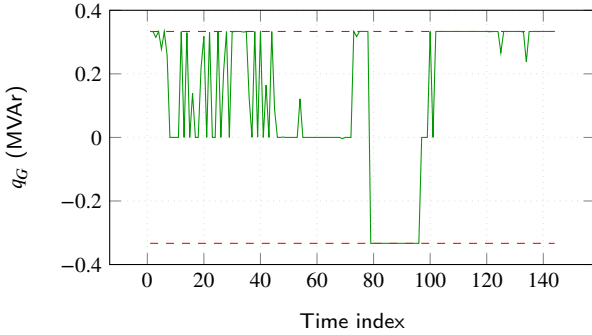




**Figure 6:** Net active power injection at maximum power point (MPP) vs. the induced signal.



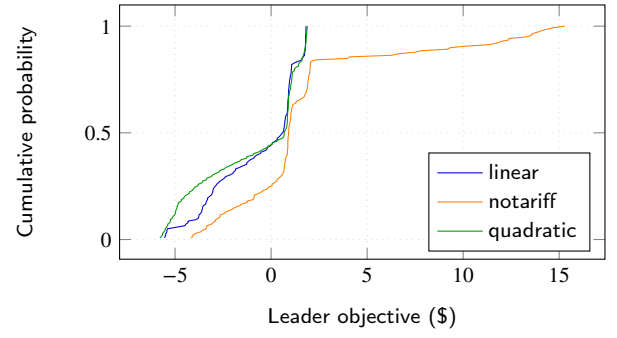
**Figure 7:** Reactive power generation of prosumer 2.



**Figure 8:** Reactive power generation of prosumer 2.

objective function over each period and then average it out.

Upon examining the NVV and NEE columns in Table 1, a comparison between the proposed approach and the constant tariff policy reveals a notable observation. While both strategies ensure the system operates safely ( $NVV = 0$ ), the proposed tariff design stands out by achieving this objective alongside maintaining higher levels of net energy export (NEE). However, it is important to note that the proposed approach does incur higher levels of ohmic losses. This can be attributed to the



**Figure 9:** Cumulative probability density function objective function of the DSO, IEEE-4bus test feeder.

increased amounts of energy being exported, which naturally leads to greater loss in the transmission process.

With the focus on operation performance, we did not consider the tariff budget in the objective function of the leader. Note that in Table 1, the TC metric changes signs from the proposed tariff to the constant tariff. The results are likely to be different if considering the budget upperbound  $M_i$  for  $i \in \mathcal{N}$  in the objective function of the leader, which can be done without changing the structure of the problem.

While these metrics offer insights into different aspects of performance, our primary focus is on minimizing the DSO's objective function, as it is indicative of the overall effectiveness of our model. The graph in Figure 9 illustrates the cumulative probability distribution of the DSO's objective function values across the three strategies we have put to the test.

## 5. Conclusions

In our research, we have successfully demonstrated the efficacy of a reverse Stackelberg game approach to demand response in power distribution networks. The proposed framework positions the DSO as the leader with the ability to impose a tariff function, to which the prosumers, interpreted as followers, react. This framework also encompasses a peer-to-peer market among the prosumers, modeled as a GNEP, and takes into account both active and reactive power generation and their effects on the network's voltage levels. This comprehensive approach addresses a commonly neglected aspect in current research, namely, the impact of market outcomes on the operation of the distribution network.

The numerical experiments indicate a marked improvement in maximizing the net power output and maintaining the voltage levels within safe margins, compared to the conventional constant time-of-use tariff, particularly when perturbations in the followers' objective function parameters are introduced. To understand the reason behind the effectiveness in the presence of perturbations in the parameters of the lower-level, we provide theoretical results ensuring that the obtained solutions are always strongly stable. Moreover, our method stands out in its ability to denoise net active and reactive power injection of prosumers, which is also attributed to the stability property mentioned above. These findings underscore the potential of our bilevel programming solution to not only streamline the

**Table 1**

Mean value of selected performance metrics.

Strategies	IEEE 4-bus test feeder				TC (\$)	IEEE 34-bus test feeder				
	NVV -	NEE (MW)	DGL (MW)	DPP (\$)		TC (\$)	NVV -	NEE (MW)	DGL (MW)	DPP (\$)
No tariff	0.087	0.351	0.028	-0.300	0.	0.101	0.118	0.089	-0.106	0.
Constant tariff	0.	0.218	0.022	-0.119	0.088	0.	0.073	0.065	-0.030	0.078
Proposed	0.	0.339	0.027	-0.292	-0.022	0.	0.108	0.077	-0.126	-0.056

objective function of the DSO but also to bring consistency and predictability to prosumer behavior, thereby reinforcing the overall efficiency and reliability of local energy markets.

We highlight the application of the Nikaido–Isoda reformulation to the lower-level GNEP to characterize the set of GNEs. Looking forward, we believe this reformulation holds significant potential for practical implementation, particularly in solving Stackelberg games with a GNEP at the lower level through a branch-and-cut strategy.

One limitation of the current approach is the assumption of fixed and known peer-to-peer prices, which should be the endogenous result of the bidding process among agents in a local market platform. Future work should relax this assumption, enabling a more dynamic and realistic modeling of price mechanisms in these markets.

Another critical area for future investigation is the incorporation of inherent uncertainties, such as solar irradiance and demand fluctuations, into our model. Addressing this right-hand side uncertainty, particularly in the lower level of the bilevel problem, is paramount. Drawing inspiration from robust optimization methods successfully applied in voltage regulation problems within active distribution networks, we propose exploring these methods within bilevel robust optimization. This approach can potentially enhance the resilience and efficacy of our model in the face of uncertainty.

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## A. The KKT reformulation of the lower-level

In this section, we construct the KKT reformulation of the lower-level GNEP (11). It is well known that if the optimization problem of each player satisfies standard constraint qualifications, solving the GNEP is equivalent to solving the concatenation of the KKT systems of all players Facchinei and Kanzow (2007). We start by constructing an inner polyhedral approximation the feasible dispatch set of each DG (1), allowing us to represent (9) as a polyhedron. In the case  $\Omega(\mathbf{u}_{-i})$  is a polyhedron, no matter the choice of  $\mathbf{u}_{-i}$ , and the objective function of each prosumer is convex quadratic as long as  $\Phi_i^k \geq 0$ , the problem of each prosumer satisfies the so-called refined Slater constraint qualification (RSCQ), presented as Theorem 28.2 in Rockafellar (1970). As a consequence, the stacked KKT conditions of all followers become necessary and sufficient for a solution  $\bar{\mathbf{u}}$  to be a solution to (11), i.e.,  $\bar{\mathbf{u}} \in S(T)$ .

First, we construct a polyhedral inner approximation of (1c) following Ben-Tal and Nemirovski (2001). Therefore, replacing (1c) by the following set of inequalities for all  $j = 0, \dots, n_{\text{SOC}}$

$$-\alpha_{G,ij}p_G - \beta_{G,ij} \leq q_G \quad (44a)$$

$$\alpha_{G,ij}p_G + \beta_{G,ij} \geq q_G \quad (44b)$$

Where

$$\beta_{G,ij} = q_{G,ij}^0 - \alpha_{G,ij}p_{G,ij}^0 \quad (45)$$

$$\alpha_{G,ij} = \frac{q_{G,i,j+1}^0 - q_{G,ij}^0}{p_{G,i,j+1}^0 - p_{G,ij}^0} \quad (46)$$

$$\left( p_{G,ij}^0, q_{G,ij}^0 \right) = \bar{s}_{G,i} \left( \sin \left( \frac{j\pi}{4n} \right), \cos \left( \frac{j\pi}{4n} \right) \right) \quad (47)$$

The KKT conditions of the GNEs are the system of equations (9b) to (9e), (48), and (49).

Similarly, we can leverage the KKT conditions for the welfare equilibria; thereby, obtaining necessary and sufficient conditions for (12). The KKT conditions for the welfare equilibria are the system of equations: (9b) to (9e), (48a) to (48g), (48i), (49), and

$$-\mu_i^k + \mu_j^k - \lambda_{EL,ij}^k = 0 \quad \forall (i, j, k) \in \mathcal{N} \times \mathcal{N}_i \times \mathcal{K} \quad (50a)$$

$$\mu_i^k + c_{E,i0}^k + \lambda_{EL,i0}^k = 0 \quad \forall (i, k) \in \mathcal{N} \times \mathcal{K} \quad (50b)$$

In other words, the KKT system of the welfare optimization problem is the same as of the GNEs, except that (48h) have to be replaced by (50).

**Proposition 3.** *The following is a necessary condition for a solution to (1) to be a solution to (2).*

$$\lambda_{EL,ij}^k + \lambda_{EL,ij}^k = 0 \quad \forall (i, j, k) \in \mathcal{N} \times \mathcal{N}_i \times \mathcal{K} \quad (51)$$

PROOF. The financial network is a symmetric directed graph, therefore if  $(i, j)$  belongs to the edges, so do  $(j, i)$ . Then, from (50a) we obtain

$$-\mu_j^k + \mu_i^k - \lambda_{EL,ji}^k = 0 \quad \forall (i, j, k) \in \mathcal{N} \times \mathcal{N}_i \times \mathcal{K} \quad (52)$$

Summing (50a) and (52), we obtain (51).  $\square$

**Proposition 4.** *The following are sufficient conditions for a GNE to be a Welfare equilibrium*

$$\mu_i^k = \mu_j^k \quad (53a)$$

$$\lambda_{EL,ij}^k = \lambda_{EL,ji}^k = 0 \quad (53b)$$

PROOF. Reorder the terms in (53a) to obtain

$$\mu_i^k - \mu_j^k = 0$$

Add a zero-valued  $\lambda_{EL,ij}^k$  on the left hand side to obtain

$$\mu_i^k - \mu_j^k + \lambda_{EL,ij}^k = 0$$

Thereby, we have arrived to (50a), hence the GNE is a welfare equilibrium.  $\square$

Stationarity conditions:

$$\Phi_{pp,i}^k p_i^k + \Phi_{pq,i}^k q_i^k + \phi_{p,i}^k + \mu_i^k + v_{P,i}^k = 0 \quad \forall(i, k) \in \mathcal{N} \times \mathcal{K} \quad (48a)$$

$$\Phi_{qq,i}^k q_i^k + \Phi_{pq,i}^k p_i^k + \phi_{q,i}^k + v_{Q,i}^k = 0 \quad \forall(i, k) \in \mathcal{N} \times \mathcal{K} \quad (48b)$$

$$v_{P,i}^k + \alpha_{G,ij} \sum_{j \in \mathcal{W}} \left( \lambda_{GL,ij}^k + \lambda_{GU,ij}^k \right) = 0 \quad \forall(i, k) \in \mathcal{N} \times \mathcal{K} \quad (48c)$$

$$v_{P,i}^k - \alpha_{G,ij} \sum_{j \in \mathcal{W}} \left( \lambda_{GU,ij}^k - \lambda_{GL,ij}^k \right) = 0 \quad \forall(i, k) \in \mathcal{N} \times \mathcal{K} \quad (48d)$$

$$v_{P,i}^k + \eta_{B,i} v_{B,i}^k + \lambda_{PBU,i}^{+,k} - \lambda_{PBL,i}^{+,k} = 0 \quad \forall(i, k) \in \mathcal{N} \times \mathcal{K} \quad (48e)$$

$$v_{P,i}^k - \frac{1}{\eta_{B,i}} v_{B,i}^k + \lambda_{PBU,i}^{-,k} - \lambda_{PBL,i}^{-,k} = 0 \quad \forall(i, k) \in \mathcal{N} \times \mathcal{K} \quad (48f)$$

$$v_{B,i}^k - v_{B,i}^{k+1} + \lambda_{SBU,i}^k - \lambda_{SBL,i}^k = 0 \quad \forall(i, k) \in \mathcal{N} \times \mathcal{K}' \quad (48g)$$

$$c_{E,ij}^k + \mu_i^k + \lambda_{EL,ij}^k = 0 \quad \forall(i, j, k) \in \mathcal{N} \times \mathcal{N}_i^0 \times \mathcal{K} \quad (48h)$$

$$c_{E,0i}^k + \mu_i^k - \lambda_{EL,0i}^k = 0 \quad \forall(i, k) \in \mathcal{N} \times \mathcal{K} \quad (48i)$$

Complementarity slackness conditions:

$$\lambda_{GL,ij}^k \geq 0 \perp -\alpha_{G,ij} p_G - \beta_{G,ij} \leq q_G \quad \forall(i, j, k) \in \mathcal{N} \times \mathcal{W}_i \times \mathcal{K} \quad (49a)$$

$$\lambda_{GU,ij}^k \geq 0 \perp \alpha_{G,ij} p_G + \beta_{G,ij} \geq q_G \quad \forall(i, j, k) \in \mathcal{N} \times \mathcal{W}_i \times \mathcal{K} \quad (49b)$$

$$\lambda_{SL,i}^k \geq 0 \perp s_{B,i}^k \geq 0 \quad \forall(i, k) \in \mathcal{N} \times \mathcal{K} \quad (49c)$$

$$\lambda_{SU,i}^k \geq 0 \perp s_{B,i}^k \leq \hat{s}_{B,i} \quad \forall(i, k) \in \mathcal{N} \times \mathcal{K} \quad (49d)$$

$$\lambda_{PBL,i}^{+,k} \geq 0 \perp p_{B,i}^{+,k} \geq 0 \quad \forall(i, k) \in \mathcal{N} \times \mathcal{K} \quad (49e)$$

$$\lambda_{PBL,i}^{-,k} \geq 0 \perp p_{B,i}^{-,k} \geq 0 \quad \forall(i, k) \in \mathcal{N} \times \mathcal{K} \quad (49f)$$

$$\lambda_{PBU,i}^{+,k} \geq 0 \perp p_{B,i}^{+,k} \leq \hat{p}_{B,i} \quad \forall(i, k) \in \mathcal{N} \times \mathcal{K} \quad (49g)$$

$$\lambda_{PBU,i}^{-,k} \geq 0 \perp p_{B,i}^{-,k} \leq \hat{p}_{B,i} \quad \forall(i, k) \in \mathcal{N} \times \mathcal{K} \quad (49h)$$

$$\lambda_{EL,ij}^k \geq 0 \perp p_{E,ij}^k \geq 0 \quad \forall(i, j, k) \in i \in \mathcal{N}_i^0 \times \mathcal{N}_i^0 \times \mathcal{K} \quad (49i)$$

**Proposition 5.** *The following are sufficient conditions for a GNE to be a Welfare equilibrium*

$$c_{E,ij}^k = c_{E,ji}^k \quad (54a)$$

$$\lambda_{EL,ij}^k = \lambda_{EL,ji}^k = 0 \quad (54b)$$

PROOF. From (48h) and (54a) we have

$$\mu_i^k + \lambda_{EL,ij}^k = \mu_j^k + \lambda_{EL,ji}^k$$

Therefore

$$\mu_i^k = \mu_j^k$$

and by Proposition 4, the GNE is also a welfare equilibrium.

Propositions 3 to 5 characterize, to some extent, the two classes of market designs considered in this work.