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ON THE NUMBER OF VERTICES OF PROJECTIVE POLYTOPES

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ABSTRACT. Let X be a set of n points in \mathbb{R}^d in general position. What is the maximum number of vertices that $\text{conv}(T(X))$ can have among all the possible permissible projective transformations T ? In this paper, we investigate this and other related questions. After presenting several upper bounds, obtained by using oriented matroid machinery, we study a closely related problem (via Gale transforms) concerning the maximal number of minimal Radon partitions of a set of points. The latter led us to a result supporting a positive answer to a question of Pach and Szegedy asking whether *balanced* 2-colorings of points in the plane maximize the number of induced multicolored Radon partitions. We also discuss a related problem concerning the size of topes in arrangements of hyperplanes as well as a tolerance-type problem of finite sets.

1 Introduction

A *projective transformation* $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a function such that $T(x) = \frac{Ax+b}{\langle c, x \rangle + \delta}$, where A is a linear transformation of \mathbb{R}^d , $b, c \in \mathbb{R}^d$ and $\delta \in \mathbb{R}$, is such that at least one of $c \neq 0$ or $\delta \neq 0$. T is said to be *permissible* for a set $X \subset \mathbb{R}^d$ if $\langle c, x \rangle + \delta \neq 0$ for all $x \in X$. T is *nonsingular* if and only if the matrix $\begin{pmatrix} A & b^t \\ c & \delta \end{pmatrix}$ is nonsingular. We refer the reader to [20, Appendix 2.6] for a nice discussion on this notion.

Consider the following question

Given a set of n points in general position $X \subset \mathbb{R}^d$, what is the maximum number of k -faces that $\text{conv}(T(X))$ can have among all the possible permissible projective transformations T ?

More precisely, let $d > k \geq 0$ be integers and let $X \subset \mathbb{R}^d$ be a set of points in general position, we define the number of *projective k -faces* of X as

$$(1) \quad h_k(X, d) = \max_T \{f_k(\text{conv}(T(X)))\},$$

where the maximum is taken over all possible permissible projective transformations T of X and $f_k(P)$ denotes the number of k -faces of a polytope P .

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We define now the function $H_k(n, d)$ which determines the *maximum* number of *projective k -faces* that *any* X configuration of n points in \mathbb{R}^d must have as,

$$H_k(n, d) = \min_{X \subset \mathbb{R}^d, |X|=n} \{h_k(X, d)\}.$$

In this paper, we focus our attention on the behavior of $H_0(n, d)$ (the number of projective vertices). It turns out that $H_0(n, d)$ is the source of several applications.

1.1 Scope/general interest

The function $H_0(n, d)$ is closely connected with different notions/problems : McMullen's problem, bounds on $H_k(n, d)$, a connection of $H_{d-1}(n, d)$ with minimal Radon partitions and its relation with an open question due to Pach and Szegedy, tolerance-type problems of finite sets and topes in arrangements of hyperplanes.

1.1.1 McMullen's problem

$H_0(n, d)$ is a natural generalization of the following well-known problem of McMullen [10]:

What is the largest integer $\nu(d)$ such that any set of $\nu(d)$ points in general position, $X \subset \mathbb{R}^d$, can be mapped by a permissible projective transformation onto the vertices of a convex polytope?

The best known bounds on McMullen's problem are:

$$(2) \quad 2d + 1 \leq \nu(d) < 2d + \left\lceil \frac{d+1}{2} \right\rceil.$$

The lower bound was given by Larman [10] while the upper bound was provided by Ramírez Alfonsín [14]. In the same spirit, the following function has also been investigated:

$\nu(d, k) :=$ the largest integer n such that any set of n points in general position in \mathbb{R}^d can be mapped, by a permissible projective transformation, onto the vertices of a k -neighborly polytope.

Recall that a d -polytope P is k -neighborly if for a fixed natural $k \leq \lfloor \frac{d}{2} \rfloor$, every subset of at most k vertices form a face of P . As a consequence of [7, Lemma 9] and [7, Equation (1)] it can be obtained that

$$(3) \quad \nu(d, k) \geq d + \left\lfloor \frac{d}{k} \right\rfloor + 1$$

This inequality will be useful later for our purposes.

Let $t \geq 0$ be an integer. We define the following function.

$n(t, d) :=$ the largest integer n such that any set of n points in general position in \mathbb{R}^d can be mapped, by a permissible projective transformation onto the vertices of a convex polytope with at most t points in its interior.

The function $n(t, d)$ will allow us to study $H_0(n, d)$ in a more general setting, that of oriented matroids.

We notice that

$$(4) \quad n(0, d) = \nu(d) \text{ and } H_0(n(t, d), d) = n(t, d) - t.$$

Our first main contribution is the following

Theorem 1.1. *Let $d, t \geq 1$ and $n \geq 2$ be integers. Then,*

$$H_0(n, d) \begin{cases} \leq 7 & \text{if } d = 3, n \geq 7, \\ \leq n - 1 - t & \text{if } d \geq 4, n \geq 2d + t(d - 2) + 2, t \geq 1. \end{cases}$$

Also, it will be proved that $H_0(n, 2) = 5$ if $n \geq 5$ (Corollary 6.2) as well as $H_0(n, 1) = 2$ if $n \geq 2$, $H_0(n, d) = n$ if $n \leq 2d + 1$ and $H_0(n, d) < n$ if $n \geq 2d + \lceil \frac{d+1}{2} \rceil$ (Proposition 2.1).

We use $C_d(n)$ to denote the d -dimensional *cyclic polytope* with n vertices. Let $C_d(n)$ be the polytope obtained as the convex hull of n distinct points in the moment curve $x(t) := (t, t^2, \dots, t^d)$. We use $P_d(n)$ to denote the d -dimensional *stacked polytope* with n vertices. Let $P_d(n)$ be the polytope formed from a simplex by repeatedly gluing another simplex onto one of its facets. We denote by $f_k(P)$ the number of k -faces of polytope P .

As a consequence, we will obtain the following.

Corollary 1.2. *Let $d, t \geq 1$, $n \geq 2$ and $1 \leq k \leq d - 1$ be integers. Then,*

$$H_k(n, d) \begin{cases} = 5 & \text{if } d = 2, n \geq 5, \\ \leq f_k(C_3(7)) & \text{if } d = 3, n \geq 7, \\ \leq f_k(C_d(n)) & \text{if } d \geq 2, n \leq 2d + 1, \\ \leq f_k(C_d(n - 1)) & \text{if } d \geq 4, n \geq 2d + \lceil \frac{d+1}{2} \rceil, \\ \leq f_k(C_d(n - 1 - t)) & \text{if } d \geq 4, n \geq 2d + t(d - 2) + 2, t \geq 1. \end{cases}$$

Moreover, $H_k(n, d) \geq f_k(P_d(n))$ for $n \leq 2d + 1, d \geq 2$. In particular, $H_1(7, 3) = 15$ and $H_2(7, 3) = 10$.

1.1.2 Minimal Radon partitions

Let $X = \{x_1, \dots, x_n\}, n \geq d+2$ be a set of points in general position in \mathbb{R}^d . We recall that $A, B \subset X$ is a *Radon partition* of X if $X = A \cup B, A \cap B = \emptyset$ and $\text{conv}(A) \cap \text{conv}(B) \neq \emptyset$.

It happens that $H_{d-1}(n, d)$ is very useful to count minimal Radon partitions. More specifically, let $X = A \cup B$ be any partition of X , we define $r_X(A, B)$ as the number of $(d + 2)$ -element subsets $S \subset X$ such that $\text{conv}(A \cap S) \cap \text{conv}(B \cap S) \neq \emptyset$, that is, as the number of *minimal (size) Radon partitions* induced by A and B .

We define the functions

$$r(X) := \max_{\{(A,B)|A \cup B = X\}} r_X(A, B) \quad \text{and} \quad r(n, d) := \min_{X \subset \mathbb{R}^d, |X|=n} r(X).$$