



HAL
open science

On the number of vertices of projective polytopes

Natalia García-Colín, Luis Pedro Montejano, Jorge Luis Ramírez Alfonsín

► **To cite this version:**

Natalia García-Colín, Luis Pedro Montejano, Jorge Luis Ramírez Alfonsín. On the number of vertices of projective polytopes. *Mathematika*, 2023, 69 (2), pp.535-561. 10.1112/mtk.12193 . hal-04525216

HAL Id: hal-04525216

<https://hal.science/hal-04525216v1>

Submitted on 28 Mar 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON THE NUMBER OF VERTICES OF PROJECTIVE POLYTOPES

NATALIA GARCÍA-COLÍN¹, LUIS PEDRO MONTEJANO²,
AND JORGE LUIS RAMÍREZ ALFONSÍN³

ABSTRACT. Let X be a set of n points in \mathbb{R}^d in general position. What is the maximum number of vertices that $\text{conv}(T(X))$ can have among all the possible permissible projective transformations T ? In this paper, we investigate this and other related questions. After presenting several upper bounds, obtained by using oriented matroid machinery, we study a closely related problem (via Gale transforms) concerning the maximal number of minimal Radon partitions of a set of points. The latter led us to a result supporting a positive answer to a question of Pach and Szegedy asking whether *balanced* 2-colorings of points in the plane maximize the number of induced multicolored Radon partitions. We also discuss a related problem concerning the size of topes in arrangements of hyperplanes as well as a tolerance-type problem of finite sets.

1 Introduction

A *projective transformation* $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a function such that $T(x) = \frac{Ax+b}{\langle c,x \rangle + \delta}$, where A is a linear transformation of \mathbb{R}^d , $b, c \in \mathbb{R}^d$ and $\delta \in \mathbb{R}$, is such that at least one of $c \neq 0$ or $\delta \neq 0$. T is said to be *permissible* for a set $X \subset \mathbb{R}^d$ if $\langle c, x \rangle + \delta \neq 0$ for all $x \in X$. T is *nonsingular* if and only if the matrix $\begin{pmatrix} A & b^t \\ c & \delta \end{pmatrix}$ is nonsingular. We refer the reader to [20, Appendix 2.6] for a nice discussion on this notion.

Consider the following question

Given a set of n points in general position $X \subset \mathbb{R}^d$, what is the maximum number of k -faces that $\text{conv}(T(X))$ can have among all the possible permissible projective transformations T ?

More precisely, let $d > k \geq 0$ be integers and let $X \subset \mathbb{R}^d$ be a set of points in general position, we define the number of *projective k -faces* of X as

$$(1) \quad h_k(X, d) = \max_T \{f_k(\text{conv}(T(X)))\},$$

where the maximum is taken over all possible permissible projective transformations T of X and $f_k(P)$ denotes the number of k -faces of a polytope P .

Date: February 8, 2023.

¹ Supported by LAISLA (CONACYT-CNRS-UNAM).³ Partially supported by Program MATH Am-Sud, Grant MATHAMSUD 18-MATH-01, Project FLaNASAGraTA and by INSMI-CNRS.

We define now the function $H_k(n, d)$ which determines the *maximum* number of *projective k -faces* that *any* X configuration of n points in \mathbb{R}^d must have as,

$$H_k(n, d) = \min_{X \subset \mathbb{R}^d, |X|=n} \{h_k(X, d)\}.$$

In this paper, we focus our attention on the behavior of $H_0(n, d)$ (the number of projective vertices). It turns out that $H_0(n, d)$ is the source of several applications.

1.1 Scope/general interest

The function $H_0(n, d)$ is closely connected with different notions/problems : McMullen's problem, bounds on $H_k(n, d)$, a connection of $H_{d-1}(n, d)$ with minimal Radon partitions and its relation with an open question due to Pach and Szegedy, tolerance-type problems of finite sets and topes in arrangements of hyperplanes.

1.1.1 McMullen's problem

$H_0(n, d)$ is a natural generalization of the following well-known problem of McMullen [10]:

What is the largest integer $\nu(d)$ such that any set of $\nu(d)$ points in general position, $X \subset \mathbb{R}^d$, can be mapped by a permissible projective transformation onto the vertices of a convex polytope?

The best known bounds on McMullen's problem are:

$$(2) \quad 2d + 1 \leq \nu(d) < 2d + \left\lceil \frac{d+1}{2} \right\rceil.$$

The lower bound was given by Larman [10] while the upper bound was provided by Ramírez Alfonsín [14]. In the same spirit, the following function has also been investigated:

$\nu(d, k) :=$ the largest integer n such that any set of n points in general position in \mathbb{R}^d can be mapped, by a permissible projective transformation, onto the vertices of a k -neighborly polytope.

Recall that a d -polytope P is k -neighborly if for a fixed natural $k \leq \lfloor \frac{d}{2} \rfloor$, every subset of at most k vertices form a face of P . As a consequence of [7, Lemma 9] and [7, Equation (1)] it can be obtained that

$$(3) \quad \nu(d, k) \geq d + \left\lfloor \frac{d}{k} \right\rfloor + 1$$

This inequality will be useful later for our purposes.

Let $t \geq 0$ be an integer. We define the following function.

$n(t, d) :=$ the largest integer n such that any set of n points in general position in \mathbb{R}^d can be mapped, by a permissible projective transformation onto the vertices of a convex polytope with at most t points in its interior.

The function $n(t, d)$ will allow us to study $H_0(n, d)$ in a more general setting, that of oriented matroids.

We notice that

$$(4) \quad n(0, d) = \nu(d) \text{ and } H_0(n(t, d), d) = n(t, d) - t.$$

Our first main contribution is the following

Theorem 1.1. *Let $d, t \geq 1$ and $n \geq 2$ be integers. Then,*

$$H_0(n, d) \begin{cases} \leq 7 & \text{if } d = 3, n \geq 7, \\ \leq n - 1 - t & \text{if } d \geq 4, n \geq 2d + t(d - 2) + 2, t \geq 1. \end{cases}$$

Also, it will be proved that $H_0(n, 2) = 5$ if $n \geq 5$ (Corollary 6.2) as well as $H_0(n, 1) = 2$ if $n \geq 2$, $H_0(n, d) = n$ if $n \leq 2d + 1$ and $H_0(n, d) < n$ if $n \geq 2d + \lceil \frac{d+1}{2} \rceil$ (Proposition 2.1).

We use $C_d(n)$ to denote the d -dimensional *cyclic polytope* with n vertices. Let $C_d(n)$ be the polytope obtained as the convex hull of n distinct points in the moment curve $x(t) := (t, t^2, \dots, t^d)$. We use $P_d(n)$ to denote the d -dimensional *stacked polytope* with n vertices. Let $P_d(n)$ be the polytope formed from a simplex by repeatedly gluing another simplex onto one of its facets. We denote by $f_k(P)$ the number of k -faces of polytope P .

As a consequence, we will obtain the following.

Corollary 1.2. *Let $d, t \geq 1$, $n \geq 2$ and $1 \leq k \leq d - 1$ be integers. Then,*

$$H_k(n, d) \begin{cases} = 5 & \text{if } d = 2, n \geq 5, \\ \leq f_k(C_3(7)) & \text{if } d = 3, n \geq 7, \\ \leq f_k(C_d(n)) & \text{if } d \geq 2, n \leq 2d + 1, \\ \leq f_k(C_d(n - 1)) & \text{if } d \geq 4, n \geq 2d + \lceil \frac{d+1}{2} \rceil, \\ \leq f_k(C_d(n - 1 - t)) & \text{if } d \geq 4, n \geq 2d + t(d - 2) + 2, t \geq 1. \end{cases}$$

Moreover, $H_k(n, d) \geq f_k(P_d(n))$ for $n \leq 2d + 1, d \geq 2$. In particular, $H_1(7, 3) = 15$ and $H_2(7, 3) = 10$.

1.1.2 Minimal Radon partitions

Let $X = \{x_1, \dots, x_n\}, n \geq d+2$ be a set of points in general position in \mathbb{R}^d . We recall that $A, B \subset X$ is a *Radon partition* of X if $X = A \cup B$, $A \cap B = \emptyset$ and $\text{conv}(A) \cap \text{conv}(B) \neq \emptyset$.

It happens that $H_{d-1}(n, d)$ is very useful to count minimal Radon partitions. More specifically, let $X = A \cup B$ be any partition of X , we define $r_X(A, B)$ as the number of $(d + 2)$ -element subsets $S \subset X$ such that $\text{conv}(A \cap S) \cap \text{conv}(B \cap S) \neq \emptyset$, that is, as the number of *minimal (size) Radon partitions* induced by A and B .

We define the functions

$$r(X) := \max_{\{(A,B)|A \cup B = X\}} r_X(A, B) \quad \text{and} \quad r(n, d) := \min_{X \subset \mathbb{R}^d, |X|=n} r(X).$$