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# Persistence Diagram Estimation of Multivariate Piecewise Hölder-continuous Signals

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## Abstract

To our knowledge, the analysis of convergence rates for persistence diagram estimation from noisy signals had predominantly relied on lifting signal estimation results through sup norm (or other functional norm) stability theorems. We believe that moving forward from this approach can lead to considerable gains. We illustrate it in the setting of Gaussian white noise model. We examine, from a minimax perspective, the inference of persistence diagram (for sublevel sets filtration). We show that for piecewise Hölder-continuous functions, with control over the reach of the discontinuities set, taking the persistence diagram coming from a simple histogram estimator of the signal, permit to achieve the minimax rates known for Hölder-continuous functions.

## Introduction

### Motivation

Inferring information from noisy signals is a central subject in statistics. Specifically, the recovery of the whole signal structure has been extensively studied by the non-parametric statistics community. When the signal is regular (e.g; belonging to a Hölder, Sobolev or Besov space) rigorous minimax study as long as tractable optimal procedures has been provided, forming a nearly exhaustive benchmark. For an overview, see [Tsybakov \(2008\)](#).

When facing more irregular signals, typically signals that are only piecewise continuous, the problem becomes significantly more difficult. Motivated by applications, later works have attempted to explore this case. For an overview, refer to [Qiu \(2005\)](#). However, proposed methods suffer from certain limitations : strong additional knowledge assumptions (e.g. suppose to known the number of jumps, their locations or their magnitudes), restrict to low dimensional cases (only univariate or bivariate signals), high computational costs or lack of rigorous and general statistical guarantees over the risk. Additionally, due to the strong sensibility to point-wise discontinuity of the sup norm, these works only consider  $L_2$  (or sometimes  $L_p$ ,  $p < +\infty$ ) metric (less sensitive to topology). All these problematic points motivate the exploration of looser descriptors that can be inferred more easily.

In the last two decades, Topological Data Analysis has emerged as a powerful approach, offering new geometric tools for characterizing complex signals. Among these tools, persistent homology has garnered significant attention. Represented through persistence diagrams (or barcodes), it has proven to be a versatile descriptor, valuable from both practical and theoretical standpoints. Recent research has focused on the estimation of such representations, opening up exciting opportunities

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to explore the statistical aspects of Topological Data Analysis. In this context, the model that has received the most attention is the density model, initiated by the work of [Bubenik and Kim \(2006\)](#) in a simple parametric setting. Subsequently, efforts have been made to extend this model to wider, non-parametric settings. Notable contributions include the work of [Balakrishnan et al. \(2012\)](#), which addresses the estimation of Betty numbers for smooth manifolds with different noise models, and [Fasy et al. \(2014\)](#), who provide confidence sets for persistence diagrams in a similar context. Additionally, [Chazal et al. \(2014\)](#) provide a minimax estimator while controlling the regularity of the density support.

The study of non-parametric regression or the Gaussian white noise model remains relatively unexplored in the context of Topological Data Analysis. Advancements in this direction include the works of [Bubenik et al. \(2009\)](#) and [Bobrowski et al. \(2017\)](#), as well as more recent contributions, such as those by [Perez \(2022\)](#), albeit in a different direction.

The general approach followed in most of these works (except [Bobrowski et al. \(2017\)](#)) involves estimating the signal (or density), quantifying the estimation error in sup-norm, Hausdorff distance, or Gromov-Hausdorff distance, and bounding the bottleneck error on the diagram using stability theorems ([Cohen-Steiner et al., 2005](#); [Chazal et al., 2009, 2016, 2012](#)). The power and importance of stability theorems are evident as they enable the direct translation of convergence rates in sup-norm (or similar metrics) to convergence rates in bottleneck distance over diagrams (under the assumption that the signal is  $q$ -tame). To further underline the significance of stability theorems, some studies, such as [Bubenik et al. \(2009\)](#) and [Chazal et al. \(2014\)](#), demonstrate that these rates are minimax for typical function classes.

However, adopting these approaches may sacrifice efficiency and generality. One of the main interest of the persistence diagram lies in its capacity to provide a more flexible representation compared to the entire signal. Consequently, in certain cases, inferring the persistence diagram should be (strictly) simpler. In this direction, [Bobrowski et al. \(2017\)](#) by breaking free from this approach, show that we can consider wider classes of functions. Unfortunately, this work does not quantify the convergence rates of the proposed estimator. This observation serves as a crucial motivation to conduct finer analysis of the convergence properties of persistence diagram estimator. Moreover, it highlights the broader appeal of utilizing topological or geometrical descriptors, especially when conventional non-parametric techniques yield unsatisfactory results. As mentioned earlier, such scenarios commonly arise when signals display irregularities.

## Framework

**Regularity assumptions.** For a set  $A \subset [0, 1]^d$ , we denote  $\bar{A}$  its adherence,  $A^\circ$  its interior,  $\partial A$  its boundary and  $A^c$  its complement. Let  $f : [0, 1]^d \rightarrow \mathbb{R}$ , we make the following assumption over  $f$  :

**A1.**  $f$  is a piecewise  $(L, \alpha)$ -Hölder-continuous function, i.e. there exist  $M_1, \dots, M_l$  open sets of  $[0, 1]^d$  such that,

$$\bigcup_{i=1}^l \bar{M}_i = [0, 1]^d$$

and for all  $i \in \{1, \dots, l\}$  and  $x, y \in M_i$ ,

$$|f(x) - f(y)| \leq L \|x - y\|_2^\alpha.$$

**A2.**  $f$  verifies,  $\forall x_0 \in [0, 1]^d$ ,

$$\liminf_{x \in \bigcup_{i=1}^l M_i \rightarrow x_0} f(x) = f(x_0)$$

In this context, two signals, differing only on a null set, are statistically undistinguishable. And persistent homology is sensitive to point-wise irregularity, two signals differing only on a null set can have very different persistence diagrams. Assumption **A2** prevents such scenario. Furthermore, note that for any piecewise Hölder-continuous function  $f$ , there exists a modification  $\tilde{f}$  verifying Assumption **A2** such that  $f$  and  $\tilde{f}$  coincide except on a null measure sets.

**A3.**  $\bigcup_{i=1}^l \partial M_i \cap ]0, 1[^d$  is a  $C^{1,1}$  hypersurface, verifying, for  $R > 0$ ,

$$\text{reach} \left( ]0, 1[^d \cap \bigcup_{i=1}^l \partial M_i \right) \geq R \text{ and } d_2 \left( \bigcap_{i=1}^l \partial M_i \cap ]0, 1[^d, \partial ]0, 1[^d \right) \geq R$$

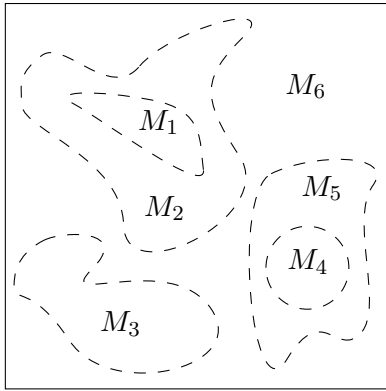
where, for a set  $A \subset \mathbb{R}^d$ ,

$$\text{reach}(A) = \sup \left\{ r \in \mathbb{R} : \forall x \in \mathbb{R}^d \setminus A \text{ with } d_2(\{x\}, A) < r, \exists ! y \in A \text{ s.t. } \|x - y\|_2 = d_2(\{x\}, A) \right\}$$

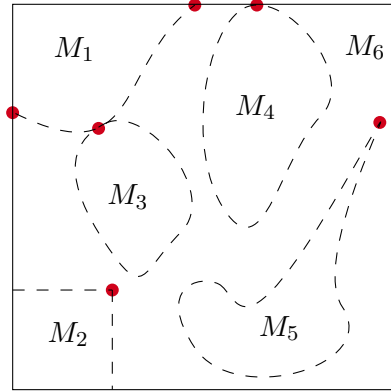
and,

$$d_2(A, B) = \max \left( \sup_{x \in B} \inf_{y \in A} \|x - y\|_2, \sup_{x \in A} \inf_{y \in B} \|x - y\|_2 \right).$$

The reach is a curvature measure introduced by [Federer \(1959\)](#). An intuitive way to approach it is that if  $A$  has a reach  $R$  we can roll a ball of radius  $R$  along the boundary of  $A$ . Positive reach assumptions are fairly common in statistical TDA ([Balakrishnan et al., 2012](#); [Niyogi et al., 2008](#)) and geometric inference ([Genovese et al., 2012](#); [Kim et al., 2016](#); [Aamari and Levrard, 2017](#); [Aamari et al., 2019](#); [Berenfeld et al., 2021](#)). Here, the first part of Assumption **A3** gives geometric control over the union of the boundary of the  $M_i$  in the interior of  $]0, 1[^d$ , for example it prevents cusps, corners and multiple points to appear. The second part ensures that discontinuities do not appear too close from the boundary of the cube  $]0, 1[^d$ .



(a) Assumption **A3** verified



(b) Assumption **A3** not verified

Figure 1: Illustration of Assumption **A3**

The combination of Assumptions **A2** and **A3** ensures that the persistence diagram of  $f$  is well-defined (see [Appendix A, Proposition 6](#)).

We denote  $S_d(L, \alpha, R)$  the set of such functions.

**Statistical model.** We considered the Gaussian white noise model given by the following stochastic equation,

$$dX_{t_1, \dots, t_d} = f(t_1, \dots, t_d) dt_1 \dots dt_d + \theta dW_{t_1, \dots, t_d} \quad (1)$$

with  $W$  a  $d$ -parameters Wiener field,  $f$  a signal in  $S_d(L, \alpha, R)$  and  $\theta \geq 0$  the level of noise. Model 1 is a classical model in non-parametric statistics.

**Estimator.** In this context, our goal is to estimate  $\text{dgm}(f)$ , the persistence diagram of  $f$  (considering singular homology with coefficient in a field). The estimation procedures consist of simply taking the persistence diagram induced by the sublevel sets of the signal estimated using histograms.

More formally, let  $h > 0$  such that  $1/h$  is an integer, consider  $G_h$  the regular orthogonal grid over  $[0, 1]^d$  of step  $h$  and  $C_h$  the collection of all the closed hypercubes of side  $h$  composing  $G_h$ . We define,  $\forall \lambda \in \mathbb{R}$ , the estimator of  $\mathcal{F}_\lambda = f^{-1}(] - \infty, \lambda])$ , by,

$$\widehat{\mathcal{F}}_\lambda = \bigcup_{H \in C_{h,\lambda}} H, \text{ with } C_{h,\lambda} = \left\{ H \in C_h \text{ such that } \int_H dX - \int_H \lambda \leq 0 \right\}.$$

It is worth noting that  $\widehat{\mathcal{F}}_\lambda$  represents the sublevel set indexed by  $\lambda$  of the histogram estimator of  $f$ . We then consider, for all  $s \in \{0, \dots, d\}$ ,  $\widehat{\mathbb{V}}_{f,s}$  the persistence module induced by the collection of homology groups  $\left( H_s \left( \widehat{\mathcal{F}}_\lambda \right) \right)_{\lambda \in \mathbb{R}}$  equipped with inclusion induced maps  $\widehat{v}_\lambda^s$  and  $\widehat{\text{dgm}}(f)$  the associated persistence diagrams. This procedure is illustrated by Figure 2, in the slightly different setting of non-parametric regression with fixed design (see Appendix C), this choice being more convenient for simulations.

A natural question is how to calibrate the window-size  $h$  for signals. From the proof of Lemma 2 (Appendix B.1), a good choice is taking  $h$  such that,

$$\frac{h^{d+\alpha}}{\sqrt{h^d \log \left( 1 + \frac{1}{h^d} \right)}} > \theta \quad (2)$$

which implies that we can take,

$$h \simeq \left( \theta^2 \log \left( \frac{1}{\theta} \right) \right)^{\frac{1}{d+2\alpha}}.$$

## Contribution

In this framework, we study the convergence properties of the estimator  $\text{dgm}(f)$ . We provide a rigorous analysis of the convergence properties for the proposed estimator, showing that it achieves the following rates for the bottleneck distance over the classes  $S_d(L, \alpha, R)$ .

**Theorem 2.** *Let  $p \geq 1$ ,*

$$\sup_{f \in S_d(L, \alpha, R)} \mathbb{E} \left( d_b \left( \widehat{\text{dgm}}(f), \text{dgm}(f) \right)^p \right) \lesssim \left( \theta^2 \log \left( \frac{1}{\theta} \right) \right)^{\frac{p\alpha}{d+2\alpha}}.$$

Furthermore, we establish that these rates are optimal, in the minimax sense, over the classes  $S_d(L, \alpha, R)$ .

**Theorem 3.** *Let  $p \geq 1$ ,*

$$\inf_{\widehat{\text{dgm}}(f)} \sup_{f \in S_d(L, \alpha, R)} \mathbb{E} \left( d_b \left( \widehat{\text{dgm}}(f), \text{dgm}(f) \right)^p \right) \gtrsim \left( \theta^2 \log \left( \frac{1}{\theta} \right) \right)^{\frac{p\alpha}{d+2\alpha}}.$$

Interestingly, these rates coincide with the well-known minimax rates obtained on Hölder spaces. Up to a multiplicative constant, there is no additional cost for considering signal in  $S_d(L, \alpha, R)$ . It

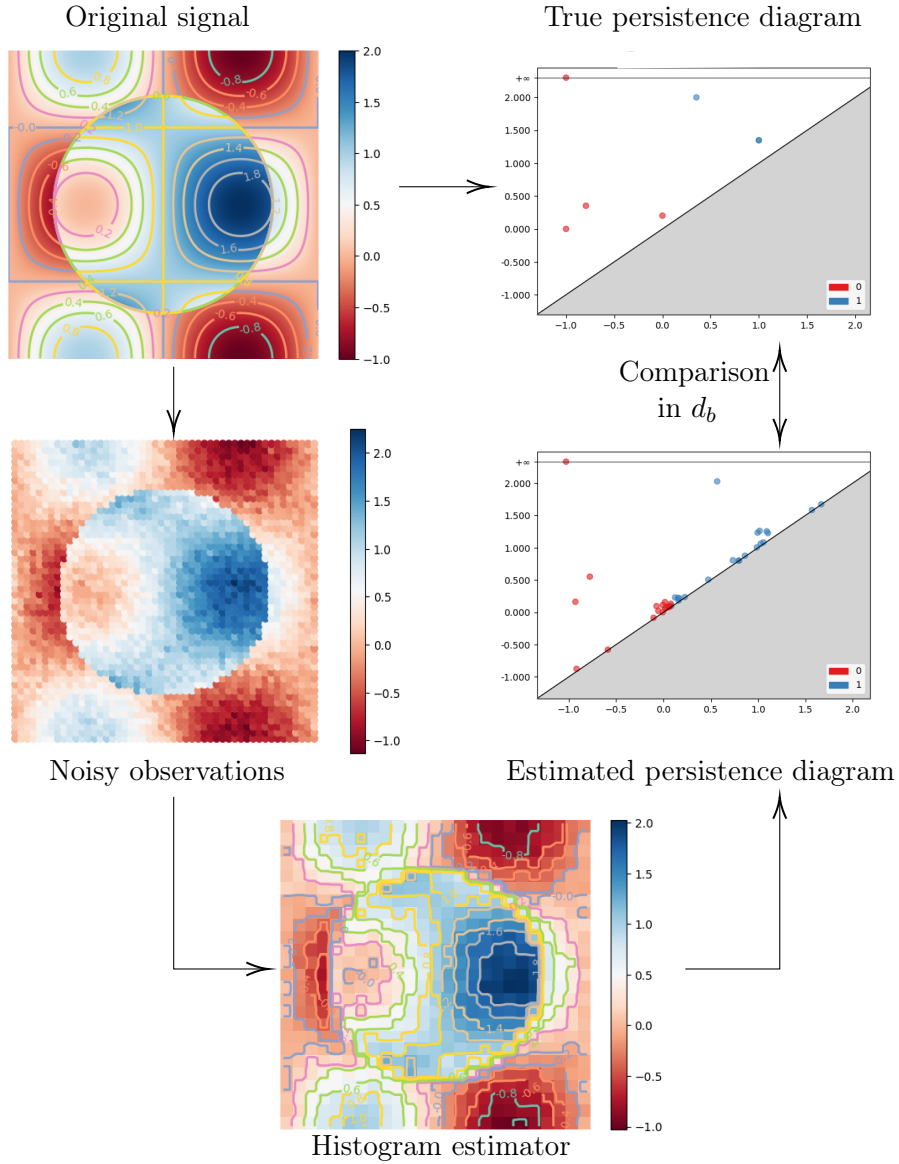


Figure 2: Numerical illustration of the estimation procedures in the setting of the non-parametric regression (see Appendix C).  $f(x, y) = \cos(2\pi x) \sin(2\pi y) + \mathbb{1}_{(x-1/2)^2 + (y-1/2)^2 < 1/8}$ ,  $\sigma = 0.1$ ,  $n = 2500$ ,  $h = 1/4(\log(n)/n)^{1/4}$ .

demonstrates the gain of breaking free from usual analysis approach in TDA and the robustness to discontinuities of persistence diagram estimation. Also, as such irregularities are challenging to handle for signal estimation, these results promote the use of persistence diagram while processing noisy (irregular) signals.

The paper is organized as follows. Section 1 provides some background on persistent homology. Section 2 is dedicated to the proof of Theorem 2. Section 3 discusses the adaptivity to the parameter  $\alpha$ . Section 4 is dedicated to the proof of Theorem 4. Appendix A and B contains proofs of technical lemmas and propositions invoked along this paper. Appendix C proposes an extension of our results in the context of the non-parametric regression.

## 1 Background on persistent homology

We first recall the required background on persistent homology, focusing on the case of persistent homology from sublevel sets of real functions. This section does not pretend to give an exhaustive exposition to persistent homology, but simply introduce the essential formalism to follow this paper. For an extensive overview, see [Chazal et al. \(2016\)](#).

The construction introduced here exploited the concept of homology, and especially singular homology. For an introduction to (singular) homology, the reader can refer to [Hatcher \(2000\)](#).

### 1.1 Filtrations and persistence modules

The idea behind persistence homology is to encode the evolution of the topology (in the homology sense) of a nested family of topological spaces, called filtration. As we are moving along indices, topological features (connected components, cycles, cavities, ...) can appear or die (existing connected components merge, cycle or cavities are filled, ...). Two keys to formalize this idea, that we use along this paper, are the notions of filtration and of persistence module.

**Definition 1.** Let  $\Lambda \subset \mathbb{R}$  be a set of indices. A filtration over  $\Lambda$  is a family  $(\mathcal{K}_\lambda)_{\lambda \in \Lambda}$  of topological spaces satisfying,  $\forall \lambda, \lambda' \in \Lambda, \lambda \leq \lambda'$

$$\mathcal{K}_\lambda \subset \mathcal{K}_{\lambda'}.$$

The typical filtration that we will consider in this paper is, for a function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ , the family of sublevel sets  $(\mathcal{F}_\lambda)_{\lambda \in \mathbb{R}}$ .

**Definition 2.** Let  $\Lambda \subset \mathbb{R}$  be a set of indices. A persistence module over  $\lambda$  is a family  $\mathbb{V} = (\mathbb{V}_\lambda)_{\lambda \in \Lambda}$  of vector spaces equipped with linear application  $v_\lambda^{\lambda'} : \mathbb{V}_\lambda \rightarrow \mathbb{V}_{\lambda'}$  such that,  $\forall \lambda \leq \lambda' \leq \lambda'' \in \Lambda$ ,

$$v_\lambda^{\lambda'} = id$$

and

$$v_{\lambda'}^{\lambda''} \circ v_\lambda^{\lambda'} = v_\lambda^{\lambda''}.$$

The typical persistence modules that we will consider in this paper is, for a function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  and  $s \in \mathbb{N}$ , the family of homology groups  $\mathbb{V}_{f,s} = (H_s(\mathcal{F}_\lambda))_{\lambda \in \mathbb{R}}$  equipped with  $v_\lambda^{\lambda'}$  the linear application induced by the inclusion  $\mathcal{F}_\lambda \subset \mathcal{F}_{\lambda'}$ . To be more precise, in this paper,  $H_s(\cdot)$  is the singular homology functor in degree  $s$  with coefficient in a field (typically  $\mathbb{Z}/2\mathbb{Z}$ ). Hence,  $H_s(\mathcal{F}_\lambda)$  is a vector space.

### 1.2 Module decompositions, persistence diagrams and $q$ -tameness

persistence diagram (or equivalently barcode) permits to summarize and represent, discretely, the algebraic structure of a persistence module. Still, this is not possible for all persistence modules.

As shown in [Chazal et al. \(2016\)](#), if  $\mathbb{V}$  verifies a  $q$ -tameness assumption, persistence diagrams can be defined. The notion of  $q$ -tameness is used in this paper to prove that the diagrams we consider are well-defined.

**Definition 3.** A persistence module  $\mathbb{V}$  is said to be  $q$ -tame if  $\forall \lambda < \lambda' \in \Lambda$ ,  $\text{rank} \left( v_{\lambda}^{\lambda'} \right)$  is finite.

By extension, when considering the persistence modules  $(\mathbb{V}_{f,s})_{s \in \mathbb{N}}$  coming from the sublevel sets filtration of a real functions  $f$ , we say that  $f$  is  $q$ -tame if  $\mathbb{V}_{f,s}$  is for all  $s \in \mathbb{N}$ .

To avoid technical definitions, in a more restrictive but illustrative case, we define persistence diagram. The basic idea being that, if we can then decompose persistence modules as a sum of elementary bricks, called interval modules. The persistence diagram can, in this case, be directly derived from this decomposition.

**Definition 4.** Let  $I$  an interval (possibly unbounded) of  $\mathbb{R}$  and  $I' = I \cap \Lambda$ . A persistence module  $\mathbb{V}$  is an interval module on  $I'$  if,

- $\mathbb{V}_{\lambda} = \mathbb{R}$  if  $\lambda \in I'$  and  $\mathbb{V}_{\lambda} = \{0\}$  otherwise
- for all  $\lambda \leq \lambda'$ ,  $v_{\lambda}^{\lambda'} = id$  if  $\lambda, \lambda' \in I'$  and  $v_{\lambda}^{\lambda'} = 0$  otherwise.

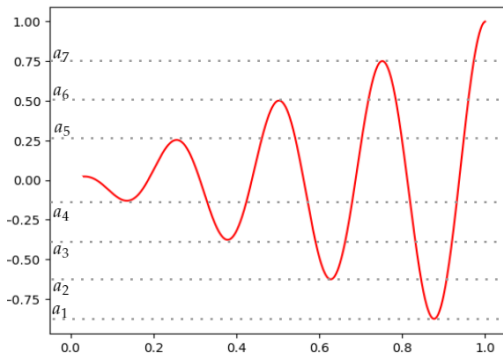
Hence, the structure of interval modules is simple and completely encoded by the extremities of  $I' = [b, d] \cap \Lambda$ . Conditions to ensure existence of a decomposition of a persistence module into sum of interval modules,

$$\mathbb{V} \simeq \bigoplus_{j \in J} \mathbb{I}_{[b_j, d_j]} \quad (3)$$

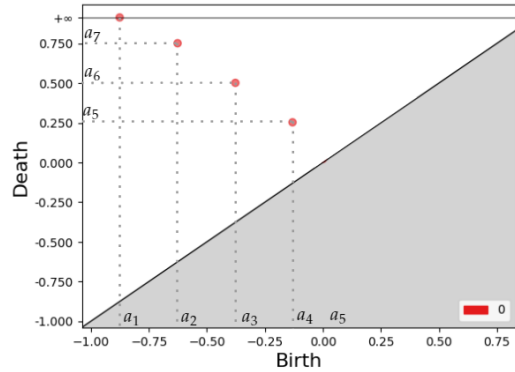
can be found in [Chazal et al. \(2016\)](#) (see theorem 1.4). Assuming we have a decomposition such as [3](#), the structure of  $\mathbb{V}$  is completely described by the extremities  $(b_j, d_j)$  of each interval in the decomposition. Thus, the associated persistence diagram can be defined simply as the collection of couples of such extremities. Intuitively, The lower extremity  $b_j$  corresponds to the birth time of a topological feature,  $d_j$  to its death time, and  $d_j - b_j$  represents its lifetime.

**Definition 5.** Let  $\mathbb{V}$  a persistence module that can be decomposed as in [3](#). The associated persistence diagram is,

$$\text{dgm}(\mathbb{V}) = \{(b_j, d_j), j \in J\} \subset \overline{\mathbb{R}}^2.$$



(a) graph of  $f$



(b)  $H_0$ -persistence diagram of  $f$

Figure 3: Graph of  $f(x) = x \cos(8\pi x)$  over  $[0, 1]$  and the persistence diagram associated to its sublevel sets filtration.  $a_1, \dots, a_4$  correspond to local minima of  $f$  and thus birth times in  $\text{dgm}(f)$ .  $a_5, \dots, a_7$  correspond to local maxima of  $f$  and thus death times in  $\text{dgm}(f)$ .



### 1.3 Bottleneck distance, interleaved modules and stability

In order to compare persistence diagrams, we need a distance. A popular such distance, due to its stability property, is the bottleneck distance. This distance is defined as the infimum over all matching between points in diagrams, of the maximal sup norm distance between two matched points. In order to be able to consider matching between diagrams not containing the same number of points, the diagonal is added to diagrams. This distance will be used in this work to evaluate the quality of our estimation procedures.

**Definition 6.** The bottleneck distance between two persistence diagrams  $D_1$  and  $D_2$  is,

$$d_b(D_1, D_2) = \inf_{\gamma \in \Gamma} \sup_{p \in D_1} \|p - \gamma(p)\|_\infty$$

with  $\Gamma$  the set of all bijection between  $D_1$  and  $D_2$  (both enriched with the diagonal).

Another notion that will be the key to prove our upper bounds, is the notion of interleaving between persistence modules. We use especially the fact that if two modules are  $\varepsilon$ -interleaved, then the bottleneck distance between their diagram is upper bounded by  $\varepsilon$  in bottleneck distance.

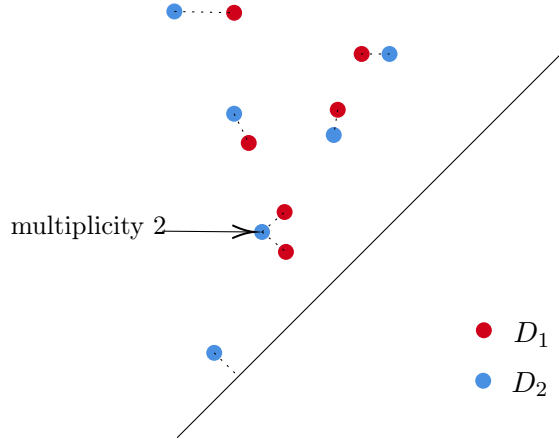


Figure 4: Optimal matching for the bottleneck distance between  $D_1$  and  $D_2$ .

**Definition 7.** Two persistence modules  $\mathbb{V} = (\mathbb{V}_\lambda)_{\lambda \in I \subset \mathbb{R}}$  and  $\mathbb{W} = (\mathbb{W}_\lambda)_{\lambda \in I \subset \mathbb{R}}$  are said to be  $\varepsilon$ -interleaved if there exists two families of applications  $\phi = (\phi_\lambda)_{\lambda \in I \subset \mathbb{R}}$  and  $\psi = (\psi_\lambda)_{\lambda \in I \subset \mathbb{R}}$  where  $\phi_\lambda : \mathbb{V}_\lambda \rightarrow \mathbb{W}_{\lambda+\varepsilon}$ ,  $\psi_\lambda : \mathbb{W}_\lambda \rightarrow \mathbb{V}_{\lambda+\varepsilon}$ , and for all  $\lambda < \lambda'$  the following diagrams commutes,

$$\begin{array}{ccc}
 \mathbb{V}_\lambda & \xrightarrow{v_\lambda^{\lambda'}} & \mathbb{V}_{\lambda'} \\
 \phi_\lambda \downarrow & & \downarrow \phi_{\lambda'} \\
 \mathbb{W}_{\lambda+\varepsilon} & \xrightarrow{w_{\lambda+\varepsilon}^{\lambda'+\varepsilon}} & \mathbb{W}_{\lambda'+\varepsilon} \\
 & & \\
 \mathbb{V}_\lambda & \xrightarrow{v_\lambda^{\lambda+2\varepsilon}} & \mathbb{V}_{\lambda+2\varepsilon} \\
 \phi_\lambda \searrow & & \nearrow \psi_{\lambda+\varepsilon} \\
 & \mathbb{W}_{\lambda+\varepsilon} & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{W}_\lambda & \xrightarrow{w_\lambda^{\lambda'}} & \mathbb{W}_{\lambda'} \\
 \psi_\lambda \downarrow & & \downarrow \psi_{\lambda'} \\
 \mathbb{V}_{\lambda+\varepsilon} & \xrightarrow{v_{\lambda+\varepsilon}^{\lambda'+\varepsilon}} & \mathbb{V}_{\lambda'+\varepsilon} \\
 & & \\
 \mathbb{W}_\lambda & \xrightarrow{w_\lambda^{\lambda+2\varepsilon}} & \mathbb{W}_{\lambda+2\varepsilon} \\
 \psi_\lambda \searrow & & \nearrow \phi_{\lambda+\varepsilon} \\
 & \mathbb{V}_{\lambda+\varepsilon} & 
 \end{array}$$

**Theorem (algebraic stability (Chazal et al., 2009)).** Let  $\mathbb{V}$  and  $\mathbb{W}$  two  $q$ -tame persistence modules. If  $\mathbb{V}$  and  $\mathbb{W}$  are  $\varepsilon$ -interleaved then,

$$d_b(\text{dgm}(\mathbb{V}), \text{dgm}(\mathbb{W})) \leq \varepsilon$$

In the context of sublevel persistence, a direct consequence of this theorem, is the following theorem. This result was already established in particular cases in [Cohen-Steiner et al. \(2005\)](#) and [Baramnikov \(1994\)](#).

**Theorem (sup norm stability).** *Let  $f$  and  $g$  two real-valued  $q$ -tame function, for all  $s \in \mathbb{N}$*

$$d_b(\text{dgm}(\mathbb{V}_{f,s}), \text{dgm}(\mathbb{V}_{g,s})) \leq \|f - g\|_\infty.$$

This property is often used to upper bounds the errors (in bottleneck distance) of "plug-in" estimators of persistence diagrams. It is important to note that this sup norm stability is weaker, and adopting such approaches may result in a loss of efficiency and generality.

## 2 Upper bounds

This section is devoted to the proof of [Theorem 2](#). The strategy is to construct an interleaving between the estimated and true persistence modules, to then apply the algebraic stability theorem ([Chazal et al., 2009](#)). In the case where for an  $\varepsilon > 0$  and all  $\lambda \in \mathbb{R}$ ,  $\mathcal{F}_{\lambda-\varepsilon} \subset \widehat{\mathcal{F}}_\lambda \subset \mathcal{F}_{\lambda+\varepsilon}$ , an  $\varepsilon$ -interleaving is directly given, taking the inclusion induced morphisms between  $\mathbb{V}_{f,s}$  and  $\widehat{\mathbb{V}}_{f,s}$ . Remark that, in this case,  $f$  and  $\hat{f}$  are  $\varepsilon$  close in sup norm. Thus, under our assumption, doing so is not possible, due to potential arbitrary large errors in neighborhoods of the discontinuity sets. But by Assumptions **A1** and **A2** we can show a slightly weaker double inclusion, given by the following proposition. Denotes, for a set  $A \subset \mathbb{R}^d$  and  $b \geq 0$ , we denote,

$$A^b = \left\{ x \in \mathbb{R}^d \text{ s.t. } d_2(x, A) \leq b \right\}$$

and

$$A^{-b} = \left( (A^c)^b \right)^c.$$

We also define,

$$\|W\|_{G(h)} = \sup_{H \in C_h} \frac{|W(H)|}{\omega(h^d)}$$

with  $\omega(r) = \sqrt{r \log(1 + 1/r)}$  and  $W(H) = \int_H dW$ .

**Proposition 1.** *Let  $f : [0, 1]^d \rightarrow \mathbb{R}$ . For all  $\lambda \in \mathbb{R}$  and  $h > 0$  verifying [\(2\)](#),*

$$\mathcal{F}_{\lambda - \|W\|_{G(h)} h^\alpha}^{-\sqrt{dh}} \subset \widehat{\mathcal{F}}_\lambda \subset \mathcal{F}_{\lambda + \|W\|_{G(h)} h^\alpha}^{\sqrt{dh}}.$$

Proof of [Proposition 1](#) can be found in [Appendix B.1](#). This double inclusion induced a morphism from the module  $(H_s(\mathcal{F}_\lambda^{-\sqrt{dh}}))_{\lambda \in \mathbb{R}}$  to  $\widehat{\mathbb{V}}_{f,s}$  and a morphism from  $\widehat{\mathbb{V}}_{f,s}$  into the module  $(H_s(\mathcal{F}_\lambda^{\sqrt{dh}}))_{\lambda \in \mathbb{R}}$ . What we now need to construct the desired interleaving is a morphism from  $\mathbb{V}_{f,s}$  into  $(H_s(\mathcal{F}_\lambda^{-\sqrt{dh}}))_{\lambda \in \mathbb{R}}$  and a morphism from  $(H_s(\mathcal{F}_\lambda^{\sqrt{dh}}))_{\lambda \in \mathbb{R}}$  into  $\mathbb{V}_{f,s}$ . The construction of those morphisms rely on the construction of deformation retract, a standard concept in algebraic topology.

**Definition 8.** A subspace  $A$  of  $X$  is called a **deformation retract** of  $X$  if there is a continuous  $F : X \times [0, 1] \rightarrow X$  (called a homotopy) such that for all  $x \in X$  and  $a \in A$ ,

- $F(x, 0) = x$
- $F(x, 1) \in A$
- $F(a, 1) = a$ .

The function  $F$  is then called a (deformation) **retraction** from  $X$  to  $A$ .

Homotopy, and thus homology, is invariant under deformation retract. Thus, a deformation retraction from  $X$  to  $A$  induces isomorphism between homology groups. More precisely, for all  $t \in [0, 1]$ ,  $F(\cdot, t)$  induces a morphism  $F(\cdot, t)^\# : C_s(X) \rightarrow C_s(X)$  between  $s$ -cycles of  $X$  defined by composing each singular  $s$ -simplex  $\sigma : \Delta^s \rightarrow X$  with  $F(\cdot, t)$  to get a singular  $s$ -simplex  $F^\#(\sigma, t) = F(\cdot, t) \circ \sigma : \Delta^s \rightarrow A$ , then extending  $F^\#(\cdot, t)$  linearly via  $F^\#(\sum_i n_i \sigma_i, t) = \sum_i n_i F^\#(\sigma_i, t) = \sum_i n_i F(\cdot, t) \circ \sigma_i$ . Then,  $F^*(\cdot, t) : H_s(X) \rightarrow H_s(X)$  defined by  $[C] \mapsto [F^\#(C, t)]$  can be shown to be an isomorphism for all  $t \in [0, 1]$  (see [Hatcher, 2000](#), pages 110-113). In particular,  $F^*(\cdot, 1) : H_s(X) \rightarrow H_s(A)$  is an isomorphism.

Theorem 4.8 of [Federer \(1959\)](#) ensure that for every  $x \in [0, 1]^d$  at (Euclidean) distance strictly smaller than  $R$  of  $]0, 1[^d \cap \bigcup_{i=1}^l \partial M_i$  there exists a unique closest point in  $]0, 1[^d \cap \bigcup_{i=1}^l \partial M_i$ , denoted  $\xi(x)$ . Furthermore,  $\xi$  is a continuous function. We exploit this property to establish the two following proposition.

Let  $h > 0$ ,  $\lambda \in \mathbb{R}$ , and,

$$\mathcal{K}_{\lambda, h} := \mathcal{F}_\lambda^h \cup \left( \bigcup_{x \in S_{\lambda, h}} [x, \xi(x)] \right) \subset \mathcal{F}_\lambda^{2h}$$

with

$$S_{\lambda, h} = \left( \left( \bigcup_{i=1}^l \partial M_i \cap ]0, 1[^d \right)^h \setminus \mathcal{F}_{\lambda + Lh^\alpha} \right) \cap \mathcal{F}_\lambda^h.$$

We define  $F_{\lambda, h} : \mathcal{K}_{\lambda, h} \times [0, 1] \rightarrow \mathcal{K}_{\lambda, h}$  by :

- If there exist  $i \in \{1, \dots, l\}$ , such that,

$$x \in \bigcup_{x \in S_{\lambda, h}} [x, \xi(x)] \cap M_i \tag{4}$$

and

$$d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda + Lh^\alpha}) \geq 2h - \|x - \xi(x)\|_2 \tag{5}$$

then, for all  $t \in [0, 1]$ ,

$$F_{\lambda, h}(x, t) = (1-t)x + t \left( \xi(x) + (2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda + Lh^\alpha}))_+ \frac{x - \xi(x)}{\|x - \xi(x)\|_2} \right).$$

- Otherwise, for all  $t \in [0, 1]$ ,  $F_{\lambda, h}(x, t) = x$ .

And we denote  $\mathcal{G}_{\lambda, h} = \text{Im}(x \mapsto F_{\lambda, h}(x, 1))$ .

**Proposition 2.** For all  $0 < h < \frac{R}{2}$  and  $\lambda \in \mathbb{R}$ ,  $F_{\lambda, h}$  is a deformation retraction of  $\mathcal{K}_{\lambda, h}$  onto  $\mathcal{G}_{\lambda, h}$ . Furthermore, we have,  $\mathcal{F}_\lambda^h \subset \mathcal{K}_{\lambda, h} \subset \mathcal{F}_\lambda^{2h}$  and  $\mathcal{F}_\lambda \subset \mathcal{G}_{\lambda, h} \subset \mathcal{F}_{\lambda + L(1+3^\alpha)h^\alpha}$ .

Proof of Proposition 2 can be found respectively in Appendix B.2. Combining the inclusion  $\mathcal{F}_\lambda^{\sqrt{dh}} \subset \mathcal{K}_{\lambda, \sqrt{dh}}$ , the retraction from  $\mathcal{K}_{\lambda, \sqrt{dh}}$  to  $\mathcal{G}_{\lambda, \sqrt{dh}}$ , and the inclusion  $\mathcal{G}_{\lambda, \sqrt{dh}} \subset \mathcal{F}_{\lambda + L(1+3^\alpha)h^\alpha}$ , furnished by proposition 2, provides a morphism from  $(H_s(\mathcal{F})_\lambda^{\sqrt{dh}})_{\lambda \in \mathbb{R}}$  into  $\mathbb{V}_{f, s}$ .

Now, let  $h > 0$ ,  $\lambda \in \mathbb{R}$ , and

$$\gamma_h(x) = \begin{cases} x + \frac{\left( h - d_2 \left( x, \bigcup_{i=1}^l \partial M_i \cap ]0, 1[^d \right) \right)}{d_2 \left( x, \bigcup_{i=1}^l \partial M_i \cap ]0, 1[^d \right)} (x - \xi(x)), & \text{if } x \in \left( \bigcup_{i=1}^l \partial M_i \cap ]0, 1[^d \right)^h \setminus \bigcup_{i=1}^l \partial M_i \\ x, & \text{if } x \notin \left( \bigcup_{i=1}^l \partial M_i \cap ]0, 1[^d \right)^h \end{cases}.$$

We denote,

$$P_{\lambda,h} = \left( \left( \bigcup_{i=1}^l \partial M_i \cap ]0, 1[^d \right)^h \setminus \mathcal{F}_{\lambda+2Lh^\alpha}^{-h} \right) \cap \mathcal{F}_\lambda.$$

The function  $\gamma_h$  can be extended continuously on  $P_{\lambda,h}$  (see proof of Proposition 3), we denote  $\gamma_{h,\lambda}$  this extension. We then define,

$$\mathcal{N}_{\lambda,h} := \mathcal{F}_\lambda \cup \left( \bigcup_{x \in P_{\lambda,h}} [x, \gamma_{\lambda,h}(x)] \right) \subset \mathcal{F}_{\lambda+Lh^\alpha}$$

and consider  $H_{\lambda,h} : \mathcal{N}_{\lambda,h} \times [0, 1] \rightarrow \mathcal{N}_{\lambda,h}$  the map defined by :

- If there exists  $i \in \{1, \dots, l\}$  such that,

$$x \in \bigcup_{x \in P_{\lambda,h}} [x, \gamma_{\lambda,h}(x)] \cap M_i \quad (6)$$

and

$$d_2(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}) \geq 3h - \|x - \gamma_{\lambda,h}(x)\|_2 \quad (7)$$

then, for all  $t \in [0, 1]$ ,

$$H_{\lambda,h}(x, t) = (1-t)x + t \left( \gamma_{\lambda,h}(x) + (3h - d_2(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}))_+ \frac{x - \gamma_{\lambda,h}(x)}{\|x - \gamma_{\lambda,h}(x)\|_2} \right).$$

- Otherwise, for all  $t \in [0, 1]$ ,  $H_{\lambda,h}(x, t) = x$ .

Finally, we denote  $\mathcal{M}_{\lambda,h} = \text{Im}(x \mapsto H_{\lambda,h}(x, 1))$ .

**Proposition 3.** *For all  $0 < h < \frac{R}{2}$  and  $\lambda \in \mathbb{R}$ ,  $H_{\lambda,h}$  is a deformation retract from  $\mathcal{N}_{\lambda,h}$  onto  $\mathcal{M}_{\lambda,h}$ . Furthermore, we have,  $\mathcal{F}_\lambda^{-h} \subset \mathcal{M}_{\lambda,h} \subset \mathcal{F}_{\lambda+(2+5^\alpha)Lh^\alpha}^{-h}$  and,  $\mathcal{F}_\lambda \subset \mathcal{N}_{\lambda,h} \subset \mathcal{F}_{\lambda+Lh^\alpha}$ .*

Proof of Proposition 3 can be found in Appendix B.4. Similarly to Proposition 2, Proposition 3 provides a morphism from  $\mathbb{V}_{f,s}$  into  $(H_s(\mathcal{F})_{\lambda}^{-\sqrt{dh}})_{\lambda \in \mathbb{R}}$ .

A last technical step to be sure that these morphisms can be used to construct the desired interleaving is to check that, for all  $x \in \mathcal{K}_{\lambda, \sqrt{dh}}$ ,  $F_{\lambda, \sqrt{dh}}(x, [0, 1])$  is included in  $\widehat{\mathcal{F}}_{\lambda+ch}^{Ch}$ , and for all  $x \in \mathcal{N}_{\lambda, \sqrt{dh}}$ ,  $H_{\lambda, \sqrt{dh}}(x, [0, 1])$  is included in  $\widehat{\mathcal{F}}_{\lambda+ch}^{Ch}$ , for some constant  $c$  (depending on the parameter of the model and the noise  $W$ ) and small  $C$ . This is ensured by the following proposition.

**Proposition 4.** *Let  $\lambda \in \mathbb{R}$ ,  $0 < C < \sqrt{d}$  and  $h \simeq (\theta^2 \log(\frac{1}{\theta}))^{\frac{1}{d+2\alpha}}$ , for sufficiently small  $\theta$ , we have, for all  $f \in S_d(L, \alpha, R)$ , for all  $x \in \widehat{\mathcal{F}}_\lambda \cap S_{\lambda+\|W\|_{G(h)}h^\alpha, \sqrt{dh}}$ ,*

$$[x, \xi(x)] \subset \widehat{\mathcal{F}}_{\lambda+(c_1\|W\|_{G(h)}+c_2)h^\alpha}^{Ch} \quad (8)$$

and for all  $x \in \widehat{\mathcal{F}}_\lambda \cap P_{\lambda+\|W\|_{G(h)}h^\alpha, \sqrt{dh}}$ ,

$$[x, \gamma_{\lambda+\|W\|_{G(h)}h^\alpha, \sqrt{dh}}(x)] \subset \widehat{\mathcal{F}}_{\lambda+(c_1\|W\|_{G(h)}+c_2)h^\alpha}^{Ch} \quad (9)$$

with  $c_1 = 2 + \lceil 2\sqrt{d} \rceil^d$  and  $c_2 = L(1 + \lceil 2\sqrt{d} \rceil^d)(5\sqrt{d})^\alpha$ .

Proof of Proposition 4 can be found in appendix B.6. The previous propositions will permit to establish an interleaving between  $\mathbb{V}_{f,s} \widehat{\mathbb{V}}_{f,s}^h$ , depending on  $\|W\|_{G(h)}$ . Hence, to complete the proof we will need concentration inequality over  $\|W\|_{G(h)}$ , given by the following proposition which proof can be found in Appendix B.7.

**Proposition 5.**

$$\mathbb{P}(\|W\|_{cube,h} \geq t) \leq 2 \left(\frac{1}{h}\right)^d \exp\left(-\frac{1}{2}t^2 \log\left(1 + \frac{1}{h^d}\right)\right).$$

Consequently, there exists two constants  $C_0$  and  $C_1$  depending only on  $d$  such that, for all  $h < 1$ ,

$$\mathbb{P}(\|W\|_{G(h)} \leq t) \leq C_0 \exp(-C_1 t^2).$$

Equipped with these propositions we can now prove Theorem 1 from which follows Theorem 2.

**Theorem 1.** Let  $h \simeq (\theta^2 \log(\frac{1}{\theta}))^{\frac{1}{d+2\alpha}}$ . There exists  $\tilde{C}_0$  and  $\tilde{C}_1$  such that, for all  $t > 0$ ,

$$\mathbb{P}\left(\sup_{f \in S_d(L,\alpha,R)} d_b(\widehat{\text{dgm}}(f), \text{dgm}(f)) \geq t \left(\theta^2 \log\left(\frac{1}{\theta}\right)\right)^{\frac{\alpha}{d+2\alpha}}\right) \leq \tilde{C}_0 \exp(-\tilde{C}_1 t^2).$$

*Proof.* It suffices to show the result for small  $\theta$  (up to rescaling  $\tilde{C}_0$ ). Hence, suppose that  $\theta$  is such that  $2\sqrt{d}h < R$  and Proposition 4 holds for  $C = 1/4$ . Note that for all  $\lambda \in \mathbb{R}$ ,  $\widehat{\mathcal{F}}_\lambda$  is a union of hypercube of  $C_h$ , hence its  $\mu$ -reach (see definition in Chazal et al. (2006)) is lower bounded by  $h/2$  for all  $\mu < 1/2$ . Hence, Theorem 12 of Kim et al. (2020) ensures that  $\widehat{\mathcal{F}}_\lambda^{h/4}$  deformation retracts onto  $\widehat{\mathcal{F}}_\lambda$ . Then, the module  $\mathbb{V}_{s,f}$  can be thought as the module induced by the filtration  $(\widehat{\mathcal{F}}_\lambda^{h/4})_{\lambda \in \mathbb{R}}$ . Let,

$$j_{0,\lambda} : H_s(\widehat{\mathcal{F}}_\lambda^{h/4}) \rightarrow H_s(\widehat{\mathcal{F}}_\lambda)$$

the map induced by the deformation retract of  $\widehat{\mathcal{F}}_\lambda^{h/4}$  onto  $\widehat{\mathcal{F}}_\lambda$ ,

$$j_{1,\lambda} : H_s(\widehat{\mathcal{F}}_\lambda) \rightarrow H_s(\mathcal{K}_{\lambda+\|W\|_{G(h)}h^\alpha, \sqrt{d}h})$$

the map induced by the inclusion  $\widehat{\mathcal{F}}_\lambda \subset \mathcal{F}_{\lambda+\|W\|_{G(h)}h^\alpha}^{\sqrt{d}h} \subset \mathcal{K}_{\lambda+\|W\|_{G(h)}h^\alpha, \sqrt{d}h}$  obtained by combining Proposition 1 and 2,

$$j_{2,\alpha} : H_s(\mathcal{K}_{\lambda+\|W\|_{G(h)}h^\alpha, \sqrt{d}h}) \rightarrow H_s(\mathcal{G}_{\lambda+\|W\|_{G(h)}h^\alpha, \sqrt{d}h})$$

induced by the deformation retract of Proposition 2, and, with  $k_1 = Ld^{\alpha/2}(1+3^\alpha) + c_2$ ,

$$j_{3,\alpha} : H_s(\mathcal{G}_{\lambda+\|W\|_{G(h)}h^\alpha, \sqrt{d}h}) \rightarrow H_s(\mathcal{F}_{\lambda+(k_1+(1+c_1)\|W\|_{G(h)}h^\alpha)})$$

the map induced by inclusion following again Proposition 2. We then define,

$$\begin{cases} \bar{\phi}_\lambda : H_s(\widehat{\mathcal{F}}_\lambda^{h/4}) \rightarrow H_s(\mathcal{F}_{\lambda+(k_1+(1+c_1)\|W\|_{G(h)}h^\alpha)}) \\ \bar{\phi}_\lambda = j_{3,\alpha} \circ j_{2,\alpha} \circ j_{1,\lambda} \circ j_{0,\lambda} \end{cases}$$

This gives us the first module morphism. Let construct the second one. Let,

$$j_{4,\lambda} : H_s(\mathcal{F}_\lambda) \rightarrow H_s(\mathcal{N}_{\lambda, \sqrt{d}h})$$

the map induced by the inclusion  $\mathcal{F}_\lambda \subset \mathcal{N}_{\lambda, \sqrt{d}h}$  from Proposition 3,

$$j_{5,\lambda} : H_s(\mathcal{N}_{\lambda, \sqrt{d}h}) \rightarrow H_s(\mathcal{M}_{\lambda, \sqrt{d}h})$$

the map induced by the deformation retract of Proposition 3, and, with  $k_2 = Ld^{\alpha/2}(2 + 5^\alpha) + c_2$

$$j_{6,\lambda} : H_s \left( \mathcal{M}_{\lambda, \sqrt{dh}} \right) \rightarrow H_s \left( \widehat{\mathcal{F}}_{\lambda + (k_2 + (1+c_1)\|W\|_{G(h)})h^\alpha}^{h/4} \right)$$

induced by the inclusion  $\mathcal{M}_{\lambda, \sqrt{dh}} \subset \widehat{\mathcal{F}}_{\lambda + (k_2 + (1+c_1)\|W\|_{G(h)})h^\alpha}^{h/4}$ , from the combination of Proposition 1 and 3. We then define,

$$\begin{cases} \bar{\psi}_\lambda : H_s(\mathcal{F}_\lambda) \longrightarrow H_s \left( \widehat{\mathcal{F}}_{\lambda + (k_2 + (1+c_1)\|W\|_{G(h)})h^\alpha}^{h/4} \right) \\ \bar{\psi}_\lambda = j_{6,\lambda} \circ j_{5,\lambda} \circ j_{4,\lambda} \end{cases}$$

We now show that  $\bar{\psi}$  and  $\bar{\phi}$  induce an interleaving between  $\widehat{\mathbb{V}}_{f,s}$  and  $\mathbb{V}_{s,f}$ . More precisely, we show that the following diagrams commute, for all  $\lambda < \lambda'$ . For compactness of notation let,  $K_1 = k_1 + (1 + c_1)\|W\|_{G(h)}$  and  $K_2 = k_2 + (1 + c_1)\|W\|_{G(h)}$ .

$$\begin{array}{ccc} H_s \left( \widehat{\mathcal{F}}_\lambda^{h/4} \right) & \xrightarrow{\hat{v}_\lambda^{\lambda'}} & H_s \left( \widehat{\mathcal{F}}_{\lambda'}^{h/4} \right) \\ \downarrow \bar{\phi}_\lambda & & \downarrow \bar{\phi}_{\lambda'} \\ H_s \left( \mathcal{F}_{\lambda + K_1 h^\alpha} \right) & \xrightarrow{v_{\lambda + K_1 h^\alpha}^{\lambda' + K_1 h^\alpha}} & H_s \left( \mathcal{F}_{\lambda' + K_1 h^\alpha} \right) \end{array} \quad (10)$$

$$\begin{array}{ccc} H_s(\mathcal{F}_\lambda) & \xrightarrow{v_\lambda^{\lambda'}} & H_s(\mathcal{F}_{\lambda'}) \\ \downarrow \bar{\psi}_\lambda & & \downarrow \bar{\psi}_{\lambda'} \\ H_s \left( \widehat{\mathcal{F}}_{\lambda + K_2 h^\alpha}^{h/4} \right) & \xrightarrow{\hat{v}_{\lambda + K_2 h^\alpha}^{\lambda' + K_2 h^\alpha}} & H_s \left( \widehat{\mathcal{F}}_{\lambda' + K_2 h^\alpha}^{h/4} \right) \end{array} \quad (11)$$

$$\begin{array}{ccc} H_s \left( \widehat{\mathcal{F}}_\lambda^{h/4} \right) & \xrightarrow{\hat{v}_\lambda^{\lambda + (K_1 + K_2)h^\alpha}} & H_s \left( \widehat{\mathcal{F}}_{\lambda + (K_1 + K_2)h^\alpha}^{h/4} \right) \\ \searrow \bar{\phi}_\lambda & & \nearrow \bar{\psi}_{\lambda + K_2 h^\alpha} \\ & H_s \left( \mathcal{F}_{\lambda + K_2 h^\alpha} \right) & \end{array} \quad (12)$$

$$\begin{array}{ccc} H_s(\mathcal{F}_\lambda) & \xrightarrow{v_\lambda^{\lambda + (K_1 + K_2)h^\alpha}} & H_s \left( \mathcal{F}_{\lambda + (K_1 + K_2)h^\alpha} \right) \\ \searrow \bar{\psi}_\lambda & & \nearrow \bar{\phi}_{\lambda + K_1 h^\alpha} \\ & H_s \left( \widehat{\mathcal{F}}_{\lambda + K_1 h^\alpha}^{h/4} \right) & \end{array} \quad (13)$$

- **Diagram 10** : We can rewrite the diagram as (unspecified maps are simply induced by set

inclusion),

$$\begin{array}{ccc}
H_s \left( \widehat{\mathcal{F}}_\lambda^{h/4} \right) & \longrightarrow & H_s \left( \widehat{\mathcal{F}}_{\lambda'}^{h/4} \right) \\
\downarrow j_{0,\lambda} & & \downarrow j_{0,\lambda'} \\
H_s \left( \widehat{\mathcal{F}}_\lambda \right) & \longrightarrow & H_s \left( \widehat{\mathcal{F}}_{\lambda'} \right) \\
\downarrow & & \downarrow \\
H_s \left( \mathcal{K}_{\lambda+\|W\|_{G(h)}h^\alpha, \sqrt{dh}} \right) & \longrightarrow & H_s \left( \mathcal{K}_{\lambda'+\|W\|_{G(h)}h^\alpha, \sqrt{dh}} \right) \\
\downarrow j_{2,\lambda} & & \downarrow j_{2,\lambda'} \\
H_s \left( \mathcal{G}_{\lambda+\|W\|_{G(h)}h^\alpha, \sqrt{dh}} \right) & \longrightarrow & H_s \left( \mathcal{G}_{\lambda'+\|W\|_{G(h)}h^\alpha, \sqrt{dh}} \right) \\
\downarrow & & \downarrow \\
H_s \left( \mathcal{F}_{\lambda+K_1h^\alpha} \right) & \longrightarrow & H_s \left( \mathcal{F}_{\lambda'+K_1h^\alpha} \right)
\end{array}$$

As  $j_{2,\lambda}$ ,  $j_{2,\lambda'}$ ,  $j_{0,\lambda}$ ,  $j_{0,\lambda'}$  comes from deformation retracts and other maps are simply induced by inclusion all faces of Diagram 10 commute and consequently Diagram 10 commutes.

- **Diagram 11** : it can be decomposed similarly to Diagram 10, one can check that the same reasoning then applies.
- **Diagram 12** : Let  $C \in C_s \left( \widehat{\mathcal{F}}_\lambda^{h/4} \right)$  and  $[C]$  its classes in  $H_s \left( \widehat{\mathcal{F}}_\lambda^{h/4} \right)$ . As  $\widehat{\mathcal{F}}_\lambda^{h/4}$  retracts by deformation onto  $\widehat{\mathcal{F}}_\lambda$ , there exists  $\bar{C} \in C_s(\widehat{\mathcal{F}}_\lambda)$  homologous to  $C$  in  $\widehat{\mathcal{F}}_\lambda^{h/4}$ , i.e.  $[C] = [\bar{C}]$  in  $H_s(\widehat{\mathcal{F}}_\lambda^{h/4})$ . Hence, without loss of generality, we can suppose  $C \in C_s(\widehat{\mathcal{F}}_\lambda)$ . The morphism  $\bar{\phi}_\lambda$  maps  $[C]$  to  $[C']$  with,

$$C' = F_{\lambda+\|W\|_{G(h)}h^\alpha, \sqrt{dh}}^\#(C, 1).$$

Similarly,  $\bar{\psi}_{\lambda+K_1h^\alpha}$  maps  $[C']$  to  $[C'']$ , with,

$$C'' = H_{\lambda+(K_2+\|W\|_{G(h)}h^\alpha, \sqrt{dh}}^\#(C', 1).$$

Assertion 8 of Proposition 4 ensures that, for all  $x \in \widehat{\mathcal{F}}_\lambda \subset \mathcal{F}_{\lambda+\|W\|_{G(h)}h^\alpha}^{\sqrt{dh}}$ ,

$$F_{\lambda+\|W\|_{G(h)}h^\alpha, \sqrt{dh}}(x, [0, 1]) \subset \widehat{\mathcal{F}}_{\lambda+K_1h^\alpha}^{h/4}$$

and thus,

$$F_{\lambda+\|W\|_{G(h)}h^\alpha, \sqrt{dh}}^\#(C, [0, 1]) \subset C_s \left( \widehat{\mathcal{F}}_{\lambda+K_1h^\alpha}^{h/4} \right).$$

As  $\widehat{\mathcal{F}}_{\lambda+K_1h^\alpha}^{h/4}$  retracts by deformation onto  $\widehat{\mathcal{F}}_\lambda + K_1h^\alpha$ , there exists  $\bar{C}' \in C_s(\widehat{\mathcal{F}}_{\lambda+K_1h^\alpha})$  homologous to  $C'$  in  $\widehat{\mathcal{F}}_{\lambda+K_1h^\alpha}^{h/4}$ , i.e.  $[C'] = [\bar{C}']$  in  $H_s(\widehat{\mathcal{F}}_{\lambda+K_1h^\alpha}^{h/4})$ . Hence, without loss of generality, we can suppose  $C' \in C_s \left( \widehat{\mathcal{F}}_{\lambda+K_1h^\alpha} \right)$ . Assertion 9 then ensures that, for all  $x \in \widehat{\mathcal{F}}_{\lambda+K_1h^\alpha}$ ,

$$H_{\lambda+(K_2+\|W\|_{G(h)}h^\alpha, \sqrt{dh}}(x, [0, 1]) \subset \widehat{\mathcal{F}}_{\lambda+(K_1+K_2)h^\alpha}^{h/4}$$

and thus,

$$H_{\lambda+(K_2+\|W\|_{G(h)}h^\alpha, \sqrt{dh}}^\#(C', [0, 1]) \subset C_s \left( \widehat{\mathcal{F}}_{\lambda+(K_1+K_2)h^\alpha}^{h/4} \right).$$

Hence,  $C$  and  $C''$  are homologous in  $\widehat{\mathcal{F}}_{\lambda+(K_1+K_2)h^\alpha}^{h/4}$  and Diagram 12 commutes.

- **Diagram 13** : Let  $C \in C_s(\mathcal{F}_\lambda)$ ,  $\bar{\psi}_\lambda$  maps  $[C]$  to  $[C']$ , with,

$$C' = H_{\lambda, \sqrt{dh}}^\#(C, 1).$$

As  $\mathcal{M}_{\lambda, \sqrt{dh}} \subset \mathcal{G}_{\lambda+K_2h^\alpha, \sqrt{dh}}$ , the morphism  $\bar{\phi}_{\lambda+K_1h^\alpha}$  behaves as an inclusion induced map, mapping  $[C']$  to  $[C']$ . From Proposition 3, we have, for all  $x \in \mathcal{F}_\lambda$ ,

$$H_{\lambda, \sqrt{dh}}(x, [0, 1]) \subset \mathcal{F}_{\lambda+(K_1+K_2)h^\alpha}.$$

and thus,

$$H_{\lambda, \sqrt{dh}}^\#(C, [0, 1]) \subset C_s(\mathcal{F}_{\lambda+(K_1+K_2)h^\alpha}).$$

Thus,  $C$  and  $C'$  are homologous in  $\mathcal{F}_{\lambda+(K_1+K_2)h^\alpha}$  and Diagram 13 commutes.

The commutativity of diagrams 11, 10, 12 and 13 means that  $\widehat{\mathbb{V}}_{f,s}$  and  $\mathbb{V}_{f,s}$  are  $(K_1 + K_2)h^\alpha$  interleaved, and thus we get from the algebraic stability theorem (Chazal et al., 2009) that,

$$d_b\left(\text{dgm}\left(\widehat{\mathbb{V}}_{f,s}\right), \text{dgm}\left(\mathbb{V}_{f,s}\right)\right) \leq (K_1 + K_2)h^\alpha$$

and as it holds for all  $s \in \mathbb{N}$ ,

$$\sup_{f \in S_d(L, \alpha, R)} d_b\left(\widehat{\text{dgm}}(f), \text{dgm}(f)\right) \leq (K_1 + K_2)h^\alpha.$$

Now, using Proposition 5, this implies that,

$$\begin{aligned} & \mathbb{P}\left(\sup_{f \in S_d(L, \alpha, R)} d_b\left(\widehat{\text{dgm}}(f), \text{dgm}(f)\right) \geq th^\alpha\right) \\ & \leq \mathbb{P}(K_1 + K_2 \geq t) \\ & = \mathbb{P}\left(\|W\|_{G(h)} \geq \frac{t - k_1 - k_2}{2(1 + c_1)}\right) \\ & \leq C_0 \exp\left(-C_1 \left(\frac{t - k_1 - k_2}{2(1 + c_1)}\right)^2\right) \\ & \leq C_0 \exp\left(2 \frac{C_1(k_1 + k_2)}{2(1 + c_1)} t\right) \exp\left(-C_1 \left(\frac{k_1 + k_2}{2(1 + c_1)}\right)^2\right) \exp\left(-\frac{C_1}{(2(1 + c_1))^2} t^2\right) \end{aligned}$$

and the result follows. □

From this result, we can derive from this result bounds in expectation.

**Theorem 2.** Let  $h \simeq (\theta^2 \log(\frac{1}{\theta}))^{\frac{1}{d+2\alpha}}$  and  $p \geq 1$ ,

$$\sup_{f \in S_d(L, \alpha, R)} \mathbb{E}\left(d_b\left(\widehat{\text{dgm}}(f), \text{dgm}(f)\right)^p\right) \lesssim \left(\theta^2 \log\left(\frac{1}{\theta}\right)\right)^{\frac{p\alpha}{d+2\alpha}}$$

*Proof.* The sub-Gaussian concentration provided by Theorem 1, gives that, for all  $t > 0$ ,

$$\mathbb{P}\left(\frac{d_b\left(\widehat{\text{dgm}}(f), \text{dgm}(f)\right)}{h^\alpha} \geq t\right) \leq \tilde{C}_0 \exp\left(-\tilde{C}_1 t^2\right)$$



Now, we have,

$$\begin{aligned}
& \mathbb{E} \left( \frac{d_b \left( \widehat{\text{dgm}}(f), \text{dgm}(f) \right)^p}{h^{p\alpha}} \right) \\
&= \int_0^{+\infty} \mathbb{P} \left( \frac{d_b \left( \widehat{\text{dgm}}(f), \text{dgm}(f) \right)^p}{h^{p\alpha}} \geq t \right) dt \\
&\leq \int_0^{+\infty} \tilde{C}_0 \exp \left( -\tilde{C}_1 t^{2/p} \right) dt < +\infty.
\end{aligned}$$

□

### 3 Adaptivity

The previous procedure depends strongly on the regularity parameter  $\alpha$  as we calibrate the window size  $h$  taking account of it. Thus, the procedure is not adaptive to the regularity. In the following, we propose an estimation procedure, based on the previous one, that is adaptive with respect to  $\alpha$ . Moreover, we show that this adaptive procedure achieves the same rates as the one given by Theorem 2.

We follow the Lepskii's method (Lepskii, 1991). Suppose that we know an upper bound on the parameter  $L$ , denoted  $\bar{L}$  and  $0 < \alpha_{\min} \leq \alpha \leq \alpha_{\max}$ . It is sufficient to work on regular grid  $\alpha_{\min} = \alpha_1 < \alpha_2 < \dots < \alpha_N = \alpha_{\max}$  with  $N \simeq \log \left( \frac{1}{\theta} \right)$ , as, for all  $1 < j \leq N$ ,

$$\log \left( \frac{h_{\alpha_{j-1}}^{\alpha_{j-1}}}{h_{\alpha_j}^{\alpha_j}} \right) = \left( \frac{\alpha_{j-1}}{2\alpha_{j-1} + d} - \frac{\alpha_j}{2\alpha_j + d} \right) \log \left( \theta \log \left( \frac{1}{\theta} \right) \right) \simeq \frac{\log \left( \theta \log \left( \frac{1}{\theta} \right) \right)}{\log(\theta)} \simeq 1$$

with  $h_\alpha = \left( \theta^2 \log \left( \frac{1}{\theta} \right) \right)^{\frac{1}{d+2\alpha}}$ . We consider the Lepskii's estimator defined by,

$$\widehat{\text{dgm}}(f)^* = \widehat{\text{dgm}}(f)_{\hat{\alpha}}$$

with

$$\hat{\alpha} = \max \left\{ \alpha \in \{\alpha_1, \dots, \alpha_N\} : \frac{d_b(\widehat{\text{dgm}}(f)_\alpha, \widehat{\text{dgm}}(f)_{\alpha'})}{h_\alpha^\alpha} < c_0 \text{ for all } \alpha' \leq \alpha \right\}.$$

$c_0$  a sufficiently large constant depending on  $d, \bar{L}, \alpha_{\min}$  and  $\alpha_{\max}$ . The notation  $\widehat{\text{dgm}}(f)_\alpha$  refer to the estimator  $\widehat{\text{dgm}}(f)$  for the window size  $h_\alpha$ , as it will play a role in this section, we highlight the dependence in  $\alpha$ .

**Theorem 3.** *Let  $p \geq 1$ ,*

$$\sup_{(L, \alpha) \in [0, \bar{L}] \times [\alpha_{\min}, \alpha_{\max}]} \sup_{f \in S_d(L, \alpha, R)} \mathbb{E} \left( \frac{d_b \left( \widehat{\text{dgm}}(f)^*, \text{dgm}(f) \right)^p}{h_\alpha^{p\alpha}} \right) \lesssim 1.$$

*Proof.* We want to apply Corollary 1 of Lepskii (1992), in our case, the only difficulty is to check assumption **A3b**. It then suffices to show that, there exists  $c_0 > 0$ , such that, for all  $\alpha' \in [\alpha_{\min}, \alpha_{\max}]$

$$\limsup_{\theta \rightarrow 0} \frac{\log(1/\theta)^2}{h_{\alpha_{\max}}^{\alpha_{\max}}} \sup_{\alpha < \alpha'} \mathbb{P} \left( \frac{d_b(\widehat{\text{dgm}}(f)_\alpha, \text{dgm}(f))}{h_\alpha^\alpha} > c_0 \right) = 0. \tag{14}$$

Now as shown in the proof of Theorem 1, for sufficiently small  $\theta$ ,

$$d_b(\widehat{\text{dgm}}(f)_\alpha, \text{dgm}(f)) \leq (K_1 + K_2) h^\alpha.$$

Thus, for sufficiently small  $\theta$ , using the concentration from Proposition 5,

$$\begin{aligned} & \mathbb{P} \left( \frac{d_b(\widehat{\text{dgm}}(f)_\alpha, \text{dgm}(f))}{h^\alpha} > c_0 \right) \\ & \mathbb{P} \left( \|W\|_{G(h)} \geq \frac{c_0 - k_1 - k_2}{2(1 + c_1)} \right) \\ & \leq 2 \left( \frac{1}{h} \right)^d \exp \left( - \left( \frac{c_0 - k_1 - k_2}{2(1 + c_1)} \right)^2 \log \left( 1 + \frac{1}{h^d} \right) \right). \\ & \lesssim \left( \theta^2 \log \left( \frac{1}{\theta} \right) \right)^{\frac{d\alpha}{d+2\alpha} \left( \left( \frac{c_0 - k_1 - k_2}{2(1 + c_1)} \right)^2 - 1 \right)} \end{aligned}$$

As  $k_1 = Ld^{\alpha/2}(1+3^\alpha) + c_2$ ,  $k_2 = Ld^{\alpha/2}(2+5^\alpha) + c_2$ ,  $c_1 = 2 + \lceil 2\sqrt{d} \rceil^d$  and  $c_2 = L(1 + \lceil 2\sqrt{d} \rceil^d)(5\sqrt{d})^\alpha$ , for sufficiently big  $c_0$  (depending only on  $\alpha_{\min}, \alpha_{\max}, \bar{L}, d$ ),

$$\mathbb{P} \left( \frac{d_b(\widehat{\text{dgm}}(f)_\alpha, \text{dgm}(f))}{h^\alpha} > c_0 \right) = o \left( \frac{h^{\alpha_{\max}}}{\log \left( \frac{1}{\theta} \right)^2} \right).$$

Hence (14) is verified and Corollary 1 of Lepskii (1992) gives the desired result.  $\square$

## 4 Lower bounds

In this section we prove that the rates obtained in the previous section are optimal, in the minimax sense in the non-adaptive and adaptive case, by proving Theorem 4.

**Theorem 4.** *Let  $p \geq 1$*

$$\inf_{\widehat{\text{dgm}}(f)} \sup_{f \in S_d(L, \alpha, R)} \mathbb{E} \left( d_b \left( \widehat{\text{dgm}}(f), \text{dgm}(f) \right)^p \right) \gtrsim \left( \theta^2 \log \left( \frac{1}{\theta} \right) \right)^{\frac{p\alpha}{d+2\alpha}}.$$

Where the infimum is taken over all the estimator of  $\text{dgm}(f)$ .

*Proof.* The proof follows standard methods to provide minimax lower bounds, as presented in section 2 of Tsybakov (2008). The idea is, for any  $r_\theta = o \left( \left( \theta^2 \log \left( \frac{1}{\theta} \right) \right)^{\frac{\alpha}{d+2\alpha}} \right)$ , to exhibit a finite collection of function in  $S_d(L, \alpha, R)$  such that their persistence diagrams are two by two at distance  $2r_\theta$  but indistinguishable, with high certainty.

We propose such a collection, let

$$f_0(x_1, \dots, x_d) = \frac{L}{2} |x_1|^\alpha$$

and for  $m$  integer in  $[0, \lfloor 1/h \rfloor]$ ,

$$f_{h,m}(x_1, \dots, x_d) = f_0 - L \left( h^\alpha - \|(x_1, \dots, x_d) - m/\lfloor 1/h \rfloor (1, \dots, 1)\|_\infty^\alpha \right)_+$$

$f_0$  and the  $f_{h,m}$  are  $(L, \alpha)$ -Hölder-continuous and thus belong to  $S_d(L, \alpha, R)$  for all  $R > 0$ .

We have  $\text{dgm}(f_0) = \{(0, +\infty)\}$  and for all  $0 < m < \lfloor 1/h \rfloor$ , integer,

$$\text{dgm}(f_{h,m}) = \left\{ (0, +\infty), \left( \frac{L}{2} \left( \frac{m}{\lfloor 1/h \rfloor} \right)^\alpha - Lh^\alpha, \frac{L}{2} \left( \frac{m}{\lfloor 1/h \rfloor} \right)^\alpha - \frac{L}{2} h^\alpha \right) \right\}.$$

Thus, for all  $0 < m \neq m' < \lfloor 1/h \rfloor$ , integers,

$$d_b(\text{dgm}(f_0), \text{dgm}(f_{m,h})) \geq \frac{Lh^\alpha}{2} \text{ and } d_b(\text{dgm}(f_{m,h}), \text{dgm}(f_{m'})) \geq \frac{Lh^\alpha}{2}.$$

We set  $r_\theta = \frac{Lh^\alpha}{4}$ , then,

$$d_b(\text{dgm}(f_0), \text{dgm}(f_{d,h,mk',\alpha})) \geq 2r_\theta \text{ and } d_b(\text{dgm}(f_{h,m}), \text{dgm}(f_{h,m'})) \geq 2r_\theta.$$

For a fixed signal  $f$ , denote  $\mathbb{P}_f^\theta$  the product distribution of the noisy trajectory  $X$  define in model 1. From section 2 of [Tsybakov \(2008\)](#), it now suffices to show that if  $r_\theta = o\left((\theta^2 \log(\frac{1}{\theta}))^{\frac{\alpha}{d+2\alpha}}\right)$ , then,

$$\frac{1}{\lfloor \frac{1}{h} \rfloor - 2} \sum_{0 < m < \lfloor 1/h \rfloor} \chi^2\left(\mathbb{P}_{f_{h,m}}^\theta \mathbb{P}_{f_0}^\theta\right) = \frac{1}{\lfloor \frac{1}{h} \rfloor - 2} \sum_{0 < m < \lfloor 1/h \rfloor} \mathbb{E}_{\mathbb{P}_{f_0}^\theta} \left[ \left( \frac{d\mathbb{P}_{f_{h,m}}^\theta}{d\mathbb{P}_{f_0}^\theta} \right)^2 \right] - 1 \quad (15)$$

converges to zero when  $\theta$  converges to zero.

By Cameron-Martin formula, for all  $0 < m < \lfloor 1/h \rfloor$ , integer,

$$\frac{d\mathbb{P}_{f_{h,m}}^\theta}{d\mathbb{P}_{f_0}^\theta} = \exp\left(\int_{[0,1]^d} \theta(f_{h,m} - f_0)(t_1, \dots, t_d) dW_{t_1, \dots, t_d} - \frac{1}{\theta^2} \|f_{h,m} - f_0\|_2^2\right).$$

We denote  $H_m$  the hypercube defined by  $\|(x_1, \dots, x_d) - m/\lfloor 1/h \rfloor(1, \dots, 1)\| \leq h$

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}_{f_0}^\theta} \left( \left( \frac{d\mathbb{P}_{f_{h,m}}^\theta}{d\mathbb{P}_{f_0}^\theta} \right)^2 \right) \\ &= \exp\left(\int_{[0,1]^d} \frac{1}{\theta^2} (f_{h,m} - f_0)^2(t_1, \dots, t_d) dt_1 \dots dt_d\right) \\ &= \exp\left(\frac{L^2}{\theta^2} \int_{H_m} (h^\alpha - \|(t_1, \dots, t_d) - m/\lfloor 1/h \rfloor(1, \dots, 1)\|_2^\alpha)^2 dt_1 \dots dt_d\right) \\ &\leq \exp\left(\frac{L^2}{\theta^2} \left( \int_{H_m} h^{2\alpha} dt_1 \dots dt_d + \int_{H_m} \|(t_1, \dots, t_d) - m/\lfloor 1/h \rfloor(1, \dots, 1)\|_2^{2\alpha} dt_1 \dots dt_d \right)\right) \\ &\leq \exp\left(\frac{2L^2}{\theta^2} \int_{H_m} h^{2\alpha} dt_1 \dots dt_d\right) \\ &\leq \exp\left(\frac{2L^2}{\theta^2} h^{2\alpha+d}\right). \end{aligned}$$

Hence, if  $(\theta^2 \log(\frac{1}{\theta}))^{\frac{1}{d+2\alpha}} \ll h$ , we have that (15) converges to zero. Consequently, if  $r_\theta = o\left((\theta^2 \log(\frac{1}{\theta}))^{\frac{\alpha}{d+2\alpha}}\right)$ , then  $(\theta^2 \log(\frac{1}{\theta}))^{\frac{1}{d+2\alpha}} \ll h$  and we get the conclusion.  $\square$

## 5 Discussion

To date, statistical studies of Topological Data Analysis tools have predominantly relied on lifting known results from signal (or density) estimation using sup norm stability. However, this work represents a step forward, breaking free from this approach. We provide a finer analysis of the plug-in histogram estimator, showing that it achieves minimax convergence rates on the classes  $S_d(L, \alpha, R)$  that coincide with the known ones for Holder-continuous signals. These classes contain irregular functions that pose challenges for conventional non-parametric techniques. Beyond the results shown here, it opens a new path to think and analyze persistent homology inference, showing that it allows relaxation of regularity assumptions over considered signals.

It then raises questions about further relaxations of the regularity assumptions. We believe that there is room to consider the potential relaxation assumption **A3**. One plausible approach involves controlling the  $\mu$ -reach, as defined in [Chazal et al. \(2006\)](#), of the discontinuities set. This would extend significantly our results, allowing to handle, for example, signals with sets of discontinuities featuring multiple points and corners. Still, as illustrated in [Figure 5](#), in this case, a plug-in estimator from histogram will fall short, we may then need to move away from plug-in approach.

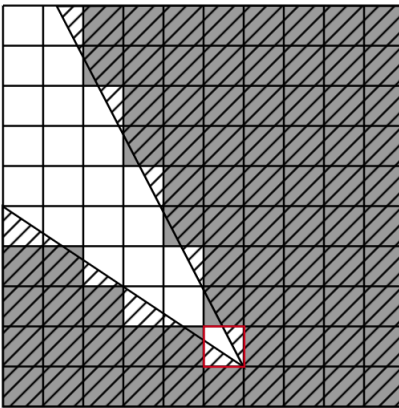


Figure 5:  $\lambda$ -sublevel cubical approximation for  $f$  the function defined as 0 on the hatched area and  $K$  outside (for arbitrarily large  $K$ ) and  $\lambda = K/4$ . The cycle in red is problematic, as it has a lifetime of  $CK$  ( $C$  an absolute constant). The discontinuity set has here a positive  $\mu$ -reach for small  $\mu$ .

One can also wonder if the methods and convergence rates established here for the Gaussian white noise model extend to other popular and richer models. In this direction, we show in [Appendix C](#) how they can be extended to the non-parametric regression model with fixed regular design. Motivated by potential application to modes detection, as sketched for example in [Genovese et al. \(2015\)](#), extending these results to the density model motivates future work in this direction.

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## A Proofs for $q$ -tameness

This section is devoted to prove the claim that the persistence diagrams we consider and estimated persistence diagrams we propose are well-defined, by proving that the underlying persistence modules are  $q$ -tame.

**Lemma 1.** *Let  $f \in S_d(L, \alpha, R)$ .  $\forall s \in \mathbb{N}$ ,  $\forall h < \frac{R}{2}$ , there exist a morphism  $\phi$  such that,  $\forall \lambda \in \mathbb{R}$ ,*

$$\begin{array}{ccc}
 H_s(\mathcal{F}_\lambda) & \xrightarrow{\quad} & H_s(\mathcal{F}_{\lambda+L(1+3^\alpha)h^\alpha}) \\
 & \searrow & \nearrow \phi_\lambda \\
 & & H_s(\mathcal{F}_\lambda^h)
 \end{array} \tag{16}$$

*is a commutative diagram (unspecified map come from set inclusions).*

*Proof.* Let  $\tilde{\phi}_\lambda : H_s(\mathcal{K}_{\lambda,h}) \rightarrow H_s(\mathcal{G}_{\lambda,h})$  the induced by the deformation retract from Proposition 2. We also denote  $i_{1,\lambda} : H_s(\mathcal{F}_\lambda^h) \rightarrow H_s(\mathcal{K}_{\lambda,h})$  the morphism induced by the inclusion  $\mathcal{F}_\lambda^h \subset \mathcal{K}_{\lambda,h}$  and  $i_{2,\lambda} : H_s(\mathcal{G}_{\lambda,h}) \rightarrow H_s(\mathcal{F}_{\lambda+L(1+3^\alpha)h^\alpha})$  the morphism induced by the inclusion  $\mathcal{G}_{\lambda,h} \subset \mathcal{F}_{\lambda+L(1+3^\alpha)h^\alpha}$ , also provided by Proposition 2. We take  $\phi_\lambda = i_{2,\lambda} \circ \tilde{\phi}_\lambda \circ i_{1,\lambda}$ . Diagram 16 then is (unspecified maps are the one induced by set inclusion),

$$\begin{array}{ccccc}
 H_s(\mathcal{F}_\lambda) & \xrightarrow{\quad} & H_s(\mathcal{F}_{\lambda+L(1+3^\alpha)h^\alpha}) & & \\
 \downarrow (F1) & \searrow (F2) & \searrow (F3) & & \uparrow i_{2,\lambda} \\
 H_s(\mathcal{F}_\lambda^h) & \xrightarrow{i_{1,\lambda}} & H_s(\mathcal{K}_{\lambda,h}) & \xleftarrow{\tilde{\phi}_\lambda} & H_s(\mathcal{G}_{\lambda,h})
 \end{array} \tag{17}$$

Faces (F1) and (F3) simply commutes by inclusion. Face (F2) commutes as  $\tilde{\phi}_\lambda$  is induced by a deformation retract. Each faces of diagram 17 are commutative, hence diagram 17 (and equivalently diagram 16) is commutative.  $\square$

**Proposition 6.** *Let  $f \in S_d(L, \alpha, R)$  then  $f$  is  $q$ -tame.*

*Proof.* Let  $s \in \mathbb{N}$  and  $\mathbb{V}_{s,f}$  the persistence module (for the  $s$ -th homology) associated to the sublevel filtration,  $\mathcal{F}$  and for fixed levels  $\lambda < \lambda'$  let denote  $v_\lambda^{\lambda'}$  the associated map. Let  $\lambda \in \mathbb{R}$  and  $h < \frac{R}{2}$ . By Lemma 1,  $v_\lambda^{\lambda+L(1+3^\alpha)h^\alpha} = \phi_\lambda \circ \tilde{i}_\lambda$ , with  $\tilde{i}_\lambda : H_s(\mathcal{F}_\lambda) \rightarrow H_s(\mathcal{F}_\lambda^h)$ . By assumption A1 and A2,  $\mathcal{F}_\lambda$  is compact. As  $[0, 1]^d$  is triangulable,  $\mathcal{F}_\lambda$  is covered by finitely many cells of the triangulation, and so there is a finite simplicial complex  $K$  such that  $\overline{\mathcal{F}_\lambda} \subset K \subset \mathcal{F}_\lambda^h$ . Consequently,  $\tilde{i}_\lambda$  factors through the finite dimensional space  $H_s(K)$  and is then of finite rank by Theorem 1.1 of [Crawley-Boevey \(2012\)](#). Thus,  $v_\lambda^{\lambda+(\sqrt{d}+1)^\alpha L h^\alpha}$  is of finite rank for all  $0 < h < \frac{R}{2}$ . As for any  $\lambda < \lambda' < \lambda''$ ,  $v_\lambda^{\lambda''} = v_\lambda^{\lambda'} \circ v_\lambda^{\lambda'}$  we then have that  $v_\lambda^{\lambda'}$  is of finite rank for all  $\lambda < \lambda'$ . Hence,  $f$  is  $q$ -tame.  $\square$

**Proposition 7.** *Let  $f \in S_d(L, \alpha, R)$  then, for all  $s \in \mathbb{N}$ ,  $\widehat{\mathbb{V}}_{s,f}^h$  is  $q$ -tame.*

*Proof.* Let  $h > 0$  and  $\lambda \in \mathbb{R}$ .  $\widehat{\mathcal{F}}_\lambda$  is a union of hypercubes of the regular grid  $G_h$ , thus,  $H_s(\widehat{\mathcal{F}}_\lambda)$  is finite dimensional. Thus  $\widehat{\mathbb{V}}_{s,f}$  is  $q$ -tame by Theorem 1.1 of [Crawley-Boevey \(2012\)](#).  $\square$

## B Proofs of technical lemmas and propositions

### B.1 Proof of Proposition 1

This section is dedicated to the proof of Proposition 1 from Section 2 which relies on the following lemma.

**Lemma 2.** *Let  $f : [0, 1]^d \rightarrow \mathbb{R}$  and  $h > 0$  verifying (2). Let  $H \subset \mathcal{F}_{\lambda + \|W\|_{G(h)} h^\alpha}^c \cap C_h$  and  $H' \subset \mathcal{F}_{\lambda - \|W\|_{G(h)} h^\alpha} \cap C_h$ . We then have that,*

$$\int_H dX - \int_H \lambda > 0 \text{ and } \int_{H'} dX - \int_{H'} \lambda < 0.$$

*Proof.* Let consider here the case where in  $H' \subset \mathcal{F}_{\lambda - \|W\|_{G(h)} h^\alpha}$  (The proof being the same in both cases). Note that,

$$\begin{aligned} & \int_{H'} dX - \int_{H'} \lambda \\ &= \int_{H'} (f - \lambda) + \theta \int_{H'} dW \\ &\leq -\|W\|_{\text{cube}, h} h^\alpha |H'| + \|W\|_{\text{cube}, h} \theta \omega(h^d) \\ &\leq \|W\|_{\text{cube}, h} \left( -h^{d+\alpha} + \theta \omega(h^d) \right) < 0 \end{aligned}$$

by the choice made for  $h$ . □

*Proof of Proposition 1.* Let  $x \in \mathcal{F}_{\lambda - \|W\|_{G(h)} h^\alpha}^{-\sqrt{d}h}$  and  $H$  the hypercube of  $C_h$  containing  $x$ . We then have,

$$H \subset \mathcal{F}_{\lambda - \|W\|_{G(h)} h^\alpha}.$$

Hence, by Lemma 2,  $\int_H dX - \int_H \lambda < 0$ , thus,

$$H \subset \widehat{\mathcal{F}}_\lambda.$$

Now, let  $x \in \left( \mathcal{F}_{\lambda + \|W\|_{G(h)} h^\alpha}^{\sqrt{d}h} \right)^c$ , and  $H$  the hypercube of  $C_h$  containing  $x$ . We then have,

$$H \subset \mathcal{F}_{\lambda + \|W\|_{G(h)} h^\alpha}^c.$$

Hence, by Lemma 2,  $\int_H dX - \int_H \lambda > 0$ , thus,

$$H \subset \widehat{\mathcal{F}}_\lambda^c$$

and Proposition 1 is proved. □

### B.2 Proof of Proposition 2

This section is devoted to the proof of Proposition 2 from Section 2.

*Proof.* First, note that if  $x$  belongs to  $\bigcup_{x \in S_{\lambda, h}} [x, \xi(x)]$  then  $x$  is at distance a most  $h$  from the union of  $(M_i)_{i=1, \dots, l}$ , and thus  $\|x - \xi(x)\|_2 \leq h$  which proves that  $\mathcal{K}_{\lambda, h} \subset \mathcal{F}_\lambda^{2h}$ .

Let's now prove that  $F_{\lambda, h}$  is a deformation retraction. As  $\mathcal{F}_\lambda \subset \left( \bigcup_{x \in S_{\lambda, h}} [x, \xi(x)] \right)^c$ ,  $\mathcal{F}_\lambda \subset \mathcal{G}_{\lambda, h}$ .



By definition of  $\mathcal{G}_{\lambda,h}$ ,

$$F_{\lambda,h}(x, 1) \in \mathcal{G}_{\lambda,h}, \quad \forall x \in \mathcal{K}_{\lambda,h}$$

and by definition of  $F_{\lambda,h}$ ,

$$F_{\lambda,h}(x, 0) = x, \quad \forall x \in \mathcal{K}_{\lambda,h}$$

Let  $x \in \mathcal{K}_{\lambda,h}$  verifying (4) and (5), we have,  $F_{\lambda,h}(x, t) \in [x, \xi(x)]$  for all  $t \in [0, 1]$ . In particular, this implies that  $\xi(F_{\lambda,h}(x, 1)) = \xi(x)$  thus  $F_{\lambda,h}(F_{\lambda,h}(x, 1), 1) = F_{\lambda,h}(x, 1)$ . Otherwise, by construction,  $F_{\lambda,h}(x, 1) = x$ . Hence,

$$F_{\lambda,h}(x, 1) = x, \quad \forall x \in \mathcal{G}_{\lambda,h}.$$

The proof for the continuity of  $F_{\lambda,h}$  is provided separately in Appendix B.3. Then  $F_{\lambda,h}$  is a deformation retract onto  $\mathcal{G}_{\lambda,h}$ .

Let's now prove that  $\mathcal{G}_{\lambda,h} \subset \mathcal{F}_{\lambda+(1+3^\alpha)Lh^\alpha}$ . Let  $x \in \mathcal{K}_{\lambda,h}$ , and suppose  $x \in \overline{M}_i \cap [0, 1]^d$ .

If  $x$  does not verify (4),  $F_{\lambda,h}(x, 1) = x$  and by definition of  $S_{\lambda,h}$ , assumption **A2** ensures that  $x \in \mathcal{F}_{\lambda+Lh^\alpha}$ .

If  $x$  verifies (4) and  $2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) \geq 0$ , as  $F_{\lambda,h}(x, 1) \in [x, \xi(x)] \subset \overline{M}_i$ , we have

$$d_2(F_{\lambda,h}(x, 1), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) \leq 3h.$$

Assumptions **A1** and **A2** then ensures that,

$$F_{\lambda,h}(x, 1) \in \mathcal{F}_{\lambda+L(1+3^\alpha)h^\alpha}.$$

If  $x$  verifies (4) and  $2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) < 0$ , then,  $F_{\lambda,h}(x, 1) = \xi(x)$ . Let  $\varepsilon > 0$ , there exists  $j \in \{1, \dots, l\}$ ,  $i \neq j$  and  $y \in \mathcal{F}_\lambda \cap M_j$ , such that  $\|x - y\|_2 \leq h + \varepsilon$ . Hence,  $\xi(x) \in \partial M_j \cap ]0, 1[^d$  and  $\|\xi(x) - y\|_2 \leq 2h + \varepsilon$ . Assumptions **A1** and **A2** then ensure that,

$$\xi(x) \in \mathcal{F}_{\lambda+L(1+(2+\varepsilon)^\alpha)h^\alpha}$$

as it holds for all  $\varepsilon > 0$ ,

$$F_{\lambda,h}(x, 1) = \xi(x) \in \mathcal{F}_{\lambda+L(1+2^\alpha)h^\alpha}.$$

Finally, combining cases,  $\mathcal{G}_{\lambda,h} \subset \mathcal{F}_{\lambda+L(1+3^\alpha)h^\alpha}$ . □

### B.3 Proof of the continuity of $F_{\lambda,h}$

This section is devoted to the proof of the deformation retract  $F_{\lambda,h}$ , introduced in the proof of Proposition 2.

**Lemma 3.** *Let  $h > 0$  and  $\lambda \in \mathbb{R}$ ,  $F_{\lambda,h}$  is continuous.*

*Proof.* Let  $\delta, \delta' > 0$ ,  $x, y \in \mathcal{K}_{\lambda,h}$  such that  $\|x - y\|_2 \leq \delta$  and  $t, s \in [0, 1]$  a such that  $|t - s| \leq \delta'$ . Let's check the different cases.

We begin by the cases where  $x \in \overline{M}_i$  and  $y \in \overline{M}_j$ ,  $i \neq j$ . Then  $\|x - \xi(x)\|_2 \leq \delta$ ,  $\|y - \xi(y)\|_2 \leq \delta$ , and thus,  $\|\xi(x) - \xi(y)\|_2 \leq 2\delta$ .

- Suppose  $x$  verifies (4) and (5) for  $i \in \{1, \dots, l\}$ . If  $(2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}))_+ > 0$ , for sufficiently small  $\delta$ , we would have,

$$\|x - \xi(x)\|_2 \leq \delta < (2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}))_+$$

which is contradictory. Hence, we can suppose  $(2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}))_+ = 0$ . Then, as  $F_{\lambda,h}(\xi(x), t) = \xi(x)$

$$\|F_{\lambda,h}(x, t) - F_{\lambda,h}(\xi(x), t)\|_2 = (1 - t)\|x - \xi(x)\|_2 \leq \delta.$$

- Otherwise,  $F_{\lambda,h}(x, t) = x$ , and directly,

$$\|F_{\lambda,h}(x, t) - F_{\lambda,h}(\xi(x), t)\|_2 = \|x - \xi(x)\|_2 \leq \delta.$$

Following the same reasoning we also have,

$$\|F_{\lambda,h}(y, s) - F_{\lambda,h}(\xi(y), s)\|_2 \leq \delta.$$

Then,

$$\begin{aligned} \|F_{\lambda,h}(x, t) - F_{\lambda,h}(y, s)\|_2 &\leq \|F_{\lambda,h}(x, t) - F_{\lambda,h}(\xi(x), t)\|_2 \\ &\quad + \|F_{\lambda,h}(\xi(x), t) - F_{\lambda,h}(\xi(y), s)\|_2 \\ &\quad + \|F_{\lambda,h}(\xi(y), s) - F_{\lambda,h}(y, s)\|_2 \\ &\leq 2\delta + \|\xi(x) - \xi(y)\|_2 \leq 4\delta \end{aligned}$$

And the conclusion follows in this case. From now, we suppose that  $x, y \in M_i$ .

- If  $x$  does not verify (4) or (5) and  $y$  does not verify (4) or (5), then, directly,

$$\|F_{\lambda,h}(x, t) - F_{\lambda,h}(y, s)\|_2 = \|x - y\|_2 \leq \delta.$$

- If  $x$  verifies (4) and (5), and  $y$  does not verify (4), then,  $y \in \mathcal{F}_{\lambda+2Lh^\alpha}$ . Thus,

$$d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) \leq \|x - \xi(x)\|_2 + \|x - y\|_2 \leq h + \delta$$

and,

$$2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) \geq h - \delta \geq \|x - \xi(x)\|_2 - \delta.$$

Consequently,

$$\left\| \xi(x) + (2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}))_+ \frac{x - \xi(x)}{\|x - \xi(x)\|_2} - x \right\|_2 \leq \delta$$

Then,

$$\begin{aligned} &\|F_{\lambda,h}(x, t) - F_{\lambda,h}(y, s)\|_2 \\ &= \left\| (1-t)x + t \left( \xi(x) + (2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}))_+ \frac{x - \xi(x)}{\|x - \xi(x)\|_2} \right) - y \right\|_2 \\ &\leq \|x - y\|_2 + \left\| \xi(x) + (2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}))_+ \frac{x - \xi(x)}{\|x - \xi(x)\|_2} - x \right\|_2 \\ &\leq 2\delta. \end{aligned}$$

- If  $x$  verifies (4) and (5), and  $y$  verifies (4) but not (5). Then,

$$\begin{aligned} 2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) &= 2h - d_2(\xi(y), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) \\ &\quad + d_2(\xi(y), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) \\ &\geq 2h - d_2(\xi(y), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) - \|\xi(x) - \xi(y)\|_2 \\ &\geq \|y - \xi(y)\|_2 - \|\xi(x) - \xi(y)\|_2 \\ &\geq \|x - \xi(x)\|_2 - 2\|\xi(x) - \xi(y)\|_2 - \|x - y\|_2. \end{aligned}$$

Hence,

$$\left\| \xi(x) + (2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}))_+ \frac{x - \xi(x)}{\|x - \xi(x)\|_2} - x \right\|_2 \leq \|x - y\|_2 + 2\|\xi(x) - \xi(y)\|_2$$

And thus, we have,

$$\begin{aligned}
& \|F_{\lambda,h}(x, t) - F_{\lambda,h}(y, s)\|_2 \\
&= \left\| (1-t)x + t \left( \xi(x) + (2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}))_+ \frac{x - \xi(x)}{\|x - \xi(x)\|_2} \right) - y \right\|_2 \\
&\leq \|x - y\|_2 + \left\| \xi(x) + (2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}))_+ \frac{x - \xi(x)}{\|x - \xi(x)\|_2} - x \right\|_2 \\
&\leq 2\|x - y\|_2 + 2\|\xi(x) - \xi(y)\|_2 \\
&\leq 2\delta + 2\|\xi(x) - \xi(y)\|_2
\end{aligned}$$

and we conclude by continuity of  $\xi$ .

- Finally, if  $x$  and  $y$  both verify (4) and (5), then,

$$\begin{aligned}
& \|F_{\lambda,h}(x, t) - F_{\lambda,h}(y, s)\|_2 \\
&= \left\| (1-t)x + t \left( \xi(x) + (2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}))_+ \frac{x - \xi(x)}{\|x - \xi(x)\|_2} \right) \right. \\
&\quad \left. - (1-t)y - t \left( \xi(y) + (2h - d_2(\xi(y), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}))_+ \frac{y - \xi(y)}{\|y - \xi(y)\|_2} \right) \right\|_2
\end{aligned}$$

and the conclusion follows again, in this case by continuity of  $\xi$ .

All possible cases have been checked, the proof is complete.  $\square$

## B.4 Proof of Proposition 3

This section is dedicated to proof of Proposition 3 from Section 2.

*Proof of Proposition 3.* First we check that  $\gamma_h$  extends continuously to  $P_{\lambda,h}$ . Let  $x \in P_{\lambda,h} \cap \bigcup_{i=1}^l \partial M_i \cap ]0, 1[^d$ . Assumption **A3**, ensures that  $P_{\lambda,h} \subset ]0, 1[^d$  and that there exists  $i, j \in \{1, \dots, l\}$  such that,

$$B_2(x, h) \subset M_i \cup M_j \cup (\partial M_i \cap \partial M_j).$$

Now, if

$$B_2(x, h) \cap \mathcal{F}_\lambda \cap M_i \neq \emptyset \text{ and } B_2(x, h) \cap \mathcal{F}_\lambda \cap M_j \neq \emptyset$$

then by assumptions **A2** and **A1**,  $B_2(x, h) \subset \mathcal{F}_{\lambda+Lh^\alpha}$  and thus  $x \in \mathcal{F}_{\lambda+Lh^\alpha}^{-h}$ . Hence,  $B_2(x, h) \cap \mathcal{P}_{\lambda,h} \cap M_j = \emptyset$  or  $B_2(x, h) \cap \mathcal{P}_{\lambda,h} \cap M_i = \emptyset$ . Without loss of generality, let suppose  $B_2(x, h) \cap \mathcal{P}_{\lambda,h} \cap M_j = \emptyset$ . Assumption **A3** impose that  $\bigcup_{i=1}^l \partial M_i \cap ]0, 1[^d$  is a  $C^{1,1}$  hypersurface and thus ensures that, for all  $x \in \overline{M_i} \cap ]0, 1[^d$ ,  $\lim_{y \in M_i \rightarrow x} \gamma_h(y)$  exists. We can then define  $\gamma_{\lambda,h}(x) = \lim_{y \in M_i \rightarrow x} \gamma_h(y)$ . And, doing so for all  $x \in P_{\lambda,h} \cap \bigcup_{i=1}^l \partial M_i \cap ]0, 1[^d$  extends continuously  $\gamma_h(x)$  to  $P_{\lambda,h}$ .

Let's now prove that  $H_{\lambda,h}$  is a deformation retraction. As  $\mathcal{F}_\lambda^{-h} \subset \left( \bigcup_{x \in P_{\lambda,h}} [x, \gamma_{\lambda,h}(x)] \right)^c$ ,  $\mathcal{F}_\lambda^{-h} \subset \mathcal{M}_{\lambda,h}$ . Note that, by definition of  $\mathcal{M}_{\lambda,h}$

$$H_{\lambda,h}(x, 1) \in \mathcal{M}_{\lambda,h}, \quad \forall x \in \mathcal{N}_{\lambda,h}$$

and by definition of  $H_{\lambda,h}$

$$H_{\lambda,h}(x, 0) = x, \quad \forall x \in \mathcal{N}_{\lambda,h}$$

Let  $x \in \mathcal{N}_{\lambda,h}$  verifying (6) and (7). By construction  $H_{\lambda,h}(x, t) \in [x, \gamma_{\lambda,h}(x)]$  for all  $t \in [0, 1]$ , in

particular this implies that  $\gamma_{\lambda,x}(H_{\lambda,h}(x,1)) = \gamma_{\lambda,h}(x)$ . Thus,  $H_{\lambda,h}(H_{\lambda,h}(x,1),1) = H_{\lambda,h}(x,1)$ . In other cases  $H_{\lambda,h}(x,1) = x$ . Hence,

$$H_{\lambda,h}(x,1) = x, \quad \forall x \in \mathcal{M}_{\lambda,\alpha}.$$

The proof of the continuity of  $H_{\lambda,h}$  is provided separately in Appendix B.5. Then  $H_{\lambda,h}$  is a deformation retract onto  $\mathcal{M}_{\lambda,h}$ .

Let's now prove that  $\mathcal{M}_{\lambda,h} \subset \mathcal{F}_{\lambda+L(2+5\alpha)h^\alpha}^{-h}$ . Let  $x \in \mathcal{N}_{\lambda,h}$ , and suppose  $x \in \overline{M}_i \cap [0,1]^d$ .

If  $x$  does not verify (6), directly,  $F(1,x) = x \in \mathcal{F}_{\lambda+2Lh^\alpha}^{-h}$ .

If  $x$  verifies (6) and  $3h - d_2(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}) \geq 0$ , then there exists  $z \in M_j \cap \mathcal{F}_{\lambda+2Lh^\alpha}$ ,  $j \neq i$  such that  $\|x - z\|_2 \leq 3h$  thus  $\|H_{\lambda,h}(x,1) - z\|_2 \leq 4h$ . Also, by assumption **A3**,  $B_2(H_{\lambda,h}(x,1), h) \subset \overline{M}_i \cup \overline{M}_j$ . Thus, by assumption **A1** and **A2**,  $B_2(H_{\lambda,h}(x,1), h) \cap \overline{M}_j \subset \mathcal{F}_{\lambda+L(2+5\alpha)h^\alpha}$  and  $B_2(H_{\lambda,h}(x,1), h) \cap \overline{M}_i \subset \mathcal{F}_{\lambda+L2^\alpha h^\alpha}$ , thus,

$$H_{\lambda,h}(x,1) \in \mathcal{F}_{\lambda+L(2+5\alpha)h^\alpha}^{-h}.$$

If  $x$  verifies (6) and  $3h - d_2(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}) < 0$ , then  $H_{\lambda,h}(x,1) = \gamma_{\lambda,h}(x)$  and thus  $B_2(H_{\lambda,h}(x,1), h) \subset \overline{M}_i$ . As  $\|x - \gamma_{\lambda,h}(x)\|_2 \leq h$ , it follows that,

$$H_{\lambda,h}(x,1) \in \mathcal{F}_{\lambda+L2^\alpha h^\alpha}^{-h}.$$

From the same reasoning, it also follows that  $[x, \gamma_{\lambda,h}(x)] \subset \mathcal{F}_{\lambda+Lh^\alpha}$  and hence  $\mathcal{N}_{\lambda,h} \subset \mathcal{F}_{\lambda+Lh^\alpha}$ .

Combining all cases, it follows that  $\mathcal{M}_{\lambda,h} \subset \mathcal{F}_{\lambda+L(2+5\alpha)h^\alpha}^{-h}$ .  $\square$

## B.5 Proof of the continuity of $H_{\lambda,h}$

This section is devoted to the proof of the deformation retract  $H_{\lambda,h}$ , introduced in the proof of Proposition 3.

**Lemma 4.** *Let  $h > 0$  and  $\lambda \in \mathbb{R}$ ,  $H_{\lambda,h}$  is continuous.*

*Proof.* Let  $\delta, \delta' > 0$ ,  $x, y \in \mathcal{K}_{\lambda,h}$  such that  $\|x - y\|_2 \leq \delta$ .  $t, s \in [0,1]$  a such that  $|t - s| \leq \delta'$ . Let's check the different cases.

- If  $x \in \overline{M}_i$  and  $y \notin \overline{M}_i$ . Assumption **A3** ensures that for sufficiently small  $\delta$ , there exists  $j \in \{1, \dots, l\}$ , with  $y \in M_j$  such that,

$$B_2(x, h) \subset B_2(y, 2h) \subset \overline{M}_i \cup \overline{M}_j.$$

By Assumption **A1** and **A2**, this implies that  $B_2(x, h) \subset \mathcal{F}_{\lambda+L2^\alpha h^\alpha}$  and thus  $x \in \mathcal{F}_{\lambda+L2^\alpha h^\alpha}^{-h}$ . From the same reasoning, it follows that  $y \in \mathcal{F}_{\lambda+L2^\alpha h^\alpha}^{-h}$ . Hence,

$$\|H_{\lambda,h}(x, t) - H_{\lambda,h}(y, s)\|_2 = \|x - y\|_2 \leq \delta.$$

From now, we can suppose that  $x, y \in M_i$ .

- If  $x$  does not verify (6) or (7), and  $y$  does not verify (6) or (7), then directly,

$$\|H_{\lambda,h}(x, t) - H_{\lambda,h}(y, s)\|_2 = \|x - y\|_2 \leq \delta.$$

- If  $x$  verifies (6) and (7) and  $y$  does not verify (6), then,  $d_2(x, \partial M_i \cap ]0, 1[^d) \geq h - \delta$  and thus  $\|x - \gamma_{\lambda, h}(x)\|_2 \leq \delta$ . As  $H_{\lambda, h}(x, 1) \in [x, \gamma_{\lambda, h}(x)]$ , we have,

$$\|H_{\lambda, h}(x, 1) - H_{\lambda, h}(y, 1)\|_2 = \|H_{\lambda, h}(x, 1) - y\|_2 \leq \|x - y\|_2 + \|x - H_{\lambda, h}(x, 1)\|_2 \leq 2\delta.$$

If  $x \in \bigcup_{x \in P_{\lambda, h}} [x, \gamma_{\lambda, h}(x)] \cap M_i$  and  $d_2(\gamma_{\lambda, h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}) \geq 3h - \|x - \gamma_{\lambda, h}(x)\|_2$  and  $y \in \left( \bigcup_{x \in P_{\lambda, h}} [x, \gamma_{\lambda, h}(x)] \right)^c \cap \left( (\partial M_i \cap ]0, 1[^d)^h \right)^\circ$ . Then,  $y \in \mathcal{F}_{\lambda+2Lh^\alpha}^{-h}$ . Thus, for sufficiently small  $\delta$  there exists  $z \in (\overline{M}_i)^c \cap B_2(y, h)$  and,

$$d_2(\gamma_{\lambda, h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}) - 2h \leq \|x - y\|_2 \leq \delta.$$

Hence,

$$\left\| \gamma_{\lambda, h}(x) + \left( 3h - d_2(\gamma_{\lambda, h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}) \right)_+ \frac{x - \gamma_{\lambda, h}(x)}{\|x - \gamma_{\lambda, h}(x)\|_2} - x \right\|_2 \leq \delta$$

and,

$$\begin{aligned} & \|H_{\lambda, h}(x, t) - H_{\lambda, h}(y, s)\|_2 \\ &= \left\| (1-t)x + t \left( \gamma_{\lambda, h}(x) + \left( 3h - d_2(\gamma_{\lambda, h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}) \right)_+ \frac{x - \gamma_{\lambda, h}(x)}{\|x - \gamma_{\lambda, h}(x)\|_2} \right) - y \right\|_2 \\ &\leq \left\| \gamma_{\lambda, h}(x) + \left( 3h - d_2(\gamma_{\lambda, h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}) \right)_+ \frac{x - \gamma_{\lambda, h}(x)}{\|x - \gamma_{\lambda, h}(x)\|_2} - x \right\|_2 + \|x - y\|_2 \\ &\leq 2\delta \end{aligned}$$

- If  $x$  verifies (6) and (7) and  $y$  verifies (6) but not (7), then,

$$\begin{aligned} 3h - d_2(\gamma_{\lambda, h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}) &\geq \|y - \gamma_{\lambda, h}(y)\|_2 - \|\gamma_{\lambda, h}(x) - \gamma_{\lambda, h}(y)\|_2 \\ &\geq \|x - \gamma_{\lambda, h}(x)\|_2 - 2\|\gamma_{\lambda, h}(x) - \gamma_{\lambda, h}(y)\|_2 - \|x - y\|_2 \end{aligned}$$

Thus,

$$\begin{aligned} & \|H_{\lambda, h}(x, t) - H_{\lambda, h}(y, s)\|_2 \\ &= \left\| (1-t)x + t \left( \gamma_{\lambda, h}(x) + \left( 3h - d_2(\gamma_{\lambda, h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}) \right)_+ \frac{x - \gamma_{\lambda, h}(x)}{\|x - \gamma_{\lambda, h}(x)\|_2} \right) - y \right\|_2 \\ &\leq \left\| \gamma_{\lambda, h}(x) + \left( 3h - d_2(\gamma_{\lambda, h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}) \right)_+ \frac{x - \gamma_{\lambda, h}(x)}{\|x - \gamma_{\lambda, h}(x)\|_2} - x \right\|_2 + \|x - y\|_2 \\ &\leq 2\|\gamma_{\lambda, h}(x) - \gamma_{\lambda, h}(y)\|_2 + 2\|x - y\|_2 \\ &\leq 2\delta + 2\|\gamma_{\lambda, h}(x) - \gamma_{\lambda, h}(y)\|_2 \end{aligned}$$

and we conclude, in this case, by continuity of  $\gamma_{\lambda, h}$ .

- Finally, if  $x$  verifies (6) and (7) and  $y$  verifies (6) and (7), then,

$$\begin{aligned} & \|H_{\lambda, h}(x, t) - H_{\lambda, h}(y, s)\|_2 \\ &= \left\| (1-t)x + t \left( \gamma_{\lambda, h}(x) + \left( 3h - d_2(\gamma_{\lambda, h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}) \right)_+ \frac{x - \gamma_{\lambda, h}(x)}{\|x - \gamma_{\lambda, h}(x)\|_2} \right) \right. \\ &\quad \left. - (1-t)y - t \left( \gamma_{\lambda, h}(y) + \left( 3h - d_2(\gamma_{\lambda, h}(y), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+Lh^\alpha}^{-h}) \right)_+ \frac{y - \gamma_{\lambda, h}(y)}{\|y - \gamma_{\lambda, h}(y)\|_2} \right) \right\|_2 \end{aligned}$$

and again the conclusion, follows in this case, by continuity of  $\gamma_{\lambda, h}$ .

All possible cases have been checked, the proof is complete.  $\square$

## B.6 Proof of Proposition 4

This section is dedicated to the proof of Proposition 4 from Section 2, which relies on the following lemma.

**Lemma 5.** *Let  $0 < K$  and  $0 < h < 1$  such that  $Kh < R/2$ . There exists a constant  $C_2$  (depending only on  $K$ ,  $d$  and  $R$ ) such that for all  $i, j \in \{1, \dots, l\}$  and  $x \in B_2((\partial M_i \cap ]0, 1[^d), Kh) \cap \overline{M}_i$  such that  $\xi(x) \in \partial M_j$ .*

$$d_H(B_2(\xi(x), Kh) \cap M_j, B_2(\xi(x), Kh) \cap \underline{P}) \leq C_2 h^2 \quad (18)$$

with

$$\underline{P} = \left\{ z \in [0, 1]^d \text{ s.t. } \left\langle z, \frac{x - \xi(x)}{\|x - \xi(x)\|_2} \right\rangle \leq \left\langle \xi(x), \frac{x - \xi(x)}{\|x - \xi(x)\|_2} \right\rangle \right\}$$

and

$$d_H(B_2(\xi(x), Kh) \cap M_i, B_2(\xi(x), Kh) \cap \overline{P}) \leq C_2 h^2 \quad (19)$$

with

$$\overline{P} = \left\{ z \in [0, 1]^d \text{ s.t. } \left\langle z, \frac{x - \xi(x)}{\|x - \xi(x)\|_2} \right\rangle \geq \left\langle \xi(x), \frac{x - \xi(x)}{\|x - \xi(x)\|_2} \right\rangle \right\}.$$

Furthermore if  $z \in \underline{P} \cap B_2(\xi(x), Kh)$  then

$$z - C_2 h^2 \frac{\xi(x) - x}{\|\xi(x) - x\|_2} \in \overline{M}_j \quad (20)$$

and if  $z \in \overline{P} \cap B_2(\xi(x), Kh)$

$$z + C_2 h^2 \frac{\xi(x) - x}{\|\xi(x) - x\|_2} \in \overline{M}_i. \quad (21)$$

*Proof.* Let  $B_1$  the Euclidean closed ball centered in  $\xi(x) - R \frac{x - \xi(x)}{\|x - \xi(x)\|_2}$  of radius  $R$  and  $B_2$  the Euclidean closed ball centered in  $\xi(x) + R \frac{x - \xi(x)}{\|x - \xi(x)\|_2}$  of radius  $R$ . By Assumption **A3**,  $B_1 \subset \overline{M}_j$  and  $B_2 \subset \overline{M}_i$ . Then, the Hausdorff distance between  $B_2(\xi(x), Kh) \cap M_j$  and  $B_2(\xi(x), Kh) \cap \underline{P}$ , and the Hausdorff distance between  $B_2(\xi(x), Kh) \cap M_i$  and  $B_2(\xi(x), Kh) \cap \overline{P}$ , are both upper bounded by the Hausdorff distance between the union of spheres  $\partial B_1 \cup \partial B_2$  intersected with  $B_2(\xi(x), Kh)$  and the intersection with  $B_2(\xi(x), Kh)$  of the hyperplane,

$$P = \left\{ z \in [0, 1]^d \text{ s.t. } \left\langle z, \frac{x - \xi(x)}{\|x - \xi(x)\|_2} \right\rangle = \left\langle \xi(x), \frac{x - \xi(x)}{\|x - \xi(x)\|_2} \right\rangle \right\}.$$

By symmetry, this distance is equal to the Hausdorff distance between  $\partial B_1 \cap B_2(\xi(x), Kh)$  and  $P \cap B_2(\xi(x), Kh)$ .

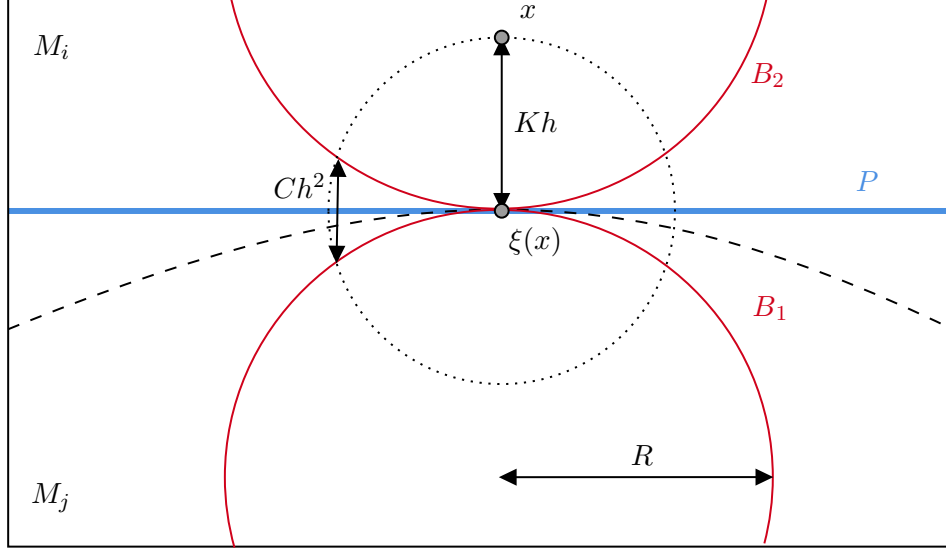


Figure 6: The tangent balls  $B_1$  and  $B_2$  of radius  $R$  bounds the region where  $\partial M_i \cap \partial M_j$  lies in. 2D illustration.

Now, let  $x \in \partial B_1 \setminus \{\xi(x)\}$ , and  $p(x)$  its projection on  $P$ . Let  $Q$  the plane containing  $x$ ,  $p(x)$  and  $\xi(x)$ ,  $Q$  intersects  $\partial B_1$  into a circle  $C$  of radius  $R$  and intersects  $P$  into a line  $D$  tangent to  $C$ . The problem then simplify to upper bounding the distance between a circle and a tangent line around the intersection point. Without loss of generality, we can suppose that we are in  $\mathbb{R}^2$ ,  $C$  being the circle of radius  $R$  centered at  $(0, R)$  and  $D$  the line  $y = 0$  (tangent to  $C$  at  $(0, 0)$ ). In  $B((0, 0), Ch)$ , as  $Kh < R/2$ ,  $C$  can be described as,

$$C = \left\{ (x, y) \in B((0, 0), Kh) \text{ s.t. } y = R - \sqrt{R^2 - x^2} \right\}.$$

Hence the distance between  $C$  and  $D$  in  $B((0, 0), Kh)$  is upper bounded by,

$$R - \sqrt{R^2 - (Kh)^2} = \frac{K^2}{2R}h^2 + O(h^3).$$

Assertions (18) and (19) then follows. Now, simply remark that, as  $Kh < R/2$ , for all  $z \in \underline{P}$ ,  $z - (R - \sqrt{R^2 - (Ch)^2})(x - \xi(x))/\|x - \xi(x)\|_2 \in B_1$  and (20) follows. And symmetrically, for all  $z \in \overline{P}$ ,  $z - (R - \sqrt{R^2 + (Ch)^2})(x - \xi(x))/\|x - \xi(x)\|_2 \in B_2$  and (21) follows.  $\square$

*Proof of Proposition 4.* We start by proving assertion (8). Let  $x \in M_i$ . By Assumption **A3**, there exists a unique  $j \neq i$  such that  $x \in (M_j \cap \mathcal{F}_{\lambda + \|W\|_{G(h)}h^\alpha})^{\sqrt{d}h}$  and thus  $\xi(x) \in \partial M_j$ . Furthermore, supposing  $\theta$  sufficiently small such that  $2\sqrt{d}h < R$ , we have,

$$([x, \xi(x)])^{\sqrt{d}h} \subset \overline{M_i} \cup \overline{M_j}$$

Let  $H_1 \in \mathcal{C}_{h,\lambda}$  the hypercube containing  $x$  and denote  $x_1$  its center. Suppose there exist  $y \in [x, \xi(x)]$  such that  $y \notin H_1^{Ch}$ . Let denote  $H_2$  the hypercube of  $C_h$  containing  $y$  and  $x_2$  its center. Suppose furthermore that  $H_2$  is adjacent to  $H_1$  (i.e.  $H_1 \cap H_2 \neq \emptyset$ ). If,

$$H_2 \cap \overline{M_j}^c \cap \mathcal{F}_{\lambda + (\|W\|_{G(h)} + 3\sqrt{d})h^\alpha} \neq \emptyset$$

then  $H_2 \subset \mathcal{F}_{\lambda + (\|W\|_{G(h)} + 5\sqrt{d})h^\alpha}$  by **A1**, **A2**. And thus, by Lemma 2,  $H_2 \subset \widehat{\mathcal{F}}_{\lambda + (2\|W\|_{G(h)} + 5\sqrt{d})h^\alpha}$ . From now, we suppose,

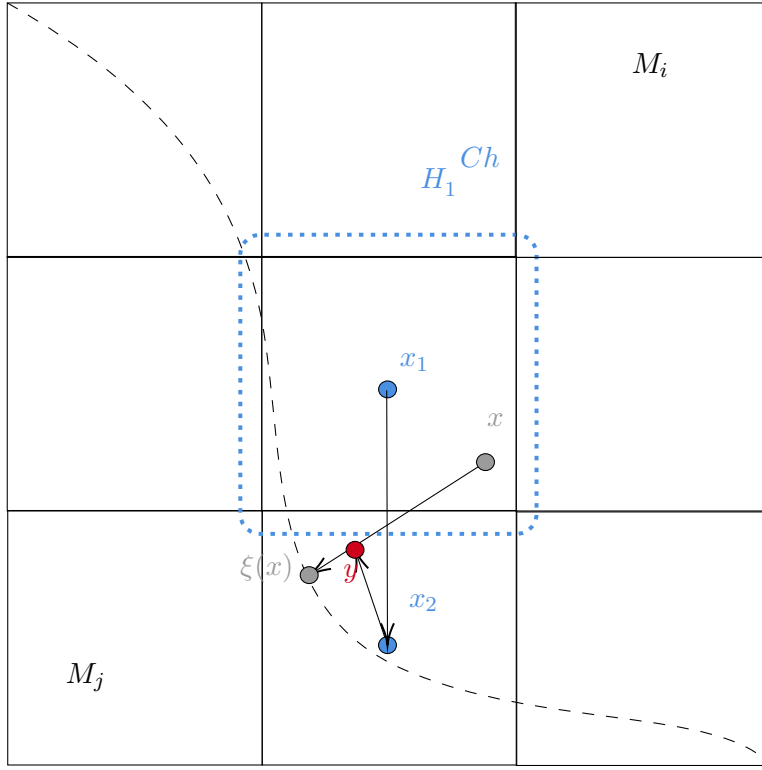


Figure 7: A 2D scenario where  $\xi(x)$  is not in  $H_1$ .

$$H_2 \cap \overline{M}_j^c \cap \mathcal{F}_{\lambda + (\|W\|_{G(h)} + 3\sqrt{d})h^\alpha} = \emptyset. \quad (22)$$

As  $x \in H_1$ , then,

$$\langle x - x_1, x_2 - x_1 \rangle \leq \frac{\|x_2 - x_1\|_2^2}{2}$$

and, as  $y \in H_2 \setminus H_1^{Ch}$ , we have,

$$\langle y - x_2, x_1 - x_2 \rangle \leq \frac{\|x_2 - x_1\|_2^2}{2} - \frac{Ch^2}{2\sqrt{d}}$$

and thus, as  $\|x - y\|_2 \leq \|x - \xi(x)\|_2 \leq \sqrt{d}h$ ,

$$\left\langle \frac{x - \xi(x)}{\|x - \xi(x)\|_2}, x_2 - x_1 \right\rangle = \left\langle \frac{x - y}{\|x - y\|_2}, x_2 - x_1 \right\rangle \leq -\frac{Ch^2}{2\sqrt{d}\|x - y\|_2} \leq -\frac{Ch}{2d}.$$

This implies that, for all  $z \in H_1 \cap \underline{P}$ ,

$$B_2 \left( z + (x_2 - x_1), \frac{Ch}{2d} \right) \subset \underline{P}. \quad (23)$$

Let  $z \in H_1^{-C_2h^2} \cap M_j$ , by assertion (18) of Lemma 5, there exists  $z'$  in  $H_1 \cap \underline{P}$  such that  $\|z - z'\|_2 \leq C_2h^2$ . And by (23),

$$B_2 \left( z' + (x_2 - x_1), \frac{Ch}{2d} \right) \subset \underline{P}.$$

Then, by assertion (20) of Lemma 5, supposing  $\theta$  sufficiently small such that  $\frac{Ch}{2d} > 3C_2h^2$ ,

$$B_2(z + (x_2 - x_1), C_2h^2) \subset B_2 \left( z' + (x_2 - x_1), \frac{Ch}{2d} - C_2h^2 \right) \subset \overline{M}_j.$$



Consequently, for  $\theta$  sufficiently small, for all  $z \in H_1 \cap \overline{M}_j$ ,  $z + (x_2 - x_1) \in H_2 \cap \overline{M}_j$ . As  $H_2 \subset H_1^{2\sqrt{d}h}$ , assumptions **A1** and **A2** implies that, for all  $z \in H_2 \cap M_j$ ,  $z \in \mathcal{F}_{\lambda + (\|W\|_{G(h)} + 3\sqrt{d})h^\alpha}$ . Hence, by (22),

$$\sup_{z \in H_2 \cap M_j} f(z) \leq \inf_{z \in H_1 \cap M_i} f(z).$$

For a set  $A$  and a vector  $u$ , we denote  $A + u = \{z + u, z \in A\}$ . From the foregoing, using again assumptions **A1** and **A2**, it follows that,

$$\begin{aligned} \int_{H_2} dX &= \int_{H_2 \cap M_j} f + \int_{H_2 \cap M_i} f + \theta \int_{H_2} dW \\ &= \int_{H_1 \cap M_j + (x_2 - x_1)} f + \int_{H_2 \cap M_j \setminus (H_1 \cap M_j + (x_2 - x_1))} f + \int_{(H_1 \cap M_i) + (x_2 - x_1) \setminus (H_2 \cap M_j)} f + \theta \int_{H_2} dW \\ &\leq \int_{H_1 \cap M_j + (x_2 - x_1)} f + \int_{H_1 \cap M_i + (x_2 - x_1)} f + \theta \int_{H_2} dW \\ &\leq \int_{H_1 \cap M_j} (f + L(3\sqrt{d}h)^\alpha) + \int_{H_1 \cap M_i} (f + L(3\sqrt{d}h)^\alpha) \\ &\quad + \theta \int_{H_1} dW + \theta \int_{H_2} dW - \theta \int_{H_1} dW \\ &\leq (\lambda + L(3\sqrt{d})^\alpha + 2\|W\|_{G(h)}h^\alpha)h^d \end{aligned}$$

by the choice made for  $h$ . Thus,  $H_2 \subset \widehat{\mathcal{F}}_{\lambda + (2\|W\|_{G(h)} + (5\sqrt{d})^\alpha L)h^\alpha}$ .

Now, if  $H_2$  is not adjacent to  $H_1$ , there exists a finite sequence  $H_3, H_4, \dots, H_n$  of cube of  $C_h$  such that for all  $k \in \{2, \dots, n\}$ ,  $[x, \xi(x)] \cap H_k \neq \emptyset$  and  $H_3$  is adjacent to  $H_2$ ,  $H_4$  adjacent to  $H_3$ , ..., and  $H_n$  adjacent to  $H_1$ . Applying the previous reasoning iteratively then gives that, for all  $k \in \{2, \dots, n\}$ ,

$$H_k \in \widehat{\mathcal{F}}_{\lambda + ((2+n-k)\|W\|_{G(h)} + (1+n-k)(5\sqrt{d})^\alpha L)h^\alpha}.$$

Remark that, as  $\|x - \xi(x)\| \leq \sqrt{d}h$ ,  $n \leq \lceil 2\sqrt{d} \rceil^d$ . Thus, for all  $k \in \{2, \dots, n\}$ ,

$$H_k \in \widehat{\mathcal{F}}_{\lambda + ((2 + \lceil 2\sqrt{d} \rceil^d)\|W\|_{G(h)} + (1 + \lceil 2\sqrt{d} \rceil^d)(5\sqrt{d})^\alpha L)h^\alpha}.$$

and assertion (8) follows.

Now we prove assertion (9). Suppose that  $x \in P_{\lambda + \|W\|_{G(h)}h, \sqrt{d}h}$ . In particular, it implies  $x \in \mathcal{F}_{\lambda + \|W\|_{G(h)}h} \cap M_i$  and  $\|x - \gamma_{\lambda + \|W\|_{G(h)}h^\alpha, \sqrt{d}h}(x)\|_2 \leq \sqrt{d}h$ . Suppose  $\theta$  sufficiently small such that  $2\sqrt{d}h < R$ . Assumption **A3**, ensures that there exist  $j \in \{1, \dots, l\}$  such that,

$$([x, \gamma_{\lambda + \|W\|_{G(h)}h^\alpha, \sqrt{d}h}(x)])^{\sqrt{d}h} \subset \overline{M}_i \cup \overline{M}_j$$

Let  $H_1 \in C_{h,\lambda}$  the hypercube containing  $x$  and denote  $x_1$  its center. Let  $y \in [x, \gamma_{\lambda + \|W\|_{G(h)}h^\alpha, \sqrt{d}h}(x)]$  such that  $y \notin H_1^{C_h}$ . Let denote  $H_2$  the hypercube of  $C_h$  containing  $y$  and  $x_2$  its center. Suppose furthermore that  $H_2$  is adjacent to  $H_1$ . If,

$$H_2 \cap \overline{M}_j \cap \mathcal{F}_{\lambda + (\|W\|_{G(h)} + 3\sqrt{d})h^\alpha} \neq \emptyset$$

then  $H_2 \in \mathcal{F}_{\lambda + (\|W\|_{G(h)} + 5\sqrt{d})h^\alpha}$  by **A1**, **A2**. And thus, by Lemma 2,  $H_2 \in \widehat{\mathcal{F}}_{\lambda + (2\|W\|_{G(h)} + 5\sqrt{d})h^\alpha}$ . From now, we suppose,

$$H_2 \cap \overline{M}_j \cap \mathcal{F}_{\lambda + (\|W\|_{G(h)} + 3\sqrt{d})h^\alpha} = \emptyset. \quad (24)$$

As  $x \in H_1$ , then,

$$\langle x - x_1, x_1 - x_2 \rangle \geq -\frac{\|x_2 - x_1\|_2^2}{2}$$

As,  $y \in H_2 \setminus H_1^{Ch}$ , for sufficiently small  $\theta$ ,

$$\langle y - x_2, x_2 - x_1 \rangle \geq -\frac{\|x_2 - x_1\|_2^2}{2} + \frac{Ch^2}{2\sqrt{d}}$$

and thus, as  $\|x - y\|_2 \leq \|x - \gamma_{\lambda+\|W\|_{G(h)}h^\alpha, \sqrt{d}h}(x)\|_2 \leq \sqrt{d}h$ ,

$$\left\langle \frac{x - \xi(x)}{\|x - \xi(x)\|_2}, x_2 - x_1 \right\rangle = \left\langle \frac{y - x}{\|x - y\|_2}, x_2 - x_1 \right\rangle \geq \frac{Ch^2}{2\sqrt{d}\|x - y\|_2} \geq \frac{Ch}{2d}.$$

Thus, this implies that,  $z \in H_1 \cap \bar{P}$

$$B_2 \left( z + (x_2 - x_1), \frac{Ch}{2d} \right) \subset \bar{P}. \quad (25)$$

Let  $z \in H_1^{-C_2h^2} \cap M_i$ , by assertion (19) of Lemma 5, there exists  $z'$  in  $H_1 \cap \bar{P}$  such that  $\|z - z'\|_2 \leq C_2h^2$ . And by (25),

$$B_2 \left( z' + (x_2 - x_1), \frac{Ch}{2d} \right) \subset \bar{P}.$$

Then, by assertion (21) of Lemma 5, supposing  $\theta$  sufficiently small such that  $\frac{Ch}{2d} > 3C_2h^2$ ,

$$B_2(z + (x_2 - x_1), C_2h^2) \subset B_2 \left( z' + (x_2 - x_1), \frac{Ch}{2d} - C_2h^2 \right) \subset \bar{M}_i.$$

Consequently, for  $\theta$  sufficiently small, for all  $z \in H_1 \cap \bar{M}_i$ ,  $z + (x_2 - x_1) \in H_2 \cap \bar{M}_i$ . As  $H_2 \subset H_1^{2\sqrt{d}h}$ , assumptions **A1** and **A2** implies that, for all  $z \in H_2 \cap M_i$ ,  $z \in \mathcal{F}_{\lambda+(\|W\|_{G(h)}+3\sqrt{d})h^\alpha}$ . Hence, by (24),

$$\sup_{z \in H_2 \cap M_i} f(z) \leq \inf_{z \in H_1 \cap M_j} f(z).$$

From the foregoing, using again assumptions **A1** and **A2**, it follows that,

$$\begin{aligned} \int_{H_2} dX &= \int_{H_2 \cap M_i} f + \int_{H_2 \cap M_j} f + \theta \int_{H_2} dW \\ &= \int_{H_1 \cap M_i + (x_2 - x_1)} f + \int_{H_2 \cap M_i \setminus (H_1 \cap M_i + (x_2 - x_1))} f + \int_{(H_1 \cap M_j) + (x_2 - x_1) \setminus (H_2 \cap M_i)} f + \theta \int_{H_2} dW \\ &\leq \int_{H_1 \cap M_i + (x_2 - x_1)} f + \int_{H_1 \cap M_i + (x_2 - x_1)} f + \theta \int_{H_2} dW \\ &\leq \int_{H_1 \cap M_j} (f + L(3\sqrt{d}h)^\alpha) + \int_{H_1 \cap M_j} (f + L(3\sqrt{d}h)^\alpha) \\ &\quad + \theta \int_{H_1} dW + \theta \int_{H_2} dW - \theta \int_{H_1} dW \\ &\leq (\lambda + L(3\sqrt{d})^\alpha + 2\|W\|_{G(h)}h^\alpha)h^d \end{aligned}$$

by the choice made for  $h$ . Thus,  $H_2 \subset \widehat{\mathcal{F}}_{\lambda+(2\|W\|_{G(h)}+(3\sqrt{d})^\alpha L)h^\alpha}$ . Applying again the iterative reasoning used for assertion (8) we obtain assertion (9).  $\square$

## B.7 Proof of Proposition 5

This section is dedicated to the proof of Proposition 5 from Section 2.

*Proof.* The proof essentially follows from the fact that for all  $h > 0$  and  $H$  hypercube of side  $h$ ,  $\frac{W(H)}{h^{d/2}}$  is a standard Gaussian.

$$\begin{aligned} \mathbb{P}\left(\sup_{H \in C_h} \frac{|W(H)|}{\omega(h^d)} > t\right) &\leq \left(\frac{1}{h}\right)^d \mathbb{P}\left(\frac{|W(H)|}{\omega(h^d)} > t\right) \\ &= \left(\frac{1}{h}\right)^d \mathbb{P}\left(\frac{|W(H)|}{h^{d/2}} > t\sqrt{\log\left(1 + \frac{1}{h^d}\right)}\right) \\ &\leq 2\left(\frac{1}{h}\right)^d \exp\left(-\frac{1}{2}t^2 \log\left(1 + \frac{1}{h^d}\right)\right) \end{aligned}$$

Now, take  $t \geq \sqrt{8}$ , then  $t^2/4 + 2 \leq t^2$ . Thus,

$$\begin{aligned} \mathbb{P}(\|W\|_{cube,h} \geq t) &\leq 2\left(\frac{1}{h}\right)^d \exp\left(-\frac{1}{2}t^2 \log\left(1 + \frac{1}{h^d}\right)\right) \\ &\leq 2\left(\frac{1}{h}\right)^d \exp\left(-(t^2/8 + 1) \log\left(1 + \frac{1}{h^d}\right)\right) \\ &\leq 2 \exp(-t^2/8). \end{aligned}$$

Hence, for all  $t > 0$ ,

$$\mathbb{P}(\|W\|_{cube,h} \geq t) \leq 2e \times \exp(-t^2/8).$$

□

## C Extension to non-parametric regression

The model 1 proves to be valuable for establishing theoretical results. However, it has a limitation as it assumes the observation of a complete trajectory, making it less popular for practical applications. In this section, we focus on proposing extensions to another essential non-parametric model with greater practical interest: non-parametric regression. The proofs of the main results are essentially the same, we detail only the few differences.

We consider the classical non-parametric regression setting (with fixed regular design), observing  $n = N^d$  points,

$$X_i = f(x_i) + \sigma \varepsilon_i$$

with  $x_i$  a point on the regular  $N^d$  grid  $G_n$  over  $[0, 1]^d$ ,  $\sigma$  the level of noise and  $\varepsilon_i$  a standard Gaussian variable. In this context, we define,

$$\widehat{\mathcal{F}}_\lambda = \bigcup_{H \in C_{h,\lambda}} H, \text{ with } C_{h,\lambda} = \left\{ H \in C_h \text{ such that } \frac{1}{|\{x_i \in H\}|} \sum_{x_i \in H} X_i \leq \lambda \right\}.$$

The key here to lift the convergence results established in Section 2 in this context is to show an analogous inclusion from the one obtained in Proposition 1, then the exact same reasoning applies. All we have to provide is similar noise control. For  $h > 0$ , let denote the variable,

$$N_h = \frac{\max_{H \in C_h} \left| \frac{1}{|\{x_i \in H\}|} \sum_{x_i \in H} \sigma \varepsilon_i \right|}{\sqrt{2\sigma^2 \frac{\log(1/h^d)}{[Nh]^d}}}$$

**Lemma 4 bis.** Let  $h > 1/N$ ,

$$\mathbb{P}(N_h \geq t) \leq 2 \left(\frac{1}{h}\right)^d \exp\left(-t^2 \log\left(1/h^d\right)\right).$$

*Proof.* Let  $h > 1/N$  and  $H \subset [0, 1]^d$  be a closed hypercube of side  $h$ . As the  $(\varepsilon_i)_{i=1, \dots, n}$  are i.i.d and standard Gaussian variables, we have, for all  $H \in C_h$ ,

$$\mathbb{P}\left(\left|\frac{1}{|\{x_i \in H\}|} \sum_{x_i \in H} \sigma \varepsilon_i\right| \geq t\right) \leq 2 \exp\left(-\frac{|\{x_i \in H\}|t^2}{2\sigma^2}\right).$$

And thus, as the number of point in any  $H \in C_h$  is at least to  $\lfloor hN \rfloor^d$ ,

$$\mathbb{P}\left(\left|\frac{1}{|\{x_i \in H\}|} \sum_{x_i \in H} \sigma \varepsilon_i\right| \geq t\right) \leq 2 \exp\left(-\frac{\lfloor hN \rfloor^d t^2}{2\sigma^2}\right).$$

Now, by union bound, using  $|C_h| = 1/h$ ,

$$\mathbb{P}\left(\max_{H \in C_h} \left|\frac{1}{|\{x_i \in H\}|} \sum_{x_i \in H} \sigma \varepsilon_i\right| \geq t\right) \leq 2 \left(\frac{1}{h}\right)^d \exp\left(-\frac{\lfloor hN \rfloor^d t^2}{2\sigma^2}\right).$$

and the result follows.  $\square$

In particular, as in Proposition 5, it follows that  $N_h$  is sub-Gaussian, more precisely there exists  $C_0$  and  $C_1$  depending only on  $d$  such that, for all  $h$ ,

$$\mathbb{P}(N_h \geq t) \leq C_0 \exp(-C_1 t^2).$$

Let now choose,  $h$  such that,

$$h^\alpha > \sqrt{\frac{\log(1/h^d)}{\lfloor Nh \rfloor^d}}$$

thus, we can choose,

$$h \simeq \left(\frac{\log(n)}{n}\right)^{\frac{1}{d+2\alpha}}.$$

With this choice we obtain the following key lemma.

**Lemma 2 bis.** Let  $f : [0, 1]^d \mapsto \mathbb{R}$ . Let  $H \subset \mathcal{F}_{\lambda + \sqrt{2\sigma^2} N_h h^\alpha}^c \cap C_h$  and  $H' \subset \mathcal{F}_{\lambda - \sqrt{2\sigma^2} N_h h^\alpha} \cap C_h$ . We then have that,

$$\frac{1}{|\{x_i \in H\}|} \sum_{x_i \in H} X_i > \lambda \text{ and } \frac{1}{|\{x_i \in H'\}|} \sum_{x_i \in H'} X_i < \lambda.$$

*Proof.* Let consider here the case where in  $H' \subset \mathcal{F}_{\lambda - \sqrt{2\sigma^2} N_h h^\alpha}$  (The proof being the same in both cases). We have,

$$\begin{aligned} & \frac{1}{|\{x_i \in H'\}|} \sum_{x_i \in H'} X_i \\ &= \frac{1}{|\{x_i \in H'\}|} \sum_{x_i \in H'} f(x_i) + \sigma \varepsilon_i \end{aligned}$$

$$\begin{aligned} &\leq \lambda - \sqrt{2\sigma^2}N_h h^\alpha + N_h \sqrt{2\sigma^2} \sqrt{\frac{\log(1/h^d)}{[Nh]^d}} \\ &< \lambda \end{aligned}$$

By the choice made for  $h$ . □

Using Lemma the Lemma 2 bis instead of Lemma 2 in the proof of Proposition 1, we obtain the following analogous proposition.

**Proposition 1 bis.** *Let  $f : [0, 1]^d \rightarrow \mathbb{R}$ . For all  $\lambda \in \mathbb{R}$ ,*

$$\mathcal{F}_{\lambda - \sqrt{2\sigma^2}N_h h^\alpha}^{-\sqrt{dh}} \subset \widehat{\mathcal{F}}_\lambda \subset \mathcal{F}_{\lambda + \sqrt{2\sigma^2}N_h h^\alpha}^{\sqrt{dh}}$$

We define  $\widehat{\mathbb{V}}_{f,s}$  and  $\widehat{\text{dgm}}(f)$  in the exact same way we did for the Gaussian White Noise model. Again, we can show that this module is  $q$ -tame applying the same ideas used in the proofs of Proposition 7.

Having the inclusion given by Proposition 1 bis, the reasoning from the proof of Theorem 1 gives,

**Proposition 1 bis.** *There exists  $\tilde{C}_0$  and  $\tilde{C}_1$  such that, for all  $t > 0$ ,*

$$\mathbb{P} \left( \sup_{f \in S_d(L, \alpha, R)} d_b \left( \widehat{\text{dgm}}(f), \text{dgm}(f) \right) \geq t \left( \frac{\log(n)}{n} \right)^{\frac{\alpha}{d+2\alpha}} \right) \leq \tilde{C}_0 \exp \left( -\tilde{C}_1 t^2 \right).$$

From this, we obtain, as in Section 2, upper bounds for estimation. This bound can be shown to be minimax also in this setting (adapting the proof of Theorem 4). Adaptivity also follows as in Section 3.