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SELF-DUAL MAPS I : ANTIPODALITY

LUIS MONTEJANO¹, JORGE L. RAMÍREZ ALFONSÍN², AND IVAN RASSKIN

ABSTRACT. A self-dual map G is said to be *antipodally self-dual* if the dual map G^* is antipodal embedded in \mathbb{S}^2 with respect to G . In this paper, we investigate necessary and/or sufficient conditions for a map to be antipodally self-dual. In particular, we present a combinatorial characterization for map G to be antipodally self-dual in terms of certain *involution labelings*. The latter lead us to obtain necessary conditions for a map to be *strongly involutive* (a notion relevant for its connection with convex geometric problems). We also investigate the relation of antipodally self-dual maps and the notion of *antipodally symmetric* maps. It turns out that the latter is a very helpful tool to study questions concerning the *symmetry* as well as the *amphicheirality* of *links*.

1. INTRODUCTION

Let G be a *map*, that is, a graph cellularly embedded in the sphere. Then $G = (V, E, F)$ has a natural *geometric dual* $G^* = (V^*, E^*, F^*)$ where each face in F correspond to a vertex in V^* and two vertices in V^* are adjacent if the corresponding faces in G share an edge. A map G is called *self-dual* if there is a bijection from V and F to V^* and F^* which reverses inclusion.

Self-dual maps have been the subject of numerous investigations in different fronts : self-dual polyhedra and ranks [3], isometries in \mathbb{S}^2 [8], eigenvalues of h -graphs [11], rigidity [7], tilings [9], etc.

A self-dual map G is said to be *antipodally self-dual* if the dual map G^* is antipodally embedded with respect to G . In other words, the map G is antipodally self-dual if the following holds for any $x \in \mathbb{S}^2$

- 1) if $x \in V(G)$ then $-x \in V(G^*)$ and
- 2) if $x \in e \in E(G)$ then $-x \in e^* \in E(G^*)$, that is, e^* is antipodally embedded in \mathbb{S}^2 with respect to the embedding of e .

Antipodally self-dual maps are closely related with the notion of *strongly involutive* maps (see beginning of Section 3.2) and thus relevant for their connection with convex geometric problems as the well-known Vázsonyi's problem on *ball polyhedra* (as reported in [2], see also [10]), the chromatic number of *distance graphs* on the sphere [4] and *Reuleaux polyhedra* [5].

The main goal of this paper is to investigate necessary and/or sufficient conditions for a map to be antipodally self-dual.

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The paper is organized as follows. In the next section, we give a brief overview of some notions on self-dual maps needed for the rest of the paper. Given a map G , we recall three special close related maps (*medial graph* $med(G)$, *square graph* G^\square and *vertex-face incidence graph* $I(G)$) that turn out to be very useful for our propose.

In Section 3, we first recall some classical results between isometries in \mathbb{S}^2 and maps. We then present a result giving necessary conditions of an antipodally self-dual map G in terms of *symmetric cycles* in $I(G)$ (Theorem 1). In order to prove this, we introduce and study the notion of *antipodally symmetric* maps. Besides many properties, we show that if G is an antipodally self-dual map then both $med(G)$ and $I(G)$ are antipodally symmetric maps (Lemma 1). Afterwards, we discuss the connection between antipodally self-dual maps and *strongly involutive* maps and give a combinatorial characterization for a map G to be antipodally self-dual in terms of certain *involutive labelings* of $I(G)^\square$ (Theorem 2). As a consequence, we obtain necessary conditions for a map G to be strongly involutive in terms of $I(G)^\square$ (Corollary 2).

In Section 4, we characterize three different infinite families of antipodally self-dual maps (Propositions 1, 2 and 3). We also present a more general construction (Theorem 3).

Finally, in Section 5, we briefly discuss some consequences on problems concerning knot theory.

2. MAPS PRELIMINARIES

Let G be a planar graph. A *map* of $G = (V, E, F)$ is the image of an embedding of G into \mathbb{S}^2 where the set of vertices are a collection of distinct points in \mathbb{S}^2 and the set of edges are a collection of Jordan curves joining two points in V satisfying that $\alpha \cap \alpha'$ is either empty or a point in the endpoints for any pair of Jordan curves α and α' . Any embedding of the topological realization of G into \mathbb{S}^2 partitions the 2-sphere into simply connected regions of $\mathbb{S}^2 \setminus G$ called the *faces* F of the embedding.

Given a map G , we may construct the *dual map* $G^* = (V^*, E^*, F^*)$ by placing a vertex f^* in the interior of each face f of G , and for each edge e of M draw a *dual edge* e^* connecting the vertices f_1^* and f_2^* (corresponding to the two faces f_1 and f_2 sharing edge e) by crossing e transversely.

Two maps $G_1 = (V_1, E_1, F_1)$ and $G_2 = (V_2, E_2, F_2)$ of the same graph are *isomorphic* if there is an isomorphism $\phi : (V_1, E_1, F_1) \rightarrow (V_2, E_2, F_2)$ preserving incidences, that is, $\{u, v\} \in E_1$ if and only if $\{\phi(u), \phi(v)\} \in E_2$ (and thus ϕ preserve facial walks). We say that a map $G = (V, E, F)$ is a *self-dual map* if the maps $G = (V, E, F)$ and $G^* = (V^*, E^*, F^*)$ are isomorphic, that is, there is an isomorphism $\phi : (V, E, F) \rightarrow (F^*, E^*, V^*)$ preserving incidences.

Given maps $G = (V, E, F)$ and $G^* = (V^*, E^*, F^*)$ we define the following auxiliaries maps.

- The *squares graph* of G is the map G^\square obtained by the simultaneous drawing of $G \cup G^*$ with all the edges split at the intersection points of an edge e with its dual edge e^* . We thus have that every face of G^\square is a square formed by half-edges of G and G^* .

For each square face in G^\square , we define two types of diagonals: the *intersecting diagonal* which is the edge joining the intersections points and the *incidence diagonal* which is the edge joining a vertex in $V(G)$ to a vertex in $V(G^*)$.

- The *vertex-face incidence graph* is the map $I(G)$ having as vertices $V \cup V^*$ and as edges are all the incidence diagonals of G^\square .
- The *medial* of G is the map $med(G)$ having as vertices the set of intersections points of $E \cap E^*$ and as edges the set of all the intersecting diagonals of G^\square .

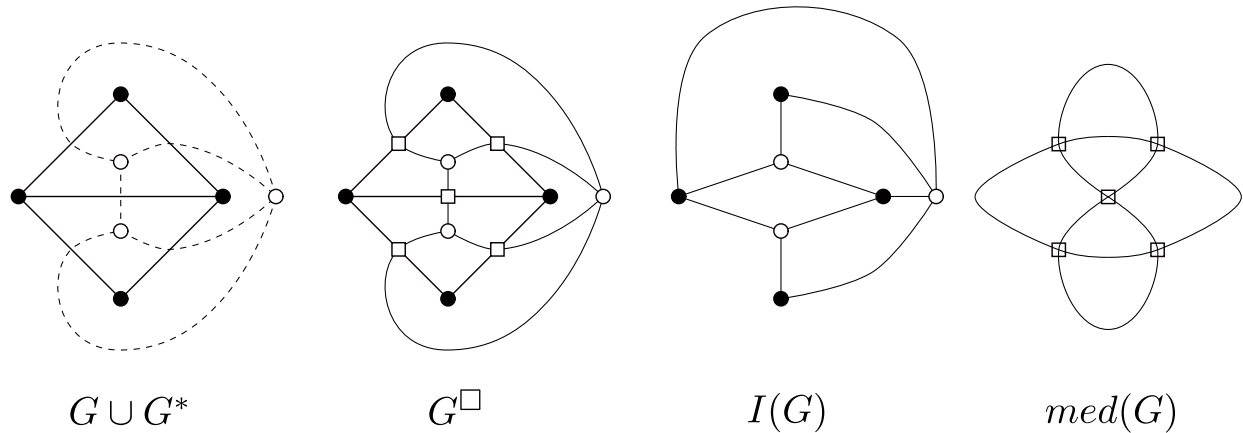


FIGURE 1. A map and its dual, the squares graph, the vertex-face incidence graph and the medial.

Throughout the paper, we will represent the vertices of G with black circles, the vertices of G^* with white circles and the intersection points with white squares and the vertices of the medial with transparent squares.

Notice that $I(G)$ and $med(G)$ are dual from each other for any map G . Hence, we can construct the squares graph of the vertex-face incidence graph which it turns out to be very useful for our propose.

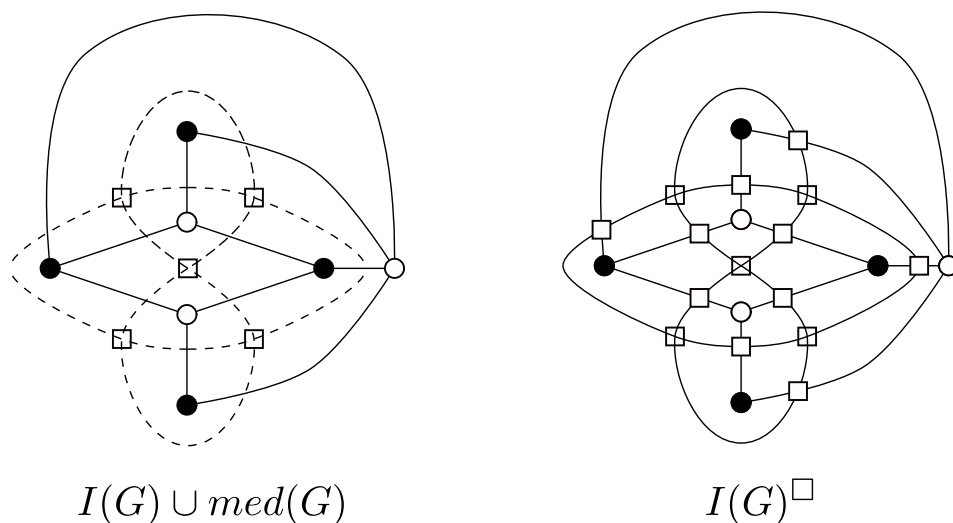


FIGURE 2. (Right) $I(G)$ (straight edges and black and vertices in white circles) and $med(G)$ (dashed edges and vertices in transparent squares) (Left) Graph $I(G)^\square$.

3. ANTIPODALLY SELF-DUAL MAPS

We recall that $Aut(G)$ is the group formed by the set of all *automorphism* of G (i.e., the set of isomorphisms of G into itself). We will denote by $Cor(G)$ the set of all *duality isomorphisms* of G into G^* . We recall that a duality isomorphism is an application ϕ which sends vertices to faces and faces to vertices while preserving incidence, that is, if vertex v is incident with face f then face $\phi(f)$ is incident with vertex $\phi(v)$. Note that ϕ^2 maps vertices to vertices and faces to faces. However, despite the fact that ‘the dual of the dual is the original’, ϕ^2 need not be the identity. We notice that $Cor(G)$ is not a group since the composition of any two of them is an automorphism.

Let us suppose that $G = (V, E, F)$ is a self-dual map so that there is a bijection $\phi : (V, E, F) \rightarrow (F^*, E^*, V^*)$. Following ϕ with the correspondence $*$ gives a permutation on $V \cup E \cup F$ which preserve incidences but reverses dimension of the elements. The collection of all such permutations or *self-dualities* generate a group $Dual(G) = Aut(G) \cup Cor(G)$ in which the automorphisms $Aut(G)$ are contained as a subgroup of index 2.

It is known [8, Lemma 1] that for a given map G there is an homeomorphism ρ of \mathbb{S}^2 to itself such that for every $\sigma \in Aut(G)$ we have that $\rho\sigma \in Iso(\mathbb{S}^2)$ where $Iso(\mathbb{S}^2)$ is the group of isometries of the 2-sphere. In other words, any planar graph G can be drawn on the 2-sphere such that any automorphism of G act as an isometry of the sphere. This was extended in [8] by showing that given any self-dual graph G there are maps G and G^* so that $Dual(G)$ is realized as a group of spherical isometries.

From now on, we will denote by $\widehat{G} = \rho(G)$ and $\widehat{\sigma} = \rho\sigma$ for a certain homeomorphism ρ satisfying the above property.

A self-dual map G is *antipodally self-dual* if $-\widehat{G} = \widehat{G}^*$ where $-G$ is the map consisting of the set of points $\{-x \in \mathbb{S}^2 \mid x \in G\}$, that is, for each $x \in \mathbb{S}^2$

- if $x \in V(\widehat{G})$ then $-x \in V(\widehat{G}^*)$
- if $x \in E(\widehat{G})$ then $-x \in E(\widehat{G}^*)$, i.e., $e^* \in E(\widehat{G}^*)$ is antipodally embedded in \mathbb{S}^2 with respect to $e \in E(\widehat{G})$
- if $x \in F(\widehat{G})$ then $-x \in F(\widehat{G}^*)$, i.e., $f^* \in F(\widehat{G}^*)$ is antipodally embedded in \mathbb{S}^2 with respect to $f \in F(\widehat{G})$

Let us present a result giving necessary combinatorial conditions for a map to be antipodally self-dual. By a *symmetric cycle* C of a planar graph G we mean that there is an automorphism $\sigma(G)$ such that $\sigma(C) = C$ and $\sigma(int(C)) = ext(C)$, that is, the induced graph in the interior of C is isomorphic to the induced graph in the exterior of C .

Theorem 1. *Let G be antipodally self-dual. Then, $I(G)$ always admits at least one symmetric cycle. Moreover, all symmetric cycles in $I(G)$ are of length $2n$ with $n \geq 1$ odd.*

The proof of Theorem 1 is presented in the next subsection where the notion of *antipodally symmetric* map (needed for the proof) is introduced and discussed.

3.1. Antipodally symmetric maps. A map $G = (V, E, F)$ is said to be *antipodally symmetric* if $-\widehat{G} = \widehat{G}$ where $-G$ is the map consisting of points $\{-x \in \mathbb{S}^2 \mid x \in G\}$, that is, for each $x \in \mathbb{S}^2$, if $x \in V(\widehat{G})$ (resp. $x \in E(\widehat{G})$ or $x \in F(\widehat{G})$) then $-x \in V(\widehat{G})$ (resp. $-x \in E(\widehat{G})$ or $-x \in F(\widehat{G})$). We thus have that any vertex, edge or face in an antipodally symmetric \widehat{G} is antipodally embedded in \mathbb{S}^2 with respect to another vertex, edge and face of \widehat{G} .

Remark 1. (a) $med(G) = med(G^*)$.

(b) If G is self-dual then $|V(med(G))|$ is even. Indeed, by Euler's formula we have $|V(G)| + |F(G)| = 2 + |E(G)|$ where $F(G)$ denote the set of faces of G . Since G is self-dual then $|V(G)| = |V(G^*)| = |F(G)|$ and thus $2|V(G)| = 2 + |E(G)|$ implying that $|E(G)| = |V(med(G))|$ is even.

Lemma 1. Let G be an antipodally self-dual map. Then, $med(G)$ and $I(G)$ are antipodally symmetric.

Proof. We first show that $med(G)$ is antipodally symmetric. For, let us consider an antipodally self-dual map G , that is, the dual map G^* is antipodally embedded with respect to the map G . The latter induces a map G^\square in which square faces of G^\square are partitioned into pairs that are antipodally embedded in \mathbb{S}^2 . Indeed, let $F = \{e_1, e_2, e_1^*, e_2^*\}$ be a face of G^\square where e_1, e_2 (resp. e_1^*, e_2^*) are the two half-edges induced by $e \in E(G)$ (resp. induced by $e^* \in E(G^*)$). Since G is antipodally self-dual then there is an edge $f^* \in G^*$ (resp. an edge $f \in G$) which is antipodally embedded to $e \in G$ (resp. to $e^* \in G^*$). We thus have that the corresponding half-edges f_1^*, f_2^* (resp. f_1, f_2) are also antipodally embedded with respect to e_1, e_2 (resp. to e_1^*, e_2^*). Obtaining an other face $F^* = \{f_1, f_2, f_1^*, f_2^*\}$ which is antipodally embedded with respect to F .

We thus have that the intersecting diagonals corresponding to faces F and F^* can also be antipodally embedded. The results follows by recalling that $med(G)$ is given by all the intersecting diagonals of G^\square , see Figure 3.

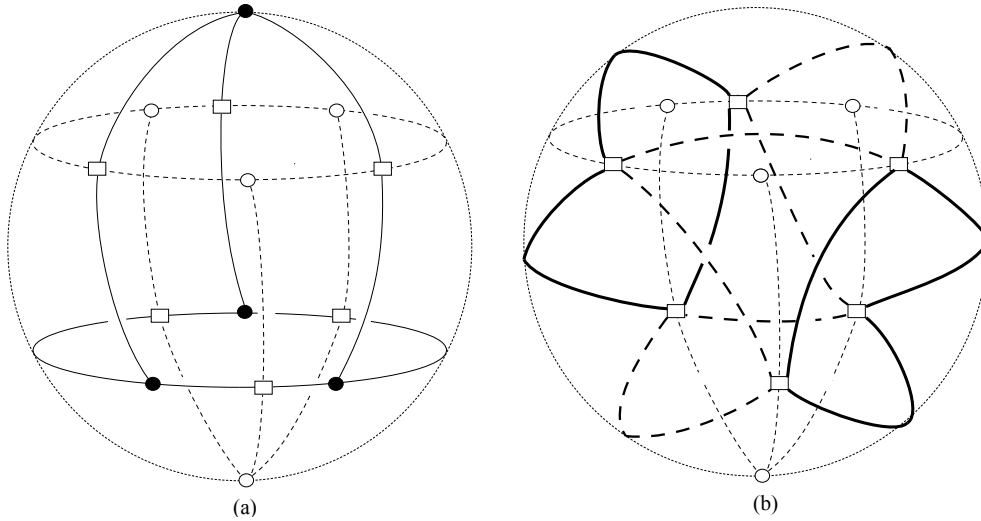


FIGURE 3. (a) Embedding of K_4 (black vertices and straight edges), its dual (white vertices and dashed edges) and the vertices of the medial graph (little squares vertices) (b) K_4 (dashed edges) and its medial graph (in bold dashed edges) with two antipodal faces (in bold).

For $I(G)$, the proof goes in the same way as above but, this time, by considering the incidence diagonals instead of the intersecting diagonals. \square

We may now prove Theorem 1.

Proof of Theorem 1. Let G be an antipodally self-dual map. Let $\widehat{med}(G)$ be the drawing of $med(G)$ where all the automorphisms are isometries and let E be the equator of \mathbb{S}^2 .

Suppose that E does not contain any vertex of $\widehat{med}(G)$; Then, E passes from a face f of $\widehat{med}(G)$ to another face f' that shares an edge with f . Since $med(G)^* = I(G)$ the pair of faces $\{f, f'\}$ corresponds to a pair of adjacent vertices $\{v, v'\}$ in $V(I(G))$. Thus, the sequence of faces $(f_1, \dots, f_n = f_1)$ intersected by E (with the order induced by E) corresponds to a cycle C in $I(G)$. Let $int(C)$ (resp. $ext(C)$) be the subgraph of $I(G)$ corresponding to the faces of $\widehat{med}(G)$ lying on the northern (resp. southern) hemisphere. By Lemma 1, $med(G)$ is antipodally symmetric so the northern faces and the southern faces of $\widehat{med}(G)$ are antipodally drawn. Thus, $int(C)$ is map isomorphic to $ext(C)$ and thus C is a symmetric cycle of $I(G)$.

Now, let us suppose that E passes through a vertex of $med(G)$. Since the set of vertices of $med(G)$ is finite there exists a point $x \in E$ such that x and $-x$ are not vertices of $med(G)$. Let E_α be the rotation of E of angle α on the line passing through x and $-x$. Let

$$\beta = \min_{\alpha > 0} \{E_\alpha \text{ contains a vertex of } \widehat{med}(G)\}.$$

Then, $E_{\beta/2}$ is a great circle of \mathbb{S}^2 which does not contain any vertex of $\widehat{med}(G)$. By taking $E_{\beta/2}$ as equator we can apply the above arguments to show that there exists a symmetric cycle of $I(G)$.

Finally, if C is a symmetric cycle of $I(G)$ then we can draw $\widehat{I(G)}$ with C being the equator of \mathbb{S}^2 . Since G is antipodally self-dual, a black vertex v of $\widehat{I(G)}$ is antipodal to a white vertex $-v$. Thus, the length of C must be $2n$ with $n \geq 1$ odd. \square

Remark 2. *An antipodally self-dual map G induces an involutive self-dual isomorphism $\sigma : V(G) \rightarrow V^*(G)$. The converse is not necessarily true, there are self-dual graphs not admitting an antipodally self-dual map. For instance, the graph G' illustrated in Figure 4 is self-dual but it is not antipodally self-dual. Indeed, it can be easily checked that $I(G')$ admits a symmetric cycle of length 8 (implying that G' is not antipodally self-dual, by Theorem 1), see Figure 4.*

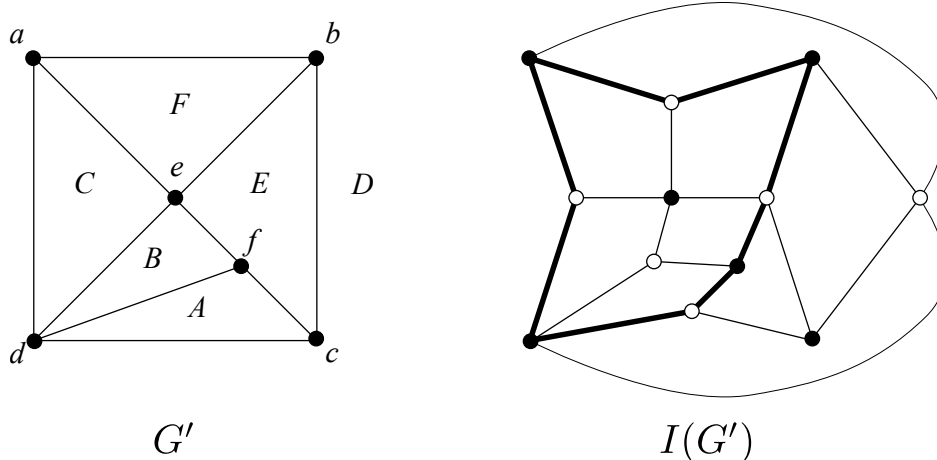


FIGURE 4. (Left) A nonstrongly involutive self-dual map G' (the isomorphism σ is given by $\sigma(a) = A, \sigma(b) = B, \sigma(c) = C$, etc.) (Right) $I(G')$ admitting a symmetric cycle of length 8 (bold edges).

3.2. Strongly involutive maps and involutive labelings. Let G be a self-dual graph with duality isomorphism $\sigma : G \rightarrow G^*$. We say that G is *strongly involutive* if the following conditions are satisfied:

- a) for each pair of vertices $u, v \in V(G)$, $u \in \sigma(v)$ if and only if $v \in \sigma(u)$ and
- b) for every vertex $v \in V(G)$, we have that $v \notin \sigma(v)$.

We notice that a) is equivalent to say that $\sigma^2 = id$.

The above conditions are the combinatorial counterpart (in the 3-dimensional case) of a more general geometric object called *strong self-dual polytopes*, first introduced by Lovász in [4]. Antipodally self-dual maps are closely related with strongly involutive isomorphism. Indeed, in [1, Theorem 9], it was proven that if G is strongly involutive then G is antipodally self-dual. As we will see below, the latter is a straightforward consequence of Theorem 2 (see Corollary 2).

Let $G = (V, E, F)$ be a map and let $X = \{x_1, \dots, x_m\}$ and $\bar{X} = \{\bar{x}_1, \dots, \bar{x}_m\}$ be two sets of labels with $1 \leq m \leq |V|$ and the property $\bar{\bar{x}}_i = x_i$. Let $\mathcal{P}(X \cup \bar{X})$ be the set of subsets of $X \cup \bar{X}$. An *involutive labeling* of G is a function $\Lambda : V \rightarrow \mathcal{P}(X \cup \bar{X})$ satisfying the following properties:

- (i) $|\Lambda(v)| \in \{1, 2\}$ for every $v \in V$.
- (ii) If $|\Lambda(v)| = 2$ then $\Lambda(v) = \{x_i, \bar{x}_i\}$ for some $1 \leq i \leq m$. In this case, we say that v is a *fixed vertex* of Λ and we write $x_i = \bar{x}_i$ (instead of $\{x_i, \bar{x}_i\}$).
- (iii) $\Lambda(u) \cap \Lambda(v) \neq \emptyset$ if and only if $u = v$.
- (iv) $\{\Lambda^{-1}(x_i), \Lambda^{-1}(x_j)\} \in E$ if and only if $\{\Lambda^{-1}(\bar{x}_i), \Lambda^{-1}(\bar{x}_j)\} \in E$ where $\Lambda^{-1}(x_i) := \{v \in V \mid x_i \in \Lambda(v)\}$.

Let $G^\square = (V^\square, E^\square, F^\square)$ be the square graph associated to a map $G = (V, E, F)$. Recall that $V^\square = V_V \cup V_E \cup V_F$ where V_V are the vertices of G , V_E are the vertices on the edges of G and V_F are the vertices of G^* (one for each face of G).

Remark 3. An involutive labeling of $I(G)^\square$ naturally induces an automorphism of $I(G)$

$$\begin{array}{ccc} \sigma_\Lambda : & V \cup V^* & \rightarrow & V \cup V^* \\ & v & \mapsto & u \end{array}$$

where $\Lambda(u) = \overline{\Lambda(v)}$ (the adjacency preserving property of σ_Λ is obtained from (ii)).

a) If vertex v was assigned labels k and \bar{k} (and thus $k = \bar{k}$) then it will be a fixed vertex under σ_Λ .

b) $\sigma_\Lambda^2 = Id$.

c) σ_Λ corresponds to an involutive duality isomorphism $\sigma : G \rightarrow G^*$ if and only if the labels of the black vertices are the opposite to those of the white vertices in $I^\square(G)$.

Remark 4. Let G be a self-dual map. We have that G is strongly involutive if and only if $I(G)$ admits an involutive labeling without edges whose ends are labeled by k and \bar{k} .

3.3. Characterizing antipodally self-dual maps. We are interested in giving necessary and sufficient combinatorial conditions for a map to be antipodally self-dual.

Remark 5. We have that any $\sigma \in Aut(G)$ naturally induces $\sigma^\square \in Aut(G^\square)$ with σ^\square preserving incidences, that is, if $v_V \in V_V$ is adjacent to $v_E \in V_E$ (resp. $v_E \in V_E$ is adjacent to $v_F \in V_F$) then $\sigma^\square(v_V)$ is adjacent to $\sigma^\square(v_E)$ (resp. $\sigma^\square(v_E)$ is adjacent to $\sigma^\square(v_F)$) and where V_V, V_E and V_F are mapped to V_V, V_E and V_F respectively. We finally notice that there might exist $\gamma \in Aut(G^\square)$ not necessarily arising from an automorphism of G .

Lemma 2. Let H be a map and let $\sigma \in Aut(H)$. Then, $\hat{\sigma}$ has a fixed point in \mathbb{S}^2 if and only if σ^\square has a fixed vertex in H^\square .

Proof. Let $x \in \mathbb{S}^2$. A point x corresponds to a vertex on H^\square , say x^\square , which lies properly on either V, E or F . If $\hat{\sigma}(x) = x$ then $\sigma^\square(x^\square) = x^\square$.

Conversely, let $v \in V^\square = \{V_V \cup V_E \cup V_F\}$ such that $\sigma^\square(v) = v$. We have three cases.

Case 1) $v \in V_V$. Then, the point $v \in \mathbb{S}^2$ is such that $\hat{\sigma}(v) = v$.

Case 2) $v \in V_E$. Suppose v lies properly on an edge e . We know that the isometry $\hat{\sigma}$ maps e into itself. Since e is topologically equivalent to \mathbb{B}^1 then $\hat{\sigma}$ is a continuous function sending \mathbb{B}^1 to itself. Therefore, by the Brouwer fixed-point theorem there is $x \in e$ such that $\hat{\sigma}(x) = x$.

Case 3) $v \in V_F$. Suppose v lies properly on a face f . We proceed as in the Case 2. The isometry $\hat{\sigma}$ maps f into itself. Since f is topologically equivalent to \mathbb{B}^2 then $\hat{\sigma}$ is a continuous function sending \mathbb{B}^2 to itself. Therefore, by the Brouwer fixed-point theorem there is $x \in f$ such that $\hat{\sigma}(x) = x$. \square

Theorem 2. Let $G = (V, E, F)$ be a self-dual map. Then, G is antipodally self-dual if and only if $I(G)^\square$ admits an involutive labeling without fixed vertices.

Proof. Suppose that G is antipodally self-dual. Therefore, there is \hat{G} isomorphic to G such that $-\hat{G} = \hat{G}^*$. Let $a : x \mapsto -x$ be the antipodal mapping of \mathbb{S}^2 . We have that a naturally induces the automorphisms $a_I \in Aut(I(\hat{G}))$ and $a^\square \in Aut(I(\hat{G})^\square)$. Furthermore, since a is the antipodal mapping then

- $a_I^2 = Id$ (implying that $I(G)^\square$ admits an involutive labeling on its vertices) and

- a_I has no fixed points of \mathbb{S}^2 . Therefore, by Lemma 2, a^\square has no fixed vertices and thus the above involutive labeling of $I(G)^\square$ has no fixed vertices.

We finally notice that an involutive labeling of $I(\widehat{G})^\square$ is also an involutive labeling of $I(G)^\square$.

Conversely, suppose that $I(G)^\square$ admits an involutive labeling without fixed vertices. By Lemma 2, $\widehat{\sigma}(I(G))$ has not a fixed point in \mathbb{S}^2 . Now, there are three sphere isometries such that $\sigma^2 = Id$: rotation of π degree, reflexion on a hyperplane and the antipodal function. Among them, it is the antipodal function the only without fixed points. Moreover, since $\sigma : G \rightarrow G^*$ then σ sends vertices of G to vertices of G^* . Therefore, G is antipodally self-dual. \square

For the involutive labelings of squares graphs, we shall use integers (and their opposites) for vertices of type V_V , letters (and their opposites) for vertices of type V_F and greek letters (and their opposites) for vertices of type V_E . On one hand Figure 5 illustrates a self-dual map G and $I(G)^\square$ together with an involutive labeling without fixed vertices. Therefore, as a consequence of Theorem 2, G is antipodally self-dual. On the other hand, Figure 6 illustrates an involutive labeling of the 4-wheel W_4 with $I(W_4)^\square$ admitting two fixed vertices. In fact, it can be checked that any involutive labeling of $I(W_4)^\square$ admits at least one fixed vertex since W_4 is not antipodally self-dual (see Proposition 1).

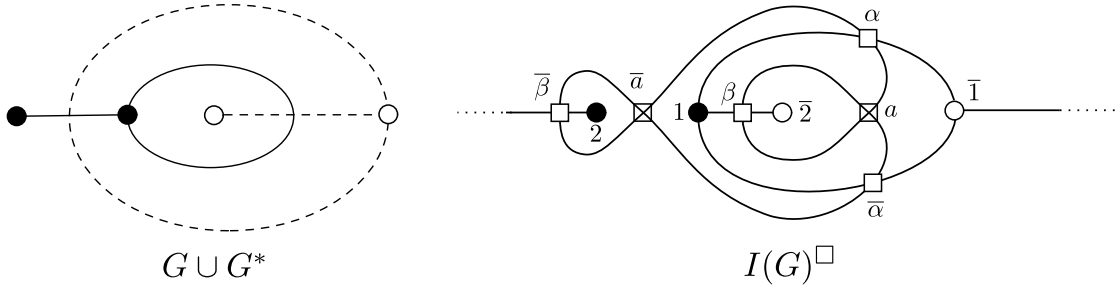


FIGURE 5. (Left) A self-dual map G (straight edges and black vertices) and G^* (dashed edges and white vertices). (Right) An involutive labeling of $I(G)^\square$ without fixed vertices.

It can easily be checked that the graph G (in Figure 5) does not admit a strongly involutive isomorphism.

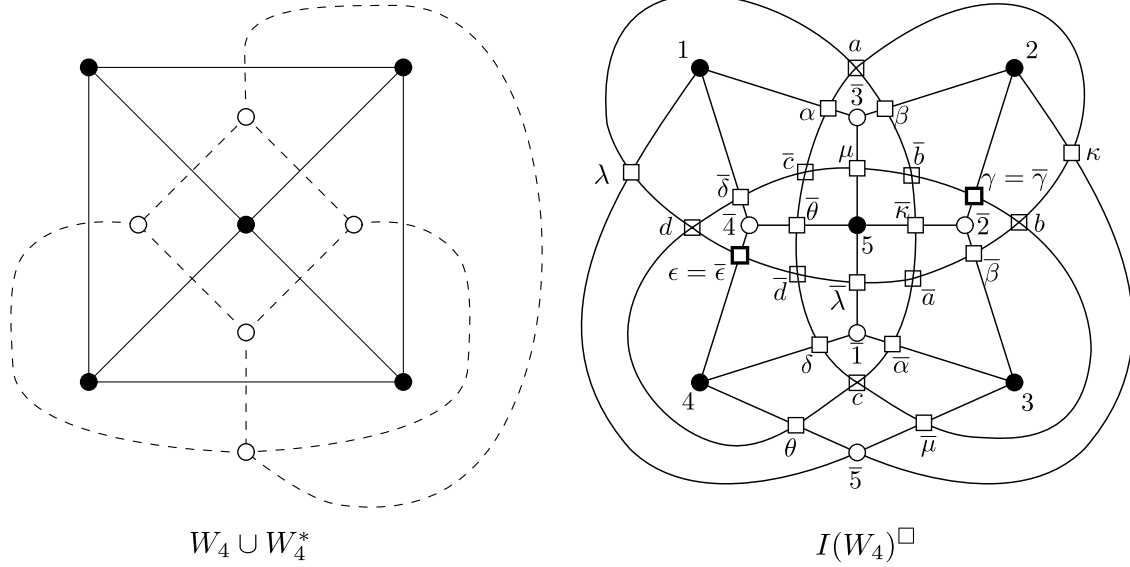


FIGURE 6. (Left) The self-dual map W_4 with its dual. (Right) $I^\square(W_4)$ together with an involutive labelling with two fixed vertices of type V_E : $\gamma = \bar{\gamma}$ and $\epsilon = \bar{\epsilon}$ (bold squares).

Corollary 1. *Let G be a self-dual map. If there is a black vertex of $I(G)$ with odd degree which is connected to all white vertices then G is not antipodally self-dual.*

Proof. Let v be such a black vertex. Since v is connected to all white vertices then for any involutive labelling Λ of $I(G)^\square$ there is an edge in $I(G)$ with ends labeled with k and \bar{k} . By Remark 3, the automorphism $\sigma_\Lambda(G)$ maps an edge with ends labeled $\{k, \bar{k}\}$ to an edge with ends labeled $\{k, \bar{k}\}$ (not necessarily the same edge, see Figure 5). Since, by hypothesis, the degree of v is odd then there must be an edge mapped to itself which correspond to a fixed vertex in $V(I(G)^\square)$. Therefore, by Theorem 2, G is not antipodally self-dual. \square

Figure 7 illustrates a graph in which $I(G)$ has a vertex in $V(G)$ adjacent to each vertex of G^* by an odd number of edges (and thus, by Corollary 1, G is not antipodally self-dual).

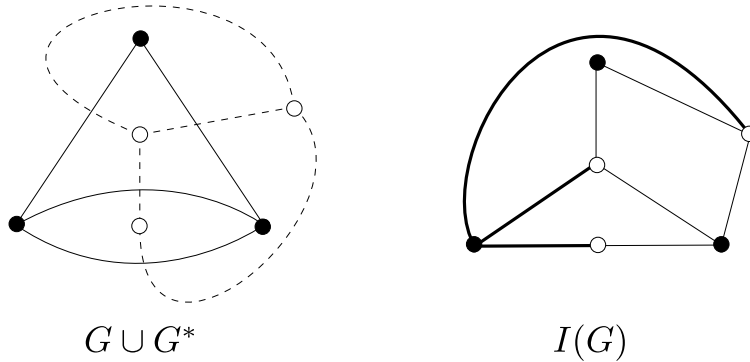


FIGURE 7. (Left) A self-dual map G with its dual. (Right) $I(G)$ with a black vertex joined to each white vertex by an odd number of edges (in bold).

Corollary 2. *Let G be a self-dual map. If G is strongly involutive then G is antipodally self-dual.*

Proof. We shall show that $I(G)^\square$ admits an involutive labeling without fixed vertices. The result then follows by Theorem 2.

Let $\sigma : G \rightarrow G^*$ be a duality isomorphism. We thus have that σ does not fix vertices. Recall that if G is strongly involutive then σ verifies

- a) for each pair of vertices $u, v \in V(G)$, $u \in \sigma(v)$ if and only if $v \in \sigma(u)$ and
- b) for every vertex $v \in V(G)$, we have that $v \notin \sigma(v)$.

As remarked above, a) is equivalent to say that $\sigma^2 = id$. We clearly have that σ does not fix vertices since it maps vertices of G to vertices of G^* . The latter implies that σ_I does not fix vertices in $I(G)$ and thus neither σ^\square in $I(G)^\square$.

Now, by combining conditions (a) and (b) we obtain that $u \notin \sigma(u)$ for every vertex u in G^* . The latter implies that $I(G)$ does not admit an edge with extremes labeled with k and \bar{k} and so σ^\square does not fix vertices of type V_E (i.e., arising from edges of $I(G)$) in $I(G)^\square$.

We finally claim that σ^\square does not fix vertices of type V_F (i.e., arising from faces of $I(G)$) in $I(G)^\square$. We proceed by contradiction, suppose that σ^\square fixes a vertex u_f arising from a face f of $I(G)$. Let f be the face in $I(G)$ corresponding to $\sigma(u_f)$. Recall that all the faces in $I(G)$ are squares, suppose that $f = \{w, x, y, z\}$ with $w, y \in V(G)$ and $x, z \in V(G^*)$ and $f' = \{w', x', y', z'\}$ with $w', y' \in V(G)$ and $x', z' \in V(G^*)$.

Since σ^\square fixes u_f then $\sigma(u_f) = u_f$ but this happens only if $\{\sigma(w), \sigma(y)\} = \{w', y'\}$ and $\{\sigma(x), \sigma(z)\} = \{x', z'\}$. The latter implies the existence of an edge with extremes labeled k and \bar{k} , which is not possible. \square

We notice that the converse of Corollary 2 is not necessarily true. Indeed, there might be a non strongly involutive map G with $I(G)^\square$ admitting an involutive labeling without fixed vertices (and thus G antipodally self-dual, by Theorem 2), see Figure 5.

4. INFINITE FAMILIES

We give below some infinite families having antipodally self-dual maps. We do so by applying the previous results.

4.1. The wheel. Let $n \geq 3$ be an integer. The n -wheel, denoted by W_n , is the graph consisting of an n -cycle with a center joined to each vertex of the cycle.

Proposition 1. *The n -wheel is antipodally self-dual if and only if $n \geq 3$ is odd.*

Proof. It can be easily checked that W_n admits a strongly involutive duality-isomorphism for any odd integer $n \geq 3$. To this end, we label the center with n and the vertices around the center with $0, \dots, n-1$. Let φ be the duality-isomorphism defined as

$$\varphi(i) = \begin{cases} \text{face with vertices } \{n, i + \frac{n-1}{2}, i + \frac{n-1}{2} + 1\} & \text{if } 0 \leq i \leq n-1, \\ \text{exterior face} & \text{if } i = n. \end{cases}$$

where the sum is taken mod n .

It can be easily checked that φ is strongly involutive, see Figure 8.

Moreover, if n is even then $I(W_n)$ admits a symmetric cycle of length $2k$ with k even, see Figure 9. Thus, by Theorem 1, W_n is not antipodally self-dual. \square

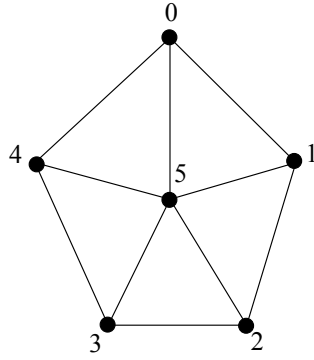


FIGURE 8. 5-wheel together with a labeling used by the strongly involutive duality-isomorphism φ . For instance, $\varphi(3)$ is the face formed by vertices $\{5, 0, 1\}$.

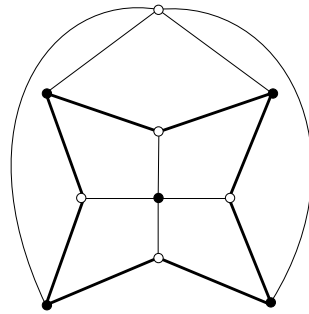


FIGURE 9. $I(W_4)$ admitting a symmetric cycle (bold edges) of length 8.

Figure 3 (a) shows that W_3 admits a antipodally self-dual map. One can easily mimic this embedding for any odd integer $n \geq 3$. Figure 10 illustrates the case $n = 5$.

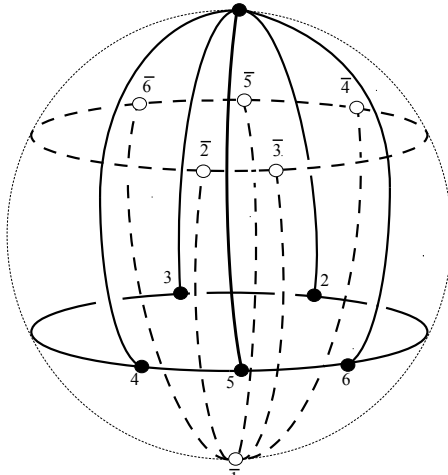


FIGURE 10. A antipodally self-dual map of W_5 (straight edges and black vertices) and its dual (dashed edges and white vertices). Antipodal vertices are given by k and \bar{k}

4.2. **The n -ear.** Let $n \geq 3$ be an integer. The n -ear, denoted by E_n is the graph consisting of a n -cycle with an ear added on each edge and a center is joined to each ear, see Figure 11

Proposition 2. *The n -ear is antipodally self-dual if and only if $n \geq 4$ is even.*

Proof. It can be easily checked that E_n admits a strongly involutive duality-isomorphism for any even integer $n \geq 4$, see Figure 11. Moreover, if n is odd then $I(E_n)$ admits a symmetric cycle of length $2k$ with k even, see Figure 12. Thus, by Theorem 1, W_n is not antipodally self-dual. \square

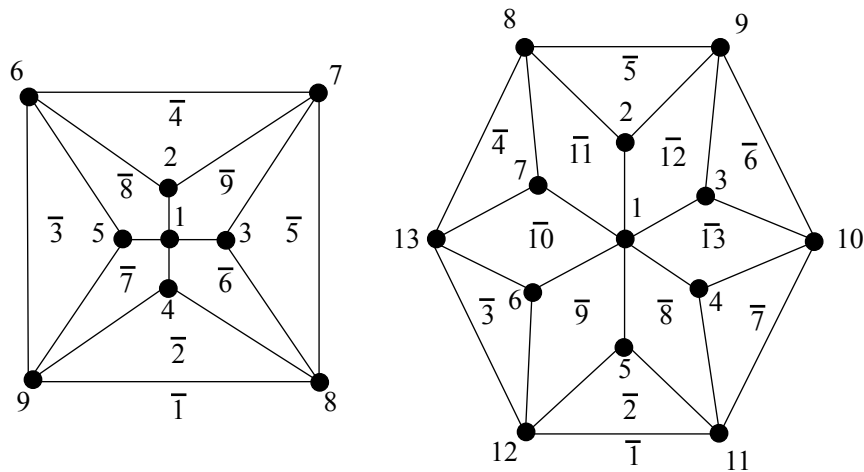


FIGURE 11. The 4-ear and 6-ear graphs together with strongly involutive duality-isomorphisms given by $\sigma(k) = \bar{k}$.

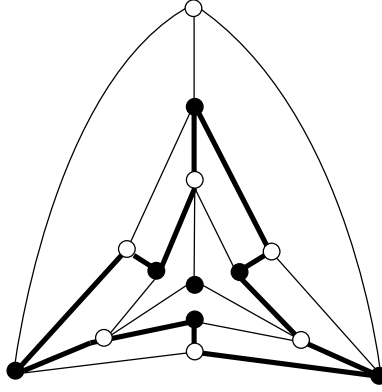


FIGURE 12. $I(3\text{-ear})$ admitting a symmetric cycle (bold edges) of length 12.

The map E_4 given in Figure 13 shows that 4-ear graph is antipodally self-dual. One can easily mimic this embedding for any even integer $n \geq 4$.

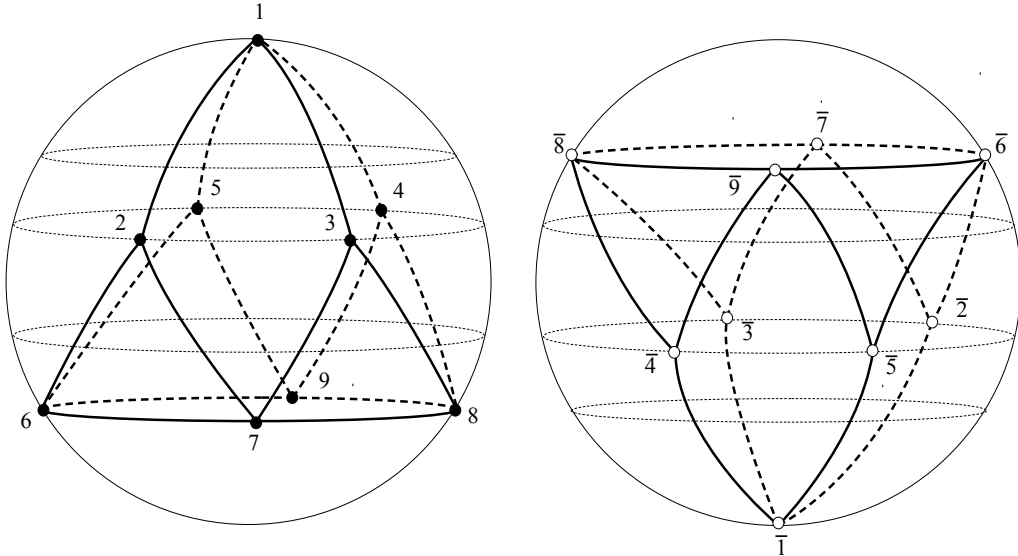


FIGURE 13. A antipodally self-dual map of E_4 (black vertices) and its dual (white vertices). Antipodal vertices are given by k and \bar{k} .

4.3. The (n, ℓ) -pancake. Let $n \geq 3$ and $\ell \geq 1$ be integers. The (n, ℓ) -pancake, denoted by P_n^ℓ , is the graph consisting of ℓ cycles $\{v_1^1, \dots, v_n^1\}, \dots, \{v_1^\ell, \dots, v_n^\ell\}$, a vertex v_i^0 and edges $\{v_i^{j-1}, v_i^j\}$ for each $j = 1, \dots, n$ and all i , see Figure 14.

Proposition 3. *The (n, ℓ) -pancake is antipodally self-dual if and only if $n \geq 3$ is odd for all $\ell \geq 1$.*

Proof. It can be easily checked that (n, ℓ) -pancake admits a strongly involutive duality-isomorphism for all integers $n \geq 3, \ell \geq 1$ with n odd, see Figure 14. Moreover, if n is even then $I((n, \ell)\text{pancake})$ admits a symmetric cycle of length $2k$ with k even, see Figure 15. Thus, by Theorem 1, (n, ℓ) -pancake is not antipodally self-dual. \square

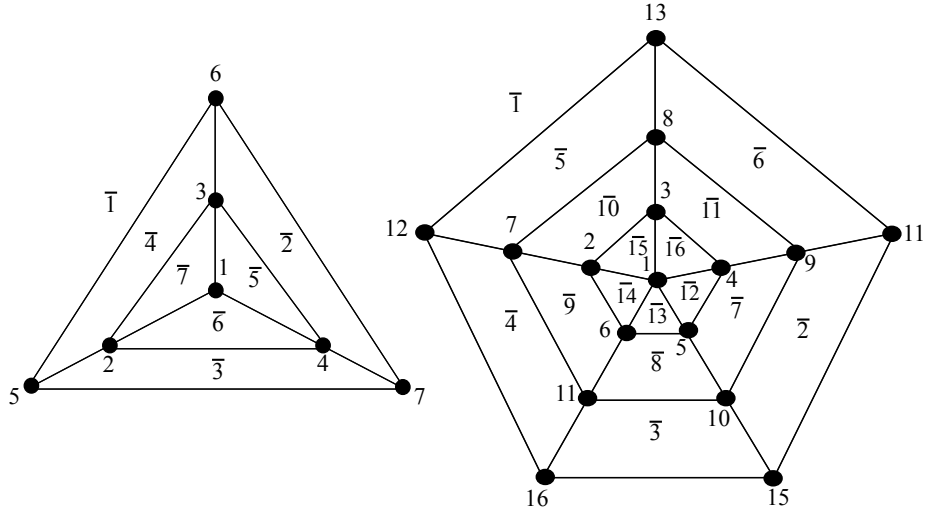


FIGURE 14. P_3^2 and P_5^3 together with a strongly involutive duality-isomorphism given by $\sigma(k) = \bar{k}$.

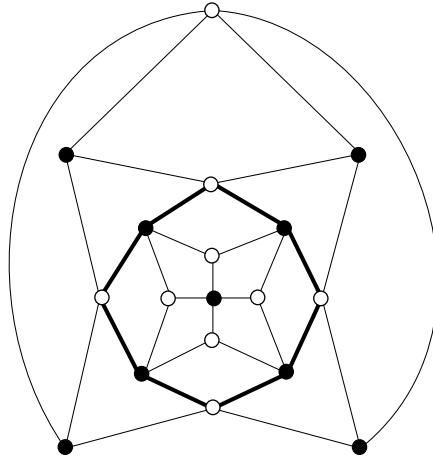


FIGURE 15. $I(P_4^2)$ admitting a symmetric cycle (bold edges) of length 8.

The map of P_5^2 given in Figure 16 shows that $(3, 2)$ -pancake is self-dual antipodal. One can easily mimic this embedding for any odd integer $n \geq 3$ and any $\ell \geq 1$.

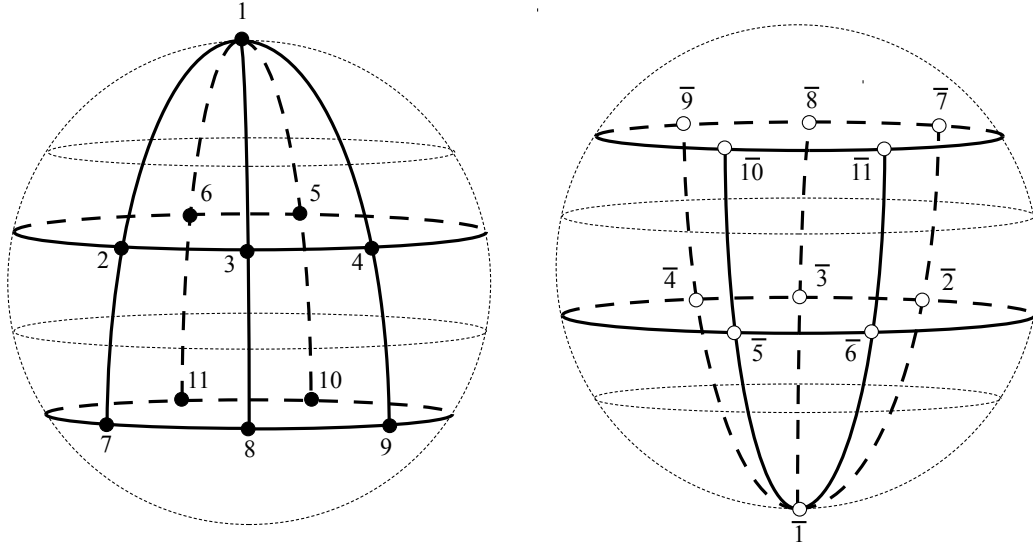


FIGURE 16. A antipodally self-dual map of P_5^2 (black vertices) and its dual (white vertices). Antipodal vertices are given by k and \bar{k} .

4.4. **Adhesion construction.** Let us give a way to construct infinite families of antipodally self-dual graphs. The latter is based on a procedure to construct self-dual graphs called the *adhesion*, given in [7]. Let G be a planar connected graph and let G^* be its geometric dual. Let x (resp. x^*) be the vertex corresponding to the exterior face of G^* (resp. exterior face of $G^{**} = G$). We define the graph $G \diamond G^*$ obtained by identifying x and x^* , see Figure 17.

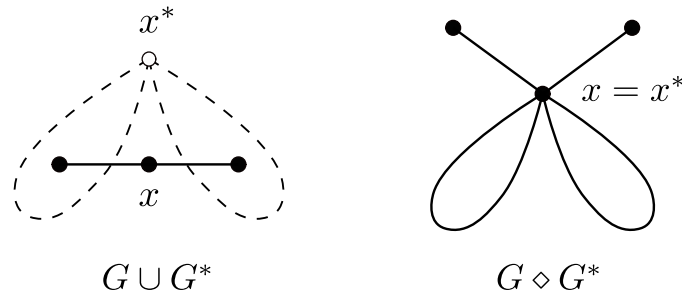


FIGURE 17. (Left) Drawing of G and its dual. (Right) The adhesion of G .

Lemma 3. [7] *Let G be a planar connected graph. Then, the graph $G \diamond G^*$ is self-dual.*

Proof. $H = G \diamond G^*$ is clearly self-dual since $H^* = (G \diamond G^*)^* = G^* \diamond G = G \diamond G^*$. □

Notice that in the construction of $G \diamond G^*$ the couple x and x^* cannot be replaced by any pair of vertices since we may end up with a not self-dual graph, see Figure 18.

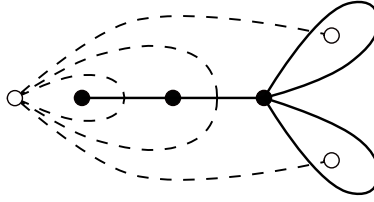


FIGURE 18. A glueing of G and G^* by another pair of vertices which leads to a map which is clearly not self-dual.

Theorem 3. *Let G be a planar connected graph. Then, $G \diamond G^*$ is antipodally self-dual.*

Proof. By Lemma 3 $H = G \diamond G^*$ is self-dual. Let us show that H admits an antipodal map. Let x (resp. x^*) be the vertex corresponding to the exterior face of G^* (resp. exterior face of $G = G^{**}$). We first draw G and its dual within a circle C such that x and x^* are antipodal points on C and no other edge or vertex (of G or G^*) lie on C , see Figure 19.

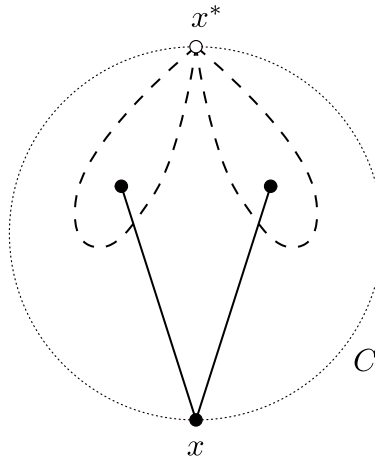


FIGURE 19. Embedding of G and G^* inside circle C with vertex b diametrically opposed to vertex b^* .

We shall construct two embeddings (one in the Northern hemisphere and the other in the Southern one) that will be glued together giving the desired antipodally self-dual embedding of H . For, we consider C as the equator of \mathbb{S}^2 and project our drawing perpendicularly to the Northern hemisphere of \mathbb{S}^2 . We then take the antipodal of the latter embedding, obtaining an embedding in the Southern hemisphere.

We finally glue together both embeddings along the equator (x and x^* are the only vertices that are identified twice on the equator). By construction, this is an antipodal map of H , see Figure 20. \square

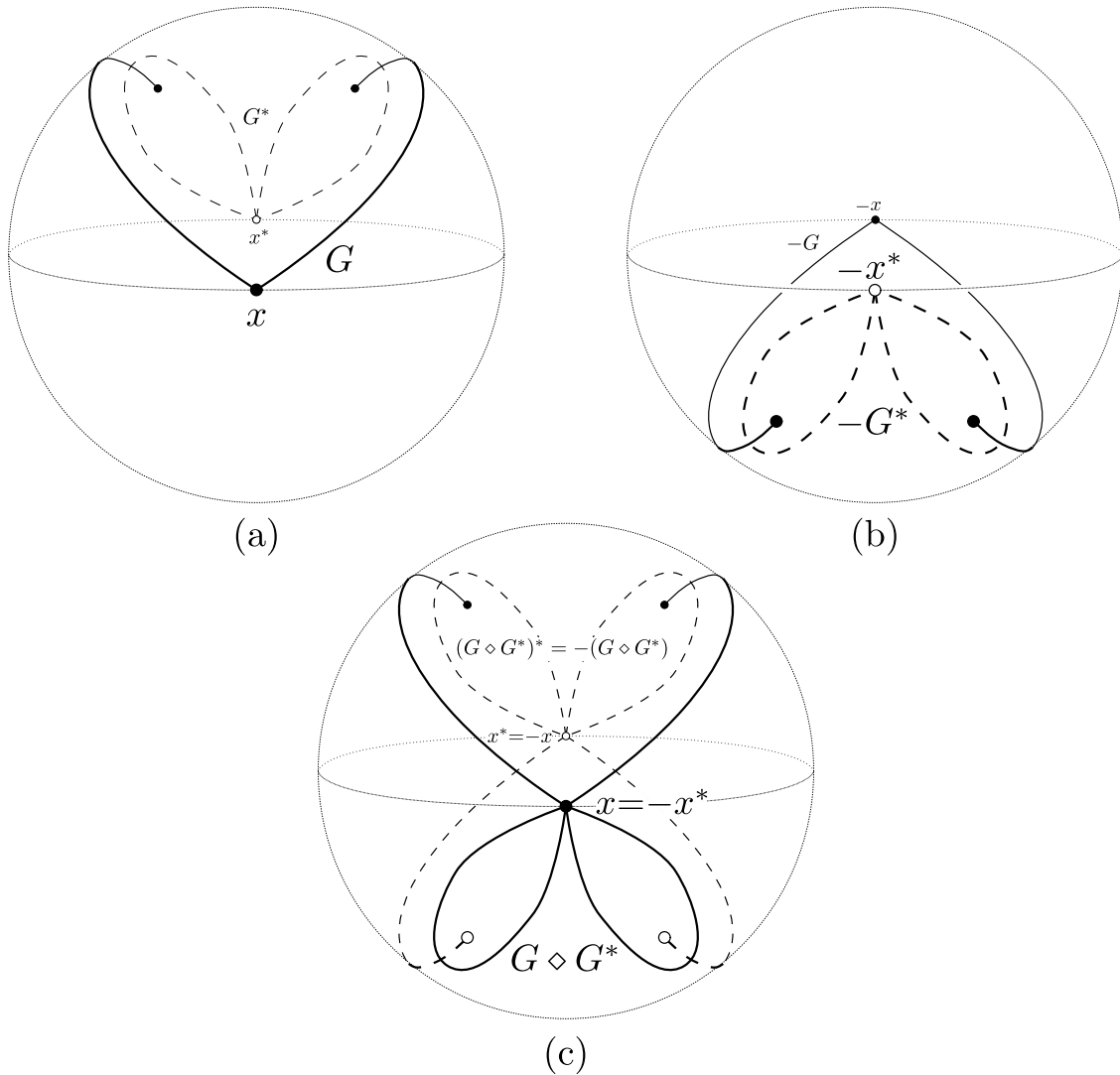


FIGURE 20. (a) Embedding of the drawings of G and G^* in the Northern hemisphere (b) Antipodal embedding of the drawing of the Northern hemisphere (c) Antipodal embedding of $H = G \diamond G^*$ (bold edges) and H^* (dashed edges).

Question 1. *Let H be a antipodally self-dual graph with a cut-vertex. Is it true that $H = G \diamond G^*$ where G is a planar connected graph and G^* its geometric dual ?*

5. LINKS: QUICK DISCUSSION

The notions of antipodally self-dual map and antipodally symmetric map turns out to be very helpful combinatorial tools to investigate special representations of links. These allow to give sufficient conditions for a link L to admit an embedding *centrally symmetric* in \mathbb{R}^3 or *antipodally symmetric* in \mathbb{S}^3 .

This approach lead to sufficient conditions for L to be *amphichiral*. Moreover, the notion of symmetric cycle is also useful to study the amphichirality of a link from another closely related point of view (also arising other combinatorial sufficient conditions).

All these require further (much technical) extra work which is done in [6].

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