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Parameters estimation of a Threshold CKLS process from continuous and discrete observations

Sara Mazzonetto* Benoit Nieto†

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Abstract. We consider a continuous-time process which is self-exciting and ergodic, called threshold Chan–Karolyi–Longstaff–Sanders (CKLS) process. We allow for the presence of several thresholds which determine changes in the dynamics. We study the asymptotic behavior of the maximum and quasi-maximum likelihood estimators of the drift parameters in the case of continuous time and discrete time observations. We show that for high frequency observations and infinite horizon the estimators satisfy the same asymptotic normality property as in the case of continuous time observations. We discuss diffusion coefficient estimation as well. Finally, we apply our estimators to simulated and real data to motivate considering (multiple) thresholds.

Keywords: CIR, CKLS, maximum likelihood, regime-switch, self-exciting process, threshold diffusion.

MSC 2020: primary: 62F12; secondary: 62F03 62M05;

1 Introduction

We deal with parameter estimation of a stochastic process, which follows different Chan–Karolyi–Longstaff–Sanders (CKLS) dynamics separately on different intervals (see equation (2.1) for a definition), that we call *threshold CKLS* (T-CKLS) process. T-CKLS is in particular a Self Exciting Threshold model [10]. It excites itself by changing dynamics according to its own position. The class of T-CKLS models includes several threshold and non-threshold models such as Ornstein–Uhlenbeck (OU) or Vasicek, Cox–Ingersoll–Ross (CIR), Black–Scholes, Merton, Constant Elasticity of Variance (CEV) model. Let us recall the equation satisfied by (non-threshold) CKLS:

$$X_t = x_0 + \int_0^t (a - bX_s) ds + \int_0^t \sigma |X_s|^\gamma dB_s \quad (1.1)$$

where $\sigma \in (0, \infty)$, $b \in \mathbb{R}$, $\gamma \in [0, 1]$. Assume also that the deterministic initial condition x_0 and the coefficient a are strictly positive. CKLS was considered for interest rate modeling in [8]. Among statistical studies of non-threshold diffusions such as CKLS and CIR, let us mention [27] and [3, 4, 5]. The approaches exploit the knowledge of the law of some

*Université de Lorraine, CNRS, Inria, IECL, F-54000 Nancy, France, sara.mazzonetto@univ-lorraine.fr

†Ecole Centrale de Lyon, ICJ UMR5208, CNRS, INSA Lyon, Université Claude Bernard Lyon 1, Université Jean Monnet, 69130 Ecully, France, benoit.nieto@ec-lyon.fr

functionals of the process. Except for very special threshold diffusions (TDs), this law is not available nor easily exploitable. Therefore, when dealing with TDs, one must consider different techniques, which would also hold for non-threshold cases (see Section 5.2 for details).

Recently, several studies have been conducted on the parametric estimation of TDs. Just to mention some results, in the case of continuous time observations [17, 30, 22], high-frequency observations on finite or infinite horizon [20, 24, 25], low frequency ones [33, 12, 25]. TDs attract attention for applications, in financial modelling e.g. [10, 21, 28], population ecology [7], etc. One of the features of the TDs is that they allow for mean reversions, even with several mean reversion levels.

In this paper we focus on ergodic T-CKLS. Note that this includes ergodic T-CIR model and allows for different dynamics on fixed intervals: OU on a region, CIR on another, and possibly other special cases of CKLS on another interval. We consider both continuous time observations and discrete observations, which are not necessarily equally spaced.

We study both maximum likelihood estimation (MLE) and a quasi maximum likelihood estimation (QMLE) for the drift coefficients and we also propose an estimator for the diffusion coefficient σ based on quadratic variation. We study the asymptotic behavior of these estimators, obtaining a law of large numbers and a central limit theorem for the drift coefficients estimators. We discuss these asymptotic results in long time in the continuous time setting, see Theorem 3.5, and in high frequency and long time in the discrete time setting, see Theorem 4.4. We also study consistency and provide a lower bound for the speed of convergence for the diffusion coefficient estimator in high frequency and long time, see Theorem 4.3.

In the continuous time setting, Theorem 3.5 is new for MLE. The drift QMLE was considered in [30]. In the same paper, numerical studies were conducted on the discretization of the QMLE. In the discrete time setting, the statistical properties of the discretized MLE and QMLE are proven in Theorem 4.4, which is the main result of this paper. Let us mention that, up to our knowledge, the results obtained in this document are also new when the process follows some special dynamics of T-CKLS process, such as T-CIR. Moreover, up to our knowledge, Theorem 4.3 about diffusion coefficient estimation is the first result of its kind in the context of TDs observed in high frequency and long time.

In the case of discrete observations, we assume that the process is in its stationary regime (X_t is distributed according to the stationary distribution for all $t \geq 0$) and it is observed on a time-grid of N observations $0 = t_0 < t_1 < \dots < t_N = T_N$ with maximal lag between two consecutive observations, say Δ_N , such that $\lim_{N \rightarrow \infty} T_N = +\infty$ and $\lim_{N \rightarrow \infty} \Delta_N \rightarrow 0$. To prove the convergence speed, we require an additional condition of the form $\lim_{N \rightarrow \infty} \Delta_N^\alpha T_N = 0$, for some power $\alpha \in (0, 1]$ that depends on the parameter vector γ . This is consistent with the existing literature in the context of parameter estimation of diffusions from discrete observations with and without threshold, e.g. [2, 3, 24] (where $\alpha = 1$). Further comments on our results, their assumptions, possible extensions and comparisons with the literature are provided in Section 5.

Outline. In Section 2, we introduce the model and the quasi-likelihood and likelihood functions associated to the T-CKLS process. In Section 3, we deal with estimation of T-CKLS from continuous observations. We provide some statistical properties related to the drift estimator. The main results of this article are provided in Section 4, which deals with drift and volatility estimation from discrete observations. We study the asymptotic

behavior in high frequency and long time for both estimators. In Section 5 we comment on the results. Numerical experiments are provided in Section 6, where the estimators are implemented and tested on simulated data and US interest rates data. Proofs are collected in Section A. Further useful results are available in Section B.

Throughout the paper, we use the notion of stable convergence, denoted $\xrightarrow{\text{stably}}$. Further details on this type of convergence can be found in [14] and [15].

2 The framework: model and assumptions

In the entire document, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ denotes a filtered probability space, and B an (\mathcal{F}_t) -standard Brownian motion. The T-CKLS process solves the following one-dimensional SDE:

$$X_t = X_0 + \int_0^t (a(X_s) - b(X_s)X_s) ds + \int_0^t \sigma(X_s)|X_s|^{\gamma(X_s)} dB_s, \quad t \geq 0, \quad (2.1)$$

with $X_0 > 0$, X_0 (either deterministic, or X_0 independent of $(B_t)_{t \geq 0}$), piecewise constant coefficients a , b , σ and γ possibly discontinuous at levels $0 = r_0 < r_1 < \dots < r_d < r_{d+1} = +\infty$, $d \in \mathbb{N}$. We focus on the case $\gamma(\mathbb{R}) \subseteq [0, 1]$ and $\gamma(0) \in [1/2, 1] \cup \{0\}$. More precisely, let $I_j := [r_j, r_{j+1})$, for $j \in \{0, \dots, d\}$, unless $\gamma(0) = 0$, in which case $I_0 = (-\infty, r_1)$. The drift coefficients are given by

$$a(x) = \sum_{j=0}^d a_j \mathbb{1}_{I_j}(x) \in \mathbb{R} \quad \text{and} \quad b(x) = \sum_{j=0}^d b_j \mathbb{1}_{I_j}(x) \in \mathbb{R},$$

and similarly, the volatility coefficients are given by

$$\sigma(x) = \sum_{j=0}^d \sigma_j \mathbb{1}_{I_j}(x) > 0 \quad \text{and} \quad \gamma(x) = \sum_{j=0}^d \gamma_j \mathbb{1}_{I_j}(x) \geq 0.$$

When $\gamma_0 \in [1/2, 1)$, we assume in addition that $a_0 > 0$. When $\gamma_0 = 1$, we allow for $a_0 \geq 0$. When $\gamma_0 = 0$, for the sake of simplicity, we keep the assumptions $r_1 > 0$ and, although unnecessary, $X_0 > 0$.

For existence of a pathwise unique strong solution to (2.1) under the assumption that $\gamma_0 \in [1/2, 1] \cup \{0\}$, we refer for instance to [19] for existence and uniqueness results for threshold diffusions. Moreover, T-CKLS is a Markov process (see e.g. [11]). When $\gamma_0 \in [1/2, 1]$ the process is always non-negative and 0 is either an unreachable point or a reflecting one (see Lemma A.1 in Section A.1).

We suppose that $(\gamma_j)_{j=0}^d$ and the thresholds $(r_j)_{j=0}^d$ are known, and we estimate drift and diffusion parameter vectors a, b, σ for continuous time observations and discrete (not necessarily equally spaced) high frequency observations and infinite horizon.

Definition 1 (Ergodicity). *We say that the process is ergodic, if it is positive recurrent.*

In this article we consider only the case in which the process is ergodic. For instance, the process is ergodic if we restrict the drift coefficients of the first interval I_0 and the last interval I_d to satisfy

$$(a_0, b_0) \in (0, +\infty) \times \mathbb{R} \quad \text{and} \quad (a_d, b_d) \in \mathbb{R} \times (0, +\infty)$$

when $\gamma_0 \neq 0$. When $\gamma_0 = 0$, the process is ergodic if, for instance,

$$(a_0, b_0) \in \mathbb{R} \times (0, +\infty) \quad \text{and} \quad (a_d, b_d) \in \mathbb{R} \times (0, +\infty).$$

More precise restrictions to the parameters are given in Table 4 in Section A.1.

When the process is ergodic, there exists a stationary distribution (invariant distribution for the transition semigroup), denoted by μ . An expression for the stationary distribution is given in Section A.1.

Definition 2 (Stationarity). *We say that the process is stationary if it is ergodic, X_0 is independent of the driving Brownian motion, and X_0 is distributed according to the stationary distribution.*

The maximum likelihood estimation method for the drift parameter requires some additional parameter restrictions that we are going to detail in the remainder of this section. Assume that we have access to an observation of an entire trajectory on the time interval $[0, T]$ of the T-CKLS. We denote by $\theta := (a, b) = (a_j, b_j)_{j=0}^d$ the drift parameters and we assume the thresholds r and the coefficients γ to be known. In the next sections, we assume σ to be unknown, and we propose an estimator. Yet, in the following lines, the reader should think as if σ is known (replaced by an estimator). We consider two different contrast functions: likelihood and quasi-likelihood. The likelihood function $\theta \mapsto \mathcal{L}_T(\theta; \sigma, \gamma)$ is related to the Girsanov weight:

$$\mathcal{L}_T(\theta; \sigma, \gamma) = \exp \left(\int_0^T \frac{a(X_s) - b(X_s)X_s}{\sigma(X_s)^2(X_s)^{2\gamma(X_s)}} dX_s - \frac{1}{2} \int_0^T \frac{(a(X_s) - b(X_s)X_s)^2}{\sigma(X_s)^2(X_s)^{2\gamma(X_s)}} ds \right). \quad (2.2)$$

Note that it is well defined if the integrals above are well defined (see Proposition B.1 for details). Hence, we further restrict the parameter space $\Theta^{(\mathcal{L})}$ to the coefficients for which the integrals above are well defined: e.g. ,if $\gamma_0 = 1/2$, we have to restrict to $a_0 \geq \sigma_0^2/2$.

We consider the quasi-likelihood function $\theta \mapsto q\text{-}\mathcal{L}_T(\theta) := \ln \mathcal{L}_T(\theta; 1, 0)$ [30]. Hence,

$$q\text{-}\mathcal{L}_T(\theta) = \int_0^T (a(X_s) - b(X_s)X_s) dX_s - \frac{1}{2} \int_0^T (a(X_s) - b(X_s)X_s)^2 ds. \quad (2.3)$$

The advantage of this contrast function is the fact that it does not depend on the diffusion's coefficients γ, σ . We denote $\Theta^{(q\text{-}\mathcal{L})}$ the set of parameters such that the quasi-likelihood is well defined. Note that $q\text{-}\mathcal{L}_T$ is always well defined because the process we consider has continuous trajectories. So, it covers a wider range of parameters than the likelihood function.

Summarizing, in the next sections, we suppose that the parameter $r = (r_j)_{j=1}^d$ and $\gamma = (\gamma_j)_{j=0}^d$ are known. We denote $\theta_\star := (a, b) = (a_j, b_j)_{j=0}^d$ and σ_\star the parameters to be estimated. We suppose that the parameters are in $\Theta^{(\mathcal{L})}$, resp. $\Theta^{(q\text{-}\mathcal{L})}$, when dealing with the likelihood, resp. quasi-likelihood function. Moreover, we assume that the process is ergodic (see Table 4 in Section A.1 for the parameter restrictions ensuring ergodicity).

3 Estimation from continuous time observations

Let $T \in (0, \infty)$, and assume we have at our disposal continuous time observations on the interval $[0, T]$ of a trajectory of the process X solution to the SDE (2.1).

First, we provide estimators which maximize likelihood and quasi-likelihood. Next, we study the asymptotic behavior of the estimators in long time under the assumption that the process is ergodic (see Definition 1).

3.1 Estimators expressions

The drift parameters estimators, MLE and QMLE, are defined as the maximal argument of the log-likelihood (2.2) and the quasi-likelihood (2.3):

$$\theta_T^{(\mathcal{L})} := \operatorname{argmax}_{\theta \in \Theta^{(\mathcal{L})}} \ln \mathcal{L}_T(\theta; \sigma, \gamma) \quad \text{and} \quad \theta_T^{(q\mathcal{L})} := \operatorname{argmax}_{\theta \in \Theta^{(q\mathcal{L})}} q\mathcal{L}_T(\theta).$$

We look for expressions for MLE and QMLE in terms of the following quantities:

$$Q_T^{j,m} := \int_0^T X_s^m \mathbb{1}_{I_j}(X_s) ds \quad \text{and} \quad M_T^{j,m} := \int_0^T X_s^m \mathbb{1}_{I_j}(X_s) dX_s \quad (3.1)$$

for $j \in \{0, \dots, d\}$ and $m \in \{-2\gamma_j, 1 - 2\gamma_j, 2 - 2\gamma_j, 2\gamma_j\} \cup \{0, 1, 2\}$. It is convenient to express log-likelihood and quasi-likelihood as follows. The log-likelihood satisfies

$$\ln \mathcal{L}_T(\theta; \sigma, \gamma) := \sum_{j=0}^d \frac{1}{\sigma_j^2} \left(a_j M_T^{j,-2\gamma_j} - b_j M_T^{j,1-2\gamma_j} - \frac{a_j^2}{2} Q_T^{j,-2\gamma_j} - a_j b_j Q_T^{j,1-2\gamma_j} - \frac{b_j^2}{2} Q_T^{j,2-2\gamma_j} \right), \quad (3.2)$$

and the quasi-likelihood $q\mathcal{L}_T(\theta) := \ln \mathcal{L}_T(\theta; 1, 0)$, which rewrites as

$$q\mathcal{L}_T(\theta) = \sum_{j=0}^d a_j M_T^{j,0} - b_j M_T^{j,1} - \frac{a_j^2}{2} Q_T^{j,0} - a_j b_j Q_T^{j,1} - \frac{b_j^2}{2} Q_T^{j,2}. \quad (3.3)$$

The following proposition provides explicit expression of MLE and QMLE, in terms of the quantities in (3.1).

Proposition 3.1. *Let $T \in (0, \infty)$, the maximum of the likelihood $\mathcal{L}_T(\theta; \sigma, \gamma)$ is achieved at $\theta_T^{(\mathcal{L})} := (a_T^{j,\gamma}, b_T^{j,\gamma})_{j=0}^d$ with*

$$(a_T^{j,\gamma}, b_T^{j,\gamma}) = \left(\frac{M_T^{j,-2\gamma_j} Q_T^{j,2-2\gamma_j} - Q_T^{j,1-2\gamma_j} M_T^{j,1-2\gamma_j}}{Q_T^{j,-2\gamma_j} Q_T^{j,2-2\gamma_j} - (Q_T^{j,1-2\gamma_j})^2}, \frac{M_T^{j,-2\gamma_j} Q_T^{j,1-2\gamma_j} - Q_T^{j,-2\gamma_j} M_T^{j,1-2\gamma_j}}{Q_T^{j,-2\gamma_j} Q_T^{j,2-2\gamma_j} - (Q_T^{j,1-2\gamma_j})^2} \right). \quad (3.4)$$

The maximum of the quasi-likelihood $q\mathcal{L}_T(\theta)$ is achieved at $\theta_T^{(q\mathcal{L})} := (a_T^{j,0}, b_T^{j,0})_{j=0}^d$, that is

$$(a_T^{j,0}, b_T^{j,0}) = \left(\frac{M_T^{j,0} Q_T^{j,2} - Q_T^{j,1} M_T^{j,1}}{Q_T^{j,0} Q_T^{j,2} - (Q_T^{j,1})^2}, \frac{M_T^{j,0} Q_T^{j,1} - Q_T^{j,0} M_T^{j,1}}{Q_T^{j,0} Q_T^{j,2} - (Q_T^{j,1})^2} \right).$$

Proof. We sketch the proof for MLE. The same works for QMLE. One shows that (3.4) is the unique singular point of the gradient (vector of the derivatives with respect to a_j and b_j for all $j \in \{0, \dots, d\}$) of (3.2) and the Hessian is negative definite. \square

Remark 3.2. *If $\gamma \equiv 0$, then the diffusion coefficient is piecewise constant, so T -CKLS is a threshold OU (T -OU) and QMLE and MLE coincide, as noticed in [24].*

Remark 3.3. *For every $j = 0, \dots, d$, $(a_T^{j,\gamma}, b_T^{j,\gamma})$ only depend on the observations of the trajectory $t \mapsto X_t$ which belong to I_j . Of course, the same holds for $(a_T^{j,0}, b_T^{j,0})$.*

MLE and QMLE do not depend on σ_* explicitly, but only on the quantities in (3.1). The following result ensures that σ_* is a.s. equal to an estimator expressed in terms of $Q_T^{j,\cdot}, M_T^{j,0}, M_T^{j,1}, X_T, X_0$.

Proposition 3.4. *Let $T \in (0, \infty)$ and $j \in \{0, \dots, d\}$. Then*

$$\sigma_j = \sqrt{\frac{\mathbf{Q}_T^j}{Q_T^{j,2\gamma_j}}} \quad \text{a.s. on the event } \{Q_T^{j,0} > 0\}$$

where

$$\begin{aligned} \mathbf{Q}_T^0 &:= (f_0(X_T))^2 - (f_0(X_0))^2 + 2(r_1 M_T^{0,0} - M_T^{0,1}) - 2r_1 \mathfrak{f}_0, \\ \mathbf{Q}_T^d &:= (f_d(X_T))^2 - (f_d(X_0))^2 + 2(r_d M_T^{d,0} - M_T^{d,1}), \end{aligned} \quad (3.5)$$

with $f_0(x) = x\mathbf{1}_{I_0}(x) + r_1\mathbf{1}_{[r_1, +\infty)}(x)$, $f_d(x) = \mathbf{1}_{I_d}(x)(x - r_d)$, and $\mathfrak{f}_0 = \min(X_T, r_1) - \min(X_0, r_1)$, and for $j \in \{1, \dots, d-1\}$:

$$\mathbf{Q}_T^j := (f_j(X_T))^2 - (f_j(X_0))^2 - 2M_T^{j,1} + 2r_j M_T^{j,0} + 2(r_{j+1} - r_j)(\mathfrak{f}_{j+1} - \sum_{i=j+1}^d M_T^{i,0}) \quad (3.6)$$

with $f_j(x) = \mathbf{1}_{I_j}(x)(x - r_j) + (r_{j+1} - r_j)\mathbf{1}_{[r_{j+1}, +\infty)}(x)$ and $\mathfrak{f}_j = \max(X_T, r_j) - \max(X_0, r_j)$.

Proof. We only consider the case $j \in \{1, \dots, d-1\}$. When $j \in \{0, d\}$ the proof works analogously. Considering the event $\{Q_T^{j,0} > 0\}$ corresponds to take trajectories, which spend some time in I_j and so $Q_T^{j,2\gamma_j}$ does not vanish. Applying Itô-Tanaka formula (see [29, Chapter VI, exercice 1.25]) ensures that

$$df_j(X_s) = \mathbf{1}_{I_j}(X_s) dX_s + 2^{-1} d(L_s^{r_j}(X) - L_s^{r_{j+1}}(X)).$$

The quadratic variation of $f_j(X)$ satisfies a.s. the equality:

$$\langle f_j(X) \rangle_T = \sigma_j^2 \int_0^T (X_s)^{2\gamma_j} \mathbf{1}_{I_j}(X_s) ds = \sigma_j^2 Q_T^{j,2\gamma_j}.$$

Since for every semi-martingale Y Itô formula ensures that a.s. $dY^2 = 2Y dY + d\langle Y \rangle$, Itô formula applied to the semi-martingale $(f_j(X_T))^2$, yields that a.s.

$$d\langle f_j(X) \rangle_s = d(f_j(X_s))^2 + 2r_j dM_s^{j,0} - 2dM_s^{j,1} + (r_{j+1} - r_j) dL_s^{r_{j+1}}(X).$$

In order to check that $\langle f_j(X) \rangle_T$ is a.s. equal to \mathbf{Q}_T^j , we exploit Itô-Tanaka formula to rewrite the local times in terms of $M^{j,0}, M^{j,1}$. Itô-Tanaka formula applied to $\max(X_T, r_j)$ yields $L_T^{r_j}(X) = 2\mathfrak{f}_j - 2\sum_{i=j}^d M_T^{i,0}$ a.s.. The proof is thus completed. \square

3.2 Asymptotic properties: long time

In this section, we explore the statistical properties as $T \rightarrow \infty$ of the MLE and QMLE from continuous time observations of a trajectory of the T-CKLS process.

We assume that the process is ergodic, μ is the stationary distribution given in Section A.1, and we introduce the following hypotheses:

- $\mathbf{H}_{\mathcal{L}}$: μ admits finite $(-2\gamma_0)$ -th and $(2 - 2\gamma_d)$ -th moment,
- $\mathbf{H}_{q\mathcal{L}}$: μ admits finite $(2 + 2\gamma_d)$ -th moment.

See Section 5.3 for comments on these assumptions. However, often the above required moments are finite. For instance, if $b_d \neq 0$, then μ admits positive moments of all order and for $\gamma_0 > 1/2$ admits negative moments of all order without any further condition. If $\gamma_0 = 1/2$ then parameters restrictions may be necessary (see Proposition B.2).

The asymptotic behavior of the MLE and QMLE are provided in the following theorem, which states that MLE and QMLE are strongly consistent and asymptotically normal estimators of the drift parameter θ_* .

Theorem 3.5. *For $\ell \in \{\mathcal{L}, q\text{-}\mathcal{L}\}$, under Hypothesis \mathbf{H}_ℓ , the MLE and QMLE are strongly consistent estimators of θ_* i.e.*

$$\theta_T^{(\ell)} \xrightarrow[T \rightarrow +\infty]{a.s.} \theta_*.$$

Furthermore, the following convergence is satisfied:

$$\sqrt{T} \left(\theta_T^{(\ell)} - \theta_* \right) \xrightarrow[T \rightarrow +\infty]{stably} \mathbf{N}^{(\ell)},$$

where $\mathbf{N}^{(\ell)} = (N_{j,a}^{(\ell)}, N_{j,b}^{(\ell)})_{j=0}^d$ are $d+1$ independent, independent of X , two-dimensional centered Gaussian random variables with covariance matrices respectively given by $\sigma_j^2 \Gamma_j^{(\ell)}$ such that

$$\Gamma_j^{(\mathcal{L})} := \Gamma_j^{(\mathcal{L}, \gamma_j)} := \begin{pmatrix} Q_\infty^{j, -2\gamma_j} & -Q_\infty^{j, 1-2\gamma_j} \\ -Q_\infty^{j, 1-2\gamma_j} & Q_\infty^{j, 2-2\gamma_j} \end{pmatrix}^{-1} \quad \text{and} \quad \Gamma_j^{(q\text{-}\mathcal{L})} = \left(\Gamma_j^{(\mathcal{L}, 0)} \right)^{-1} \Gamma_j^{(\mathcal{L}, -\gamma_j)} \left(\Gamma_j^{(\mathcal{L}, 0)} \right)^{-1}$$

where $Q_\infty^{j,\cdot}$ are real constants defined in Lemma 3.6.

Proof. The proof is similar to the one in [22, 24], nevertheless we summarize the steps and stress the specificity of the case we consider in this article. Note that MLE and QMLE rewrite as follows:

$$\theta_j^{(\mathcal{L})} \stackrel{a.s.}{=} \theta_j + \sigma_j \left(\mathfrak{M}_T^{j, -\gamma_j}, -\mathfrak{M}_T^{j, 1-\gamma_j} \right) \Gamma^{(\mathcal{L})} \quad \text{and} \quad \theta_j^{(q\text{-}\mathcal{L})} \stackrel{a.s.}{=} \theta_j + \sigma_j \left(\mathfrak{M}_T^{j, \gamma_j}, -\mathfrak{M}_T^{j, 1+\gamma_j} \right) \Gamma^{(\mathcal{L}, 0)}$$

where $\mathfrak{M}_T^{j,k} = \int_0^T (X_s)^k \mathbf{1}_{I_j}(X_s) dB_s$ for $k \in \{-\gamma_j, 1-\gamma_j, \gamma_j, 1+\gamma_j\}$ are martingales. Indeed, note that $Q_T^{j, 2k}$ is the quadratic variation of $M_T^{j, k-\gamma_j}$ which is the one of $\mathfrak{M}_T^{j,k}$ up to a multiplicative factor. We can now exploit martingale theorems. The consistency of the estimator (MLE and QMLE) follows directly from [23, Theorem 1] and the ergodicity of the process which implies, for instance, Lemma 3.6. The asymptotic normality property follows from [9, Theorem 2.2]. Hypotheses $\mathbf{H}_\mathcal{L}$ and $\mathbf{H}_{q\text{-}\mathcal{L}}$ are necessary for the application of [23, Theorem 1]. \square

Lemma 3.6 (Ergodic properties). *For $j \in \{0, \dots, d\}$ and $m \in \mathbb{R}$, if the m -th moment of μ is finite on the set I_j , then*

$$Q_\infty^{j,m} \stackrel{a.s.}{=} \lim_{T \rightarrow \infty} \frac{Q_T^{j,m}}{T} = \int_{I_j} x^m \mu(dx),$$

are non-vanishing constants.

Remark 3.7. *The above asymptotic normality of the estimators implies the local asymptotic normality (LAN) property (see [18]). The LAN property, is a fundamental concept in the asymptotic theory of statistics. For instance, when it is satisfied, it can be combined with the Minimax theorem to establish a lower bound for the asymptotic variance of estimators.*

Since statement and proof are analogous to the one of the T-OU process (corresponding to $\gamma = 0$) given in [24, Theorem 1.(iv)], we just provide a short statement.

Let $\ell \in \{\mathcal{L}, q\text{-}\mathcal{L}\}$ and assume that \mathbf{H}_ℓ holds. The LAN property holds for the ℓ -function with rate of convergence $1/\sqrt{T}$. Furthermore, the asymptotic Fisher information is given by

$$\Gamma^{(\ell)} = \begin{bmatrix} \frac{1}{\sigma_0^2} \left(\Gamma_0^{(\ell)} \right)^{-1} & 0_{\mathbb{R}^{2 \times 2}} & \dots & 0_{\mathbb{R}^{2 \times 2}} \\ 0_{\mathbb{R}^{2 \times 2}} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \frac{1}{\sigma_{d-1}^2} \left(\Gamma_{d-1}^{(\ell)} \right)^{-1} & 0_{\mathbb{R}^{2 \times 2}} \\ 0_{\mathbb{R}^{2 \times 2}} & \dots & 0_{\mathbb{R}^{2 \times 2}} & \frac{1}{\sigma_d^2} \left(\Gamma_d^{(\ell)} \right)^{-1} \end{bmatrix}.$$

4 Estimation from discrete observations

In this section, we assume to observe the process on a discrete time grid $0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T_N < \infty$, for $N \in \mathbb{N}$.

First, we provide estimators, which maximize a discretized versions of likelihood and quasi-likelihood. Next, we study the asymptotic behavior of the estimators in high frequency and long time under the assumption that the process is stationary (see Definition 2).

4.1 Estimators expressions

There is no exploitable explicit expression for the transition densities of the T-CKLS process, nor for the finite dimensional distributions. This is also true in the well known special case of threshold Brownian motion with piecewise constant drift. Hence, instead of considering the likelihood function associated to the sample $(X_{t_k})_{k=0}^N$, we considered a discretization of the likelihood \mathcal{L}_T (2.2) and quasi-likelihood $q\text{-}\mathcal{L}_T$ (2.3). Further comments on this choice are given in Section 5. Once these discretizations introduced, we denote them respectively by $\mathcal{L}_{T_N, N}$ and $q\text{-}\mathcal{L}_{T_N, N}$ and we compute the estimators

$$\theta_{T_N, N}^{(\mathcal{L})} = \operatorname{argmax}_{\theta \in \Theta^{(\mathcal{L})}} \ln \mathcal{L}_{T_N, N}(\theta; \sigma, \gamma) \quad \text{and} \quad \theta_{T_N, N}^{(q\text{-}\mathcal{L})} = \operatorname{argmax}_{\theta \in \Theta^{(q\text{-}\mathcal{L})}} q\text{-}\mathcal{L}_{T_N, N}(\theta).$$

Let us denote by $Q_{T_N, N}^{j, m}$ and $M_{T_N, N}^{j, m}$ the discrete versions of $Q_{T_N}^{j, m}$ and $M_{T_N}^{j, m}$ in (3.1):

$$Q_{T_N, N}^{j, m} := \sum_{i=0}^{N-1} X_{t_i}^m \mathbf{1}_{I_j}(X_{t_i})(t_{i+1} - t_i) \quad \text{and} \quad M_{T_N, N}^{j, m} := \sum_{i=0}^{N-1} X_{t_i}^m \mathbf{1}_{I_j}(X_{t_i})(X_{t_{i+1}} - X_{t_i}) \quad (4.1)$$

for $j \in \{0, \dots, d\}$ and $m \in \{-2\gamma_j, 1 - 2\gamma_j, 2 - 2\gamma_j, 2\gamma_j\} \cup \{-1, 0, 1, 2\}$. To obtain the discretized likelihood and quasi-likelihood, it would be natural to replace the above quantities in the continuous-time observations likelihood and quasi-likelihood functions given

in (3.2)-(3.3). The discretized quasi-likelihood is then

$$q\text{-}\mathcal{L}_{T_N,N}(\theta) := \sum_{j=0}^d a_j M_{T_N,N}^{j,0} - b_j M_{T_N,N}^{j,1} - \frac{a_j^2}{2} Q_{T_N,N}^{j,0} - a_j b_j Q_{T_N,N}^{j,1} - \frac{b_j^2}{2} Q_{T_N,N}^{j,2}, \quad (4.2)$$

and we *could* do similarly for the log-likelihood. We would get

$$\begin{aligned} \ln \mathcal{L}_{T_N,N}(\theta; \sigma, \gamma) \\ = \sum_{j=0}^d \frac{1}{\sigma_j^2} \left(a_j M_{T_N,N}^{j,-2\gamma_j} - b_j M_{T_N,N}^{j,1-2\gamma_j} - \frac{a_j^2}{2} Q_{T_N,N}^{j,-2\gamma_j} - a_j b_j Q_{T_N,N}^{j,1-2\gamma_j} - \frac{b_j^2}{2} Q_{T_N,N}^{j,2-2\gamma_j} \right). \end{aligned} \quad (4.3)$$

Actually, we do not choose the latter quantity as discretized-log-likelihood. Instead, we consider a different discretization of $M^{j,m}$ based on an alternative expression, which depends on $M^{j,0}$ and $Q^{j,0}$ and $Q^{j,-1}$ (see Lemma 5.1 in Section 5, where we also explain this choice for the discretization). For every $j \in \{0, \dots, d\}$ we define $\mathcal{M}_{T_N,N}^{j,0} := M_{T_N,N}^{j,0}$ and for $m \in \{-2\gamma_j, 1 - 2\gamma_j\} \setminus \{0\}$, then

$$\mathcal{M}_{T_N,N}^{0,m} = f_{0,m+1}(X_0) - f_{0,m+1}(X_{T_N}) - \frac{m}{2} \sigma_0^2 Q_{T_N,N}^{0,m+2\gamma_0-1} + r_1^m (M_{T_N,N}^{0,0} + \mathbf{f}_0),$$

$$\mathcal{M}_{T_N,N}^{d,m} = f_{d,m+1}(X_{T_N}) - f_{d,m+1}(X_0) - \frac{m}{2} \sigma_d^2 Q_{T_N,N}^{d,m+2\gamma_d-1} + r_d^m (M_{T_N,N}^{d,0} - \mathbf{f}_d),$$

where $f_{0,m}(x) = \int_x^{r_1} y^{m-1} dy \mathbf{1}_{I_0}(x)$, $f_{d,m}(x) = \int_{r_d}^x y^{m-1} dy \mathbf{1}_{I_d}(x)$, and if $j \in \{1, \dots, d-1\}$, then

$$\begin{aligned} \mathcal{M}_{T_N,N}^{j,m} &= f_{j,m+1}(X_{T_N}) - f_{j,m+1}(X_0) - \frac{m}{2} \sigma_j^2 Q_{T_N,N}^{j,m+2\gamma_j-1} \\ &\quad + r_j^m M_{T_N,N}^{j,0} + r_{j+1}^m \mathbf{f}_{j+1} - r_j^m \mathbf{f}_j - (r_{j+1}^m - r_j^m) \sum_{k=j+1}^d M_{T_N,N}^{k,0} \end{aligned} \quad (4.4)$$

where $f_{j,m}(x) = \int_{r_j}^{x \wedge r_{j+1}} y^{m-1} dy \mathbf{1}_{(r_j, +\infty)}(x)$.

Finally, we consider the following discretized log-likelihood:

$$\ln \mathcal{L}_{T_N,N}(\theta; \sigma, \gamma) = \sum_{j=0}^d \frac{1}{\sigma_j^2} \left(a_j \mathcal{M}_{T_N,N}^{j,-2\gamma_j} - b_j \mathcal{M}_{T_N,N}^{j,1-2\gamma_j} - \frac{a_j^2}{2} Q_{T_N,N}^{j,-2\gamma_j} - a_j b_j Q_{T_N,N}^{j,1-2\gamma_j} - \frac{b_j^2}{2} Q_{T_N,N}^{j,2-2\gamma_j} \right).$$

The following proposition establishes an explicit expression of the discretized MLE and discretized QMLE.

Proposition 4.1. *Let $(T_N)_{N \in \mathbb{N}}$ be a sequence in $(0, \infty)$ and let $N \in \mathbb{N}$. The maximum of the discretized likelihood is achieved at $\theta_{T_N,N}^{(\mathcal{L})} = (a_{T_N,N}^{j,\gamma_j}, b_{T_N,N}^{j,\gamma_j})_{j=0}^d$ with*

$$(a_{T_N,N}^{j,\gamma_j}, b_{T_N,N}^{j,\gamma_j}) := \left(\frac{\mathcal{M}_{T_N,N}^{j,-2\gamma_j} Q_{T_N,N}^{j,2-2\gamma_j} - Q_{T_N,N}^{j,1-2\gamma_j} \mathcal{M}_{T_N,N}^{j,1-2\gamma_j}}{Q_{T_N,N}^{j,-2\gamma_j} Q_{T_N,N}^{j,2-2\gamma_j} - (Q_{T_N,N}^{j,1-2\gamma_j})^2}, \frac{\mathcal{M}_{T_N,N}^{j,-2\gamma_j} Q_{T_N,N}^{j,1-2\gamma_j} - Q_{T_N,N}^{j,-2\gamma_j} \mathcal{M}_{T_N,N}^{j,1-2\gamma_j}}{Q_{T_N,N}^{j,-2\gamma_j} Q_{T_N,N}^{j,2-2\gamma_j} - (Q_{T_N,N}^{j,1-2\gamma_j})^2} \right). \quad (4.5)$$

The maximum of discretized quasi-likelihood is achieved at $\theta_{T_N,N}^{(q\text{-}\mathcal{L})} = (a_{T_N,N}^{j,0}, b_{T_N,N}^{j,0})_{j=0}^d$ with

$$(a_{T_N,N}^{j,0}, b_{T_N,N}^{j,0}) = \left(\frac{M_{T_N,N}^{j,0} Q_{T_N,N}^{j,2} - Q_{T_N,N}^{j,1} M_{T_N,N}^{j,1}}{Q_{T_N,N}^{j,0} Q_{T_N,N}^{j,2} - (Q_{T_N,N}^{j,1})^2}, \frac{M_{T_N,N}^{j,0} Q_{T_N,N}^{j,1} - Q_{T_N,N}^{j,0} M_{T_N,N}^{j,1}}{Q_{T_N,N}^{j,0} Q_{T_N,N}^{j,2} - (Q_{T_N,N}^{j,1})^2} \right). \quad (4.6)$$

The proof is omitted because it is analogous to the one of Proposition 3.1.

Note that the QMLE does not depend on the parameter vector $\sigma_\star = (\sigma_j)_{j=0}^d$, instead the MLE does because so do the expressions $\mathcal{M}_{T_N, N}^{j, m}$. Since we assume σ_\star is not known, we replace it by an estimator. For $j \in \{0, \dots, d\}$,

$$\sigma_{T_N, N}^j = \sqrt{\frac{\mathbf{Q}_{T_N, N}^j}{Q_{T_N, N}^{j, 2\gamma_j}}}, \quad (4.7)$$

where $\mathbf{Q}_{T_N, N}^j$ is obtained by discretizing the right hand side of formula (3.5) and (3.6) in Proposition 3.4. Note that $\mathbf{Q}_{T_N, N}^j$ depends on $M_{T_N, N}^{j, 1}$ and $M_{T_N, N}^{j, 0}$ defined in (4.1). For instance,

$$\mathbf{Q}_{T_N, N}^d := \mathbb{1}_{I_d}(X_T)(X_T - r_d)^2 - \mathbb{1}_{I_d}(X_0)(X_0 - r_d)^2 + 2 \left(r_d M_{T_N, N}^{d, 0} - M_{T_N, N}^{d, 1} \right).$$

4.2 Asymptotic properties: high frequency - long time

In this section, we state the statistical properties of the discretized MLE and QMLE.

Let $\Delta_N := \max_{k=1, \dots, N} (t_k - t_{k-1})$ denote the maximal lag between two consecutive observations. We assume that the observation time window goes to infinity (long time) and the maximal lag between consecutive observations vanishes (high frequency):

$$\lim_{N \rightarrow +\infty} T_N = +\infty \quad \text{and} \quad \lim_{N \rightarrow +\infty} \Delta_N = 0. \quad (4.8)$$

Moreover, we assume that the process is stationary, see Definition 2.

The results of this section require additional assumptions on the moments of the stationary distribution and on Δ_N . Let $\mathcal{H}_{\mathcal{L}}$ and $\mathcal{H}_{q-\mathcal{L}}$ be the following assumptions:

- $\mathcal{H}_{\mathcal{L}}$: μ admits finite p -th moment and $(-q)$ -th moments with
 - $p = 2 + 2\gamma_d$ and $q = 2\gamma^{\max}$ if $\gamma_0 = 0$ and $\gamma^{\max} \in [0, 1/2)$,
 - $p = \max(2 + 2\gamma_d, p_{\mathcal{L}})$ and $q = q_{\mathcal{L}}$ if $\gamma^{\max} \in (0, 1/2)$ and $\gamma_0 \neq 0$ or $\gamma^{\max} = 1/2$,
 - $p = \max(2 + 2\gamma_d, p_{\mathcal{L}}, p'\gamma^{\max})$ and $q = \max(q_{\mathcal{L}}, 2q'\gamma^{\max})$ if $\gamma^{\max} \in (1/2, 1]$,
- $\mathcal{H}_{q-\mathcal{L}} = \mathbf{H}_{q-\mathcal{L}}$: μ admits finite $(2 + 2\gamma_d)$ -th moment,

where $\gamma^{\max} := \max\{\gamma_j : j = 0, \dots, d\}$, and $p_{\mathcal{L}}, q_{\mathcal{L}}, p', q' \geq 1$ such that $1 = 1/p_{\mathcal{L}} + 2/q_{\mathcal{L}} = 1/p' + 3/(2q')$. Observe that, if $p' = 4$ then $q' = 2$ and $p'\gamma^{\max} = 2q'\gamma^{\max} = 4\gamma^{\max}$. Note that, the conditions on moments of μ may lead to parameters restrictions for our asymptotic results (see Proposition B.2), e.g. if $\gamma_0 = 1/2$. Nevertheless, if we restrict to $b_d \neq 0$ and $\gamma_0 > 1/2$ then μ admits positive and negative moments of all order. The interplay between $p_{\mathcal{L}}, q_{\mathcal{L}}, p', q'$ may come into play to reduce the required negative moments if, for instance, μ admits positive moments of all orders. See Section 5.3 for more details and comments on these assumptions.

Without loss of generality we assume $\Delta_N \in (0, 1)$ for all N . Then, we also introduce the following quantities $g_N^{(q-\mathcal{L})} = \Delta_N$,

$$g_N^{(\mathcal{L})} = \max_{j=0, \dots, d} \begin{cases} \Delta_N^{2-2\gamma_j} & \text{if } \gamma_j \in (\frac{3}{4}, 1), \\ \Delta_N^{2\gamma_j-1} & \text{if } \gamma_j \in (\frac{1}{2}, \frac{3}{4}], \\ \Delta_N^{1-2\gamma_j} & \text{if } \gamma_j \in (\frac{1}{4}, \frac{1}{2}), \\ \Delta_N^{2\gamma_j} & \text{if } \gamma_j \in (0, \frac{1}{4}], \\ \Delta_N & \text{if } \gamma_j \in \{0, \frac{1}{2}, 1\}, \end{cases} \quad g_N^{(\sigma)} := \max_{j=0, \dots, d} \begin{cases} \Delta_N^{\gamma_j} & \text{if } \gamma_j \in (\frac{1}{2}, 1), \\ \Delta_N^{2\gamma_j} & \text{if } \gamma_j \in (0, \frac{1}{2}), \\ \Delta_N & \text{if } \gamma_j \in \{0, \frac{1}{2}, 1\}. \end{cases}$$

Remark 4.2. $\Delta_N = g_N^{(q-\mathcal{L})} \leq g_N^{(\sigma)} \leq g_N^{(\mathcal{L})}$. If $\gamma^{\max} \in \{0, 1/2, 1\}$, then $g_N^{(\mathcal{L})} = \Delta_N$.

We are now ready to provide our first convergence result in the discrete setting. We consider the volatility estimator in (4.7) and we prove consistency, and show that the speed of convergence is larger than $\sqrt{T_N}$.

Theorem 4.3. *Assume that (4.8) holds, that the T-CKLS X is stationary and that Hypothesis $\mathcal{H}_{q-\mathcal{L}}$ holds. Then, the estimator $\sigma_{T_N, N}^2 = ((\sigma_{T_N, N}^j)^2)_{j=0}^d$ in (4.7) is a consistent estimator of the diffusion coefficient vector $\sigma_\star^2 = ((\sigma_j)^2)_{j=0}^d$, i.e.*

$$\sigma_{T_N, N}^2 \xrightarrow[N \rightarrow +\infty]{\mathbb{P}} \sigma_\star^2.$$

Under the additional assumption that $\lim_{N \rightarrow +\infty} T_N g_N^{(\sigma)} = 0$, it holds that

$$\sqrt{T_N} (\sigma_{T_N, N}^2 - \sigma_\star^2) \xrightarrow[N \rightarrow +\infty]{\mathbb{P}} 0_{\mathbb{R}^{d+1}}.$$

Since we assume that the diffusion coefficient σ_\star vector is unknown, we replace it by estimator (4.7) in the expression of $\mathcal{M}_{T_N, N}^{j,\cdot}$.

The following theorems state the asymptotic properties in high frequency and long time observations of the discretized MLE and QMLE of θ_\star .

Theorem 4.4. *Assume that (4.8) holds and that the T-CKLS X is stationary. For $\ell \in \{\mathcal{L}, q-\mathcal{L}\}$, under Hypothesis \mathcal{H}_ℓ , the MLE and QMLE are consistent estimators of θ_\star i.e.*

$$\theta_{T_N, N}^{(\ell)} \xrightarrow[N \rightarrow +\infty]{\mathbb{P}} \theta_\star.$$

Furthermore, if $\lim_{N \rightarrow +\infty} T_N g_N^{(\ell)} = 0$, then under the same hypothesis, we have

$$\sqrt{T_N} (\theta_{T_N, N}^{(\ell)} - \theta_\star) \xrightarrow[N \rightarrow +\infty]{\text{stably}} \mathbf{N}^{(\ell)},$$

where $\mathbf{N}^{(\cdot)}$ is defined in Theorem 3.5.

The proof of Theorem 4.3 and Theorem 4.4 are postponed to Section A. Nevertheless we give the main ideas and tools in Section 5.

Remark 4.5. *For $\ell \in \{\mathcal{L}, q-\mathcal{L}\}$, it follows from the previous results that the estimator $(\theta_{T_N, N}^{(\ell)}, \sigma_{T_N, N})$ is a consistent estimator of $(\theta_\star, \sigma_\star)$. By the stable convergence properties, $\sqrt{T_N}(\theta_{T_N, N}^{(\ell)}, \sigma_{T_N, N})$ converges stably to the vector $(\mathbf{N}^{(\ell)}, 0_{\mathbb{R}^{d+1}})$. If σ_\star is known, the analogous to the LAN property in Remark 3.7 holds for the discretized MLE and QMLE.*

5 Comments on the results and their proofs

In this section, we comment the results of the previous sections and we summarize the key elements of the proofs of consistency and asymptotic normality of the continuous time and discretized MLE and QMLE. Moreover, we discuss assumptions and extensions, and compare with related literature.

The main results of this article are the asymptotic results in the context of a T-CKLS process observed in high frequency and long time: Theorems 4.3 and 4.4. As we already

mentioned, these results are also new in the context of other threshold diffusions such as T-CIR, and for mixed dynamics: CIR on a space interval, OU on another, CKLS on another one. We exploit this feature in Section 6. In Section 5.1 we comment on the estimators. In particular, we discuss the discretization choice \mathcal{M} and the novelty on estimation of σ_* .

Since the process is ergodic, the proofs of the asymptotic results in the context of continuous time observations follow from standard martingale central limit theorems. This is the case in other results for TDs, e.g. [30, 22, 24]. Therefore, in this section, we only focus on the proofs in the case of discrete observations. This is done in Section 5.2. Nevertheless, in Section 5.3, we provide some comments on the assumptions of the results both in the context of discrete and continuous time observations and we compare our assumptions with those considered in the literature.

5.1 On the estimators

On the volatility estimator. Estimator (4.7) is inspired by the results in [20]. The latter reference studies some estimators for the volatility of oscillating Brownian motion (null drift, $\gamma = 0$, one threshold) from high frequency discrete observations over a finite time horizon. The estimators are based on quadratic variation, but on two separate intervals over which the volatility is constant. Our estimator exploits a different discretization choice for the quadratic variation over the intervals I_j , which allows to obtain information about its behavior in high frequency and long time. Up to our knowledge, this is the first time an estimator of the volatility for TDs (with $\gamma \equiv 0$ as well) is analyzed in high frequency and long time.

Likelihood discretization: $M_{T_N, N}^{j, m}$ versus $\mathcal{M}_{T_N, N}^{j, m}$. Let us first introduce an expression \mathbb{P} -a.s. equal to $M_T^{j, m}$, which would not involve any term $M^{j, k}$ except if $k = 0$. The notation, in particular the functions $f_{j, m}$, has been introduced for equation (4.4).

Lemma 5.1. *Let $T \in (0, \infty)$, $j \in \{1, \dots, d\}$ and $m \in \{-2\gamma_j, 1 - 2\gamma_j\} \setminus \{0\}$. It holds \mathbb{P} -a.s. that*

$$M_T^{0, m} = f_{0, m+1}(X_0) - f_{0, m+1}(X_T) - \frac{m}{2} \sigma_0^2 Q_T^{0, m+2\gamma_0-1} + r_1^m (M_T^{0, 0} + \mathfrak{f}_0), \quad (5.1)$$

$$M_T^{d, m} = f_{d, m+1}(X_T) - f_{d, m+1}(X_0) - \frac{m}{2} \sigma_d^2 Q_T^{d, m+2\gamma_d-1} + r_d^m (M_T^{d, 0} - \mathfrak{f}_d), \quad (5.2)$$

and for $j \in \{1, \dots, d-1\}$:

$$\begin{aligned} M_T^{j, m} &= f_{j, m+1}(X_T) - f_{j, m+1}(X_0) - \frac{m}{2} \sigma_j^2 Q_T^{j, m+2\gamma_j-1} \\ &\quad + r_j^m M_T^{j, 0} + r_{j+1}^m \mathfrak{f}_{j+1} - r_j^m \mathfrak{f}_j - (r_{j+1}^m - r_j^m) \sum_{k=j+1}^d M_T^{k, 0}, \end{aligned} \quad (5.3)$$

where $\mathfrak{f}_0 = \min(X_T, r_1) - \min(X_0, r_1)$ and $\mathfrak{f}_j = \max(X_T, r_j) - \max(X_0, r_j)$ for $j \geq 1$.

Proof. We prove only the case $j \in \{1, \dots, d-1\}$, and the case $j \in \{0, d\}$ works similarly. Applying Itô-Tanaka formula (see e.g. [29, Chapter VI, exercice 1.25]), yields the a.s. equality

$$f_{j, m+1}(X_T) = f_{j, m+1}(X_0) + \frac{m}{2} \sigma_j^2 Q_T^{j, m+2\gamma_j-1} + M_T^{j, m} + \frac{1}{2} (r_j^m L_T^{r_j}(X) - r_{j+1}^m L_T^{r_{j+1}}(X)). \quad (5.4)$$

Applying a second time Itô-Tanaka formula to $\max(X_T, r_j)$ shows that a.s. $L_T^{r_j}(X) = 2\mathfrak{f}_j - 2 \sum_{i=j}^d M_T^{i,0}$. Combining the latter equation with (5.4) completes the proof. \square

We used the latter result to obtain $\mathcal{M}_{T_N, N}^{j,m}$: we just considered the discretized versions of the right hand side of (5.1), (5.2) and (5.3) in Lemma 5.1 by replacing the quantities $M^{j,0}$ and $Q^{j,m+2\gamma_j-1}$ by their discretized versions (4.1). Without this choice, proceeding as in the next sections, we would have obtained a more restrictive hypothesis $\mathcal{H}_{\mathcal{L}}$.

5.2 Key elements of proofs, comments and extensions.

We now comment on the proofs of consistency and asymptotic normality for drift and diffusion coefficients, namely Theorems 4.3 and 4.4. They rely on the next Lemma 5.2 and on the continuous time results Proposition 3.4 and Theorem 3.5. More precisely, one shows that the rescaled difference between the discrete-time and continuous-time estimators (e.g. $\sqrt{T_N} \left(\theta_{T_N, N}^{(\mathcal{L})} - \theta_{T_N}^{(\mathcal{L})} \right)$) vanishes as $N \rightarrow \infty$. Since all estimators depend on $M_{T_N, N}^{j,m}$ and $Q_{T_N, N}^{j,k}$ for suitable k, m , the proof is based on the following result.

Lemma 5.2. *Let $\lambda \in \{1, 2\}$. Assume that (4.8) holds and that the T-CKLS X is stationary (see Definition 2). Then*

$$\lim_{N \rightarrow \infty} T_N^{-1/\lambda} \mathbb{E} \left[|Q_{T_N, N}^{j,k} - Q_{T_N, N}^{j,k}| \right] = 0 \quad \text{and} \quad \lim_{N \rightarrow \infty} T_N^{-1/\lambda} \mathbb{E} \left[|M_{T_N, N}^{j,m} - M_{T_N, N}^{j,m}| \right] = 0 \quad (5.5)$$

for all $j \in \{0, \dots, d\}$ in each one of the following cases

- (a) (drift QMLE) for every $k \in \{0, 1, 2\}$ and $m \in \{0, 1\}$,
under the assumptions: $\lim_{N \rightarrow +\infty} T_N^{\lambda-1} g_N^{(q-\mathcal{L})} = 0$ and hypothesis $\mathcal{H}_{q-\mathcal{L}}$.
- (b) (volatility estimation) for every $k = 2\gamma_j$ and $m \in \{0, 1\}$,
under the assumptions: $\lim_{N \rightarrow +\infty} T_N^{\lambda-1} g_N^{(\sigma)} = 0$ and hypothesis $\mathcal{H}_{q-\mathcal{L}}$.
- (c) (drift MLE, σ_* unknown) for every $k \in \{-2\gamma_j, 1 - 2\gamma_j, 2 - 2\gamma_j, 2\gamma_j\} \cup \{-1, 0\}$ and $m \in \{0, 1\}$, under the assumptions: $\lim_{N \rightarrow +\infty} T_N^{\lambda-1} g_N^{(\mathcal{L})} = 0$ and hypothesis $\mathcal{H}_{\mathcal{L}}$.

Remark 5.3. *If $\lambda = 1$, then $\lim_{N \rightarrow \infty} T_N^{\lambda-1} g_N^{(\cdot)} = 0$ is equivalent to $\lim_{N \rightarrow \infty} \Delta_N = 0$.*

The proof of Lemma 5.2 is provided in Section A.4. It is quite technical so we give the main ideas and tools here. It relies on two auxiliary results for T-CKLS processes: Propositions 5.4-5.5, whose proofs are postponed to Section A.5 and Section A.6 respectively. Proposition 5.4 is a property very commonly used in statistics for diffusion processes.

Proposition 5.4. *Assume that the T-CKLS X is stationary. Let $j \in \{0, \dots, d\}$ and $m \geq 1$. Assume that the m -th moment of μ is finite. Then there exists a constant $C \in (0, \infty)$ such that for all $0 \leq s < t$ it holds $\mathbb{E} [|X_t - X_s|^m] \leq C(t-s)^{m/2}$.*

Proposition 5.5, instead, is the remedy to the lack of knowledge of the finite dimensional distributions of TDs. Indeed, the quantities $M^{j\cdot}, Q^{j\cdot}$ consider only observations taking values on I_j together with their following observation. Hence, in the proof of Lemma 5.2, one needs to bound the probability that the process crossed a threshold between two consecutive observations.

Proposition 5.5. *Assume that the T-CKLS X is stationary. Let $j \in \{0, \dots, d\}$ and $m \in \mathbb{R}$, such that the m -th moment of μ is finite on the set I_j . Then there exists a constant $C \in (0, \infty)$ such that for all $0 \leq s < t$, we have:*

$$\mathbb{E} \left[|X_s|^m \mathbb{P}_{X_s} \left(\tau_{I_j}^{\xi^{(j)}} < t - s \right) \mathbf{1}_{I_j}(X_s) \right] \leq C(t - s)^{1/2},$$

where $\xi^{(j)}$ is a CKLS process with parameters $(a_j, b_j, \sigma_j, \gamma_j)$ starting at X_s and driven by a Brownian motion independent of \mathcal{F}_s (denoted by B as well), and $\tau_{I_j}^{\xi^{(j)}}$ is the first hitting time of the interval I_j for the process $\xi^{(j)}$.

Differences with respect to related literature. Let us first consider standard diffusions related to T-CKLS, for instance CIR process. The proofs in [3, 4] rely on the knowledge of the law of the process, and some other quantities such as the integral $\int_0^t (X_s)^{-1} ds$. We do not have access to the law of T-CKLS, nor even of T-drifted Brownian motion. Under the ergodicity assumption, in the context of discrete observations our Theorem 4.4, recovers and improves the existing result for CIR.

The proof strategy that we just illustrated above, is analogous to the one exploited in [24] for drift estimation of T-OU. Nevertheless, we believe that the way we deal with controlling the probability of crossing a threshold between two consecutive observations (proof of Proposition 5.5) is the key to extend the results of this paper to more general TDs. Hence, in our opinion, Proposition 5.5 is one of the most relevant results of this document.

5.3 Comments on assumptions

Assumptions in the continuous setting: $\mathbf{H}_{q-\mathcal{L}}$ and $\mathbf{H}_{\mathcal{L}}$. In the case of T-OU process (solution to (2.1) with $\gamma \equiv 0$), $\mathbf{H}_{q-\mathcal{L}}$ and $\mathbf{H}_{\mathcal{L}}$ rewrite as follows: μ admits finite second moment. When T-OU is ergodic (see Table 4), μ admits moments of all order (see Proposition B.2), so $\mathbf{H}_{q-\mathcal{L}}$ and $\mathbf{H}_{\mathcal{L}}$ hold. Indeed, for the asymptotic results in [24], no additional assumptions have been introduced.

For CIR process (solution to the SDE (1.1), with $\gamma = 1/2$). The asymptotic behavior of the MLE of an ergodic standard CIR process with $a, b > 0$ is studied in [4] under the additional condition $a > \sigma^2/2$. This condition ensures that μ admits finite (-1) -moment (see Proposition B.2) and so $\mathbf{H}_{\mathcal{L}}$ is satisfied. Indeed, since $b > 0$ the stationary measure of the CIR process admits positive moments of all order (see Proposition B.2).

In a similar way, in [27], the author studies the asymptotic behavior of the MLE of a CKLS process (solution to the SDE (1.1)) in the ergodic case with $a, b > 0$ and $\gamma \in (1/2, 1)$. In this case, the stationary distribution of the CKLS process admits moments of all order (see [27, Proposition 2.1] or next Proposition B.2). Therefore, $\mathbf{H}_{q-\mathcal{L}}$ and $\mathbf{H}_{\mathcal{L}}$ are satisfied.

We improve the conditions in [30], where consistency and asymptotic normality of the QMLE have been proven. More precisely, in [30], the process is supposed to be stationary and geometrically ergodic and they require finiteness of the fourth order moment. We only assume ergodicity, and, for the QMLE, assumption $\mathbf{H}_{q-\mathcal{L}}$ is less restrictive than the existence of the fourth order moment.

Assumptions in the discrete setting: the discretization choice. Let us note that in the likelihood function (4.3) there could be a term such as $M_T^{0,-1}$ (take $\gamma_0 = 1/2$). If

one takes $M_{T,N}^{0,-1}$ instead of $\mathcal{M}_{T,N}^{0,-1}$ for the discretization of the likelihood function, then one should prove the convergence in Lemma 5.2 for it. Following our proof, this requires stronger hypothesis than the ones we consider in $\mathcal{H}_{\mathcal{L}}$. Let us mention that the idea of considering a different expression is inspired by [4], which deals with non-threshold CIR process.

In the case of the QMLE estimator, the replacement of $M_{T,N}^{j,1}$ by $\mathcal{M}_{T,N}^{j,1}$ makes no sense if σ_* is unknown because the estimator (4.7) of σ_* depends on $M_{T,N}^{j,1}$, and, if σ_* is known, the replacement would not allow for better assumptions (see Remark A.5).

Assumptions in the discrete setting: asymptotic normality. In the case $\gamma \equiv 0$ (T-OU) or $\gamma \equiv \frac{1}{2}$ (T-CIR), the hypotheses $\lim_{N \rightarrow \infty} T_N \Delta_N = 0$ has been considered for the asymptotic normality of estimators for CIR in [4], T-OU process and T-drifted Brownian motion in [24, 25]. The condition degrades when the diffusion coefficient is non-linear in some interval, *i.e.* when there exists $j \in \{0, \dots, d\}$ such that γ_j is not equal to 0, 1/2, or 1. Indeed, $\Delta_N = g_N^{(q-\mathcal{L})} \leq g_N^{(\sigma)} \leq g_N^{(\mathcal{L})}$.

The quantities $g_N^{(\mathcal{L})}$ and $g_N^{(\sigma)}$ involve the maximum over all j of some powers of Δ_N depending on γ_j . Indeed, for each interval I_j we get a condition for convergence on Δ_N depending on γ_j (for a rigorous proof, see Section A.4), and, taking the most restrictive condition, corresponds to taking the maximum.

Let us comment on the Hypothesis $\mathcal{H}_{\mathcal{L}}$. Finiteness of moments for the stationary measure μ is summarized in Proposition B.2. When $\gamma_0 = 0$ and $\gamma^{\max} < 1/2$, $\mathcal{H}_{\mathcal{L}}$ is satisfied. When $\gamma_0 = 1/2$ and $b_d > 0$, $\mathcal{H}_{\mathcal{L}}$ is satisfied if $a_0 > \sigma_0^2$. Indeed, $q_{\mathcal{L}}$ in hypothesis $\mathcal{H}_{\mathcal{L}}$ can be taken such that $2 < q_{\mathcal{L}} < 2a_0/\sigma_0^2$. This condition is the one in [4, Proposition 5] in the case of the CIR process (solution to (1.1) with $\gamma = 1/2$).

Theorem 4.4 when σ_* is known. When σ_* is known, there is no need to replace it by its estimator. Theorem 4.4 holds with weaker assumptions, we show this in Section A.4.3. The case of the QMLE is discussed in Remark A.5. Nevertheless, the assumption $\mathcal{H}_{q-\mathcal{L}}$ have to be more restrictive than $\mathbf{H}_{q-\mathcal{L}}$ and it is already equal to it. The case of the MLE is developed in details in the paragraph for the proof of Item (c) (see Section A.4.3).

6 Numerical Experiments

In this section, we implement the MLE and QMLE based on discrete observations on simulated and US interest rates data.

6.1 Simulated Data

In this section, we investigate the efficiency of our estimators on simulated data. We simulate the T-CKLS process combining known Euler-Type schemes on different intervals such as the scheme in [1, equation (3)] when the process is a CIR or a drifted version of the scheme in [32] when the diffusion coefficient is non-linear. More precisely we use the following scheme. Given $X_0 \in (0, \infty)$ and $(G_k)_{k \in \mathbb{N}}$ a sequence of i.i.d. standard Gaussian random variables, next we set $X_0^{(n)} = X_0$ and, we define for all $k \in \mathbb{N}$

$$X_{(k+1)/n}^{(n)} := \left| X_{k/n}^{(n)} + \frac{1}{n} \left(a(X_{k/n}^{(n)}) - b(X_{k/n}^{(n)})X_{k/n}^{(n)} \right) + \frac{1}{\sqrt{n}} \sigma(X_{k/n}^{(n)}) (X_{k/n}^{(n)})^{\gamma(X_{k/n}^{(n)})} G_k \right|.$$

Further discussion about the most suitable numerical scheme for T-CKLS is beyond the purpose of this document. To estimate the parameters from the simulated data, we use the estimators from discrete observations in Section 4.1. The implementation has been done using Matlab and the parameters are as in Table 1.

a_0	b_0	σ_0	γ_0	a_1	b_1	σ_1	γ_1
0.3	0.2	0.2	0.5	0	0	0.4	0
a_2	b_2	σ_2	γ_2	r_1	r_2		
0.3	0.2	0.2	0.5	1	1.5		

Table 1: Simulations parameters.

Firstly, we illustrate Theorem 4.4 for the drift parameters $\theta_\star = (a_i, b_i)_{i=0}^{d-2}$, diffusion parameters σ_\star, γ and two thresholds r_1, r_2 given in Table 1. We consider a process, which follows a CIR dynamic close to 0 and far away from 0 and is a BM on an intermediate bounded interval. We simulate 10^4 trajectories of the T-CKLS with two threshold. The set of numerical parameters is $(T, N) = (10^3, 10^6)$ with starting condition determined as follows. As the process is supposed to be stationary, we first simulate one trajectory starting from $X_0 > 0$ chosen arbitrarily, say $X_0 = 1$, with the scheme $X^{(n)}$ given above with $n = kN/T$ for some $k \in \mathbb{N} \setminus \{0\}$ (we took $k = 1$). Then, we consider the final value of the latter trajectory as initial condition of the 10^4 trajectories.

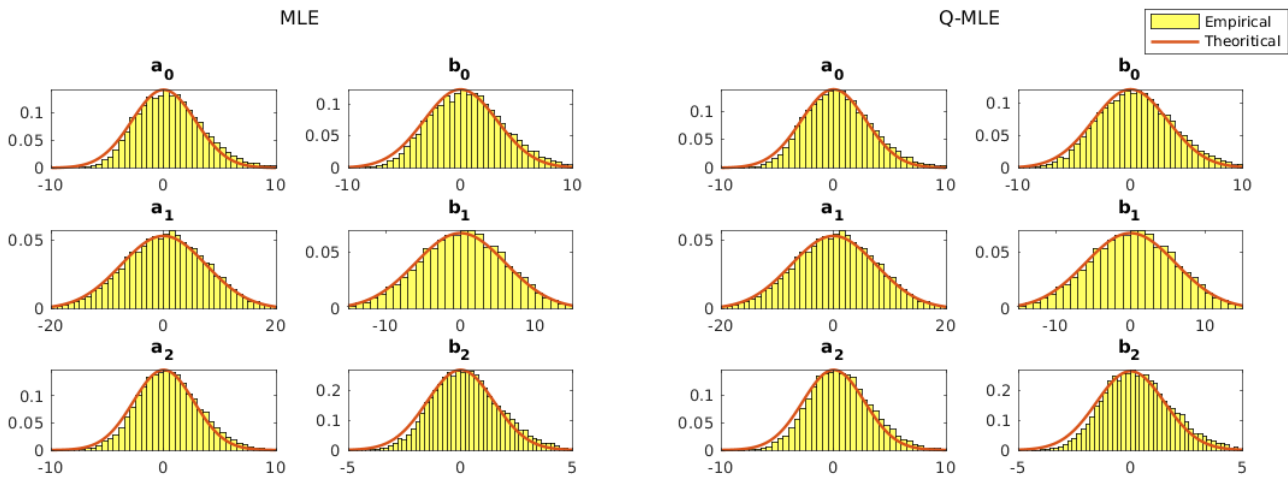


Figure 1: Asymptotic normality property in Theorem 4.4, with parameters as in Table 1. We plot the theoretical distribution using (A.1) and compare with the empirical distribution on 10^4 trajectories.

Remark that, despite the fact that the set of numerical parameters (T, N) does not satisfy the conditions for the asymptotic normality in Theorem 4.4 ($T^2 = N$ instead of $T^2 \ll N$), numerics show good results.

We compare the estimators by means of relative root-mean-square error (RMSE) and mean-error (ME) in Table 2.

Estimator	Relative RMSE	ME
$(a_{T_N,N}^{0,(\mathcal{L})}, a_{T_N,N}^{1,(\mathcal{L})}, a_{T_N,N}^{2,(\mathcal{L})})$	(0.3206, 0.2341, 0.2921)	(0.0154, 0.0092, 0.0156)
$(b_{T_N,N}^{0,(\mathcal{L})}, b_{T_N,N}^{1,(\mathcal{L})}, b_{T_N,N}^{2,(\mathcal{L})})$	(0.5500, 0.1857, 0.2439)	(0.0173, 0.0069, 0.0093)
$(a_{T_N,N}^{0,(q-\mathcal{L})}, a_{T_N,N}^{1,(q-\mathcal{L})}, a_{T_N,N}^{2,(q-\mathcal{L})})$	(0.3233, 0.2341, 0.2965)	(0.0170, 0.0092, 0.0191)
$(b_{T_N,N}^{0,(q-\mathcal{L})}, b_{T_N,N}^{1,(q-\mathcal{L})}, b_{T_N,N}^{2,(q-\mathcal{L})})$	(0.5555, 0.1857, 0.2478)	(0.0182, 0.0069, 0.0112)
$(\sigma_{T_N,N}^0, \sigma_{T_N,N}^1, \sigma_{T_N,N}^2)$	(0.0088, 0.0087, 0.0015)	(0.0087, -0.0034, 0.001)

Table 2: Table of the relative RMSE and ME for the estimator of (θ_*, σ_*) using the MLE (4.5), QMLE (4.6) and volatility estimator (4.7).

Observe that the MLE gives a better estimation of the drift parameters, which is easily explained by the fact that the likelihood contains more information about the model. In general the MLE tends to have a better RMSE and ME than the QMLE. Applying the estimator on several data sets, we remark that the QMLE has a greater variance than the MLE.

Remark 6.1 (Threshold estimation). *The thresholds can be estimated using the method proposed in [30] based on continuous observations. We use this procedure in the next section. It is a QMLE-based method without an explicit expression for the threshold estimator. Consequently, the numerical cost of this method increases with the number of thresholds. However, satisfactory results can still be achieved in the case of two thresholds. Mathematical study of the estimator for discrete observations should be conducted.*

Remark 6.2 (Diffusion exponent estimation). *The parameter γ was estimated in [27] for non-threshold CKLS. A study on the estimator properties for discrete observations in the context of CKLS or its generalisation to T-CKLS is necessary. However, we expect that the estimator converges quite slowly, especially around 0.*

6.2 Interest rates analysis

In this section, we apply our estimators to the ten year US treasury rate based on the Federal Reserve Bank's H15 data set. We apply a discretized version of the test to evaluate the existence of one threshold (see [30], see Remark 6.1) or more thresholds (see [31]) in this dataset. In [31], the authors introduce a test statistic to identify thresholds in the drift term of a diffusion model: detect their presence and estimate them. They also develop a computationally efficient approach to calibrate the p-value and extend the test to detect multiple thresholds. Inspired by [33], we exploit here a discretized version of the method presented in [31] and combine with the drift MLE and σ_* estimator considered in this document.

General step. Let us describe the test to be applied at a general step and then the procedure describing the steps. Suppose that there are m thresholds, and that we look for the presence of an additional threshold on the k -th interval $I_k = (r_{k-1}, r_k)$ (we know the value of r_{k-1}, r_k : either known or estimated in previous steps). We consider the

hypothesis:

$$\begin{cases} H_0 : \text{Null hypothesis} & m \text{ thresholds;} \\ H_1 : \text{Alternative hypothesis} & (m + 1) \text{ thresholds.} \end{cases} \quad (6.1)$$

Under the null hypothesis H_0 , the model has m thresholds. Under hypothesis H_1 , there is an additional threshold \bar{r} in the k -th regime, meaning that the sequence of thresholds becomes $-\infty = r_0 < r_1 < \dots < r_{k-1} < \bar{r} < r_k < \dots < r_m$. The quasi-likelihood ratio test statistic is given by

$$T = \sup_{\bar{r} \in [a, b]} T(\bar{r})$$

with

$$T(\bar{r}) := 2 \left(q\mathcal{L}_{T_N, N} \left(\theta_{T_N, N}^{(H_1)}(m + 1, \bar{r}) \right) - q\mathcal{L}_{T_N, N} \left(\theta_{T_N, N}^{(H_0)}(m) \right) \right) \quad (6.2)$$

where a and b are 20 and 80 percentiles of the data in $[r_{k-1}, r_k]$, the value $\theta_{T_N, N}^{(H_1)}(m + 1, \bar{r})$ is the drift MLE under hypothesis H_1 with the additional threshold given by \bar{r} , and $\theta_{T_N, N}^{(H_0)}(m)$ is the drift MLE of the model under hypothesis H_0 . The MLE is given in Proposition 4.1, with volatility parameter σ_* estimated by (4.7).

We compute the statistics for some values of \bar{r} , say $\bar{r}_j := a(1 - j/n) + bj/n$, $j \in \{0, 1, \dots, n\}$ (we choose $n = 10^3$). Next, we take as an estimator for the threshold r_j the \bar{r}_j which maximises $T(\bar{r})$ and the observed test statistics T_{data} is then the quantity $T(\bar{r}_j)$.

The distribution of the test statistic (6.2) is obtained using a bootstrap method. To compute the p-value we simulate 10^3 trajectories of the process with the parameters under H_0 , we compute T_j the statistics on the j -th trajectory. Then the p-value is given by $\#\{j: T_{data} < T_j\}/10^3$.

We fix the significance level at the conventional 5%.

Procedure. We apply the test above as follows, in a sequential procedure. We first test for the presence of a threshold on the data: applying this test for $m = 0$. If the test is significant, then we take as an estimator for the threshold \hat{r}_1 the \bar{r} which realises the maximum in the test statistics. This threshold divides the state space into two intervals. We then test the presence of thresholds on each of the two intervals, starting from the left to the right. On each interval, if the test is significant, we keep dividing the interval into two sub-intervals and so on. Once we do not have evidence of new thresholds in the interval we are considering, we go to the next interval.

Application to ten year US treasury rate. We consider the ten year US Treasury rate and we adopt the convention that the daily time intervals is $dt = 0.046$, where one unit of time represents one month. We assume that the data follow a T-CIR dynamics, *i.e.* $\gamma \equiv 1/2$.

We consider the ten year US Treasury rate for two different time window: Jan 2016 - Dec 2019, and Jan 2020 - Jan 2024 represented in Figure 2.

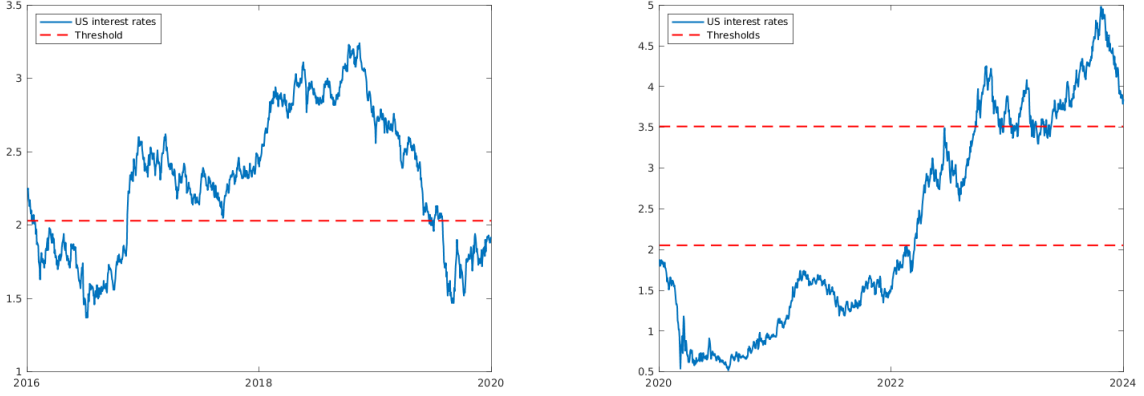


Figure 2: The figure shows the interest rate daily data (solid line) for the time window Jan 2016 - Dec 2019, and Jan 2020 - Jan 2024. The fitted thresholds are represented by the dashed lines.

Let us consider the time window, Jan 2016 - Dec 2019. The threshold test (6.1) for $m = 0$ is significant and the threshold estimation is $r_1 = 2.0303$. We apply the test (6.1) to detect a threshold on $(0, r_1)$. It is not significant. The same conclusion holds testing for threshold presence on $(r_1, +\infty)$.

On the time window, Jan 2020 - Jan 2024, the threshold test (6.1) for $m = 0$ is significant and we estimate the threshold $r_1 = 2.0507$. There is no evidence of additional thresholds on $(0, r_1)$, and the null hypothesis is rejected for the existence of a threshold $r_2 = 3.5112$ on $(r_1, +\infty)$. Instead, for the tests (6.1) with $m = 2$ for finding a threshold on (r_1, r_2) and $(r_2, +\infty)$, the null hypothesis H_0 is not-rejected.

Jan 2016 - Dec 2019		Jan 2020 - Jan 2024	
Estimator	Value	Estimator	Value
$(a_{T_N, N}^{0,(\mathcal{L})}, a_{T_N, N}^{1,(\mathcal{L})})$	(1.6434 , 0.1713)	$(a_{T_N, N}^{0,(\mathcal{L})}, a_{T_N, N}^{1,(\mathcal{L})}, a_{T_N, N}^{2,(\mathcal{L})})$	(0.2013, 0.5826, -0.0207)
$(b_{T_N, N}^{0,(\mathcal{L})}, b_{T_N, N}^{1,(\mathcal{L})})$	(0.9410, 0.0723)	$(b_{T_N, N}^{0,(\mathcal{L})}, b_{T_N, N}^{1,(\mathcal{L})}, b_{T_N, N}^{2,(\mathcal{L})})$	(0.1556, 0.0670, 0.0236)
$(\sigma_{T_N, N}^0, \sigma_{T_N, N}^1)$	(0.1616, 0.1053)	$(\sigma_{T_N, N}^0, \sigma_{T_N, N}^1, \sigma_{T_N, N}^2)$	(0.2129, 0.2091, 0.1807)
$r_{T_N, N}^1$	(2.0303)	$(r_{T_N, N}^1, r_{T_N, N}^2)$	(2.0507, 3.5112)

Table 3: Estimated parameters corresponding to Figure 2.

Therefore, we conclude that there is a single threshold in the time window, Jan 2016 - Dec 2019, and two thresholds in Jan 2020 - Jan 2024. In Table 3, we summarize the values obtained for each of the fitted parameters using the estimators.

A Appendix: Proofs

In this section, recalling some well known results, we show under which conditions, on the parameters, the process admits a stationary distribution. Then, we prove Theorem 4.3, Theorem 4.4, Lemma 5.2, Proposition 5.4 and Proposition 5.5.

A.1 The process: properties of solutions and conditions for the stationary distribution

T-CLKS shows several behaviors, it may behave as an OU process on some intervals, a CIR or CKLS in others. The state space of T-CKLS is determined by the process behavior around 0, in particular at I_0 where it behaves as a standard CKLS process. The regime of the process (transient, recurrent, positive-recurrent) is also determined by the behavior at I_0 and I_d .

Let us introduce the scale function S and the speed measure $m(x) dx$. The interested reader could refer to [6, II.4] for a summary or find more details e.g. in [29, Chapter VII, Section 3]. The scale function is continuous, unique up to a multiplicative constant, and its derivative satisfies $S'(x) = \exp\left(-\int_{r_1}^x \frac{2(a(y)-b(y)y)}{\sigma(y)^2 y^{2\gamma(y)}} dy\right)$. The speed measure is given by $m(x) dx = \frac{2}{(\sigma(x))^2 |x|^{2\gamma(x)} S'(x)} dx$. The state of space of the T-CKLS process, denoted $I = \cup_{j=0}^d I_j$, depends on the value of the parameters in I_0 (a_0 , σ_0 and γ_0). When $\gamma_0 \in [1/2, 1]$, a suitable comparison theorem for SDEs ensures that the process is non-negative. This, together with an adaptation of the proof of Proposition 1.5 in [29, Chapter XI] and the Feller boundary classification criteria (see e.g. [29]), imply the following lemma.

Lemma A.1. *Let X be the solution to the SDE (2.1).*

- *If $\gamma_0 = 0$, the state of space of the process is $I = \mathbb{R}$.*
- *If $\gamma_0 = 1/2$ and $0 < a_0 < \sigma_0^2/2$, the state space is $I = [0, +\infty)$ and the point 0 is instantaneously reflecting.*
- *If $\gamma_0 \in (1/2, 1]$ or if $\gamma_0 = 1/2$ and $a_0 \geq \sigma_0^2/2$, then $I = (0, +\infty)$ and 0 is an unattainable boundary.*

The regime of the process can be obtained by properties on the scale function and the speed measure (see e.g. [29, Exercice 3.15 in Chapter X] and [16, Theorem 20.15]). We recall that, in the recurrent case, the measure $m(x) dx$ is a stationary measure and the fact that $m(x) dx$ is a finite measure corresponds to positive recurrence (*ergodicity*) of the process. In the ergodic case, the stationary measure can be renormalized to the stationary distribution:

$$\mu(dx) = \frac{m(x)}{\int_I m(y) dy} dx. \quad (\text{A.1})$$

The recurrent positivity property of the process only depends on the parameters below the first threshold (on I_0) and above the last threshold (on I_d). In the following table, we give conditions on the parameters a_0, a_d, b_0, b_d and σ_0, σ_d depending on the value of γ_0, γ_d such that the process is ergodic (admits a stationary distribution).

$\gamma_0 = 0$	$a_0 \in \mathbb{R}$ and $b_0 > 0$, or $a_0 > 0$ and $b_0 = 0$		$a_d \in \mathbb{R}$ and $b_d > 0$, or
$\gamma_0 \in [1/2, 1)$	$a_0 > 0$ and $b_0 \in \mathbb{R}$	$\gamma_d \in [0, 1/2]$	$a_d < 0$ and $b_d = 0$
$\gamma_0 = 1$	$a_0 > 0$ and $b_0 \in \mathbb{R}$, or $a_0 = 0$ and $b_0 < -\sigma_0^2/2$	$\gamma_d \in (1/2, 1)$	$a_d \in \mathbb{R}$ and $b_d = 0$
		$\gamma_d = 1$	$a_d \in \mathbb{R}$ and $b_d \in (-\sigma_d^2/2, 0]$

Table 4: Parameter conditions for ergodicity of T-CKLS process X solution to (2.1).

A.2 Proof of Theorem 4.3

This proof relies on Lemma 5.2 whose proof is provided in Section A.4.

We study the asymptotic behavior of $(\sigma_{T_N,N})^2 - \sigma_\star^2$. For $j \in \{0, \dots, d\}$, by Proposition 3.4, on the event $\{Q_T^{j,0} > 0\}$, we have:

$$(\sigma_{T_N,N}^j)^2 - \sigma_j^2 = \sigma_j^2 \frac{Q_{T_N}^{j,2\gamma_j} - Q_{T_N,N}^{j,2\gamma_j}}{Q_{T_N,N}^{j,2\gamma_j}} + \frac{\mathbf{Q}_{T_N,N}^j - \mathbf{Q}_{T_N}^j}{Q_{T_N,N}^{j,2\gamma_j}}.$$

By equations (3.5)-(3.6), we have:

$$|\mathbf{Q}_{T_N}^j - \mathbf{Q}_{T_N,N}^j| \leq 2 |M_{T_N}^{j,1} - M_{T_N,N}^{j,1}| + 2 \max(|r_j|, |r_{j+1}|) \sum_{i=j}^d |M_{T_N}^{i,0} - M_{T_N,N}^{i,0}|,$$

for $j \in \{1, \dots, d-1\}$ and for $j \in \{0, d\}$

$$|\mathbf{Q}_{T_N}^j - \mathbf{Q}_{T_N,N}^j| \leq 2 |M_{T_N}^{j,1} - M_{T_N,N}^{j,1}| + 2|r_{(j+1) \wedge d}| |M_{T_N}^{j,0} - M_{T_N,N}^{j,0}|.$$

Then, we conclude by making use of item (b) in Lemma 5.2 and the fact that $\mathbb{P}(\lim_{T \rightarrow +\infty} Q_T^{j,0} > 0) = 1$.

A.3 Proof of Theorem 4.4

This proof relies on Lemma 5.2 that we prove in the next section. For all $N \in \mathbb{N}$ and $\ell \in \{\mathcal{L}, q\mathcal{L}\}$ it holds that

$$\left(\theta_{T_N,N}^{(\ell)} - \theta_\star\right) = \left(\theta_{T_N,N}^{(\ell)} - \theta_{T_N}^{(\ell)}\right) + \left(\theta_{T_N}^{(\ell)} - \theta_\star\right).$$

The second term, on the right hand side of the equality, provides the asymptotic behavior by applying Theorem 3.5.

In the case $\ell = \mathcal{L}$, for $j \in \{0, \dots, d\}$, using equations (3.4) and (4.5), each component of the first term can be rewritten as follows:

$$\begin{aligned} & \left(\frac{Q_{T_N,N}^{j,k}}{Q_{T_N,N}^{j,-2\gamma_j} Q_{T_N,N}^{j,2-2\gamma_j} - (Q_{T_N,N}^{j,1-2\gamma_j})^2} - \frac{Q_{T_N}^{j,k}}{Q_{T_N}^{j,-2\gamma_j} Q_{T_N}^{j,2-2\gamma_j} - (Q_{T_N}^{j,1-2\gamma_j})^2} \right) \mathcal{M}_{T_N}^{j,m} \\ & + \frac{Q_{T_N,N}^{j,k} (\mathcal{M}_{T_N,N}^{j,m} - \mathcal{M}_{T_N}^{j,m})}{Q_{T_N,N}^{j,-2\gamma_j} Q_{T_N,N}^{j,2-2\gamma_j} - (Q_{T_N,N}^{j,1-2\gamma_j})^2}, \end{aligned}$$

with $k \in \{-2\gamma_j, 1-2\gamma_j, 2-2\gamma_j\}$ and $m \in \{-2\gamma_j, 1-2\gamma_j\}$. Let $\bar{r} := \max\{|r_j| : j = 1, \dots, d\}$. Then, using formula (5.1), (5.2) and (5.3), for $j \in \{1, \dots, d-1\}$, we have

$$|\mathcal{M}_{T_N}^{j,m} - \mathcal{M}_{T_N,N}^{j,m}| \leq \bar{r}^m \sum_{i=j}^d |M_{T_N}^{i,0} - M_{T_N,N}^{i,0}| + \frac{|m|}{2} \left| (\sigma_{T_N,N}^j)^2 - \sigma_j^2 \right| \left| Q_{T_N}^{j,m+2\gamma_j-1} - Q_{T_N,N}^{j,m+2\gamma_j-1} \right|,$$

and for $j \in \{0, d\}$:

$$|\mathcal{M}_{T_N}^{j,m} - \mathcal{M}_{T_N,N}^{j,m}| \leq \bar{r}^m |M_{T_N}^{j,0} - M_{T_N,N}^{j,0}| + \frac{|m|}{2} \left| (\sigma_{T_N,N}^j)^2 - \sigma_j^2 \right| \left| Q_{T_N}^{j,m+2\gamma_j-1} - Q_{T_N,N}^{j,m+2\gamma_j-1} \right|.$$

This, Item (c) in Lemma 5.2, and Theorem 4.3 ensure that

$$T_N^{-1/\lambda} \left| Q_{T_N,N}^{j,k} - Q_{T_N}^{j,k} \right| \xrightarrow[N \rightarrow +\infty]{\mathbb{P}} 0 \quad \text{and} \quad T_N^{-1/\lambda} \left| \mathcal{M}_{T_N,N}^{j,m} - \mathcal{M}_{T_N}^{j,m} \right| \xrightarrow[N \rightarrow +\infty]{\mathbb{P}} 0$$

with $\lambda = 1$ to get the consistency of the MLE and $\lambda = 2$ for the speed of convergence.

The case $\ell = q\mathcal{L}$, the first term works analogously, exploiting Item (a) in Lemma 5.2.

A.4 Proof of Lemma 5.2

We prove (5.5) under the assumptions corresponding to the different possible values of k and m appearing in the different estimators we consider.

Let us recall that (4.8) holds and that the process is stationary, where μ denotes the stationary distribution. Let us introduce the round ground notation $\lfloor t \rfloor_{\Delta_N} := t_k$ for $t \in [t_k, t_{k+1}) \subseteq [t_k, t_k + \Delta_N]$. Moreover, without loss of generality, we assume $T_N \leq N$ and $\Delta_N \leq 1$ for all $N \in \mathbb{N}$.

Analogously to [24], the proof relies on Proposition 5.4 and 5.5, that we prove in the next sections. Basically, for $\lambda \in \{1, 2\}$, the proof of Lemma 5.2 reduces to prove that the following integrals are $o(T_N^{1/\lambda})$

$$\int_0^{T_N} \mathbb{E} \left[|X_t - X_{\lfloor t \rfloor_{\Delta_N}}|^p \right] dt \quad \text{and} \quad (\text{A.2})$$

$$\int_0^{T_N} \mathbb{E} \left[|X_{\lfloor t \rfloor_{\Delta_N}}|^q \mathbf{1}_{X_t \notin I_j, X_{\lfloor t \rfloor_{\Delta_N}} \in I_j} \right] dt \quad \text{and} \quad \int_0^{T_N} \mathbb{E} \left[|X_{\lfloor t \rfloor_{\Delta_N}}|^q \mathbf{1}_{X_t \in I_j, X_{\lfloor t \rfloor_{\Delta_N}} \notin I_j} \right] dt \quad (\text{A.3})$$

for some suitable $p > 0$ and $q \in \mathbb{R}$ and specific assumptions, all depending on the value of m and k appearing in $M^{j,m}$ and $Q^{j,k}$.

Proposition 5.4 and Proposition 5.5, once they can be applied (i.e. if μ admits finite moments of order $\max(p, 1)$ and q respectively), yield upper bounds involving T_N and Δ_N for (A.2) and (A.3) :

$$\int_0^{T_N} \mathbb{E} \left[|X_t - X_{\lfloor t \rfloor_{\Delta_N}}|^p \right] dt \leq \int_0^{T_N} \mathbb{E} \left[|X_t - X_{\lfloor t \rfloor_{\Delta_N}}|^{\max(p,1)} \right]^{\min(p,1)} dt \leq CT_N \Delta_N^{p/2} \quad (\text{A.4})$$

and

$$\int_0^{T_N} \mathbb{E} \left[|X_{\lfloor t \rfloor_{\Delta_N}}|^q (\mathbf{1}_{X_t \notin I_j, X_{\lfloor t \rfloor_{\Delta_N}} \in I_j} + \mathbf{1}_{X_t \in I_j, X_{\lfloor t \rfloor_{\Delta_N}} \notin I_j}) \right] dt \leq CT_N \sqrt{\Delta_N} \quad (\text{A.5})$$

for some positive constant C independent of N . The assumptions on $g_N^{(\cdot)}$ ensure that these quantities are $o(T_N^{1/\lambda})$.

The proof of each item of Lemma 5.2 is then reduced to determine the corresponding values of p and q in (A.4) and (A.5) and so deduce the assumptions $\mathcal{H}_{\mathcal{L}}$ and $\mathcal{H}_{q-\mathcal{L}}$ on moments of μ and the conditions on $g_N^{(\cdot)}$. This is done in the next sections. We consider separately the two quantities in (5.5) and then we summarize the items of Lemma 5.2 in Section A.4.3.

To conclude the section, for the reader's convenience, let us rewrite equation (A.3) in such a way that it can be more easily recognised that Proposition 5.5 can be applied. The tower property of conditional expectation and Markov property imply that

$$\mathbb{E} \left[|X_{\lfloor t \rfloor_{\Delta_N}}|^q \mathbf{1}_{X_t \notin I_j, X_{\lfloor t \rfloor_{\Delta_N}} \in I_j} \right] \leq \mathbb{E} \left[|X_{\lfloor t \rfloor_{\Delta_N}}|^q \mathbb{P}_{X_{\lfloor t \rfloor_{\Delta_N}}} \left(\tau_{I_j}^{\xi^{(j)}} < \Delta_N \right) \mathbf{1}_{I_j}(X_{\lfloor t \rfloor_{\Delta_N}}) \right],$$

and

$$\mathbb{E} \left[|X_{\lfloor t \rfloor_{\Delta_N}}|^q \mathbf{1}_{X_t \in I_j, X_{\lfloor t \rfloor_{\Delta_N}} \notin I_j} \right] \leq \sum_{\substack{i=0 \\ i \neq j}}^d \mathbb{E} \left[|X_{\lfloor t \rfloor_{\Delta_N}}|^q \mathbb{P}_{X_{\lfloor t \rfloor_{\Delta_N}}} \left(\tau_{I_i}^{\xi^{(i)}} < \Delta_N \right) \mathbf{1}_{I_i}(X_{\lfloor t \rfloor_{\Delta_N}}) \right],$$

where $\xi^{(j)}$ is a CKLS process with parameter $(a_j, b_j, \sigma_j, \gamma_j)$ starting at $\xi_0^{(j)} = X_{\lfloor t \rfloor_{\Delta_N}}$ but driven by a BM, (denoted by B as well), independent of $\mathcal{F}_{\lfloor t \rfloor_{\Delta_N}}$ and $\tau_{I_j}^{\xi^{(j)}}$ is the first hitting time of the interval I_j for the process $\xi^{(j)}$. The right-hand-side of the latter equations is of the same kind of (A.3).

A.4.1 Dealing with $\mathbb{E} \left[|Q_{T_N}^{j,k} - Q_{T_N,N}^{j,k}| \right]$.

In this section, exploiting (A.4)-(A.5), we precise under which assumptions on the moments of μ and on Δ_N , it holds that $\mathbb{E} \left[|Q_{T_N}^{j,k} - Q_{T_N,N}^{j,k}| \right]$ is $o(T_N^{1/\lambda})$ for $k \in [-2, 2]$, $j \in \{0, \dots, d\}$. Let us remind that we are interested in $k \in \{-2\gamma_j, 1 - 2\gamma_j, 2 - 2\gamma_j, 2\gamma_j\} \cup \{-1, 0, 1, 2\}$.

Let us first note that

$$\begin{aligned} Q_{T_N}^{j,k} - Q_{T_N,N}^{j,k} &= \int_0^{T_N} (X_t^k - X_{[t]_{\Delta_N}}^k) \mathbb{1}_{I_j}(X_t) dt + \int_0^{T_N} X_{[t]_{\Delta_N}}^k (\mathbb{1}_{I_j}(X_t) - \mathbb{1}_{I_j}(X_{[t]_{\Delta_N}})) dt \\ &= \int_0^{T_N} X_{[t]_{\Delta_N}}^k (\mathbb{1}_{X_t \in I_j, X_{[t]_{\Delta_N}} \notin I_j} - \mathbb{1}_{X_t \notin I_j, X_{[t]_{\Delta_N}} \in I_j}) dt + \int_0^{T_N} (X_t^k - X_{[t]_{\Delta_N}}^k) \mathbb{1}_{I_j}(X_t) dt, \end{aligned}$$

therefore

$$\begin{aligned} \mathbb{E} \left[|Q_{T_N}^{j,k} - Q_{T_N,N}^{j,k}| \right] &\leq \int_0^{T_N} \mathbb{E} \left[|X_t^k - X_{[t]_{\Delta_N}}^k| \mathbb{1}_{I_j}(X_t) \right] dt \\ &\quad + \int_0^{T_N} \mathbb{E} \left[|X_{[t]_{\Delta_N}}^k|^k \mathbb{1}_{X_t \in I_j, X_{[t]_{\Delta_N}} \notin I_j} \right] dt + \int_0^{T_N} \mathbb{E} \left[|X_{[t]_{\Delta_N}}^k|^k \mathbb{1}_{X_t \notin I_j, X_{[t]_{\Delta_N}} \in I_j} \right] dt. \end{aligned} \quad (\text{A.6})$$

Inequality (A.6) involves only terms of the kind (A.3) with $q = k$ (leading to the bounds in (A.5)) and a term similar to (A.2):

$$\int_0^{T_N} \mathbb{E} \left[|X_t^k - X_{[t]_{\Delta_N}}^k| \mathbb{1}_{I_j}(X_t) \right] dt. \quad (\text{A.7})$$

This term is trivial if $k = 0$, it is bounded from above by (A.2) if $k = 1$, and, in all other cases, we manage to bound (A.7) with functions as (A.2). We then derive the desired conditions by (A.4)-(A.5). Our bounds are not necessarily optimal.

The case $k = 0, 1$. μ admits finite k -th moment and $\lim_{N \rightarrow \infty} T_N^{\lambda-1} \Delta_N = 0$.

The case $k \in (0, 1)$. μ admits finite first moment and $\lim_{N \rightarrow \infty} T_N^{\lambda-1} \Delta_N^k = 0$.

Note that $k \in (0, 1)$ if $\gamma_j \in (0, 1/2)$ for $k = 2\gamma_j$, $k = 1 - 2\gamma_j$ and if $\gamma_j \in (1/2, 1)$ for $k = 2(1 - \gamma_j)$. Sub-additivity of $x \mapsto x^k$ yields (A.4).

Remark A.2. *The inequalities in the next paragraph work for all $k \in (0, 2]$. Nevertheless, when $k \in (0, 1)$ the previous paragraph, requires less restrictive assumptions: $\Delta_N^k \leq \Delta_N^{k/2}$.*

The case $k \in (1, 2]$. μ admits finite k -th moment and $\lim_{N \rightarrow \infty} T_N^{\lambda-1} \Delta_N^{k/2} = 0$.

This is the case if $\gamma_j \in (1/2, 1]$ when $k = 2\gamma_j$, and if $\gamma_j \in [0, 1/2)$ when $k = 2(1 - \gamma_j)$. Cauchy-Schwarz inequality implies

$$\mathbb{E} \left[|X_t^k - X_{[t]_{\Delta_N}}^k| \mathbb{1}_{I_j}(X_t) \right] \leq \sqrt{\mathbb{E} \left[\left(X_t^{k/2} - X_{[t]_{\Delta_N}}^{k/2} \right)^2 \right] \left(\mathbb{E} [X_t^k] + \mathbb{E} [X_{[t]_{\Delta_N}}^k] \right)}.$$

Moreover, the fact that $x \mapsto x^{k/2}$ is sub-additive ensures that

$$\mathbb{E} \left[\left(X_t^{k/2} - X_{[t]_{\Delta_N}}^{k/2} \right)^2 \right] \leq \mathbb{E} \left[|X_t - X_{[t]_{\Delta_N}}|^k \right].$$

The case $k \in [-1, 0)$. μ admits finite moments of order $-q_{\mathcal{L}}$ and $p_{\mathcal{L}}$ with $p_{\mathcal{L}}, q_{\mathcal{L}} \geq 1$ such that $1 = 1/p_{\mathcal{L}} + 2/q_{\mathcal{L}}$, and $\lim_{N \rightarrow \infty} T_N^{\lambda-1} \Delta_N^{|k|} = 0$.

This is the case if $k = -1$ or $k = 1 - 2\gamma_j \in [-1, 0)$ with $\gamma_j \in (1/2, 1]$ or $k = -2\gamma_j$ with $\gamma_j \in (0, 1/2)$. By sub-additivity of $x \mapsto x^{-k}$ and Jensen's inequality, we have

$$\mathbb{E} \left[|X_t^k - X_{[t]_{\Delta_N}}^k| \mathbf{1}_{I_j}(X_t) \right] \leq \mathbb{E} \left[|X_t^{-1} - X_{[t]_{\Delta_N}}^{-1}|^{|k|} \mathbf{1}_{I_j}(X_t) \right] \leq \mathbb{E} \left[|X_t^{-1} - X_{[t]_{\Delta_N}}^{-1}| \mathbf{1}_{I_j}(X_t) \right]^{|k|}.$$

Hölder's inequality with $1 = 1/p_{\mathcal{L}} + 2/q_{\mathcal{L}}$ gives

$$\mathbb{E} \left[|X_t^{-1} - X_{[t]_{\Delta_N}}^{-1}| \mathbf{1}_{I_j}(X_t) \right] \leq \mathbb{E} \left[|X_t - X_{[t]_{\Delta_N}}|^{p_{\mathcal{L}}} \right]^{1/p_{\mathcal{L}}} \mathbb{E} \left[|X_t^{-1}|^{q_{\mathcal{L}}} \right]^{1/q_{\mathcal{L}}} \mathbb{E} \left[|X_{[t]_{\Delta_N}}^{-1}|^{q_{\mathcal{L}}} \right]^{1/q_{\mathcal{L}}}.$$

The case $k \in [-2, -1)$. μ admits finite moments of order $-q'|k|$ and $\max(1, p'|k|/2)$ with $p', q' \geq 1$ such that $1 = 1/p' + 1/q' + 1/(2q')$ and $\lim_{N \rightarrow \infty} T_N^{\lambda-1} \Delta_N^{|k|/2} = 0$.

This is the case when $k = -2\gamma_j$ with $\gamma_j \in (1/2, 1]$. Note that what follows holds for all $k \in [-2, 0)$ as well. Observe that

$$\mathbb{E} \left[|X_t^k - X_{[t]_{\Delta_N}}^k| \right] \leq \mathbb{E} \left[|X_t^{k/2} - X_{[t]_{\Delta_N}}^{k/2}| |X_t^{k/2} + X_{[t]_{\Delta_N}}^{k/2}| \right]$$

This rewrites $\mathbb{E} \left[|X_t^{|k|/2} - X_{[t]_{\Delta_N}}^{|k|/2}| |X_t^{k/2} X_{[t]_{\Delta_N}}^{k/2} + X_t^{k/2} X_{[t]_{\Delta_N}}^k| \right]$. The fact that $x \mapsto x^{|k|/2}$ is sub-additive and Hölder's inequality (with $p', q' \geq 1$ such that $1 = 1/p' + 1/q' + 1/(2q')$, e.g. $p' = 4, q' = 2$) yield

$$\begin{aligned} \mathbb{E} \left[|X_t^k - X_{[t]_{\Delta_N}}^k| \right] &\leq \mathbb{E} \left[|X_t - X_{[t]_{\Delta_N}}|^{p'|k|/2} \right]^{1/p'} \left(\mathbb{E} \left[X_t^{q'k} \right]^{1/q'} \mathbb{E} \left[X_{[t]_{\Delta_N}}^{2q'k/2} \right]^{1/(2q')} \right. \\ &\quad \left. + \mathbb{E} \left[X_t^{2q'k/2} \right]^{1/(2q')} \mathbb{E} \left[X_{[t]_{\Delta_N}}^{q'k} \right]^{1/q'} \right). \end{aligned}$$

Remark A.3. If $k \in [-1, 0)$, considering the same bounds as in the last paragraph, we get a more restrictive condition on Δ_N than with the previously considered bounds in the case $k \in [-1, 0)$ but a possibly less restrictive condition on moments. For instance, taking $p' = 4$ and $q' = 2$, if $k = -1$ then finite moments of order -2 and 2 finite for μ suffice, while $q_{\mathcal{L}} \geq 2$.

Remark A.4. Let $k \in [-2, 0]$ and let $j \neq 0$. Note that $X_t^k \mathbf{1}_{I_j}(X) \leq r_1^k \mathbf{1}_{I_j}(X)$. Hence, the integrand of (A.7) in k can be bounded from above by

$$r_1^{2k} \mathbb{E} \left[|X_t^{|k|} - X_{[t]_{\Delta_N}}^{|k|}| \mathbf{1}_{I_j}(X_t) \mathbf{1}_{I_j}(X_{[t]_{\Delta_N}}) \right] + \mathbb{E} \left[|X_t^k - X_{[t]_{\Delta_N}}^k| \mathbf{1}_{I_j}(X_t) \mathbf{1}_{I_j^c}(X_{[t]_{\Delta_N}}) \right].$$

The first term is the integrand of (A.7) in $|k| \in [0, 2]$ and the second is bounded by

$$r_1^k \mathbb{E} \left[\mathbf{1}_{I_j}(X_t) \mathbf{1}_{I_j^c}(X_{[t]_{\Delta_N}}) \right] + \mathbb{E} \left[X_{[t]_{\Delta_N}}^k \mathbf{1}_{I_j}(X_t) \mathbf{1}_{I_j^c}(X_{[t]_{\Delta_N}}) \right]$$

which is sum of terms of the kind (A.3) with $q = 0$ and $q = k$. If, e.g., $k \in [0, 1]$ then the conditions are μ admits finite first and $(-|k|)$ -th moment and $\lim_{N \rightarrow \infty} T_N^{\lambda-1} \Delta_N^{|k|} = 0$.

A.4.2 Dealing with $\mathbb{E} [|M_{T_N}^{j,m} - M_{T_N,N}^{j,m}|]$

As in the previous section, we exploit (A.4)-(A.5) to obtain the assumptions on the moments of μ and on Δ_N under which $\mathbb{E} [|M_{T_N}^{j,m} - M_{T_N,N}^{j,m}|]$ is $o(T_N^{1/\lambda})$ for $m \in \{0, 1\}$, $j \in \{0, \dots, d\}$.

As for $Q^{j,k}$, we can rewrite $M_{T_N,N}^{j,m} - M_{T_N}^{j,m}$ as follows

$$\begin{aligned} M_{T_N,N}^{j,m} - M_{T_N}^{j,m} &= \int_0^{T_N} \left(X_t^m \mathbb{1}_{I_j}(X_t) - X_{[t]_{\Delta_N}}^m \mathbb{1}_{I_j}(X_{[t]_{\Delta_N}}) \right) (a(X_t) - b(X_t)X_t) dt \\ &\quad + \int_0^{T_N} \left(X_t^m \mathbb{1}_{I_j}(X_t) - X_{[t]_{\Delta_N}}^m \mathbb{1}_{I_j}(X_{[t]_{\Delta_N}}) \right) \sigma(X_t) (X_t)^{\gamma(X_t)} dB_t. \end{aligned}$$

Using Triangular inequality, Hölder's inequality, and Itô-isometry, we obtain

$$\begin{aligned} \mathbb{E} [|M_{T_N}^{j,m} - M_{T_N,N}^{j,m}|] &\leq \int_0^{T_N} \mathbb{E} \left[|X_t^m - X_{[t]_{\Delta_N}}^m| (|a_j| + b_j |X_{[t]_{\Delta_N}}| + b_j |X_t - X_{[t]_{\Delta_N}}|) \right] dt \\ &\quad + \int_0^{T_N} \mathbb{E} \left[|X_{[t]_{\Delta_N}}|^m \mathbb{1}_{X_t \in I_j, X_{[t]_{\Delta_N}} \notin I_j} \left(\max_{i=0,\dots,d} |a_i| + \max_{i=0,\dots,d} |b_i| (|X_{[t]_{\Delta_N}}| + |X_t - X_{[t]_{\Delta_N}}|) \right) \right] dt \\ &\quad + \int_0^{T_N} \mathbb{E} \left[|X_{[t]_{\Delta_N}}|^m \mathbb{1}_{X_t \notin I_j, X_{[t]_{\Delta_N}} \in I_j} \left(\max_{i=0,\dots,d} |a_i| + \max_{i=0,\dots,d} |b_i| (|X_{[t]_{\Delta_N}}| + |X_t - X_{[t]_{\Delta_N}}|) \right) \right] dt \\ &\quad + \sqrt{2} \max_{i=0,\dots,d} (\sigma_i) \left(\int_0^{T_N} \mathbb{E} \left[(X_t^m - X_{[t]_{\Delta_N}}^m)^2 (X_t)^{2\gamma(X_t)} + X_{[t]_{\Delta_N}}^{2m} (X_t)^{2\gamma(X_t)} \mathbb{1}_{X_t \in I_j, X_{[t]_{\Delta_N}} \notin I_j} \right. \right. \\ &\quad \left. \left. + X_{[t]_{\Delta_N}}^{2m} (X_t)^{2\gamma(X_t)} \mathbb{1}_{X_t \notin I_j, X_{[t]_{\Delta_N}} \in I_j} \right] dt \right)^{1/2}. \end{aligned} \quad (\text{A.8})$$

The case $m = 0$. μ has finite $\max(1, 2\gamma_d)$ -th moment and $\lim_{N \rightarrow 0} T_N \Delta_N = 0$.

Inequality (A.8) involves terms of the kind (A.2) with $p = 1$ and (A.3) with $q \in \{0, 1\}$ but also

$$\left(\int_0^{T_N} \mathbb{E} \left[(X_t)^{2\gamma(X_t)} (\mathbb{1}_{X_t \in I_j, X_{[t]_{\Delta_N}} \notin I_j} + \mathbb{1}_{X_t \notin I_j, X_{[t]_{\Delta_N}} \in I_j}) \right] dt \right)^{1/2}.$$

Since I_j is not bounded only if $j = d$ or $j = 0$ with $\gamma_0 = 0$, the above quantity is bounded from above for all $j \neq d$ and for $j = d$ if $\gamma_d = 0$ by terms (A.3) with $q = 0$. If $j = d$ and $\gamma_d \neq 0$, its square is bounded by a sum of terms as (A.2) with $p = 2\gamma_d$ and (A.3) with $q = 2\gamma_d$:

$$\mathbb{E} \left[(X_t)^{2\gamma_d} \mathbb{1}_{X_t \in I_d, X_{[t]_{\Delta_N}} \notin I_d} \right] \leq C \mathbb{E} \left[|X_t - X_{[t]_{\Delta_N}}|^{2\gamma_d} + X_{[t]_{\Delta_N}}^{2\gamma_d} \mathbb{1}_{X_t \in I_d, X_{[t]_{\Delta_N}} \notin I_d} \right]$$

for some positive constant C . The latter inequality is derived from sub-additivity (if $\gamma_d \leq 1/2$) or Jensen's inequality (if $\gamma_d > 1/2$).

Inequalities (A.4)-(A.5) ensure that

$$\mathbb{E} [|M_{T_N}^{j,0} - M_{T_N,N}^{j,0}|] \leq C_1 T_N \sqrt{\Delta_N} + C_2 \sqrt{T_N \Delta_N^{\max(1, 2\gamma_d)/2}} \leq (C_1 + C_2) T_N \sqrt{\Delta_N}$$

for some positive constants C_1, C_2 .

The case $m = 1$. μ admits finite $2(1 + \gamma_d)$ -th moment and $\lim_{N \rightarrow 0} T_N \Delta_N = 0$. Inequality (A.8) involves terms of the kind (A.2) with $p \in \{1, 2\}$ and (A.3) with $q \in \{1, 2\}$ but also the square root of the following term

$$\int_0^{T_N} \mathbb{E} \left[(X_t - X_{[t]_{\Delta_N}})^2 X_t^{2\gamma(X_t)} + X_{[t]_{\Delta_N}}^2 X_t^{2\gamma(X_t)} (\mathbb{1}_{X_t \in I_j, X_{[t]_{\Delta_N}} \notin I_j} + \mathbb{1}_{X_t \notin I_j, X_{[t]_{\Delta_N}} \in I_j}) \right] dt.$$

Similarly to the case $m = 0$, by the fact that $X_s^{2\gamma_j} \mathbb{1}_{I_j}(X_s)$ is not bounded from above (up to multiplicative constant) by $\mathbb{1}_{I_j}(X_s)$ only if $j = d$, the fact that the same holds for $X_s^2 \mathbb{1}_{I_j}(X_s)$ unless $j = d$ or $j = 0$ and $\gamma_0 = 0$, and by sub-additivity of the square-root, we can reduce to

$$\begin{aligned} & \left(\int_0^{T_N} \mathbb{E} \left[(X_t - X_{[t]_{\Delta_N}})^2 + \mathbb{1}_{X_t \in I_j, X_{[t]_{\Delta_N}} \notin I_j} + \mathbb{1}_{X_t \notin I_j, X_{[t]_{\Delta_N}} \in I_j} \right] dt \right)^{1/2} \\ & + \left(\int_0^{T_N} \mathbb{E} \left[(X_t - X_{[t]_{\Delta_N}})^2 X_t^{2\gamma_d} \mathbb{1}_{I_d}(X_t) + X_{[t]_{\Delta_N}}^2 X_t^{2\gamma_d} \mathbb{1}_{X_t \in I_d, X_{[t]_{\Delta_N}} \in I_0} \right] \right)^{1/2} \\ & + \left(\int_0^{T_N} \mathbb{E} \left[X_{[t]_{\Delta_N}}^2 \mathbb{1}_{X_t \notin I_d, X_{[t]_{\Delta_N}} \in I_d} + X_t^{2\gamma_d} \mathbb{1}_{X_t \in I_d, X_{[t]_{\Delta_N}} \notin I_d \cup I_0} \right] \right)^{1/2}. \end{aligned}$$

The first line involves terms like (A.2) with $p = 2$ and (A.3) with $q = 0$. The last line shows a term (A.3) with $q = 2$ and a term appearing for the case $m = 0$. The second line, by subadditivity and Hölder's inequality ($(1 + \gamma_d)^{-1} + (1 + \gamma_d)^{-1} \gamma_d = 1$), is bounded from above by

$$\begin{aligned} & \left(\int_0^{T_N} \mathbb{E} \left[(X_t - X_{[t]_{\Delta_N}})^{2(1+\gamma_d)} \right]^{1/(1+\gamma_d)} \mathbb{E} \left[X_t^{2(1+\gamma_d)} \right]^{\gamma_d/(1+\gamma_d)} dt \right)^{1/2} \\ & + \left(\int_0^{T_N} \mathbb{E} \left[X_{[t]_{\Delta_N}}^{2(1+\gamma_d)} \mathbb{1}_{X_t \in I_d, X_{[t]_{\Delta_N}} \in I_0} \right]^{1/(1+\gamma_d)} \mathbb{E} \left[X_t^{2(1+\gamma_d)} \right]^{\gamma_d/(1+\gamma_d)} dt \right)^{1/2} \end{aligned}$$

where we recognise (A.2) and (A.3) with $p = q = 2(1 + \gamma_d)$.

By (A.4) and (A.5) we deduce $\mathbb{E} [|M_{T_N}^{j,1} - M_{T_N, N}^{j,1}|]$ is bounded from above by

$$C_1 (T_N \sqrt{\Delta_N} + T_N \Delta_N + \sqrt{T_N \Delta_N} + \sqrt{T_N \Delta_N^{1/2(1+\gamma_j)}}) \leq 4C_1 T_N \sqrt{\Delta_N}$$

for some positive constant C_1 .

A.4.3 End of the proof of Lemma 5.2

In this section, we complete the proof of Lemma 5.2 by summarising the assumptions of the previous sections for each item.

Proof of Item (a) (QMLE $\theta_{T_N, N}^{(q-\mathcal{L})}$). The QMLE (4.6) involves the statistics $Q^{j,k}$ and $M^{j,m}$, where $m \in \{0, 1\}$, $k \in \{0, 1, 2\}$, and $j \in \{0, \dots, d\}$. The most restrictive assumptions on the moments of μ is obtained for $m = 1$: μ has finite moment of order $2(1 + \gamma_d)$, that is $\mathcal{H}_{q-\mathcal{L}}$. Instead, the unique condition on Δ_N is $\lim_{N \rightarrow \infty} T_N^{\lambda-1} \Delta_N = 0$. Hence, $g_N^{(q-\mathcal{L})} = \Delta_N$.

Remark A.5. If σ_* is known, one could replace $M_{T_N, N}^{j,1}$ by $\mathcal{M}_{T_N, N}^{j,1}$ in (4.2) obtaining the analogous of (4.6). However, in order to exploit the asymptotic properties of the estimator from continuous time observations, $\mathbf{H}_{q-\mathcal{L}}$ has to be included in $\mathcal{H}_{\mathcal{L}}$, hence it cannot be relaxed. Moreover, the most restrictive condition on Δ_N would come from $k = 2\gamma_j$ when $\gamma_j \in (0, 1/2) \cup (1/2, 1)$, for all $j \in \{0, \dots, d\}$, leading to the condition $\lim_{N \rightarrow \infty} T_N^{\lambda-1} g_N^{(\sigma)} = 0$, which is worse than $\lim_{N \rightarrow \infty} T_N^{\lambda-1} \Delta_N = 0$.

Proof of Item (b) (Volatility estimation). Only the statistics $Q^{j,k}$, $M^{j,m}$, and $M^{i,0}$ with $m \in \{0, 1\}$, $k = 2\gamma_j$, and $i, j \in \{0, \dots, d\}$ appear in the volatility estimator (4.7). We observe that the most restrictive assumptions on the moments of μ is obtained for $m = 1$: μ has finite moment of order $2(1 + \gamma_d)$, that is $\mathcal{H}_{q-\mathcal{L}}$. Instead the unique condition on Δ_N more restrictive than $\lim_{N \rightarrow \infty} T_N \Delta_N = 0$ comes from $k = 2\gamma_j$ when $\gamma_j \in (0, 1/2) \cup (1/2, 1)$, for all $j \in \{0, \dots, d\}$, leading to the condition $\lim_{N \rightarrow \infty} T_N^{\lambda-1} g_N^{(\sigma)} = 0$.

Proof of Item (c) (MLE $\theta_{T_N, N}^{(\mathcal{L})}$ with σ_* unknown). First, assume that σ_* is known. To deal with the case σ_* is unknown, one takes the most restrictive conditions among the ones in Item (b) and the ones obtained in the case σ_* known.

The MLE (4.5) involves the statistics $Q^{j,k}$, $M^{i,0}$, where $k \in \{-2\gamma_j, 1 - 2\gamma_j, 2 - 2\gamma_j\} \cup \{-1, 0\}$, and $i, j \in \{0, \dots, d\}$. This case is more subtle than the others. The more restrictive conditions come from the different values of k . Let us consider separately the cases $\gamma_j \in (0, 1/2]$ and $\gamma_j \in (1/2, 1]$. Let $p_{\mathcal{L}}, q_{\mathcal{L}}, p', q'$ given in $\mathcal{H}_{\mathcal{L}}$.

If $\gamma_j \in (0, 1/2]$, the unique negative moment condition comes from $k = -1$ (and $k = -2\gamma_j$), while for the positive moments the most restrictive conditions come from $k = -1$ and $k = 2(1 - \gamma_j)$: finite moments of order $-q_{\mathcal{L}}$ and $\max(p_{\mathcal{L}}, 2(1 - \gamma_j))$. The condition on Δ_N follows from the fact that $\lim_{N \rightarrow \infty} T_N^{\lambda-1} (\Delta_N + \Delta_N^{2\gamma_j} + \Delta_N^{1-2\gamma_j} + \Delta_N^{1-\gamma_j}) \leq \lim_{N \rightarrow \infty} T_N^{\lambda-1} g_N^{(\mathcal{L})} = 0$.

If $\gamma_j \in (1/2, 1]$, the most restrictive moment conditions comes from $k = -1$ and $k = -2\gamma_j$: $\max(p_{\mathcal{L}}, p'\gamma_j)$ and $-\max(q_{\mathcal{L}}, 2q'\gamma_j)$. The most restrictive condition on Δ_N is given by $\lim_{N \rightarrow \infty} T_N^{\lambda-1} (\Delta_N + \Delta_N^{\gamma_j} + \Delta_N^{2\gamma_j-1} + \Delta_N^{2(1-\gamma_j)}) \leq \lim_{N \rightarrow \infty} T_N^{\lambda-1} g_N^{(\mathcal{L})} = 0$.

In conclusion, since $j \in \{0, \dots, 1\}$, we have $\lim_{N \rightarrow \infty} T_N^{\lambda-1} g_N^{(\mathcal{L})} = 0$ and μ has finite moments of orders

$$-\max(q_{\mathcal{L}}, 2q'\gamma^{\max} \mathbf{1}_{(1/2, 1]}(\gamma^{\max})) \quad \text{and} \quad \max(p_{\mathcal{L}}, p'\gamma^{\max} \mathbf{1}_{(1/2, 1]}(\gamma^{\max}), 2(1-\gamma^{\min}) \mathbf{1}_{(0, 1/2]}(\gamma^{\min})).$$

Remark A.6. If $\gamma^{\max} \leq 1/2$, taking $p' = 4$ and $q' = 2$, the orders are $-q_{\mathcal{L}}$ and $p_{\mathcal{L}}$.

We consider separately the case $\gamma_0 = 0$ and $\gamma^{\max} \leq 1/2$. If $\gamma_0 = 0$ then we deal with positive moments up to the one of order 2, and with negative moments when $k = -2\gamma_j$. By Remark A.4, the most restrictive moment conditions are finite 2-nd and $-2\gamma_j$ -th moment and the following one on Δ_N : $\lim_{N \rightarrow 0} T_N^{\lambda-1} g_N^{(\mathcal{L})} = 0$.

A.5 Proof of Proposition 5.4

Given s and t such that $0 \leq s < t$, we show that for every $m \geq 1$ such that μ admits finite m -th moment, there exists a constant $C \in (0, \infty)$ depending only on m and the parameters of the process such that $\mathbb{E}[|X_t - X_s|^m] \leq C(t - s)^{m/2}$.

By the triangular inequality,

$$\begin{aligned} |X_t - X_s| &\leq \int_s^t |a(X_u) - b(X_u)X_u| \, du + \left| \int_s^t \sigma(X_u)(X_u)^{\gamma(X_u)} \, dB_u \right| \\ &\leq (t-s) \max_{i=0,\dots,d} |a_i| + \max_{i=0,\dots,d} |b_i| \int_s^t |X_u| \, du + \left| \int_s^t \sigma(X_u)(X_u)^{\gamma(X_u)} \, dB_u \right|. \end{aligned}$$

Then, Jensen's inequality ensures that for $m \geq 1$ it holds that

$$\begin{aligned} \mathbb{E}[|X_t - X_s|^m] &\leq 2^{2m-2} \max_{i=0,\dots,d} |a_i| (t-s)^m + 2^{2m-2} \max_{i=0,\dots,d} |b_i| (t-s)^{m-1} \int_s^t \mathbb{E}[|X_u|^m] \, du \\ &\quad + 2^{m-1} \mathbb{E} \left[\left| \int_s^t \sigma(X_u)(X_u)^{\gamma(X_u)} \, dB_u \right|^m \right]. \end{aligned}$$

Since X_0 is distributed as the stationary distribution μ , which admits finite m -th moment, then $\sup_{u \in [s,t]} \mathbb{E}[|X_u|^m \mathbf{1}_{I_d}(X_u)] < \infty$.

Burkholder-Davis-Gundy inequality implies that

$$\mathbb{E} \left[\left| \int_s^t \sigma(X_u)(X_u)^{\gamma(X_u)} \, dB_u \right|^m \right] \leq \mathbb{E} \left[\left(\int_s^t \max_{i=0,\dots,d} |\sigma_i|(X_u)^{2\gamma(X_u)} \, du \right)^{m/2} \right].$$

Then, we distinguish the case $m \geq 2$ and $m \in [1, 2)$. In both cases we apply Hölder's inequality but in a different way. If $m \geq 2$, we obtain

$$\begin{aligned} \mathbb{E} \left[\left| \int_s^t \sigma(X_u)(X_u)^{\gamma(X_u)} \, dB_u \right|^m \right] &\leq (t-s)^{m/2-1} \left(\max_{i=0,\dots,d} |\sigma_i| \right)^{m/2} \int_s^t \mathbb{E}[|X_u|^{m\gamma_d}] \, du \\ &\leq C(t-s)^{m/2}, \end{aligned}$$

since, $\sup_{u \in [s,t]} \mathbb{E}[|X_u|^{m\gamma_d} \mathbf{1}_{I_d}(X_u)] < \infty$. If $m \in [1, 2)$, we reduce to the previous case ($m \geq 2$):

$$\mathbb{E} \left[\left| \int_s^t \sigma(X_u)(X_u)^{\gamma(X_u)} \, dB_u \right|^m \right] \leq \left(\mathbb{E} \left[\left| \int_s^t \sigma(X_u)(X_u)^{\gamma(X_u)} \, dB_u \right|^{2m} \right] \right)^{1/2} \leq C(t-s)^{m/2}.$$

The proof is thus completed.

A.6 Proof of the key result: Proposition 5.5

For all $j \in \{0, \dots, d\}$, let $\xi^{(j)}$ denote a standard CKLS process with parameters $(a_j, b_j, \sigma_j, \gamma_j)$ starting at X_s . Let $s, t \in [0, \infty)$ be fixed such that $0 \leq s < t$. Note that:

$$\begin{aligned} &\mathbb{E} \left[|X_s|^m \mathbb{P}_{X_s} \left(\tau_{I_j}^{\xi^{(j)}} < t-s \right) \right] \\ &\leq \mathbb{E} \left[|X_s|^m \left(\mathbb{P}_{X_s} \left(\tau_{r_j, \searrow}^{\xi^{(j)}} < t-s \right) + \mathbb{P}_{X_s} \left(\tau_{r_{j+1}, \nearrow}^{\xi^{(j)}} < t-s \right) \right) \mathbf{1}_{I_j}(X_s) \right], \end{aligned}$$

where $\tau_{r_j, \searrow}^{\xi^{(j)}}$ is the first hitting time from above of the level r_j , $\tau_{r_{j+1}, \nearrow}^{\xi^{(j)}}$ is the first hitting time from below of the level r_{j+1} of the process $\xi^{(j)}$.

Without loss of generality, we reduce to show that:

$$\mathbb{E} \left[|X_s|^m \mathbb{P}_{X_s} \left(\tau_{r_d, \searrow}^{\xi^{(d)}} < t-s \right) \mathbf{1}_{I_d}(X_s) \right] \leq C_2(t-s)^{1/2}, \quad (\text{A.9})$$

and

$$\mathbb{E} \left[|X_s|^m \mathbb{P}_{X_s} \left(\tau_{r_1, \nearrow}^{\xi^{(0)}} < t - s \right) \mathbb{1}_{I_0}(X_s) \right] \leq C_1 (t - s)^{1/2}, \quad (\text{A.10})$$

where C_1 and C_2 are strictly positive constant.

Indeed, in the other cases, X_s belongs in I_j for $j \notin \{0, d\}$, which is compact, and the desired inequality can be deduced using a similar reasoning.

A.6.1 Bounds on the first hitting time from above of the level r_d : (A.9)

We focus on the case $\gamma_d \in [0, 1)$. The case $\gamma_d = 1$ can be proven using a similar reasoning, the proof is thus omitted. Let us recall that the parameters $(a_d, b_d, \sigma_d, \gamma_d)$ satisfy the ergodicity conditions in Table 4 in Section A.1, in particular $b_d \geq 0$.

The main idea of this proof is to bound the first hitting time by the hitting times of some drifted Brownian motions. To do so, we apply the Lamperti transform and we bound the process, over a well chosen time interval.

We define the process $(Y_u)_{u \geq 0}$ as follows. For all $u \geq 0$ let $Y_u = \psi(\xi_u^{(d)})$ where $\psi(x) = \int_0^x \frac{1}{\sigma_d y^{\gamma_d}} dy = \frac{x^{1-\gamma_d}}{\sigma_d(1-\gamma_d)}$ (Lamperti transform). We denote ψ^{-1} as the inverse function of ψ , then Y is solution to the following SDE:

$$dY_u = \frac{a_d}{\sigma_d} (\psi^{-1}(Y_u))^{-\gamma_d} - b_d(1 - \gamma_d) Y_u - \frac{1}{2} \sigma_d \gamma_d (\psi^{-1}(Y_u))^{\gamma_d - 1} du + dB_u.$$

Let $\varepsilon > 0$ be fixed and let $\tau_{[\psi(r_d), \psi(X_s) + \varepsilon]}^Y$ denote the first hitting time of the boundary of $[\psi(r_d), \psi(X_s) + \varepsilon]$ of the process Y . Then, we have

$$\mathbb{P}_{X_s} \left(\tau_{r_d, \searrow}^{\xi^{(d)}} < t - s \right) = \mathbb{P}_{\psi(X_s)} \left(\tau_{\psi(r_d), \searrow}^Y < t - s \right) \leq \mathbb{P}_{\psi(X_s)} \left(\tau_{[\psi(r_d), \psi(X_s) + \varepsilon]}^Y < t - s \right).$$

By applying the Comparison Theorem (e.g. [13, p352]) until time $\tau_{[\psi(r_d), \psi(X_s) + \varepsilon]}^Y$, $Y - \psi(X_s)$ is bounded from above by a drifted BM $B^{\nu+}$ starting from 0 and from below by a drifted BM $B^{\nu-}$ starting from 0 with parameters:

$$\begin{cases} \nu_+ = \frac{|a_d|}{\sigma_d} r_d^{-\gamma_d} \\ \nu_- = -\frac{|a_d|}{\sigma_d} r_d^{-\gamma_d} - b_d(1 - \gamma_d)(\psi(X_s) + \varepsilon) - \frac{1}{2} \sigma_d \gamma_d r_d^{\gamma_d - 1}. \end{cases}$$

Hence, the following inequality holds:

$$\mathbb{P}_{X_s} \left(\tau_{r_d, \searrow}^{\xi^{(d)}} < t - s \right) \leq \mathbb{P}_0 \left(\tau_{\psi(r_d) - \psi(X_s), \searrow}^{B^{\nu-}} < t - s \right) + \mathbb{P}_0 \left(\tau_{\varepsilon, \nearrow}^{B^{\nu+}} < t - s \right),$$

and classical results on the first hitting of a drifted Brownian motion (see [6]) yield

$$\mathbb{P}_0 \left(\tau_{\varepsilon, \nearrow}^{B^{\nu+}} < t - s \right) \leq K_1 e^{-\frac{\varepsilon^2}{2(t-s)}},$$

and,

$$\mathbb{P}_0 \left(\tau_{\psi(r_d) - \psi(X_s), \searrow}^{B^{\nu-}} < t - s \right) \leq K_2 e^{-\frac{|\psi(r_d) - \psi(X_s)|^2}{2(t-s)} + b_d(1 - \gamma_d)\psi(X_s)^2} f(X_s),$$

where K_1 and K_2 are two strictly positive constants, f is an explicit function which depends only on X_s and such that $\lim_{x \rightarrow +\infty} f(x) e^{-\psi(x)^2} = 0$.

Let us note that

$$\mu(x) \mathbb{1}_{x \geq r_d} = K_3 \frac{2}{\sigma_d^2 x^{1 - 2\frac{a_d}{\sigma_d^2}}} \exp(-b_d(1 - \gamma_d)\psi(x)^2) \mathbb{1}_{x \geq r_d},$$

with K_3 a strictly positive constant. Since μ admits finite m -th moment by assumption, inequality (A.9) holds. We avoid details here, but the interested reader could appreciate the following remarks. When $t - s$ is close to 0, $e^{-K/(t-s)}$ decreases faster than any polynomial of $(t - s)$. So, by splitting the integral to distinguish between when $\psi(x) - \psi(r_d)$ is small (e.g. $O(\sqrt{t - s})$) and when it isn't, we can easily compute the bounds. To avoid repetitions, we do not mention anymore this remark.

A.6.2 Bounds on the first hitting time from below of the level r_1 : (A.10)

The parameters $(a_0, b_0, \sigma_0, \gamma_0)$ satisfy the conditions ensuring ergodicity in Table 4. We remark that, on a suitable time interval, the process $(\xi^{(0)})^{2(1-\gamma_0)}$ can be bounded from above by the norm of a multi-dimensional Brownian motion. This leads to obtaining (A.10) under the assumption that $\gamma_0 \in \{0, 1/2\}$. Instead, if $\gamma_0 \in (1/2, 1]$, this bound is not enough. Thus, we additionally bound from below the Lamperti transform of the process $\xi^{(0)}$ by a drifted Brownian motion.

The case $\gamma_0 = 1/2$. We remind the following results, for $i \in \{1, \dots, n\}$, we denote $\tilde{B} := (\tilde{B}^i)_{i \leq n}$ a n -dimensional Brownian motion. Itô formula and Levy characterization imply that

$$\forall u \geq 0, \quad \left\| \frac{\sigma_0}{2} \tilde{B}_u - \sqrt{\frac{X_s}{n}} \right\|_2^2 = Y_{u,n} = X_s + n \frac{\sigma_0^2}{4} u + \int_0^u \sigma_0 \sqrt{Y_{v,n}} dW_v,$$

where W is a Brownian motion. Then $Y_{\cdot,n}$ is a CIR process whose coefficients satisfy the conditions for ergodicity in Table 4. Moreover we take n such that $0 < a_0 < n\sigma_0^2/4$. So, by the Comparison Theorem, it holds a.s. that $\xi_u^{(0)} \leq Y_{u,n}$ for all $0 \leq u \leq \tau_{r_1, \nearrow}^{\xi^{(0)}}$ and then

$$\mathbb{P}_{X_s} \left(\tau_{r_1, \nearrow}^{\xi^{(0)}} \leq t - s \right) \leq \mathbb{P}_{X_s} \left(\tau_{r_1, \nearrow}^{Y_{\cdot,n}} \leq t - s \right).$$

Moreover, we have

$$\{\tau_{r_1, \nearrow}^{Y_{\cdot,n}} \leq t - s\} \subseteq \bigcup_{i=1}^n \left\{ \forall u \leq t - s, -\sqrt{\frac{r_1}{n}} < \left(\frac{\sigma_0}{2} \tilde{B}_u^i - \sqrt{\frac{X_s}{n}} \right) < \sqrt{\frac{r_1}{n}} \right\}^c.$$

Then, by the Comparison Theorem and the symmetry of BM, we obtain:

$$\mathbb{P}_{X_s} \left(\tau_{r_1, \nearrow}^{\xi^{(0)}} \leq t - s \right) \leq 2n \mathbb{P}_0 \left(\tau_{\frac{2\sqrt{r_1} - \sqrt{X_s}}{\sigma_0 \sqrt{n}}, \nearrow}^{\tilde{B}^1} \leq t - s \right) \leq K_4 e^{-\frac{2(\sqrt{r_1} - \sqrt{X_s})^2}{\sigma_0^2 n(t-s)}},$$

with K_4 a strictly positive constant. This, and the stationary distribution μ (A.1), if μ admits finite m -th moment, yield inequality (A.10).

The case $\gamma_0 = 0$. By applying Itô formula on $Y := (\xi^{(0)})^2$, for all $u \geq 0$, we have:

$$dY_u = 2 \left(\text{sgn}(\xi^{(0)}) a_0 \sqrt{Y_u} - b_0 Y_u + \frac{\sigma_0^2}{2} \right) du + 2\sigma_0 \sqrt{Y_u} d\tilde{B}_u$$

with $Y_0 = X_s^2$ and \tilde{B}_u another Brownian motion. Since $2 \text{sgn}(\xi^{(0)}) a_0 \sqrt{Y_u} - 2b_0 Y_u + \sigma_0^2 \leq 2|a_0|r_1 + \sigma_0^2 \leq n\sigma_0^2$ for some $n \in \mathbb{N}$, the Comparison Theorem ensures that for all $u \leq \tau_{r_1, \nearrow}^{Y_{\cdot,n}}$

the process Y is bounded from above by the norm of a n -dimensional Brownian motion. Hence, similarly to the case $\gamma_0 = \frac{1}{2}$, we have:

$$\mathbb{P}_{X_s} \left(\tau_{r_1, \nearrow}^{\xi^{(0)}} \leq t - s \right) \leq 2n \mathbb{P}_0 \left(\tau_{\frac{r_1 - X_s}{\sigma_0 \sqrt{n}}, \nearrow}^{\tilde{B}^1} \leq t - s \right) \leq K_5 e^{-\frac{(r_1 - X_s)^2}{2\sigma_0^2 n(t-s)}},$$

with K_5 a strictly positive constant. We conclude analogously to the case $\gamma_0 = 1/2$.

The case $\gamma_0 \in (1/2, 1)$. By Lamperti transform, $Y_u := d\psi(\xi_u^{(0)})$ with $\psi(x) = \frac{x^{1-\gamma_0}}{\sigma_0(1-\gamma_0)}$ (in particular $Y_0 = \frac{X_s^{1-\gamma_0}}{\sigma_0(1-\gamma_0)}$). So,

$$dY_u = \left[\frac{a_0}{\sigma_0} ((1-\gamma_0)Y_u)^{1-\frac{1}{(1-\gamma_0)}} - b_0(1-\gamma_0)Y_u - \frac{\gamma_0}{2(1-\gamma_0)}Y_u^{-1} \right] du + dB_u.$$

By the Comparison Theorem, it holds a.s. for all $u \in [0, \tau_{[\frac{\psi(X_s)}{2}, \psi(r_1)]}]$ that

$$B_u^{\nu_-} \leq Y_u \quad \text{and} \quad Y_u^2 \leq \left\| \frac{1}{2} \tilde{B}_u - \frac{\psi(X_s)}{\sqrt{n}} \right\|_2^2,$$

where B^{ν_-} is drifted Brownian motion and $\tilde{B} := (\tilde{B}^i)_{i \leq n}$ a n -dimensional Brownian motion. Here the drift parameter ν_- and the dimension n are given by

$$\begin{cases} \nu_- = -\frac{|b_0|r_1^{1-\gamma_0}}{\sigma_0} - \frac{\gamma_0\sigma_0}{X_s^{1-\gamma_0}}, \\ n(X_s) = \left\lceil \frac{a_0}{\sigma_0^2(1-\gamma_0)} X_s^{1-2\gamma_0} + |b_0|(1-\gamma_0) \frac{r_1^{2(1-\gamma_0)}}{\sigma_0^2(1-\gamma_0)^2} \right\rceil. \end{cases}$$

Finally, we obtain the following inequality:

$$\begin{aligned} \mathbb{P}_{X_s} \left(\tau_{r_1, \nearrow}^{\xi^{(0)}} < t - s \right) &\leq \mathbb{P}_0 \left(\tau_{-\frac{\psi(X_s)}{2}, \searrow}^{B^{\nu_-}} < t - s \right) + 2n(X_s) \mathbb{P}_0 \left(\tau_{2\frac{\psi(r_1)^2 - \psi(X_s)^2}{\sqrt{n(X_s)}}, \nearrow}^{\tilde{B}^1} < t - s \right) \\ &\leq K_6 \left(e^{-\frac{|\psi(r_1) - \psi(X_s)|^2}{2(t-s)}} + n(X_s) e^{-\frac{2(\psi(r_1)^2 - \psi(X_s)^2)^2}{n(X_s)(t-s)}} \right), \end{aligned}$$

with K_6 a strictly positive constant.

Similarly to the previous cases, this, and the stationary distribution μ (A.1), if μ admits finite m -th moment, yield inequality (A.10).

The case $\gamma_0 = 1$. One can conclude by bounding directly the Lamperti transform from above by a drifted Brownian motion as it has been done for the bounds on the first hitting time from above of the level r_d .

B Appendix: Auxiliary results

In this section, we provide some auxiliary results on well posedness of some integrals appearing in the likelihood in Proposition B.1 and on the finiteness of the moments of the stationary measure μ in Proposition B.2. More precisely, we give some properties of the

moments of the T-CKLS process. Some are straightforward applications of the ergodic properties.

The following proposition describes the behavior of various integrals of the T-CKLS process. It establishes whether the likelihood (2.2) is well defined or not. Properties of this kind have also been considered in [26].

Proposition B.1. *Let X be solution to the SDE (2.1).*

(a) *If $\gamma_0 = 1/2$ and $a_0 \geq \sigma_0^2/2$ or if $\gamma_0 \in (1/2, 1] \cup \{0\}$, then*

$$\forall t \geq 0, \quad \int_0^t \frac{1}{X_s^{2\gamma_0}} \mathbb{1}_{I_0}(X_s) ds < \infty \quad \mathbb{P}_{x_0} - a.s..$$

(b) *If $\gamma_0 = 1/2$ and $a_0 < \sigma_0^2/2$, then*

$$\forall t \geq 0, \quad \mathbb{P}_{x_0} \left(\int_0^t \frac{1}{X_s^{2\gamma_0}} \mathbb{1}_{I_0}(X_s) ds = \infty \right) > 0.$$

Proof. If $\gamma_0 = 0$, it is trivial. When $\gamma_0 \neq 0$, the first item follows from the fact that 0 is an unattainable boundary and continuity of the trajectories: the image of $[0, t]$ through each trajectory $s \mapsto X_s(\omega)$ is a compact of $]0, \infty[$. The second item, for $\gamma_0 = 1/2$ has been proven in [4] by using properties of the Laplace transform. \square

The following proposition describes the behaviour of the moments from the stationary distribution of the T-CKLS process in the ergodic regime. We recall that the stationary distribution μ is given by (A.1).

Proposition B.2. *Let $m \in (0, \infty)$ and assume that the conditions in Table 4 (ensuring that μ is the stationary distribution) hold. Then μ admits finite m -th moment unless it holds simultaneously $b_d = 0$ and $\gamma_d \in [1/2, 1]$ in which case the m -th moment is finite if*

- $\gamma_d = 1/2$, $b_d = 0$ and $a_d < -m \sigma_d^2/2$.
- $\gamma_d \in (1/2, 1)$, $b_d = 0$ and $m < \gamma_d - 1/2 < 1$.
- $\gamma_d = 1$ and $m \leq 1$.

The measure μ admits finite $-m$ -th moment unless $\gamma_0 \in \{0, 1/2\}$ in which case the $(-m)$ -th moment is finite if

- $\gamma_0 = 1/2$ and $a_0 > m \sigma_0^2/2$.
- $\gamma_0 = 0$ and $m < 1$.

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