Self-avoiding space-filling folding curves in dimension 3

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Various examples of folding curves have been considered in the plane. For each integer $n \geq 1$, a square n-curve such as the dragon curve is obtained by folding n times a strip of paper in 2, then unfolding it, leaving a $\pi/2$ angle at each fold. A triangular n-curve such as the terdragon curve is obtained by folding n times a strip of paper in 3, then unfolding it, leaving a $\pi/3$ angle at each fold. Peano-Gosper curves, which are based on regular hexagons, have similar properties.

Folding curves were used to construct fractals. They have two characteristic properties: First, they are self-avoiding. Second, n-curves of the same type completely cover larger and larger regions of the plane when n becomes larger.

Square (resp. triangular) folding curves are finite or infinite sequences of consecutive segments taken among the edges of squares (resp. equilateral triangles) which form a regular tiling of the plane. They are obtained from 1 segment by applying repeatedly an inflation process: multiply the length of each segment by k, then replace all the segments with the same curve which consists of k^2 segments. The same process can be applied uniformly, starting with a covering of the plane by segments which are the edges of squares (resp. equilateral triangles) in a regular tiling.

According to [3], [4] and [5], the inductive limit of such a process gives a covering of the plane by curves without endpoint which satisfies the local isomorphism property. Any infinite folding curve can be extended into such a covering in an essentially unique way. Two coverings obtained by the same process are locally isomorphic.

Such coverings of the plane consist of a small number of curves, at most 6 in the examples that we considered until present. Each local isomorphism class contains some coverings which consist of only 1 curve.

We also proved in [5] that the same results are true for Peano-Gosper curves.

In order to obtain similar examples in dimension 3, one natural idea was to fold a wire in 2 several times according to the 3 directions, then unfold it, leaving a $\pi/2$ angle at each fold. However, it was proved in [2] that the curves obtained in that way are generally not self-avoiding. Another possible direction of research was mentioned in [1].

Presently, we have an example with the properties above in dimension 3. Here, the curves consist of consecutive segments of length 1 taken among the edges of cubes which form a regular tiling of the space. The inflation process is as follows: multiply the length of the segments by 2, then replace each segment with the image through a positive isometry of the curve with 8 segments passing through the points $(0,0,0)$, $(0,\overline{1},0)$, $(0,\overline{1},\overline{1})$, $(0,0,\overline{1})$, $(1,0,\overline{1})$, $(1,\overline{1},\overline{1})$, $(1,\overline{1},0)$, $(1,0,0)$, $(2,0,0)$, where $\overline{1}=-1$.

References

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