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Abstract

We study the PSPACE-complete k-Canadian Traveller Problem, where a weighted graph $G = (V, E, \omega)$ with a source $s \in V$ and a target $t \in V$ are given. This problem also has a hidden input $E_* \subsetneq E$ of cardinality at most k representing blocked edges. The objective is to travel from s to t with the minimum distance. At the beginning of the walk, the blockages E_* are unknown: the traveller discovers that an edge is blocked when visiting one of its endpoints. Online algorithms, also called strategies, have been proposed for this problem and assessed with the competitive ratio, *i.e.*, the ratio between the distance actually traversed by the traveller divided by the distance we would have traversed knowing the blockages in advance.

Even though the optimal competitive ratio is $2k + 1$ even on unit-weighted planar graphs of treewidth 2, we design a polynomial-time strategy achieving competitive ratio 9 on unit-weighted outerplanar graphs. This value 9 also stands as a lower bound for this family of graphs as we prove that, for any $\varepsilon > 0$, no strategy can achieve a competitive ratio $9 - \varepsilon$. Finally, we show that it is not possible to achieve a constant competitive ratio (independent of G and k) on weighted outerplanar graphs.

1 Introduction

The k-*Canadian Traveller Problem* (k-CTP) was introduced by Papadimitriou and Yannakakis [21]. It models the travel through a graph where some obstacles may appear suddenly. Given an undirected weighted graph $G = (V, E, \omega)$ and two of its vertices $s, t \in V$, a traveller walks from s to t on graph G in the shortest way despite the existence of blocked edges $E_* \subsetneq E$ (also called *blockages*). The traveller does not know which edges are blocked when he begins his journey. He discovers that an edge $e = (u, v)$ is blocked, *i.e.*, belongs to E_*, when he visits one of its endpoints u or v. The parameter k is an upper bound on the number of blocked edges: $|E_*| \leq k$. The k-CTP is known to be PSPACE-complete [4, 21]. Variants of the k-CTP have been studied: where edges are blocked with a certain probability [1, 4, 9, 16], with multiple travellers [7, 22], where we can pay to sense remote edges [16], or where we seek the shortest tour [18, 19]. This problem has applications in robot routing for various types of logistics [1, 2, 5, 14, 20].

The graph $G = (V, E, \omega)$ has edge weights given by the function $\omega : E \to \mathbb{Q}^+$. Our objective is to make the traveller reach the target t with a minimum cost (also called distance), which is the sum of the weights of the traversed edges. A pair (G, E_*) is called a *road map*. All the road maps considered are feasible: there exists an (s, t)-path in $G \setminus E_*$, the graph G deprived of the blocked edges E_*. In other words, there always is a way to reach target t from source s despite the blockages.

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A solution to the k-CTP is an online algorithm, called a *strategy*, which guides the traveller through his walk on the graph. Its quality can be assessed with competitive analysis [10]. Roughly speaking, the *competitive ratio* is the quotient between the distance actually traversed by the traveller and the distance he would have traversed, knowing which edges are blocked before beginning his walk. Westphal [24] proved that no deterministic strategy achieves a competitive ratio better than $2k+1$ on all road maps satisfying $|E_*| \leq k$. Said differently, for any deterministic strategy A, there is at least one k-CTP road map for which the competitive ratio of A is at least $2k+1$. Two strategies proposed in the literature reach this optimal ratio: REPOSITION [24] and COMPARISON [25]. The REPOSITION strategy consists in trying to traverse the shortest (s,t)-path (exploration phase) of G deprived of the blockages revealed: if a blocked edge is found on this path and thus prevents us from reaching t, we update the set of blockages discovered, go back to s (backtracking phase) and restart the process until reaching t. The COMPARISON strategy is a trade-off between the greedy strategy (trying to take the shortest path between the current position and the target t) and REPOSITION.

Randomized strategies, *i.e.*, strategies in which choices of directions depend on a random draw, have also been studied. Westphal [24] proved that there is no randomized strategy achieving a ratio lower than $k+1$. Bender *et al.* [6] studied graphs composed only of vertex-disjoint (s,t)-paths and proposed a polynomial-time strategy of ratio $k+1$. A slight revision of this strategy is reported in [23]. Demaine *et al.* proposed a pseudo-polynomial-time randomized strategy on general graphs which achieves a competitive ratio $(1+\frac{\sqrt{2}}{2})k + O(1)$ [13].

In this article, we focus only on deterministic strategies. Our objective is to be able to distinguish between graph classes on which the k-CTP has competitive ratio $2k+1$ (the optimal ratio for general graphs) and the ones for which this bound can be improved. This direction of research has already been explored in [8]: there is a polynomial-time strategy which achieves ratio $\sqrt{2}k + O(1)$ on graphs with bounded-size maximum (s,t)-cuts. We pursue this study by focusing on a well-known family of graphs: outerplanar graphs. In [8], an outcome dedicated to a superclass of weighted outerplanar graphs implies that there is a strategy with ratio $2\frac{3}{4}k + O(1)$ on them.

Observe however that slightly larger families of graphs have $2k+1$ as their optimal ratio. By adapting the instance proposed by Westphal [24] and only using weight 1, it was shown [11] that the ratio $2k+1$ cannot be outperformed on unit-weighted graphs consisting only of disjoint (s,t)-paths. These graphs are planar and of treewidth 2. With small effort, we can transform them into an equivalent tree-like structure and state that ratio $2k+1$ is also optimal for unit-weighted planar graphs with maximum degree 3. In the remainder of this article, we put in evidence a very large gap between unit-weighted outerplanar and planar graphs, as we show that there exists a strategy with ratio 9, *i.e.*, independent from k, on unit-weighted outerplanar graphs.

Our results and outline. After some preliminaries (Section 2), we describe in Section 3 a polynomial-time strategy achieving a competitive ratio 9 on instances where the input graph is a unit-weighted outerplanar graph:

Theorem 1. *There is a strategy with competitive ratio 9 for unit-weighted outerplanar graphs.*

In the input outerplanar graph, vertices s and t lie on the outerface. The latter can be seen (provided the graph is 2-connected) as a cycle embedded in the plane, allowing to explore two sides when we travel from s to t. The strategy exploits the existence of these two sides: it consists in a so-called *exponential balancing*. More precisely, we explore some distance D on one side, then we explore $2D$ on the other side, then we come back to the original side with budget $4D$, then $8D$ on the second side, and so on. Moreover, the strategy handles the chords linking both sides and maintains an invariant ensuring that the travelled distance is upper-bounded by 9 times the optimal distance towards the farthest vertices visited. At the end of the execution, this provides us with a competitive ratio at most 9.

Observe that this outcome on unit-weighted outerplanar graphs can be directly extended to equal-weight outerplanar graphs, as such a modification has no impact on the competitive ratio. More generally, if we focus on weighted outerplanar graphs where the quotient between the maximum weight and the minimum one is bounded by some constant stretch S, we have a ratio $9S$ with the proposed strategy.

We then prove in Section 4 that, on unit-weighted outerplanar graphs, the competitive ratio stated in Theorem 1 is optimal:

Theorem 2. *For any $\varepsilon > 0$, no deterministic strategy can achieve a competitive ratio $9 - \varepsilon$ on all road maps (G, E_*), where G is a unit-weighted outerplanar graph.*

This lower bound comes from studying a specific pathological outerplanar graph, the so-called *shell graph*. We encode every deterministic strategy on this graph as a sequence of explored distances on each side. The existence of a deterministic strategy with ratio $9 - \varepsilon$, with $\varepsilon > 0$, is translated into a system of linear equalities which has to be satisfied. We prove that this system has no solution using Farkas' lemma [15], a well-known tool in the area of linear programming.

Our last contribution in this article is presented in Section 5. We show that no constant competitive ratio can be achieved on the more general family of graphs, where the graph is outerplanar but weights can be selected arbitrarily.

Theorem 3. *There is no constant C, independent from G and k, such that a deterministic strategy achieves competitive ratio C on all road maps (G, E_*) where G is a weighted outerplanar graph.*

The proof consists in showing that there is no strategy achieving a certain constant ratio C on a trivial weighted outerplanar graph. Then, we extend it inductively: assume there is a outerplanar graph G on which we cannot achieve some ratio C, we are able to use G to build a bigger outerplanar graph on which ratio $C + 1$ cannot be achieved. We end this article with concluding remarks and directions for future research (Section 6).

We summarize in Table 1 below the state-of-the-art of the competitive analysis of deterministic strategies for the k-CTP, giving for each family of graphs an upper bound of competitiveness (*i.e.*, a strategy with such ratio exists) and a lower bound (*i.e.*, no strategy can achieve a smaller ratio). Our contributions are framed.

Family of graphs	upper bound	lower bound
unit-weighted planar of treewidth 2	$2k + 1$ [24]	$2k + 1$ [11, 24]
bounded maximum edge (s, t)-cuts	$\sqrt{2}k + O(1)$ [8]	?
outerplanar	$2\frac{3}{4}k + O(1)$ [8]	not constant
unit-weighted outerplanar	9	9

Table 1: How deterministic strategies perform for the k-CTP on specific families of graphs.

2 Definitions and first observations

2.1 Graph preliminaries

We work on undirected connected weighted graphs $G = (V, E, \omega)$, where $\omega : E \to \mathbb{Q}^+$. A graph is called *equal-weighted* if the value of $\omega(e)$ is the same for every edge $e \in E$. A special case of equal-weighted graphs are *unit-weighted* graphs, where $\omega(e) = 1$ for every edge e.

A subgraph G' of G is a graph $G' = (V', E', \omega')$, where $V' \subseteq V$, $E' \subseteq E \cap (V' \times V')$, and $\omega' = \omega_{|E'}$. For any $U \subseteq V$, we denote by $E[U]$ the set of edges of G with two endpoints in U. We denote by $G[U]$ the subgraph of G induced by U: $G[U] = \big(U, E[U], \omega_{|E[U]}\big)$. We denote by $G \setminus U$ the graph deprived of vertices in U: $G \setminus U = G[V \setminus U]$. Similarly, for any set of edges $E' \subseteq E$, the graph G deprived of E' is denoted by $G \setminus E' = \big(V, E \setminus E', \omega_{|E \setminus E'}\big)$.

A *simple path* P is a sequence of pairwise different vertices $v_1 \cdot v_2 \cdots v_i \cdot v_{i+1} \cdots v_\ell$, with departure v_1 and arrival v_ℓ, where $v_i v_{i+1} \in E$ for $i \in \{1, \ldots, \ell - 1\}$. A simple path between vertices u and v is called a (u, v)-*path*. In a (u, v)-*walk* , however, vertices can be repeated. We abuse notations: $v_1 \in P$ and $v_1 v_2 \in P$ mean that vertex v_1 and edge $v_1 v_2$ are on path P, respectively.

An (s, t)-*separator* $X \subsetneq V \setminus \{s, t\}$ in graph G is a set of vertices such that s and t are disconnected (*i.e.*, there is no path between them) in graph $G \setminus X$. We denote by $R_G(s, X)$ (resp. $R_G(t, X)$) the *source component* (resp. *target component*) of separator X, which is a set made up of the vertices of X together with all vertices reachable from s (resp. t) in $G \setminus X$. When there is no ambiguity on the graph treated, we might use the simpler notations $R(s, X)$ and $R(t, X)$.

2.2 Problem definition and competitive analysis

Let $G = (V, E, \omega)$ be a graph and E_* represents a set of blocked edges. We define below the concept of *road maps* which are the instances of the k-CTP problem.

Definition 4 (Road maps)**.** *A pair (G, E_*) is a road map if s and t are connected in $G \setminus E_*$.*

In other words, there must be an (s,t)-path in the graph G deprived of the blocked edges E_*. We can now formally introduce the k-Canadian Traveller Problem.

Definition 5 (k-CTP)**.**
Input: *A graph $G = (V, E, \omega)$, two vertices $s, t \in V$, and a set E_* of blocked edges which are unknown and such that $|E_*| = k$ and (G, E_*) is a road map.*
Objective: *Traverse graph G from s to t with minimum distance.*

A partial solution is a walk $v_0 \cdots v_\ell$ where $v_0 = s$, the v_i are non-necessarily distinct vertices of G, and $v_i v_{i+1} \in E \setminus E_*$. A solution for the k-CTP is a partial solution where $v_\ell = t$.

The set of blocked edges E_* is a hidden input at the beginning of the walk. We say an edge is *revealed* when one of its endpoints has already been visited. A *discovered blocked edge* is a revealed edge which is blocked. At any moment of the walk, we usually denote by E'_* the set of discovered blocked edges, in other words the set of blocked edges for which we visited at least one endpoint. Naturally, $E'_* \subseteq E_*$ and, at the beginning of the walk, we can assume that E'_* is empty since the presence of blockages incident with s amounts to consider the graph without these edges. At any moment, we are in fact working on graph $G \setminus E'_*$ as discovered blocked edges can be withdrawn from the input graph G.

We say a path is *blocked* if we are sure that it contains a blocked edge, *i.e.*, one of its edges was discovered blocked. Similarly, we say a path is *open* if we are sure that it does not contain any blocked edge: either all of its edges were revealed open, or it is apparently open and $|E'_*| = k$, or by connectivity considerations since s and t must stay connected in road maps. As an example: if at some moment of the walk, $G \setminus E'_*$ is narrowed to a simple (s,t)-path, we know that this path is open even if its edges have not all been revealed, due to the (s,t)-connectivity of $G \setminus E_*$ in road maps. Finally, we say a path is *apparently open* if no blocked edge has been discovered on it for now. However, it may contain a blocked edge which has not been discovered yet.

For any subset of blocked edges $F \subseteq E_*$ and two vertices x, y, let $d_F(x, y)$ be the cost of the shortest (x, y)-path in graph $G \setminus F$. If $F = E'_*$ refers to the set of blocked edges revealed by the traveller since the beginning of its walk, then $d_F(x, y)$ is the cost of the shortest apparently open (x, y)-path in G. If needed, we may add the considered graph into this notation, for example $d_F(G, x, y)$.

Definition 6 (Optimal offline path and cost)**.** *We denote by P_{opt} some* optimal offline path *of road map (G, E_*): it is one of the shortest (s,t)-paths in the graph $G \setminus E_*$. Its cost, the* optimal offline cost*, is given by $d_{\text{opt}} = d_{E_*}(s, t)$. Concretely, this is the distance the traveller would have traversed if he had known the blockages in advance.*

The *competitive ratio* is defined in [10]. We denote by $d_A^{\text{Tr}}(G, E_*)$ the distance traversed by the traveller guided by a strategy A on graph G from source s to target t with blocked edges E_*.

Definition 7 (Competitive ratio)**.** *The competitive ratio $c_A(G, E_*)$ of A over a road map (G, E_*) is defined as the ratio between $d_A^{Tr}(G, E_*)$ and the optimal offline cost d_{opt}. For $k \in \mathbb{N}$, the competitive ratio c_A of a strategy A for the k-CTP is the maximum over all road maps with at most k blocked edges. Formally:*

$$c_A(G, E_*) = \frac{d_A^{Tr}(G, E_*)}{d_{opt}} \qquad\qquad c_A = \max_{\substack{road\ map\ (G,E_*) \\ |E_*| \leq k}} c_A(G, E_*)$$

An intermediary indicator $c_A(G)$ is the competitive ratio of strategy A over a graph G, assessing all road maps containing this input graph G.

Given a monotone family of graphs $\mathcal{F}$, we say that a strategy A admits a competitive ratio $c(k)$ for the family $\mathcal{F}$ if it is an upper bound for all values $c_A(G, E_*)$ over all k-CTP road maps (G, E_*) such that $G \in \mathcal{F}$. Conversely, we say that some ratio $c(k)$ cannot be achieved for family $\mathcal{F}$ for every strategy A, there is a graph $G \in \mathcal{F}$ such that $c_A(G) > c(k)$.

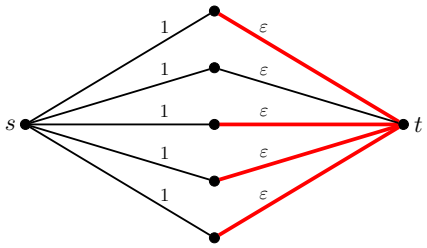


Figure 1: Graph W_4, as defined in [24]

Observe that, if strategy A admits some competitive ratio c on some road map (G, E_*) where G is unit-weighted, then it also achieves ratio c on the same graph G with equal weights different from 1. Indeed, multiplying unit weights with an arbitrary positive value has the same influence on both the traversed distance of our walk and the offline optimal path. Hence, unit-weighted and equal-weighted graphs can be considered equivalent under competitive analysis.

We remind the state of the art on the competitive ratio of deterministic strategies for k-CTP. We present here some known worst-case road maps, *i.e.*, road maps on which the competitive ratio $2k + O(1)$ cannot be beaten. Westphal [24] identified, for any integer k, a relatively trivial family of graphs for which any deterministic strategy achieves at least ratio $2k + 1$ (an example is shown on Figure 1). This family contains the graphs W_k made up of $k + 1$ disjoint (s, t)-paths, *i.e.*, they only pairwise intersect in s and t. Each path has two edges su_i and u_it, with $\omega(su_i) = 1$ and $\omega(u_it) = \varepsilon \ll 1$.

As the $k+1$ disjoint (s, t)-paths are indistinguishable, any deterministic strategy has no choice but to arbitrarily select the first path traversed. In this situation, there exists a configuration of k blocked edges such that the only open path is the last one visited. In this case, the total distance traversed is $2k+1+\varepsilon$ while $d_{\text{opt}} = 1+\varepsilon$. Making ε tend to 0 produces the bound $2k+1$. Conversely, there are two strategies in the literature achieving competitive ratio $2k+1$ on general graphs: REPOSITION [24] and COMPARISON [25].

Observe that the monotone family induced by graphs W_k (*i.e.* the closure of $(W_k)_{k\geq 1}$ by taking subgraphs) is included into series-parallel graphs. As a consequence, a competitive ratio smaller than $2k + 1$ cannot be achieved on planar graphs of treewidth 2. Moreover, one can subdivide each edge of weight 1 in W_k plenty of times and put weight ε everywhere, without modifying the analysis. Hence, the lower bound $2k + 1$ is naturally generalized to equal-weighted (and thus unit-weighted) series-parallel graphs. In summary, the optimal competitive ratio for unit-weighted planar graphs of treewidth 2 is also $2k+1$. Our objective in the remainder is to prove that the lower bound $2k + 1$ can be strongly outperformed on unit-weighted outerplanar graphs, a well-known sub-family of planar graphs of treewidth 2.

2.3 Articulation points

An *articulation point* of a connected graph G is a vertex such that $G \setminus \{v\}$ is not connected. For the k-CTP, if, at some moment of the walk, the current discovered graph $G \setminus E'_*$ contains an articulation point, then either the induced biconnected component can be withdrawn or the problem can be decomposed independently into several biconnected components of $G \setminus E'_*$. In particular, this allows for a preliminary decomposition and simplification of a graph, before even exploring, depending on whether a given articulation point is an (s, t)-separator or not, as shown on Figure 2. More formally:

Lemma 8. *Let $\mathcal{F}$ be a monotone family of graphs, and assume that we have a strategy A achieving competitive ratio C on graphs of $\mathcal{F}$ that do not contain any articulation point. Then, there exists a strategy A' achieving the same competitive ratio C on all graphs of $\mathcal{F}$.*

Proof. The strategy A' goes as follows: let (G, E_*) be a road map with $G \in \mathcal{F}$. If G does not contain any articulation point, apply strategy A. Otherwise, let z be an articulation point of G. If $\{z\}$ is not an (s, t)-separator, then, recursively apply strategy A' on $R_G(s, \{z\}) \cup \{z\}$, which is both the source and the target component, to reach t from s. Otherwise (so $\{z\}$ is an (s, t)-separator),

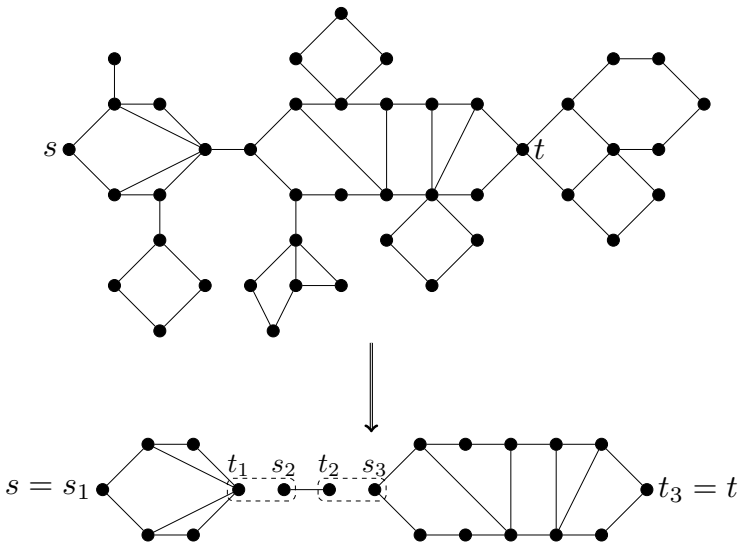


Figure 2: Decomposing the graph into components with no articulation points and removing the useless components (the vertices in a dashed rectangle are the same in the original graph).

recursively apply strategy A' on the source component $R_G(s, \{z\}) \cup \{z\}$ to reach z from s, then recursively apply strategy A' on the target component $R_G(s, \{z\}) \cup \{z\}$ to reach t from z.

We prove by induction on the number p of articulation points that A' terminates and achieves competitive ratio C. The base case $p = 0$ holds by property of A. For the inductive step, we distinguish two cases. If $\{z\}$ is not an (s,t)-separator, the walk we obtain is of length at most Cd_{opt}, which gives competitive ratio C. Otherwise, the length of the whole walk at most $Cd_{E_*}(s,z) + Cd_{E_*}(z,t)$. Since z is an (s,t)-separator, $z \in P_{\text{opt}}$ and we have $d_{\text{opt}} = d_{E_*}(s,z) + d_{E_*}(z,t)$, which concludes the proof. □

2.4 Outerplanar graphs

An outerplanar graph is a graph that can be drawn in the plane in such a way that all vertices are on the outer face. In other words, there exists a planar embedding of the graph where all vertices are placed on the exterior boundary. Outerplanar graphs can also be characterized by a set of forbidden minors: a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$ as a minor [12].

An outerplanar graph is 2-connected if and only if the outer face forms a cycle. Given an embedding of a 2-connected outerplanar graph $G = (V, E)$ and two vertices s and t, let $s \cdot p_1 \cdot p_2 \cdots p_h \cdot t \cdot q_1 \cdot q_2 \cdots q_\ell \cdot s$ be the closed walk along the outer face of G and let $S_1 = \{p_1, p_2, \ldots, p_h\}$ and $S_2 = \{q_1, q_2, \ldots, q_\ell\}$ with $V = \{s, t\} \cup S_1 \cup S_2$. We can deform slightly the embedding so that s and t are aligned along the horizontal axis; since the outer face forms a cycle, we will refer to the set S_1 (resp. S_2) as the *upper* (resp. *lower*) *side* of G. A chord xy of the cycle formed by the outer face is said to be (s,t)-*vertical* (resp. (s,t)-*horizontal*) if x and y belong to different sides (resp. to the same side), see Figure 3. When $x = s$ and/or $y = t$, the chord is considered as (s,t)-horizontal and not (s,t)-vertical. In the rest of the paper, if the context is clear, we will simply refer to these types of chords as *vertical chords* and *horizontal chords*.

Each vertical chord uv forms an (s,t)-separator $\{u, v\}$ as, by planarity, every (s,t)-path has to go through at least one of u or v. Hence, each vertical chord naturally induces both a source and a target component. Considering a set of vertical chords, we say that the *rightmost* one is the one with the minimal inclusion-wise target component. Due to planarity, the rightmost vertical chord is unique for any such set.

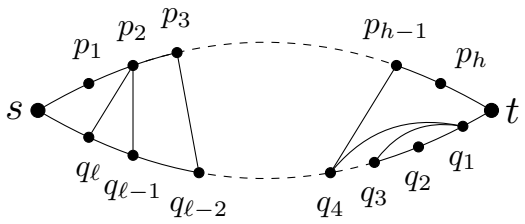


Figure 3: An outerplanar graph with s and t axis aligned, the p_i's vertices as the upper side S_1 and the q_i's vertices as the lower side S_2. Edges p_2q_ℓ, $p_2q_{\ell-1}$, $p_3q_{\ell-2}$, and $p_{h-1}q_4$ are vertical chords and edges q_1q_3, q_1q_4 are horizontal chords; $p_{h-1}q_4$ is the rightmost vertical chord.

3 Competitive ratio 9 for unit-weighted outerplanar graphs

We propose a polynomial-time strategy called ExpBalancing dedicated to unit-weighted outerplanar graphs. We show that it achieves competitive ratio 9 for this family of graphs, which we will later prove is optimal (see Theorem 2).

3.1 Presentation of the strategy

First, note that Lemma 8 allows us to work on outerplanar graphs without articulation points. The input is a unit-weighted 2-connected outerplanar graph G and two vertices s and t. We provide a detailed description of the strategy ExpBalancing that we follow to explore the graph G.

1. **Reaching** t. If, at any point in our exploration, we reach t, then we exit the algorithm and return the processed walk.
2. **Horizontal chords treatment**. If, at any point in our exploration, we visit a vertex $u \in S_i$, $i \in \{1, 2\}$, incident with an open horizontal chord uv revealed for the first time, then we can remove all the vertices on side S_i that lie between u and v on the outer face. Said differently, we get rid of the vertices which are surrounded by the chord uv. If several horizontal chords incident with u are open, then one can only apply this rule to the one which surrounds all others. This procedure comes from the observation that, due to both unit weights and planarity, the open horizontal chord uv with the rightmost v is necessarily the shortest way to go from u to t on side S_i and thus visiting the vertices surrounded by it will be extra costly.
3. **Exponential balancing**. The core *exponential balancing* principle of the strategy consists in alternately exploring sides within a given budget that doubles each time we switch sides. The budget is initialized to 1. Hence, we walk first on side S_1 with budget 1, second on side S_2 with budget 2, then on side S_1 with budget 4, and so on. We say each budget corresponds to an *attempt*. During each attempt, we traverse a path starting from the source s and stay exclusively on a side S_i. As evoked in the previous step, at each newly visited vertex, we use an open horizontal chord from our position which brings us as close as possible to t on our side. Either a horizontal chord is open and we use the one which surrounds all other open chords, or if no such chord is open, we pursue our walk on the outer face.

   This balancing process can be described on an automaton depicted in Figure 4 which will be particularly useful in the analysis of this strategy. Here, we assume that we neither are completely blocked on one side nor reveal an open vertical chord. We will handle these cases in Steps 4-6.

   We start our walk on s (state $\mathbf{E_1}$) and make an attempt on an arbitrary side (say S_1) with budget 1 (state $\mathbf{E_2}$). During our first attempt on side S_2 with budget 2, we cross a first edge

and reach state **A**. Then, we cross a second edge if we are not blocked, but this part of the journey corresponds to the transition between states **A** and **B**. The automaton works as follows:

- In state **A**, we have explored D vertices on each side (hence $D = 1$ when we first arrive in state **A**). Call x and y the last explored vertices on each side, assume we are on x. The current budget is $2D$ and we pursue our attempt on the side of x.
- We then explore at most D more vertices on the side of x. We reach state **B**.
- We then go back to y through s, reaching state **C**.
- We explore at most D more vertices on the side of y. We go back to state **A** with an updated value of D that is doubled, update x and y, and the sides were switched.

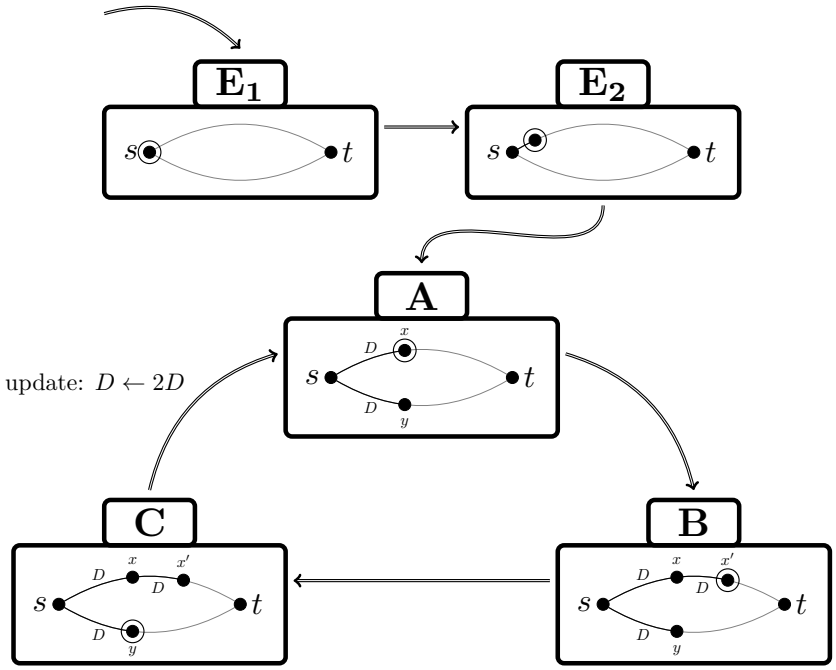
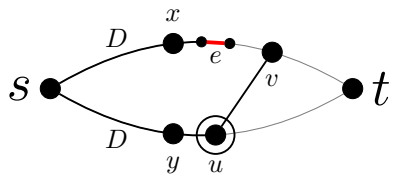
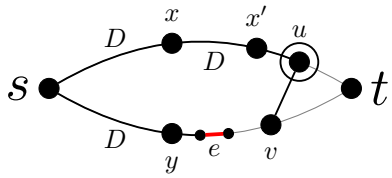


Figure 4: Representation of the exponential balancing divided into three different states. The circled vertex is the one we are currently exploring.

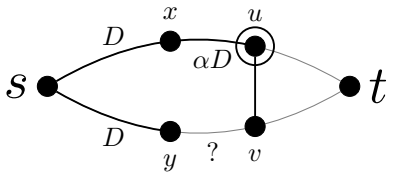
4. **Bypassing a blocked side**. If, during some attempt on side S_i, we are completely blocked (there is no open (s,t)-path on $G[S_i] \setminus E'_*$) before reaching the budget, hence exploring αD ($\alpha < 1$) instead of D (see Figures 5a and 5b), then we backtrack to s and pursue the balancing on the other side S_j ($j \in \{1,2\}, j \neq i$). However, we forget any budget consideration: we travel until we either reach t or visit the endpoint u of some open vertical chord uv. In case there are several open vertical chords incident with u revealed at the same time, we consider the rightmost one. At this moment, we update the current graph $G \setminus E'_*$ by keeping only the target component of separator $\{u, v\}$ and considering u as a new source. Concretely, we concatenate the current walk computed before arriving at u with a recursive call of ExpBalancing on input $(G[R_G(t, \{u, v\})], u, t)$.
5. **Handling open vertical chords between states A and B**. If, during some attempt on side S_i, especially in the transition between states **A** and **B**, we reveal an open vertical chord uv, $u \in S_i$, after having explored distance αD (parameter α is rational, $0 < \alpha \leq 1$, but αD is an integer), then we go to the other side S_j, $j \neq i$, through uv and explore side S_j from v towards s until we:
   - either "see" a vertex y already visited after distance βD (we fix $\beta D \leq \alpha D - 1$, so $0 \leq \beta < \alpha$),



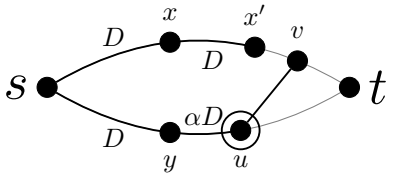
(a) Step 4 : blocked edge e between states **A** and **B**.



(b) Step 4 : blocked edge e between states **C** and **A**.



(c) Step 5 : open vertical chord uv between states **A** and **B**.



(d) Step 6 : open vertical chord uv between states **C** and **A**.

Figure 5: Four situations potentially met with ExpBalancing on some unit-weighted outerplanar graph.

- or explore distance $\alpha D - 1$ and do not see any already visited vertex,
- or are completely blocked on S_j before we reach distance $\alpha D - 1$.

By "see", we mean that we can reach - or not - a neighbor of y which reveals the status of the edge between them: in this way, we actually know the distance to reach y from v even if we did not visit it. Figure 5c describes this rule with an example.

If we see, after distance $\beta D = \alpha D - 1$, an already visited vertex (denoted by y in Figure 5c) at distance αD from v, then, we continue the exponential balancing: we go back to vertex v and thus to state **A** in the automaton, update the budget value D which now becomes $D + \alpha D$, and switch sides.

Otherwise, we update G by keeping only the target component of separator $\{u, v\}$. The current graph becomes $G' = G[R(t, \{u, v\})]$. If we saw an already visited vertex $y \in S_j$ by exploring distance $\beta D < \alpha D - 1$, then the new source becomes $s' = v$. Otherwise, the new source is $s' = u$. We concatenate the current walk with the walk returned by applying ExpBalancing on input (G', s', t).

6. **Handling open vertical chords between states C and A**. If, during the transition between states **C** and **A** (when some attempt is launched on the side of y and the traversed distance on the other side is larger, see Figure 4), an open vertical chord uv is revealed (see Figure 5d), then we keep only the target component of $\{u, v\}$ and restate u as the new source. More formally, we concatenate the current walk with the walk returned by applying ExpBalancing on input (G', u, t), where $G' = G[R_G(t, \{u, v\})]$.

Steps 4-6 can be summarized in this way: when we reveal an open vertical chord uv such that $d_{E_*}(s, v) = d_{E_*}(s, u) + 1$, we launch a recursive call on the target component of separator $\{u, v\}$ with source u and target t. Indeed, an optimal offline path must pass through separator $\{u, v\}$ and, as $d_{E_*}(s, v) = d_{E_*}(s, u) + 1$, we can say there is one optimal offline path P_{opt} such that $u \in P_{\text{opt}}$. For this reason, it makes sense to select u as our new source. Furthermore, there is now no interest in visiting vertices different from $\{u, v\}$ which belong to their source component.

Examples of executions of ExpBalancing are given in Figures 6 and 7.

3.2 Competitive analysis

We show now that the strategy ExpBalancing presented above has a competitive ratio 9 on unit-weighted outerplanar graphs. We prove this statement by minimal counterexample. In this

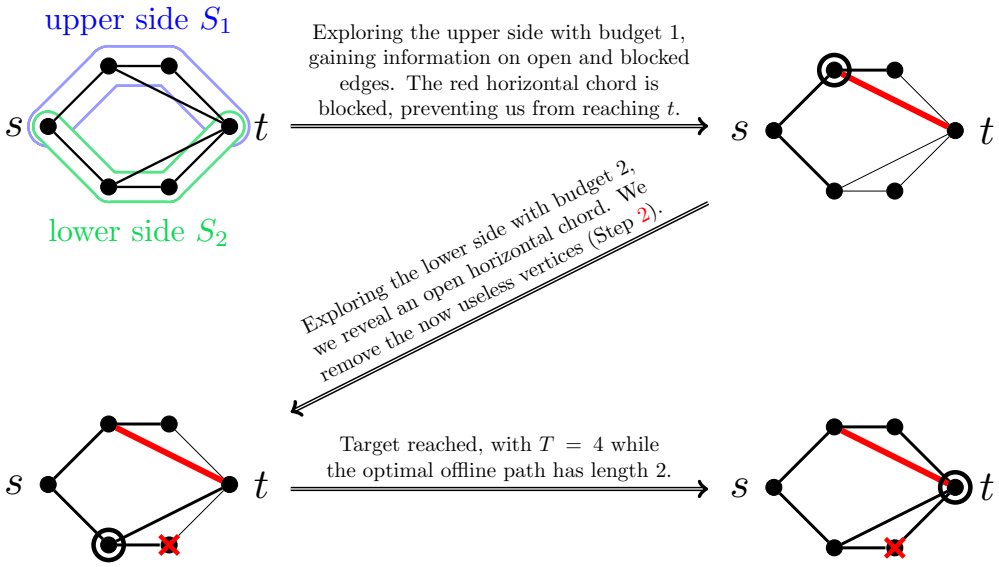


Figure 6: Application of ExpBalancing on the first graph of the decomposition of Figure 2. At each step, the circled vertex is the one we are currently exploring, and we know the status of the bold edges: black is open, red is blocked.

subsection, let G denote the smallest (by number of vertices, then number of edges) unit-weighted outerplanar graph on which ExpBalancing does not achieve competitive ratio 9. We will see that the existence of such graph G necessarily implies a contradiction.

We begin with a first observation dealing with the recursive call on some vertical chord of G.

Lemma 9. *Assume that we are executing* ExpBalancing *on graph G. Assume also that, at some moment of the execution, a recursive call is launched after revealing the vertical chord uv with new source u. Let T be the distance traversed before the recursive call. Then, either $T > 9d_{E_*}(s,u)$ or $d_{E_*}(s,v) < d_{E_*}(s,u) + 1$.*

Proof. If $d_{E_*}(s,v) \geq d_{E_*}(s,u) + 1$, following the rules established in Steps 4-6, we will launch a recursive call on the target component of $\{u,v\}$ with new source u. Hence, we will have $d^{\text{Tr}}_{\text{EXP}}(G, E_*) = T + T'$, where $T' \leq 9d_{E_*}(u,t)$ by minimality of G and EXP abbreviates ExpBalancing. By way of contradiction, suppose that $T \leq 9d_{E_*}(s,u)$. The optimal offline path P_{opt} necessarily goes through the separator $\{u,v\}$ in graph G and, since $d_{E_*}(s,v) = d_{E_*}(s,u) + 1$, u belongs to some optimal offline path. Consequently, $T + T' \leq 9(d_{E_*}(s,u) + d_{E_*}(u,t)) = 9d_{E_*}(s,t)$. □

In fact, ExpBalancing should launch at least one recursive call when applied to G, otherwise we obtain a contradiction:

Lemma 10. *During the execution of* ExpBalancing*, let T be the distance travelled at a given point before the first recursive call (if any). Then, $T \leq 9d_{\text{opt}}$. Moreover, if we are in state* **A***, let x and y be the last two vertices explored on each side during the exponential balancing. Then: (i) $d_{E_*}(s,x) = D$, (ii) $d_{E_*}(s,y) = D$ and (iii) $T \leq 5D$.*

Proof. Assume that we applied ExpBalancing on G until a certain point and that no recursive call was launched so far. We first focus on the second part of the invariant we want to show :

$$\text{In state } \mathbf{A},\ (i)\ d_{E_*}(s,x) = D,\ (ii)\ d_{E_*}(s,y) = D \text{ and } (iii)\ T \leq 5D.$$

Items (i) and (ii) are true, since no shortcut between s and either x or y can exist: any open horizontal chord is used, and an open vertical chord opening up a shortcut lead to a recursive call (Steps 5 and 6).

Item (*iii*) is trivially true when we kick-start the exponential balancing: when entering **A** from $\mathbf{E_2}$, we have $T = 3$ and $d_{E_*}(s, x) = d_{E_*}(s, y) = 1$. Assume that it is true for a given $D \geq 1$, and let T_0 be the value of T at this point. When we reach state **B**, we have $T = T_0 + D \leq 6D$. When we reach state **C**, we have $T = T_0 + D + 3D \leq 9D$. In brief, from state **A** to **C**, we have $d_{\text{opt}} \geq D$ as distance D was explored on both sides without reaching t. The largest ratio of T by D on these phases is 9 at state **C**, where we have $T \leq 9d_{\text{opt}}$.

During the transition from **C** to **A**, if $D + \alpha D$ denotes the traversed distance on current side at any moment (see Figure 4), then $d_{\text{opt}} \geq D + \alpha D$ and $T = 9D + \alpha D$. The ratio $\frac{T}{d_{\text{opt}}}$ admits a decreasing upper bound, from 9 in state **C** to 5 in **A**. Indeed, when we are back to state **A**, we have $T = T_0 + D + 3D + D$, but the value of D is updated. Let $D' = 2D$. We have $T = T_0 + 5D \leq 5D + 5D = 5D'$, and so item (*iii*) remains true during the core loop.

We also have to check that it is true when we met an open vertical chord uv between states **A** and **B** which satisfies $d_{E_*}(s, v) = d_{E_*}(s, u)$ (case $\beta D = \alpha D - 1$ in Step 5). In this case, the new value of D is $D' = D + \alpha D$ and we have $T \leq 5D + \alpha D + 1 + 2\alpha D \leq 5(D + \alpha D) = 5D'$ (since $\alpha D \geq 1$), so item (*iii*) remains true.

In summary, assuming that no recursive call is used on G leads to the conclusion that the competitive ratio of ExpBalancing on G is at most 9, a contradiction. □

We are now ready to prove the major contribution of this article.

Theorem 1. *There is a strategy with competitive ratio 9 for unit-weighted outerplanar graphs.*

Proof. From Lemma 10, we know that ExpBalancing will, during some attempt, launch a recursive call on G (otherwise, it has competitive ratio 9, a contradiction). Lemma 9 also has an important consequence: if we launch a recursive call on the open vertical chord uv with new source u and can guarantee that both $d_{E_*}(s, v) = d_{E_*}(s, u) + 1$ and $T \leq 9d_{E_*}(s, u)$, then, we have a contradiction. According to the description of ExpBalancing, a recursive call is launched when we are sure that $d_{E_*}(s, v) = d_{E_*}(s, u) + 1$: this concerns Step 4, Step 5 when $\beta D < \alpha D - 1$ and Step 6.

Assume first that we are blocked on one side between states **A** and **B** in Step 4 (see Figure 5a). We know that $d_{E_*}(s, v) = d_{E_*}(s, u) + 1$ because u is an articulation point of $G \backslash E'_*$ and $d_{E_*}(s, u) = D + d_{E_*}(y, u)$. Let T be the distance traversed before the recursive call, then using the invariant in state **A** we have:

$$\begin{array}{rcll} T & \leq & (5D + \alpha D) + (\alpha D + 2D) + d_{E_*}(y, u) & \\ & \leq & (7 + 2\alpha)D + d_{E_*}(y, u) & \\ & \leq & 9(D + d_{E_*}(y, u)) & (\alpha \leq 1) \\ & \leq & 9d_{E_*}(s, u) & \end{array}$$

which, by Lemma 9, leads to a contradiction.

Assume now that we are blocked on one side between states **C** and **A** in Step 4 (see Figure 5b). Let x' be the last vertex reached at the end of state **A**, we know that $d_{E_*}(s, v) = d_{E_*}(s, u) + 1$ because u is an articulation point of $G \backslash E'_*$ and $d_{E_*}(s, u) = 2D + d_{E_*}(x', u)$, and we have:

$$\begin{array}{rcll} T & \leq & (9D + \alpha D) + (\alpha D + 3D) + d_{E_*}(x', u) & \\ & \leq & (12 + 2\alpha)D + d_{E_*}(x', u) & \\ & \leq & 9(2D + d_{E_*}(x', u)) & (\alpha \leq 1) \\ & \leq & 9d_{E_*}(s, u) & \end{array}$$

which, by Lemma 9, leads to a contradiction.

Assume now that we reveal an open vertical chord uv between states **A** and **B** in Step 5 (see Figure 5c). Recall that $d_{E_*}(s, u) \leq D + \alpha D$, and we explore up to distance $\alpha D - 1$ towards y. There are two possibilities: either we see y by exploring distance βD (with $\beta D < \alpha D - 1$), or we do not see y even if we explore distance $\alpha D - 1$.

If we see y, then, we know that $d_{E_*}(s, u) = d_{E_*}(s, v) + 1$ since going to u through x will yield distance $D + \alpha D$ while going through y and v will yield distance at most $D + \beta D + 2$, and we

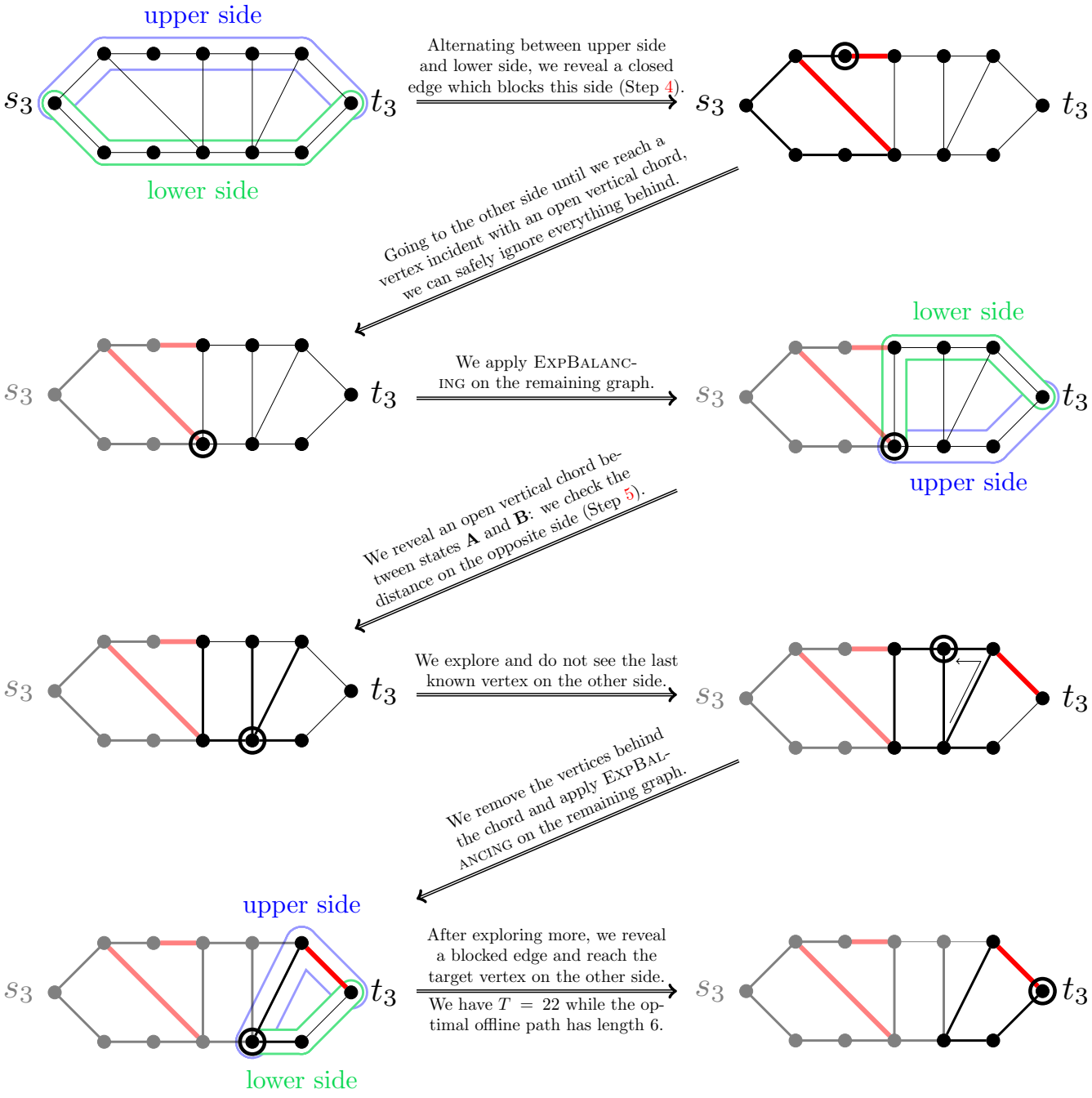


Figure 7: Application of ExpBalancing on the third graph of the decomposition of Figure 2. At each step, the circled vertex is the one we are currently exploring, and we know the status of the bold edges: black is open, red is blocked.

know that $\beta D < \alpha D - 1$ and $\beta D \geq 0$. So, $d_{E_*}(s, v) = D + \beta D + 1$ and we have:

$$\begin{array}{rcll} T & \leq & (5D + \alpha D) + (1 + 2(\beta D + 1)) & \\ & \leq & (5 + \alpha + 2\beta)D + 3 & \\ & \leq & 9(D + \beta D + 1) & (\beta < \alpha \leq 1) \\ & \leq & 9d_{E_*}(s, v) & \end{array}$$

which, by Lemma 9 leads to a contradiction (the roles of u and v are reversed here, since v is the new source).

If we do not reach y, either by blocked edges or because we have explored distance $\alpha D - 1$ without reaching it, then, we know that $d_{E_*}(s, v) = d_{E_*}(s, u) + 1$, so we have:

$$\begin{array}{rcll} T & \leq & (5D + \alpha D) + (1 + 2(\alpha D - 1) + 1) & \\ & \leq & (5 + 3\alpha)D & \\ & \leq & 9(D + \alpha D) & (\alpha \leq 1) \\ & \leq & 9d_{E_*}(s, u) & \end{array}$$

which, by Lemma 9, leads to a contradiction.

Finally, assume that we reveal an open vertical chord uv between states **C** and **A** after having explored αD vertices in Step 6 (see Figure 5d). Since uv was not revealed before, this implies that the shortest path from s to v goes through u, and so $d_{E_*}(s, v) = d_{E_*}(s, u) + 1$, so we have:

$$\begin{array}{rcll} T & \leq & 9D + \alpha D & \\ & \leq & 9(D + \alpha D) & (\alpha \leq 1) \\ & \leq & 9d_{E_*}(s, u) & \end{array}$$

which, by Lemma 9, leads to a contradiction.

Hence, all the possible cases lead to contradictions, and so such G cannot exist. ExpBalancing thus achieves competitive ratio 9 on unit-weighted outerplanar graphs. □

Strategy ExpBalancing can thus naturally be applied on outerplanar graphs where the stretch of weights is bounded.

Corollary 11. *There is a strategy with competitive ratio $9S$ on outerplanar graphs of stretch S.*

Proof. Apply strategy ExpBalancing as if the graph was unit-weighted. Let α be the minimum weight of the input graph and W_{opt} be the number of edges of the optimal offline path. The total distance traversed is upper-bounded by $9S\alpha W_{\text{opt}}$ while $d_{\text{opt}} \geq \alpha W_{\text{opt}}$. □

4 Lower bound 9 for unit-weighted outerplanar graphs

In this section we prove that the competitive ratio achieved with the ExpBalancing strategy presented above is optimal on unit-weighted outerplanar graphs.

Theorem 2. *For any $\varepsilon > 0$, no deterministic strategy can achieve a competitive ratio $9 - \varepsilon$ on all road maps (G, E_*), where G is a unit-weighted outerplanar graph.*

The proof follows this plan: we introduce an infinite family of outerplanar graphs and show that any potential strategy on these can be encoded as a sequence of positive integers. Then for any given $\varepsilon > 0$ we find a large enough graph of this family such that there is no strategy reaching ratio $9 - \varepsilon$ on it. This non-existence is obtained by using Farkas' lemma on a set of inequalities which would all be satisfied if a strategy with ratio $9 - \varepsilon$ existed.

The *shell graph* on $2n$ vertices, denoted by Sh_n, is the graph obtained from a cycle on $2n$ vertices $\{v_0, v_1, \ldots, v_{2n-1}\}$ with all possible chords incident with vertex v_n, except v_0v_n. In our setting, we shall consider v_0 as the source s and v_n as the target t (see Figure 8). It is clearly outerplanar. All weights are naturally fixed to 1. It is 2-connected, so it contains an upper side S_1 and a lower side S_2. We will focus on specific road maps (Sh_n, E_*) where E_* is made up only of edges incident with t. Said differently, the traveller cannot be blocked on the outer face on some edge v_iv_{i+1}. We will see that even in this tight configuration, ratio $9 - \varepsilon$ cannot be achieved.

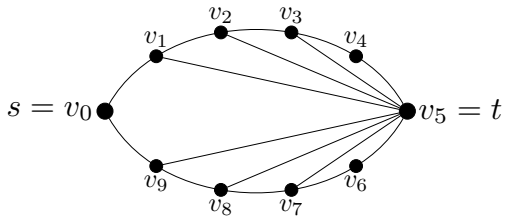


Figure 8: The shell graph on 10 vertices Sh_5.

Given some positive integer n, let A_n be an optimal strategy for graph Sh_n, *i.e.*, a strategy which minimizes the competitive ratio obtained on road maps (Sh_n, E_*). A first observation is that, following A_n, when the traveller stands on some vertex v_i, he should always traverse a chord v_it if it is open (the ratio can only increase if the traveller decides to explore the graph a bit longer). Furthermore, when traversing an already visited section of a side, A_n should do it directly, with a simple walk, and avoid multiple crossings of the same edge. Also note that if the traveller, following A_n, starts switching sides by coming back to s, he should not change his mind and go back to exploring his side, since doing so would incur a cost for no additional information (no edge can be revealed this way). Finally, if the traveller has already explored ℓ vertices on a side, when he goes back to exploring this same side, he should always traverse at least $\ell+1$ vertices (so reveal the edges incident with at least one more vertex) before switching again. Doing otherwise would incur a cost for no additional information.

Thus strategy A_n can be described as a sequence of integers $(x_i)_{i\geq 1}$ where x_i represents the budget distance we afford ourselves after coming back to s and switching side. Concretely, first, the traveller selects one side arbitrarily (say S_1) and traverses a distance of x_1 on the outer face. If an open chord is revealed during his walk, he reaches t. Otherwise, after being forced to stay on the outer face, he has no choice but backtrack towards s and traverse the other side S_2 for some distance x_2. Generally, value x_i denotes the distance budget we allow ourselves to traverse for the ith attempt, (on upper side S_1 for odd i and bottom side S_2 for even i) before coming back to s if no horizontal chord to t was found. The strategy ends whenever an open edge incident with t is found and traversed. Observe that all values x_i are at least 1. Furthermore, by the last observation of the previous paragraph, we always have $x_{i+2} > x_i$.

Assume the traveller reaches target t on attempt $j+1$. Let S_i be the last side visited, $i \in \{1, 2\}$. Compatible with such travel hypothesis, we consider the road map (Sh_n, E_*) with the following blocked edges in E_*.

- all chords ut where u is on S_i and is at distance at most x_{j-1} from s,
- all chords ut where u is not on S_i.

In this way, we force the traveller to reach t via the vertex which was placed just after the last one he visited during attempt $j-1$. We have $d_{\mathrm{opt}} = x_{j-1} + 2$ and the total traversed distance from the beginning of the walk is $2(\sum_{i=1}^{j} x_i) + x_{j-1} + 2$.

Assume that A_n achieves a ratio strictly less than 9 on the road map (Sh_n, E_*). Then, there must be a strictly positive ε such that (we set $x_0 = 0$):

$$\forall j \geq 1, \quad \frac{2\sum_{i=1}^{j} x_i + x_{j-1} + 2}{x_{j-1} + 2} \leq 9 - 2\varepsilon.$$

These inequalities can be rewritten as:

$$\forall j \geq 1, \quad \left(\sum_{i=0, i\neq j-1}^{j} x_i\right) - (3-\varepsilon)x_{j-1} \leq 8 - 2\varepsilon.$$

They form a system of linear inequalities with a lower triangular matrix. All entries equal 1 on the diagonal and lower, except for elements on the subdiagonal (lower diagonal) which all

equal $-(3-\varepsilon)$. Our system is $M_{j,\varepsilon} \cdot \mathbf{x}_j \leq \mathbf{b}_{j,\varepsilon}$ with $\mathbf{b}_{j,\varepsilon} = (8-2\varepsilon)\cdot\mathbf{1}$, where $\mathbf{x}_j$ is the vector made of first j values of sequence $(x_i)_{i\geq 1}$ and matrix $M_{j,\varepsilon}$ is the following $j \times j$ matrix:

$$M_{j,\varepsilon} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -(3-\varepsilon) & 1 & 0 & \cdots & 0 \\ 1 & -(3-\varepsilon) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & -(3-\varepsilon) & 1 \end{pmatrix}$$

When no ambiguity is present, we shall omit subscripts and write M, $\mathbf{b}$ and $\mathbf{x}$. Farkas' lemma (hereafter recalled) deals with the existence (or not) of nonnegative solutions for a system of linear inequalities. Since all values x_i's of vector $\mathbf{x}$ are at least 1, we may shift our vector by 1 and still be nonnegative. In other words, there should exist a nonnegative vector $\mathbf{x}' = \mathbf{x} - \mathbf{1}$ such that $M \cdot (\mathbf{x}' + \mathbf{1}) \leq \mathbf{b}$. After rewriting, we get $M\mathbf{x}' \leq \mathbf{b}'$ where $\mathbf{b}'$ is the vector $\mathbf{b} - M \cdot \mathbf{1}$. Note that $M \cdot \mathbf{1}$ has the following coordinates: $(1, -2+\varepsilon, -1+\varepsilon, \varepsilon, 1+\varepsilon, 2+\varepsilon, 3+\varepsilon, \ldots)$. Thus the coordinate b'_i of vector $\mathbf{b}'$ is negative for any $i \geq 12$.

$$\mathbf{b}' = \mathbf{b} - M \cdot \mathbf{1} = \begin{pmatrix} 7-2\varepsilon \\ 10-3\varepsilon \\ 9-3\varepsilon \\ 8-3\varepsilon \\ \vdots \\ 12-j-3\varepsilon \end{pmatrix}$$

We are now ready to establish a relationship between a system of linear inequalities and the competitiveness of strategies A_n.

Proposition 12. *Assume there exists a positive integer j and some real $\varepsilon > 0$ such that the system $M_{j,\varepsilon} \cdot \mathbf{x}' \leq \mathbf{b}'_{j,\varepsilon}$, $\mathbf{x}' \geq \mathbf{0}$ has no solution. Then, there exists an integer $n_{j,\varepsilon}$ such that strategy A_n has ratio at least $9-2\varepsilon$.*

Proof. From observations above, if all strategies A_n have ratio at most $9-2\varepsilon$ for some $\varepsilon > 0$, then, for any positive integer j, the system $M_{j,\varepsilon} \cdot \mathbf{x}' \leq \mathbf{b}'_{j,\varepsilon}$, $\mathbf{x}' \geq \mathbf{0}$ has necessarily a solution. Using the contraposition for any $\varepsilon > 0$ gives the proof. □

Proposition 12 implies that our lower bound of competitiveness for unit-weighted outerplanar graphs can be proved by showing that some system of linear inequalities has no solution. We now recall the statement of Farkas' lemma in our context:

Lemma 13 (Farkas [15], see [17, Prop 6.4.3])**.** *Exactly one of the following holds: either the system $M \cdot \mathbf{x}' \leq \mathbf{b}'$ has a solution with $\mathbf{x}' \geq \mathbf{0}$, or the system $M^T \cdot \mathbf{y} \geq \mathbf{0}^T$ has a nonnegative solution $\mathbf{y}$ with $\mathbf{b}'^T \cdot \mathbf{y} < 0$.*

We now find a nonnegative vector $\mathbf{y}$ of size j such that $M^T_{j,\varepsilon} \cdot \mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}'^T_{j,\varepsilon} \cdot \mathbf{y} < 0$, which will allow us to obtain a contradiction.

Proposition 14. *For any $\varepsilon > 0$, there exists a positive integer j and a nonnegative vector $\mathbf{y}$ of size j such that $M^T_{j,\varepsilon} \cdot \mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}'^T_{j,\varepsilon} \cdot \mathbf{y} < 0$.*

Proof. For a given $\varepsilon > 0$, we fix an integer j depending on ε. The choice of value j will be made clear hereafter. For now, consider that j is only greater than 12. Our construction of $\mathbf{y}$ consists in identifying a solution $M^T \cdot \mathbf{y}$ very close to vector $\mathbf{0}$ and then verifying whether $\mathbf{b}'^T \cdot \mathbf{y} < 0$.

Let us consider the equation $M^T \cdot \mathbf{y} = \mathbf{0}$, where the coordinates of $\mathbf{y}$ are $y_1, \ldots y_j$. For any $1 \leq i \leq j-1$, we have:

$$y_i - (3-\varepsilon)y_{i+1} + y_{i+2} + y_{i+3} + \ldots + y_j = 0.$$

By subtracting two consecutive such equations, we get that, for any $1 \leq i \leq j-3$:

$$y_i = (4-\varepsilon)y_{i+1} - (4-\varepsilon)y_{i+2}.$$

Reversing the indices, we recognize a linear recurrence relation of depth 2: $u_{n+2} = (4-\varepsilon)u_{n+1} - (4-\varepsilon)u_n$. The characteristic equation is $\lambda^2 - (4-\varepsilon)\lambda + (4-\varepsilon) = 0$ with roots:

$$\lambda_{1,2} = \frac{(4-\varepsilon) \pm \sqrt{-\varepsilon(4-\varepsilon)}}{2}.$$

Observe that if $\varepsilon = 0$, then the characteristic equation has a single root and the sequence is exponentially increasing. For $\varepsilon > 0$, however, both roots are complex numbers.

Since there are two roots, we could rewrite it as $u_n = (\sqrt{4-\varepsilon})^n(c_1\cos(\alpha n) + c_2\sin(\alpha n))$, where $\alpha = \operatorname{atan}\left(\sqrt{\frac{\varepsilon}{4-\varepsilon}}\right)$. Note that when ε tends to zero, the period of oscillation tends to infinity. The coordinates of the vector $\mathbf{y}$ will follow, starting from the bottom, the scheme of this sequence u_n. We fix as an initial condition $y_j = 1$. Consequently, $y_{j-1} = 3-\varepsilon$ and the following terms follow the scheme of the sequence. With these initial conditions, we obtain:

$$y_{j-p} = \frac{2(\sqrt{4-\varepsilon})^{p-1}}{\sqrt{\varepsilon}}\cos(\alpha p - \beta). \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_{j-2} \\ y_{j-1} \\ y_j \end{pmatrix} = \begin{pmatrix} u_{j-1} \\ \vdots \\ u_3 \\ u_2 \\ u_0 \end{pmatrix} = \begin{pmatrix} u_{j-1} \\ \vdots \\ (3-\varepsilon)^2 - 1 \\ 3-\varepsilon \\ 1 \end{pmatrix} \tag{1}$$

where $\beta = \operatorname{atan}\left(\frac{2-\varepsilon}{\sqrt{\varepsilon(4-\varepsilon)}}\right)$. In this way, $M^T \cdot \mathbf{y}$ is the vector made of zeros except the last coordinate being equal to 1.

Equation (1) shows us that we start from positive values $(y_j, y_{j-1}, \ldots)$ and then alternate between positive and negative with an increasing amplitude. For any small $\varepsilon > 0$, we select j such that terms $u_0, \ldots, u_{j-12}$ are positive and the last eleven terms $u_{j-11}, \ldots, u_{j-1}$ are negative. This choice of j is possible since, for sufficiently small ε, the "angular speed" α and the "shift" β are negligible compared to $\frac{\pi}{2}$. Hence, let j be the integer such that $\alpha(j-11) - \beta \leq \frac{\pi}{2}$, while $\alpha(j-10) - \beta > \frac{\pi}{2}$. On vector $\mathbf{y}$, it means that the 11 first terms $y_1, \ldots, y_{11}$ are negative while all others are positive. Observe that the more ε decreases towards 0, the largest the period of sequence (u_i) is and hence the largest this integer j is.

Values $y_1, \ldots, y_{11}$, which are the negative values of $\mathbf{y}$, are replaced by 0: we obtain a new vector $\mathbf{y}'$. In this way, value $\mathbf{b}'^T \cdot \mathbf{y}$ is negative: first values $y'_1, \ldots, y'_{11}$ are zeros, second values $y'_{12}, \ldots, y'_j$ are positive while $b'_{12}, \ldots, b'_j$ are negative. Moreover, obviously, $\mathbf{y}' \geq \mathbf{0}$. Let us check that even with this modification on 11 entries, we still have $M^T \cdot \mathbf{y}' \geq \mathbf{0}$, allowing us to use Farkas' lemma.

Suppose by way of contradiction that some coordinate of $M^T \cdot \mathbf{y}'$ is negative: say $y'_i - (3-\varepsilon)y'_{i+1} + y'_{i+2} + \ldots + y'_j < 0$. As all y'_i are nonnegative, then necessarily y'_{i+1} is positive and thus $y'_{i+1} = y_{i+1}$. Either $y'_i > 0$, so $y'_i = y_i$ and the linear sum is nonnegative from the definition of sequence $(u_i)_{i\geq 1}$, a contradiction; or $y'_i = 0$ but then again the previous equation is positive since $y_i - (3-\varepsilon)y'_{i+1} + y'_{i+2} + \ldots + y'_j = 0$ and we shifted it positively by replacing y_i by y'_i which is greater. In brief, vector $\mathbf{y}'$ verifies $M^T \cdot \mathbf{y}' \geq 0$.

As a conclusion, we identified a nonnegative vector $\mathbf{y}'$ satisfying the requirements of the proposition. □

So now, by Farkas's lemma, we prove that the initial system of Proposition 12 of inequalities does not have any solution.

Proof of Theorem 2. Simply fix $\varepsilon' = 2\varepsilon > 0$. From Lemma 13 and Proposition 14, we know that there exists an integer j such that there is no nonnegative solution $\mathbf{x}'$ of system $M_{j,\varepsilon} \cdot \mathbf{x}' \leq \mathbf{b}'_{j,\varepsilon}$. From Proposition 12, there is no deterministic strategy achieving ratio $9-\varepsilon'$ on every road map of the family of graphs Sh_n, and hence on the super-family of unit-weighted outerplanar graphs. □

5 The case of arbitrarily weighted outerplanar graphs

Given our results on the unit-weighted case (and which could generalized to fixed stretch naturally), a natural question is whether we can design a deterministic strategy achieving a constant competitive ratio for the more general family of arbitrarily weighted outerplanar graphs. In this

section, we prove that this is impossible since, for any constant $C \geq 1$, there exists a weighted outerplanar graph on which the competitive ratio obtained is greater than C.

Let us introduce a sub-family of outerplanar graphs that will be useful here.

Definition 15. *An outerplanar graph G containing s and t is said to be* (s,t)-unbalanced *if either it is a single st edge or one of its sides contains all vertices V of the graph.*

In other words, an (s,t)-unbalanced outerplanar graph is such that s and t are neighbors on the outerface. While one side contains all vertices (say w.l.o.g. the lower side), the upper one only contains s and t and simply consists in a single edge st. Such graph thus does not admit vertical chords. We show in the remainder that, in fact, constant competitive ratio cannot be obtained even on weighted (s,t)-balanced outerplanar graphs.

We begin with the definition of a graph transformation $\mathcal{T}$ which takes as input: a weighted (s,t)-unbalanced outerplanar graph $H = (V, E, \omega)$, three positive rational values α, C, and η, and an integer N. The construction of the output graph $\mathcal{T}(H, \alpha, C, \eta, N)$ works as follows:

- Create two vertices s and t with an edge st of weight C. This edge will stand as the upper side of the graph.
- Add N copies of the graph αH, where $\alpha H = (V_\alpha, E_\alpha, \omega_\alpha)$ is a graph such that $V_\alpha = V$, $E_\alpha = E$ and $\omega_\alpha(e) = \alpha\omega(e)$ for every edge $e \in E$. These copies are denoted by $\alpha H^{(1)}, \ldots, \alpha H^{(N)}$ and the source/target pair of each $\alpha H^{(j)}$ is denoted by (s_j, t_j).
- Connect in series all copies $\alpha H^{(1)}, \ldots, \alpha H^{(N)}$ from s to t in order to form the lower side of the graph, using their source/target as input/output vertices. In brief, merge s with s_1, t_i with s_{i+1} for $i \in \{1, \ldots, N-1\}$, and t_N with t.
- Add all edges $t_j t$ for $1 \leq j \leq N-1$ with weight η.

Figure 9 illustrates the graph $\mathcal{T}(H, \alpha, C, \eta, N)$ obtained. Observe that it is an (s,t)-unbalanced outerplanar graph because the lower side of each αH contains all its own vertices. Therefore, all vertices of $\mathcal{T}(H, \alpha, C, \eta, N)$ lie on its lower side. We also set $t_0 = s$.

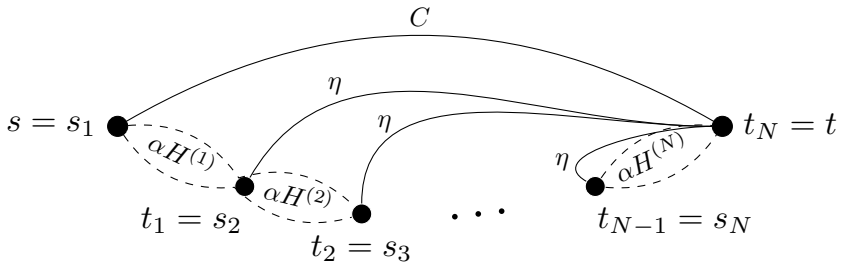


Figure 9: The graph $\mathcal{T}(H, \alpha, C, \eta, N)$ with its outerplanar embedding.

For the remainder, we define a trivial arithmetic sequence generating all positive half-integers: for any integer $i \geq 0$, let $C_i = \frac{1}{2} + i$. For any value C_i, we are able to construct a collection of road maps for which ratio C_i cannot be achieved by any deterministic strategy.

Proposition 16. *For any nonnegative integer i, there exists a family $\mathcal{R}_i$ of road maps which satisfies the following properties:*

- *all the road maps of $\mathcal{R}_i$ are defined on the same weighted (s,t)-unbalanced outerplanar graph,*
- *no deterministic strategy can achieve ratio C_i on family $\mathcal{R}_i$.*

Proof. We proceed by induction on i.

Base case. Let H_0 be the weighted graph with a single edge st of weight 1. By Definition 15, H_0 is (s,t)-unbalanced. Moreover, we necessarily have $E_* = \emptyset$ since s and t cannot be disconnected by blockages (Definition 4). Thus, $\mathcal{R}_0$ contains the single road map $(H_0, \emptyset)$. Obviously, by definition, the competitive ratio of any strategy on any road map cannot be less than 1. So, naturally, no deterministic strategy can achieve ratio $C_0 = \frac{1}{2}$ on $(H_0, \emptyset)$.

Induction step. We suppose that the statement holds for some integer $i \geq 0$. Let H_i be the (s,t)-unbalanced outerplanar graph on which the road maps of $\mathcal{R}_i$ are defined. We will construct the graph H_{i+1} by applying transformation $\mathcal{T}$ on graph H_i. Let α and η be two positive rationals and N a positive integer: we define $G = \mathcal{T}(H_i, \alpha, C_i, \eta, N)$. We will see, after some competitive analysis on G, which values of α, η, and N can be selected in such a way that ratio C_{i+1} cannot be performed on such graph. Eventually, we will fix $H_{i+1} = G$ with the relevant values for parameters α, η, and N. Together with G, we consider sets of blocked edges E_* which contain certain edges $t_j t$ but also some edges lying inside the copies $\alpha H_i^{(j)}$ of G. However, st will never belong to E_*.

Let $L_{\min}$ be the cost of the shortest (s,t)-path in H_i: $L_{\min} = d_\emptyset(H_i, s, t)$. Also, let $L_{\max}$ be the cost of the longest simple (s,t)-path in H_i: $L_{\max} = \max_{E_*}(d_{E_*}(H_i, s, t))$.

Intuitively, when fixing $H_{i+1} = G$ at the end of our analysis, parameters α and η will be selected small enough to satisfy several criteria. Similarly, N will be selected large enough. We already impose certain conditions on these three parameters.

- α and η are small enough so that the cost of the longest simple (s,t)-path of H_i is negligible compared to C_i: we impose
$$\alpha L_{\max} + \eta < C_i(C_i + 1). \tag{2}$$
- N is large enough so that the shortest (s,t)-path by passing through all $H_i^{(j)}$ copies is very large compared to C_{i+1}: we impose:
$$N\alpha L_{\min} > C_i + 1 = C_{i+1}. \tag{3}$$

Let A be a deterministic strategy dedicated to graph G. The behaviour of a traveller guided by strategy A on G is rather limited: either he directly traverses the st edge of weight C_i, or he explores the lower side. With this second option, either he reaches t directly via the lower side, or he decides at some step to come back to s in order to traverse the open st edge. Let q be the maximum index $0 \leq q \leq N$ such that the vertex t_q was visited by the traveller during his exploration of the lower side. If some edge $t_j t$, $1 \leq j \leq q$ was open, the traveller would have chosen to traverse it. For this reason, we assume $\{t_1 t, t_2 t, \ldots, t_q t\} \subseteq E_*$ in the remainder.

We fixed N large enough so that having $t_q = t$ would imply that the traversed distance is arbitrarily large compared to the open edge st of weight C_i. Indeed, from the induction hypothesis, the distance traversed on graph H_i is at least C_i times the optimal offline cost. This statement can be generalized to a series of graph H_i, from Lemma 8. For this reason, the traversed distance is at least $C_i N \alpha L_{\min} > C_i C_{i+1}$. Hence, having $q = N$ cannot give competitive ratio at most C_{i+1}. So, we assume $q < N$.

Let D be the optimal distance from s to t_q without using t (and thus only traversing $\alpha H_i^{(1)}$, …, $\alpha H_i^{(q)}$). Formally, $D = d_{E_*}(G \setminus \{t\}, s, t_q)$. The total distance traversed during the walk of the traveller (denoted by d_A^{Tr}) on G satisfies: $d_A^{\mathrm{Tr}} > C_i D + D + C_i$. The term $C_i D$ is a lower bound for the travel from s to t_q, the term D is a lower bound for the return trip towards s, and C_i is the cost of edge st. We now provide a competitive analysis and distinguish three cases.

Case A. If $D + \alpha L_{\min} + \eta \geq C_i$. The optimal offline cost of the road map (G, E_*) is C_i. Indeed, as t_q is the last target vertex t_j visited on the lower side, we know that $t_q t$ is blocked and, at best, the shortest (s,t)-path passing through the lower side has cost at least $D + \alpha L_{\min} + \eta$. Hence, in this case, the competitive ratio of A over this road map satisfies:

$$c_A(H_{i+1}, E_*) > \frac{C_i D + D + C_i}{C_i} \geq 1 + \left(1 + \frac{1}{C_i}\right) D \geq 1 + \left(1 + \frac{1}{C_i}\right)(C_i - \alpha L_{\min} - \eta)$$

We impose restrictions on α and η such that the following conditions holds:

$$\left(1 + \frac{1}{C_i}\right)(C_i - \alpha L_{\min} - \eta) > C_i \tag{4}$$

Case B. Else, if $D + \alpha L_{\max} + \eta \leq C_i$, then $d_{\mathrm{opt}} < D + \alpha L_{\max} + \eta$. Indeed, observe that the traveller did not reach t_{q+1}, so the edge $t_{q+1} t$ could be open without influencing the initial decision of the traveller to come back. If $q = 0$ and $D = 0$, the competitive ratio is C_i divided

by $d_{E_*}(\alpha H_i^{(1)}, s, t) + \eta$. This ratio is larger than $\frac{C_i}{\alpha L_{\max}+\eta}$, and according to condition (2), it is greater than C_{i+1}. So, let us assume $D > 0$. In this case, the competitive ratio of A satisfies:

$$c_A(H_{i+1}, E_*) > \frac{C_i D + D + C_i}{D + \alpha L_{\max} + \eta} \geq 1 + (C_i + 1)\frac{D}{D + \alpha L_{\max} + \eta}$$

We impose restrictions on α and η such that the following conditions holds:

$$(C_i + 1)\frac{D}{D + \alpha L_{\max} + \eta} > C_i \tag{5}$$

Case C. Else, $D + \alpha L_{\min} + \eta < C_i < D + \alpha L_{\max} + \eta$. The optimal offline cost is for sure at most C_i as st is open, and the competitive ratio satisfies:

$$c_A(H_{i+1}, E_*) > \frac{C_i D + D + C_i}{C_i} \geq 1 + \left(1 + \frac{1}{C_i}\right) D > 1 + \left(1 + \frac{1}{C_i}\right)(C_i - \alpha L_{\max} - \eta)$$

We impose restrictions on α and η such that the following conditions holds:

$$\left(1 + \frac{1}{C_i}\right)(C_i - \alpha L_{\max} - \eta) > C_i \tag{6}$$

We can select small enough rationals α and η such that conditions (2), (4), (5) and (6) are all satisfied, and in turn a large enough integer N such that condition (3) also holds. After doing so, we construct the graph $H_{i+1} = \mathcal{T}(H_i, \alpha, C_i, \eta, N)$ which is clearly (s,t)-unbalanced outerplanar and define the family $\mathcal{R}_{i+1}$ as the road maps given by (H_{i+1}, E_*) for all sets of blocked edges E_* verifying $\{t_1 t, \ldots, t_q t\} \subseteq E_*$ and $st \notin E_*$. The competitive analysis that we conducted proves that strategy A does not have competitive ratio $C_i + 1 = C_{i+1}$ on $\mathcal{R}_{i+1}$, and thus that the induction hypothesis holds. □

Hence, no deterministic strategy can achieve constant competitive ratio on weighted outerplanar graphs since integer i can take arbitrarily large values:

Theorem 3. *There is no constant C, independent from G and k, such that a deterministic strategy achieves competitive ratio C on all road maps (G, E_*) where G is a weighted outerplanar graph.*

Proof. Assume by contradiction that there is a constant C such that a strategy A achieves competitive ratio C on all such road maps. Let $i = \lceil C \rceil$. By Proposition 16, there is a family of road maps $\mathcal{R}_i$ defined on outerplanar graphs such that A cannot achieve competitive ratio $C_i = \lceil C \rceil + \frac{1}{2}$ (and thus C) on $\mathcal{R}_i$, a contradiction. □

6 Perspectives

The main contribution of this article is the highlighting of a non-trivial unit-weighted family of graphs for which there exists a deterministic strategy, called ExpBalancing, with constant competitive ratio. This family is the well-known class of outerplanar graphs. We also proved that the ratio 9 obtained with strategy ExpBalancing is optimal. From this statement, several questions rise:

- Is there a non-trivial sub-family of unit-weighted outerplanar graphs for which the optimal ratio of deterministic strategies is strictly between 1 and 9? For now, the only sub-family for which some optimal bounds are known are the trivial classes of trees (with ratio 1) and cycles (with ratio 3, which can be extended to cacti by Lemma 8). It would be interesting to further investigate the gap between those classes and outerplanar graphs.
- Is there a super-family of unit-weighted outerplanar graphs which admit constant competitive ratio? In fact, the gap is quite tight, given that planar graphs of treewidth 2 admit an optimal ratio $2k+1$. Future research could focus on the natural extension of p-outerplanar graphs [3], on which all vertices can be deleted by successively removing at most p outerfaces.

- Are there other families of graphs, not necessarily related to outerplanar, where the competitive ratio is unbounded for arbitrary weights, but for which the unit-weighted version admits constant competitive ratio?

For the general weighted case, we established that no constant competitive ratio can be obtained with deterministic strategies on arbitrarily weighted outerplanar graphs. A natural question is the dependence on k we can achieve for this family of graphs. Indeed, we know from [8] that there is some strategy with ratio linear in k: $\rho k + O(1)$ with $\rho = 2^{\frac{3}{4}}$. Is it possible to achieve a ratio $O(k^r)$, with $r < 1$, on weighted outerplanar graphs?

We also wonder whether there exists a polynomial-time deterministic strategy with constant competitive ratio on some non-trivial family of graphs. A good candidate would be the family of graphs with bounded-sized minimal edge (s,t)-cuts. We already know that a ratio $\sqrt{2}k + O(1)$ can be achieved on this family [8]. Moreover, observe that the construction $\mathcal{T}$ used to reject the existence of constant competitive ratio for outerplanar graphs, produces graphs with edge (s,t)-cuts of unbounded size. It seems at first sight that such a criterion is necessary if we want to make deterministic strategies fail on outerplanar graphs. All in all, we conjecture that there exists a polynomial-time deterministic strategy achieving constant competitive ratio on graphs with edge (s,t)-cuts of bounded size.

References

[1] V. Aksakalli, O. F. Sahin, and I. Ari. An AO* based exact algorithm for the canadian traveler problem. *INFORMS Journal on Computing*, 28(1):96–111, 2016.

[2] A. F. Alkaya, S. Yildirim, and V. Aksakalli. Heuristics for the Canadian Traveler Problem with neutralizations. *Comput. Ind. Eng.*, 159:107488, 2021.

[3] B. S. Baker. Approximation algorithms for NP-complete problems on planar graphs. *J. ACM*, 41(1):153–180, 1994.

[4] A. Bar-Noy and B. Schieber. The Canadian Traveller Problem. In *Proc. of ACM/SIAM SODA*, pages 261–270, 1991.

[5] J. Becker and R. Batta. Canadian prize collection problem. *Military Operations Research*, 28(2):55–92, 2023.

[6] M. Bender and S. Westphal. An optimal randomized online algorithm for the k-Canadian Traveller Problem on node-disjoint paths. *J. Comb. Optim.*, 30(1):87–96, 2015.

[7] P. Bergé, J. Desmarchelier, W. Guo, A. Lefebvre, A. Rimmel, and J. Tomasik. Multiple canadians on the road: minimizing the distance competitive ratio. *J. Comb. Optim.*, 38(4):1086–1100, 2019.

[8] P. Bergé and L. Salaün. The influence of maximum $(s,\ t)$-cuts on the competitiveness of deterministic strategies for the Canadian Traveller Problem. *Theor. Comput. Sci.*, 941:221–240, 2023.

[9] Z. Bnaya, A. Felner, and S. E. Shimony. Canadian traveler problem with remote sensing. In *Proc. of IJCAI*, pages 437–442, 2009.

[10] A. Borodin and R. El-Yaniv. *Online computation and competitive analysis*. Cambridge Univ. Press, 1998.

[11] H. Chan, J. Chang, H. Wu, and T. Wu. The k-Canadian Traveller Problem on Equal-Weight Graphs. *Proc. of WCMCT*, pages 135–137, 2015.

[12] G. Chartrand and F. Harary. Planar permutation graphs. *Ann. Inst. H. Poincaré Probab. Statist. B*, 3:433–438, 1967.

[13] E. D. Demaine, Y. Huang, C. Liao, and K. Sadakane. Canadians Should Travel Randomly. *Proc. of ICALP*, pages 380–391, 2014.

[14] P. Eyerich, T. Keller, and M. Helmert. High-quality policies for the Canadian Traveler's Problem. In *Proces. of AAAI*, pages 51–58. AAAI Press, 2010.

[15] J. Farkas. Theorie der einfachen ungleichungen. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 1902(124):1–27, 1902.

[16] D. Fried, S. E. Shimony, A. Benbassat, and C. Wenner. Complexity of canadian traveler problem variants. *Theor. Comput. Sci.*, 487:1–16, 2013.

[17] Jiří Gärtner, Bernd; Matoušek. *Understanding and Using Linear Programming.*, page 81–104. Berlin: Springer., 2006.

[18] N. Hahn and M. Xefteris. The covering canadian traveller problem revisited. In *Procs. of MFCS*, volume 272 of *LIPIcs*, pages 53:1–53:12, 2023.

[19] C. Liao and Y. Huang. The covering canadian traveller problem. *Theor. Comput. Sci.*, 530:80–88, 2014.

[20] L. V. Lita, J. Schulte, and S. Thrun. A system for multi-agent coordination in uncertain environments. In *Procs. of AGENTS*, pages 21–22. ACM, 2001.

[21] C. Papadimitriou and M. Yannakakis. Shortest paths without a map. *Theor. Comput. Sci.*, 84(1):127–150, 1991.

[22] D. Shiri and F. S. Salman. On the online multi-agent $O - D$ k-Canadian Traveler Problem. *J. Comb. Optim.*, 34(2):453–461, 2017.

[23] D. Shiri and F. S. Salman. On the randomized online strategies for the k-Canadian traveler problem. *J. Comb. Opt.*, pages 1–14, 2019.

[24] S. Westphal. A note on the k-Canadian Traveller Problem. *Inform. Proces. Lett.*, 106(3):87–89, 2008.

[25] Y. Xu, M. Hu, B. Su, B. Zhu, and Z. Zhu. The Canadian traveller problem and its competitive analysis. *J. Comb. Optim.*, 18(2):195–205, 2009.