



**HAL**  
open science

## A note on Sarnak processes

Mariusz Lemańczyk, Michal D. Lemańczyk, Thierry de La Rue

► **To cite this version:**

Mariusz Lemańczyk, Michal D. Lemańczyk, Thierry de La Rue. A note on Sarnak processes. 2024.  
hal-04523555

**HAL Id: hal-04523555**

**<https://hal.science/hal-04523555>**

Preprint submitted on 28 Mar 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A note on Sarnak processes

Mariusz Lemańczyk, Michał D. Lemańczyk, Thierry de la Rue

March 27, 2024

## Abstract

Basic properties of stationary processes called Sarnak processes are studied. As an application, a combinatorial reformulation of Sarnak's conjecture on Möbius orthogonality is provided.

## 1 Introduction

### 1.1 Notation and definitions

Throughout this paper (unless stressed otherwise) random variables are defined on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We consider double sided stochastic processes  $\mathbf{X} = (X_i)_{i \in \mathbb{Z}} \in \mathcal{X}$ , where  $X_i \in \mathcal{X}$  belong to the common *state space*  $\mathcal{X}$  and  $\mathcal{X}$  stands for the state space of  $\mathbf{X}$ . We assume that the common state space  $|\mathcal{X}|$  is **finite** (and  $\mathcal{X} \subset \mathcal{X}^{\mathbb{Z}}$ ). For the convenience sake, for any sequence  $\mathbf{x} = (x_i)_{i \in \mathbb{Z}}$  and  $m, n \in \mathbb{Z}$  such that  $m \leq n$ , we will write

$$(1) \quad \mathbf{x}_{\leq n} = (x_i)_{i \leq n}, \quad \mathbf{x}_{< n} = (x_i)_{i < n}, \quad \mathbf{x}_m^n = (x_i)_{m \leq i \leq n}$$

Moreover, we extend each function or operation acting on  $\mathcal{X}$  to the sequences by applying it coordinate-wise so that, e.g., for any sequences  $\mathbf{x} = (x_i)_{i \in \mathbb{Z}}$  and  $\mathbf{y} = (y_i)_{i \in \mathbb{Z}}$ , we have

$$(2) \quad (\mathbf{x}, \mathbf{y}) = (x_i, y_i)_{i \in \mathbb{Z}}, \quad \mathbf{xy} = (x_i y_i)_{i \in \mathbb{Z}}$$

the latter when  $\mathbf{x}$  and  $\mathbf{y}$  are complex-valued.

We denote by  $S : \mathcal{X}^{\mathbb{Z}} \rightarrow \mathcal{X}^{\mathbb{Z}}$  the left shift map acting on sequences  $\mathbf{x}$  via  $S\mathbf{x} = (x_{i+1})_{i \in \mathbb{Z}}$ . We say that a process  $\mathbf{X}$  is stationary if  $\mathbf{X}$  and  $S\mathbf{X}$  share the same distribution  $\nu$  (for short  $\mathbf{X} \sim S\mathbf{X} \sim \nu$ ). Recall that with every stationary processes we can associate the corresponding dynamical system  $(\mathcal{X}, \nu, S)$  where  $\mathbf{X} \in \mathcal{X}$ ,  $\mathbf{X} \sim \nu$ . Conversely, with every given dynamical system  $(\mathcal{X}, \nu, T)$  and a function  $f : \mathcal{X} \rightarrow \mathcal{Y}$ , we can associate a stationary process (with  $\Omega = \mathcal{X}$ ),  $\mathbf{Y} = (Y_i)_{i \in \mathbb{Z}}$ , where  $Y_i := f \circ T^i \in \mathcal{Y}$ .

Remembering that  $\mathcal{X}$  is finite, we denote by  $\mathbb{H}(X) = -\sum_{x \in \mathcal{X}} \mathbb{P}(X = x) \log \mathbb{P}(X = x)$  the **Shannon entropy** of  $X$  (note that in this paper we use base 2 version of logarithm). If additionally an arbitrary random variable  $Y$  is given such that the regular conditional probability  $p_{X|Y}$  exists,  $\mathbb{H}(X | Y) = \mathbb{E}_Y \mathbb{H}(X^{(Y)})$ , stands for the **conditional Shannon entropy** ( $\mathbb{E}_Y$  is the integration with respect to  $Y$ ) and  $X^{(y)} \sim p_{X|Y}(\cdot | y)$ . For a given stationary process  $\mathbf{X}$ ,  $\mathbb{H}(\mathbf{X}) = \mathbb{H}(X_0 | X_{\leq -1})$  stands for the **process entropy**. We say that process is **deterministic** (or of **zero entropy**) if  $\mathbb{H}(\mathbf{X}) = 0$ . Similarly, given a dynamical system  $(\mathcal{X}, \nu, T)$  and a finite (measurable) partition  $\mathcal{A} = (\mathbf{A}_i)_i$  of  $\mathcal{X}$  we denote by  $\mathbb{H}(\nu, T, \mathcal{A}) = \mathbb{H}(\mathbf{X}_{\mathcal{A}})$  the metric entropy corresponding to the partition  $\mathcal{A}$  (here  $\mathbf{X}_{\mathcal{A}}$  stands for the process determined by  $f(\mathbf{x}) = i$  iff  $\mathbf{x} \in \mathbf{A}_i$ ). For the (topological and metric) entropy theory of dynamical systems (and relations with the entropy of stationary processes), we refer the reader to [5], [22].

The *past tail  $\sigma$ -algebra* (or *remote past*) of  $\mathbf{X}$  is defined by

$$(3) \quad \Pi(\mathbf{X}) := \bigcap_{i \geq 0} \sigma(X_{\leq -i}).$$

### Remark 1.1: Pinsker $\sigma$ -algebra

The past tail  $\sigma$ -algebra of  $\mathbf{X}$  coincides with the the Pinsker  $\sigma$ -algebra of the corresponding dynamical system. Recall that the Pinsker  $\sigma$ -algebra is the largest invariant  $\sigma$ -algebra such that the corresponding factor of the dynamical system determined by  $\nu \sim \mathbf{X} \in \mathcal{X}$  has zero entropy. In fact, if for any measurable  $\mathbf{B}$  we denote by  $\mathcal{A}_{\mathbf{B}}$  the binary partition generated by  $\mathbf{B} \subset \mathcal{X}$  then

$$(4) \quad \Pi(\mathbf{X}) = \sigma \{ \mathbf{B} : \mathbb{H}(\nu, S, \mathcal{A}_{\mathbf{B}}) = 0 \},$$

In connection with the celebrated Sarnak's conjecture on Möbius orthogonality (see below), in [13], the following notion of Sarnak process has been introduced.

### Definition 1.2: Sarnak process

A stationary process  $\mathbf{X}$  such that  $X_i \in \mathcal{X} \subset \mathbb{C}$ ,  $|\mathcal{X}| < \infty$ , is called a *Sarnak process* if  $\mathbb{E}(X_0 | X_{\leq -n}) \xrightarrow{n \rightarrow \infty} 0$  in  $L^2$ . Equivalently,  $\mathbb{E}(X_0 | \Pi(\mathbf{X})) = 0$ .

### Remark 1.3: Basic properties of a Sarnak process

Sarnak processes are centered, that is  $\mathbb{E}X_0 = 0$ . Moreover, every non-zero Sarnak process is, by the very definition, of positive entropy but it need not be even ergodic<sup>a</sup> (see Section 5). On the other hand, we will show that the class of Sarnak processes is stable under multiplication by deterministic processes.

<sup>a</sup>In fact, the dynamical system given by  $\mathbf{X}$  is ergodic if and only if so is its Pinsker factor.

## 2 Results

In this section we assume that every process  $\mathbf{X}$  is **stationary** and  $X_i \in \mathcal{X} \subset \mathbb{C}$ , where  $|\mathcal{X}| < \infty$ . Sometimes additionally we assume that  $\mathcal{X} = \{-1, 1\}$ . Note that in such a case, if  $\mathbb{E}X_0 = 0$  (in particular this holds if  $\mathbf{X}$  is Sarnak), then  $\mathbb{P}(X_0 = \pm 1) = 1/2$  and hence  $X_i$ 's are random signs. In this section we present a bunch of results. The most important ones are formulated in Theorem 2.3 and Theorem 2.8. The first one can be viewed as a number theoretic reformulation of Sarnak's conjecture. The second one is of slightly different flavor, namely, it shows that every deterministic process can be embedded as the Pinsker factor in some Sarnak process. Before we present the theorems, let us recall the formulation of Sarnak's conjecture.

Given an arithmetic function  $\mathbf{u} : \mathbb{N} \rightarrow \mathcal{X}$ , where  $\mathcal{X} \subset \mathbb{C}$  is finite, and whose mean

$$M(\mathbf{u}) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i \leq n} u_i$$

exists and equals 0, we ask whether

$$(5) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i \leq n} f(T^i x) u_i = 0$$

for each zero entropy topological dynamical system  $(X, T)$ , all  $f \in C(X)$  and all  $x \in X$ . The most known instance of (5) is Sarnak's conjecture [18] which predicts that (5) holds for  $\mathbf{u} = \boldsymbol{\lambda}$ , where  $\boldsymbol{\lambda} = (\lambda_n)$  stands for the Liouville function ( $\lambda_n = 1$  if  $n$  has an even number of primes divisors counted with multiplicity, and -1 otherwise).

### Remark 2.1: Logarithmic Sarnak's conjecture

The logarithmic version, originated by Tao [20], in which instead of Cesàro averages  $1/n \sum_{i \leq n} a_i$  we consider  $1/\log n \sum_{i \leq n} a_i/i$  is equally interesting. In fact, Tao proved that the logarithmic Sarnak's conjecture is equivalent to the logarithmic Chowla conjecture.

### Remark 2.2

Originally, Sarnak's conjecture has been formulated for the Möbius function,  $\mathbf{u} = \boldsymbol{\mu}$ , but it is known that Sarnak's conjecture for  $\boldsymbol{\lambda}$  and  $\boldsymbol{\mu}$  are equivalent, [6], [20].

## 2.1 Main results

Our first result links the notion of Sarnak process with the classical concept of weakly Bernoulli processes (see e.g. [19]). Recall that a stationary process  $\mathbf{X} \sim \nu$  is *weakly Bernoulli* if, given  $\varepsilon > 0$ , there is a gap  $g \geq 0$  such that for each  $n \geq 1$  and  $m \geq 0$ ,

$$\mathbb{E}_{X_{-g-m}^{-g}} \left\| \mathbb{P} \left( X_0^{n-1} \in \cdot \mid X_{-g-m}^{-g} \right) - \mathbb{P} \left( X_0^{n-1} \in \cdot \right) \right\|_{TV} < \varepsilon,$$

where  $\mathbb{E}_{X_{-g-m}^{-g}}$  denotes the integration with respect to  $X_{-g-m}^{-g}$  and, for any complex measure  $\mu$  defined on a  $\sigma$ -algebra  $\mathcal{A}$ ,

$$(6) \quad \|\mu\|_{TV} := \sup_{A \in \mathcal{A}} |\mu(A)| + |\mu(A^c)|$$

stands for the total variation of measure  $\mu$ . It turns out that in the special case of  $X_i \in \{-1, 1\}$ , if we do not require in the above condition uniformity in  $n$  then we recover the class of Sarnak processes.

### Theorem 2.3: Total variation formulation of the Sarnak property

Let  $\mathbf{X}$  be a stationary process taking values  $\pm 1$ . Then  $\mathbf{X}$  is Sarnak if and only if, given  $\varepsilon > 0$ , there is a gap  $g \geq 0$  such that for all  $m \geq 0$ ,

$$(7) \quad \mathbb{E}_{X_{-g-m}^{-g}} \left\| \mathbb{P} \left( X_0 \in \cdot \mid X_{-g-m}^{-g} \right) - \mathbb{P} \left( X_0 \in \cdot \right) \right\|_{TV} < \varepsilon.$$

Let us give some intuition behind (7). Think about large gap  $g$ . Then (7) can be roughly be interpreted as follows. Almost none of typical realizations of block in the far past  $X_{-g-m}^{-g} = x_{-g-m}^{-g}$  affects the probability of occurrence of given sign at a given non-negative coordinate. Note that Sarnak's conjecture is of similar flavour, namely, it roughly states that the Liouville function (counterpart of our  $\mathbf{X}$ ) is uncorrelated with deterministic sequences (thus, in some sense, Liouville function behaves like a random sign). Thus, there is no surprise that as a corollary of Theorem 2.3, we obtain the following description of Sarnak's conjecture.

**Corollary 2.4: Number-theoretic reformulation of Sarnak's conjecture**

Let  $\mathbf{u} : \mathbb{N} \rightarrow \pm 1$ . Then  $\mathbf{u}$  satisfies Sarnak's conjecture (i.e. it satisfies (5)) if and only if for every increasing sequence  $(n_k)_{k \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$  such that  $1/n_k \sum_{0 \leq n \leq n_k-1} \delta_{S^n \mathbf{u}} \xrightarrow{k \rightarrow \infty} \kappa$  and for each  $\varepsilon > 0$  there exists a gap  $g \geq 1$  such that for all  $m \geq 0$ ,

$$\sum_{q \in \{-1, 1\}^m} \varepsilon_g(q) < \varepsilon,$$

where for any block  $q \in \{-1, 1\}^m$ ,

$$\varepsilon_g(q) := \lim_{k \rightarrow \infty} \frac{1}{n_k} \left| \left| \{n + g \in A(q, n_k) : u_n = 1\} \right| - \frac{1}{2} |A(q, n_k)| \right|$$

and  $A(q, n) := \{i \leq n : u_i^{i+m-1} = q\}$  is the set of appearances of the block  $q$  in  $u_n^{n+m-1}$ .

**Remark 2.5: Magical constant 1/2 in the definition of  $\varepsilon_g$** 

Note that the constant 1/2 appearing in the definition of  $\varepsilon_g$  corresponds to the fact that we randomly choose a sign ( $\varepsilon$ -independently of the given remote past).

**Remark 2.6: Logarithmic version**

Logarithmic version of Corollary 2.4 holds as well. It suffices to replace all arithmetic weights by their logarithmic counterparts.

One may wonder under what assumptions put on  $\mathbf{Y}$  the following statement holds: if  $\mathbf{X}$  is Sarnak then so is  $\mathbf{XY}$ . The following theorem states that this statement is valid if we consider the class of zero entropy processes.

**Theorem 2.7: Closure under multiplication by a deterministic process**

Let  $(\mathbf{X}, \mathbf{Y}) = (X_i, Y_i)_{i \in \mathbb{Z}}$  be a stationary process. If  $\mathbf{X}$  is Sarnak and  $\mathbf{Y}$  is deterministic, then  $\mathbf{XY}$  is Sarnak.

The next fact states that each deterministic system can be realized as the Pinsker factor of some Sarnak process. It shows as well that Sarnak processes  $\mathbf{X}$  (even restricted to the case of  $\mathcal{X} = \{-1, 1\}$ ) are ubiquitous.

**Theorem 2.8: Embedding of a deterministic process into a Sarnak one**

For each aperiodic zero entropy dynamical system  $R$  there is a  $\pm 1$ -valued Sarnak process  $\mathbf{X}$  such that the Pinsker factor of the dynamical system given by  $\mathbf{X}$  is (measure-theoretically) isomorphic to  $R$ .

While Sarnak's processes need not be ergodic, see e.g. Example 4.3, in fact, we have the following result.

**Theorem 2.9**

Let  $\mathbf{X}$  be a stationary process. Then  $\mathbf{X}$  is Sarnak if and only if the stationary processes in its ergodic decomposition are a.a. Sarnak.

### Corollary 2.10

Let  $\mathbf{X}$  be a Sarnak process. Then for a.a. realizations  $(X_n(\omega))$  of the process, the sequence  $(X_n(\omega))$  is orthogonal to all topological dynamical systems of zero entropy.

## 3 Motivation

### 3.1 Chowla conjecture

Sarnak's conjecture is motivated by the Chowla conjecture [3] from 1965 about vanishing of all auto-correlations of  $\lambda$ :

$$(8) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i \leq n} \lambda_n \lambda_{n+r_1} \cdots \lambda_{n+r_k} = 0$$

for each choice of  $1 \leq r_1 < \dots < r_k$ .

### Remark 3.1: Chowla conjecture implies Sarnak's conjecture

The implication (8)  $\Rightarrow$  (5) (for  $\mathbf{u} = \lambda$ ) has been shown by Sarnak. See also the post [21] for the original proof by Sarnak and [1] for a dynamical proof of this implication.

It was noticed by Sarnak [18] that Chowla conjecture has a dynamical reformulation, indeed, (8) is equivalent to the fact that  $\lambda$  is a generic point for the Bernoulli measure  $\mathcal{B}(1/2, 1/2)$  on the full shift  $\{-1, 1\}^{\mathbb{Z}}$ , see [1], [6].

### 3.2 Why Sarnak processes are interesting from the analytic number theory point of view?

As we have seen the Chowla conjecture can be seen from both: number-theoretic (combinatorial) and dynamical points of view, while Sarnak's conjecture, by its nature, is a purely dynamical statement. The natural question arises:

$$(9) \quad \text{Is there a number-theoretic reformulation of Sarnak's conjecture?}$$

This question has been partially answered in [13], where Veech's conjecture has been proved. Namely, it has been proved that Sarnak's conjecture is equivalent to the fact that for each Furstenberg system  $(X_\lambda, \kappa, S)$  of  $\lambda$ , the corresponding stationary process  $\pi = (\pi_n)$  is Sarnak (in the sense of Definition 1.2), where  $\pi_n(\mathbf{x}) = x_n$ .

### Remark 3.2: Dynamical approach (Furstenberg systems), see e.g. [7] and the surveys [6], [14]

Let us only mention that in this approach through Furstenberg systems  $\mathbf{u}$  is treated as a two-sided sequence, for example via  $u_{-n} := u_n$  and we consider the subshift  $X_{\mathbf{u}} := \overline{\{S^k \mathbf{u} : k \in \mathbb{Z}\}} \subset \mathcal{X}^{\mathbb{Z}}$ . The set of shift invariant measures  $\kappa$  obtained as weak\*-limits of  $\frac{1}{n_k} \sum_{i \leq n_k} \delta_{S^i \mathbf{u}}$ ,  $k \geq 1$ , is denoted by  $V(\mathbf{u})$  and each such measure makes the coordinate projection process  $\pi = (\pi_0 \circ S^n)$  stationary.

**Remark 3.3: Mixing property approach, see [13]**

In this approach the condition  $\pi_0 \perp L^2(\Pi(\boldsymbol{\pi}))$  is equivalent to a (“relative”) uniform mixing property of the function  $\pi_0$  which finally yields a certain combinatorial (hardly satisfactory) condition on  $\mathbf{u}$ . Let us add that the idea is borrowed from the classical ergodic theory, namely, we use the classical fact that so called Kolmogorov automorphisms are those satisfying the uniform mixing property, see e.g. [4].

The main aim of this note is to consider the special case  $\mathbf{u} \in \{-1, 1\}^{\mathbb{Z}}$ . Sarnak processes resulting from this situation satisfy  $\pi_0 \perp \Pi(\boldsymbol{\pi})$  which allows us to use some entropy techniques and to obtain a rather clear combinatorial condition given in Corollary 2.4. Note finally that if in Corollary 2.4, given a Furstenberg system  $\kappa \in V(\mathbf{u})$ , for some  $g \geq 1$  we have  $\varepsilon_g(q) = 0$  for all blocks  $q$  then for the stationary process  $(\pi_n)$ , we have  $\pi_0$  is independent from the  $\sigma$ -algebra generated by  $(X_g, X_{g+1}, \dots)$ . In particular, the Chowla conjecture holds if and only if  $\varepsilon_1(q) = 0$  for all blocks  $q$ .

## 4 Examples of Sarnak processes

Firstly, let us make some remarks about properties of the class of Sarnak processes. It follows from the very definition that for each Sarnak process  $\mathbf{X} = (X_n)$ , we have  $\mathbb{E}X_0 = 0$ . Moreover, it has positive entropy, as otherwise  $X_0 \in L^2(\Pi(\mathbf{X}))$ , whence  $X_0 = 0$ . By the Rokhlin-Sinai theorem, see e.g. [22] Thm. 4.36, the spectral measure  $\sigma_{X_0}$  of a Sarnak process  $\mathbf{X}$ , i.e. the (symmetric, Borel, positive) measure determined by  $\hat{\sigma}_{X_0}(n) = \mathbb{E}X_n \overline{X_0}$ ,  $n \in \mathbb{Z}$ , has to be absolutely continuous. Therefore, if the spectral measure of a stationary process is not absolutely continuous, the process is not Sarnak. To see positive entropy processes whose spectral measure is partly singular notice that (contrary to the multiplication, cf. Theorem 2.7), Sarnak processes are not stable under addition of deterministic processes. Indeed, suppose that  $\mathbf{X}$  is a Sarnak process, and let  $\mathbf{Y} = (Y_n)$  be a (finite-valued) centered, stationary process with zero entropy such that the dynamical systems generated by these processes are disjoint.<sup>1</sup> Then, in view of [8], the process  $\mathbf{X} + \mathbf{Y} = (X_n + Y_n)$  is generating for the Cartesian product of the dynamical systems corresponding to  $\mathbf{X}$  and  $\mathbf{Y}$ , so it has positive entropy. On the other hand, the process  $\mathbf{X} + \mathbf{Y}$  is not Sarnak, as  $X_0 + Y_0$  is not orthogonal to  $Y_0$ , so it is not orthogonal to  $L^2(\Pi(\mathbf{X} + \mathbf{Y}))$ . Moreover, the spectral measure of the process  $\mathbf{X} + \mathbf{Y}$  is  $\sigma_{X_0} + \sigma_{Y_0}$ . Varying  $\mathbf{Y}$ , we can now obtain a positive entropy non-Sarnak process whose spectral measure is partly singular, as well as a process whose spectral measure is Lebesgue.

---

<sup>1</sup>For example, we can take for  $\mathbf{X}$  a Kolmogorov process and for  $\mathbf{Y}$  a finite valued process representing an irrational rotation on the circle.

#### Example 4.1: When factors of a Sarnak process are Sarnak?

Assume that  $\mathbf{X}$  is a Sarnak process. One can ask which functions (depending on finitely<sup>a</sup> many coordinates) of this process yield Sarnak processes? To get a partial answer to this question, assume that  $f : \mathbb{R}^k \rightarrow \mathbb{R}$  is measurable and let  $\mathbf{Y} = (Y_n)$  with

$$Y_n := f(X_n, X_{n+1}, \dots, X_{n+k-1}), \quad n \in \mathbb{Z}.$$

Note that

$$\sigma(Y_{-n}, Y_{-n-1}, \dots) \subset \sigma(X_{-n+k-1}, X_{-n+k-2}, \dots),$$

whence  $\Pi(\mathbf{Y}) \subset \Pi(\mathbf{X})$ . If the function  $f$  is linear then  $Y_0 \perp L^2(\Pi(\mathbf{X}))$ . Hence, under linearity,  $\mathbf{Y}$  is Sarnak. A similar argument shows that, for any  $k \geq 1$ , the jumping process  $(X_{kn})_n$  is Sarnak.

---

<sup>a</sup>If we admit infinitely many coordinates then the Pinsker factor can be expressed as a function of a generating process.

#### Remark 4.2

A large source of examples of Sarnak processes is given by the replacement of the orthogonality requirement in Definition 1.2 with the independence condition:  $X_0 \perp\!\!\!\perp \Pi(\mathbf{X})$ . Note that each  $\pm 1$ -valued Sarnak process satisfies this property. Take, for example, any case in which  $\Pi(\mathbf{X})$  is trivial, i.e.  $\mathbf{X}$  is Kolmogorov. Then  $\mathbf{X}$  is Sarnak as soon as it is centered.

#### Example 4.3: Non-ergodic Sarnak process

Take the Bernoulli process  $\mathcal{B}(1/2, 1/2)$  on  $\{-1, 1\}^{\mathbb{Z}}$  and let  $\mathbf{u}$  be any generic sequence for it. Let  $\mathbf{v} \in \{0, 1\}^{\mathbb{N}}$  be any generic point for a zero entropy measure. Then, in view of Theorem 2.7,  $\mathbf{uv}$  generates a Sarnak process and since  $(\mathbf{uv})^2 = \mathbf{v}$ , clearly, the zero entropy system is a factor of the system given by  $\mathbf{uv}$ , and the zero entropy system need not be ergodic. In fact, as proved in [15] (using [9]), once  $\mathbf{X}$  is Kolmogorov, and  $\mathbf{Y}$  is deterministic with  $\mathcal{Y} = \{0, 1\}$ , we can always filter out the zero entropy system from the dynamical system given by  $\mathbf{XY}$ .<sup>a</sup>

---

<sup>a</sup>Assuming the Chowla conjecture, the above applies to  $\mu = \lambda \cdot \mu^2$  as  $\mu^2$  is generic for a zero entropy measure. As Sarnak's conjecture for  $\mu$  and  $\lambda$  are equivalent, e.g. [6], the processes determined by  $\lambda$  are Sarnak iff the processes determined by  $\mu$  so are.

#### Example 4.4: Sarnak process with non-trivial Pinsker $\sigma$ -algebra

We will show that there are  $\pm 1$ -valued Sarnak processes with non-trivial remote past. Consider the space  $\{0, 1\}$  with  $\mathcal{B}(1/2)$  measure and  $Ti = 1 - i$ , and the full shift  $S$  on  $\{-1, 1\}^{\mathbb{Z}}$  considered with the Bernoulli measure  $\mathcal{B}(1/2, 1/2)$ . Let  $\tilde{T}$  be the corresponding skew product (considered with the product measure)

$$\tilde{T}(i, \mathbf{x}) := (1 - i, S^i \mathbf{x}).$$

Clearly,  $\tilde{T}$  has  $T$  as factor, so its Pinsker algebra is non-trivial. Let  $Y_0(i, \mathbf{x}) := x_0$ . We claim that the process  $(Y_n)$ , with  $Y_n := Y_0 \circ \tilde{T}^n$ , is Sarnak.



### Remark 4.5

Note that  $T$  produces sequences of alternating 0's and 1's. In the preceding example, we have  $I \sim \mathcal{B}(1/2) \perp\!\!\!\perp B \sim \mathcal{B}(1/2, 1/2)$ . If  $Z$  is given by  $Z_i = B_{\lfloor i/2 \rfloor}$ , then  $Y$  is given by  $Y_i = Z_{i+1}$  for every  $i \in \mathbb{Z}$ .

*Proof of the statement from Example 4.4.* We need three facts:

- (10)  $Y_0 \perp L^2(\sigma(I))$
- (11)  $(Y_n)$  generates the full  $\sigma$ -algebra
- (12)  $\Pi(\tilde{T}) =$  the first coordinate  $\sigma$ -algebra.

While (10) is obvious, to establish (11) note that the consecutive values of the process at  $(0, x)$  are:

$$Y_0(0, x) = x_0, Y_1(0, x) = x_0, Y_2(0, x) = x_1, Y_3(0, x) = x_1, Y_4(0, x) = x_2, Y_5(0, x) = x_2, \dots,$$

so it is clear that the process separates points and therefore (11) holds. Finally, note that  $\tilde{T}^2$  is the Cartesian product of two factors: of the first coordinate  $\sigma$ -algebra (on which that action is the identity) and of the second coordinate  $\sigma$ -algebra on which it acts as the shift  $S$  (indeed,  $\tilde{T}(\{0, 1\} \times \{x\}) = \{0, 1\} \times \{Sx\}$ ). The latter is just the Bernoulli  $\mathcal{B}(1/2, 1/2)$ , and since the Pinsker  $\sigma$ -algebra of the product is the product of Pinsker  $\sigma$ -algebras, we see that  $\Pi(\tilde{T}^2) =$  first coordinate  $\sigma$ -algebra. To conclude, i.e. to obtain (12), it is enough to notice that the Pinsker  $\sigma$ -algebras of non-zero powers of an automorphism are all equal.  $\square$

### Remark 4.6: Combinatorial intuition for the above example

Suppose that  $u$  satisfies (5). Set  $v$  via

$$v_{2n} = v_{2n+1} := u_n \text{ for all } n \in \mathbb{Z}.$$

Then  $v$  also satisfies (5), as  $T^2$  has also zero entropy,

$$\frac{1}{N} \sum_{n \leq N} f(T^n x) v_n = \frac{1}{N} \sum_{m \leq N/2} u_m (f(T^{2m} x) + f(T^{2m+1} x)) + o(1) \rightarrow 0 \text{ when } N \rightarrow \infty.$$

It follows that the Furstenberg systems of  $v$  yield Sarnak processes. However, if we consider the subshift  $X_v := \overline{\{S^n v : n \in \mathbb{Z}\}}$  then for each  $y \in X_v$  either

- (a)  $y_{2n} = y_{2n+1}$  for all  $n \in \mathbb{Z}$  holds or
- (b)  $y_{2n} = y_{2n-1}$  for all  $n \in \mathbb{Z}$  is satisfied.

Note that if a point  $y$  satisfies (a) and (b) simultaneously, then  $y$  is a fixed point. Notice that no Furstenberg system of  $v$  gives positive measure to the set of fixed points.<sup>a</sup> Then if  $A \subset X_v$  denote the set of point satisfying (a) and  $B$  stands for the set of points satisfying (b), then  $A \cap B = \emptyset$ ,  $B = SA$ , whence -1 is an eigenvalue of the dynamical system given by any Furstenberg system of  $v$ , whence the remote past cannot be trivial.

<sup>a</sup>This is a property of all Sarnak processes not taking the value zero. Indeed, let  $F$  be the set of fixed points of the process  $\mathbf{X}$  (it is a finite set naturally identified with a subset of  $\mathcal{X}$ ). Of course this set belongs to  $\Pi(\mathbf{X})$ , moreover, each subset of  $F$  is  $\Pi(\mathbf{X})$ -measurable. It follows that the function  $X_0 \cdot 1_F$  is  $\Pi(\mathbf{X})$ -measurable. Hence  $\mathbb{E}(X_0 \cdot \overline{(X_0 \cdot 1_F)}) = 0$ . On the other hand, this integral equals  $\int_F |X_0|^2 > 0$ , a contradiction.

## 5 Proofs

The organization is as follows. Each section containing a proof of one of our results is preceded by a section which provides some necessary background.

## 5.1 Background for Theorem 2.3

In this part all random variables take only **finitely many values**. For any random variable  $X$ , we denote by  $p_X$  its distribution. The main goal of this section is to prove the following fact.

### Lemma 5.1: Pinsker two-sided inequality in finite state case

Fix random variables  $X, Y$ . Define  $1/\beta := \sup_{x,y} \frac{p_{X,Y}(x,y)}{p_X(x)p_Y(y)}$ . Then

$$(13) \quad \frac{2\sqrt{\beta}}{\log e} [\mathbb{H}(X) - \mathbb{H}(X|Y)] \leq \mathbb{E} \|p_{X|Y}(\cdot|Y) - p_X\|_{TV} \leq \sqrt{2 [\mathbb{H}(X) - \mathbb{H}(X|Y)]}.$$

Note that, as a direct result, we immediately obtain the following fact.

### Corollary 5.2: Pinsker inequality for random sign

Fix random variables  $X, Y$  such that  $X$  is a symmetric random sign. Then

$$(14) \quad \frac{\sqrt{2}}{\log e} [1 - \mathbb{H}(X|Y)] \leq \mathbb{E} \|p_{X|Y}(\cdot|Y) - p_X\|_{TV} \leq \sqrt{2 [1 - \mathbb{H}(X|Y)]}.$$

*Proof.* If  $X$  is a random symmetric sign (i.e.  $X$  takes values  $\pm 1$  with equal probability) then  $p_X \equiv 1/2$  and thus  $\mathbb{H}(X) = \log 2 = 1$  and

$$(15) \quad 1/\beta = 2 \sup_{x,y} p_{X|Y}(x|y) \leq 2.$$

□

The idea of the proof Lemma 5.1 is to use the Pinsker and reversed Pinsker inequalities. In order to do so, we need reformulate quantities appearing in (14) in terms of KL-divergence and get rid of the integral  $\mathbb{E}$ . To this end, recall that given two probability distributions  $p$  and  $q$  on a finite space  $\mathcal{X}$ ,

$$(16) \quad \mathbb{D}(p \parallel q) = \sum_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)}$$

stands the KL-divergence between distributions  $p_X$  and  $p_Y$  with the convention that  $0/0 = 0$ . Now, one easily checks that

$$(17) \quad \mathbb{D}(p_{X,Y} \parallel p_X \otimes p_Y) = \mathbb{H}(X) - \mathbb{H}(X|Y)$$

The last ingredient needed for the proof is the following one.

### Proposition 5.3: Mean conditional variation norm equals to joint total variation

For any random variables  $X, Y$ ,

$$(18) \quad \mathbb{E} \|p_{X|Y}(\cdot|Y) - p_X\|_{TV} = \|p_{X,Y} - p_X \otimes p_Y\|_{TV}$$

*Proof.* By the very definitions,

$$\begin{aligned} \mathbb{E} \|p_{X|Y}(\cdot|Y) - p_X\|_{TV} &= \mathbb{E}_{p_Y} \|p_{X|Y}(\cdot|Y) - p_X\|_{TV} = \sum_{y \in \mathcal{Y}} p_Y(y) \sum_{x \in \mathcal{X}} |p_{X|Y}(x|y) - p_X(x)| \\ &= \sum_{x,y} |p_{X,Y}(x,y) - p_X(x)p_Y(y)| = \|p_{X,Y} - p_X \otimes p_Y\|_{TV}. \end{aligned}$$

□

*Proof of Lemma 5.1.* The upper bound is obtained using the Pinsker inequality. Indeed, by Proposition 5.3,

$$(19) \quad \mathbb{E} \|p_{X|Y}(\cdot|Y) - p_X\|_{TV} = \|p_{X,Y} - p_X \otimes p_Y\|_{TV} \leq \sqrt{2D(p_{X,Y} \| p_X \otimes p_Y)} = \sqrt{2[\mathbb{H}(X) - \mathbb{H}(X|Y)]}.$$

On the other hand, using the reverse Pinsker inequality, see e.g. [23] Theorem 7, we obtain

$$\mathbb{E} \|p_{X|Y}(\cdot|Y) - p_X\|_{TV} \geq \frac{2\sqrt{\beta}}{\log e} D(p_{X,Y} \| p_X \otimes p_Y) = \frac{2\sqrt{\beta}}{\log e} [\mathbb{H}(X) - \mathbb{H}(X|Y)],$$

where  $1/\beta := \sup_{x,y} \frac{p_{X,Y}(x,y)}{p_X(x)p_Y(y)}$ .  $\square$

Let us now make a remark on the monotonicity of the three terms which appear in (14). It is well known that given two probability distributions on the same state space  $\mathcal{X}$ ,

$$(20) \quad \|p - q\|_{TV} = \inf_{X \sim p, Y \sim q} \mathbb{P}(X \neq Y)$$

Thus, intuitively,  $\|p - q\|_{TV}$  tells us what is the best (that is, closest to diagonal) coupling of  $p$  and  $q$ . This fact immediately yields monotonicity property of the total variation norm.

**Proposition 5.4: Monotonicity property of the total variation norm**

Let  $X, Y, Z \in \mathcal{X}$  be some random variables. Then

$$(21) \quad \|p_{X,Y,Z} - p_X \otimes p_{Y,Z}\|_{TV} \geq \|p_{X,Y} - p_X \otimes p_Y\|_{TV}.$$

*Proof.* Take the realization of (20) with  $p = p_{X,Y,Z}$  and  $q = p_X \otimes p_{Y,Z}$  to obtain random variables  $(X, Y, Z) \sim p_{X,Y,Z}$  and  $(X', Y', Z') \sim p_X \otimes p_{Y,Z}$  such that  $\mathbb{P}(X \neq X', Y \neq Y', Z \neq Z') = \|p_{X,Y,Z} - p_X \otimes p_{Y,Z}\|_{TV}$ . Clearly,

$$(22) \quad \mathbb{P}(X \neq X', Y \neq Y', Z \neq Z') \leq \mathbb{P}(X \neq X', Y \neq Y')$$

and thus, once more using (20),

$$\|p_{X,Y} - p_X \otimes p_Y\|_{TV} \leq \mathbb{P}(X \neq X', Y \neq Y') \leq \mathbb{P}(X \neq X', Y \neq Y', Z \neq Z') = \|p_{X,Y,Z} - p_X \otimes p_{Y,Z}\|_{TV}.$$

$\square$

**Remark 5.5: Monotonicity of terms in (14)**

Since the Shannon entropy is monotonic, we have

$$(23) \quad \mathbb{H}(X) - \mathbb{H}(X|Y) \leq \mathbb{H}(X) - \mathbb{H}(X|Y, Z)$$

Combining this observation with the previous remark, we see that both terms in (14)  $\mathbb{E} \|p_{X|Y}(\cdot|Y) - p_X\|_{TV}$  and  $1 - \mathbb{H}(X|Y)$  possess monotonicity property: if  $Y$  is a vector random variable in (14) then dropping of any of its coordinates forces all terms in (14) to decrease.

Last but not least let us present the well known fact which says that in case of a random variable attaining only two values and being  $L^2$  perpendicular to some  $\sigma$ -algebra is equivalent to being independent of that algebra. Note that this observation can be used for Sarnak processes  $\mathbf{X}$  taking at most two values, namely, in such a case  $X_0 \perp\!\!\!\perp \Pi(\mathbf{X})$ .

**Proposition 5.6**

Let  $X$  be a two valued random variable and  $\mathcal{G}$  be some sub- $\sigma$ -algebra such that  $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}X$ . Then  $X \perp\!\!\!\perp \mathcal{G}$ .

*Proof.* Clearly, our assumption is equivalent to:  $\mathbb{E}X\mathbf{1}_G = \mathbb{E}X\mathbb{E}\mathbf{1}_G$  for any  $G \in \mathcal{G}$ . In other words,  $\text{Cov}(X, \mathbf{1}_G) = 0$ . Now, it is enough to recall standard fact that if two random variables taking only two values are uncorrelated then they are independent. Applying this fact, we get that for any  $G \in \mathcal{G}$ ,  $X \perp\!\!\!\perp \mathbf{1}_G$ . Hence  $X \perp\!\!\!\perp \mathcal{G}$ .  $\square$

**Remark 5.7: When uncorrelated random variables are independent**

In the above proof we used the fact that if  $X \in \mathcal{X}$  and  $Y \in \mathcal{Y}$  are uncorrelated and  $|\mathcal{X}| = |\mathcal{Y}| = 2$  then they are independent. Here we provide a sketch of a proof. Firstly, without loss of generality we can assume that  $\mathcal{X} = \mathcal{Y} = \{0, 1\}$ . To see this consider

$$(24) \quad X' = \frac{X - x_0}{x_1 - x_0}, \quad Y' = \frac{Y - y_0}{y_1 - y_0},$$

where  $\mathcal{X} = \{x_0, x_1\}$  and  $\mathcal{Y} = \{y_0, y_1\}$ . For binary random variables, the assumption is equivalent to

$$(25) \quad \mathbb{P}(X = 1, Y = 1) = \mathbb{P}(X = 1)\mathbb{P}(Y = 1).$$

It remains to use the fact that if events  $A$  and  $B$  are independent then so are  $A^c$  and  $B$ .

**5.2 Proof of Theorem 2.3**

Firstly, since  $X_0$  is perpendicular to  $\Pi(\mathbf{X})$  and  $X_0$  takes only two values, by Proposition 5.6, we in fact have  $X_0 \perp\!\!\!\perp \Pi(\mathbf{X})$ . In particular, Sarnak property is equivalent to the  $\mathbb{H}(X_0 | X_{\leq -g}) \xrightarrow{g \rightarrow \infty} \mathbb{H}(X_0) = 1$ . By the monotonicity property of conditional entropy this is equivalent to the statement that for every  $\varepsilon > 0$  there is a gap  $g \geq 1$  such that for all  $m \in \mathbb{N}$

$$(26) \quad 1 - \mathbb{H}(X_0 | X_{-g-m}^-) \leq \varepsilon.$$

Indeed, if (26) holds then taking  $m \rightarrow \infty$  yields the result. Conversely, for every  $\varepsilon$  there is  $g \in \mathbb{N}$  such that  $1 - \mathbb{H}(X_0 | X_{\leq -g}) \leq \varepsilon$ . However, for any  $m \geq 0$ ,  $1 - \mathbb{H}(X_0 | X_{\leq -g}) \geq 1 - \mathbb{H}(X_0 | X_{-g-m}^-)$ .

Now, an application of Corollary 5.2 with  $X = X_0$  and  $Y = X_{-m-g}^-$  yields,

$$(27) \quad \frac{\sqrt{2}}{\log e} \left[ 1 - \mathbb{H}(X_0 | X_{-m-g}^-) \right] \leq \mathbb{E} |p_{X_0 | X_{-m-g}^-}(\cdot | X_{-m-g}^-) - p_{X_0}| \leq \sqrt{2 \left[ 1 - \mathbb{H}(X_0 | X_{-m-g}^-) \right]}$$

and the result follows.

**Remark 5.8: Rate of convergence**

Note that, thanks to the (27), if we know the rate of convergence for  $1 - \mathbb{H}(X_0 | X_{-m-g}^-)$  then (27) controls the rate for the term  $\mathbb{E} |p_{X_0 | X_{-m-g}^-}(\cdot | X_{-m-g}^-) - p_{X_0}|$ .

*Proof of Corollary 2.4.* Let  $\mathbf{X} \sim \kappa$ , Since  $1/n_k \sum_{0 \leq n \leq n_k - 1} \delta_{S^n \mathbf{u}} \xrightarrow{k \rightarrow \infty} \kappa$ ,

$$(28) \quad \lim_{k \rightarrow \infty} \frac{1}{n_k} |A(q, n_k)| = \lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \mathbf{1}_{S^i \mathbf{u} \text{ starts with } q} = \kappa(q).$$

Similarly,

$$(29) \quad \frac{1}{n_k} |\{n + g \in A(q, n_k) \mid u_n = 1\}| = \frac{1}{n_k} \sum_{i=0}^{n_k-1} \mathbf{1}_{S^i \mathbf{u}(0)=1} \mathbf{1}_{S^{i+g} \mathbf{u} \text{ starts with } q} \rightarrow \mathbb{P}(X_0 = 1, X_g^{+m} = q).$$

Combining (28) and (29),

$$\varepsilon_g(q) = \left| \frac{1}{2} \mathbb{P}(X_g^{g+m} = q) - \mathbb{P}(X_0 = 1, X_g^{g+m} = q) \right|$$

and the result follows from Theorem 2.3.  $\square$

### Remark 5.9

In the above proof we have used the symmetric version of Theorem 2.3 with the past replaced by the future, which is valid as well because in the finite case the past tail  $\sigma$ -field of process equals to the future tail  $\sigma$ -field one. In particular, the Sarnak property can be stated equivalently with reversed roles of the past and the future.

## 5.3 Background for Theorem 2.7

### Remark 5.10: Relatively independent coupling above a $\sigma$ -field

Later on we use the following construction. Given a probability space  $(Z, \mathcal{D}, \kappa)$  and a sub- $\sigma$ -algebra  $\mathcal{E} \subset \mathcal{D}$ , the formula

$$\lambda(D_1 \times D_2) := \int \mathbb{E}(D_1 | \mathcal{E}) \mathbb{E}(D_2 | \mathcal{E}) d\kappa$$

determines a coupling on the space  $(Z_1 \times Z_2, \mathcal{D}_1 \otimes \mathcal{D}_2)$  ( $(Z_j, \mathcal{D}_j) = (Z, \mathcal{D})$  for  $j = 1, 2$ ). A characteristic property of this coupling is that if we have two measurable functions  $f, g : Z \rightarrow \mathbb{R}$  then the following conditions are equivalent:

- (i)  $f(z_1) = g(z_2)$  for  $\lambda$ -a.e.  $(z_1, z_2) \in Z_1 \times Z_2$ ,
- (ii)  $f \stackrel{\kappa}{=} g$ , and  $f$  is  $\mathcal{E}$ -measurable.

In the following theorem a similar type of joining is used, for more information we refer the reader to [10], Examples 6.3 and Theorem 6.8.

To prove Theorem 2.7, let us recall the classical theorem (see e.g. [17], the result is a consequence of the basic lemma on non-disjointness proved in [11] and [16]) about joinings with deterministic systems.

**Theorem 5.1.** *Let  $(Y, \mathcal{C}, \nu, S)$  be a dynamical system and  $(Z, \mathcal{D}, \kappa, R)$  has entropy zero. Assume that  $\rho \in J(S, R)$  is a joining of  $\nu$  and  $\kappa$ . Then*

$$(30) \quad \int_{Y \times Z} f(y)g(z) d\rho(y, z) = \int_{(Y/\Pi(S)) \times Z} \mathbb{E}(f | \Pi(S))(x)g(z) d\rho|_{(Y/\Pi(S)) \times Z}(x, z),$$

*i.e. each such joining has to be the relatively independent extension of its restriction to  $(Y/\Pi(S)) \times Z$ .*

## 5.4 Proof of Theorem 2.7

By assumption, the dynamical system  $(\mathcal{X}^{\mathbb{Z}} \times \mathcal{Y}^{\mathbb{Z}}, \rho, S)$  associated to the stationary process  $(\mathbf{X}, \mathbf{Y})$  is a joining of the dynamical systems  $(\mathcal{X}^{\mathbb{Z}}, \nu, S)$  and  $(\mathcal{Y}^{\mathbb{Z}}, \kappa, S)$  given by  $\mathbf{X}$  and  $\mathbf{Y}$ , respectively. Let us fix  $F \in L^2(\Pi(\rho))$ . Note that  $\rho$  induces a joining of  $(\mathcal{X}^{\mathbb{Z}}, \nu, S)$  with the zero-entropy system  $((\mathcal{X}^{\mathbb{Z}} \times \mathcal{Y}^{\mathbb{Z}})/\Pi(\rho), \rho|_{\Pi(\rho)}, S \times S)$ . Therefore, by (30), and using the assumption that  $\mathbf{X}$  is Sarnak, we get

$$\int X_0(\mathbf{x})F(\mathbf{x}, \mathbf{y}) d\rho(\mathbf{x}, \mathbf{y}) = \int \mathbb{E}(X_0 | \Pi(\nu))(\mathbf{x})F(\mathbf{x}, \mathbf{y}) d\rho(\mathbf{x}, \mathbf{y}) = 0.$$

Since  $Y_0 \in L^2(\Pi(\rho))$ , we also have

$$\int X_0(\mathbf{x}) \left( Y_0(\mathbf{y}) F(\mathbf{x}, \mathbf{y}) \right) d\rho(\mathbf{x}, \mathbf{y}) = 0,$$

which shows that  $X_0 \otimes Y_0 \perp L^2(\Pi(\rho))$ .

## 5.5 Background for Theorem 2.8

In order to prove Theorem 2.8, we will need the result from [9] and [15] which was already employed in Example 4.3, and whose proof (we provide it for the sake of completeness) in case of  $\{0, 1\}$ -valued processes is a short compilation of the arguments from the aforementioned papers.

### Lemma 5.11: Retrieval of a deterministic process from its product with a K-process

Let  $(\mathbf{X}, \mathbf{Y})$  be a stationary process such that both  $\mathbf{X}$  and  $\mathbf{Y}$  are binary,  $\mathbf{X} \neq \mathbf{0}$  is Kolmogorov and  $\mathbf{Y}$  is deterministic (note that this implies  $\mathbf{X} \perp\!\!\!\perp \mathbf{Y}$ ). Then  $\mathbf{Y}$  is measurable with respect to  $\sigma(\mathbf{X}\mathbf{Y})$ .

*Proof.* Consider two copies  $(\mathbf{X}', \mathbf{Y}')$  and  $(\mathbf{X}'', \mathbf{Y}'')$  of  $(\mathbf{X}, \mathbf{Y})$ , which are relatively independent over  $\mathbf{X}\mathbf{Y}$ . Then,  $X'_n Y'_n = X''_n Y''_n$  for all  $n \in \mathbb{Z}$ . Multiply both sides of this equality by  $\mathbf{1}_{Y'_n=0} \mathbf{1}_{Y''_n=1}$  to obtain  $0 = X''_n \mathbf{1}_{Y'_n=0} \mathbf{1}_{Y''_n=1}$ . Since  $(\mathbf{Y}, \mathbf{Y}')$  is deterministic and  $\mathbf{X}''$  is Kolmogorov, it follows that these two processes are disjoint. In particular,  $X''_n \perp\!\!\!\perp \mathbf{1}_{Y'_n=0} \mathbf{1}_{Y''_n=1}$ . Hence  $0 = \mathbb{P}(Y'_n = 0, Y''_n = 1) \mathbb{E} X''_n$  which implies that (recall  $\mathbf{X} \neq \mathbf{0}$ )  $0 = \mathbb{P}(Y'_n = 0, Y''_n = 1)$ . By symmetry it follows that  $Y'_n = Y''_n$ . Thus,  $\mathbf{Y}' = \mathbf{Y}''$  which concludes the proof.  $\square$

### Remark 5.12: Random variables concentrated on the graph of a function

The following simple observation allows one to generalize Lemma 5.11 to the case in which  $\mathbf{X}$  is not necessarily Kolmogorov. Suppose that  $(Y, Z) \sim (X, f(X))$  for some random variables  $X, Y, Z$  and some measurable function  $f$ . Then  $Z = f(Y)$ . Indeed, we have  $p_{Z,Y}(Z \neq f(Y)) = p_{X,f(X)}(f(X) \neq f(X)) = 0$  and the result follows.

### Lemma 5.13: Modify and retrieve

Let  $(\mathbf{X}, \mathbf{Y})$  be a stationary process such that both  $\mathbf{X} \neq \mathbf{0}$ ,  $\mathbf{Y}$  are binary and  $\mathbf{Y}$  is deterministic. Suppose additionally that we can find a Kolmogorov binary process  $\mathbf{X}' \perp\!\!\!\perp \mathbf{Y}$  such that  $\mathbf{X}'\mathbf{Y} \sim \mathbf{X}\mathbf{Y}$  conditionally on  $\mathbf{Y} = \mathbf{y}$ . Then  $\mathbf{Y}$  is measurable with respect to  $\sigma(\mathbf{X}\mathbf{Y})$ .

*Proof.* By Lemma 5.11 we know that there is a function such that  $\mathbf{Y} = f(\mathbf{X}'\mathbf{Y})$ . Moreover,  $(\mathbf{X}\mathbf{Y}, \mathbf{Y}) \sim (\mathbf{X}'\mathbf{Y}, \mathbf{Y}) = (\mathbf{X}'\mathbf{Y}, f(\mathbf{X}'\mathbf{Y}))$ . It remains to use Remark 5.12 with  $X = \mathbf{X}'\mathbf{Y}$ ,  $Y = \mathbf{X}\mathbf{Y}$  and  $Z = \mathbf{Y}$ .  $\square$

Let us present now a simple observation needed in the proof of the lemma below.

### Proposition 5.14: Example of a factor of a Bernoulli which is Bernoulli

Suppose that  $\mathbf{X}$  is a Bernoulli  $\mathcal{B}(1/2, 1/2)$  process with two point state space  $\mathcal{X}$ . Then  $\mathbf{Y}$  given by

$$(31) \quad Y_i = \mathbf{1}_{X_i \neq X_{i+1}}$$

is Bernoulli  $\mathcal{B}(1/2, 1/2)$  as well.

*Proof.* Since  $\mathbf{Y}$  is binary it is enough to check if

$$(32) \quad \mathbb{P}(Y_0^{n-1} = 1) = \prod_{i=0}^{n-1} \mathbb{P}(Y_i = 1) = 2^{-n}.$$

By the very definition,

$$(33) \quad \mathbb{P}(Y_0^{n-1} = 1) = \mathbb{P}(\forall_{0 \leq i \leq n-1} X_i \neq X_{i+1}).$$

But there are only two alternating sequences (on  $\mathcal{Y}^{n+1}$ ) of length  $n+1$ . The result follows.  $\square$

**Lemma 5.15: Skew process  $\mathbf{X}'$**

Let  $\mathbf{X} \perp\!\!\!\perp \mathbf{Y}$  be stationary processes such that  $\mathbf{X}$  is i.i.d.,  $X_i \in \mathcal{X} = \{a, b\}$  ( $a, b \in \mathbb{C}$ ,  $a \neq b$ ) and the binary process  $\mathbf{Y}$  is deterministic. Let  $\mathbf{K} = (K_n)_{n \in \mathbb{Z}}$  be the non-decreasing sequence of random times defined by  $K_0 := 0$ , and for  $n \geq 1$ ,  $K_n := \sum_{i=0}^{n-1} Y_i$  and  $K_{-n} := -\sum_{i=1}^n Y_{-i}$  (so that  $K_{n+1} = K_n + Y_n$  for all  $n$ ). We assume that we have almost surely

$$\lim_{n \rightarrow \infty} K_{-n} = -\infty \text{ and } \lim_{n \rightarrow \infty} K_n = \infty.$$

We define the process  $\mathbf{X}'$  via  $X'_i = X_{K_i}$ . Then  $(\mathbf{X}', \mathbf{Y})$  is stationary, and

$$(34) \quad \Pi(\mathbf{X}', \mathbf{Y}) = \Pi(\mathbf{X}') = \sigma(\mathbf{Y}).$$

**Remark 5.16: Why the name skew process?**

Clearly, by the very definition, we can look at  $\mathbf{X}'$  as a version of  $\mathbf{X}$  skewed by  $\mathbf{Y}$ . Moreover,  $\mathbf{X}'$  arises as a coordinate in a skew product of dynamical systems (see (40) below, which also justifies that  $(\mathbf{X}', \mathbf{Y})$  is stationary).

**Remark 5.17:  $\mathbf{X}'$  is Sarnak**

The above lemma is a recipe for producing Sarnak processes: indeed the centered skew process  $\mathbf{X}' - \mathbb{E}X_0$  is Sarnak. To see this, note that  $X'_0 = X_0$ , and

$$(35) \quad \mathbb{E}(X'_0 | \Pi(\mathbf{X}')) \stackrel{(34)}{=} \mathbb{E}(X_0 | \Pi(\mathbf{Y})) \stackrel{\mathbf{X} \perp\!\!\!\perp \mathbf{Y}}{=} \mathbb{E}X_0.$$

*Proof of Lemma 5.15.* Firstly, we show  $\Pi(\mathbf{X}', \mathbf{Y}) = \sigma(\mathbf{Y})$ . Clearly,  $\Pi(\mathbf{X}', \mathbf{Y}) \supset \Pi(\mathbf{Y}) = \sigma(\mathbf{Y})$ . Hence it remains to show that  $\Pi(\mathbf{X}', \mathbf{Y}) \subset \sigma(\mathbf{Y})$ . Intuitively this is clear because  $\Pi(\mathbf{X})$  is trivial so as it comes to some remote past of  $(\mathbf{X}', \mathbf{Y})$ , the past of  $\mathbf{X}'$  brings no additional information to the past of  $\mathbf{Y}$ . Formally, take some random variable  $F$  measurable with respect to  $\Pi(\mathbf{X}', \mathbf{Y})$ . By the definition of the tail  $\sigma$ -algebra, for every  $n \in \mathbb{N}$ , there exists a measurable function  $f_n$  such that

$$(36) \quad F = f_n(X'_{\leq -n}, Y_{\leq -n}).$$

Consider this equation conditionally on  $\mathbf{Y} = \mathbf{y}$ . Then, by the very definition of  $\mathbf{X}'$ , there exists  $f_{n, \mathbf{y}}$  (naturally defined via bijection correspondence  $f_{n, \mathbf{y}}(X_{\leq K_{-n}}) = f_n(X'_{\leq -n}, Y_{\leq -n})$ ) such that

$$(37) \quad F = f_{n, \mathbf{y}}(X_{\leq k_{-n}}),$$

where  $k$  is the value of  $\mathbf{K}$  under  $\mathbf{Y} = \mathbf{y}$ . However,  $k_{-n} \xrightarrow{n \rightarrow \infty} -\infty$ . Since  $\mathbf{X} \perp\!\!\!\perp \mathbf{Y}$  and (37) holds for any  $n$ , we conclude that conditionally on  $\mathbf{Y} = \mathbf{y}$ ,  $F \in \Pi(\mathbf{X})$ . However,  $\Pi(\mathbf{X})$  is trivial and hence conditionally on  $\mathbf{Y} = \mathbf{y}$ ,  $F$  is a constant. Therefore, (unconditionally)  $F$  is  $\sigma(\mathbf{Y})$ -measurable.

Now, we take care of  $\Pi(\mathbf{X}') = \sigma(\mathbf{Y})$ . By the previous step,  $\Pi(\mathbf{X}') \subset \Pi(\mathbf{X}', \mathbf{Y}) = \sigma(\mathbf{Y})$ . It remains to show that  $\Pi(\mathbf{X}') \supset \sigma(\mathbf{Y})$ . To this end, define a process  $\mathbf{C}$  which checks when  $\mathbf{X}'$  changes, that is,

$$(38) \quad C_n := \mathbf{1}_{X'_{n-1} \neq X'_n}.$$

Note that, if we could recover the process  $\mathbf{Y}$  from  $\mathbf{C}$ , then it would be the end of the proof. Indeed, in such a case,  $\mathbf{C}$  is a function of  $\mathbf{X}'$  and thus  $\mathbf{Y} = f(\mathbf{X}')$ . To achieve this, using the fact that  $C_n = 1$  iff  $Y_n = 1$  and  $X_{K_{n-1}} \neq X_{K_n}$ , we can express  $C_n$  as

$$(39) \quad C_n = Y_n \mathbf{1}_{X_{K_{n-1}} \neq X_{K_n}}.$$

We claim that, conditionally on  $\mathbf{Y} = \mathbf{y}$ , the process  $\mathbf{C}$  is distributed as  $\mathbf{B}\mathbf{Y}$  where  $\mathcal{B}(1/2, 1/2) \sim \mathbf{B} \perp \perp \mathbf{Y}$ . For this, for a fixed  $n \in \mathbb{N}$ , and fixed  $y, w \in \{0, 1\}^{\mathbb{Z}}$ , we compare  $\mathbb{P}(C_{-n}^n = w_{-n}^n | Y = y)$  and  $\mathbb{P}((\mathbf{B}\mathbf{Y})_{-n}^n = w_{-n}^n | Y = y)$ . Clearly, both vanish if there exists  $j \in \{-n, \dots, n\}$  such that  $y_j = 0$  but  $w_j = 1$ . Otherwise, let

$$\{j_1 < j_2 < \dots < j_\ell\} := \{j \in \{-n, \dots, n\} : y_j = 1\}.$$

Since we assume now that  $w_{-n}^n \leq y_{-n}^n$ , we have

$$\mathbb{P}((\mathbf{B}\mathbf{Y})_{-n}^n = w_{-n}^n | Y = y) = \mathbb{P}(B_{j_i} = w_{j_i} \text{ for all } i \in \{1, \dots, \ell\}) = \frac{1}{2^\ell}.$$

On the other hand, denoting again by  $\mathbf{k}$  the value of  $\mathbf{K}$  under  $\mathbf{Y} = \mathbf{y}$ , since  $k_{j_1} < k_{j_2} < \dots < k_{j_\ell}$ , by Proposition 5.14 we also have

$$\mathbb{P}(C_{-n}^n = w_{-n}^n | Y = y) = \mathbb{P}(\mathbf{1}_{X_{k_{j_i-1}} \neq X_{k_{j_i}}} = w_{j_i} \text{ for all } i \in \{1, \dots, \ell\}) = \frac{1}{2^\ell}.$$

This completes the proof of the claim, and then it is enough to apply Lemma 5.13 to conclude.  $\square$

## 5.6 Proof of Theorem 2.8

By a non-ergodic version of Jewett-Krieger theorem (see [2], Theorem 1.2), each aperiodic, zero entropy system can be realized as a binary deterministic process. Thus, in view of Remark 5.17, to finish the proof, it is enough to construct an appropriate processes from Lemma 5.15. To do so consider the full shift  $S$  on  $\{-1, 1\}^{\mathbb{Z}}$  with the Bernoulli measure  $\nu := \mathcal{B}(1/2, 1/2)$  and some aperiodic zero-entropy system  $(\{0, 1\}^{\mathbb{Z}}, \mu, S)$ . On the product space  $\{-1, 1\}^{\mathbb{Z}} \times \{0, 1\}^{\mathbb{Z}}$ , let  $\bar{T}$  be the corresponding skew product (considered with the product measure  $\nu \otimes \mu$ ):

$$(40) \quad \bar{T}(\mathbf{x}, \mathbf{y}) := (S^{y_0} \mathbf{x}, S\mathbf{y}).$$

Let  $Y_n$ ,  $n \in \mathbb{Z}$ , (respectively  $X_k$ ,  $k \in \mathbb{Z}$ ) denote the projection on the  $n$ -coordinate on  $\{0, 1\}^{\mathbb{Z}}$  (respectively, on the  $k$ -coordinate on  $\{-1, 1\}^{\mathbb{Z}}$ ). Then, we construct the stationary process  $\mathbf{X}'$  by setting  $X'_n := X_0 \circ \bar{T}^n$ . Note that for all  $n \in \mathbb{Z}$  we can write  $X'_n = X_{K_n}$ , where  $\mathbf{K}$  is  $\mathbf{Y}$ -measurable, and recursively defined by

- $K_0 := 0$ ,
- For each  $n \geq 1$ ,  $K_n := K_{n-1} + Y_n$ ,
- For each  $n \leq -1$ ,  $K_n := K_{n+1} - Y_{n+1}$ .

It remains to note that, by aperiodicity of the system we started from, we have  $\mu(y_n = 0 \text{ for all } n) = 0$ . Therefore, for almost every ergodic component  $\lambda$  of the system  $(\{0, 1\}^{\mathbb{Z}}, \mu, S)$ , we have  $\lambda(y_0 = 1) > 0$ , and then by the pointwise ergodic theorem we get that  $K_n \xrightarrow{n \rightarrow \pm\infty} \pm\infty$  almost surely.



## 5.7 Background for the proof of Theorem 2.9

The following result has been proved in [12]:

### Proposition 5.18

Let  $R$  be an automorphism of  $(Z, \mathcal{D}, \kappa)$ . Let  $\kappa = \int_{\Gamma} \kappa_{\gamma} dQ(\gamma)$  be the ergodic decomposition of  $\kappa$ . Then there exists an  $R$ -invariant  $\sigma$ -algebra  $\mathcal{C} \subset \mathcal{D}$  such that  $\Pi(\kappa) = \mathcal{C}$   $\kappa$ -a.e. and for  $Q$ -a.a.  $\gamma \in \Gamma$ , we have  $\Pi(\kappa_{\gamma}) = \mathcal{C}$   $\kappa_{\gamma}$ -a.e. Moreover, for each  $f \in L^1(\kappa)$ , there exists a  $\mathcal{C}$ -measurable  $g$  such that  $\mathbb{E}_{\kappa}(f|\mathcal{C}) = g$   $\kappa$ -a.e. and, for  $Q$ -a.a.  $\gamma \in \Gamma$ , we have  $\mathbb{E}_{\kappa_{\gamma}}(f|\mathcal{C}) = g$   $\kappa_{\gamma}$ -a.e.

Assume now that  $\mathcal{X} \subset \mathbb{C}$  is finite and let  $\mu$  be an  $S$ -invariant measure on  $\mathcal{X}^{\mathbb{Z}}$ . Then the coordinate projection process  $(\pi_n) = ((\pi_n), \mu)$  is stationary (and has distribution  $\mu$ ). Assume that  $\mu = \int_{\Gamma} \mu_{\gamma} dQ(\gamma)$  is the ergodic decomposition of  $\mu$ .

### Lemma 5.19

The stationary process  $((\pi_n), \mu)$  is Sarnak if and only if the stationary processes  $((\pi_n), \mu_{\gamma})$  are Sarnak for  $Q$ -a.e.  $\gamma \in \Gamma$ .

*Proof.*  $\Rightarrow$  We use Proposition 5.18 for  $(\mathcal{X}^{\mathbb{Z}}, \mu, S)$  and  $f = \pi_0$ . We have

$$\begin{aligned} 0 &= \|\mathbb{E}_{\mu}(\pi_0|\Pi(\mu))\|_{L^2}^2 = \|\mathbb{E}_{\mu}(\pi_0|\mathcal{C})\|_{L^2}^2 = \int \mathbb{E}_{\mu}(\pi_0|\mathcal{C})\bar{\pi}_0 d\mu = \int g\bar{\pi}_0 d\mu = \\ &= \int_{\Gamma} \left( \int g\bar{\pi}_0 d\mu_{\gamma} \right) dQ(\gamma) = \int_{\Gamma} \left( \int \mathbb{E}_{\mu_{\gamma}}(g\bar{\pi}_0|\mathcal{C}) d\mu_{\gamma} \right) dQ(\gamma) = \\ &= \int_{\Gamma} g\mathbb{E}_{\mu_{\gamma}}(\bar{\pi}_0|\mathcal{C}) d\mu_{\gamma} dQ(\gamma) = \int_{\Gamma} \left( \int |\mathbb{E}_{\mu_{\gamma}}(\pi_0|\mathcal{C})|^2 d\mu_{\gamma} \right) dQ(\gamma), \end{aligned}$$

whence, for  $Q$ -a.e.  $\gamma$ , we have

$$0 = \mathbb{E}_{\mu_{\gamma}}(\pi_0|\mathcal{C}) = \mathbb{E}_{\mu_{\gamma}}(\pi_0|\Pi(\mu_{\gamma})),$$

so our claim follows.

$\Leftarrow$  The same by reading in the reversed order.<sup>2</sup>  $\square$

## 5.8 Proofs of Theorem 2.9 and of Corollary 2.10

Theorem 2.9 follows directly from Lemma 5.19.

*Proof of Corollary 2.10* Assume first that  $\mathbf{X}$  (with distribution  $\mu$ ) is ergodic. Then almost every realization  $(X_n(\omega))$  of the process is generic for the measure  $\mu$ . But since  $\mathbf{X}$  is Sarnak,  $X_0 \perp \Pi(\mu)$ , so the Veech condition is satisfied for the (unique) Furstenberg system of  $\mathbf{u} = (X_n(\omega))$ , and therefore by [13],  $\mathbf{u}$  is orthogonal to all deterministic sequences. If  $\mathbf{X}$  is not ergodic then, by Theorem 2.9, we pass to ergodic components to which we apply the above argument.

<sup>2</sup>Or using only the first property of  $\mathcal{C}$ : for each  $h \in L^{\infty}(\mathcal{C})$ ,

$$\begin{aligned} \int \mathbb{E}_{\mu}(\pi_0|\mathcal{C})h d\mu &= \int \pi_0 h d\mu = \int_{\Gamma} \left( \int \pi_0 h d\mu_{\gamma} \right) dQ(\gamma) = \\ &= \int_{\Gamma} \left( \int \mathbb{E}_{\mu_{\gamma}}(\pi_0|\mathcal{C})h d\mu_{\gamma} \right) dQ(\gamma) = 0 \end{aligned}$$

since  $\mathbb{E}_{\mu_{\gamma}}(\pi_0|\mathcal{C}) = 0$ . Since  $h$  was arbitrary,  $\mathbb{E}_{\mu}(\pi_0|\mathcal{C}) = \mathbb{E}_{\mu}(\pi_0|\Pi(\mu)) = 0$ .

## References

- [1] H. El Abdalauoi, J. Kułaga-Przymus, M. Lemańczyk, T. de la Rue, *The Chowla and the Sarnak conjectures from ergodic theory point of view*, Discrete and Continuous Dynamical Systems **37** (2017), 2899–2944.
- [2] A. Alpeev, B. Seward, *Krieger’s finite generator theorem for actions of countable groups III*, Ergodic Theory Dynam. Systems 41 (2021), 2881–2917.
- [3] S. Chowla, *The Riemann hypothesis and Hilbert’s tenth problem*, Mathematics and Its Applications, Vol. 4, Gordon and Breach Science Publishers, New York, 1965.
- [4] I. P. Cornfeld, S. V. Fomin, and Y. G. Sinai, *Ergodic theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 245, Springer-Verlag, New York, 1982
- [5] T. Downarowicz, *Entropy in dynamical systems*, New Mathematical Monographs, 18. Cambridge University Press, Cambridge, 2011
- [6] S. Ferenczi, J. Kułaga-Przymus, M. Lemańczyk, S. Ferenczi, J. Kułaga-Przymus, M. Lemańczyk, *Sarnak’s Conjecture – what’s new*, in: Ergodic Theory and Dynamical Systems in their Interactions with Arithmetics and Combinatorics, CIRM Jean-Morlet Chair, Fall 2016, Editors: S. Ferenczi, J. Kułaga-Przymus, M. Lemańczyk, Lecture Notes in Mathematics 2213, Springer International Publishing, pp. 418.
- [7] N. Frantzikinakis, B. Host, *The logarithmic Sarnak conjecture for ergodic weights*, Annals Math. (2), 187(3) (2018), 869–931.
- [8] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory 1, 1–49.
- [9] H. Furstenberg, Y. Perez, B. Weiss, *Perfect filtering and double disjointness*, Ann. Inst. H. Poincaré Probab. Statist. 31 (1995), no. 3, 453–465.
- [10] E. Glasner, *Ergodic Theory via Joinings*, Mathematical Surveys and Monographs, vol. 101, American Mathematical Society, 2003.
- [11] E. Glasner, J.-P. Thouvenot, B. Weiss, *Entropy theory without a past*, Ergodic Theory Dynam. Systems 20 (2000), 1355–1370.
- [12] M. Górska, M. Lemańczyk, T. de la Rue, *On orthogonality to uniquely ergodic systems*, preprint.
- [13] A. Kanigowski, J. Kułaga-Przymus, M. Lemańczyk, T. de la Rue, *On arithmetic functions orthogonal to deterministic sequences*, Advances Math. **428** (2023), 109138.
- [14] J. Kułaga-Przymus, M. Lemańczyk, *Sarnak’s Conjecture from the Ergodic Theory Point of View*, Ergodic Theory, 293–311, Encycl. Complex. Syst. Sci., Springer, New York, 2023.
- [15] J. Kułaga-Przymus, M.D. Lemańczyk, *Entropy rate of product of independent processes*, Monatshefte Math. 200 (2023), no. 1, 131–162.
- [16] M. Lemańczyk, F. Parreau, J.-P. Thouvenot, *Gaussian automorphisms whose ergodic self-joinings are Gaussian*, Fundamenta Math. 164 (2000), 253–293.
- [17] T. de la Rue, *Notes on Austin’s multiple ergodic theorem*, [hal-00400975](https://hal.archives-ouvertes.fr/hal-00400975).

- [18] P. Sarnak, *Three lectures on the Möbius function, randomness and dynamics*, [publications.ias.edu/sarnak/paper/506](http://publications.ias.edu/sarnak/paper/506).
- [19] P. Shields, *The Ergodic Theory of the Discrete Sample Paths*, Graduate Studies in Mathematics, volume 13, American Math. Soc. 1996.
- [20] T. Tao, *Equivalence of the logarithmically averaged Chowla and Sarnak conjectures*, Number Theory – Diophantine Problems, Uniform Distribution and Applications: Festschrift in Honour of Robert F. Tichy’s 60th Birthday (C. Elsholtz and P. Grabner, eds.), Springer International Publishing, Cham, 2017, pp. 391–421.
- [21] T. Tao, *The Chowla conjecture and the Sarnak conjecture*, [terrytao.wordpress.com/2012/10/14/the-chowla-conjecture-and-the-sarnak-conjecture](http://terrytao.wordpress.com/2012/10/14/the-chowla-conjecture-and-the-sarnak-conjecture).
- [22] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York, 1982.
- [23] Verdú, Sergio, *Total variation distance and the distribution of relative information*, 2014 Information Theory and Applications Workshop (ITA). IEEE, 2014.

Faculty of Mathematics and Computer Science  
Nicolaus Copernicus University,  
Chopin street 12/18, 87-100 Toruń, Poland  
mlem@mat.umk.pl

Faculty of Physics, Astronomy and Informatics  
Nicolaus Copernicus University,  
Grudziądzka street 5, 87-100 Toruń, Poland  
m.lemanczyk@umk.pl

CNRS, Univ Rouen Normandie  
LMRS, UMR6085 F76000 Rouen, France  
Thierry.de-la-Rue@univ-rouen.fr