



HAL
open science

Gyrokinetic limit of the 2D Hartree equation in a large magnetic field

Denis Périce, Nicolas Rougerie

► **To cite this version:**

Denis Périce, Nicolas Rougerie. Gyrokinetic limit of the 2D Hartree equation in a large magnetic field. 2025. [⟨hal-04523429v3⟩](#)

HAL Id: hal-04523429

<https://hal.science/hal-04523429v3>

Preprint submitted on 12 Dec 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

GYROKINETIC LIMIT OF THE 2D HARTREE EQUATION IN A LARGE MAGNETIC FIELD

DENIS PÉRICE AND NICOLAS ROUGERIE

ABSTRACT. We study the dynamics of two-dimensional interacting fermions submitted to a homogeneous transverse magnetic field. We consider a large magnetic field regime, with the gap between Landau levels set to the same order as that of potential energy contributions. Within the mean-field approximation, i.e. starting from Hartree's equation for the first reduced density matrix, we derive a drift equation for the particle density. We use vortex coherent states and the associated Husimi function to define a semi-classical density almost satisfying the limiting equation. We then deduce convergence of the density of the true Hartree solution by a Dobrushin-type stability estimate.

CONTENTS

1. Introduction	2
2. Main results	4
2.1. Model and scaling	4
2.2. Results	7
2.3. Organisation of the paper	8
3. Landau quantization and the magnetic Hartree equation	8
3.1. Landau quantization	9
3.2. Coherent states	11
3.3. Conservation properties	14
4. Husimi functions and semi-classical densities	15
4.1. Semi-classical density	15
4.2. Improved convergence with confinement	19
5. Semi-classical dynamics of the densities	25
5.1. Dynamics of the truncated semi-classical density	26
5.2. Dynamics of the first reduced density	38
6. Stability for perturbed classical flows	44
6.1. A Dobrushin-type Estimate	44
6.2. Application: proof of Theorem 2.4	48
References	53

1. INTRODUCTION

Motivated in particular by the physical context of the quantum Hall effect [30, 22] we study the dynamics of many interacting 2D fermions in a large perpendicular magnetic field. At the many-body level the set-up would be that of the N -body Schrödinger equation

$$\begin{aligned} i\hbar\partial_t\Psi_N &= H_N\Psi_N \\ H_N &= \sum_{j=1}^N \left\{ \left(i\hbar\nabla_j + \frac{b}{2}\mathbf{x}_j^\perp \right)^2 + NV(\mathbf{x}_j) \right\} + \sum_{j<k} w(\mathbf{x}_j - \mathbf{x}_k) \end{aligned} \quad (1.1)$$

for $\Psi_N(t) \in L^2_{\text{asym}}(\mathbb{R}^{2N})$ an antisymmetric many-body wave-function. The second and third terms of the Hamiltonian were chosen to each formally weigh $O(N^2)$ in the large N limit. The first term, because of the Pauli exclusion principle encoded in the wave-function's antisymmetry and the nature of the spectrum of the magnetic Laplacian, will weigh $\simeq \max(N^2, bN)$ (say for fixed \hbar). What we mean by a large magnetic field limit is a scaling where $b \gtrsim N \rightarrow \infty$ with fixed \hbar , with time possibly rescaled appropriately. This will result in a combined mean-field and semi-classical¹ limit, as usual for many-fermions systems. Indeed the Pauli principle will impose the occupancy of a large number of single-particle quantum states, the hallmark of semi-classical regimes. The most crucial feature of the large magnetic field regime is that the appropriate classical phase-space is *not* the position/momentum (\mathbf{x}, \mathbf{p}) space. This being perhaps the most novel aspect of the problem, we shall for now bypass the justification of the mean-field approximation to focus on semi-classics.

Hence we start from the mean-field approximation of the above. This means replacing Ψ_N e.g. by a Slater determinant of N orthogonal one-body wave-functions and consider the time evolution of the projector on the subspace thus spanned. More generally, and with the above scaling conventions, this leads to Hartree's equation

$$i\hbar\partial_t\gamma = [H_\gamma, \gamma] \quad (1.2)$$

with $0 \leq \gamma(t) \leq \mathbb{1}$ a trace-class operator on $L^2(\mathbb{R}^2)$, that one should think of as being related to Ψ_N by a partial trace

$$\gamma = \text{Tr}_{2 \rightarrow N} |\Psi_N\rangle\langle\Psi_N|.$$

The mean-field Hamiltonian H_γ is given as

$$H_\gamma = \left(i\hbar\nabla + \frac{b}{2}\mathbf{x}^\perp \right)^2 + NV + w \star \rho_\gamma \quad (1.3)$$

where

$$\rho_\gamma(\mathbf{x}) = \gamma(\mathbf{x}, \mathbf{x})$$

is the density of γ , defined in terms of its' operator kernel. The Pauli principle (antisymmetry of Ψ_N) translates into the operator constraint [33, Chapter 3]

$$0 \leq \gamma(t) \leq \mathbb{1}$$

while the number of particles is set as

$$\text{Tr}\gamma = N.$$

The limiting dynamics can be guessed by studying that of a classical particle of charge -1 in a transverse magnetic field of amplitude b and a force field F . Newton's fundamental equation of dynamics gives

$$Z''(t) = F(t, Z(t)) + bZ'(t)^\perp \quad (1.4)$$

¹Observe the respective roles of \hbar and b in the kinetic energy operator $\left(i\hbar\nabla + \frac{b}{2}\mathbf{x}^\perp \right)^2$.

For a constant and homogeneous force field, the motion is split into a cyclotron orbit and a drift (of the orbit's center) term

$$Z(t) = \underbrace{\frac{|Z'_c(0)|}{b} \begin{pmatrix} \cos(bt) \\ \sin(bt) \end{pmatrix}}_{=:Z_c(t)} + \underbrace{\frac{F^\perp}{b} t}_{=:Z_d(t)} \quad (1.5)$$

where we assumed

$$\begin{aligned} Z_d(0) &= (0, 0) \\ Z_c(0) &= \frac{|Z'_c(0)|}{b} (1, 0) \end{aligned}$$

The characteristic time for the cyclotron orbit is b^{-1} , that for the drift is of order b . This suggests, for $b \rightarrow \infty$, to observe the motion over a time scale of order b , and that the cyclotron motion will be averaged over in the limit equation, its' radius being given by

$$r_c := \frac{|Z'_c(0)|}{b}.$$

Then, if we assume a more general force field F , slowly varying on the scale of the cyclotron orbit, we should expect to leading order an effective equation

$$Z'_d(t) = \frac{F^\perp}{b}$$

for the motion of the orbit center. Following (1.3) we should set

$$F = \nabla(NV + w \star \rho)$$

with ρ the density of particles. This leads, via the method of characteristics, to a transport-type equation

$$\partial_t \rho + \nabla^\perp(NV + w \star \rho) \cdot \nabla \rho = 0. \quad (1.6)$$

Our goal is to rigorously connect (1.2) to (1.6) in the limit $b \sim N \rightarrow \infty$. At the classical level this is a gyrokinetic limit.

At the quantum level, the cyclotron radius gets quantized in multiples of \sqrt{n} where $n \in \mathbb{N}$ corresponds to the Landau level index labelling the eigenstates of the magnetic Laplacian. I.e. we write the spectral decomposition of the latter as

$$\left(i\nabla + \frac{b}{2}\mathbf{x}^\perp\right)^2 = \sum_{n \in \mathbb{N}} 2b \left(n + \frac{1}{2}\right) \Pi_n$$

where Π_n is an orthogonal projection (see below for more details). The main difference between the large magnetic field limit we consider here and more common analysis over the (\mathbf{x}, \mathbf{p}) semi-classical phase-space is that the gap $2b$ between Landau levels will be of the same order as other energetic contributions. As a consequence our phase-space will be parameterised by $\mathbf{z}, n \in \mathbb{R}^2 \times \mathbb{N}$, corresponding to the center \mathbf{z} of the cyclotron orbit's (that we will identify with the complex number z) and the Landau level index/quantized cyclotron motion parameter n .

Corresponding static problems have been considered at the level of energy ground states in the series of works [35, 36, 51] for large atoms (see also [19]) and, more related to our context, for quantum dots in [37] (see also [40]). In particular it is found that for $b \ll N$, the problem reduces to leading order to a (\mathbf{x}, \mathbf{p}) semi-classical one, similar to the weak magnetic field situation of [34, 50, 18]. The limit energy is the usual Vlasov/Thomas-Fermi functional. By contrast, for $b \gtrsim N$, one finds a

magnetic Thomas-Fermi theory set on the (\mathbf{z}, n) phase-space. This is the case under consideration here, seemingly for the first time at the dynamical level.

Indeed, dynamical studies of large fermionic systems we are aware of, even for non-zero magnetic fields, all proceed on the (\mathbf{x}, \mathbf{p}) phase-space. The classical counterpart of this work, i.e. the gyrokinetic limit of the Vlasov equation, has been well studied [26, 8, 20, 6, 28, 21, 38, 27, 48, 49, 7]. Some results start from Newton's dynamics [29]. In the quantum literature it is known that the Hartree equation can be obtained by a mean field limit from the N -body Schrödinger dynamics [5, 25, 24, 41]. It is also known that the Vlasov equation can be derived in a semi-classical limit from the Hartree equation [1, 2, 4, 32, 31, 47]. The mean field and semi-classical limits can be coupled to obtain directly the Vlasov equation from the N -body Schrödinger dynamics [12, 13, 14]. More recent results have been dealing with singular potentials [46, 42, 15, 16].

Closer to our setting, we mention the recent [3] where Euler's equation in vorticity form is obtained from the N -body Schrödinger dynamics with large magnetic field and repulsive 2D Coulomb interaction $w = -\log |\cdot|$. This corresponds to the drift equation (1.6) in this context. The crucial difference between [3] and the present contribution is that the former is set in a regime where the gap between Landau levels is small compared to the interactions. The classical phase-space is consequently again the position/momentum one. However, [3] can deal with a much more singular interaction potential, leveraging its' coercivity.

As regards the approach to the semi-classical limit, in the (\mathbf{x}, \mathbf{p}) phase-space, the use of the Wigner function is often the privileged angle of attack. We do not see that such a tool is available for the (\mathbf{z}, n) phase-space we have to consider here. Hence we rely on appropriate magnetic coherent states $\psi_{z,n} \in L^2(\mathbb{R}^2)$ with Landau level index n , approximately localized around a guiding center position \mathbf{z} . We associate to γ a Husimi function

$$m_\gamma(z, n) \propto \langle \psi_{z,n}, \gamma \psi_{z,n} \rangle$$

and study the dynamics thereof. A related approach, based on (\mathbf{x}, \mathbf{p}) coherent states, has been implemented previously in [12, 13, 14]. As it now stands, our method is rather demanding in terms of regularity of the potentials V and w . This can be improved if the boundedness of some moments of the magnetic kinetic energy are propagated in time (see Remark 5.2 below). We cannot prove this at present: we are in effect dealing with long-time asymptotics, for which it is difficult to keep moments under control, even at the level of the classical Vlasov equation.

Acknowledgments. Work financially supported by the European Union's Horizon 2020 Research and Innovation Programme (Grant agreement CORFRONMAT No. 758620). The present contribution is an expanded version of a chapter of the first author's PhD thesis [39]. We are grateful to the members of the thesis committee (Thierry Champel, Mathieu Lewin, Sören Petrat, Nicolas Raymond, Laure Saint-Raymond, Chiara Saffirio) for their careful reading of, and helpful suggestions on, the first version of this text. We also thank Laurent Laffèche for interesting discussions and an anonymous referee for spotting mistakes in a previous version of the text.

2. MAIN RESULTS

2.1. Model and scaling. It will be convenient to work in a slightly different scaling than sketched in the introduction. We set this up first.

Notation 2.1 (Model).

We work on \mathbb{R}^2 . The one body kinetic energy operator is the magnetic Laplacian

$$\mathcal{L}_b := (i\hbar\nabla + b\mathbf{A})^2$$

With

$$\text{Dom}(\mathcal{L}_b) := \{\psi \in L^2(\mathbb{R}^2) \mid \mathcal{L}_b \psi \in L^2(\mathbb{R}^2)\}$$

We work in symmetric gauge, namely the vector potential is

$$\mathbf{A} = \frac{1}{2} \mathbf{x}^\perp. \quad (2.1)$$

We denote by b the magnetic field amplitude, associated to the magnetic length

$$l_b := \sqrt{\frac{\hbar}{b}}$$

Let V be the external potential and w the interaction potential, assumed to be radial.

We study a solution $\gamma \in L^\infty(\mathbb{R}_+, \mathcal{L}^1(L^2(\mathbb{R}^2)))$ to the Hartree equation

$$\boxed{i\hbar \partial_t \gamma = [\mathcal{L}_b + V + w \star \rho_\gamma, \gamma]} \quad (2.2)$$

where \mathcal{L}^p is the p -th Schatten class,

$$\rho_\gamma(t, \mathbf{x}) := \gamma(t)(\mathbf{x}, \mathbf{x}) \quad (2.3)$$

the density associated to γ that we identify with its integral kernel. We will denote

$$H_b(t) := \mathcal{L}_b + V + \frac{1}{2} w \star \rho_{\gamma_b(t)}. \quad (2.4)$$

◇

Our goal is to obtain from the Hartree equation (2.2) the following drift equation for a density $\rho : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}_+$,

$$\partial_t \rho + \nabla^\perp(V + w \star \rho) \cdot \nabla \rho = 0. \quad (2.5)$$

We will denote

$$\text{DRIFT}_\rho(\mu)(t, \mathbf{z}) := \partial_t \mu(t, \mathbf{z}) + \nabla^\perp(V + w \star \rho(t))(\mathbf{z}) \cdot \nabla \mu(t, \mathbf{z})$$

so that our target equation (2.5) takes the form

$$\text{DRIFT}_\rho(\rho) = 0.$$

Our plan is to examine a truly large magnetic field regime where all the terms in the Hamiltonian (2.4) are of order 1. As recalled in Section 3 below, the order of magnitude of the kinetic energy is $\hbar b$, which we will henceforth fix to unity. As discussed in the introduction, the time-scale we work on is of order b . Since we consider fermionic particles, constraints on the density matrix are imposed to enforce the Pauli exclusion principle. We summarize these conventions below:

Notation 2.2 (Scaling).

We work in a large magnetic field/semi-classical limit

$$b \rightarrow +\infty, \quad \hbar \xrightarrow{b \rightarrow \infty} 0$$

such that the magnetic kinetic energy is of order 1:

$$\hbar b \xrightarrow{b \rightarrow \infty} 1. \quad (2.6)$$

Let $\gamma \in L^\infty(\mathbb{R}_+, \mathcal{L}^1(L^2(\mathbb{R}^2)))$, such that

$$\text{Tr}[\gamma(0)] = 1 \quad \text{and} \quad 0 \leq \gamma(0) \leq 2\pi l_b^2 = 2\pi \frac{\hbar}{b} \quad (2.7)$$

and define the time rescaled density matrix

$$\forall t \in \mathbb{R}_+, \quad \gamma_b(t) := \gamma(bt) \quad (2.8)$$

◇

If γ satisfies (2.2), the equation for the time-rescaled density matrix is

$$\partial_t \gamma_b = \frac{b}{i\hbar} \left[\mathcal{L}_b + V + w \star \rho_{\gamma_b}, \gamma_b \right] = \frac{1}{il_b^2} \left[\mathcal{L}_b + V + w \star \rho_{\gamma_b}, \gamma_b \right] \quad (2.9)$$

The Pauli principle $\gamma_b \leq 2\pi l_b^2$ (which propagates in time, see Lemma 3.8 below) guarantees that the system occupies a volume of order 1 in the limit

$$l_b \xrightarrow{b \rightarrow \infty} 0$$

Indeed it is known, [30, Chapter 3] or [40, Subsection I.4], that the degeneracy per area inside a Landau level is of order l_b^{-2} . A typical fermionic state satisfying (2.7) is a projection onto a N -body Slater determinant of N orthonormal one body wave-functions with

$$N := \mathcal{O} \left(\frac{1}{2\pi l_b^2} \right)$$

Such a N -particles state occupies a volume of order

$$\frac{N}{l_b^{-2}} = \mathcal{O}(1)$$

Hence with (2.6) this confirms that all the terms in the Hamiltonian $\mathcal{L}_b + V + w \star \rho_\gamma$ are of order 1. As a remark, we give an equivalent formulation of this scaling, making connections to the introductory section. If one takes exactly $\hbar = 1/b$, then (2.9) is equivalent to

$$i\partial_t \gamma = \left[(i\nabla + b^2 \mathbf{A}) + b^2 (V + w \star \rho_\gamma), \gamma \right]$$

In other words with the new scaling

$$\begin{aligned} \tilde{b} &:= b^2 \\ \tilde{\gamma}_b &:= \frac{b^2}{2\pi} \gamma \end{aligned}$$

we have

$$\begin{aligned} \text{Tr} [\tilde{\gamma}_b] &= \frac{\tilde{b}}{2\pi}, \quad \tilde{\gamma}_b \leq 1 \\ i\partial_t \tilde{\gamma}_b &= \left[(i\nabla + \tilde{b} \mathbf{A})^2 + \tilde{b} V + w \star \rho_{\tilde{\gamma}_b}, \tilde{\gamma}_b \right] \end{aligned}$$

where all the terms in the Hamiltonian

$$(i\nabla + \tilde{b} \mathbf{A})^2 + \tilde{b} V + w \star \rho_{\tilde{\gamma}_b}$$

are of order \tilde{b} . The above is the equivalent of the scaling “particle number proportional to magnetic field”, $\tilde{b} \propto N$ originally studied in [37] at the level of ground states, where a magnetic Thomas-Fermi theory emerges as the relevant effective description.

2.2. Results. We may now state our main results. We first prove that the density of the Hartree solution almost satisfies the weak form of the drift equation:

Theorem 2.3 (Dynamics of the Hartree solution).

Let $V, w \in W^{4,\infty}(\mathbb{R}^2)$ and $\gamma_b \in L^\infty(\mathbb{R}_+, \mathcal{L}^1(L^2(\mathbb{R}^2)))$ be a solution of (2.9) with initial datum satisfying

$$\mathrm{Tr}[\gamma_b(0)] = 1, \quad 0 \leq \gamma_b(0) \leq 2\pi l_b^2, \quad \mathrm{Tr}[\gamma_b(0)H_b(0)] < C$$

with C independent of b . Let the associated drift operator be as in (2.5). Then, $\forall \varphi \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$ with compact support

$$\left| \int_{\mathbb{R}_+ \times \mathbb{R}^2} \rho_{\gamma_b}(t, \mathbf{z}) \mathrm{DRIFT}_{\rho_{\gamma_b}}(\varphi)(t, \mathbf{z}) dt d\mathbf{z} - \int_{\mathbb{R}^2} \varphi(0, \mathbf{z}) \rho_{\gamma_b}(0, \mathbf{z}) d\mathbf{z} \right| \leq C(\varphi, V, w) l_b^{\frac{7}{2}}$$

for some fixed constant $C(\varphi, V, w)$.

The proof of the above consists of two main parts:

- We define a semi-classical measure on phase-space approximating the exact quantum density. Using the vortex coherent state $\psi_{z,n}$ (approximately) localized around the position $z \in \mathbb{C} \leftrightarrow \mathbf{z} \in \mathbb{R}^2$ and (exactly) localized in the n -th Landau level of the magnetic kinetic energy (see Section 3.2 below for more details) we form the Husimi function

$$(z, n) \mapsto \frac{1}{2\pi l_b^2} \langle \psi_{z,n}, \gamma_b \psi_{z,n} \rangle =: m_{\gamma_b}(z, n).$$

Selecting a suitably large cut-off $M \gg 1$ for the Landau level index we mimic the true density by summing the above for $n \leq M$

$$\rho_{\gamma_b}^{sc, \leq M} := \sum_{n \leq M} m_{\gamma_b}(z, n) \simeq \rho_{\gamma_b} \quad (2.10)$$

under suitable, mild assumptions.

- Combining the Hartree equation (2.2) with the algebraic properties of vortex coherent states we find that $\rho_{\gamma_b}^{sc, \leq M}$ approximately solves (for suitably large b and tuned $M \gg 1$) the drift equation (2.5). The control of the implied error terms is the core analytical part of the proof.

Next, using appropriate stability estimates for solutions of the limiting drift equation, we can lift the above theorem to an estimate between ρ_{γ_b} and the classical solution. We denote W_1 the Monge-Kantorovitch-Wasserstein (MKW) metric

$$W_1(\mu, \nu) := \inf_{\pi \in \Gamma(\mu, \nu)} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y| d\pi(\mathbf{x}, \mathbf{y}) = \sup_{\|\nabla \varphi\|_{L^\infty(\mathbb{R}^2)} \leq 1} \left| \int_{\mathbb{R}^2} \varphi d(\mu - \nu) \right| \quad (2.11)$$

where $\Gamma(\mu, \nu)$ is the set of couplings between $\mu, \nu \in \mathcal{P}(\mathbb{R}^2)$, namely $\mathcal{P}(\mathbb{R}^4) \ni \pi \in \Gamma(\mu, \nu)$ if

$$\int_{\mathbb{R}^2} \pi(\mathbf{x}, \mathbf{y}) d\mathbf{y} = \mu(\mathbf{x}), \quad \int_{\mathbb{R}^2} \pi(\mathbf{x}, \mathbf{y}) d\mathbf{x} = \nu(\mathbf{y}). \quad (2.12)$$

We then have

Theorem 2.4 (Convergence of densities).

We make the same assumptions as in Theorem 2.3, with in addition

$$\nabla w \in L^1(\mathbb{R}^2), \quad w \in H^2(\mathbb{R}^2), \quad V, w \in W^{9,\infty}(\mathbb{R}^2), \quad d^9 V, d^9 w \in L^1(\mathbb{R}^2)$$

and, with the position operator denoted by \mathbf{X} ,

$$\mathrm{Tr}[\gamma_b(0) |X|^p] < C \quad (2.13)$$

independently of b , for some $p > 7$.

Let $\rho \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^2))$ solve the drift equation (2.5). Then $\forall t \in \mathbb{R}_+$ and every test function φ over \mathbb{R}^2

$$\left| \int_{\mathbb{R}^2} \varphi \left(\rho_{\gamma_b}(t) - \rho(t) \right) \right| \leq \tilde{C}(p, t, V, w) \left(\|\varphi\|_{W^{1,\infty}} + \|\nabla \varphi\|_{L^2} \right) \left(W_1 \left(\rho_{\gamma_b}(0), \rho(0) \right) + I_b^{\min\left(2\frac{p-7}{4p-7}, \frac{2}{7}\right)} \right) \quad (2.14)$$

with

$$\begin{aligned} \tilde{C}(p, t, V, w) := & \left| \text{Tr} [\gamma_b(0) H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \\ & + e^{2(\|w\|_{W^{2,\infty}} + \|V\|_{W^{2,\infty}})t} \left(1 + C(p) (1 + \text{Tr} [\gamma_b(0) |X|^p]) \left(\left| \text{Tr} [\gamma_b(0) H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \right) \right. \\ & + Ct^2 \left(\|w\|_{H^2} + \|V\|_{W^{9,\infty}} + \|d^9 V\|_{L^1} + \|w\|_{W^{9,\infty}} + \|d^9 w\|_{L^1} \right) \\ & \left. \left(\|\nabla w\|_{L^1} + \|w\|_{W^{2,\infty}} e^{(\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}})t} \right) \left(\left| \text{Tr} [\gamma_b(0) H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \right) \right) \end{aligned}$$

Theorem 2.4 follows from a stability estimate à la Dobrushin [17] for classical Hamiltonian equations. We treat the error between the quantum and classical equations obtained from Theorem 2.3 as a source term for the limiting equation, and show that the stability theory of the latter (as reviewed e.g. in [23, Section 1.4]) survives this addition. When considering the limit of the semi-classical density (2.10), an estimate is obtained directly in Wasserstein-1 metric, see Proposition 6.4 below. The additional initial trapping assumption (2.13), and the slight modification of the norm in which the convergence is measured (2.14), arise when vindicating the approximation in (2.10). As regards the additional assumption (2.13), we note that it is fairly natural for typical initial data. For example, equilibria of systems with an additional trapping external potential included in the Hamiltonian usually decay exponentially in space, in which case one may think that $p = \infty$ formally.

2.3. Organisation of the paper. Section 3 covers preliminaries, focusing on the magnetic Laplacian and the conserved properties of the dynamics. In Section 4, we introduce Husimi functions, the associated semi-classical densities and prove that they approximate the physical density. Then we study the dynamics of these semi-classical densities in Section 5 and prove they approximately follow the drift equation. The conclusion of the proof of Theorem 2.3 is given in Section 5.2 by combining this with the results of Section 4. We study perturbed classical flows associated with (2.5) in Section 6. This leads to the Dobrushin-like stability estimate allowing to conclude the proof of Theorem 2.4 in Section 6.1.

3. LANDAU QUANTIZATION AND THE MAGNETIC HARTREE EQUATION

We here recall the usual formalism for describing the magnetic Laplacian in terms of annihilation and creation operators. Further details about these operators and the properties of Landau levels are reviewed e.g. in [45] or [40] and references therein. This formalism provides a basis (3.4) of eigenstates indexed by two quantum numbers n and m , with n denoting the index describing the Landau level and m representing “angular momentum minus Landau level index”. To obtain a projector on a point in phase space, we use coherent states for a fixed n . In two dimensions, the complex parameter in the definition of coherent states can be identified with a position. Consequently, following e.g. [9, 11, 10], we construct a one-particle state localized at a specific point in space, see Definition 3.4. Then, we provide some properties of the associated projector. We conclude this section with a brief recap of the conserved quantities of the Hartree equation (2.2).

3.1. Landau quantization.

Notation 3.1 (Magnetic momentum and kinetic energy).

We denote by p_1, p_2 the coordinates of the magnetic momentum

$$\mathcal{P}_{\hbar, b} := \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} := - \begin{pmatrix} i\hbar\partial_1 + bA_1 \\ i\hbar\partial_2 + bA_2 \end{pmatrix}$$

with $\mathbf{A} = \frac{\mathbf{x}^\perp}{2}$. Define then the annihilation, creation and number operators respectively as

$$a := \frac{p_1 + ip_2}{\sqrt{2\hbar b}}, \quad a^\dagger := \frac{p_1 - ip_2}{\sqrt{2\hbar b}}, \quad \mathcal{N} := a^\dagger a$$

◇

We have the commutation relations:

$$\begin{aligned} [p_1, p_2] &= i\hbar b \\ [a, a^\dagger] &= \mathbb{1} \text{ (canonical commutation relation)} \end{aligned}$$

and may express the magnetic Laplacian as

$$\mathcal{L}_b = 2\hbar b \left(\mathcal{N} + \frac{\text{Id}}{2} \right)$$

Notation 3.2 (Landau levels).

We define the n^{th} Landau level as the eigenspace associated to $n \in \mathbb{N}$:

$$\text{nLL} := \{ \psi \in \text{Dom}(\mathcal{L}_{\hbar, b}) \text{ such that } \mathcal{N}\psi = n\psi \}$$

The ground level, denoted LLL for *Lowest Landau Level* has energy $E_0 = \hbar b$.

◇

The Landau levels are isomorphic, and the operator $a^\dagger/\sqrt{n+1}$ is a unitary mapping from nLL to (n+1)LL of inverse $a/\sqrt{n+1}$. Therefore we may, using a^\dagger , extend a basis of LLL to higher Landau levels. The Lowest Landau level consists of holomorphic functions pondered by a Gaussian factor, see e.g. [45].

The Landau level quantization of the kinetic energy corresponds to the quantization of the cyclotron orbit. To complete this aspect, we associate an operator to the motion of the guiding center of the orbit.

Notation 3.3 (Guiding center oscillator).

For the rest of the text, we will identify a vector

$$\mathbf{x} := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$$

with the corresponding complex number

$$x := x_1 + ix_2.$$

It will be convenient to use an upper case \mathbf{X} for the position operator (multiplication by \mathbf{x}) in \mathbb{R}^2 and X the position operator in \mathbb{C} (multiplication by x).

We introduce the following ‘‘cyclotron motion’’ and ‘‘orbit center’’ operators [45]

$$\begin{aligned} \mathbf{r} &:= \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} := \frac{\mathcal{P}_{\hbar, b}^\perp}{b} = \frac{1}{b} \begin{pmatrix} -p_2 \\ p_1 \end{pmatrix} \\ \mathbf{R} &:= \mathbf{X} - \mathbf{r} \end{aligned}$$

and associate to \mathbf{R} the creation and annihilation operators

$$c = \frac{R_1 - iR_2}{\sqrt{2}l_b}, \quad c^\dagger = \frac{R_1 + iR_2}{\sqrt{2}l_b}.$$

◇

The operator \mathbf{r} represents the position of a particle in the center of orbit frame. The classical physics meaning of this definition is that, during cyclotron motion, the momentum is perpendicular to the position relative to the center to the orbit. Electrons are describing clockwise orbits, thus the momentum rotated of $\pi/2$ gives us \mathbf{r} . Moreover, \mathbf{r} is related to the quantization of the cyclotron pulsation of the orbit because

$$a = \frac{p_1 + ip_2}{\sqrt{2\hbar b}} = \frac{r_2 - ir_1}{\sqrt{2}l_b} = \frac{-ir}{\sqrt{2}l_b}, \quad a^\dagger = \frac{i\bar{r}}{\sqrt{2}l_b}.$$

From the definition of \mathbf{r} , the position \mathbf{R} of the orbit center is indeed

$$\mathbf{X} = \mathbf{R} + \mathbf{r}$$

and related to the second harmonic oscillator

$$c = \frac{\bar{R}}{\sqrt{2}l_b}, \quad c^\dagger = \frac{R}{\sqrt{2}l_b}. \quad (3.1)$$

The components of \mathbf{r} , \mathbf{R} and \mathbf{X} commute with one another. Moreover

$$\begin{aligned} [r_1, r_2] &= il_b^2 \\ [R_1, R_2] &= -il_b^2 \\ [c, c^\dagger] &= \text{Id} \\ [a, c] &= [a, c^\dagger] = [a^\dagger, c] = [a^\dagger, c^\dagger] = 0 \end{aligned} \quad (3.2)$$

We therefore have two independent harmonic oscillators. By successively applying the creation operators a^\dagger and c^\dagger we obtain the desired eigenbasis of the magnetic Laplacian.

It is useful to rewrite the creation and annihilation operators using complex notation:

$$\begin{aligned} a &= -i \left(\frac{X}{2\sqrt{2}l_b} + \sqrt{2}l_b \partial_{\bar{X}} \right), \quad a^\dagger = i \left(\frac{\bar{X}}{2\sqrt{2}l_b} - \sqrt{2}l_b \partial_X \right) \\ c &= \frac{\bar{X}}{2\sqrt{2}l_b} + \sqrt{2}l_b \partial_X, \quad c^\dagger = \frac{X}{2\sqrt{2}l_b} - \sqrt{2}l_b \partial_{\bar{X}} \end{aligned} \quad (3.3)$$

In symmetric gauge (2.1), the family defined by

$$\varphi_{n,m} := \frac{(a^\dagger)^n (c^\dagger)^m}{\sqrt{n!m!}} \varphi_{0,0} \quad (3.4)$$

with

$$\varphi_{0,0}(\mathbf{x}) = \frac{1}{\sqrt{2\pi l_b^2}} e^{-\frac{|\mathbf{x}|^2}{4l_b^2}}$$

is an orthonormal Hilbert basis of $L^2(\mathbb{R}^2)$. The full expression, obtained from (3.3) or see [9, 11, 10, 45], is

$$\varphi_{n,m}(\mathbf{x}) = \frac{\left(i\frac{\bar{X}}{2} - 2il_b^2\partial_X\right)^n X^m e^{-\frac{|x|^2}{4l_b^2}}}{\sqrt{2\pi l_b^2 n! m!} \left(\sqrt{2}l_b\right)^{n+m}} \quad (3.5)$$

The orthogonal projector on the n -th Landau level from Notation 3.2 is recovered as

$$\Pi_n := \sum_{m \in \mathbb{N}} |\varphi_{n,m}\rangle \langle \varphi_{n,m}|.$$

3.2. Coherent states. We next define coherent states in order to have wave functions localized at a precise point in the phase-space “position of the orbit center \times Landau level index”.

Definition 3.4 (Vortex coherent states).

Let $z \in \mathbb{C} \leftrightarrow \mathbb{R}^2$, $n \in \mathbb{N}$. We define the associated coherent state

$$\psi_{z,n} := e^{\frac{\bar{z}c^\dagger - zc}{\sqrt{2}l_b}} \varphi_{n,0} = e^{-\frac{|z|^2}{4l_b^2} + \frac{\bar{z}c^\dagger}{\sqrt{2}l_b}} \varphi_{n,0} = e^{-\frac{|z|^2}{4l_b^2}} \sum_{m \in \mathbb{N}} \frac{1}{\sqrt{m!}} \left(\frac{\bar{z}}{\sqrt{2}l_b}\right)^m \varphi_{n,m} \quad (3.6)$$

and the associated projectors

$$\begin{aligned} \Pi_{z,n} &:= |\psi_{z,n}\rangle \langle \psi_{z,n}| \\ \Pi_z &= \sum_{n \in \mathbb{N}} \Pi_{z,n}. \end{aligned} \quad (3.7)$$

Finally we define for $M \in \mathbb{N}$ or $N_1 \leq N_2$ the truncated projectors

$$\Pi_{N_1:N_2} := \sum_{n=N_1}^{N_2} \Pi_n, \quad \Pi_{\leq M} := \Pi_{0:M}, \quad \Pi_{z,\leq M} := \sum_{n=0}^M \Pi_{z,n}$$

with similar definitions for $\Pi_{>M}$ and $\Pi_{z,>M}$. By convention, if $N_1 < 0$ we set

$$\Pi_{N_1:N_2} := \Pi_{\leq N_2}.$$

◇

By construction we have that

$$a^\dagger \psi_{z,n} = \sqrt{n+1} \psi_{z,n+1}, \quad a \psi_{z,n} = \sqrt{n} \psi_{z,n-1}. \quad (3.8)$$

Also,

$$c \psi_{z,n} = \frac{\bar{z}}{\sqrt{2}l_b} \psi_{z,n} \quad (3.9)$$

and thus, recalling (3.1) we note that $\psi_{z,n}$ is localised around z in the sense that

$$\bar{R} \psi_{z,n} = \bar{z} \psi_{z,n}$$

with fluctuations of order l_b (as can be seen from the next lemma). The following explicit expressions will be important for us:

Lemma 3.5 (Expression of vortex coherent states).

We have

$$\psi_{z,n}(\mathbf{x}) = \frac{i^n}{\sqrt{2\pi n!} l_b} \left(\frac{\overline{x-z}}{\sqrt{2} l_b} \right)^n e^{-\frac{|x-z|^2 - 2iz^\perp \cdot \mathbf{x}}{4l_b^2}} \quad (3.10)$$

$$\begin{aligned} \Pi_{z,n}(\mathbf{x}, \mathbf{y}) &= \frac{1}{2\pi n! l_b^2} \left(\frac{(\overline{x-z})(y-z)}{2l_b^2} \right)^n e^{-\frac{|x-z|^2 + |y-z|^2 - 2iz^\perp \cdot (\mathbf{x}-\mathbf{y})}{4l_b^2}} \\ \Pi_z(\mathbf{x}, \mathbf{y}) &= \frac{1}{2\pi l_b^2} e^{-\frac{|x-y|^2 - 2i(\mathbf{x}^\perp \cdot \mathbf{y} + 2z^\perp \cdot (\mathbf{x}-\mathbf{y}))}{4l_b^2}}. \end{aligned} \quad (3.11)$$

Consequently

$$\nabla_z^\perp \Pi_z(\mathbf{x}, \mathbf{y}) = \frac{\mathbf{y} - \mathbf{x}}{il_b^2} \Pi_z(\mathbf{x}, \mathbf{y}) \quad (3.12)$$

or, as an operator identity,

$$\nabla_z^\perp \Pi_z = \frac{1}{il_b^2} [\Pi_z, \mathbf{X}] \quad (3.13)$$

We refer to [9] again for the derivation of the above exact expressions, from which (3.12) immediately follows. We will rely heavily on the following closure relations [9, 45]:

Lemma 3.6 (Coherent states partition of unity).

With Definition 3.4,

$$\frac{1}{2\pi l_b^2} \int_{\mathbb{R}^2} \Pi_{z,n} d\mathbf{z} = \Pi_n \quad (3.14)$$

$$\frac{1}{2\pi l_b^2} \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^2} \Pi_{z,n} d\mathbf{z} = \mathbb{1}_{L^2(\mathbb{R}^2)} \quad (3.15)$$

Proof. We prove (3.14) with a direct computation

$$\begin{aligned} & \frac{1}{2\pi l_b^2} \int_{\mathbb{R}^2} |\psi_{z,n}\rangle \langle \psi_{z,n}| d\mathbf{z} \\ &= \sum_{m_1, m_2 \in \mathbb{N}} \frac{1}{2\pi l_b^2 \sqrt{m_1! m_2!}} \int_{\mathbb{R}^2} \left(\frac{\overline{z}}{\sqrt{2} l_b} \right)^{m_1} \left(\frac{z}{\sqrt{2} l_b} \right)^{m_2} e^{-\frac{|z|^2}{2l_b^2}} d\mathbf{z} |\varphi_{n, m_1}\rangle \langle \varphi_{n, m_2}| \\ &= \sum_{m \in \mathbb{N}} \frac{1}{2\pi l_b^2 m!} \int_{\mathbb{R}^2} \left(\frac{|z|^2}{2l_b^2} \right)^m e^{-\frac{|z|^2}{2l_b^2}} d\mathbf{z} |\varphi_{n, m}\rangle \langle \varphi_{n, m}| = \sum_{m \in \mathbb{N}} |\varphi_{n, m}\rangle \langle \varphi_{n, m}| = \Pi_n. \end{aligned}$$

In the second line we have used polar coordinates to observe that the integrals over z vanish when $m_1 \neq m_2$. Summing over n , (3.15) follows. \square

Formula (3.13) will play a key role in the computation of the spacial derivative of the density in Section 5 below. We will also rely heavily on an approximation thereof applying to the truncated projector.

Lemma 3.7 (Spatial derivatives of coherent state projectors).

For the localized projector (3.7) we have

$$\begin{aligned} \nabla_z^\perp \Pi_{z,n} &= \frac{1}{il_b^2} [\Pi_{z,n}, \mathbf{X}] - \frac{\sqrt{n+1}}{\sqrt{2}l_b} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \left(\begin{array}{l} |\psi_{z,n}\rangle \langle \psi_{z,n+1}| \\ |\psi_{z,n+1}\rangle \langle \psi_{z,n}| \end{array} \right) \\ &\quad + \frac{\sqrt{n}}{\sqrt{2}l_b} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \left(\begin{array}{l} |\psi_{z,n-1}\rangle \langle \psi_{z,n}| \\ |\psi_{z,n}\rangle \langle \psi_{z,n-1}| \end{array} \right) \end{aligned} \quad (3.16)$$

and, summing over $n \leq M$,

$$\nabla_z^\perp \Pi_{z,\leq M} = \frac{1}{il_b^2} [\Pi_{z,\leq M}, \mathbf{X}] - \frac{\sqrt{M+1}}{\sqrt{2}l_b} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \left(\begin{array}{l} |\psi_{z,M}\rangle \langle \psi_{z,M+1}| \\ |\psi_{z,M+1}\rangle \langle \psi_{z,M}| \end{array} \right). \quad (3.17)$$

Proof. Starting from (3.10) we have

$$\psi_{z,n}(\mathbf{x}) = \frac{-i}{\sqrt{n}} \frac{\overline{z-x}}{\sqrt{2}l_b} \psi_{z,n-1}(\mathbf{x}), \quad \overline{\psi_{z,n}(\mathbf{y})} = \frac{i}{\sqrt{n}} \frac{z-y}{\sqrt{2}l_b} \overline{\psi_{z,n-1}(\mathbf{y})}. \quad (3.18)$$

Hence

$$\begin{aligned} \Pi_{z,n}(\mathbf{x}, \mathbf{y}) &= \psi_{z,n}(\mathbf{x}) \overline{\psi_{z,n}(\mathbf{y})} = \frac{1}{2\pi n! l_b^2} \left(\frac{(\overline{x-z})(y-z)}{2l_b^2} \right)^n e^{-\frac{|x-z|^2 + |y-z|^2 - 2iz \cdot (x-y)}{4l_b^2}} \\ &= \frac{1}{2\pi n! l_b^2} \left(\frac{(\overline{x-z})(y-z)}{2l_b^2} \right)^n e^{-\frac{2|z|^2 + |x|^2 + |y|^2 - 2(\overline{z}x + z\overline{y})}{4l_b^2}} \end{aligned}$$

and we deduce

$$\begin{aligned} \partial_z \Pi_{z,n}(\mathbf{x}, \mathbf{y}) &= \frac{\overline{y-z}}{2l_b^2} \Pi_{z,n}(\mathbf{x}, \mathbf{y}) + \frac{\overline{z-x}}{2l_b^2} \Pi_{z,n-1}(\mathbf{x}, \mathbf{y}) \\ &= \frac{\overline{y-x}}{2l_b^2} \psi_{z,n}(\mathbf{x}) \overline{\psi_{z,n}(\mathbf{y})} + \frac{\overline{x-z}}{2l_b^2} \psi_{z,n}(\mathbf{x}) \overline{\psi_{z,n}(\mathbf{y})} + \frac{\overline{z-x}}{2l_b^2} \psi_{z,n-1}(\mathbf{x}) \overline{\psi_{z,n-1}(\mathbf{y})} \\ &= \frac{\overline{y-x}}{2l_b^2} \psi_{z,n}(\mathbf{x}) \overline{\psi_{z,n}(\mathbf{y})} - i \frac{\sqrt{n+1}}{\sqrt{2}l_b} \psi_{z,n+1}(\mathbf{x}) \overline{\psi_{z,n}(\mathbf{y})} + i \frac{\sqrt{n}}{\sqrt{2}l_b} \psi_{z,n}(\mathbf{x}) \overline{\psi_{z,n-1}(\mathbf{y})} \end{aligned}$$

together with

$$\begin{aligned} \partial_{\overline{z}} \Pi_{z,n}(\mathbf{x}, \mathbf{y}) &= \frac{x-z}{2l_b^2} \Pi_{z,n}(\mathbf{x}, \mathbf{y}) + \frac{z-y}{2l_b^2} \Pi_{z,n-1}(\mathbf{x}, \mathbf{y}) \\ &= \frac{x-y}{2l_b^2} \psi_{z,n}(\mathbf{x}) \overline{\psi_{z,n}(\mathbf{y})} + \frac{y-z}{2l_b^2} \psi_{z,n}(\mathbf{x}) \overline{\psi_{z,n}(\mathbf{y})} + \frac{z-y}{2l_b^2} \psi_{z,n-1}(\mathbf{x}) \overline{\psi_{z,n-1}(\mathbf{y})} \\ &= \frac{x-y}{2l_b^2} \psi_{z,n}(\mathbf{x}) \overline{\psi_{z,n}(\mathbf{y})} + i \frac{\sqrt{n+1}}{\sqrt{2}l_b} \psi_{z,n}(\mathbf{x}) \overline{\psi_{z,n+1}(\mathbf{y})} - i \frac{\sqrt{n}}{\sqrt{2}l_b} \psi_{z,n-1}(\mathbf{x}) \overline{\psi_{z,n}(\mathbf{y})}. \end{aligned}$$

This leads to

$$\begin{aligned} \partial_{z_1} \Pi_{z,n}(\mathbf{x}, \mathbf{y}) &= (\partial_z + \partial_{\overline{z}}) \Pi_{z,n}(\mathbf{x}, \mathbf{y}) \\ &= i \frac{x_2 - y_2}{l_b^2} \Pi_{z,n}(\mathbf{x}, \mathbf{y}) + i \frac{\sqrt{n+1}}{\sqrt{2}l_b} \left(\psi_{z,n}(\mathbf{x}) \overline{\psi_{z,n+1}(\mathbf{y})} - \psi_{z,n+1}(\mathbf{x}) \overline{\psi_{z,n}(\mathbf{y})} \right) \\ &\quad - i \frac{\sqrt{n}}{\sqrt{2}l_b} \left(\psi_{z,n-1}(\mathbf{x}) \overline{\psi_{z,n}(\mathbf{y})} - \psi_{z,n}(\mathbf{x}) \overline{\psi_{z,n-1}(\mathbf{y})} \right) \end{aligned}$$

and

$$\begin{aligned} \partial_{z_2} \Pi_{z,n}(\mathbf{x}, \mathbf{y}) &= i (\partial_z - \partial_{\bar{z}}) \Pi_{z,n}(\mathbf{x}, \mathbf{y}) \\ &= i \frac{y_1 - x_1}{l_b^2} \Pi_{z,n}(\mathbf{x}, \mathbf{y}) + \frac{\sqrt{n+1}}{\sqrt{2}l_b} \left(\psi_{z,n}(\mathbf{x}) \overline{\psi_{z,n+1}(\mathbf{y})} + \psi_{z,n+1}(\mathbf{x}) \overline{\psi_{z,n}(\mathbf{y})} \right) \\ &\quad - \frac{\sqrt{n}}{\sqrt{2}l_b} \left(\psi_{z,n-1}(\mathbf{x}) \overline{\psi_{z,n}(\mathbf{y})} + \psi_{z,n}(\mathbf{x}) \overline{\psi_{z,n-1}(\mathbf{y})} \right) \end{aligned}$$

It follows that

$$\begin{aligned} \nabla_z^\perp \Pi_{z,n}(\mathbf{x}, \mathbf{y}) &= i \frac{x-y}{l_b^2} \Pi_{z,n}(\mathbf{x}, \mathbf{y}) - \frac{\sqrt{n+1}}{\sqrt{2}l_b} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \psi_{z,n}(\mathbf{x}) \overline{\psi_{z,n+1}(\mathbf{y})} \\ \psi_{z,n+1}(\mathbf{x}) \overline{\psi_{z,n}(\mathbf{y})} \end{pmatrix} \\ &\quad + \frac{\sqrt{n}}{\sqrt{2}l_b} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \psi_{z,n-1}(\mathbf{x}) \overline{\psi_{z,n}(\mathbf{y})} \\ \psi_{z,n}(\mathbf{x}) \overline{\psi_{z,n-1}(\mathbf{y})} \end{pmatrix}, \end{aligned}$$

which is (3.16). The summation over n cancels telescopic terms, leading to

$$\nabla_z^\perp \Pi_{z, \leq M}(\mathbf{x}, \mathbf{y}) = i \frac{x-y}{l_b^2} \Pi_{z, \leq M}(\mathbf{x}, \mathbf{y}) - \frac{\sqrt{M+1}}{\sqrt{2}l_b} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \psi_{z,M}(\mathbf{x}) \overline{\psi_{z,M+1}(\mathbf{y})} \\ \psi_{z,M+1}(\mathbf{x}) \overline{\psi_{z,M}(\mathbf{y})} \end{pmatrix}.$$

□

3.3. Conservation properties. We next state some basic properties of the Hartree dynamics (2.2).

Lemma 3.8 (Conservation of mass and Pauli principle).

Assume $\gamma_b \in L^\infty(\mathbb{R}_+, \mathcal{L}^1(L^2(\mathbb{R}^2)))$ solves

$$\partial_t \gamma_b(t) = \frac{1}{il_b^2} \left[\mathcal{L}_b + V + w \star \rho_{\gamma_b(t)}, \gamma_b \right]$$

and satisfies

$$\mathrm{Tr} [\gamma_b(0)] = 1, \quad 0 \leq \gamma_b(0) \leq 2\pi l_b^2$$

then $\forall t \in \mathbb{R}_+$,

$$\mathrm{Tr} [\gamma_b(t)] = 1, \quad 0 \leq \gamma_b(t) \leq 2\pi l_b^2$$

Proof. This follows from the fact that the dynamics is Hamiltonian. □

We also have

Lemma 3.9 (Energy conservation).

Assume $\gamma_b \in L^\infty(\mathbb{R}_+, \mathcal{L}^1(L^2(\mathbb{R}^2)))$ solves

$$\partial_t \gamma_b(t) = \frac{1}{il_b^2} \left[\mathcal{L}_b + V + w \star \rho_{\gamma_b(t)}, \gamma_b \right]$$

with initial datum satisfying

$$\mathrm{Tr} [\gamma_b(0)] = 1, \quad \mathrm{Tr} [\gamma_b(0) H_b(0)] < C$$

then

$$\frac{d}{dt} \mathrm{Tr} [\gamma_b(t) H_b(t)] = 0.$$

Moreover, $\forall W \in L^\infty(\mathbb{R}^2)$,

$$\left| \text{Tr} [\gamma_b \mathcal{L}_b] - \text{Tr} [\gamma_b (\mathcal{L}_b + W)] \right| \leq \|W\|_{L^\infty}$$

Proof. The equation being Hamiltonian, the total energy is certainly conserved. Then, the kinetic energy is bounded by

$$\text{Tr} [\gamma_b \mathcal{L}_b] = \text{Tr} [\gamma_b (\mathcal{L}_b + W)] - \text{Tr} [\gamma_b W] \leq \left| \text{Tr} [\gamma_b (\mathcal{L}_b + W)] \right| + \|W\|_{L^\infty}$$

□

4. HUSIMI FUNCTIONS AND SEMI-CLASSICAL DENSITIES

In this section, we provide the first set of tools mentioned below Theorem 2.3, namely the construction and properties of Husimi functions and associated semi-classical densities.

4.1. Semi-classical density. Given a coherent state basis over a Hilbert space and a trace-class operator acting on the latter, the notion of Husimi function/lower symbol is fairly standard (see e.g. [44, Section 3.3] and references therein):

Definition 4.1 (Husimi function and associated density).

For $\gamma \in \mathcal{L}^1(L^2(\mathbb{R}^2))$, let

$$m_\gamma(z, n) := \frac{1}{2\pi l_b^2} \langle \psi_{z,n} | \gamma \psi_{z,n} \rangle = \frac{1}{2\pi l_b^2} \text{Tr} [\Pi_{z,n} \gamma] \quad (4.1)$$

with the associated semi-classical density

$$\rho_\gamma^{sc}(z) := \sum_{n=0}^{\infty} m_\gamma(z, n).$$

For $M \in \mathbb{N}$ such that $1 \ll M \ll l_b^{-2}$, we define the truncated version thereof

$$\rho_\gamma^{sc, \leq M}(z) := \frac{1}{2\pi l_b^2} \text{Tr} [\gamma \Pi_{z, \leq M}] \quad (4.2)$$

◇

The parameter M represents the number of Landau levels we take into account for the semi-classical approximations. It will be important to have $M \gg 1$ when $b \rightarrow \infty$ to recover the true quantum density ρ_γ of a general γ (with reasonable magnetic kinetic energy). On the other hand, the larger n the less the coherent state $\psi_{z,n}$ is localized around z , making the approximation less efficient. For this reason we will mostly use the truncated (4.2) for a suitable $1 \ll M \ll l_b^{-2}$, and use moments of the kinetic energy to discard the contribution to the density of Landau levels with index $n > M$.

The main estimate we will rely on is as follows:

Proposition 4.2 (Convergence of the truncated semi-classical density). ,

Let $k \geq 0$, $\gamma_b \in \mathcal{L}^1(L^2(\mathbb{R}^2))$ and assume

$$\text{Tr} [\gamma_b] = 1, \quad 0 \leq \gamma_b \leq 2\pi l_b^2, \quad \text{Tr} [\gamma_b \mathcal{L}_b^k] < \infty \quad (4.3)$$

then $\forall \varphi \in L^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$,

$$\left| \int_{\mathbb{R}^2} \varphi \left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc, \leq M} \right) \right| \leq \|\varphi\|_{L^\infty} M^{-\frac{k}{2}} \sqrt{\text{Tr} [\gamma_b \Pi_{>M} \mathcal{L}_b^k]}$$

$$+ C \|\nabla\varphi\|_{L^2} \sqrt{\text{Tr} [\gamma_b \mathcal{L}_b^k]} \cdot \begin{cases} M^{1-\frac{k}{2}} l_b & \text{if } k < 2 \\ \sqrt{\ln(M)} l_b & \text{if } k = 2 \\ l_b & \text{if } k > 2 \end{cases} \quad (4.4)$$

The first term on the right-hand side of (4.4) corresponds to the contribution of high Landau levels. In our approach to the semi-classical limit we will rely solely on one moment of the magnetic kinetic energy being bounded uniformly in time, $k = 1$. Observe that then the error terms will be small if $\sqrt{M}l_b \ll 1$. This constraint has a physical meaning. Specifically, from the expression of the coherent state (3.10), we can infer that the characteristic localization length for particles in nLL is $\sqrt{n}l_b$. Therefore, $\rho_{\gamma_b}^{sc, \leq M}$ with $\sqrt{M}l_b \ll 1$ represents the semi-classical density of particles localized with precision better than what is aimed at in the classical equation. The fluctuations of the position operator will be small in the limit for this contribution.

We will have to test γ_b against multiplication operators by nice functions φ . Using the resolution of the identity from Lemma 3.6, the gist of the estimate consists in vindicating that

$$\frac{1}{2\pi l_b^2} \sum_{n \geq 0} \int_{\mathbb{R}^2} \iint_{\mathbb{R}^4} (\varphi(\mathbf{x}) - \varphi(\mathbf{z})) \gamma_b(\mathbf{x}, \mathbf{y}) \Pi_{z,n}(\mathbf{y}, \mathbf{x}) dx dy dz \underset{b \rightarrow \infty}{\simeq} 0$$

using that the coherent projector's kernel

$$\Pi_{z,n}(\mathbf{x}, \mathbf{y}) = \psi_{z,n}(\mathbf{x}) \overline{\psi_{z,n}(\mathbf{y})}$$

is strongly localized, for moderate values of n , around $\mathbf{z} = \mathbf{y} = \mathbf{x}$. We start with a lemma that will deal with the part of the sum bearing on low Landau levels, $A(\mathbf{z})$ below playing the role of the multiplication operator by $\varphi(\bullet) - \varphi(\mathbf{z})$.

Lemma 4.3 (Bounds on expectations of truncated operators).

Let $\forall \mathbf{z} \in \mathbb{R}^2$, $(A_n(\mathbf{z}))_{n \in \mathbb{N}}$ be a sequence of operators on $L^2(\mathbb{R}^2)$, $k \leq 0$, $\gamma_b \in \mathcal{L}^1(L^2(\mathbb{R}^2))$ and assume

$$\text{Tr} [\gamma_b] = 1, \quad 0 \leq \gamma_b \leq 2\pi l_b^2. \quad (4.5)$$

Then

$$\frac{1}{2\pi l_b^2} \sum_{n=0}^M \int_{\mathbb{R}^2} |\text{Tr} [A_n(\mathbf{z}) \gamma_b \Pi_{z,n}]| d\mathbf{z} \leq \frac{1}{2} \sqrt{\text{Tr} [\gamma_b \mathcal{L}_b^k]} \left(\sum_{n=0}^M \frac{1}{(n+1)^k} \int_{\mathbb{R}^2} \text{Tr} [|A_n(\mathbf{z})|^2 \Pi_{z,n}] d\mathbf{z} \right)^{\frac{1}{2}} \quad (4.6)$$

and $\forall \varphi \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$,

$$\begin{aligned} & \frac{1}{2\pi l_b^2} \sum_{n=0}^M \int_{\mathbb{R}^2} |\varphi(\mathbf{z}) \text{Tr} [A_n(\mathbf{z}) \gamma_b \Pi_{z,n}]| d\mathbf{z} \\ & \leq \frac{\sqrt{\|\varphi\|_{L^1} \|\varphi\|_{L^\infty}}}{2} \sqrt{\text{Tr} [\gamma_b \mathcal{L}_b^k]} \left(\sum_{n=0}^M (n+1)^{-k} \sup_{\mathbf{z} \in \mathbb{R}^2} \|A_n(\mathbf{z})\|_{L^2}^2 \right)^{\frac{1}{2}} \quad (4.7) \end{aligned}$$

Proof. The main idea is to exploit the sum over Landau levels to introduce moments of the kinetic energy. From (2.6), we notice that $\forall n \in \mathbb{N}$,

$$(n+1)^k \text{Tr} [\gamma_b \Pi_n] = \left(\frac{n+1}{2\hbar b \left(n + \frac{1}{2}\right)} \right)^k \text{Tr} [\gamma_b \Pi_n \mathcal{L}_b^k] \leq \text{Tr} [\gamma_b \Pi_n \mathcal{L}_b^k] \quad (4.8)$$

so

$$\sum_{n=0}^M (n+1)^k \text{Tr} [\gamma_b \Pi_n] \leq \text{Tr} [\gamma_b \mathcal{L}_b^k] \quad (4.9)$$

Applying the Cauchy-Schwarz inequality, using $\Pi_{z,n}^2 = \Pi_{z,n}$ and (4.5) followed by Young's inequality we find

$$\begin{aligned} \sum_{n=0}^M \left| \text{Tr} [A_n(\mathbf{z}) \gamma_b \Pi_{z,n}] \right| &\leq \sum_{n=0}^M \sqrt{\text{Tr} [|A_n(\mathbf{z})|^2 \Pi_{z,n}]} \sqrt{\text{Tr} [\gamma_b^2 \Pi_{z,n}]} \\ &\leq \sqrt{2\pi l_b^2} \sum_{n=0}^M \sqrt{\text{Tr} [|A_n(\mathbf{z})|^2 \Pi_{z,n}]} \sqrt{\text{Tr} [\gamma_b \Pi_{z,n}]} \\ &\leq \sqrt{\frac{\pi}{2}} l_b \sum_{n=0}^M \left(\frac{1}{\epsilon_n} \text{Tr} [|A_n(\mathbf{z})|^2 \Pi_{z,n}] + \epsilon_n \text{Tr} [\gamma_b \Pi_{z,n}] \right) \end{aligned} \quad (4.10)$$

where we will choose $\epsilon_n := \epsilon(n+1)^k$ for some $\epsilon > 0$. Integrating in z , using (3.14) and inserting (4.9) gives

$$\begin{aligned} &\frac{1}{2\pi l_b^2} \sum_{n=0}^M \int_{\mathbb{R}^2} \left| \text{Tr} [A_n(\mathbf{z}) \gamma_b \Pi_{z,n}] \right| d\mathbf{z} \\ &\leq \sqrt{\frac{\pi}{2}} l_b \sum_{n=0}^M \left(\frac{1}{2\pi l_b^2 \epsilon_n} \int_{\mathbb{R}^2} \text{Tr} [|A_n(\mathbf{z})|^2 \Pi_{z,n}] d\mathbf{z} + \epsilon_n \text{Tr} [\gamma_b \Pi_n] \right) \\ &\leq \sqrt{\frac{\pi}{2}} l_b \left(\sum_{n=0}^M \frac{1}{2\pi l_b^2 \epsilon(n+1)^k} \int_{\mathbb{R}^2} \text{Tr} [|A_n(\mathbf{z})|^2 \Pi_{z,n}] d\mathbf{z} + \epsilon \text{Tr} [\gamma_b \mathcal{L}_b^k] \right) \end{aligned}$$

Choosing now

$$\epsilon := \left(\frac{1}{2\pi l_b^2} \sum_{n=0}^M \frac{1}{(n+1)^k} \int_{\mathbb{R}^2} \text{Tr} [|A_n(\mathbf{z})|^2 \Pi_{z,n}] d\mathbf{z} \frac{1}{\text{Tr} [\gamma_b \mathcal{L}_b^k]} \right)^{\frac{1}{2}}$$

leads to (4.6):

$$\begin{aligned} &\frac{1}{2\pi l_b^2} \sum_{n=0}^M \int_{\mathbb{R}^2} \left| \text{Tr} [A_n(\mathbf{z}) \gamma_b \Pi_{z,n}] \right| d\mathbf{z} \\ &\leq \sqrt{\frac{\pi}{2}} l_b \left(\frac{1}{2\pi l_b^2} \sum_{n=0}^M \frac{1}{(n+1)^k} \int_{\mathbb{R}^2} \text{Tr} [|A_n(\mathbf{z})|^2 \Pi_{z,n}] d\mathbf{z} \right)^{\frac{1}{2}} \sqrt{\text{Tr} [\gamma_b \mathcal{L}_b^k]} \\ &= \frac{1}{2} \sqrt{\text{Tr} [\gamma_b \mathcal{L}_b^k]} \left(\sum_{n=0}^M \frac{1}{(n+1)^k} \int_{\mathbb{R}^2} \text{Tr} [|A_n(\mathbf{z})|^2 \Pi_{z,n}] d\mathbf{z} \right)^{\frac{1}{2}} \end{aligned}$$

As regards (4.7), we start again from (4.10) to find

$$\frac{1}{2\pi l_b^2} \sum_{n=0}^M \int_{\mathbb{R}^2} \left| \varphi(\mathbf{z}) \text{Tr} [A_n(\mathbf{z}) \gamma_b \Pi_{z,n}] \right| d\mathbf{z}$$

$$\begin{aligned}
&\leq \sqrt{\frac{\pi}{2}} l_b \left(\sum_{n=0}^M \frac{1}{2\pi l_b^2 \epsilon (n+1)^k} \int_{\mathbb{R}^2} |\varphi(\mathbf{z})| \operatorname{Tr} \left[|A_n(\mathbf{z})|^2 \Pi_{z,n} \right] d\mathbf{z} + \epsilon \|\varphi\|_{L^\infty} \operatorname{Tr} [\gamma_b \mathcal{L}_b^k] \right) \\
&\leq \sqrt{\frac{\pi}{2}} l_b \left(\sum_{n=0}^M \frac{1}{2\pi l_b^2 \epsilon (n+1)^k} \|\varphi\|_{L^1} \sup_{z \in \mathbb{R}^2} \|A_n(\mathbf{z})\|_{L^2}^2 + \epsilon \|\varphi\|_{L^\infty} \operatorname{Tr} [\gamma_b \mathcal{L}_b^k] \right).
\end{aligned}$$

We conclude the proof by choosing

$$\epsilon := \sqrt{\frac{\|\varphi\|_{L^1}}{2\pi l_b^2 \|\varphi\|_{L^\infty} \operatorname{Tr} [\gamma_b \mathcal{L}_b^k]} \sum_{n=0}^M \frac{1}{(n+1)^k} \sup_{z \in \mathbb{R}^2} \|A_n(\mathbf{z})\|_{L^2}^2}$$

□

We may next proceed to the

Proof of Proposition 4.2. For $\varphi \in C_c^\infty(\mathbb{R}^2)$, we write

$$\begin{aligned}
\int_{\mathbb{R}^2} \varphi \left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc, \leq M} \right) &= \operatorname{Tr} [\varphi \gamma_b] - \frac{1}{2\pi l_b^2} \int_{\mathbb{R}^2} \varphi(\mathbf{z}) \operatorname{Tr} [\Pi_{z, \leq M} \gamma_b] d\mathbf{z} \\
&= \frac{1}{2\pi l_b^2} \int_{\mathbb{R}^2} \operatorname{Tr} [(\varphi - \varphi(\mathbf{z})) \Pi_{z, \leq M} \gamma_b] d\mathbf{z} + \operatorname{Tr} [\varphi \Pi_{> M} \gamma_b] \quad (4.11)
\end{aligned}$$

Step 1, low Landau levels. Using the change of variables

$$\frac{\mathbf{x} - \mathbf{z}}{\sqrt{2} l_b} \mapsto \mathbf{x}$$

and Taylor's theorem, we get

$$\begin{aligned}
\int_{\mathbb{R}^2} \operatorname{Tr} [|\varphi - \varphi(\mathbf{z})|^2 \Pi_{z,n}] d\mathbf{z} &= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |\varphi(\mathbf{x}) - \varphi(\mathbf{z})|^2 \Pi_{z,n}(\mathbf{x}, \mathbf{z}) d\mathbf{x} d\mathbf{z} \\
&= \frac{1}{2\pi n! l_b^2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |\varphi(\mathbf{x}) - \varphi(\mathbf{z})|^2 \left| \frac{\mathbf{x} - \mathbf{z}}{\sqrt{2} l_b} \right|^{2n} e^{-\frac{|\mathbf{x}-\mathbf{z}|^2}{2l_b^2}} d\mathbf{x} d\mathbf{z} \\
&= \frac{1}{\pi n!} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left| \varphi(\mathbf{z} + \sqrt{2} l_b \mathbf{x}) - \varphi(\mathbf{z}) \right|^2 |\mathbf{x}|^{2n} e^{-|\mathbf{x}|^2} d\mathbf{x} d\mathbf{z} \\
&= \frac{1}{\pi n!} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left| \int_0^1 \nabla \varphi(\mathbf{z} + \sqrt{2} l_b s \mathbf{x}) \cdot \sqrt{2} l_b \mathbf{x} ds \right|^2 |\mathbf{x}|^{2n} e^{-|\mathbf{x}|^2} d\mathbf{x} d\mathbf{z} \\
&\leq \frac{2l_b^2}{\pi n!} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \int_0^1 \left| \nabla \varphi(\mathbf{z} + \sqrt{2} l_b s \mathbf{x}) \right|^2 |\mathbf{x}|^{2(n+1)} e^{-|\mathbf{x}|^2} ds d\mathbf{x} d\mathbf{z} \\
&\leq \frac{2l_b^2}{\pi n!} \|\nabla \varphi\|_{L^2}^2 \int_{\mathbb{R}^2} |\mathbf{x}|^{2(n+1)} e^{-|\mathbf{x}|^2} d\mathbf{x} = 2 \|\nabla \varphi\|_{L^2}^2 (n+1) l_b^2 \quad (4.12)
\end{aligned}$$

Introducing the notation

$$p_\lambda(n) := n^{-\lambda} \mathbb{1}_{\lambda < 0} + \ln(n) \mathbb{1}_{\lambda = 0} + \mathbb{1}_{\lambda > 0} \quad (4.13)$$

we have the asymptotics

$$\sum_{n=0}^M \frac{1}{(n+1)^{k-1}} = \begin{cases} \mathcal{O}(M^{2-k}) & \text{if } k < 2 \\ \mathcal{O}(\ln(M)) & \text{if } k = 2 \\ \mathcal{O}(1) & \text{if } k > 2 \end{cases} = \mathcal{O}(p_{k-2}(M))$$

Hence (4.12) gives

$$\sum_{n=0}^M \frac{1}{(n+1)^k} \int_{\mathbb{R}^2} \text{Tr} [|\varphi - \varphi(\mathbf{z})|^2 \Pi_{z,n}] \, d\mathbf{z} \leq 2l_b^2 \|\nabla\varphi\|_{L^2}^2 \sum_{n=0}^M (n+1)^{1-k} = Cl_b^2 \|\nabla\varphi\|_{L^2}^2 p_{k-2}(M)$$

We next apply (4.6) to $A_n(\mathbf{z}) := \varphi - \varphi(\mathbf{z})$ and obtain

$$\begin{aligned} & \left| \frac{1}{2\pi l_b^2} \int_{\mathbb{R}^2} \text{Tr} [(\varphi - \varphi(\mathbf{z})) \gamma_b \Pi_{z, \leq M}] \, d\mathbf{z} \right| \\ & \leq \frac{1}{2} \sqrt{\text{Tr} [\gamma_b \mathcal{L}_b^k]} \left(\sum_{n=0}^M \frac{1}{(n+1)^k} \int_{\mathbb{R}^2} \text{Tr} [|\varphi - \varphi(\mathbf{z})|^2 \Pi_{z,n}] \, d\mathbf{z} \right)^{\frac{1}{2}} \\ & \leq C \|\nabla\varphi\|_{L^2} \sqrt{\text{Tr} [\gamma_b \mathcal{L}_b^k]} l_b \sqrt{p_{k-2}(M)} \end{aligned} \quad (4.14)$$

Step 2, high Landau levels. We remark that

$$\left| \text{Tr} [\varphi \Pi_{>M} \gamma_b] \right| \leq \sqrt{\text{Tr} [\gamma_b |\varphi|^2]} \sqrt{\text{Tr} [\gamma_b \Pi_{>M}]} \leq \|\varphi\|_{L^\infty} \sqrt{\text{Tr} [\gamma_b \Pi_{>M}]}$$

and

$$\text{Tr} [\gamma_b \Pi_{>M}] \leq \sum_{n>M} \frac{n^k}{M^k} \text{Tr} [\gamma_b \Pi_n] \leq \frac{1}{M^k} \sum_{n>M} \text{Tr} [\gamma_b \Pi_n \mathcal{L}_b^k] = \frac{1}{M^k} \text{Tr} [\gamma_b \Pi_{>M} \mathcal{L}_b^k]. \quad (4.15)$$

Hence

$$\left| \text{Tr} [\varphi \Pi_{>M} \gamma_b] \right| \leq \|\varphi\|_{L^\infty} M^{-\frac{k}{2}} \sqrt{\text{Tr} [\gamma_b \Pi_{>M} \mathcal{L}_b^k]} \quad (4.16)$$

We conclude by combining (4.14), (4.16) and (4.11). \square

4.2. Improved convergence with confinement. To complete the proof of Theorem 2.4 we need to improve (4.4) to an estimate in Wasserstein-1 distance. We may do this at the price of an extra confining assumption:

Proposition 4.4 (Convergence of the truncated semi-classical density with confinement).

Let $\beta > 0, p > 3$. We make the same assumptions as in Proposition 4.2, with in addition

$$\text{Tr} [\gamma_b |X|^p] < \infty \quad (4.17)$$

If we denote

$$\rho_b := \frac{\rho_{\gamma_b}^{sc, \leq M}}{\text{Tr} [\gamma_b \Pi_{\leq M}]} \quad (4.18)$$

and assume $4\sqrt{M}l_b \leq l_b^{-\beta}$,

$$W_1(\rho_{\gamma_b}, \rho_b) \leq C(p) (1 + \text{Tr} [\gamma_b |X|^p]) (\text{Tr} [\gamma_b \Pi_{>M} \mathcal{L}_b]) M^{-k} \quad (4.19)$$

$$+\sqrt{\operatorname{Tr} [\gamma_b \mathcal{L}_b^k]} l_b^{1-\beta} \left\{ \begin{array}{ll} M^{1-\frac{k}{2}} & \text{if } k < 2 \\ \sqrt{\ln(M)} & \text{if } k = 2 \\ 1 & \text{if } k > 2 \end{array} \right. + M^{-\frac{k}{2}} \sqrt{\operatorname{Tr} [\gamma_b \Pi_{>M} \mathcal{L}_b^k]} + l_b^{\beta(p-1)} + M l_b^{\beta(p-3)-2} \quad (4.20)$$

We need a technical lemma containing some basic estimates:

Lemma 4.5 (Technical integration results).

Recalling the definition (3.10),

$$\|\Psi_{z,n}\|_{L^1} \leq C(n+1)^{\frac{1}{4}} l_b. \quad (4.21)$$

Let $n \in \mathbb{N}$, $a > 0$ and

$$I_n(a) := \int_a^\infty t^n e^{-\frac{t^2}{2}} dt$$

then

$$I_{2n+1}(a) = 2^n n! e^{-\frac{a^2}{2}} \sum_{i=0}^n \frac{1}{i!} \left(\frac{a^2}{2}\right)^i.$$

Proof. Using Stirling's formula

$$\begin{aligned} & \|\Psi_{z,n}\|_{L^1} \\ &= \frac{1}{\sqrt{2\pi n!} l_b} \int_{\mathbb{R}^2} \left| \frac{x-z}{\sqrt{2} l_b} \right|^n e^{-\frac{|x-z|^2}{4l_b^2}} d\mathbf{x} = \sqrt{\frac{2}{\pi n!}} l_b \int_{\mathbb{R}^2} |x|^n e^{-\frac{|x|^2}{2}} d\mathbf{x} = \sqrt{\frac{2\pi}{n!}} l_b \int_{\mathbb{R}_+} 2t^{n+1} e^{-\frac{t^2}{2}} dt \\ &= \sqrt{\frac{2\pi}{n!}} l_b \int_{\mathbb{R}_+} t^{\frac{n}{2}} e^{-\frac{t}{2}} dt = \sqrt{\frac{2\pi}{n!}} l_b 2^{\frac{n}{2}+1} \int_{\mathbb{R}_+} t^{\frac{n}{2}} e^{-t} dt = 2^{\frac{n+3}{2}} \sqrt{\pi} l_b \frac{\Gamma\left(\frac{n}{2}+1\right)}{\sqrt{n!}} \\ &\underset{n \rightarrow \infty}{\sim} 2^{\frac{n+3}{2}} \sqrt{\pi} l_b \frac{\sqrt{2\pi}^{\frac{n}{2}} \left(\frac{n}{2e}\right)^{\frac{n}{2}}}{(2\pi n)^{\frac{1}{4}} \left(\frac{n}{e}\right)^{\frac{n}{2}}} = 2^{\frac{5}{4}} \pi^{\frac{3}{4}} n^{\frac{1}{4}} l_b \underset{n \rightarrow \infty}{\sim} 2^{\frac{5}{4}} \pi^{\frac{3}{4}} (n+1)^{\frac{1}{4}} l_b. \end{aligned}$$

As for the second claim in the lemma, an integration by parts shows

$$I_{n+2}(a) = (n+1) I_n(a) + a^{n+1} e^{-\frac{a^2}{2}}$$

so for odd integers

$$I_{2(n+1)+1}(a) = I_{2n+1+2}(a) = 2(n+1) I_{2n+1}(a) + a^{2(n+1)} e^{-\frac{a^2}{2}}$$

hence by induction,

$$I_{2n+1}(a) = 2^n n! \left(I_1(a) + e^{-\frac{a^2}{2}} \sum_{i=1}^n \frac{1}{i!} \left(\frac{a^2}{2}\right)^i \right)$$

and

$$I_1(a) = e^{-\frac{a^2}{2}}$$

corresponds to the index $i = 0$ in the sum. \square

We now turn to the

Proof of Proposition 4.4. Using the dual representation of the Wasserstein-1 metric, we aim at estimating

$$\left| \int_{\mathbb{R}^2} \varphi \left(\rho_{\gamma_b} - \rho_b \right) \right|$$

for a Lipschitz function φ . Since ρ_{γ_b} and ρ_b have the same integral we assume without loss of generality that $\varphi(0) = 0$, and hence

$$|\varphi(\mathbf{x})| \leq \|\nabla\varphi\|_{L^\infty} |\mathbf{x}|. \quad (4.22)$$

Let

$$R := l_b^{-\beta}$$

and define a partition of unity, $\chi_R, \eta_R \in C^\infty(\mathbb{R}^2, \mathbb{R}_+)$ such that

$$\begin{aligned} \chi_R + \eta_R &= 1 \\ |z| \leq R &\implies \chi_R = 1, \eta_R = 0 \\ |z| \geq 2R &\implies \chi_R = 0, \eta_R = 1 \\ \|\nabla\eta_R\|_{L^\infty} &\leq \frac{2}{R} \end{aligned} \quad (4.23)$$

Using (4.11) for $\varphi\chi_R$ instead of φ , we decompose

$$\begin{aligned} \int_{\mathbb{R}^2} \varphi \left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc, \leq M} \right) &= \frac{1}{2\pi l_b^2} \int_{\mathbb{R}^2} \text{Tr} \left[(\varphi\chi_R - \varphi(\mathbf{z})\chi_R(\mathbf{z})) \Pi_{z, \leq M} \gamma_b \right] d\mathbf{z} + \text{Tr} \left[\varphi\chi_R \Pi_{>M} \gamma_b \right] \\ &\quad + \int_{\mathbb{R}^2} \varphi\eta_R \left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc, \leq M} \right) \end{aligned} \quad (4.24)$$

Step 1, low Landau levels. With the bounds (4.22) and (4.23) we find that, $\forall |z| \leq 2R$,

$$\left| \nabla \left(\varphi\chi_R \right) (\mathbf{z}) \right| \leq |\nabla\varphi(\mathbf{z})| + \frac{2}{R} |\varphi(\mathbf{z})| \leq |\nabla\varphi(\mathbf{z})| + \frac{2}{R} |\mathbf{z}| \|\nabla\varphi\|_{L^\infty} \leq 5 \|\nabla\varphi\|_{L^\infty}$$

Hence

$$\left\| \nabla \left(\varphi\chi_R \right) (\mathbf{z}) \right\|_{L^2} \leq 5 \|\nabla\varphi\|_{L^\infty} \sqrt{|B(0, 2R)|} \leq C \|\nabla\varphi\|_{L^\infty} R$$

Using (4.14) we deduce

$$\left| \frac{1}{2\pi l_b^2} \int_{\mathbb{R}^2} \text{Tr} \left[(\varphi\chi_R - \varphi(\mathbf{z})\chi_R(\mathbf{z})) \gamma_b \Pi_{z, \leq M} \right] d\mathbf{z} \right| \leq C R l_b \|\nabla\varphi\|_{L^\infty} \sqrt{\text{Tr} [\gamma_b \mathcal{L}_b^k]} \sqrt{p_{k-2}(M)} \quad (4.25)$$

Step 2, higher Landau levels. Since

$$|\varphi(\mathbf{z})\chi_R(\mathbf{z})| \leq \|\nabla\varphi\|_{L^\infty} |\mathbf{z}|$$

it follows from (4.22) that

$$\left| \text{Tr} \left[\varphi\chi_R \Pi_{>M} \gamma_b \right] \right| \leq \sqrt{\text{Tr} [\gamma_b |\varphi\chi_R|^2]} \sqrt{\text{Tr} [\gamma_b \Pi_{>M}]} \leq \|\nabla\varphi\|_{L^\infty} \sqrt{\text{Tr} [\gamma_b |X|^2]} \sqrt{\text{Tr} [\gamma_b \Pi_{>M}]}.$$

Inserting (4.15), we deduce

$$\left| \text{Tr} \left[\varphi\chi_R \Pi_{>M} \gamma_b \right] \right| \leq \|\nabla\varphi\|_{L^\infty} \sqrt{\text{Tr} [\gamma_b |X|^2]} M^{-\frac{k}{2}} \sqrt{\text{Tr} [\gamma_b \Pi_{>M} \mathcal{L}_b^k]} \quad (4.26)$$

Step 3, tails of the densities. We next estimate the third term in (4.24). First, using (4.22),

$$\begin{aligned} & \left| \int_{\mathbb{R}^2} \varphi \eta_R \rho_{\gamma_b} \right| \\ & \leq \|\nabla \varphi\|_{L^\infty} \int_{|z| \geq R} |z| \rho_{\gamma_b}(\mathbf{z}) d\mathbf{z} \leq \frac{\|\nabla \varphi\|_{L^\infty}}{R^{p-1}} \int_{|z| \geq R} |z|^p \rho_{\gamma_b}(\mathbf{z}) d\mathbf{z} \leq \frac{\|\nabla \varphi\|_{L^\infty}}{R^{p-1}} \text{Tr} [\gamma_b |X|^p]. \end{aligned} \quad (4.27)$$

On the other hand, using (4.22) again,

$$\begin{aligned} & \left| \int_{\mathbb{R}^2} \varphi \eta_R \rho_{\gamma_b}^{sc, \leq M} \right| \leq \|\nabla \varphi\|_{L^\infty} \int_{|z| \geq R} |z| \rho_{\gamma_b}^{sc, \leq M}(\mathbf{z}) d\mathbf{z} \\ & = \frac{\|\nabla \varphi\|_{L^\infty}}{2\pi l_b^2} \sum_{n=0}^M \int_{|z| \geq R} |z| \text{Tr} [\gamma_b \Pi_{z,n}] d\mathbf{z} \end{aligned} \quad (4.28)$$

We write the spectral decomposition of γ_b in the manner

$$\gamma_b = \sum_{i \in \mathbb{N}} \lambda_i |u_i\rangle \langle u_i|$$

with $0 \leq \lambda_i \leq (2\pi)^{-1} l_b^{-2}$ and $(u_i)_i$ an orthonormal basis of L^2 . Let $|z| \geq R, n \leq M$, using the Cauchy-Schwarz inequality, we estimate

$$\begin{aligned} \text{Tr} [\gamma_b \Pi_{z,n}] & = \sum_{i \in \mathbb{N}} \lambda_i \left| \langle \psi_{z,n} | u_i \rangle \right|^2 \leq \sum_{i \in \mathbb{N}} \lambda_i \left(\int_{\mathbb{R}^2} |u_i| \sqrt{|\psi_{z,n}|} \cdot \sqrt{|\psi_{z,n}|} \right)^2 \\ & \leq \|\psi_{z,n}\|_{L^1} \sum_{i \in \mathbb{N}} \lambda_i \int_{\mathbb{R}^2} |u_i|^2 |\psi_{z,n}| = \|\psi_{z,n}\|_{L^1} \int_{\mathbb{R}^2} \rho_{\gamma_b} |\psi_{z,n}| \end{aligned} \quad (4.29)$$

Hence

$$\begin{aligned} \int_{\mathbb{R}^2} \rho_{\gamma_b} |\psi_{z,n}| & = \int_{\mathbb{R}^2} \rho_{\gamma_b}(\mathbf{x}) (1 + |x|^p) \frac{|\psi_{z,n}(\mathbf{x})|}{1 + |x|^p} d\mathbf{x} \leq \text{Tr} [\gamma_b (1 + |X|^p)] \left\| \frac{\psi_{z,n}}{1 + |\bullet|^p} \right\|_{L^\infty} \\ & = \frac{1 + \text{Tr} [\gamma_b |X|^p]}{\sqrt{2\pi n} l_b} \sup_{\mathbf{x} \in \mathbb{R}^2} \frac{|x|^n e^{-\frac{|x|^2}{2}}}{1 + |z + \sqrt{2} l_b x|^p} \end{aligned} \quad (4.30)$$

The function $t \mapsto t^n e^{-\frac{t^2}{2}}$ attains its' global maximal on \mathbb{R}^+ at $t = \sqrt{n}$ with maximal value $\left(\frac{n}{e}\right)^{\frac{n}{2}}$ and is decreasing for $t > \sqrt{n}$.

Since we assume $4\sqrt{M} l_b \leq R$, if $4l_b |x| \geq |z|$, we have that

$$|x| \geq \frac{|z|}{4l_b} \geq \frac{R}{4l_b} \geq \sqrt{M} \geq \sqrt{n}$$

and hence

$$\frac{|x|^n e^{-\frac{|x|^2}{2}}}{1 + |z + \sqrt{2} l_b x|^p} \leq \left(\frac{|z|}{4l_b}\right)^n e^{-\frac{1}{2} \left(\frac{|z|}{4l_b}\right)^2} \quad (4.31)$$

If instead $4l_b |x| \leq |z|$, we have

$$|z + \sqrt{2} l_b x| \geq |z| - \sqrt{2} l_b |x| \geq |z| \left(1 - \frac{\sqrt{2}}{4}\right) \geq \frac{|z|}{2}$$

and thus

$$\frac{|x|^n e^{-\frac{|x|^2}{2}}}{1 + \left|z + \sqrt{2}l_b x\right|^p} \leq \frac{\left(\frac{n}{e}\right)^{\frac{n}{2}}}{1 + \left|\frac{z}{2}\right|^p} \quad (4.32)$$

Putting (4.31) (4.32) and (4.30) together leads to

$$\int_{\mathbb{R}^2} \rho_{\gamma_b} |\psi_{z,n}| \leq \frac{(1 + \text{Tr} [\gamma_b |X|^p])}{\sqrt{2\pi n} l_b} \left(\left(\frac{|z|}{4l_b}\right)^n e^{-\frac{1}{2}\left(\frac{|z|}{4l_b}\right)^2} + \frac{\left(\frac{n}{e}\right)^{\frac{n}{2}}}{1 + \left|\frac{z}{2}\right|^p} \right)$$

Inserting this and (4.21) in (4.29), combining with Stirling's formula again gives

$$\begin{aligned} \text{Tr} [\gamma_b \Pi_{z,n}] &\leq C (1 + \text{Tr} [\gamma_b |X|^p]) \frac{(n+1)^{\frac{1}{4}}}{\sqrt{n!}} \left(\left(\frac{|z|}{4l_b}\right)^n e^{-\frac{1}{2}\left(\frac{|z|}{4l_b}\right)^2} + \frac{\left(\frac{n}{e}\right)^{\frac{n}{2}}}{1 + \left|\frac{z}{2}\right|^p} \right) \\ &\leq C (1 + \text{Tr} [\gamma_b |X|^p]) \left(\frac{(n+1)^{\frac{1}{4}}}{\sqrt{n!}} \left(\frac{|z|}{4l_b}\right)^n e^{-\frac{1}{2}\left(\frac{|z|}{4l_b}\right)^2} + \frac{1}{1 + \left|\frac{z}{2}\right|^p} \right) \end{aligned}$$

With the notation of Lemma 4.5,

$$\int_{|z| \geq R} |z| \left(\frac{|z|}{4l_b}\right)^n e^{-\frac{1}{2}\left(\frac{|z|}{4l_b}\right)^2} \mathbf{d}z = 128\pi l_b^3 \int_{\frac{R}{4l_b}}^{\infty} r^{n+2} e^{-\frac{1}{2}r^2} = 128\pi l_b^3 I_{n+2} \left(\frac{R}{4l_b}\right)$$

Since $p > 3$, we can integrate

$$\int_{|z| \geq R} \frac{|z|}{1 + \left|\frac{z}{2}\right|^p} \mathbf{d}z \leq 2^p \int_{|z| \geq R} |z|^{1-p} \mathbf{d}z = 2^{p+1} \pi \int_R^{\infty} r^{2-p} dr = \frac{2^{p+1} \pi}{(p-3) R^{p-3}}$$

Collecting the above considerations gives

$$\int_{|z| \geq R} |z| \text{Tr} [\gamma_b \Pi_{z,n}] \mathbf{d}z \leq C(p) (1 + \text{Tr} [\gamma_b |X|^p]) \left(\frac{(n+1)^{\frac{1}{4}}}{\sqrt{n!}} l_b^3 I_{n+2} \left(\frac{R}{4l_b}\right) + \frac{1}{R^{p-3}} \right)$$

that we may combine with (4.28) to get

$$\left| \int_{\mathbb{R}^2} \varphi \eta_R \rho_{\gamma_b}^{sc, \leq M} \right| \leq \frac{\|\nabla \varphi\|_{L^\infty}}{2\pi l_b^2} \sum_{n=0}^M \int_{|z| \geq R} |z| \text{Tr} [\gamma_b \Pi_{z,n}] \mathbf{d}z \quad (4.33)$$

$$\leq C(p) \|\nabla \varphi\|_{L^\infty} (1 + \text{Tr} [\gamma_b |X|^p]) \left(l_b \sum_{n=0}^M \frac{(n+1)^{\frac{1}{4}}}{\sqrt{n!}} I_{n+2} \left(\frac{R}{4l_b}\right) + \frac{M}{l_b^2 R^{p-3}} \right) \quad (4.34)$$

But $R \geq 4\sqrt{M}l_b \gg 4l_b$ so $\left(I_n \left(\frac{R}{4l_b}\right)\right)_{n \in \mathbb{N}}$ is increasing, hence it follows from Lemma 4.5 that

$$\sum_{n=0}^M \frac{(n+1)^{\frac{1}{4}}}{\sqrt{n!}} I_{n+2} \left(\frac{R}{4l_b}\right) \leq \sum_{n=0}^{2\lfloor \frac{M}{2} \rfloor + 1} \frac{(n+1)^{\frac{1}{4}}}{\sqrt{n!}} I_{n+2} \left(\frac{R}{4l_b}\right)$$

$$\begin{aligned}
&= \sum_{n=0}^{\lfloor \frac{M}{2} \rfloor} \frac{(2n+1)^{\frac{1}{4}}}{\sqrt{(2n)!}} I_{2n+2} \left(\frac{R}{4l_b} \right) + \sum_{n=0}^{\lfloor \frac{M}{2} \rfloor} \frac{(2n+2)^{\frac{1}{4}}}{\sqrt{(2n+1)!}} I_{2n+3} \left(\frac{R}{4l_b} \right) \\
&\leq 2 \sum_{n=0}^{\lfloor \frac{M}{2} \rfloor} \frac{(2n+2)^{\frac{1}{4}}}{\sqrt{(2n)!}} I_{2n+3} \left(\frac{R}{4l_b} \right) \\
&= 2e^{-\frac{R^2}{32l_b^2}} \sum_{n=0}^{\lfloor \frac{M}{2} \rfloor} \frac{(2n+2)^{\frac{1}{4}}}{\sqrt{(2n)!}} 2^{n+1} (n+1)! \sum_{i=0}^{n+1} \frac{1}{i!} \left(\frac{R^2}{32l_b^2} \right)^i
\end{aligned}$$

Then using $(2n)! \geq (n!)^2$, $i! \geq 1$ and $\frac{R^2}{32l_b^2} \geq \frac{M}{2} \geq 1$,

$$\begin{aligned}
&\sum_{n=0}^M \frac{(n+1)^{\frac{1}{4}}}{\sqrt{n!}} I_{n+2} \left(\frac{R}{4l_b} \right) \\
&\leq 2e^{-\frac{R^2}{32l_b^2}} \sum_{n=0}^{\lfloor \frac{M}{2} \rfloor} (2n+2)^{\frac{1}{4}} (n+1)(n+2) 2^{n+1} \left(\frac{R^2}{32l_b^2} \right)^{n+1} \\
&\leq 2e^{-\frac{R^2}{32l_b^2}} \left(\lfloor \frac{M}{2} \rfloor + 1 \right) \left(2 \lfloor \frac{M}{2} \rfloor + 2 \right)^{\frac{1}{4}} \left(\lfloor \frac{M}{2} \rfloor + 1 \right) \left(\lfloor \frac{M}{2} \rfloor + 2 \right) \left(\frac{R^2}{16l_b^2} \right)^{\lfloor \frac{M}{2} \rfloor + 1} \\
&\leq CM^{\frac{13}{4}} \left(\frac{R^2}{16l_b^2} \right)^{\frac{M}{2} + 1} e^{-\frac{R^2}{32l_b^2}} = CM^{\frac{13}{4}} e^{(M+2) \ln \left(\frac{l_b^{-(1+\beta)}}{16} \right) - \frac{l_b^{-2(1+\beta)}}{32}} \leq CM^{\frac{13}{4}} e^{(1+\beta)(M+2) \ln(l_b^{-1}) - \frac{l_b^{-2(1+\beta)}}{32}}
\end{aligned}$$

Since $M \ll l_b^{-2}$ for large enough b ,

$$\sum_{n=0}^M \frac{(n+1)^{\frac{1}{4}}}{\sqrt{n!}} I_{n+2} \left(\frac{R}{4l_b} \right) \leq Cl_b^{-\frac{13}{2}} e^{l_b^{-2} \ln(l_b^{-1}) - \frac{l_b^{-2(1+\beta)}}{32}} = Cl_b^{-\frac{13}{2}} e^{-l_b^{-2(1+\beta)} \left(\frac{1}{32} - l_b^{-2\beta} \ln(l_b^{-1}) \right)}.$$

Inserting this in (4.33), we get

$$\left| \int_{\mathbb{R}^2} \varphi \eta_R \rho_{\gamma_b}^{sc, \leq M} \right| \leq C(p) \|\nabla \varphi\|_{L^\infty} \left(1 + \text{Tr} [\gamma_b |X|^p] \right) \left(l_b^{-\frac{11}{2}} e^{-l_b^{-2(1+\beta)} \left(\frac{1}{32} - l_b^{-2\beta} \ln(l_b^{-1}) \right)} + M l_b^{\beta(p-3)-2} \right) \quad (4.35)$$

Step 4, normalization error. We write, using complex notation,

$$\begin{aligned}
\int_{\mathbb{R}^2} \varphi \left(\rho_b - \rho_{\gamma_b}^{sc, \leq M} \right) &= \int_{\mathbb{R}^2} \varphi \rho_b (1 - \text{Tr} [\gamma_b \Pi_{\leq M}]) = \text{Tr} [\gamma_b \Pi_{>M}] \int_{\mathbb{R}^2} \int_0^1 \nabla \varphi(\tau \mathbf{z}) \cdot \mathbf{z} \rho_b(\mathbf{z}) d\tau d\mathbf{z} \\
&= \frac{\text{Tr} [\gamma_b \Pi_{>M}]}{\text{Tr} [\gamma_b \Pi_{\leq M}]} \int_0^1 \int_{\mathbb{R}^2} \nabla \varphi(\tau \mathbf{z}) \cdot \mathbf{z} \text{Tr} [\gamma_b \Pi_{z, \leq M}] \frac{d\mathbf{z}}{2\pi l_b^2} d\tau \\
&= \frac{\text{Tr} [\gamma_b \Pi_{>M}]}{\text{Tr} [\gamma_b \Pi_{\leq M}]} \int_0^1 \text{Re} \int_{\mathbb{R}^2} 2\partial_z \varphi(\tau \mathbf{z}) \mathbf{z} \text{Tr} [\gamma_b \Pi_{z, \leq M}] \frac{d\mathbf{z}}{2\pi l_b^2} d\tau
\end{aligned}$$

$$= \frac{\text{Tr} [\gamma_b \Pi_{>M}]}{\text{Tr} [\gamma_b \Pi_{\leq M}]} \int_0^1 \text{Re} \int_{\mathbb{R}^2} 2\partial_{\bar{z}}\varphi(\tau z) \text{Tr} \left[\gamma_b \Pi_{z,\leq M} \sqrt{2} l_b c^\dagger \right] \frac{dz}{2\pi l_b^2} d\tau$$

where, in the last step, we used (3.9).

But (3.3) implies that

$$\sqrt{2} l_b c^\dagger = X - i\sqrt{2} l_b a$$

and thus using the Cauchy-Schwarz inequality,

$$\begin{aligned} \left| \text{Tr} \left[\gamma_b \Pi_{z,\leq M} \sqrt{2} l_b c^\dagger \right] \right| &\leq \text{Tr} [\gamma_b \Pi_{z,\leq M}] + \frac{1}{2} \text{Tr} \left[\gamma_b \bar{X} \Pi_{z,\leq M} X \right] + l_b^2 \text{Tr} [\gamma_b a^\dagger \Pi_{z,\leq M} a] \\ &= (1 + (M+1)l_b^2) \text{Tr} [\gamma_b \Pi_{z,\leq M+1}] + \frac{1}{2} \text{Tr} \left[\gamma_b \bar{X} \Pi_{z,\leq M} X \right] \end{aligned}$$

Inserting in the above and recalling that $Ml_b^2 \leq 1$, we deduce

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \varphi \left(\rho_b - \rho_{\gamma_b}^{sc,\leq M} \right) \right| &\leq C \|\nabla\varphi\|_{L^\infty} \frac{\text{Tr} [\gamma_b \Pi_{>M}]}{\text{Tr} [\gamma_b \Pi_{\leq M}]} \left(\text{Tr} [\gamma_b \Pi_{\leq M+1}] + \text{Tr} \left[\gamma_b \bar{X} \Pi_{\leq M} X \right] \right) \\ &\leq C (1 + \text{Tr} [\gamma_b |X|^2]) \|\nabla\varphi\|_{L^\infty} \text{Tr} [\gamma_b \Pi_{>M}] \\ &\leq C (1 + \text{Tr} [\gamma_b |X|^2]) \|\nabla\varphi\|_{L^\infty} \text{Tr} [\gamma_b \Pi_{>M} \mathcal{L}_b^k] M^{-k} \end{aligned} \quad (4.36)$$

Step 5, conclusion. We insert (4.25) (4.26) (4.27) (4.35) in (4.24):

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \varphi \left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leq M} \right) \right| &\leq C(p) \|\nabla\varphi\|_{L^\infty} \left(\sqrt{\text{Tr} [\gamma_b \mathcal{L}_b^k]} l_b^{1-\beta} \sqrt{p_{k-2}(M)} \right. \\ &+ \sqrt{\text{Tr} [\gamma_b |X|^2]} M^{-\frac{k}{2}} \sqrt{\text{Tr} [\gamma_b \Pi_{>M} \mathcal{L}_b^k]} + \text{Tr} [\gamma_b |X|^p] l_b^{\beta(p-1)} \\ &\left. + (1 + \text{Tr} [\gamma_b |X|^p]) \left(l_b^{-\frac{11}{2}} e^{-l_b^{-2(1+\beta)}} \left(\frac{1}{32} l_b^{2\beta} \ln(l_b^{-1}) \right) + M l_b^{\beta(p-3)-2} \right) \right) \end{aligned}$$

We further notice that, since $p > 3$,

$$\sqrt{\text{Tr} [\gamma_b |X|^2]} \leq \sqrt{1 + \text{Tr} [\gamma_b |X|^p]} \leq 1 + \text{Tr} [\gamma_b |X|^p].$$

Hence, using the triangular inequality along with (4.36) leads to

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \varphi \left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leq M} \right) \right| &\leq C(p) \|\nabla\varphi\|_{L^\infty} (1 + \text{Tr} [\gamma_b |X|^p]) \left(\text{Tr} [\gamma_b \Pi_{>M} \mathcal{L}_b] M^{-k} \right. \\ &\left. + \sqrt{\text{Tr} [\gamma_b \mathcal{L}_b^k]} l_b^{1-\beta} \begin{cases} M^{1-\frac{k}{2}} & \text{if } k < 2 \\ \sqrt{\ln(M)} & \text{if } k = 2 \\ 1 & \text{if } k > 2 \end{cases} + M^{-\frac{k}{2}} \sqrt{\text{Tr} [\gamma_b \Pi_{>M} \mathcal{L}_b^k]} + l_b^{\beta(p-1)} + M l_b^{\beta(p-3)-2} \right) \end{aligned}$$

and we obtain (4.19) using Kantorovitch-Rubinstein duality (2.11). \square

5. SEMI-CLASSICAL DYNAMICS OF THE DENSITIES

This section contains the proof of Theorem 2.3. We first provide the second main ingredient mentioned after the statement, namely the study of the dynamics of the truncated semi-classical density. We next combine this with the bounds of Section 4 to conclude the proof.

5.1. Dynamics of the truncated semi-classical density. We now prove that $\rho_{\gamma_b}^{sc, \leq M}$, as defined in Section 4, almost satisfies the weak formulation of (2.5) modulo a suitable choice of $1 \ll M \ll l_b^{-2}$. Here W is a (possibly time-dependent) generic potential that we will replace with $V + w \star \rho_{\gamma_b(t)}$ later on.

Proposition 5.1 (Drift equation for the truncated semi-classical density).

Let $t \in \mathbb{R}_+$, $k \geq 0$, $l \geq 1$, $\gamma_b(t) \in \mathcal{L}^1(L^2(\mathbb{R}^2))$, $W \in W^{l+1, \infty}(\mathbb{R}^2)$ and assume

$$\begin{aligned} \operatorname{Tr} [\gamma_b(t)] &= 1, 0 \leq \gamma_b(t) \leq 2\pi l_b^2 \\ \operatorname{Tr} [\gamma_b(t) \mathcal{L}_b^k] &< \infty \\ \partial_t \gamma_b(t) &= \frac{1}{il_b^2} [\mathcal{L}_b + W, \gamma_b(t)]. \end{aligned}$$

Then, $\forall \varphi \in L^1 \cap W^{1, \infty}(\mathbb{R}^2)$,

$$\begin{aligned} & \left| \int_{\mathbb{R}^2} \varphi(\mathbf{z}) \left(\partial_t \rho_{\gamma_b}^{sc, \leq M}(t, \mathbf{z}) + \nabla^\perp W(\mathbf{z}) \cdot \nabla \rho_{\gamma_b}^{sc, \leq M}(t, \mathbf{z}) \right) d\mathbf{z} \right| \\ & \leq C(l) \|\varphi\|_{L^1 \cap W^{1, \infty}} \|W\|_{W^{l+1, \infty}} \left(\sqrt{\operatorname{Tr} [\gamma_b \mathcal{L}_b^k]} l_b^{l-1} \begin{cases} M^{1+\frac{l-k}{2}} & \text{if } k < l+2 \\ \sqrt{\ln(M)} & \text{if } k = l+2 \\ 1 & \text{if } k > l+2 \end{cases} \right. \\ & \quad \left. + \frac{\operatorname{Tr} [\gamma_b \mathcal{L}_b^k \Pi_{M-l+1: M+l}]}{l_b M^{k-\frac{1}{2}}} + \operatorname{Tr} [\gamma_b \mathcal{L}_b^k] \sum_{p=2}^l l_b^{p-1} \begin{cases} M^{\frac{p+1}{2}-k} & \text{if } k < \frac{p+1}{2} \\ \ln(M) & \text{if } k = \frac{p+1}{2} \\ 1 & \text{if } k > \frac{p+1}{2} \end{cases} \right) \end{aligned}$$

The proof will proceed from a Taylor expansion of the potential W at order l . Several bounds will be proved by induction on l and are not particularly more complicated to write in the above generality. However our choice of l will be set by the a priori bound available to us, as we explain first:

Remark 5.2 (Choice of the expansion parameters).

In the proof of Theorem 2.3 we take $k = 1$ to be able to use Lemma 3.9. Then Proposition 5.1 gives

$$\begin{aligned} & \left| \int_{\mathbb{R}^2} \varphi \left(\partial_t \rho_{\gamma_b}^{sc, \leq M}(t) + \nabla^\perp W \cdot \nabla \rho_{\gamma_b}^{sc, \leq M}(t) \right) \right| \leq C(l) \|\varphi\|_{L^1 \cap W^{1, \infty}} \|W\|_{W^{l+1, \infty}} \\ & \left(\sqrt{\operatorname{Tr} [\gamma_b(t) \mathcal{L}_b]} l_b^{l-1} M^{\frac{l+1}{2}} + \frac{\operatorname{Tr} [\gamma_b(t) \mathcal{L}_b \Pi_{M-l+1: M+l}]}{l_b \sqrt{M}} + \operatorname{Tr} [\gamma_b(t) \mathcal{L}_b] l_b \sqrt{M} \right) \quad (5.1) \end{aligned}$$

The first error above is of order $l_b^{l-1} M^{\frac{l+1}{2}}$, the third is of order $l_b \sqrt{M}$. In Lemma 5.7 we will see that $M \gg 1$ may be chosen so that the second error weighs $l_b^{-1} M^{-\frac{3}{2}}$. We are able to control all the errors if

$$\left(\frac{1}{l_b} \right)^{\frac{2}{3}} \ll M \ll \left(\frac{1}{l_b} \right)^{2\frac{l-1}{l+1}}$$

which is possible if and only if $l > 2$. Hence the above will be applied to obtain Theorem 2.3 with the choices $k = 1, l = 3$. This will yield error terms

$$l_b^2 M^2 + l_b^{-1} M^{-\frac{3}{2}} + l_b M^{1/2}.$$

Since the second term must be $o(1)$, the first one dominates the third, and optimization in M leads to a choice $M \sim l_b^{-6/7}$ and a final error of order $O(l_b^{2/7})$.

If more moments of the kinetic energy are bounded for positive times, we may use the above with a larger value of k and hence get an efficient estimate for lower l , i.e. asking for less regularity of the external and interaction potentials. \diamond

The first step in our proof consists in deriving an explicit equation satisfied by $\rho_{\gamma_b}^{sc, \leq M}$. Most bounds are then obtained by writing quantum expectations (traces) using integrals of operator kernels. A general term will be an integral in $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$ where \mathbf{x}, \mathbf{y} are the arguments of the operator kernel and \mathbf{z} the guiding-center coordinate of the coherent state at hand. As per the observations of Section 3.2 such integrals are concentrated around $\mathbf{x} \sim \mathbf{y} \sim \mathbf{z}$. Principal terms corresponding to Equation (2.5) are easily identified by replacing $W(\mathbf{x}) - W(\mathbf{y})$ (coming from the commutator with W) by $(\mathbf{x} - \mathbf{y}) \cdot \nabla W(\mathbf{z})$ and then using (3.17). Some care in controlling the expansion of W is needed:

- Using (3.12) (or more precisely, the first term in (3.16)) a factor of $(\mathbf{x} - \mathbf{y})$ can make us gain² a factor of l_b^2 , at the price of integrations by parts (that we can perform using the regularity of the test function).
- A factor of $|x - z|$ or $|y - z|$ makes us gain at best a factor of $l_b \sqrt{n}$, the localization length of the coherent state wave-function in the n -th Landau level. Cf the discussion below Proposition 4.2.
- There is a truncation error due to the second term in (3.12), calling for some optimization in M .

In (5.5) below the remainder is in a form allowing to leverage the first observation above. Indeed, the commutator naturally brings factors of $(\mathbf{x} - \mathbf{y})$ (think of the commutator with the position operator \mathbf{X}). The remainders in (5.6) are estimated using the second observation, which we formalize in Lemma 5.4 below. Finally (5.7) is the truncation error, whose control will demand a proper choice of M later in the proof, see Lemma 5.7.

Lemma 5.3 (Equation for the semi-classical density and potential expansion).

Let $t \in \mathbb{R}_+, l \in \mathbb{N}, \gamma_b(t) \in \mathcal{L}^1(L^2(\mathbb{R}^2)), W \in W^{l, \infty}(\mathbb{R}^2)$ and assume

$$\partial_t \gamma_b(t) = \frac{1}{i l_b^2} [\mathcal{L}_b + W, \gamma_b(t)]. \quad (5.2)$$

Denote $d^p W_{\mathbf{z}}$ the p^{th} differential of W at $\mathbf{z} \in \mathbb{R}^2$, meaning that $d^p W_{\mathbf{z}}$ is a p -linear form on \mathbb{R}^2 .

Then, with $\rho_{\gamma_b}^{sc, \leq M}$ as in Definition 4.1,

$$\partial_t \rho_{\gamma_b}^{sc, \leq M} + \nabla^\perp W \cdot \nabla \rho_{\gamma_b}^{sc, \leq M} = \mathcal{E}_0^l + \sum_{p=2}^l \sum_{q=1}^p \mathcal{E}_I^{p,q} + \sum_{p=1}^l \sum_{q=1}^p \mathcal{E}_{II}^{p,q} \quad (5.3)$$

where

$$\mathcal{V}_{z,l}(\mathbf{x}) := W(\mathbf{x}) - \sum_{p=0}^l \frac{1}{p!} d^p W_{\mathbf{z}}(\mathbf{x} - \mathbf{z})^{\otimes p} \quad (5.4)$$

$$\mathcal{E}_0^l(\mathbf{z}) := \frac{1}{2i\pi l_b^4} \text{Tr} [\gamma_b [\Pi_{z, \leq M}, \mathcal{V}_{z,l}]] \quad (5.5)$$

²Note that this aspect is algebraic in nature: it is not clear that a factor of $|x - y|$ would gain us a factor of l_b^2 .

and

$$\mathcal{E}_I^{p,q}(\mathbf{z}) := \frac{1}{2\pi l_b^2 p!} \text{Tr} \left[\gamma_b d^p W_{\mathbf{z}} (\mathbf{X} - \mathbf{z})^{\otimes(q-1)} \otimes (\nabla_{\mathbf{z}}^\perp \Pi_{z, \leq M}) \otimes (\mathbf{X} - \mathbf{z})^{\otimes(p-q)} \right] \quad (5.6)$$

$$\begin{aligned} \mathcal{E}_{II}^{p,q}(\mathbf{z}) &:= \frac{\sqrt{M+1}}{2\sqrt{2}\pi l_b^3 p!} \\ &\text{Tr} \left[\gamma_b d^p W_{\mathbf{z}} (\mathbf{X} - \mathbf{z})^{\otimes(q-1)} \otimes \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} |\psi_{z,M}\rangle \langle \psi_{z,M+1}| \\ |\psi_{z,M+1}\rangle \langle \psi_{z,M}| \end{pmatrix} \otimes (\mathbf{X} - \mathbf{z})^{\otimes(p-q)} \right] \end{aligned} \quad (5.7)$$

Proof. Step 1 : Direct computation. We start from (4.2) and Equation (5.2):

$$\begin{aligned} \partial_t \rho_{\gamma_b}^{sc, \leq M}(\mathbf{z}) &= \frac{1}{2\pi l_b^2} \text{Tr} [\partial_t \gamma_b \Pi_{z, \leq M}] = \frac{1}{2i\pi l_b^4} \text{Tr} [[\mathcal{L}_b + W, \gamma_b] \Pi_{z, \leq M}] \\ &= \frac{1}{2i\pi l_b^4} \text{Tr} [\gamma_b [\Pi_{z, \leq M}, \mathcal{L}_b + W]] = \frac{1}{2i\pi l_b^4} \text{Tr} [\gamma_b [\Pi_{z, \leq M}, W]] \end{aligned} \quad (5.8)$$

On the other hand, using (3.17),

$$\begin{aligned} \nabla^\perp W(\mathbf{z}) \cdot \nabla_{\mathbf{z}} \rho_{\gamma_b}^{sc, \leq M}(\mathbf{z}) &= -\nabla W(\mathbf{z}) \cdot \nabla_{\mathbf{z}}^\perp \rho_{\gamma_b}^{sc, \leq M}(\mathbf{z}) = \frac{-1}{2\pi l_b^2} \nabla W(\mathbf{z}) \cdot \text{Tr} [\gamma_b \nabla_{\mathbf{z}}^\perp \Pi_{z, \leq M}] \\ &= \frac{-1}{2i\pi l_b^4} \text{Tr} [\gamma_b [\Pi_{z, \leq M}, \mathbf{X} \cdot \nabla W(\mathbf{z})]] \\ &\quad + \frac{\sqrt{M+1}}{2\sqrt{2}\pi l_b^3} \nabla W(\mathbf{z}) \cdot \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \text{Tr} [\gamma_b |\psi_{z,M}\rangle \langle \psi_{z,M+1}|] \\ \text{Tr} [\gamma_b |\psi_{z,M+1}\rangle \langle \psi_{z,M}|] \end{pmatrix} \end{aligned} \quad (5.9)$$

Putting together Equation (5.8) and Equation (5.9) yields

$$\begin{aligned} \partial_t \rho_{\gamma_b}^{sc, \leq M}(\mathbf{z}) + \nabla^\perp W(\mathbf{z}) \cdot \nabla_{\mathbf{z}} \rho_{\gamma_b}^{sc, \leq M}(\mathbf{z}) &= \frac{1}{2i\pi l_b^4} \text{Tr} [\gamma_b [\Pi_{z, \leq M}, W - \mathbf{X} \cdot \nabla W(\mathbf{z})]] \\ &\quad + \frac{\sqrt{M+1}}{2\sqrt{2}\pi l_b^3} \nabla W(\mathbf{z}) \cdot \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \text{Tr} [\gamma_b |\psi_{z,n}\rangle \langle \psi_{z,M+1}|] \\ \text{Tr} [\gamma_b |\psi_{z,M+1}\rangle \langle \psi_{z,n}|] \end{pmatrix} \end{aligned} \quad (5.10)$$

Step 2 : Taylor expansion. We compute

$$\begin{aligned} \mathcal{V}_{z,l}(\mathbf{y}) - \mathcal{V}_{z,l}(\mathbf{x}) &= W(\mathbf{y}) - W(\mathbf{x}) - \sum_{p=0}^l \frac{1}{p!} d^p W_{\mathbf{z}} ((\mathbf{y} - \mathbf{z})^{\otimes p} - (\mathbf{x} - \mathbf{z})^{\otimes p}) \\ &= W(\mathbf{y}) - W(\mathbf{x}) - (\mathbf{y} - \mathbf{x}) \cdot \nabla W(\mathbf{z}) - \sum_{p=2}^l \frac{1}{p!} d^p W_{\mathbf{z}} ((\mathbf{y} - \mathbf{z})^{\otimes p} - (\mathbf{x} - \mathbf{z})^{\otimes p}) \end{aligned}$$

and notice the telescopic expression

$$\begin{aligned} (\mathbf{y} - \mathbf{z})^{\otimes p} - (\mathbf{x} - \mathbf{z})^{\otimes p} &= \sum_{q=0}^{p-1} (\mathbf{x} - \mathbf{z})^{\otimes q} \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p-q)} - \sum_{q=1}^p (\mathbf{x} - \mathbf{z})^{\otimes q} \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p-q)} \\ &= \sum_{q=1}^p (\mathbf{x} - \mathbf{z})^{\otimes(q-1)} \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p+1-q)} - \sum_{q=1}^p (\mathbf{x} - \mathbf{z})^{\otimes q} \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p-q)} \end{aligned}$$

$$\begin{aligned}
&= \sum_{q=1}^p (\mathbf{x} - \mathbf{z})^{\otimes(q-1)} \otimes (\mathbf{y} - \mathbf{z} - (\mathbf{x} - \mathbf{z})) \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p-q)} \\
&= \sum_{q=1}^p (\mathbf{x} - \mathbf{z})^{\otimes(q-1)} \otimes (\mathbf{y} - \mathbf{x}) \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p-q)}
\end{aligned}$$

Combining the above we express the integral kernel

$$\begin{aligned}
&[\Pi_{z, \leq M}, W - \mathbf{X} \cdot \nabla W(\mathbf{z})](\mathbf{x}, \mathbf{y}) = \Pi_{z, \leq M}(\mathbf{x}, \mathbf{y}) (W(\mathbf{y}) - W(\mathbf{x}) - (\mathbf{y} - \mathbf{x}) \cdot \nabla W(\mathbf{z})) \\
&= \Pi_{z, \leq M}(\mathbf{x}, \mathbf{y}) \left(\mathcal{V}_{z,l}(\mathbf{y}) - \mathcal{V}_{z,l}(\mathbf{x}) + \sum_{p=2}^l \frac{1}{p!} d^p W_{\mathbf{z}} ((\mathbf{y} - \mathbf{z})^{\otimes p} - (\mathbf{x} - \mathbf{z})^{\otimes p}) \right) \\
&= \Pi_{z, \leq M}(\mathbf{x}, \mathbf{y}) \left(\mathcal{V}_{z,l}(\mathbf{y}) - \mathcal{V}_{z,l}(\mathbf{x}) + \sum_{p=2}^l \sum_{q=1}^p \frac{1}{p!} d^p W_{\mathbf{z}} (\mathbf{x} - \mathbf{z})^{\otimes(q-1)} \otimes (\mathbf{y} - \mathbf{x}) \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p-q)} \right)
\end{aligned}$$

Inserting (3.17) we obtain the operator equality

$$\begin{aligned}
&[\Pi_{z, \leq M}, W - \mathbf{X} \cdot \nabla W(\mathbf{z})] \\
&= [\Pi_{z, \leq M}, \mathcal{V}_{z,l}] + \sum_{p=2}^l \sum_{q=1}^p \frac{1}{p!} d^p W_{\mathbf{z}} (\mathbf{X} - \mathbf{z})^{\otimes(q-1)} \otimes [\Pi_{z, \leq M}, \mathbf{X}] \otimes (\mathbf{X} - \mathbf{z})^{\otimes(p-q)} \\
&= [\Pi_{z, \leq M}, \mathcal{V}_{z,l}] + il_b^2 \sum_{p=2}^l \sum_{q=1}^p \frac{1}{p!} d^p W_{\mathbf{z}} (\mathbf{X} - \mathbf{z})^{\otimes(q-1)} \otimes (\nabla_{\mathbf{z}}^{\perp} \Pi_{z, \leq M}) \otimes (\mathbf{X} - \mathbf{z})^{\otimes(p-q)} \\
&+ il_b \sqrt{\frac{M+1}{2}} \sum_{p=2}^l \sum_{q=1}^p \frac{1}{p!} d^p W_{\mathbf{z}} (\mathbf{X} - \mathbf{z})^{\otimes(q-1)} \otimes \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} |\psi_{z,M}\rangle \langle \psi_{z,M+1}| \\ |\psi_{z,M+1}\rangle \langle \psi_{z,M}| \end{pmatrix} \otimes (\mathbf{X} - \mathbf{z})^{\otimes(p-q)}
\end{aligned} \tag{5.11}$$

Step 3 : Conclusion. Inserting (5.11) in Equation (5.10) we find

$$\begin{aligned}
&\partial_t \rho_{\gamma_b}^{sc, \leq M} + \nabla^{\perp} W \cdot \nabla \rho_{\gamma_b}^{sc, \leq M} = \frac{1}{2i\pi l_b^4} \text{Tr} \left[\gamma_b [\Pi_{z, \leq M}, \mathcal{V}_{z,l}] \right] \\
&+ \sum_{p=2}^l \sum_{q=1}^p \frac{1}{2\pi l_b^2 p!} \text{Tr} \left[\gamma_b d^p W_{\mathbf{z}} (\mathbf{X} - \mathbf{z})^{\otimes(q-1)} \otimes (\nabla_{\mathbf{z}}^{\perp} \Pi_{z, \leq M}) \otimes (\mathbf{X} - \mathbf{z})^{\otimes(p-q)} \right] \\
&+ \sum_{p=1}^l \sum_{q=1}^p \frac{\sqrt{M+1}}{2\sqrt{2}\pi l_b^3 p!} \\
&\cdot \text{Tr} \left[\gamma_b d^p W_{\mathbf{z}} (\mathbf{X} - \mathbf{z})^{\otimes(q-1)} \otimes \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} |\psi_{z,M}\rangle \langle \psi_{z,M+1}| \\ |\psi_{z,M+1}\rangle \langle \psi_{z,M}| \end{pmatrix} \otimes (\mathbf{X} - \mathbf{z})^{\otimes(p-q)} \right]
\end{aligned}$$

Note that the high Landau level error term in Equation (5.10) is exactly the term indexed by $p = 1$ in the second sum above. Finally, defining the errors terms (5.5), (5.6), (5.7), we get the decomposition Equation (5.3). \square

In the proof of Proposition 5.1 we will use repeatedly the following bound to dispatch the error terms in (5.6) and (5.7).

Lemma 5.4 (A general error term).

Let γ_b be a non negative trace-class operator, $\mathbf{X} = (X_1, X_2)$ the position operator, $r, M \in \mathbb{N}$, $\mathbf{z} = (z_1, z_2) \in \mathbb{R}^2$ and $\Pi_{z,M}$ the coherent state projector (3.7). Define

$$\mathcal{E}_{r,M}(\mathbf{z}) := \sum_{i_1 \dots i_r \in \{1,2\}^r} \text{Tr} \left[\gamma_b \prod_{j=1}^r (X_{i_j} - z_{i_j}) \Pi_{z,M} \prod_{j=1}^r (X_{i_j} - z_{i_j}) \right]. \quad (5.12)$$

Then, we have the bound

$$\int_{\mathbb{R}^2} |\mathcal{E}_{r,M}(\mathbf{z})| \frac{d\mathbf{z}}{2\pi l_b^2} \leq C(r) l_b^{2r} (M+1)^{r-k} \text{Tr} [\gamma_b \mathcal{L}_b^k \Pi_{M-r;M+r}] \quad (5.13)$$

with $\Pi_{M-r;M+r}$ as in Definition 3.4.

Proof. Following the proof of the binomial theorem, we get by induction on r

$$\begin{aligned} & \sum_{i_1 \dots i_r \in \{1,2\}^r} \prod_{j=1}^r (X_{i_j} - z_{i_j}) \Pi_{z,M} \prod_{j=1}^r (X_{i_j} - z_{i_j}) \\ &= 2^{-r} \sum_{s=0}^r \binom{r}{s} (X-z)^s \overline{(X-z)^{r-s}} \Pi_{z,M} (X-z)^{r-s} \overline{(X-z)^s}. \end{aligned} \quad (5.14)$$

Indeed, for $r=0$, (5.14) reduces to $\Pi_{z,M} = \Pi_{z,M}$. Then, noticing that

$$\sum_{i \in \{1,2\}} (X_i - z_i) [\dots] (X_i - z_i) = \frac{(X-z) [\dots] \overline{(X-z)} + \overline{(X-z)} [\dots] (X-z)}{2}$$

and using (5.14), the induction argument is obtained through

$$\begin{aligned} & \sum_{i_1 \dots i_{r+1} \in \{1,2\}^{r+1}} \prod_{j=1}^{r+1} (X_{i_j} - z_{i_j}) \Pi_{z,M} \prod_{j=1}^{r+1} (X_{i_j} - z_{i_j}) \\ &= 2^{-(r+1)} \sum_{s=0}^r \binom{r}{s} (X-z)^{s+1} \overline{(X-z)^{r-s}} \Pi_{z,M} (X-z)^{r-s} \overline{(X-z)^{s+1}} \\ & \quad + 2^{-(r+1)} \sum_{s=0}^r \binom{r}{s} (X-z)^s \overline{(X-z)^{r+1-s}} \Pi_{z,M} (X-z)^{r+1-s} \overline{(X-z)^s} \\ &= 2^{-(r+1)} \sum_{s=0}^{r+1} \binom{r+1}{s} (X-z)^s \overline{(X-z)^{r+1-s}} \Pi_{z,M} (X-z)^{r+1-s} \overline{(X-z)^s} \end{aligned}$$

with a change of variable $s+1 \mapsto s$ in the first sum. But, in view of (3.3),

$$X-z = \sqrt{2} l_b \left(c^\dagger - \frac{z}{\sqrt{2} l_b} + ia \right), \quad \overline{X-z} = \sqrt{2} l_b \left(c - \frac{\bar{z}}{\sqrt{2} l_b} - ia^\dagger \right).$$

Combining this with (5.14), (3.9), (3.2) and (3.8) we have

$$\begin{aligned} & \sum_{i_1 \dots i_r \in \{1,2\}^r} \prod_{j=1}^r (X_{i_j} - z_{i_j}) \Pi_{z,M} \prod_{j=1}^r (X_{i_j} - z_{i_j}) \\ &= l_b^{2r} \sum_{s=0}^r \binom{r}{s} \left(c^\dagger - \frac{z}{\sqrt{2} l_b} + ia \right)^s (-ia^\dagger)^{r-s} \Pi_{z,M} (ia)^{r-s} \left(c - \frac{\bar{z}}{\sqrt{2} l_b} - ia^\dagger \right)^s \end{aligned}$$

$$= l_b^{2r} \sum_{s=0}^r \binom{r}{s} \frac{(M+r-s)!}{M!} \left(c^\dagger - \frac{z}{\sqrt{2}l_b} + ia \right)^s \Pi_{z, M+r-s} \left(c - \frac{\bar{z}}{\sqrt{2}l_b} - ia^\dagger \right)^s \quad (5.15)$$

Then, by the Cauchy-Schwarz inequality,

$$\begin{aligned} & \left(c^\dagger - \frac{z}{\sqrt{2}l_b} + ia \right)^s \Pi_{z, M+r-s} \left(c - \frac{\bar{z}}{\sqrt{2}l_b} - ia^\dagger \right)^s \\ &= \sum_{n, m=0}^s \binom{s}{n} \binom{s}{m} \left(c^\dagger - \frac{z}{\sqrt{2}l_b} \right)^n (ia)^{s-n} \Pi_{z, M+r-s} (-ia^\dagger)^{s-m} \left(c - \frac{\bar{z}}{\sqrt{2}l_b} \right)^m \\ &\leq 2^s \sum_{n=0}^s \binom{s}{n} \left(c^\dagger - \frac{z}{\sqrt{2}l_b} \right)^n (ia)^{s-n} \Pi_{z, M+r-s} (-ia^\dagger)^{s-n} \left(c - \frac{\bar{z}}{\sqrt{2}l_b} \right)^n \\ &= 2^s \sum_{n=0}^s \frac{(M+r-s)!}{(M+r+n-2s)!} \left(c^\dagger - \frac{z}{\sqrt{2}l_b} \right)^n \Pi_{z, M+r+n-2s} \left(c - \frac{\bar{z}}{\sqrt{2}l_b} \right)^n \end{aligned} \quad (5.16)$$

As regards the combinatorial factors we have that, $\forall s \in \llbracket 0, r \rrbracket$, $n \in \llbracket 0, s \rrbracket$,

$$\begin{aligned} \binom{r}{s} \frac{(M+r-s)!}{M!} 2^s \frac{(M+r-s)!}{(M+r+n-2s)!} &\leq 2^r (M+r-s)^{r-s} 2^r (M+r-s)^{s-n} \\ &\leq 4^r (M+r)^{r-n} \leq 4^r (1+r)^{r-n} (M+1)^{r-n} \\ &\leq 4^r (1+r)^r (M+1)^{r-n}. \end{aligned} \quad (5.17)$$

Hence, inserting (5.16) and (5.17) in (5.15), we get

$$\begin{aligned} & \sum_{i_1 \dots i_r \in \{1,2\}^r} \prod_{j=1}^r (X_{i_j} - z_{i_j}) \Pi_{z, M} \prod_{j=1}^r (X_{i_j} - z_{i_j}) \\ &\leq C(r) l_b^{2r} \sum_{s=0}^r \sum_{n=0}^s (M+1)^{r-n} \left(c^\dagger - \frac{z}{\sqrt{2}l_b} \right)^n \Pi_{z, M+r+n-2s} \left(c - \frac{\bar{z}}{\sqrt{2}l_b} \right)^n \end{aligned} \quad (5.18)$$

Expanding the powers using (3.9), $\forall j \in \mathbb{N}$,

$$0 \leq \left(c^\dagger - \frac{z}{\sqrt{2}l_b} \right)^n \Pi_{z, j} \left(c - \frac{\bar{z}}{\sqrt{2}l_b} \right)^n = \sum_{p, q=0}^n \binom{n}{p} \binom{n}{q} c^{\dagger n-p} (-c)^q \Pi_{z, j} (-c^\dagger)^p c^{n-q}$$

Using $[\Pi_j, c] = [\Pi_j, c^\dagger] = 0$ together with (3.14) and the following normal ordering formula (cf. for example [43, Lemma 4.7])

$$c^q c^{\dagger p} = \sum_{i=0}^{\min(p, q)} i! \binom{p}{i} \binom{q}{i} c^{\dagger p-i} c^{q-i}$$

we obtain

$$\begin{aligned} & \int_{\mathbb{R}^2} \left| \text{Tr} \left[\gamma_b \left(c^\dagger - \frac{z}{\sqrt{2}l_b} \right)^n \Pi_{z, j} \left(c - \frac{\bar{z}}{\sqrt{2}l_b} \right)^n \right] \right| dz \\ &= 2\pi l_b^2 \sum_{p, q=0}^n (-1)^{p+q} \binom{n}{p} \binom{n}{q} \text{Tr} \left[\gamma_b \Pi_j c^{\dagger n-p} c^q c^{\dagger p} c^{n-q} \right] \end{aligned}$$

$$\begin{aligned}
&= 2\pi l_b^2 \sum_{i=0}^n \sum_{p,q=i}^n (-1)^{p+q} \binom{n}{p} \binom{n}{q} i! \binom{p}{i} \binom{q}{i} \operatorname{Tr} \left[\gamma_b \Pi_j c^{\dagger n-i} c^{n-i} \right] \\
&= 2\pi l_b^2 \sum_{i=0}^n i! \left(\sum_{p=i}^n \binom{n}{p} \binom{p}{i} (-1)^p \right)^2 \operatorname{Tr} \left[\gamma_b \Pi_j c^{\dagger n-i} c^{n-i} \right] = 2\pi n! l_b^2 \operatorname{Tr} \left[\gamma_b \Pi_j \right]
\end{aligned}$$

noticing that

$$\sum_{p=i}^n \binom{n}{p} \binom{p}{i} (-1)^p = \sum_{p=i}^n \binom{n-i}{p-i} \binom{n}{i} (-1)^p = (-1)^i \binom{n}{i} \sum_{p=0}^{n-i} \binom{n-i}{p} (-1)^p = \delta_{n,i} (-1)^n$$

There remains to insert this in (5.18) and recall (4.8):

$$\begin{aligned}
\int_{\mathbb{R}^2} |\mathcal{E}_{r,M}(\mathbf{z})| d\mathbf{z} &\leq C(r) l_b^{2(r+1)} \sum_{s=0}^r \sum_{n=0}^s (M+1)^{r-n} n! \operatorname{Tr} \left[\gamma_b \Pi_{M+r+n-2s} \right] \\
&\leq C(r) l_b^{2(r+1)} (M+1)^r \operatorname{Tr} \left[\gamma_b \Pi_{M-r:M+r} \right] \\
&\leq C(r) l_b^{2(r+1)} (M+1)^{r-k} \operatorname{Tr} \left[\gamma_b \mathcal{L}_b^k \Pi_{M-r:M+r} \right]
\end{aligned}$$

□

We next note a technical bound

Lemma 5.5 (Some integrals).

Let $\alpha \in \mathbb{R}_+$, then

$$I_n(\alpha) := \frac{1}{\pi n!} \int_{\mathbb{R}^2} |x|^{\alpha+2n} e^{-|x|^2} d\mathbf{x} \underset{n \rightarrow \infty}{\sim} n^{\frac{\alpha}{2}}$$

and $\exists C > 0$ such that

$$\forall n \in \mathbb{N}, I_n(\alpha) \leq C(n+1)^{\frac{\alpha}{2}}$$

Proof. With polar coordinates and the change of variable $u^2 \mapsto u$

$$\frac{1}{\pi} \int_{\mathbb{R}^2} |x|^\alpha e^{-|x|^2} d\mathbf{x} = 2 \int_{\mathbb{R}_+} u^{\alpha+1} e^{-u^2} du = \int_{\mathbb{R}_+} u^{\frac{\alpha}{2}} e^{-u} du = \Gamma\left(\frac{\alpha}{2} + 1\right)$$

where Γ is the Euler Gamma function,

$$\Gamma(z) := \int_{\mathbb{R}_+} t^{z-1} e^{-t} dt$$

We have the following equivalent for the Euler Gamma function (as a direct consequence of the Stirling formula)

$$\frac{\Gamma(n+x)}{\Gamma(n)} \underset{n \rightarrow \infty}{\sim} n^x.$$

Thus

$$I_n(\alpha) = \frac{1}{n!} \Gamma\left(n + \frac{\alpha}{2} + 1\right) = \frac{\Gamma\left(n + \frac{\alpha}{2} + 1\right)}{\Gamma(n+1)}$$

and then

$$I_n(\alpha) \underset{n \rightarrow \infty}{\sim} (n+1)^{\frac{\alpha}{2}} \underset{n \rightarrow \infty}{\sim} n^{\frac{\alpha}{2}}.$$

□

We may now turn to the

Proof of Proposition 5.1. We start from (5.3) and estimate the right-hand side term by term.

Step 1: Estimate of \mathcal{E}_0^l . We introduce another projector:

$$\begin{aligned} \text{Tr} [\gamma_b [\Pi_{z, \leq M}, \mathcal{V}_{z,l}]] &= \sum_{n=0}^M \text{Tr} [\gamma_b [\Pi_{z,n}, \mathcal{V}_{z,l}]] \\ &= \sum_{n=0}^M (\text{Tr} [\gamma_b \Pi_{z,n} [\Pi_{z,n}, \mathcal{V}_{z,l}]] + \text{Tr} [\Pi_{z,n} \gamma_b [\Pi_{z,n}, \mathcal{V}_{z,l}]]) \end{aligned}$$

so with (4.7) applied to $A_n(\mathbf{z}) := [\Pi_{z,n}, \mathcal{V}_{z,l}]$, recalling (5.5),

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \varphi(\mathbf{z}) \mathcal{E}_0^l(\mathbf{z}) d\mathbf{z} \right| &= \left| \frac{1}{4i\pi l_b^4} \sum_{n=0}^M \int_{\mathbb{R}^2} \varphi(\mathbf{z}) (\text{Tr} [\gamma_b \Pi_{z,n} [\Pi_{z,n}, \mathcal{V}_{z,l}]] + \text{Tr} [\Pi_{z,n} \gamma_b [\Pi_{z,n}, \mathcal{V}_{z,l}]]) d\mathbf{z} \right| \\ &\leq \frac{2}{l_b^2} \left| \frac{1}{2\pi l_b^2} \sum_{n=0}^M \int_{\mathbb{R}^2} |\varphi(\mathbf{z}) \text{Tr} [\gamma_b \Pi_{z,n} [\Pi_{z,n}, \mathcal{V}_{z,l}]]| d\mathbf{z} \right| \\ &\leq \frac{\sqrt{\|\varphi\|_{L^1} \|\varphi\|_{L^\infty}}}{l_b^2} \sqrt{\text{Tr} [\gamma_b \mathcal{L}_b^k]} \left(\sum_{n=0}^M \frac{\sup_{z \in \mathbb{R}^2} \|\Pi_{z,n}, \mathcal{V}_{z,l}\|_{\mathcal{L}^2}^2}{(n+1)^k} \right)^{\frac{1}{2}} \end{aligned} \quad (5.19)$$

We estimate the Hilbert-Schmidt norm with the changes of variables

$$\frac{\mathbf{x} - \mathbf{z}}{\sqrt{2}l_b} \rightarrow \mathbf{x}, \quad \frac{\mathbf{y} - \mathbf{z}}{\sqrt{2}l_b} \rightarrow \mathbf{y}.$$

This gives

$$\begin{aligned} \|\Pi_{z,n}, \mathcal{V}_{z,l}\|_{\mathcal{L}^2}^2 &= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |[\Pi_{z,n}, \mathcal{V}_{z,l}](\mathbf{x}, \mathbf{y})|^2 d\mathbf{x} d\mathbf{y} = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (\mathcal{V}_{z,l}(\mathbf{x}) - \mathcal{V}_{z,l}(\mathbf{y}))^2 |\Pi_{z,n}(\mathbf{x}, \mathbf{y})|^2 d\mathbf{x} d\mathbf{y} \\ &= \frac{1}{(2\pi n! l_b^2)^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} (\mathcal{V}_{z,l}(\mathbf{x}) - \mathcal{V}_{z,l}(\mathbf{y}))^2 \left| \frac{(x-z)(y-z)}{2l_b^2} \right|^{2n} e^{-\frac{|x-z|^2 + |y-z|^2}{2l_b^2}} d\mathbf{x} d\mathbf{y} \\ &= \frac{1}{(\pi n!)^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} (\mathcal{V}_{z,l}(\mathbf{z} + \sqrt{2}l_b \mathbf{x}) - \mathcal{V}_{z,l}(\mathbf{z} + \sqrt{2}l_b \mathbf{y}))^2 |xy|^{2n} e^{-|x|^2 - |y|^2} d\mathbf{x} d\mathbf{y} \end{aligned}$$

Using the expansion from Equation (5.4),

$$|\mathcal{V}_{z,l}(\mathbf{z} + \sqrt{2}l_b \mathbf{x})| \leq \frac{1}{(l+1)!} \|d^{l+1} W\|_{L^\infty} |\sqrt{2}l_b x|^{l+1} \leq e^{\sqrt{2}} \|d^{l+1} W\|_{L^\infty} |l_b x|^{l+1}$$

and similarly with y instead of x . With Lemma 5.5,

$$\begin{aligned} \|\Pi_{z,n}, \mathcal{V}_{z,l}\|_{\mathcal{L}^2}^2 &\leq C \frac{\|d^{l+1} W\|_{L^\infty}^2}{n!^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} l_b^{2(l+1)} (|x|^{2(l+1)} + |y|^{2(l+1)}) |xy|^{2n} e^{-|x|^2 - |y|^2} d\mathbf{x} d\mathbf{y} \\ &\leq C \|d^{l+1} W\|_{L^\infty}^2 ((n+1)l_b^2)^{l+1} \end{aligned}$$

Inserting this in (5.19),

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \varphi(\mathbf{z}) \mathcal{E}_0^l(\mathbf{z}) d\mathbf{z} \right| &\leq C \sqrt{\|\varphi\|_{L^1} \|\varphi\|_{L^\infty}} \|d^{l+1} W\|_{L^\infty} \frac{1}{l_b^2} \sqrt{\sum_{n=0}^M (n+1)^{l+1-k} l_b^{2(l+1)}} \sqrt{\text{Tr} [\gamma_b \mathcal{L}_b^k]} \\ &= C \sqrt{\|\varphi\|_{L^1} \|\varphi\|_{L^\infty}} \|d^{l+1} W\|_{L^\infty} \sqrt{\text{Tr} [\gamma_b \mathcal{L}_b^k]} l_b^{l-1} \sqrt{p_{k-(l+2)}(M)} \end{aligned} \quad (5.20)$$

with $p_{k-(l+2)}(M)$ as in (4.13).

Step 2: Estimate of $\mathcal{E}_{\text{II}}^{p,q}$. From (5.7),

$$\begin{aligned} \mathcal{E}_{\text{II}}^{p,q}(\mathbf{z}) &:= \frac{\sqrt{M+1}}{2\sqrt{2\pi} l_b^3 p!} \sum_{i_1, \dots, i_p \in \{1,2\}^p} \partial_{i_1} \dots \partial_{i_p} W(\mathbf{z}) i^{i_q-1} \\ &\text{Tr} \left[\gamma_b \left(\prod_{j=1}^{q-1} (X_{i_j} - z_{i_j}) \right) \left((-1)^{i_q-1} |\psi_{z,M}\rangle \langle \psi_{z,M+1}| + |\psi_{z,M+1}\rangle \langle \psi_{z,M}| \right) \left(\prod_{j=q+1}^p (X_{i_j} - z_{i_j}) \right) \right] \end{aligned}$$

Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \varphi(\mathbf{z}) \mathcal{E}_{\text{II}}^{p,q}(\mathbf{z}) d\mathbf{z} \right| &\leq C(p) \|d^p W\|_{L^\infty} \frac{\sqrt{M}}{l_b^3} \\ &\int_{\mathbb{R}^2} |\varphi(\mathbf{z})| \left(\epsilon \mathcal{E}_{q-1,M}(\mathbf{z}) + \epsilon \mathcal{E}_{q-1,M+1}(\mathbf{z}) + \frac{1}{\epsilon} \mathcal{E}_{p-q,M}(\mathbf{z}) + \frac{1}{\epsilon} \mathcal{E}_{p-q,M+1}(\mathbf{z}) \right) d\mathbf{z} \end{aligned} \quad (5.21)$$

in the notation of Lemma 5.4. We insert the bound (5.13), replacing $M+1$ by M , since $M \gg 1$:

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \varphi(\mathbf{z}) \mathcal{E}_{\text{II}}^{p,q}(\mathbf{z}) d\mathbf{z} \right| &\leq C(p) \|d^p W\|_{L^\infty} \|\varphi\|_{L^\infty} \frac{M^{\frac{1}{2}-k}}{l_b} \left(\epsilon (l_b^2 M)^{q-1} \text{Tr} [\gamma_b \mathcal{L}_b^k \Pi_{M-q+1; M+q}] \right. \\ &\quad \left. + \frac{1}{\epsilon} (l_b^2 M)^{p-q} \text{Tr} [\gamma_b \mathcal{L}_b^k \Pi_{M-p+q; M+p-q+1}] \right) \end{aligned}$$

Since $1 \leq q \leq p$, choosing $\epsilon := (l_b^2 M)^{\frac{p+1}{2}-q}$, we conclude

$$\left| \int_{\mathbb{R}^2} \varphi(\mathbf{z}) \mathcal{E}_{\text{II}}^{p,q}(\mathbf{z}) d\mathbf{z} \right| \leq C(p) \|\nabla W\|_{L^\infty} \|\varphi\|_{L^\infty} M^{\frac{p}{2}-k} l_b^{p-2} \text{Tr} [\gamma_b \mathcal{L}_b^k \Pi_{M-p+1; M+p}] \quad (5.22)$$

Step 3: Estimate of $\mathcal{E}_{\text{I}}^{p,q}$. In order to estimate Equation (5.6), we start with an integration by parts for which the following preparations will be helpful.

Let \odot denote the tensor contraction defined for $n, m \geq k$ by

$$u_1 \otimes \dots \otimes u_n \odot^k v_1 \otimes \dots \otimes v_m := \langle u_n | v_1 \rangle \dots \langle u_{n-k+1} | v_k \rangle u_1 \otimes \dots \otimes u_{n-k} \otimes v_{k+1} \otimes \dots \otimes v_m$$

Identifying $d^p W(\mathbf{z})$ with the associated rank p tensor, we notice that

$$\begin{aligned} &d^p W_{\mathbf{z}}(\mathbf{x} - \mathbf{z})^{\otimes(q-1)} \otimes \nabla_{\mathbf{z}}^\perp \Pi_{z, \leq M}(\mathbf{x}, \mathbf{y}) \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p-q)} \\ &= \nabla^{\otimes p} W(\mathbf{z}) \odot^p (\mathbf{x} - \mathbf{z})^{\otimes(q-1)} \otimes \nabla_{\mathbf{z}}^\perp \Pi_{z, \leq M}(\mathbf{x}, \mathbf{y}) \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p-q)} \\ &= \nabla_{\mathbf{z}}^\perp \Pi_{z, \leq M}(\mathbf{x}, \mathbf{y}) \cdot \nabla^{\otimes p} W(\mathbf{z}) \odot^{p-1} (\mathbf{x} - \mathbf{z})^{\otimes(q-1)} \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p-q)} \end{aligned}$$

and

$$\nabla_{\mathbf{z}}^\perp \cdot \nabla^{\otimes p} W(\mathbf{z}) \odot^{p-1} (\mathbf{x} - \mathbf{z})^{\otimes(q-1)} \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p-q)}$$

$$\begin{aligned}
&= (\nabla^\perp \cdot \nabla^{\otimes p}) W(\mathbf{z}) \odot^{p-1} (\mathbf{x} - \mathbf{z})^{\otimes(q-1)} \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p-q)} \\
&\quad + \nabla^{\otimes p} W(\mathbf{z}) \odot^p (\nabla_z^\perp \otimes (\mathbf{x} - \mathbf{z})) \otimes (\mathbf{x} - \mathbf{z})^{\otimes(q-2)} \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p-q)} \\
&\quad + \dots \\
&\quad + \nabla^{\otimes p} W(\mathbf{z}) \odot^p (\mathbf{x} - \mathbf{z})^{\otimes(q-1)} \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p-q-1)} \otimes (\nabla_z^\perp \otimes (\mathbf{y} - \mathbf{z}))
\end{aligned}$$

but because $\nabla^\perp \cdot \nabla = 0$, $\nabla_z^\perp \otimes (\mathbf{x} - \mathbf{z}) = \nabla_z^\perp \otimes (\mathbf{y} - \mathbf{z}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

$$\begin{aligned}
&\nabla_z^\perp \cdot \nabla^{\otimes p} W(\mathbf{z}) \odot^{p-1} (\mathbf{x} - \mathbf{z})^{\otimes(q-1)} \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p-q)} \\
&= (\nabla^\perp \cdot \nabla) \nabla^{\otimes(p-1)} W(\mathbf{z}) \odot^{p-1} (\mathbf{x} - \mathbf{z})^{\otimes(q-1)} \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p-q)} \\
&\quad + \nabla^{\otimes p} W(\mathbf{z}) \odot^p \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes (\mathbf{x} - \mathbf{z})^{\otimes(q-2)} \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p-q)} \\
&\quad + \dots \\
&\quad + \nabla^{\otimes p} W(\mathbf{z}) \odot^p (\mathbf{x} - \mathbf{z})^{\otimes(q-1)} \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p-q-1)} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \left(\nabla^{\otimes p} W(\mathbf{z}) \odot^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \odot^{p-2} (\mathbf{x} - \mathbf{z})^{\otimes(q-2)} \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p-q)} \\
&\quad + \dots \\
&\quad + \left(\nabla^{\otimes p} W(\mathbf{z}) \odot^{p-2} (\mathbf{x} - \mathbf{z})^{\otimes(q-1)} \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p-q-1)} \right) \odot^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= 0.
\end{aligned}$$

Indeed, since

$$\nabla^{\otimes p} W(\mathbf{z}), \dots, \nabla^{\otimes p} W(\mathbf{z}) \odot^{p-2} (\mathbf{x} - \mathbf{z})^{\otimes(q-1)} \otimes (\mathbf{y} - \mathbf{z})^{\otimes(p-q-1)}$$

are symmetric tensors and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is antisymmetric their contraction product is null.

In view of these considerations, an integration by parts gives

$$\begin{aligned}
&\int_{\mathbb{R}^2} \varphi(\mathbf{z}) \mathcal{E}_1^{p,q}(\mathbf{z}) \, d\mathbf{z} \\
&= -\frac{1}{2\pi l_b^2 p!} \int_{\mathbb{R}^2} \text{Tr} \left[\gamma_b d^p W_{\mathbf{z}} (\mathbf{X} - \mathbf{z})^{\otimes(q-1)} \otimes \nabla^\perp \varphi(\mathbf{z}) \Pi_{z, \leq M} \otimes (\mathbf{X} - \mathbf{z})^{\otimes(p-q)} \right] \, d\mathbf{z} \\
&= -\frac{1}{2\pi l_b^2 p!} \sum_{i_1, \dots, i_p \in \{1, 2\}^p} \int_{\mathbb{R}^2} \partial_{i_1: p} W(\mathbf{z}) \nabla_{i_q}^\perp \varphi(\mathbf{z}) \\
&\quad \cdot \text{Tr} \left[\gamma_b \left(\prod_{j=1}^{q-1} (X_{i_j} - z_{i_j}) \right) \Pi_{z, \leq M} \left(\prod_{j=q+1}^p (X_{i_j} - z_{i_j}) \right) \right] \, d\mathbf{z} \tag{5.23}
\end{aligned}$$

Using the Cauchy-Schwarz inequality and (5.13) leads to

$$\begin{aligned}
\left| \int_{\mathbb{R}^2} \varphi(\mathbf{z}) \mathcal{E}_1^{p,q}(\mathbf{z}) \, d\mathbf{z} \right| &\leq \frac{C(p)}{2\pi l_b^2} \|d^p W\|_{L^\infty} \|\nabla \varphi\|_{L^\infty} \sum_{n=0}^M \int_{\mathbb{R}^2} \left(\epsilon \mathcal{E}_{q-1, n}(\mathbf{z}) + \frac{1}{\epsilon} \mathcal{E}_{p-q, n}(\mathbf{z}) \right) \, d\mathbf{z} \\
&\leq C(p) \|d^p W\|_{L^\infty} \|\nabla \varphi\|_{L^\infty}
\end{aligned}$$

$$\sum_{n=0}^M \left(\epsilon_n l_b^{2(q-1)} (n+1)^{q-1-k} + \frac{1}{\epsilon_n} l_b^{2(p-q)} (n+1)^{p-q-k} \right) \text{Tr} [\gamma_b \mathcal{L}_b^k \Pi_{\leq M+p-1}]$$

Choosing $\epsilon_n := (l_b^2 n)^{\frac{p+1}{2}-q}$ and recalling (4.13), we conclude

$$\left| \int_{\mathbb{R}^2} \varphi(\mathbf{z}) \mathcal{E}_1^{p,q}(\mathbf{z}) \, d\mathbf{z} \right| \leq C(p) \|d^p W\|_{L^\infty} \|\nabla \varphi\|_{L^\infty} l_b^{p-1} p_{k-\frac{p+1}{2}}(M) \text{Tr} [\gamma_b \mathcal{L}_b^k \Pi_{\leq M+p-1}] \quad (5.24)$$

Step 4: Conclusion. Putting Equations (5.3), (5.20), (5.22) and (5.24) together we obtain

$$\begin{aligned} & \left| \int_{\mathbb{R}^2} \varphi(\mathbf{z}) \left(\partial_t \rho_{\gamma_b}^{sc, \leq M}(t, \mathbf{z}) + \nabla^\perp W(\mathbf{z}) \cdot \nabla \rho_{\gamma_b}^{sc, \leq M}(t, \mathbf{z}) \right) d\mathbf{z} \right| \\ & \leq \left(C \sqrt{\|\varphi\|_{L^1} \|\varphi\|_{L^\infty}} \|d^{l+1} W\|_{L^\infty} \sqrt{\text{Tr} [\gamma_b \mathcal{L}_b^k] l_b^{l-1}} \sqrt{p_{k-(l+2)}(M)} \right. \\ & \quad + \sum_{p=1}^l \sum_{q=1}^p C(p) \|\nabla W\|_{L^\infty} \|\varphi\|_{L^\infty} M^{\frac{p}{2}-k} l_b^{p-2} \text{Tr} [\gamma_b \mathcal{L}_b^k \Pi_{M-p+1:M+p}] \\ & \quad \left. + \sum_{p=2}^l \sum_{q=1}^p C(p) \|d^p W\|_{L^\infty} \|\nabla \varphi\|_{L^\infty} l_b^{p-1} p_{k-\frac{p+1}{2}}(M) \text{Tr} [\gamma_b \mathcal{L}_b^k \Pi_{\leq M+p-1}] \right) \\ & \leq C(l) \|\varphi\|_{L^1 \cap W^{1,\infty}} \|W\|_{W^{l+1,\infty}} \left(\sqrt{\text{Tr} [\gamma_b \mathcal{L}_b^k] l_b^{l-1}} \sqrt{p_{k-(l+2)}(M)} \right. \\ & \quad \left. + \text{Tr} [\gamma_b \mathcal{L}_b^k \Pi_{M-l+1:M+l}] \sum_{p=1}^l M^{\frac{p}{2}-k} l_b^{p-2} + \text{Tr} [\gamma_b \mathcal{L}_b^k] \sum_{p=2}^l l_b^{p-1} p_{k-\frac{p+1}{2}}(M) \right) \end{aligned}$$

and then

$$\begin{aligned} [\dots] & \leq C(l) \|\varphi\|_{L^1 \cap W^{1,\infty}} \|W\|_{W^{l+1,\infty}} \left(\sqrt{\text{Tr} [\gamma_b \mathcal{L}_b^k] l_b^{l-1}} \begin{cases} M^{1+\frac{l-k}{2}} & \text{if } k < l+2 \\ \sqrt{\ln(M)} & \text{if } k = l+2 \\ 1 & \text{if } k > l+2 \end{cases} \right. \\ & \quad \left. + \frac{\text{Tr} [\gamma_b \mathcal{L}_b^k \Pi_{M-l+1:M+l}]}{l_b M^{k-\frac{1}{2}}} + \text{Tr} [\gamma_b \mathcal{L}_b^k] \sum_{p=2}^l l_b^{p-1} \begin{cases} M^{\frac{p+1}{2}-k} & \text{if } k < \frac{p+1}{2} \\ \ln(M) & \text{if } k = \frac{p+1}{2} \\ 1 & \text{if } k > \frac{p+1}{2} \end{cases} \right) \end{aligned}$$

where we used $l_b^2 M \leq 1$ for the last inequality. \square

In addition to the estimate of Proposition 5.1 we will need to control the time derivative of the normalization of $\rho_{\gamma_b}^{sc, \leq M}$, with similar ideas as above:

Corollary 5.6 (Controlling the normalization).

Under the assumptions of Proposition 5.1 with in addition $l \geq 2$, $d^{l+1} W \in L^1(\mathbb{R}^2)$, we have

$$\left| \partial_t \text{Tr} [\gamma_b \Pi_{\leq M}] \right| \leq C(l) \left(\|W\|_{W^{l+1,\infty}} + \|d^{l+1} W\|_{L^1} \right) \left(l_b^{l-2} M^{\frac{l+3}{2}} + \frac{\text{Tr} [\gamma_b \mathcal{L}_b^k \Pi_{M-l+1:M+l}]}{l_b M^{k-\frac{1}{2}}} \right)$$

$$+ \operatorname{Tr} \left[\gamma_b \mathcal{L}_b^k \right] \sum_{p=2}^l l_b^{p-1} \left(\begin{array}{ll} M^{\frac{p+1}{2}-k} & \text{if } k < \frac{p+1}{2} \\ \ln(M) & \text{if } k = \frac{p+1}{2} \\ 1 & \text{if } k > \frac{p+1}{2} \end{array} \right) \quad (5.25)$$

Proof. An integration by parts gives

$$\int_{\mathbb{R}^2} \nabla^\perp W(\mathbf{z}) \cdot \nabla_{\mathbf{z}} \rho_{\gamma_b}^{sc, \leq M}(\mathbf{z}) d\mathbf{z} = 0$$

so that we may estimate

$$\begin{aligned} \partial_t \operatorname{Tr} [\gamma_b \Pi_{\leq M}] &= \partial_t \int_{\mathbb{R}^2} \operatorname{Tr} [\gamma_b \Pi_{\leq z, M}] \frac{d\mathbf{z}}{2\pi l_b^2} = \int_{\mathbb{R}^2} \partial_t \rho_{\gamma_b}^{sc, \leq M}(\mathbf{z}) d\mathbf{z} \\ &= \int_{\mathbb{R}^2} \left(\partial_t \rho_{\gamma_b}^{sc, \leq M}(\mathbf{z}) + \nabla^\perp W(\mathbf{z}) \cdot \nabla_{\mathbf{z}} \rho_{\gamma_b}^{sc, \leq M}(\mathbf{z}) \right) d\mathbf{z} \end{aligned} \quad (5.26)$$

We will apply ideas similar to the proof of Proposition 5.1, but this time with $\varphi = 1$. As in the previous proof we start from (5.3). Our bounds on the $\mathcal{E}_I^{p,q}$ and $\mathcal{E}_{II}^{p,q}$ apply mutatis mutandis, for they were estimated in terms of $\|\varphi\|_{L^\infty}$ only. Hence we only need to adapt the estimate of \mathcal{E}_0^l to get rid of the dependence on $\|\varphi\|_{L^1}$.

We write

$$\int_{\mathbb{R}^2} \mathcal{E}_0^l(\mathbf{z}) d\mathbf{z} = \frac{1}{il_b^2} \int_{\mathbb{R}^2} \operatorname{Tr} [\gamma_b [\Pi_{z, \leq M}, \mathcal{V}_{z,l}]] \frac{d\mathbf{z}}{2\pi l_b^2} = \frac{2}{l_b^2} \operatorname{Im} \int_{\mathbb{R}^2} \operatorname{Tr} [\gamma_b \Pi_{z, \leq M} \mathcal{V}_{z,l}] \frac{d\mathbf{z}}{2\pi l_b^2} \quad (5.27)$$

using Taylor's formula

$$\mathcal{V}_{z,l}(\mathbf{x}) := W(\mathbf{x}) - \sum_{p=0}^l \frac{1}{p!} d^p W_{\mathbf{z}}(\mathbf{x} - \mathbf{z})^{\otimes p} = \frac{1}{l!} \int_0^1 (1-\tau)^l d^{l+1} W_{\mathbf{z}+\tau(\mathbf{x}-\mathbf{z})}(\mathbf{x} - \mathbf{z})^{\otimes(l+1)} d\tau$$

Let $n \in \mathbb{N}$, we bound

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \operatorname{Tr} [\gamma_b \Pi_{z,n} \mathcal{V}_{z,l}] \frac{d\mathbf{z}}{2\pi l_b^2} \right| &\leq \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} |\gamma_b(\mathbf{x}, \mathbf{y}) \Pi_{z,n}(\mathbf{y}, \mathbf{x}) \mathcal{V}_{z,l}(\mathbf{x})| d\mathbf{x} d\mathbf{y} \frac{d\mathbf{z}}{2\pi l_b^2} \\ &\leq \frac{1}{l!} \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} |\gamma_b(\mathbf{x}, \mathbf{y})| |\Pi_{z,n}(\mathbf{y}, \mathbf{x})| \int_0^1 (1-\tau)^l |d^{l+1} W_{\mathbf{z}+\tau(\mathbf{x}-\mathbf{z})}| d\tau |\mathbf{x} - \mathbf{z}|^{l+1} d\mathbf{x} d\mathbf{y} \frac{d\mathbf{z}}{2\pi l_b^2} \\ &\leq \frac{1}{l!} \left(\int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} |\gamma_b(\mathbf{x}, \mathbf{y})|^2 \int_0^1 (1-\tau)^l |d^{l+1} W_{\mathbf{z}+\tau(\mathbf{x}-\mathbf{z})}| d\tau d\mathbf{x} d\mathbf{y} \frac{d\mathbf{z}}{2\pi l_b^2} \right)^{\frac{1}{2}} \\ &\quad \left(\int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} |\Pi_{z,n}(\mathbf{y}, \mathbf{x})|^2 \int_0^1 (1-\tau)^l |d^{l+1} W_{\mathbf{z}+\tau(\mathbf{x}-\mathbf{z})}| d\tau |\mathbf{x} - \mathbf{z}|^{2(l+1)} d\mathbf{x} d\mathbf{y} \frac{d\mathbf{z}}{2\pi l_b^2} \right)^{\frac{1}{2}} \end{aligned} \quad (5.28)$$

As $l \geq 2$, we can estimate the first factor by performing the following integral with the change of variable $(1-\tau)\mathbf{z} \mapsto \mathbf{z}$

$$\int_{\mathbb{R}^2} \int_0^1 (1-\tau)^l |d^{l+1} W_{\mathbf{z}+\tau(\mathbf{x}-\mathbf{z})}| d\tau \frac{d\mathbf{z}}{2\pi l_b^2} = \frac{\|d^{l+1} W\|_{L^1}}{2\pi l_b^2} \int_0^1 (1-\tau)^{l-2} d\tau = \frac{\|d^{l+1} W\|_{L^1}}{2(l-1)\pi l_b^2}$$

so with the Pauli principle

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} |\gamma_b(\mathbf{x}, \mathbf{y})|^2 \int_0^1 (1-\tau)^l \left| d^{l+1} W_{\mathbf{z}+\tau(\mathbf{x}-\mathbf{z})} \right| d\tau d\mathbf{x} d\mathbf{y} \frac{d\mathbf{z}}{2\pi l_b^2} \leq \frac{\|d^{l+1} W\|_{L^1}}{2(l-1)\pi l_b^2} \text{Tr} [\gamma_b^2] \leq \frac{\|d^{l+1} W\|_{L^1}}{l-1} \quad (5.29)$$

For the second factor in (5.28), after the $d\mathbf{y}$ integration and the change of variable $\frac{\mathbf{x}-\mathbf{z}}{\sqrt{2}l_b} \mapsto x$,

$$\begin{aligned} & \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} |\Pi_{z,n}(\mathbf{y}, \mathbf{x})|^2 \int_0^1 (1-\tau)^l \left| d^{l+1} W_{\mathbf{z}+\tau(\mathbf{x}-\mathbf{z})} \right| d\tau |x-z|^{2(l+1)} d\mathbf{x} d\mathbf{y} \frac{d\mathbf{z}}{2\pi l_b^2} \\ &= \frac{(2l_b^2)^{l+1}}{\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left| \psi_{z,n}(\mathbf{z} + \sqrt{2}l_b \mathbf{x}) \right|^2 \int_0^1 (1-\tau)^l \left| d^{l+1} W_{\mathbf{z}+\sqrt{2}l_b \tau \mathbf{x}} \right| d\tau |x|^{2(l+1)} d\mathbf{x} d\mathbf{z} \\ &= \frac{(2l_b^2)^{l+1}}{\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{2\pi n! l_b^2} |x|^{2(n+l+1)} e^{-|x|^2} \int_0^1 (1-\tau)^l \left| d^{l+1} W_{\mathbf{z}+\sqrt{2}l_b \tau \mathbf{x}} \right| d\tau d\mathbf{x} d\mathbf{z} \\ &= \frac{2(2l_b^2)^l \|d^{l+1} W\|_{L^1}}{\pi(l+1)n!} \int_{\mathbb{R}_+} u^{2(n+l+1)+1} e^{-u^2} du = \frac{(2l_b^2)^l \|d^{l+1} W\|_{L^1} (n+l+1)!}{\pi(l+1)n!} \\ &\leq C(l) \left\| d^{l+1} W \right\|_{L^1} l_b^{2l} (n+1)^{l+1} \end{aligned} \quad (5.30)$$

Inserting (5.29) and (5.30) in (5.28) leads to

$$\left| \int_{\mathbb{R}^2} \text{Tr} [\gamma_b \Pi_{z,n} \mathcal{V}_{z,l}] \frac{d\mathbf{z}}{2\pi l_b^2} \right| \leq C(l) \left\| d^{l+1} W \right\|_{L^1} l_b^l (n+1)^{\frac{l+1}{2}}$$

and thus, combining with (5.27) we get

$$\left| \int_{\mathbb{R}^2} \mathcal{E}_0^l(\mathbf{z}) d\mathbf{z} \right| \leq C(l) \left\| d^{l+1} W \right\|_{L^1} l_b^{l-2} \sum_{n=0}^M (n+1)^{\frac{l+1}{2}} \leq C(l) \left\| d^{l+1} W \right\|_{L^1} l_b^{l-2} M^{\frac{l+3}{2}}$$

We conclude by using (5.26) and proceeding as in the proof of Proposition 5.1, picking $\varphi = 1$ and replacing by the above the estimate of the $\mathcal{E}_0^l(\mathbf{z})$ term therein. \square

5.2. Dynamics of the first reduced density. In this part we conclude the proof of Theorem 2.3. What is left to do is to put together the estimates of Sections 4 and 5.1. We start by explaining how to fix the Landau level cut-off M , as hinted at in Remark 5.2.

Lemma 5.7 (Fixing the Landau level cut-off).

Let $\alpha > 0$, $k \geq 0$, $\varphi \in L^1(\mathbb{R}_+)$, $\gamma_b \in L^\infty(\mathbb{R}_+, \mathcal{L}^1(L^2(\mathbb{R}^2)))$ and assume

$$\begin{aligned} & \forall t \in \mathbb{R}_+, \gamma_b(t) \geq 0, \text{Tr} [\gamma_b(t)] = 1 \\ & \int_{\mathbb{R}_+} |\varphi(t)| \text{Tr} [\gamma_b(t) \mathcal{L}_b^k] dt < \infty \end{aligned}$$

then $\exists M(\varphi) \in \llbracket l_b^{-\alpha}, 2l_b^{-\alpha} \rrbracket$ such that

$$\int_{\mathbb{R}_+} |\varphi(t)| \text{Tr} [\gamma_b(t) \mathcal{L}_b^k \Pi_{M(\varphi)-l+1: M(\varphi)+l}] dt \leq C(l) l_b^\alpha \int_{\mathbb{R}_+} |\varphi(t)| \text{Tr} [\gamma_b(t) \mathcal{L}_b^k] dt$$

Proof. Assume for contradiction that $\forall M \in \llbracket \lfloor l_b^{-\alpha} \rfloor, 2 \lfloor l_b^{-\alpha} \rfloor \rrbracket$,

$$\int_{\mathbb{R}_+} |\varphi(t)| \operatorname{Tr} [\gamma_b(t) \mathcal{L}_b^k \Pi_{M-l+1:M+l}] dt > \frac{4l}{M} \int_{\mathbb{R}_+} |\varphi(t)| \operatorname{Tr} [\gamma_b(t) \mathcal{L}_b^k] dt$$

then

$$\begin{aligned} \int_{\mathbb{R}_+} |\varphi(t)| \operatorname{Tr} [\gamma_b(t) \mathcal{L}_b^k] dt &\geq \frac{1}{2l} \sum_{M=\lfloor l_b^{-\alpha} \rfloor}^{2\lfloor l_b^{-\alpha} \rfloor} \int_{\mathbb{R}_+} |\varphi(t)| \operatorname{Tr} [\gamma_b(t) \mathcal{L}_b^k \Pi_{M-l+1:M+l}] dt \\ &> \int_{\mathbb{R}_+} |\varphi(t)| \operatorname{Tr} [\gamma_b(t) \mathcal{L}_b^k] dt \sum_{M=\lfloor l_b^{-\alpha} \rfloor}^{2\lfloor l_b^{-\alpha} \rfloor} \frac{2}{M} \\ &\geq \frac{\lfloor l_b^{-\alpha} \rfloor + 1}{\lfloor l_b^{-\alpha} \rfloor} \int_{\mathbb{R}_+} |\varphi(t)| \operatorname{Tr} [\gamma_b(t) \mathcal{L}_b^k] dt \end{aligned}$$

which yields the desired contradiction. \square

Let us now summarize some findings of the previous subsection:

Lemma 5.8 (Summary of semi-classical approximation).

Let $\frac{2}{3} < \alpha < 1$ and $\gamma_b \in L^\infty(\mathbb{R}_+, \mathcal{L}^1(L^2(\mathbb{R}^2)))$ be a solution of (2.9) with initial datum satisfying

$$\operatorname{Tr} [\gamma_b(0)] = 1, \quad 0 \leq \gamma_b(0) \leq 2\pi l_b^2, \quad \operatorname{Tr} [\gamma_b(0) H_b(0)] < C$$

with $V, w \in W^{4,\infty}(\mathbb{R}^2)$, then $\forall \varphi \in L^1(\mathbb{R}_+, L^1 \cap W^{1,\infty}(\mathbb{R}^2))$, $\exists M = \mathcal{O}(l_b^{-\alpha})$ such that

- $\forall t \geq 0, \forall \mu \in L^\infty \cap H^1(\mathbb{R}^2)$,

$$\left| \int_{\mathbb{R}^2} \mu \left(\rho_{\gamma_b}(t) - \rho_{\gamma_b}^{\text{sc}, \leq M}(t) \right) \right| \leq C \left(\|\mu\|_{L^\infty} + \|\nabla \mu\|_{L^2} \right) \left(\left| \operatorname{Tr} [\gamma_b(0) H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \right) l_b^{\frac{\alpha}{2}} \quad (5.31)$$

$$\left| \int_{\mathbb{R}^2} \mu \left(\rho_{\gamma_b}(t) - \rho_b(t) \right) \right| \leq C \left(\|\mu\|_{L^\infty} + \|\nabla \mu\|_{L^2} \right) \left(\left| \operatorname{Tr} [\gamma_b(0) H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \right) l_b^{\frac{\alpha}{2}} \quad (5.32)$$

- moreover

$$\begin{aligned} \left| \int_{\mathbb{R}_+ \times \mathbb{R}^2} \varphi \operatorname{DRIFT}_{\rho_{\gamma_b}} \left(\rho_{\gamma_b}^{\text{sc}, \leq M} \right) \right| &\leq C \|\varphi\|_{L^1(\mathbb{R}_+, L^1 \cap W^{1,\infty}(\mathbb{R}^2))} \left(\|V\|_{W^{4,\infty}} + \|w\|_{W^{4,\infty}} \right) \\ &\left(\left| \operatorname{Tr} [\gamma_b(0) H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \right) \left(l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha-1} \right) \end{aligned} \quad (5.33)$$

- if $p > 3$ and $\operatorname{Tr} [\gamma_b(t) |X|^p] < \infty$, then

$$W_1 \left(\rho_{\gamma_b}(t), \rho_b(t) \right) \leq C_1(t, p, V, w) \left(l_b^{1-\frac{\alpha}{2}-\frac{6+\alpha}{2p-4}} + l_b^{\frac{\alpha}{2}} \right) \quad (5.34)$$

with

$$C_1(t, p, V, w) := C(p) \left(1 + \operatorname{Tr} [\gamma_b(t) |X|^p] \right) \left(\left| \operatorname{Tr} [\gamma_b(0) H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \right)$$

- if $V, w \in W^9$ and $d^9 V, d^9 W \in L^1(\mathbb{R}^2)$,

$$\left| \int_{\mathbb{R}_+ \times \mathbb{R}^2} \varphi \text{DRIFT}_{\rho_{\gamma_b}}(\rho_b) \right| \leq C \|\varphi\|_{L^1(\mathbb{R}_+, L^1 \cap W^{1,\infty}(\mathbb{R}^2))} \left(\left| \text{Tr} [\gamma_b(0) H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \right) \\ \left(\|V\|_{W^{9,\infty}} + \|d^9 V\|_{L^1} + \|w\|_{W^{9,\infty}} + \|d^9 w\|_{L^1} \right) \left(l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha-1} \right) \quad (5.35)$$

Proof. By Lemma 3.8,

$$\text{Tr} [\gamma_b(t)] = 1 \text{ and } 0 \leq \gamma_b(t) \leq 2\pi l_b^2$$

then with Lemma 3.9 applied to $W := V + \frac{1}{2}w \star \rho_{\gamma_b(t)}$,

$$\begin{aligned} \text{Tr} [\gamma_b(t) \mathcal{L}_b] &\leq \left| \text{Tr} [\gamma_b(t) H_b(t)] \right| + \left\| V + \frac{1}{2}w \star \rho_{\gamma_b(t)} \right\|_{L^\infty} \\ &= \left| \text{Tr} [\gamma_b(0) H_b(0)] \right| + \left\| V + \frac{1}{2}w \star \rho_{\gamma_b(t)} \right\|_{L^\infty} \\ &\leq \left| \text{Tr} [\gamma_b(0) H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \end{aligned} \quad (5.36)$$

Moreover

$$\left\| V + w \star \rho_{\gamma_b}(t) \right\|_{W^{4,\infty}} \leq \|V\|_{W^{4,\infty}} + \|w\|_{W^{4,\infty}} \quad (5.37)$$

For this proof we choose $M := M(t \mapsto \|\varphi(t)\|_{L^1 \cap W^{1,\infty}(\mathbb{R}^2)})$ according to Lemma 5.7.

Proof of (5.31): using (4.4) for $k = 1$, (5.36), $\sqrt{\text{Tr} [\gamma_b \mathcal{L}_b]} \leq \text{Tr} [\gamma_b \mathcal{L}_b]$ and $M = \mathcal{O}(l_b^{-\alpha})$,

$$\begin{aligned} &\left| \int_{\mathbb{R}^2} \mu \left(\rho_{\gamma_b}(t) - \rho_{\gamma_b}^{sc, \leq M}(t) \right) \right| \\ &\leq C (\|\mu\|_{L^\infty} + \|\nabla \mu\|_{L^2}) \left(M^{-\frac{1}{2}} + l_b \sqrt{M} \right) \text{Tr} [\gamma_b(t) \mathcal{L}_b] \\ &\leq C (\|\mu\|_{L^\infty} + \|\nabla \mu\|_{L^2}) \left(\left| \text{Tr} [\gamma_b(0) H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \right) \left(l_b^{\frac{\alpha}{2}} + l_b^{1-\frac{\alpha}{2}} \right) \end{aligned}$$

we obtain (5.31) by noticing that

$$\alpha < 1 \implies \frac{\alpha}{2} \leq 1 - \frac{\alpha}{2}$$

To obtain (5.32) we only need to incorporate

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \mu \left(\rho_b - \rho_{\gamma_b}^{sc, \leq M} \right) \right| &= \left| \int_{\mathbb{R}^2} \mu \rho_b (1 - \text{Tr} [\gamma_b \Pi_{\leq M}]) \right| \leq \|\mu\|_{L^\infty} \text{Tr} [\gamma_b \Pi_{>M}] \\ &\leq \|\mu\|_{L^\infty} M^{-k} \text{Tr} [\gamma_b \Pi_{>M} \mathcal{L}_b^k] \end{aligned}$$

in the argument and discard this extra error term since $M \geq \sqrt{M}$.

Proof of (5.33): integrating Equation (5.1) in time with $W := V + w \star \rho_{\gamma_b}$, $l := 3$ and using (5.37), we find

$$\left| \int_{\mathbb{R}_+ \times \mathbb{R}^2} \varphi \text{DRIFT}_{\rho_{\gamma_b}} \left(\rho_{\gamma_b}^{sc, \leq M} \right) \right| \leq C (\|V\|_{W^{4,\infty}} + \|w\|_{W^{4,\infty}})$$

$$\int_{\mathbb{R}_+} \|\varphi(t)\|_{L^1 \cap W^{1,\infty}} \left(\text{Tr} [\gamma_b(t) \mathcal{L}_b] \left(l_b^2 M^2 + l_b \sqrt{M} \right) + \frac{\text{Tr} [\gamma_b(t) \mathcal{L}_b \Pi_{M-2; M+3}]}{l_b \sqrt{M}} \right) dt$$

Then, using Lemma 5.7 and (5.36),

$$\begin{aligned} & \left| \int_{\mathbb{R}_+ \times \mathbb{R}^2} \varphi \text{DRIFT}_{\rho_{\gamma_b}} \left(\rho_{\gamma_b}^{sc, \leq M} \right) \right| \\ & \leq C (\|V\|_{W^{4,\infty}} + \|w\|_{W^{4,\infty}}) \left(l_b^{2-2\alpha} + l_b^{1-\frac{\alpha}{2}} + l_b^{\frac{3}{2}\alpha-1} \right) \int_{\mathbb{R}_+} \|\varphi(t)\|_{L^1 \cap W^{1,\infty}} \text{Tr} [\gamma_b(t) \mathcal{L}_b] dt \\ & \leq C \|\varphi\|_{L^1(\mathbb{R}_+, L^1 \cap W^{1,\infty}(\mathbb{R}^2))} (\|V\|_{W^{4,\infty}} + \|w\|_{W^{4,\infty}}) \left(\left| \text{Tr} [\gamma_b(0) H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \right) \\ & \quad \left(l_b^{2-2\alpha} + l_b^{1-\frac{\alpha}{2}} + l_b^{\frac{3}{2}\alpha-1} \right) \end{aligned}$$

This leads to (5.33) after noticing that

$$\frac{2}{3} < \alpha \implies 2 - 2\alpha \leq 1 - \frac{\alpha}{2}$$

Proof of (5.34): let $\beta > 0$, we have

$$\frac{2}{3} < \alpha < 1 \implies 4\sqrt{M}l_b = \mathcal{O} \left(l_b^{1-\frac{\alpha}{2}} \right) \ll l_b^{\frac{\alpha}{2}} \ll l_b^{-\beta}$$

so we can apply Proposition (4.4) with $k = 1$ and obtain

$$\begin{aligned} W_1 \left(\rho_{\gamma_b}(t), \rho_b(t) \right) & \leq C(p) \left(1 + \text{Tr} [\gamma_b(t) |X|^p] \right) \text{Tr} [\gamma_b(t) \mathcal{L}_b] \\ & \quad \left(M^{-1} + l_b^{1-\beta} \sqrt{M} + M^{-\frac{1}{2}} + l_b^{\beta(p-1)} + M l_b^{\beta(p-3)-2} \right) \\ & \leq C(p) \|\mu\|_{W^{1,\infty}} \left(1 + \text{Tr} [\gamma_b(t) |X|^p] \right) \left(\left| \text{Tr} [\gamma_b(0) H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \right) \\ & \quad \left(l_b^\alpha + l_b^{1-\frac{\alpha}{2}-\beta} + l_b^{\frac{\alpha}{2}} + l_b^{\beta(p-1)} + l_b^{\beta(p-3)-2-\alpha} \right) \end{aligned}$$

Remark that $\alpha > \frac{\alpha}{2}$ and

$$\beta(p-1) \geq \beta(p-3) - 2 - \alpha$$

so that optimizing the above with respect to β leads to

$$\beta(p-3) - 2 - \alpha = 1 - \frac{\alpha}{2} - \beta \implies \beta = \frac{3 + \frac{\alpha}{2}}{p-2}$$

which finally gives (5.34).

Proof of (5.35): with (4.8) and then (5.36), we note that ρ_b is well defined for large enough M as

$$\begin{aligned} \text{Tr} [\gamma_b(t) \Pi_{\leq M}] & = 1 - \text{Tr} [\gamma_b(t) \Pi_{> M}] \geq 1 - \frac{\text{Tr} [\gamma_b(t) \mathcal{L}_b \Pi_{> M}]}{M} \geq 1 - \frac{\text{Tr} [\gamma_b(t) \mathcal{L}_b]}{M} \\ & \geq 1 - \frac{\left| \text{Tr} [\gamma_b(0) H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty}}{M} \end{aligned} \tag{5.38}$$

Recalling (4.18) we compute

$$\text{DRIFT}_{\rho_{\gamma_b}}(\rho_b) = \frac{\text{DRIFT}_{\rho_{\gamma_b}}(\rho_{\gamma_b}^{sc, \leq M})}{\text{Tr}[\gamma_b \Pi_{\leq M}]} - \rho_b \frac{\partial_t \text{Tr}[\gamma_b \Pi_{\leq M}]}{\text{Tr}[\gamma_b \Pi_{\leq M}]} \quad (5.39)$$

With (5.25) applied to $W := V + w \star \rho_{\gamma_b}(t)$, $k = 1$, $l \geq 3$, noticing that

$$\|d^{l+1}W\|_{L^1} \leq \|d^{l+1}V\|_{L^1} + \|d^{l+1}w\|_{L^1}$$

and following the same computations as that leading to (5.33) above, we find

$$\begin{aligned} & \left| \int_{\mathbb{R}_+ \times \mathbb{R}^2} \varphi \rho_b \partial_t \text{Tr}[\gamma_b \Pi_{\leq M}] \right| \\ & \leq \int_{\mathbb{R}_+} \|\varphi(t)\|_{L^\infty} \left| \partial_t \text{Tr}[\gamma_b \Pi_{\leq M}] \right| dt \leq \int_{\mathbb{R}_+} \|\varphi(t)\|_{L^1 \cap W^{1,\infty}} \left| \partial_t \text{Tr}[\gamma_b \Pi_{\leq M}] \right| dt \\ & \leq C \|\varphi\|_{L^1(\mathbb{R}_+, L^1 \cap W^{1,\infty}(\mathbb{R}^2))} \left(\|V\|_{W^{l+1,\infty}} + \|d^{l+1}V\|_{L^1} + \|w\|_{W^{l+1,\infty}} + \|d^{l+1}w\|_{L^1} \right) \\ & \quad \left(\left| \text{Tr}[\gamma_b(0)H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \right) \left(l_b^{l-2-\alpha\frac{l+3}{2}} + l_b^{1-\frac{\alpha}{2}} + l_b^{\frac{3}{2}\alpha-1} \right) \end{aligned}$$

Combining this with (5.39) and (5.33)

$$\begin{aligned} & \left| \int_{\mathbb{R}_+ \times \mathbb{R}^2} \varphi \text{DRIFT}_{\rho_{\gamma_b}}(\rho_b) \right| \leq C \frac{\|\varphi\|_{L^1(\mathbb{R}_+, L^1 \cap W^{1,\infty}(\mathbb{R}^2))}}{\text{Tr}[\gamma_b \Pi_{>M}]} \left(\left| \text{Tr}[\gamma_b(0)H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \right) \\ & \quad \left(\|V\|_{W^{l+1,\infty}} + \|d^{l+1}V\|_{L^1} + \|w\|_{W^{l+1,\infty}} + \|d^{l+1}w\|_{L^1} \right) \left(l_b^{l-2-\alpha\frac{l+3}{2}} + l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha-1} \right) \end{aligned}$$

taking $l = 8$,

$$\alpha < 1 \implies l - 2 - \alpha \frac{l+3}{2} \geq \frac{l-7}{2} = \frac{1}{2} \geq \frac{3}{2}\alpha - 1$$

we can discard an error term and conclude with (5.38). \square

Next we turn to the

Proof of Theorem 2.3. Let $\varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$ and choose M according to Lemma 5.8.

Step 1: decomposition. With an integration by parts,

$$\begin{aligned} & \int_{\mathbb{R}^2} \varphi(0) \rho_{\gamma_b}(0) - \int_{\mathbb{R}_+ \times \mathbb{R}^2} \rho_{\gamma_b} \text{DRIFT}_{\rho_{\gamma_b}}(\varphi) \\ & = \int_{\mathbb{R}^2} \varphi(0) \left(\rho_{\gamma_b}(0) - \rho_{\gamma_b}^{sc, \leq M}(0) \right) + \int_{\mathbb{R}^2} \varphi(0) \rho_{\gamma_b}^{sc, \leq M}(0) - \int_{\mathbb{R}_+ \times \mathbb{R}^2} \left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc, \leq M} \right) \text{DRIFT}_{\rho_{\gamma_b}}(\varphi) \\ & \quad - \int_{\mathbb{R}_+ \times \mathbb{R}^2} \rho_{\gamma_b}^{sc, \leq M} \text{DRIFT}_{\rho_{\gamma_b}}(\varphi) \\ & = \int_{\mathbb{R}^2} \varphi(0) \left(\rho_{\gamma_b}(0) - \rho_{\gamma_b}^{sc, \leq M}(0) \right) + \int_{\mathbb{R}_+ \times \mathbb{R}^2} \varphi \text{DRIFT}_{\rho_{\gamma_b}}(\rho_{\gamma_b}^{sc, \leq M}) \\ & \quad - \int_{\mathbb{R}_+ \times \mathbb{R}^2} \left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc, \leq M} \right) \text{DRIFT}_{\rho_{\gamma_b}}(\varphi) \end{aligned} \quad (5.40)$$

Step 2: third term in (5.40). For $t \in \mathbb{R}_+$, (5.31) implies

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \left(\rho_{\gamma_b}(t) - \rho_{\gamma_b}^{sc, \leq M}(t) \right) \text{DRIFT}_{\rho_{\gamma_b}}(\varphi)(t) \right| &\leq C \left(\left| \text{Tr} [\gamma_b(0) H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \right) \\ &\left(\left\| \text{DRIFT}_{\rho_{\gamma_b}}(\varphi)(t) \right\|_{L^\infty} + \left\| \nabla \text{DRIFT}_{\rho_{\gamma_b}}(\varphi)(t) \right\|_{L^2} \right) l_b^{\frac{\alpha}{2}} \end{aligned} \quad (5.41)$$

Hence we estimate

$$\begin{aligned} \left\| \text{DRIFT}_{\rho_{\gamma_b}}(\varphi)(t) \right\|_{L^\infty} &= \left\| \partial_t \varphi(t) + \nabla^\perp (V + w \star \rho_{\gamma_b(t)}) \cdot \nabla \varphi(t) \right\|_{L^\infty} \\ &\leq \left\| \partial_t \varphi(t) \right\|_{L^\infty} + (\| \nabla V \|_{L^\infty} + \| \nabla w \|_{L^\infty}) \| \nabla \varphi(t) \|_{L^\infty} \end{aligned}$$

and

$$\begin{aligned} &\left\| \nabla \text{DRIFT}_{\rho_{\gamma_b}}(\varphi)(t) \right\|_{L^2} \\ &= \left\| \partial_t \nabla \varphi(t) + \nabla \left(\nabla^\perp (V + w \star \rho_{\gamma_b(t)}) \cdot \nabla \varphi(t) \right) \right\|_{L^2} \\ &\leq \left\| \partial_t \nabla \varphi(t) \right\|_{L^2} + (\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}}) \| \nabla \varphi(t) \|_{L^2} + (\|dV\|_{L^\infty} + \|dw\|_{L^\infty}) \left\| d^2 \varphi(t) \right\|_{L^2} \end{aligned}$$

so

$$\begin{aligned} &\int_{\mathbb{R}_+} \left(\left\| \text{DRIFT}_{\rho_{\gamma_b}}(\varphi)(t) \right\|_{L^\infty} + \left\| \nabla \text{DRIFT}_{\rho_{\gamma_b}}(\varphi)(t) \right\|_{L^2} \right) dt \\ &\leq (1 + \|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}}) \\ &\int_{\mathbb{R}_+} \left(\left\| \partial_t \varphi(t) \right\|_{L^\infty} + \| \nabla \varphi(t) \|_{L^\infty} + \left\| \partial_t \nabla \varphi(t) \right\|_{L^2} + \| \nabla \varphi(t) \|_{L^2} + \left\| d^2 \varphi(t) \right\|_{L^2} \right) dt \\ &\leq (1 + \|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}}) \| \varphi \|_{W^{1,1}(\mathbb{R}_+, W^{1,\infty} \cap H^2(\mathbb{R}^2))} \end{aligned}$$

thus (5.41) implies

$$\begin{aligned} \left| \int_{\mathbb{R}_+ \times \mathbb{R}^2} \left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc, \leq M} \right) \text{DRIFT}_{\rho_{\gamma_b}}(\varphi) \right| &\leq C \| \varphi \|_{W^{1,1}(\mathbb{R}_+, W^{1,\infty} \cap H^2(\mathbb{R}^2))} (1 + \|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}}) \\ &\left(\left| \text{Tr} [\gamma_b(0) H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \right) l_b^{\frac{\alpha}{2}} \end{aligned} \quad (5.42)$$

Step 3: conclusion. Inserting (5.31) for $\mu := \varphi(0)$, (5.33) and (5.42) in (5.40), we get

$$\left| \int_{\mathbb{R}^2} \varphi(0) \rho_{\gamma_b}(0) - \int_{\mathbb{R}_+ \times \mathbb{R}^2} \rho_{\gamma_b} \text{DRIFT}_{\rho_{\gamma_b}}(\varphi) \right| \leq C(\varphi, V, w) \left(l_b^{\frac{\alpha}{2}} + l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha-1} \right)$$

Finally,

$$\alpha \mapsto \min \left(\frac{\alpha}{2}, 2 - 2\alpha, \frac{3}{2}\alpha - 1 \right)$$

is maximal at $2 - 2\alpha = -1 + \frac{3}{2}\alpha$, so we conclude by taking

$$\alpha := \frac{6}{7}.$$

□

6. STABILITY FOR PERTURBED CLASSICAL FLOWS

Our limit model, the drift equation (2.5), enjoys stability estimates with respect to the initial data, by transport arguments à la Dobrushin [17] (see e.g. [23, Section 1.4] for exposition of this material). In this section we include a small source term in this formalism, with the goal of treating the error obtained in Theorem 2.3 in this way. This will provide estimates on the difference between the density of the solution to the quantum evolution (2.2) and the solution of the limit equation, leading to the proof of Theorem 2.4.

6.1. A Dobrushin-type Estimate. We aim at comparing a solution to the drift equation (2.5) to a solution to a similar equation with a small source term and a possibly different initial datum.

Let S_b be function of time and space. Let ρ_b be the solution to

$$\begin{aligned} \partial_t \rho_b + \nabla^\perp V \cdot \nabla \rho_b + \nabla^\perp w \star \rho_b \cdot \nabla \rho_b &= S_b \\ \rho_b(t_0, \bullet) &=: \rho_{b,0} \in L^1(\mathbb{R}^2, \mathbb{R}_+), \|\rho_{b,0}\|_{L^1} = 1 \end{aligned} \quad (6.1)$$

and ρ the solution to

$$\begin{aligned} \partial_t \rho + \nabla^\perp V \cdot \nabla \rho + \nabla^\perp w \star \rho \cdot \nabla \rho &= 0 \\ \rho(t_0, \bullet) &=: \rho_0 \in L^1(\mathbb{R}^2, \mathbb{R}_+), \|\rho_0\|_{L^1} = 1 \end{aligned} \quad (6.2)$$

Let $Z_\rho(t, t_0, \mathbf{z}_0)$ be the flow defined by

$$\partial_t Z_\rho(t, t_0, \mathbf{z}_0) = \nabla^\perp V(Z_\rho(t, t_0, \mathbf{z}_0)) + \nabla^\perp w \star \rho(t, \bullet)(Z_\rho(t, t_0, \mathbf{z}_0)) \quad (6.3)$$

$$Z_\rho(t_0, t_0, \mathbf{z}_0) = \mathbf{z}_0 \quad (6.4)$$

We claim the following, which is classical for $S_b(t, \mathbf{z}) \equiv 0$

Proposition 6.1 (A Dobrushin-type Estimate).

With the notation above,

$$W_1(\rho_b(t), \rho(t)) \leq e^{2(\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}})|t-t_0|} \left(W_1(\rho_{b,0}, \rho_0) + \mathcal{E}_{S_b}(t) \right)$$

with

$$\mathcal{E}_{S_b}(t) := \left| \int_{t_0}^t \int_{t_0}^\tau \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla^\perp w \left(Z_{\rho_b}(\tau, t_0, \mathbf{y}) - Z_{\rho_b}(\tau, \nu, \mathbf{x}) \right) S_b(\nu, \mathbf{x}) \rho_{b,0}(\mathbf{y}) d\tau d\nu d\mathbf{x} d\mathbf{y} \right| \quad (6.5)$$

We will use characteristics for the above equations. For a general, time-dependent, potential $W \in L^\infty(\mathbb{R}_+, W^{2,\infty}(\mathbb{R}^2))$ the PDE

$$\partial_t \rho(t, \mathbf{z}) + \nabla^\perp W(t, \mathbf{z}) \cdot \nabla_{\mathbf{z}} \rho(t, \mathbf{z}) = 0$$

is a transport equation with velocity field $\nabla^\perp W$, thus we define the flow

$$Z : \begin{array}{ccc} \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 & \rightarrow & \mathbb{R}^2 \\ t, t_0, \mathbf{z}_0 & \mapsto & Z(t, t_0, \mathbf{z}_0) \end{array}$$

as the unique (by the Cauchy-Lipschitz theorem) solution of the ODEs

$$\begin{aligned} \partial_t Z(t, t_0, \mathbf{z}_0) &= \nabla^\perp W(t, Z(t, t_0, \mathbf{z}_0)) \\ Z(t_0, t_0, \mathbf{z}_0) &= \mathbf{z}_0 \end{aligned} \quad (6.6)$$

We denote $Z(t, t_0, \bullet)_* \sigma$ the push-forward of a measure σ by the flow. We then have the classical

Lemma 6.2 (Characteristics).

Let

$$\rho(t, \bullet) := Z(t, t_0, \bullet)_* \rho_0 + \int_{t_0}^t Z(t, \tau, \bullet)_* S(\tau, \bullet) d\tau \quad (6.7)$$

then

$$\begin{aligned} \partial_t \rho(t, \mathbf{z}) + \nabla^\perp W(t, \mathbf{z}) \cdot \nabla_{\mathbf{z}} \rho(t, \mathbf{z}) &= S(t, \mathbf{z}) \\ \rho(t_0, \bullet) &= \rho_0 \end{aligned}$$

Proof. Let $t, \tau \in \mathbb{R}$ the flow satisfies

$$Z(t, \tau, Z(\tau, t, \bullet)) = \text{Id}_{\mathbb{R}^2} \quad (6.8)$$

so

$$Z(t, \tau, \bullet)^{-1} = Z(\tau, t, \bullet)$$

Thus (6.7) can be rewritten

$$\rho(t, \mathbf{z}) = \rho_0 (Z(t_0, t, \mathbf{z})) + \int_{t_0}^t S(\tau, Z(\tau, t, \mathbf{z})) d\tau$$

Now we compute

$$\begin{aligned} \partial_t \rho(t, \mathbf{z}) &= \partial_{t_0} Z(t_0, t, \mathbf{z}) \cdot \nabla \rho_0 (Z(t_0, t, \mathbf{z})) + S(t, Z(t, t, \mathbf{z})) \\ &\quad + \int_{t_0}^t \partial_{t_0} Z(\tau, t, \mathbf{z}) \cdot \nabla_{\mathbf{z}} S(\tau, Z(\tau, t, \mathbf{z})) d\tau \\ &= S(t, \mathbf{z}) + \partial_{t_0} Z(t_0, t, \mathbf{z}) \cdot \nabla \rho_0 (Z(t_0, t, \mathbf{z})) + \int_{t_0}^t \partial_{t_0} Z(\tau, t, \mathbf{z}) \cdot \nabla_{\mathbf{z}} S(\tau, Z(\tau, t, \mathbf{z})) d\tau \end{aligned}$$

and

$$\begin{aligned} \nabla^\perp W(t, \mathbf{z}) \cdot \nabla_{\mathbf{z}} \rho(t, \mathbf{z}) &= d\rho(t, \bullet)_z \nabla^\perp W(t, \mathbf{z}) \\ &= d\rho_0(Z(t_0, t, \mathbf{z})) dZ(t_0, t, \bullet)_z \nabla^\perp W(t, \mathbf{z}) \\ &\quad + \int_{t_0}^t dS(\tau, \bullet)_{Z(\tau, t, \mathbf{z})} dZ(\tau, t, \bullet)_z \nabla^\perp W(t, \mathbf{z}) d\tau \\ &= \nabla \rho_0(Z(t_0, t, \mathbf{z})) \cdot dZ(t_0, t, \bullet)_z \nabla^\perp W(t, \mathbf{z}) \\ &\quad + \int_{t_0}^t \nabla_{\mathbf{z}} S(\tau, Z(\tau, t, \mathbf{z})) \cdot dZ(\tau, t, \bullet)_z \nabla^\perp W(t, \mathbf{z}) d\tau \end{aligned}$$

Put together, the above give

$$\begin{aligned} \partial_t \rho(t, \mathbf{z}) + \nabla^\perp W(t, \mathbf{z}) \cdot \nabla_{\mathbf{z}} \rho(t, \mathbf{z}) &= S(t, \mathbf{z}) \\ &\quad + \nabla \rho_0(Z(t_0, t, \mathbf{z})) \cdot \left(\partial_{t_0} Z(t_0, t, \mathbf{z}) + dZ(t_0, t, \bullet)_z \nabla^\perp W(t, \mathbf{z}) \right) \\ &\quad + \int_{t_0}^t \nabla_{\mathbf{z}} S(\tau, Z(\tau, t, \mathbf{z})) \cdot \left(\partial_{t_0} Z(\tau, t, \mathbf{z}) + dZ(\tau, t, \bullet)_z \nabla^\perp W(t, \mathbf{z}) \right) d\tau \end{aligned} \quad (6.9)$$

But (6.8) also implies

$$\begin{aligned} dZ(t, \tau, \bullet)_{Z(\tau, t, \mathbf{z})} dZ(\tau, t, \bullet)_z &= \text{Id}_{\mathbb{R}^2} \\ \frac{d}{dt} Z(t, \tau, Z(\tau, t, \mathbf{z})) &= \partial_t Z(t, \tau, Z(\tau, t, \mathbf{z})) + dZ(t, \tau, \bullet)_{Z(\tau, t, \mathbf{z})} \partial_{t_0} Z(\tau, t, \mathbf{z}) = 0 \end{aligned}$$

therefore, combining with (6.6) and (6.8) leads to

$$\begin{aligned}\partial_{t_0} Z(\tau, t, \mathbf{z}) &= - \left(dZ(t, \tau, \bullet)_{Z(\tau, t, \mathbf{z})} \right)^{-1} \partial_t Z(t, \tau, Z(\tau, t, \mathbf{z})) = -dZ(\tau, t, \bullet)_{\mathbf{z}} \partial_t Z(t, \tau, Z(\tau, t, \mathbf{z})) \\ &= -dZ(\tau, t, \bullet)_{\mathbf{z}} \nabla^\perp W(t, Z(t, \tau, Z(\tau, t, \mathbf{z}))) = -dZ(\tau, t, \bullet)_{\mathbf{z}} \nabla^\perp W(t, \mathbf{z})\end{aligned}$$

so we conclude by seeing that the last two terms in (6.9) are null. \square

Next we introduce couplings as in (2.12) to state the

Lemma 6.3 (Coupling).

Let ρ_b and ρ be as in (6.1) and (6.2). Let π_b be a coupling between $\rho_{b,0}$ and ρ_0 and introduce the notation

$$D_{\pi_b}(\tau) := \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left| Z_{\rho_b}(\tau, t_0, \mathbf{x}) - Z_\rho(\tau, t_0, \mathbf{y}) \right| d\pi_b(\mathbf{x}, \mathbf{y})$$

then

$$D_{\pi_b}(t) \leq \left(D_{\pi_b}(t_0) + \mathcal{E}_{S_b}(t) \right) e^{2(\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}})|t-t_0|}$$

with the same notation as in (6.5).

Proof. Starting from (6.3), (6.4),

$$\begin{aligned}& Z_{\rho_b}(t, t_0, \mathbf{z}_{b,0}) - Z_\rho(t, t_0, \mathbf{z}_0) - (\mathbf{z}_{b,0} - \mathbf{z}_0) \\ &= \int_{t_0}^t \nabla^\perp (V + w \star \rho_b(\tau, \bullet)) (Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0})) d\tau - \int_{t_0}^t \nabla^\perp (V + w \star \rho(\tau, \bullet)) (Z_\rho(\tau, t_0, \mathbf{z}_0)) d\tau \\ &= \int_{t_0}^t \left(\nabla^\perp V (Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0})) - \nabla^\perp V (Z_\rho(\tau, t_0, \mathbf{z}_0)) \right) d\tau \\ &+ \int_{t_0}^t \int_{\mathbb{R}^2} \nabla^\perp w (Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0}) - \mathbf{x}) \rho_b(\tau, \mathbf{x}) d\tau d\mathbf{x} - \int_{t_0}^t \int_{\mathbb{R}^2} \nabla^\perp w (Z_\rho(\tau, t_0, \mathbf{z}_0) - \mathbf{y}) \rho(\tau, \mathbf{y}) d\tau d\mathbf{y}\end{aligned}$$

Using Lemma 6.2,

$$\rho_b(\tau, \mathbf{x}) = \rho_{b,0} (Z_{\rho_b}(t_0, \tau, \mathbf{x})) + \int_{t_0}^\tau S_b(v, Z_{\rho_b}(v, \tau, \mathbf{x})) dv, \quad \rho(\tau, \mathbf{y}) = \rho_0 (Z_\rho(t_0, \tau, \mathbf{y}))$$

and inserting this in the above leads to

$$\begin{aligned}& Z_{\rho_b}(t, t_0, \mathbf{z}_{b,0}) - Z_\rho(t, t_0, \mathbf{z}_0) - (\mathbf{z}_{b,0} - \mathbf{z}_0) \\ &= \int_{t_0}^t \left(\nabla^\perp V (Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0})) - \nabla^\perp V (Z_\rho(\tau, t_0, \mathbf{z}_0)) \right) d\tau \\ &+ \int_{t_0}^t \int_{\mathbb{R}^2} \nabla^\perp w (Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0}) - \mathbf{x}) \rho_{b,0} (Z_{\rho_b}(t_0, \tau, \mathbf{x})) d\tau d\mathbf{x} \\ &- \int_{t_0}^t \int_{\mathbb{R}^2} \nabla^\perp w (Z_\rho(\tau, t_0, \mathbf{z}_0) - \mathbf{y}) \rho_0 (Z_\rho(t_0, \tau, \mathbf{y})) d\tau d\mathbf{y} \\ &+ \int_{t_0}^t \int_{t_0}^\tau \int_{\mathbb{R}^2} \nabla^\perp w (Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0}) - \mathbf{x}) S_b(v, Z_{\rho_b}(v, \tau, \mathbf{x})) d\tau dv d\mathbf{x}\end{aligned}$$

Since the flow is divergence free it preserves volume, thus with the changes of variable

$$Z_{\rho_b}(t_0, \tau, \mathbf{x}) \mapsto \mathbf{x}, \quad Z_\rho(t_0, \tau, \mathbf{y}) \mapsto \mathbf{y}$$

and again $Z_{\rho_b}(\nu, \tau, \mathbf{x}) \mapsto \mathbf{x}$ for the last integral,

$$\begin{aligned}
& Z_{\rho_b}(t, t_0, \mathbf{z}_{b,0}) - Z_{\rho}(t, t_0, \mathbf{z}_0) - (\mathbf{z}_{b,0} - \mathbf{z}_0) \\
&= \int_{t_0}^t \left(\nabla^\perp V \left(Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0}) \right) - \nabla^\perp V \left(Z_{\rho}(\tau, t_0, \mathbf{z}_0) \right) \right) d\tau \\
&\quad + \int_{t_0}^t \int_{\mathbb{R}^2} \nabla^\perp w \left(Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0}) - Z_{\rho_b}(\tau, t_0, \mathbf{x}) \right) \rho_{b,0}(\mathbf{x}) d\tau d\mathbf{x} \\
&\quad - \int_{t_0}^t \int_{\mathbb{R}^2} \nabla^\perp w \left(Z_{\rho}(\tau, t_0, \mathbf{z}_0) - Z_{\rho}(\tau, t_0, \mathbf{y}) \right) \rho_0(\mathbf{y}) d\tau d\mathbf{y} \\
&\quad + \int_{t_0}^t \int_{t_0}^{\tau} \int_{\mathbb{R}^2} \nabla^\perp w \left(Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0}) - Z_{\rho_b}(\tau, \nu, \mathbf{x}) \right) S_b(\nu, \mathbf{x}) d\tau d\nu d\mathbf{x}
\end{aligned}$$

So

$$\begin{aligned}
& Z_{\rho_b}(t, t_0, \mathbf{z}_{b,0}) - Z_{\rho}(t, t_0, \mathbf{z}_0) - (\mathbf{z}_{b,0} - \mathbf{z}_0) \\
&= \int_{t_0}^t \left(\nabla^\perp V \left(Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0}) \right) - \nabla^\perp V \left(Z_{\rho}(\tau, t_0, \mathbf{z}_0) \right) \right) d\tau \\
&\quad \int_{t_0}^t \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left(\nabla^\perp w \left(Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0}) - Z_{\rho_b}(\tau, t_0, \mathbf{x}) \right) - \nabla^\perp w \left(Z_{\rho}(\tau, t_0, \mathbf{z}_0) - Z_{\rho}(\tau, t_0, \mathbf{y}) \right) \right) \\
&\quad d\tau d\pi_b(\mathbf{x}, \mathbf{y}) + \int_{t_0}^t \int_{t_0}^{\tau} \int_{\mathbb{R}^2} \nabla^\perp w \left(Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0}) - Z_{\rho_b}(\tau, \nu, \mathbf{x}) \right) S_b(\nu, \mathbf{x}) d\tau d\nu d\mathbf{x}
\end{aligned}$$

Next, using

$$\begin{aligned}
& \left| \nabla^\perp w \left(Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0}) - Z_{\rho_b}(\tau, t_0, \mathbf{x}) \right) - \nabla^\perp w \left(Z_{\rho}(\tau, t_0, \mathbf{z}_0) - Z_{\rho}(\tau, t_0, \mathbf{y}) \right) \right| \\
&\leq \left| \nabla^\perp w \left(Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0}) - Z_{\rho_b}(\tau, t_0, \mathbf{x}) \right) - \nabla^\perp w \left(Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0}) - Z_{\rho}(\tau, t_0, \mathbf{y}) \right) \right| \\
&\quad + \left| \nabla^\perp w \left(Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0}) - Z_{\rho}(\tau, t_0, \mathbf{y}) \right) - \nabla^\perp w \left(Z_{\rho}(\tau, t_0, \mathbf{z}_0) - Z_{\rho}(\tau, t_0, \mathbf{y}) \right) \right| \\
&\leq \|w\|_{W^{2,\infty}} \left(\left| Z_{\rho_b}(\tau, t_0, \mathbf{x}) - Z_{\rho}(\tau, t_0, \mathbf{y}) \right| + \left| Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0}) - Z_{\rho}(\tau, t_0, \mathbf{z}_0) \right| \right)
\end{aligned}$$

we obtain

$$\begin{aligned}
& \left| Z_{\rho_b}(t, t_0, \mathbf{z}_{b,0}) - Z_{\rho}(t, t_0, \mathbf{z}_0) \right| \leq |\mathbf{z}_{b,0} - \mathbf{z}_0| + \|V\|_{W^{2,\infty}} \int_{t_0}^t \left| Z_{\rho}(\tau, t_0, \mathbf{z}_0) - Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0}) \right| d\tau \\
&\quad + \|w\|_{W^{2,\infty}} \int_{t_0}^t \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left| Z_{\rho_b}(\tau, t_0, \mathbf{x}) - Z_{\rho}(\tau, t_0, \mathbf{y}) \right| d\pi_b(\mathbf{x}, \mathbf{y}) d\tau \\
&\quad + \|w\|_{W^{2,\infty}} \int_{t_0}^t \left| Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0}) - Z_{\rho}(\tau, t_0, \mathbf{z}_0) \right| d\tau \\
&\quad + \int_{t_0}^t \int_{t_0}^{\tau} \int_{\mathbb{R}^2} \left| \nabla^\perp w \left(Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0}) - Z_{\rho_b}(\tau, \nu, \mathbf{x}) \right) S_b(\nu, \mathbf{x}) \right| d\tau d\nu d\mathbf{x}
\end{aligned}$$

and integrating against $d\pi_b(\mathbf{z}_{b,0}, \mathbf{z}_0)$ we obtain

$$\begin{aligned}
D_{\pi_b}(t) &\leq D_{\pi_b}(0) + (\|V\|_{W^{2,\infty}} + 2\|w\|_{W^{2,\infty}}) \int_{t_0}^t D_{\pi_b}(\tau) d\tau \\
&\quad + \left| \int_{t_0}^t \int_{t_0}^{\tau} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla^\perp w \left(Z_{\rho_b}(\tau, t_0, \mathbf{z}_{b,0}) - Z_{\rho_b}(\tau, \nu, \mathbf{x}) \right) \mathcal{S}_b(\nu, \mathbf{x}) \rho_{b,0}(\mathbf{z}_{b,0}) d\tau d\nu d\mathbf{x} d\mathbf{z}_{b,0} \right| \\
&\leq D_{\pi_b}(0) + 2(\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}}) \int_{t_0}^t D_{\pi_b}(\tau) d\tau \\
&\quad + \left| \int_{t_0}^t \int_{t_0}^{\tau} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla^\perp w \left(Z_{\rho_b}(\tau, t_0, \mathbf{y}) - Z_{\rho_b}(\tau, \nu, \mathbf{x}) \right) \mathcal{S}_b(\nu, \mathbf{x}) \rho_{b,0}(\mathbf{y}) d\tau d\nu d\mathbf{x} d\mathbf{y} \right|
\end{aligned}$$

□

We conclude this subsection by giving the

Proof of Proposition 6.1. Let $\pi_b \in \Gamma(\rho_{b,0}, \rho_0)$ and define

$$\begin{aligned}
\phi_t(\mathbf{x}, \mathbf{y}) &:= \left(Z_{\rho_b}(t, t_0, \mathbf{x}), Z_\rho(t, t_0, \mathbf{y}) \right) \\
\pi_b(t) &:= \phi_{t*} \pi_b
\end{aligned}$$

Then $\forall A \subseteq \mathbb{R}^2$ Borel set,

$$\begin{aligned}
\pi_b(t)(\mathbb{R}^2, A) &= \pi_b \left(Z_{\rho_b}(t_0, t, \mathbb{R}^2), Z_\rho(t_0, t, A) \right) = \pi_b(\mathbb{R}^2, Z_\rho(t_0, t, A)) = \rho_0(Z_\rho(t_0, t, A)) \\
&= Z_\rho(t, t_0, \bullet)_* \rho_0(A) = \rho(t, A)
\end{aligned}$$

and similarly for the second variable thus $\pi_b(t)$ is a coupling for $\rho_b(t)$ and $\rho(t)$. Exchanging t and t_0 and using lemma 6.3 we get

$$\begin{aligned}
W_1(\rho_b(t), \rho(t)) &= \inf_{\pi_b \in \Gamma(\rho_b(t), \rho(t))} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y| d\pi_b(\mathbf{x}, \mathbf{y}) \\
&= \inf_{\pi_b \in \Gamma(\rho_{b,0}, \rho_0)} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y| d\phi_{t*} \pi_b(\mathbf{x}, \mathbf{y}) \\
&= \inf_{\pi_b \in \Gamma(\rho_{b,0}, \rho_0)} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left| Z_{\rho_b}(t, t_0, \mathbf{x}) - Z_\rho(t, t_0, \mathbf{y}) \right| d\pi_b(\mathbf{x}, \mathbf{y}) \\
&= \inf_{\pi_b \in \Gamma(\rho_{b,0}, \rho_0)} D_{\pi_b}(t) \leq e^{2(\|w\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}})|t-t_0|} \left(\inf_{\pi_b \in \Gamma(\rho_{b,0}, \rho_0)} D_{\pi_b}(0) + \mathcal{E}_{S_b}(t) \right) \\
&= e^{2(\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}})|t-t_0|} \left(W_1(\rho_{b,0}, \rho_0) + \mathcal{E}_{S_b}(t) \right)
\end{aligned}$$

□

6.2. Application: proof of Theorem 2.4. As in Section 5 it is convenient to first consider the dynamics of the truncated semi-classical density:

Proposition 6.4 (Convergence of the semi-classical density in Wasserstein metric).

Let $\frac{2}{3} < \alpha < 1$. Under the same assumptions as in Theorem 2.3, with in addition

$$\nabla w \in L^1(\mathbb{R}^2), \quad w \in H^2(\mathbb{R}^2), \quad V, w \in W^{9,\infty}(\mathbb{R}^2), \quad d^9 V, d^9 w \in L^1(\mathbb{R}^2)$$

if $\rho \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^2))$ solves the drift equation (2.5) then $\exists M = \mathcal{O}(l_b^{-\alpha})$ such that

$$W_1(\rho_b(t), \rho(t)) \leq e^{2(\|w\|_{W^{2,\infty}} + \|V\|_{W^{2,\infty}})t} \left(W_1(\rho_b(0), \rho(0)) + C_2(t, V, w) \left(l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha-1} \right) \right)$$

with ρ_b as defined in (4.18) and

$$C_2(t, V, w) := Ct^2 \left(\|w\|_{H^2} + \|V\|_{W^{9,\infty}} + \|d^9 V\|_{L^1} + \|w\|_{W^{9,\infty}} + \|d^9 w\|_{L^1} \right) \\ \left(\|\nabla w\|_{L^1} + \|w\|_{W^{2,\infty}} e^{(\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}})t} \right) \left(\left| \text{Tr} [\gamma_b(0) H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \right)$$

Proof. Our goal is to apply Proposition 6.1 with $t_0 = 0$ and

$$S_b := \text{DRIFT}_{\rho_b}(\rho_b)$$

To this end, we need to estimate the error term $\mathcal{E}_{S_b}(t)$ coming from (6.5). We define

$$\varphi(t, \mathbf{z}) := \int_0^t \int_{\mathbb{R}^2} \nabla^\perp w \left(Z_{\rho_b}(\tau, 0, \mathbf{x}) - Z_{\rho_b}(\tau, t, \mathbf{z}) \right) \rho_b(0, \mathbf{x}) d\tau d\mathbf{x}$$

so that

$$\begin{aligned} \mathcal{E}_{S_b}(T) &= \left| \int_0^T \int_0^\tau \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla^\perp w \left(Z_{\rho_b}(\tau, 0, \mathbf{x}) - Z_{\rho_b}(\tau, t, \mathbf{z}) \right) S_b(t, \mathbf{z}) \rho_b(0, \mathbf{x}) d\tau dt d\mathbf{z} d\mathbf{x} \right| \quad (6.10) \\ &\leq \left| \int_{\mathbb{R}_+ \times \mathbb{R}^2} \varphi \mathbb{1}_{[0,T]} S_b \right| \\ &\leq \left| \int_{\mathbb{R}_+ \times \mathbb{R}^2} \varphi \mathbb{1}_{[0,T]} \text{DRIFT}_{\rho_b}(\rho_b) \right| + \left| \int_{\mathbb{R}_+ \times \mathbb{R}^2} \varphi \mathbb{1}_{[0,T]} \nabla^\perp w \star (\rho_{\gamma_b} - \rho_b) \cdot \nabla \rho_b \right| \\ &=: \mathcal{E}_{S_b}^1(T) + \mathcal{E}_{S_b}^2(T). \quad (6.11) \end{aligned}$$

Step 1: we estimate $\varphi \in L^1([0, T], L^1(\mathbb{R}^2) \cap W^{1,\infty}(\mathbb{R}^2))$. Let $t \in [0, T]$. With the changes of variable $Z_{\rho_b}(\tau, t, \mathbf{z}) \mapsto \mathbf{z}$,

$$\begin{aligned} \|\varphi(t, \bullet)\|_{L^1} &\leq \int_0^t \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left| \nabla^\perp w \left(Z_{\rho_b}(\tau, 0, \mathbf{x}) - Z_{\rho_b}(\tau, t, \mathbf{z}) \right) \rho_b(0, \mathbf{x}) \right| d\tau d\mathbf{x} d\mathbf{z} \\ &= \int_0^t \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left| \nabla^\perp w \left(Z_{\rho_b}(\tau, 0, \mathbf{x}) - \mathbf{z} \right) \rho_b(0, \mathbf{x}) \right| d\tau d\mathbf{x} d\mathbf{z} \leq T \|\nabla w\|_{L^1} \quad (6.12) \end{aligned}$$

by performing the \mathbf{z} integral first. Moreover

$$|\varphi(t, \mathbf{z})| \leq T \|\nabla w\|_{L^\infty} \quad (6.13)$$

and

$$|\nabla_{\mathbf{z}} \varphi(t, \mathbf{z})| \leq \|w\|_{W^{2,\infty}} \int_0^t \left| dZ_{\rho_b}(\tau, t, \bullet)_{\mathbf{z}} \right| d\tau. \quad (6.14)$$

But, using

$$Z_{\rho_b}(\tau, t, \mathbf{z}) = \mathbf{z} + \int_t^\tau \nabla^\perp (V + w \star \rho_b(s, \bullet)) (Z_{\rho_b}(s, t, \mathbf{z})) ds$$

we get

$$dZ_{\rho_b}(\tau, t, \bullet)_\mathbf{z} = \text{Id}_{\mathbb{R}^2} + \int_t^\tau d\nabla^\perp (V + w \star \rho_b(s, \bullet))_{Z_{\rho_b}(s, t, \mathbf{z})} dZ_{\rho_b}(s, t, \bullet)_\mathbf{z} ds$$

so

$$\left| dZ_{\rho_b}(\tau, t, \bullet)_\mathbf{z} \right| \leq |\text{Id}_{\mathbb{R}^2}| + (\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}}) \int_t^\tau \left| dZ_{\rho_b}(s, t, \bullet)_\mathbf{z} \right| ds$$

Hence, applying Grönwall's lemma,

$$\left| dZ_{\rho_b}(\tau, t, \bullet)_\mathbf{z} \right| \leq \sqrt{2} e^{(\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}})|\tau-t|}$$

With (6.14), we conclude that

$$\begin{aligned} |\nabla_\mathbf{z} \varphi(t, \mathbf{z})| &\leq \sqrt{2} \|w\|_{W^{2,\infty}} \int_0^T e^{(\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}})|\tau-t|} d\tau \\ &\leq \sqrt{2} \|w\|_{W^{2,\infty}} T e^{(\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}})T} \end{aligned} \quad (6.15)$$

Collecting (6.12), (6.13) and (6.15) we find

$$\|\varphi(t, \bullet)\|_{L^1 \cap W^{1,\infty}(\mathbb{R}^2)} \leq CT \left(\|\nabla w\|_{L^1} + \|w\|_{W^{2,\infty}} e^{(\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}})T} \right)$$

and thus

$$\begin{aligned} \left\| \varphi \mathbb{1}_{[0,T]} \right\|_{L^1(\mathbb{R}_+, L^1 \cap W^{1,\infty}(\mathbb{R}^2))} &= \|\varphi\|_{L^1([0,T], L^1 \cap W^{1,\infty}(\mathbb{R}^2))} \\ &\leq CT^2 \left(\|\nabla w\|_{L^1} + \|w\|_{W^{2,\infty}} e^{(\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}})T} \right) \end{aligned} \quad (6.16)$$

Step 2: bound on $\mathcal{E}_{S_b}^2(T)$. We choose M according to Lemma 5.8. Integrating by parts and using the symmetry of w ,

$$\mathcal{E}_{S_b}^2(T) = \left| \int_{\mathbb{R}_+ \times \mathbb{R}^2} \mathbb{1}_{[0,T]} \nabla \varphi \cdot \nabla^\perp w \star (\rho_{\gamma_b} - \rho_b) \rho_b \right| = \left| \int_{\mathbb{R}_+ \times \mathbb{R}^2} \mathbb{1}_{[0,T]} (\rho_{\gamma_b} - \rho_b) (\nabla \varphi \rho_b) \star \nabla^\perp w \right|.$$

Then, using (5.32) with

$$\mu(t, x) := (\nabla \varphi \rho_b)(t, \bullet) \star \nabla^\perp w(\mathbf{x}) := \int \rho_b(t, \mathbf{y}) \nabla \varphi(t, \mathbf{y}) \cdot \nabla^\perp w(\mathbf{x} - \mathbf{y}) d\mathbf{y}$$

leads to

$$\mathcal{E}_{S_b}^2(T) \leq C \int_0^T (\|\mu(t)\|_{L^\infty} + \|\nabla \mu(t)\|_{L^2}) dt \left(\left| \text{Tr} [\gamma_b(0) \mathbf{H}_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \right) l_b^{\frac{\alpha}{2}}.$$

With Young's convolution inequality,

$$\begin{aligned} \|\mu(t)\|_{L^\infty} &\leq \|\nabla \varphi(t)\|_{L^\infty} \|\nabla w\|_{L^\infty} \\ \|\nabla \mu(t)\|_{L^2} &\leq \left\| \left(\nabla \varphi(t) \rho_{\gamma_b(t)}^{sc, \leq M} \right) \right\|_{L^1} \|w\|_{H^2} \leq \|\nabla \varphi(t)\|_{L^\infty} \|w\|_{H^2} \end{aligned}$$

so

$$\mathcal{E}_{S_b}^2(T) \leq C \left\| \varphi \mathbb{1}_{[0,T]} \right\|_{L^1(\mathbb{R}_+, W^{1,\infty}(\mathbb{R}^2))} (\|\nabla w\|_{L^\infty} + \|w\|_{H^2})$$

$$\left(\left| \text{Tr} [\gamma_b(0)H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \right) l_b^{\frac{\alpha}{2}}. \quad (6.17)$$

Step 3: conclusion. There remains to estimate the error term $\mathcal{E}_{S_b}^1(T)$ from (6.11). With (5.35),

$$\begin{aligned} \mathcal{E}_{S_b}^1(T) &\leq C \left\| \mathbb{1}_{[0,T]} \varphi \right\|_{L^1(\mathbb{R}_+, L^1 \cap W^{1,\infty}(\mathbb{R}^2))} \left(\left| \text{Tr} [\gamma_b(0)H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \right) \\ &\quad \left(\|V\|_{W^{9,\infty}} + \|d^9 V\|_{L^1} + \|w\|_{W^{9,\infty}} + \|d^9 w\|_{L^1} \right) \left(l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha-1} \right) \end{aligned}$$

We combine this with (6.17) and insert the resulting bound in (6.11):

$$\begin{aligned} \mathcal{E}_{S_b}(T) &\leq C \left\| \mathbb{1}_{[0,T]} \varphi \right\|_{L^1(\mathbb{R}_+, L^1 \cap W^{1,\infty}(\mathbb{R}^2))} \left(\left| \text{Tr} [\gamma_b(0)H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \right) \\ &\quad \left(\|w\|_{H^2} + \|V\|_{W^{9,\infty}} + \|d^9 V\|_{L^1} + \|w\|_{W^{9,\infty}} + \|d^9 w\|_{L^1} \right) \left(l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha-1} \right) \end{aligned}$$

noticing that

$$\alpha < 1 \implies \frac{3}{2}\alpha - 1 \leq \frac{\alpha}{2}, \quad (6.18)$$

we obtain the desired conclusion by using Proposition 6.1 and (6.16). \square

Finally we turn to the

Proof of Theorem 2.4. Let $\frac{2}{3} < \alpha < 1$, $l \geq 3$ and M be chosen as in Proposition 6.4. Using Proposition 6.4 and then (5.34) for $t = 0$,

$$\begin{aligned} &\left| \int_{\mathbb{R}^2} \varphi(\rho_b(t) - \rho(t)) \right| \\ &\leq \|\nabla \varphi\|_{L^\infty} W_1(\rho_b(t), \rho(t)) \\ &\leq \|\nabla \varphi\|_{L^\infty} e^{2(\|w\|_{W^{2,\infty}} + \|V\|_{W^{2,\infty}})t} \left(W_1(\rho_b(0), \rho(0)) + C_2(t, V, w) \left(l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha-1} \right) \right) \\ &\leq \|\nabla \varphi\|_{L^\infty} e^{2(\|w\|_{W^{2,\infty}} + \|V\|_{W^{2,\infty}})t} \\ &\quad \left(W_1(\rho_{\gamma_b}(0), \rho(0)) + C_1(0, p, V, w) \left(l_b^{1-\frac{\alpha}{2}-\frac{6+\alpha}{2p-4}} + l_b^{\frac{\alpha}{2}} \right) + C_2(t, V, w) \left(l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha-1} \right) \right) \\ &\leq \|\nabla \varphi\|_{L^\infty} e^{2(\|w\|_{W^{2,\infty}} + \|V\|_{W^{2,\infty}})t} \left(1 + C_1(0, p, V, w) + C_2(t, v, w) \right) \\ &\quad \left(W_1(\rho_{\gamma_b}(0), \rho(0)) + l_b^{1-\frac{\alpha}{2}-\frac{6+\alpha}{2p-4}} + l_b^{\frac{\alpha}{2}} + l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha-1} \right) \end{aligned}$$

On the other hand, from (5.32),

$$\left| \int_{\mathbb{R}^2} \varphi(\rho_{\gamma_b}(t) - \rho_b(t)) \right| \leq C (\|\varphi\|_{L^\infty} + \|\nabla \varphi\|_{L^2}) \left(\left| \text{Tr} [\gamma_b(0)H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \right) l_b^{\frac{\alpha}{2}}$$

Recalling (6.18) and using the triangle inequality,

$$\begin{aligned} &\left| \int_{\mathbb{R}^2} \varphi(\rho_{\gamma_b}(t) - \rho(t)) \right| \\ &\leq \tilde{C}(p, t, V, w) (\|\varphi\|_{W^{1,\infty}} + \|\nabla \varphi\|_{L^2}) \left(W_1(\rho_{\gamma_b}(0), \rho(0)) + l_b^{1-\frac{\alpha}{2}-\frac{6+\alpha}{2p-4}} + l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha-1} \right) \end{aligned}$$

With

$$\begin{aligned} \tilde{C}(p, t, V, w) = & \left| \text{Tr} [\gamma_b(0) H_b(0)] \right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \\ & + e^{2(\|w\|_{W^{2,\infty}} + \|V\|_{W^{2,\infty}})t} (1 + C_1(0, p, V, w) + C_2(t, v, w)) \end{aligned}$$

We conclude with the following optimisation:

$$\alpha \mapsto \min \left(1 - \frac{\alpha}{2} - \frac{6 + \alpha}{2p - 4}, 2 - 2\alpha, \frac{3}{2}\alpha - 1 \right)$$

is maximal at

$$\alpha := \min \left(2\frac{2p - 7}{4p - 7}, \frac{6}{7} \right)$$

with maximal value

$$\min \left(2\frac{p - 7}{4p - 7}, \frac{2}{7} \right).$$

□

REFERENCES

- [1] L. AMOUR, M. KHODJA, AND J. NOURRIGAT, *The semiclassical limit of the time dependent Hartree–Fock equation: the Weyl symbol of the solution*, *Analysis & PDEs*, 6 (2013), pp. 1649–1674.
- [2] ———, *The semiclassical limit of the time dependent Hartree–Fock equation: the Wick symbol of the solution*, *Mathematical Research Letters*, 20 (2013), pp. 119–139.
- [3] I. BEN PORAT, *Derivation of Euler’s equations of perfect fluids from von Neumann’s equation with magnetic field*, *J. of Stat. Phys.*, 190 (2023), p. 121.
- [4] N. BENEDIKTER, M. PORTA, C. SAFFIRIO, AND B. SCHLEIN, *From the Hartree Dynamics to the Vlasov Equation*, *Archive for Rational Mechanics and Analysis*, 221 (2015), pp. 273–334.
- [5] N. BENEDIKTER, M. PORTA, AND B. SCHLEIN, *Mean-field evolution of fermionic systems*, *Communications in Mathematical Physics*, 331 (2014), pp. 1087 – 1131.
- [6] M. BOSTAN, *The Vlasov-Poisson system with strong external magnetic field. Finite Larmor radius regime*, *Asymptot. Anal.*, 61 (2009), pp. 91–123.
- [7] M. BOSTAN AND A.-T. VU, *Asymptotic behavior of the two-dimensional Vlasov-Poisson-Fokker-Planck equation with a strong external magnetic field*. arXiv:2303.15009, 2023.
- [8] Y. BRENIER, *Convergence of the Vlasov-Poisson system to the incompressible Euler equations*, *Communications in Partial Differential Equations*, (2007).
- [9] T. CHAMPEL AND S. FLORENS, *Quantum transport properties of two-dimensional electron gases under high magnetic fields*, *Phys. Rev. B*, 75 (2007), p. 245326.
- [10] T. CHAMPEL AND S. FLORENS, *Local density of states in disordered two-dimensional electron gases at high magnetic field*, *Phys. Rev. B*, 80 (2009), p. 161311.
- [11] T. CHAMPEL, S. FLORENS, AND L. CANET, *Microscopics of disordered two-dimensional electron gases under high magnetic fields: Equilibrium properties and dissipation in the hydrodynamic regime*, *Phys. Rev. B*, 78 (2008), p. 125302.
- [12] L. CHEN, J. LEE, AND M. LIEW, *Combined mean-field and semiclassical limits of large fermionic systems*, *J. Stat. Phys.*, 182 (2021), p. 24.
- [13] ———, *Convergence towards the Vlasov-Poisson equation from the N -fermionic Schrödinger equation*, *Annales Henri Poincaré*, 23 (2022), pp. 555–593.
- [14] ———, *A mixed-norm estimate of two-particle reduced density matrix of many-body Schrödinger dynamics*, *Journal of Statistical Physics*, 190 (2023), p. 109.
- [15] J. CHONG, L. LAFLÈCHE, AND C. SAFFIRIO, *From Schrödinger to Hartree-Fock and Vlasov equations with singular potentials*, arXiv, (2021).
- [16] J. J. CHONG, L. LAFLÈCHE, AND C. SAFFIRIO, *On the l^2 rate of convergence in the limit from the Hartree to the Vlasov–Poisson equation*, *Journal de l’École polytechnique — Mathématiques*, 10 (2023), pp. 703–726.
- [17] R. DOBRUSHIN, *Vlasov equations*, *Funct. Anal. Appl.*, 13 (1979), pp. 115–123.
- [18] S. FOURNAIS, M. LEWIN, AND J.-P. SOLOVEL, *The semi-classical limit of large fermionic systems*, *Calculus of Variations and Partial Differential Equations*, 57 (2018), p. 105.
- [19] S. FOURNAIS AND P. MADSEN, *Semi-classical limit of confined fermionic systems in homogeneous magnetic fields*, *Annales Henri Poincaré*, 21 (2019), pp. 1401–1449.
- [20] E. FRÉNOD AND E. SONNENDRÜCKER, *Long time behaviour of the two-dimensional Vlasov equation with a strong external magnetic field*, *Mathematical Models and Methods in Applied Sciences*, 10 (2000), pp. 539–553.
- [21] P. GHENDRIH, M. HAURAY, AND A. NOURI, *Derivation of a gyrokinetic model. Existence and uniqueness of specific stationary solution*, *Kinetic and Related Models*, 2 (2009), pp. 707–725.
- [22] M. GOERBIG, *Quantum Hall effects*. lecture notes, 2009.
- [23] F. GOLSE, *On the Dynamics of Large Particle Systems in the Mean Field Limit*, ArXiv e-prints 1301.5494, (2013). Lecture notes for a course at the NDNS+ Applied Dynamical Systems Summer School "Macroscopic and large scale phenomena", Universiteit Twente, Enschede (The Netherlands).
- [24] F. GOLSE, C. MOUHOT, AND T. PAUL, *On the Mean Field and Classical Limits of Quantum Mechanics*, *Comm. Math. Phys.*, 343 (2013), pp. 165–205.
- [25] F. GOLSE, T. PAUL, AND M. PULVIRENTI, *On the derivation of the Hartree equation from the N -body Schrödinger equation: Uniformity in the Planck constant*, *Journal of Functional Analysis*, 275 (2018), pp. 1603–1649.
- [26] F. GOLSE AND L. SAINT-RAYMOND, *The Vlasov-Poisson system with strong magnetic field*, *Journal de Mathématiques Pures et Appliquées*, 78 (1999), pp. 791–817.
- [27] M. GRIFFIN-PICKERING AND M. IACOBELLI, *Recent developments on quasineutral limits for Vlasov-type equations*, arXiv, (2020).

- [28] D. HAN-KWAN, *The three-dimensional finite Larmor radius approximation*, *Asymptot. Anal.*, 66 (2008), pp. 9–33.
- [29] D. HAN-KWAN AND M. IACOBELLI, *From Newton’s second law to Euler’s equations of perfect fluids*, *Proceedings of the American Mathematical Society*, 149 (2020).
- [30] J. JAIN, *Composite fermions*, Cambridge University Press, 2009.
- [31] L. LAFLÈCHE, *Global semiclassical limit from Hartree to Vlasov equation for concentrated initial data*, *Annales de l’Institut Henri Poincaré C, Analyse non linéaire*, 38 (2021), pp. 1739–1762.
- [32] L. LAFLÈCHE AND C. SAFFIRIO, *Strong semiclassical limit from Hartree and Hartree-Fock to Vlasov-Poisson equation*, arXiv: Mathematical Physics, (2020).
- [33] E. LIEB AND R. SEIRINGER, *The Stability of Matter in Quantum Mechanics*, Cambridge Univ. Press, 2010.
- [34] E. LIEB AND B. SIMON, *The Thomas-Fermi theory of atoms, molecules and solids*, *Adv. Math.*, 23 (1977), pp. 22–116.
- [35] E. LIEB, J.-P. SOLOVEJ, AND J. YNGVASON, *Asymptotics of heavy atoms in high magnetic fields: I. Lowest Landau band regions*, *Comm. Pure Appl. Math.*, 47 (1994), pp. 513–591.
- [36] ———, *Asymptotics of heavy atoms in high magnetic fields: II. Semi-classical regions*, *Comm. Math. Phys.*, 161 (1994), pp. 77–124.
- [37] ———, *Ground states of large quantum dots in magnetic fields*, *Phys. Rev. B*, 51 (1995), pp. 10646–10665.
- [38] E. MIOT, *On the gyrokinetic limit for the two-dimensional Vlasov-Poisson system*, arXiv: Analysis of PDEs, (2016).
- [39] D. PÉRICE, *Semi-classical limits of 2D fermions under high magnetic fields*, PhD thesis, ENS Lyon, 2023.
- [40] ———, *Multiple Landau level filling for a mean field limit of 2D fermions*, *J. Math. Phys.*, 65 (2024), p. 021902.
- [41] S. PETRAT AND P. PICKL, *A new method and a new scaling for deriving fermionic mean-field dynamics*, *Mathematical Physics, Analysis and Geometry*, 19 (2016).
- [42] M. PORTA, S. RADEMACHER, C. SAFFIRIO, AND B. SCHLEIN, *Mean field evolution of fermions with Coulomb interaction*, *Journal of Statistical Physics*, 166 (2016), pp. 1345–1364.
- [43] N. ROUGERIE, *De Finetti theorems, mean-field limits and Bose-Einstein condensation*, LMU Lecture Notes, arXiv:1506.05263 (2014)
- [44] N. ROUGERIE, *Scaling limits of bosonic ground states, from many-body to nonlinear Schrödinger*, *EMS Surveys in Mathematical Sciences*, 7 (2020), pp. 253–408.
- [45] N. ROUGERIE AND J. YNGVASON, *Holomorphic quantum Hall states in higher Landau levels*, *J. Math. Phys.*, 61 (2020), p. 041101.
- [46] C. SAFFIRIO, *Mean-field evolution of fermions with singular interaction*, arXiv: Mathematical Physics, (2017).
- [47] C. SAFFIRIO, *From the Hartree Equation to the Vlasov–Poisson system: Strong convergence for a class of mixed states*, *SIAM Journal on Mathematical Analysis*, 52 (2020), pp. 5533–5553.
- [48] L. SAINT-RAYMOND, *The gyrokinetic approximation for the Vlasov-Poisson system*, *Mathematical Models and Methods in Applied Science*, 10 (2000), pp. 1305–1332.
- [49] ———, *Control of large velocities in the two-dimensional gyrokinetic approximation*, *Journal de Mathématiques Pures et Appliquées*, 81 (2002), pp. 379–399.
- [50] W. THIRRING, *A lower bound with the best possible constant for Coulomb Hamiltonians*, *Comm. Math. Phys.*, 79 (1981), pp. 1–7.
- [51] J. YNGVASON, *Thomas-Fermi theory for matter in a magnetic field as a limit of quantum mechanics*, *Lett. Math. Phys.*, 22 (1991), pp. 107–117.

CONSTRUCTOR UNIVERSITY BREMEN

Email address: dperice@constructor.university

ECOLE NORMALE SUPÉRIEURE DE LYON & CNRS, UMPA (UMR 5669)

Email address: nicolas.rougerie@ens-lyon.fr