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Adaptive Boundary Control of the Kuramoto-Sivashinsky Equation Under Intermittent Sensing

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Abstract

We study in this paper boundary stabilization, in the L^2 sense, of the perturbed Kuramoto-Sivashinsky (KS) equation subject to intermittent sensing. We assume that we measure the state on a given spatial subdomain during certain time intervals, while we measure the state on the remaining spatial subdomain during the remaining time intervals. We assign a feedback law at the boundary of the spatial domain and force to zero the value of the state at the junction of the two subdomains. Throughout the study, the equation's destabilizing coefficient is assumed to be unknown and possibly space dependent but bounded. As a result, adaptive boundary controllers are designed under different assumptions on the perturbation. In particular, we guarantee input-to-state stability (ISS) when an upperbound on the perturbation's size is known. Otherwise, only global uniform ultimate boundedness (GUUB) is guaranteed. In contrast, when the state is measured at every spatial point all the time (full state measurement), convergence to an arbitrarily-small neighborhood of the origin is guaranteed, even if the perturbation's maximal size is unknown. Numerical simulations are performed to illustrate our results.

Key words: KS equation, boundary control, intermittent sensing, Lyapunov methods, adaptive design.

1 Introduction

In control loops, inputs are often subject to sensor limitations. For systems governed by partial differential equations (PDEs) evolving in both space and time, a major limitation is the spatial range of sensing, as it can be impractical to measure the state simultaneously at every spatial point. Another limitation is the energy cost associated with always-on sensors, as it can be unaffordable to run the sensor all the time. In this work, we propose an intermittent-sensing scenario that takes into account the aforementioned limitations, in which the system's state is measured at certain spatial subdomains over specific time intervals, rather than continuously across the entire spatial domain. This scenario depicts situations of restricted-energy sensors [1,2], network control systems [3,4], and mobile sensors [5].

The proposed sensing scenario is considered in the context of the perturbed KS equation given by [6,7]

$$u_t + uu_x + \lambda(x)u_{xx} + u_{xxxx} = f(x, t), \quad (1)$$

where $x \in (0, 1)$ is the one-dimensional space variable,

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$\lambda > 0$ is known as the *destabilizing coefficient*, and f is the perturbation. This equation has been used to model various physical phenomena including wildfires [8], turbulence in reaction-diffusion systems [6], and plasma instabilities [9], among others.

Since [10], several boundary controllers are proposed to stabilize the origin for (1) when $f \equiv 0$; see [11–13]. These works, however, either assume a small λ , in which case boundary measurements are enough to ensure global asymptotic/exponential stability, or assume arbitrary λ but small initial conditions while requiring full state measurement. Additionally, λ is often assumed constant, though it could be space-dependent according to [14,15]. To the best of our knowledge, the first work to study (1) without constraining the range of the initial condition nor the size of λ is [16]. The state in the latter reference is assumed to be measured intermittently over the subdomains $(0, Y)$ and $(Y, 1)$. Furthermore, feedback controllers are applied at $x = 0$ and $x = 1$, while enforcing a zero state at $x = Y$, to ensure L^2 global exponential stability of the origin. However, the destabilizing coefficient λ is assumed constant and known. This requirement is relaxed in [17] by allowing λ to be unknown, and proposing an adaptive design of the control parameters, to guarantee the same stability conclusions. On the other hand, the perturbed KS equation is considered in [18], where a boundary controller using boundary measure-

ments is shown to guarantee an input-to-state stability with respect to f , under small λ .

In this work, we study boundary stabilization for (1) under intermittent sensing, in the presence of the bounded space- and time-dependent perturbation f . More specifically, we extend the adaptive Lyapunov-based approach in [17] to the perturbed case, when an upper bound on the norm of f is either known or unknown. In the former case, we establish L^2 -input-to-state stability [19] with respect to f , while in the latter case, we establish only L^2 -global uniform ultimate boundedness. On the other hand, under full state measurement, we simplify our design to achieve convergence to arbitrarily-small neighborhoods of the origin, even if we ignore an upper bound on the norm of f . Notably, the presence of perturbations prevents us from establishing the same properties under full and intermittent sensing, different from the unperturbed case, where L^2 global exponential stability of the origin is guaranteed under both scenarios.

The rest of the paper is organized as follows. Problem formulation is in Section 2. The proposed adaptive controller as well as key Lyapunov inequalities are in Section 3. The results are in Sections 4 and 5. Finally, numerical simulations are in Section 6.

Notation. Depending on the context, a.e. means either almost every or almost everywhere. We denote by $L^2(a, b)$, for $b > a$, the space of functions $u : [a, b] \rightarrow \mathbb{R}$ such that $\int_a^b u(x)^2 dx < +\infty$. Furthermore, we let $|u|_\infty := \text{ess sup}_{x \in (a, b)} |u(x)| := \inf\{M \geq 0 : |u(x)| \leq M \text{ for a.e. } x \in (a, b)\}$. For $(x, t) \mapsto u(x, t)$, the partial derivative of u with respect to t is denoted by u_t , the first partial derivative with respect to x is denoted by u_x , the second partial derivative with respect to x is denoted by u_{xx} (and so on), and we may write $u(x)$ instead of $u(x, t)$. We denote the time derivative of a function $t \mapsto V(t)$ by \dot{V} . We also denote the space derivative of a function $x \mapsto \lambda(x)$ of a scalar variable by λ' . A continuous function $\Phi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class \mathcal{K} if it vanishes at zero and is strictly increasing. Finally, for $x \in \mathbb{R}$, $\text{sgn}(x) = 1$ if $x > 0$, $= 0$ if $x = 0$ and $= -1$ if $x < 0$.

2 Intermittent sensing and control location

In this section, we formulate the proposed sensing scenario and the in-domain and boundary conditions.

2.1 Intermittent sensing

Consider equation (1), let $Y \in (0, 1)$, and consider a sequence $\{t_i\}_{i=1}^\infty \subset \mathbb{R}_{\geq 0}$, where $t_1 = 0$ and $t_{i+1} > t_i$ for all $i \in \{1, 2, \dots\}$, such that

- S1) $u(x, t)$ is measured for all $t \in I_1 := \bigcup_{k=1}^\infty [t_{2k-1}, t_{2k}]$ and for a.e. $x \in (0, Y)$.
- S2) $u(x, t)$ is measured for all $t \in I_2 := \bigcup_{k=1}^\infty [t_{2k}, t_{2k+1}]$ and for a.e. $x \in (Y, 1)$.

Associated with the proposed sensing scenario, we consider the following assumption.

Assumption 1 *There exist four constants $\bar{T}_1, \bar{T}_2, \underline{T}_1, \underline{T}_2 > 0$ such that, for each $k \in \mathbb{N}^*$, we have*

$$\underline{T}_1 \leq t_{2k} - t_{2k-1} \leq \bar{T}_1, \quad \underline{T}_2 \leq t_{2k+1} - t_{2k} \leq \bar{T}_2.$$

The proposed sensing scenario could represent situations where two battery-powered sensors are used (one measuring u over $(0, Y)$, and the other one over $(Y, 1)$), in case where their simultaneous activation can be costly; see [1, 2]. Furthermore, in the context of network control systems [20], we can assume that the two sensors share the same channel to transfer measurements to the controller. Hence, each sensor is allowed to use the channel only over certain time intervals. Another motivation emerges when using a single mobile sensor [21]. Indeed, when the sensor's commuting speed (between two subdomains $(0, Y)$ and $(Y, 1)$) is fast enough, the proposed sensing scenario can offer a fair approximation.

2.2 Control locations

In the case of intermittent sensing, we propose to control (1) at $x = 0$ and $x = 1$. Also, we set the u and its spatial derivative u_x at $x = Y$ to a null value. Therefore, we assimilate (1) to a system of two PDEs interconnected through boundary constraints at $x = Y$. That is, we introduce the system of PDEs

$$\begin{aligned} w_t + ww_x + \lambda w_{xx} + w_{xxx} &= f & x \in (0, Y), \\ v_t + vv_x + \lambda v_{xx} + v_{xxx} &= f & x \in (Y, 1), \end{aligned} \quad (2a)$$

$$\begin{aligned} w(Y) &= w_x(Y) = w_x(0) = 0, \\ v(Y) &= v_x(Y) = v_x(1) = 0, \\ w(0) &= u_1, \quad v(1) = u_2, \end{aligned} \quad (2b)$$

where (u_1, u_2) are control inputs.

As a consequence, we define a solution $u : [0, 1] \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ to (1) subject to the boundary conditions

$$\begin{aligned} u(Y) &= u_x(0) = u_x(Y) = u_x(1) = 0, \\ u(0) &= u_1, \quad u(1) = u_2, \end{aligned} \quad (3)$$

for some inputs (u_1, u_2) , as $u(x, t) := w(x, t)$ for a.e. $(x, t) \in (0, Y) \times \mathbb{R}_{>0}$ and $u(x, t) := v(x, t)$ for a.e. $(x, t) \in (Y, 1) \times \mathbb{R}_{>0}$, with (w, v) a strong solution to (2) subject to the same inputs (u_1, u_2) ; see [22, Chapter 9] for the concept of strong solutions. According to this concept of solutions, we study (1) under (3) by fully focusing on (2).

3 General approach

We follow in this work a Lyapunov-based approach to design (u_1, u_2) for (2). To do so, we start introducing the Lyapunov function candidates

$$V_1(w) := \frac{1}{2} \int_0^Y w(x)^2 dx, \quad V_2(v) := \frac{1}{2} \int_Y^1 v(x)^2 dx. \quad (4)$$

The following lemma establishes key Lyapunov inequalities along (2), that hold under the following assumption.

Assumption 2 *The function λ is absolutely continuous on $(0, Y) \cup (Y, 1)$, and there exist \bar{f} , $\bar{\lambda}_l$, $\bar{\lambda}_r$, $\bar{\lambda}'_l$, $\bar{\lambda}'_r > 0$ such that $|f|_\infty \leq \bar{f}$ and*

$$\begin{aligned} \operatorname{ess\,sup}_{x \in (0, Y)} |\lambda(x)| &\leq \bar{\lambda}_l, & \operatorname{ess\,sup}_{x \in (Y, 1)} |\lambda(x)| &\leq \bar{\lambda}_r, \\ \operatorname{ess\,sup}_{x \in (0, Y)} |\lambda'(x)| &\leq \bar{\lambda}'_l, & \operatorname{ess\,sup}_{x \in (Y, 1)} |\lambda'(x)| &\leq \bar{\lambda}'_r. \end{aligned}$$

Lemma 1 *Along (2), it holds that*

$$\dot{V}_1 \leq \theta_1 V_1 + C_1 \sqrt{V_1} + \frac{u_1^3}{3} + u_1 w_{xxx}(0), \quad (5)$$

$$\dot{V}_2 \leq \theta_2 V_2 + C_2 \sqrt{V_2} - \frac{u_2^3}{3} - u_2 v_{xxx}(1), \quad (6)$$

where $C_1 := \sqrt{2Y}\bar{f}$, $C_2 := \sqrt{2(1-Y)}\bar{f}$, and

$$\begin{aligned} \theta_1 &:= \bar{\lambda}'_l{}^2 + 2 \left(\bar{\lambda}_l + \frac{1}{2} \right) \left(\left(\bar{\lambda}_l + \frac{1}{2} \right) + \frac{12}{Y^2} \right), \\ \theta_2 &:= \bar{\lambda}'_r{}^2 + 2 \left(\bar{\lambda}_r + \frac{1}{2} \right) \left(\left(\bar{\lambda}_r + \frac{1}{2} \right) + \frac{12}{(1-Y)^2} \right). \end{aligned}$$

Proof. The proof is in the Appendix. \blacksquare

In view of (5)-(6), the control input u_1 affects the (worst-case) behavior of V_1 only and since $(V_1, w_{xxx}(0))$ is available on I_1 only, we propose to design u_1 to guarantee a stable behavior for V_1 over I_1 and set $u_1 = 0$ otherwise. For the same reasons, we design u_2 to guarantee a stable behavior for V_2 over I_2 and set $u_2 = 0$ otherwise. That is, we let

$$(u_1, u_2) := \begin{cases} (\kappa(V_1, w_{xxx}(0), \hat{\theta}_1), 0) & \text{on } I_1, \\ (0, -\kappa(V_2, v_{xxx}(1), \hat{\theta}_2)) & \text{on } I_2, \end{cases} \quad (7)$$

where $\hat{\theta}_1, \hat{\theta}_2 \geq 0$ are adaptation parameters that we design to handle the fact that λ , λ' , and $(\bar{T}_1, \bar{T}_2, \underline{T}_1, \underline{T}_2)$ from Assumption 1 can be unknown. Furthermore, we design $\kappa : \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ to verify

$$\frac{\kappa(V, \omega, \hat{\theta})^3}{3} + \kappa(V, \omega, \hat{\theta})\omega \leq -\hat{\theta}V. \quad (8)$$

The latter inequality combined to (5)-(6) yields

$$\begin{cases} \dot{V}_1 \leq (\theta_1 - \hat{\theta}_1)V_1 + C_1\sqrt{V_1} & \text{a.e. on } I_1, \\ \dot{V}_2 \leq \theta_2 V_2 + C_2\sqrt{V_2} & \\ \dot{V}_1 \leq \theta_1 V_1 + C_1\sqrt{V_1} & \text{a.e. on } I_2, \\ \dot{V}_2 \leq (\theta_2 - \hat{\theta}_2)V_2 + C_2\sqrt{V_2} & \end{cases} \quad (9)$$

As a result, the stability analysis of the origin for the system of PDEs (2) is achieved through the stability analysis of the origin for the switched system (9).

We provide in the next lemma an example of a feedback law κ verifying (8).

Lemma 2 *Given $\varepsilon, \delta > 0$ such that $\delta\varepsilon \geq 1$, $\varepsilon(\delta\varepsilon) \geq 1$, and $\varepsilon^3\delta^2 - (\varepsilon/3) - \varepsilon\delta - 1 \geq 0$, the function $\kappa : \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ given by*

$$\kappa(V, \omega, \hat{\theta}) := \begin{cases} -\operatorname{sgn}(\omega)\sqrt[3]{V} & \text{if } |\omega| \geq l(V, \hat{\theta}), \\ -\varepsilon(\hat{\theta} + \delta)\sqrt[3]{V} & \text{otherwise,} \end{cases} \quad (10)$$

where $l(V, \hat{\theta}) := (1/3)(1 + 3\hat{\theta})V^{2/3}$, verifies (8).

Proof. The proof is in the Appendix. \blacksquare

Remark 1 The discontinuity of κ is key to verify (8) while guaranteeing boundedness of κ independently of how large ω can be, and as long as V and $\hat{\theta}$ are bounded. This will allow us to conclude that, as long as $(V_1, V_2, \hat{\theta}_1, \hat{\theta}_2)$ are bounded, the control inputs (u_1, u_2) will remain bounded independently of how large $(w_{xxx}(0), v_{xxx}(1))$ may be. Another discontinuity in (u_1, u_2) is due to transitioning from a time interval in I_1 to a time interval in I_2 , and vice versa; see (7). Discontinuous boundary control of PDEs brings challenges in terms of numerical and practical implementation as well as to ensure well-posedness; see [23,24].

4 Results under intermittent sensing

In this section, we design the parameters $(\hat{\theta}_1, \hat{\theta}_2)$ in (7) when \bar{f} is unknown. Then, to ensure stronger properties, we assume that \bar{f} is known.

4.1 Unknown perturbation range

In this case, we update $(\hat{\theta}_1, \hat{\theta}_2)$ according to the following algorithm.

Algorithm 1 *Given $\Delta_1, \Delta_2, \sigma > 0$, the coefficients $(\hat{\theta}_1, \hat{\theta}_2)$ are dynamically updated as follows.*

R1) On every interval $[t_{2k-1}, t_{2k}) \subset I_1$, we set $\dot{\hat{\theta}}_2 := 0$. Moreover, if

$$\begin{aligned} V_1(t_{2k-1}) &> V_1(t_{2k-3}) \exp^{-\sigma(t_{2k-1}-t_{2k-3})} \\ &+ \left(\hat{\theta}_1(t_{2k-3}) + \frac{\hat{\theta}_1(t_{2k-3})^2}{4} \right) \exp^{(\hat{\theta}_1(t_{2k-3})+1)(t_{2k-1}-t_{2k-3})}, \end{aligned}$$

we set $\dot{\hat{\theta}}_1 := \Delta_1$; otherwise, we set $\dot{\hat{\theta}}_1 := 0$.

R2) On every interval $[t_{2k}, t_{2k+1}) \subset I_2$, we set $\dot{\hat{\theta}}_1 := 0$. Moreover, if

$$\begin{aligned} V_2(t_{2k}) &> V_2(t_{2k-2}) \exp^{-\sigma(t_{2k}-t_{2k-2})} \\ &+ \left(\hat{\theta}_2(t_{2k-2}) + \frac{\hat{\theta}_2(t_{2k-2})^2}{4} \right) \exp^{(\hat{\theta}_2(t_{2k-2})+1)(t_{2k}-t_{2k-2})}, \end{aligned}$$

we set $\hat{\theta}_2 := \Delta_2$; otherwise, we set $\hat{\theta}_2 := 0$.

R3) On $[t_1, t_3]$, $\hat{\theta}_1 = \hat{\theta}_1(0) \geq 0$ and $\hat{\theta}_2 = \hat{\theta}_2(0) \geq 0$.

Remark 2 According to Algorithm 1, if we assume that $t \mapsto \hat{\theta}_1(t)$ is bounded (which we prove later), being non-decreasing, it will become equal to a constant θ_1^* after some finite time. Hence, according to R1), we verify that $k \mapsto V_1(t_{2k-1})$ converges to the ball centered at the origin of radius $\left(\theta_1^* + \frac{\theta_1^{*2}}{4}\right) \exp^{(\theta_1^*+1)(\bar{T}_1+\bar{T}_2)}$. As we will show that θ_1^* admits an upperbound that is independent of the PDEs initial conditions, we will be able to conclude that V_1 converges to a neighborhood of the origin whose size is independent of the PDEs initial condition. The same reasoning applies to $(\hat{\theta}_2, V_2)$.

Remark 3 The tuning parameters Δ_1 and Δ_2 represent the adaptation gains for $\hat{\theta}_1$ and $\hat{\theta}_2$, respectively. Specifically, Δ_1 (respectively, Δ_2) determines the rate at which $t \mapsto \hat{\theta}_1(t)$ (respectively, $t \mapsto \hat{\theta}_2(t)$) increases during intervals in I_1 (respectively, I_2) until specific decay of $t \mapsto V_1(t)$ (respectively, $t \mapsto V_2(t)$) is detected, which marks the end of the adaptation phase. Hence, larger values of Δ_1 (respectively, Δ_2) will result in $\hat{\theta}_1$ (respectively, $\hat{\theta}_2$) quickly compensating the destabilizing effect of θ_1 (respectively, θ_2). It is to be noted that, since the growth rate of $t \mapsto \hat{\theta}_1(t)$ (respectively, $t \mapsto \hat{\theta}_2(t)$) is updated at specific time instants, large values for Δ_1 (respectively, Δ_2) can lead to large values for $\hat{\theta}_1$ (respectively, $\hat{\theta}_2$) and thus large control inputs. On the other hand, the gain σ defines the desired decay rate for $t \mapsto (V_1(t) + V_2(t))$, once the adaptation phase is over. That is, larger values for σ would result in faster decay rates.

Theorem 1 Consider system (2) such that Assumption 2 holds. Consider the intermittent sensing scenario in S1)-S2) in Section 2.1 such that Assumption 1 holds. Let (V_1, V_2) be defined in (4), κ be defined in (10), and $(\hat{\theta}_1, \hat{\theta}_2)$ be governed by Algorithm 1. Then, system (2) in closed loop with (u_1, u_2) as in (7) satisfies:

- L^2 -global uniform ultimate boundedness: For any $(\hat{\theta}_1(0), \hat{\theta}_2(0))$, there exists a constant $r > 0$ such that for any $R > 0$, there exists a finite time $T(R) \geq 0$ such that, along the closed-loop solutions, we have $V_1(0) + V_2(0) \leq R \Rightarrow V_1(t) + V_2(t) \leq r, \forall t \geq T(R)$.
- For each $(\hat{\theta}_1(0), \hat{\theta}_2(0))$, there exists $M > 0$ such that $\max\{|\hat{\theta}_1|_\infty, |\hat{\theta}_2|_\infty\} \leq M$ for all $(V_1(0), V_2(0))$.
- The inputs (u_1, u_2) remain bounded.

Proof. The proof relies on the stability analysis of the origin $\{(V_1, V_2) = 0\}$ for the switched system (9) and is divided into three steps. In Step 1, we prove that $(\hat{\theta}_1, \hat{\theta}_2)$ become constant after some finite time $T \geq 0$. Furthermore, $\hat{\theta}_1$ and $\hat{\theta}_2$ never exceed $\max\{\hat{\theta}_1(0), M_1\}$ and $\max\{\hat{\theta}_2(0), M_2\}$, respectively, where $M_1, M_2 > 0$ depend only on $(\sigma, \theta_1, \theta_2, C_1, C_2, \bar{T}_1, \bar{T}_2, \underline{T}_1, \underline{T}_2, \Delta_1, \Delta_2)$. In Step 2, we analyze the function $V_1 + V_2$ and show that the

L^2 -global uniform ultimate boundedness property holds. Finally, in Step 3, we use the structure of κ and the fact that $(\hat{\theta}_1, \hat{\theta}_2, V_1, V_2)$ are bounded to conclude that (u_1, u_2) remains bounded.

Step 1: We first show that $\hat{\theta}_1$ and $\hat{\theta}_2$ become constant after some $T \geq 0$ using contradiction. Namely, we assume that there is not such a $T \geq 0$ such that, for all $t \geq T$, $\dot{\hat{\theta}}_1(t) = \dot{\hat{\theta}}_2(t) = 0$. As a result, according to R1)-R3) in Algorithm 1, we conclude that $\lim_{t \rightarrow \infty} \hat{\theta}_1(t) = \lim_{t \rightarrow \infty} \hat{\theta}_2(t) = \infty$. Therefore, there exists $T \geq 0$ such that, for all $t \geq T$, we have

$$\hat{\theta}_1(t) \geq \theta_1 + C_1 + \frac{(\theta_1 + 1)\bar{T}_2 + \sigma(\bar{T}_1 + \bar{T}_2)}{\underline{T}_1} + 1, \quad (11)$$

$$\hat{\theta}_2(t) \geq \theta_2 + C_2 + \frac{(\theta_2 + 1)\bar{T}_1 + \sigma(\bar{T}_1 + \bar{T}_2)}{\underline{T}_2} + 1. \quad (12)$$

Let $k > 1$ be such that $t_{2k-3} \geq T$. Using Lemma 4 in the Appendix, while replacing therein $(V, \hat{\theta}, \theta, C)$ by $(V_1, \hat{\theta}_1, \theta_1, C_1)$, we find

$$V_1(t_{2k-2}) \leq V_1(t_{2k-3}) \exp^{-\sigma(t_{2k-2}-t_{2k-3})} + \hat{\theta}_1(t_{2k-3}). \quad (13)$$

Next, we shall prove the following inequality

$$V_1(t_{2k-1}) \leq V_1(t_{2k-2}) \exp^{(\theta_1+1)(t_{2k-1}-t_{2k-2})} + \frac{\hat{\theta}_1(t_{2k-3})^2}{4} \exp^{(\hat{\theta}_1(t_{2k-3})+1)(t_{2k-1}-t_{2k-2})}. \quad (14)$$

To obtain the latter inequality, we start using

$$\dot{V}_1 \leq \theta_1 V_1 + C_1 \sqrt{V_1} \quad \text{a.e. on } [t_{2k-2}, t_{2k-1}) \subset I_1.$$

Then, applying Lemma 5 in the Appendix while replacing $(\theta, C, [0, T], \delta)$ therein by $(\theta_1, C_1, [t_{2k-2}, t_{2k-1}], 1)$, we obtain

$$\begin{aligned} V_1(t_{2k-1}) &\leq V_1(t_{2k-2}) \exp^{(\theta_1+1)(t_{2k-1}-t_{2k-2})} \\ &\quad + \frac{C_1^2}{4(\theta_1+1)} \exp^{(\theta_1+1)(t_{2k-1}-t_{2k-2})} \\ &\leq \left(V_1(t_{2k-2}) + \frac{C_1^2}{4} \right) \exp^{(\theta_1+1)(t_{2k-1}-t_{2k-2})}. \end{aligned}$$

Using the fact that $\hat{\theta}_1(t_{2k-3}) \geq \theta_1$, we obtain

$$V_1(t_{2k-1}) \leq V_1(t_{2k-2}) \exp^{(\theta_1+1)(t_{2k-1}-t_{2k-2})} + \frac{C_1^2}{4} \exp^{(\hat{\theta}_1(t_{2k-3})+1)(t_{2k-1}-t_{2k-2})}. \quad (15)$$

Finally, using (15) and the fact that $\hat{\theta}_1(t_{2k-3}) \geq C_1$, we obtain (14).

By combining (13) and (14), we obtain

$$V_1(t_{2k-1}) \leq V_1(t_{2k-3}) \exp^{-\sigma(t_{2k-1}-t_{2k-3})} + \left(\hat{\theta}_1(t_{2k-3}) + \frac{\hat{\theta}_1(t_{2k-3})^2}{4} \right) \exp^{(\hat{\theta}_1(t_{2k-3})+1)(t_{2k-1}-t_{2k-3})}, \quad (16)$$

which implies, according to R1) in Algorithm 1, that $\dot{\hat{\theta}}_1(t) = 0$ for all $t \geq t_{2k-1}$. We show in a similar way that $\dot{\hat{\theta}}_2(t) = 0$ for all $t \geq t_{2k}$, which leads to a contradiction.

To show the uniform boundedness of $\hat{\theta}_1$ with respect to the closed-loop trajectories, we first suppose the existence of $T \geq 0$ such that

$$\hat{\theta}_1(T) = \theta_1 + C_1 + \frac{(\theta_1 + 1)\bar{T}_2 + \sigma(\bar{T}_1 + \bar{T}_2)}{\underline{T}_1} + 1. \quad (17)$$

Since $\hat{\theta}_1$ is non-decreasing, then either $\hat{\theta}_1$ is smaller than the right-hand side of (17), or $\hat{\theta}_1(0)$ is greater than the right-hand side of (17), or there exists T such that (17) holds. In the case where such a T exists, we can always pick it to be in I_1 , as $\hat{\theta}_1$ is constant over each interval in I_2 . Thus, we let $T \in [t_{2k'-3}, t_{2k'-2}) \subset I_1$ for some $k' > 1$. Note that inequality (17) implies (11), with $t \geq T$. We have already shown that if (11) holds, then (16) also holds for all $k > 1$ such that $t_{2k-3} \geq T$. Thus, we can write that

$$V_1(t_{2k+1}) \leq V_1(t_{2k-1}) \exp^{-\sigma(t_{2k+1}-t_{2k-1})} + \left(\hat{\theta}_1(t_{2k-1}) + \frac{\hat{\theta}_1(t_{2k-1})^2}{4} \right) \exp^{(\hat{\theta}_1(t_{2k-1})+1)(t_{2k+1}-t_{2k-1})},$$

for all $k > 1$ such that $t_{2k-1} \geq T$. Furthermore, using R1) in Algorithm 1, we conclude that $0 \leq \dot{\hat{\theta}}(t) \leq \Delta_1$ for all $t \geq 0$. As a result,

$$\hat{\theta}_1(T) \leq \hat{\theta}_1(t_{2k'+1}) \leq \hat{\theta}_1(T) + 2\Delta_1\bar{T}_1.$$

On the other hand, note that, using R1), we have $\dot{\hat{\theta}}_1(t) = 0$ for all $t \geq t_{2k'+1}$. As a result,

$$\hat{\theta}_1(t) \leq \hat{\theta}_1(T) + 2\Delta_1\bar{T}_1, \quad \forall t \geq t_{2k'+1}.$$

We thus conclude that $|\hat{\theta}_1| \leq \max\{\hat{\theta}_1(0), M_1\}$, with

$$M_1 := \theta_1 + C_1 + \frac{(\theta_1 + 1)\bar{T}_2 + \sigma(\bar{T}_1 + \bar{T}_2)}{\underline{T}_1} + 1 + 2\Delta_1\bar{T}_1. \quad (18)$$

Similarly, we can show that $|\hat{\theta}_2| \leq \max\{\hat{\theta}_2(0), M_2\}$, with

$$M_2 := \theta_2 + C_2 + \frac{(\theta_2 + 1)\bar{T}_1 + \sigma(\bar{T}_1 + \bar{T}_2)}{\underline{T}_2} + 1 + 2\Delta_2\bar{T}_2. \quad (19)$$

Step 2: To study the function $V_1 + V_2$, we first define the sequences $\{T_i\}_{i=0}^\infty$ and $\{T'_i\}_{i=1}^\infty$, such that $T_i := t_{2i+1}$ and $T'_i := t_{2i}$. Since $\hat{\theta}_1$ and $\hat{\theta}_2$ are nondecreasing and become constant after some finite time, we conclude the existence of at most a finite number of intervals $[t_{2k-1}, t_{2k}) \subset I_1$, on which, $\hat{\theta}_1$ may increase. On the latter intervals, we know that V_1 is governed by the inequality $\dot{V}_1 \leq \theta_1 V_1 + C_1 \sqrt{V_1}$, while on the remaining intervals, V_1 does not verify the inequality in R1). The same reasoning applies to $\hat{\theta}_2$ and V_2 . More precisely, for each initial condition $(\hat{\theta}_1(0), \hat{\theta}_2(0))$, there exist two integers $N_1^*, N_2^* \in \mathbb{N}$ such that, for each locally absolutely continuous solution (V_1, V_2) to (9), there exist two finite increasing subsequences $\{i_1, i_2, \dots, i_{N_1^*}\} \subset \mathbb{N}$ and $\{j_1, j_2, \dots, j_{N_2^*}\} \subset \mathbb{N}^*$ such that

- For each $i \in \{i_1, i_2, \dots, i_{N_1^*}\}$, we have $V_1(T_{i+1}) \leq \left(V_1(T_i) + \frac{M_1^2}{4} \right) \exp^{(M_1+1)(T_{i+1}-T_i)}$.
- For each $j \in \{j_1, j_2, \dots, j_{N_2^*}\}$, we have $V_2(T'_{j+1}) \leq \left(V_2(T'_j) + \frac{M_2^2}{4} \right) \exp^{(M_2+1)(T'_{j+1}-T'_j)}$.
- For each $i \in \mathbb{N} \setminus \{i_1, i_2, \dots, i_{N_1^*}\}$, we have $V_1(T_{i+1}) \leq V_1(T_i) \exp^{-\sigma(T_{i+1}-T_i)} + \left(M_1 + \frac{M_1^2}{4} \right) \exp^{(M_1+1)(T_{i+1}-T_i)}$.
- For each $j \in \mathbb{N}^* \setminus \{j_1, j_2, \dots, j_{N_2^*}\}$, we have $V_2(T'_{j+1}) \leq V_2(T'_j) \exp^{-\sigma(T'_{j+1}-T'_j)} + \left(M_2 + \frac{M_2^2}{4} \right) \exp^{(M_2+1)(T'_{j+1}-T'_j)}$.

Using Lemma 6 in the Appendix, while replacing (V, M, ψ, N^*) therein by $(V_1, M_1, M_1 + 1, N_1^*)$, we obtain

$$V_1(T_i) \leq \gamma_1 V_1(0) \exp^{-\sigma T_i} + \Phi_1(M_1) \quad \forall i \in \mathbb{N},$$

for some $\gamma_1 > 0$ and $\Phi_1 \in \mathcal{K}$. Similarly, using Lemma 6 in the Appendix, while replacing $(V, M, \psi, N^*, \{T_i\}_{i=1}^\infty)$ therein by $(V_2, M_2, M_2 + 1, N_2^*, \{T'_i\}_{i=1}^\infty)$, we obtain

$$V_2(T'_i) \leq \gamma_2 V_2(0) \exp^{-\sigma T'_i} + \Phi_2(M_2) \quad \forall i \in \mathbb{N}^*,$$

for some $\gamma_2 > 0$ and $\Phi_2 \in \mathcal{K}$. As a consequence, for each $t \in [T_i, T_{i+1}]$,

$$V_1(t) \leq \gamma_1 \exp^{(\theta_1+1+\sigma)(\bar{T}_1+\bar{T}_2)} V_1(0) \exp^{-\sigma t} + \left[\Phi_1(M_1) + \frac{M_1^2}{4(\theta_1+1)} \right] \exp^{(\theta_1+1)(\bar{T}_1+\bar{T}_2)}. \quad (20)$$

Moreover, for each $t \in [T'_i, T'_{i+1}]$,

$$V_2(t) \leq \gamma_2 \exp^{(\theta_2+2+\sigma)(\bar{T}_1+\bar{T}_2)} V_2(0) \exp^{-\sigma t} + \left[\Phi_2(M_2) + \frac{M_2^2}{4(\theta_2+1)} \right] \exp^{(\theta_2+1)(\bar{T}_1+\bar{T}_2)}. \quad (21)$$

Defining

$$\gamma := \max \left\{ \gamma_1 \exp^{(\theta_1+1+\sigma)(\bar{T}_1+\bar{T}_2)}, \gamma_2 \exp^{(\theta_2+2+\sigma)(\bar{T}_1+\bar{T}_2)} \right\},$$

$$\Phi := \left[\Phi_1(M_1) + \frac{M_1^2}{4(\theta_1+1)} \right] \exp^{(\theta_1+1)(\bar{T}_1+\bar{T}_2)} + \left[\Phi_2(M_2) + \frac{M_2^2}{4(\theta_2+1)} \right] \exp^{(\theta_2+1)(\bar{T}_1+\bar{T}_2)},$$

and summing (20) and (21), which are valid for all $t \geq 0$, we obtain

$$V_1(t) + V_2(t) \leq \gamma(V_1(0) + V_2(0)) \exp^{-\sigma t} + \Phi.$$

Let $r := \Phi + \epsilon$, where ϵ is any positive constant, and suppose that $V_1(0) + V_2(0) \leq R$. We conclude that $V_1(t) + V_2(t) \leq \gamma R \exp^{-\sigma t} + \Phi$ for all $t \geq 0$. Hence, to guarantee that $\gamma R \exp^{-\sigma t} + \Phi \leq r$, it is sufficient to have $t \geq T(R) := \frac{1}{\sigma} \log \left(\frac{\gamma R}{\epsilon} \right)$.

Step 3: The boundedness of (u_1, u_2) follows from the boundedness of $(\hat{\theta}_1, \hat{\theta}_2)$, the L^2 -global uniform ultimate boundedness property of the closed-loop solutions, and the structure of κ in (10) which implies that $|u_i| \leq \max\{\epsilon(\hat{\theta}_i + \delta), 1\} \sqrt[3]{V_i}$ for $i \in \{1, 2\}$. ■

4.2 Known perturbation range

In this section, we use the knowledge of (C_1, C_2) in (9) to design $(\hat{\theta}_1, \hat{\theta}_2)$ according to the following algorithm.

Algorithm 2 Given $\Delta_1, \Delta_2, \sigma > 0$, we update $(\hat{\theta}_1, \hat{\theta}_2)$ according to the following rules.

R1) On each interval $[t_{2k-1}, t_{2k}) \subset I_1$, we set $\dot{\hat{\theta}}_2 := 0$. Moreover, if

$$V_1(t_{2k-1}) > V_1(t_{2k-3}) \exp^{-\sigma(t_{2k-1}-t_{2k-3})} + \left(C_1 + \frac{C_1^2}{4} \right) \exp^{(\hat{\theta}_1(t_{2k-3})+1)(t_{2k-1}-t_{2k-3})},$$

we set $\dot{\hat{\theta}}_1 := \Delta_1$; otherwise, we set $\dot{\hat{\theta}}_1 := 0$.

R2) On each interval $[t_{2k}, t_{2k+1}) \subset I_2$, we set $\dot{\hat{\theta}}_1 := 0$. Moreover, if

$$V_2(t_{2k}) > V_2(t_{2k-2}) \exp^{-\sigma(t_{2k}-t_{2k-2})} + \left(C_2 + \frac{C_2^2}{4} \right) \exp^{(\hat{\theta}_2(t_{2k-2})+1)(t_{2k}-t_{2k-2})},$$

we set $\dot{\hat{\theta}}_2 := \Delta_2$; otherwise, we set $\dot{\hat{\theta}}_2 := 0$.

R3) On $[t_1, t_3]$, $\hat{\theta}_1 = \hat{\theta}_1(0) \geq 0$ and $\hat{\theta}_2 = \hat{\theta}_2(0) \geq 0$.

The main difference between Algorithms 1 and 2, is that in the latter case, the inequality in R1) uses the constant C_1 . As a result, along the lines of Remark 2, we show that the map $k \mapsto V_1(t_{2k-1})$ converges to the ball centered at

the origin of radius $\left(C_1 + \frac{C_1^2}{4} \right) \exp^{(\theta_1^*+1)(\bar{T}_1+\bar{T}_2)}$. Hence, since $C_1 := \sqrt{2Y} \bar{f}$, we conclude that the latter radius is a class \mathcal{K} function of \bar{f} . The same reasoning applies for the behavior of V_2 .

Theorem 2 Consider system (2) such that Assumption 2 holds. Consider the intermittent sensing scenario in S1)-S2) in Section 2.1 such that Assumption 1 holds. Let (V_1, V_2) be defined in (4), κ be defined in (10), and the parameters $(\hat{\theta}_1, \hat{\theta}_2)$ governed by Algorithm 2. Then, system (2) in closed loop with (u_1, u_2) as in (7) satisfies:

- L^2 -input-to-state stability with respect to \bar{f} ; namely, for each $(\hat{\theta}_1(0), \hat{\theta}_2(0))$, there exists $\gamma \geq 1$ and a class \mathcal{K} function Φ such that, for all $t \geq 0$, we have

$$V_1(t) + V_2(t) \leq \gamma(V_1(0) + V_2(0)) e^{-\sigma t} + \Phi(\bar{f}),$$

where $\sigma > 0$ comes from Algorithm 2.

- For each $(\hat{\theta}_1(0), \hat{\theta}_2(0))$, there exists $M > 0$ such that $|\hat{\theta}_1|_\infty, |\hat{\theta}_2|_\infty \leq M$ for all $(V_1(0), V_2(0))$.
- The inputs (u_1, u_2) remain bounded. Additionally, if $\bar{f} = 0$, then (u_1, u_2) converge to zero.

Proof. The proof is divided into three steps. In Step 1, we prove that $\hat{\theta}_1$ and $\hat{\theta}_2$ become constant after some $T > 0$ and are bounded uniformly with respect to (V_1, V_2) . In Step 2, we analyze the function $V_1 + V_2$ and show that the L^2 -input-to-state stability property with respect to f is verified. Finally, the boundedness of (u_1, u_2) follows, as in Step 3 in the proof of Theorem 3, from the structure of κ and the boundedness of $(V_1, V_2, \hat{\theta}_1, \hat{\theta}_2)$.

Step 1: We show that $\hat{\theta}_1$ and $\hat{\theta}_2$ are constant after some $T \geq 0$ using contradiction. Assume that there is not such a finite time $T \geq 0$ such that $\dot{\hat{\theta}}_1(t) = \dot{\hat{\theta}}_2(t) = 0$ for all $t \geq T$. As a result, according to R1)-R3) in Algorithm 2, $\lim_{t \rightarrow \infty} \hat{\theta}_1(t) = \lim_{t \rightarrow \infty} \hat{\theta}_2(t) = \infty$. Therefore, there exists $T \geq 0$ such that, for all $t \geq T$, inequalities (11) and (12) hold. Let $k > 1$ be such that $t_{2k-3} \geq T$. Inequality (15) hold. Furthermore, using Lemma 4 in the Appendix while replacing therein $(V, \hat{\theta}, \theta, C)$ by $(V_1, \hat{\theta}_1, \theta_1, C_1)$, we obtain,

$$V_1(t_{2k-2}) \leq V_1(t_{2k-3}) \exp^{-\sigma(t_{2k-2}-t_{2k-3})} + C_1. \quad (22)$$

By combining (15) and (22), we obtain

$$V_1(t_{2k-1}) \leq V_1(t_{2k-3}) \exp^{-\sigma(t_{2k-1}-t_{2k-3})} + (C_1 + C_1^2/4) \exp^{(\hat{\theta}_1(t_{2k-3})+1)(t_{2k-1}-t_{2k-3})},$$

which implies, according to R1) in Algorithm 2, that $\dot{\hat{\theta}}_1(t) = 0$ for all $t \geq t_{2k-1}$. We show in a similar way that $\dot{\hat{\theta}}_2(t) = 0$ for all $t \geq t_{2k}$, which leads to a contradiction. We follow the exact same steps as in the proof of Theorem 1 to conclude that $|\hat{\theta}_1| \leq \max\{\hat{\theta}_1(0), M_1\}$ and

$|\hat{\theta}_2| \leq \max\{\hat{\theta}_2(0), M_2\}$, where M_1 and M_2 are defined in (18) and (19) respectively.

Step 2: To study the function $V_1 + V_2$, we introduce the sequences $\{T_i\}_{i=0}^\infty$ and $\{T'_i\}_{i=1}^\infty$, such that $T_i := t_{2i+1}$ and $T'_i := t_{2i}$. As in the proof of Theorem 1, for each initial conditions $(\hat{\theta}_1(0), \hat{\theta}_2(0))$, there exist two integers $N_1^*, N_2^* \in \mathbb{N}$ such that, for each locally absolutely continuous solution (V_1, V_2) to (9), there exist two finite increasing subsequences $\{i_1, i_2, \dots, i_{N_1^*}\} \subset \mathbb{N}$ and $\{j_1, j_2, \dots, j_{N_2^*}\} \subset \mathbb{N}^*$ such that

- For each $i \in \{i_1, i_2, \dots, i_{N_1^*}\}$, we have

$$V_1(T_{i+1}) \leq \left(V_1(T_i) + \frac{C_1^2}{4} \right) \exp^{(M_1+1)(T_{i+1}-T_i)}.$$
- For each $j \in \{j_1, j_2, \dots, j_{N_2^*}\}$, we have

$$V_2(T'_{j+1}) \leq \left(V_2(T'_j) + \frac{C_2^2}{4} \right) \exp^{(M_2+1)(T'_{j+1}-T'_j)}.$$
- For each $i \in \mathbb{N}/\{i_1, i_2, \dots, i_{N_1^*}\}$, we have

$$V_1(T_{i+1}) \leq V_1(T_i) \exp^{-\sigma(T_{i+1}-T_i)} + \left(C_1 + \frac{C_1^2}{4} \right) \exp^{(M_1+1)(T_{i+1}-T_i)}.$$
- For each $j \in \mathbb{N}^*/\{j_1, j_2, \dots, j_{N_2^*}\}$, we have

$$V_2(T'_{j+1}) \leq V_2(T'_j) \exp^{-\sigma(T'_{j+1}-T'_j)} + \left(C_2 + \frac{C_2^2}{4} \right) \exp^{(M_2+1)(T'_{j+1}-T'_j)}.$$

Using Lemma 6 in the Appendix, while replacing (V, M, ψ, N^*) therein by $(V_1, C_1, M_1 + 1, N_1^*)$, we obtain the inequality

$$V_1(T_i) \leq \gamma_1 V_1(0) \exp^{-\sigma T_i} + \Phi_1(C_1) \quad \forall i \in \mathbb{N}, \quad (23)$$

for some $\gamma_1 > 0$ and $\Phi_1 \in \mathcal{K}$. Similarly, using Lemma 6 in the Appendix, while replacing $(V, M, \psi, N^*, \{T_i\}_{i=1}^\infty)$ therein by $(V_2, C_2, M_2 + 1, N_2^*, \{T'_i\}_{i=1}^\infty)$, we obtain

$$V_2(T'_i) \leq \gamma_2 V_2(0) \exp^{-\sigma T'_i} + \Phi_2(C_2) \quad \forall i \in \mathbb{N}^*, \quad (24)$$

for some $\gamma_2 > 0$ and $\Phi_2 \in \mathcal{K}$. As a consequence, for each $t \in [T_i, T_{i+1}]$, we have

$$V_1(t) \leq \gamma_1 \exp^{(\theta_1+1+\sigma)(\bar{T}_1+\bar{T}_2)} V_1(0) \exp^{-\sigma t} + \left[\Phi_1(C_1) + \frac{C_1^2}{4(\theta_1+1)} \right] \exp^{(\theta_1+1)(\bar{T}_1+\bar{T}_2)}. \quad (25)$$

Similarly, for each $t \in [T'_i, T'_{i+1}]$, we have

$$V_2(t) \leq \gamma_2 \exp^{(\theta_2+2+\sigma)(\bar{T}_1+\bar{T}_2)} V_2(0) \exp^{-\sigma t} + \left[\Phi_2(C_2) + \frac{C_2^2}{4(\theta_2+1)} \right] \exp^{(\theta_2+1)(\bar{T}_1+\bar{T}_2)}. \quad (26)$$

Defining

$$\begin{aligned} \gamma &:= \\ &\max \left\{ \gamma_1 \exp^{(\theta_1+1+\sigma)(\bar{T}_1+\bar{T}_2)}, \gamma_2 \exp^{(\theta_2+2+\sigma)(\bar{T}_1+\bar{T}_2)} \right\}, \\ \Phi(\bar{f}) &:= \left[\Phi_1(C_1) + \frac{C_1^2}{4(\theta_1+1)} \right] \exp^{(\theta_1+1)(\bar{T}_1+\bar{T}_2)} \\ &\quad + \left[\Phi_2(C_2) + \frac{C_2^2}{4(\theta_2+1)} \right] \exp^{(\theta_2+1)(\bar{T}_1+\bar{T}_2)}, \end{aligned}$$

and combining (25) and (26), we obtain, for all $t \geq 0$,

$$V_1(t) + V_2(t) \leq \gamma(V_1(0) + V_2(0)) \exp^{-\sigma t} + \Phi(\bar{f}). \quad (27)$$

■

Remark 4 The main result in [17] follows now as a direct corollary of Theorem 2. Indeed, using the latter theorem with $f \equiv 0$, we recover the L^2 global exponential stability of the origin.

5 Result under full sensing

In this section, we consider (1) subject to

$$u(0) = u_1, \quad u_x(0) = u(1) = u_x(1) = 0, \quad (28)$$

where u_1 is a control input to be designed. We assume that λ is absolutely continuous on $(0, 1)$ and that u is measured a.e. on $(0, 1)$ and for all time. As a result, defining the Lyapunov function candidate

$$V(u) := \frac{1}{2} \int_0^1 u(x)^2 dx, \quad (29)$$

and using Lemma 1, we conclude that, along (1) and (28),

$$\dot{V} \leq \theta V + C\sqrt{V} + \frac{u_1^3}{3} + u_1 u_{xxx}(0), \quad C := \sqrt{2} \bar{f} \quad (30)$$

where $\theta := |\lambda|_\infty^2 + 2(|\lambda|_\infty + \frac{1}{2})((|\lambda|_\infty + \frac{1}{2}) + 12)$.

Now, by letting $u_1 := \kappa(V, u_{xxx}(0), \hat{\theta})$, where κ is defined in (10) and $\hat{\theta} > 0$ to be designed, we conclude using (30) and Lemma 2 that

$$\dot{V} \leq (\theta - \hat{\theta})V + C\sqrt{V}.$$

Finally, using Young inequality, we obtain

$$\dot{V} \leq \left(\theta + \frac{C^2}{\epsilon} - \hat{\theta} \right) V + \epsilon \quad \forall \epsilon > 0. \quad (31)$$

At this point, we propose to update the parameter $\hat{\theta}$ according to the following algorithm.

Algorithm 3 Given $\Delta, \tau, \epsilon, \sigma > 0$, the coefficient $\hat{\theta}$ is dynamically updated, on each interval $[k\tau, (k+1)\tau]$ with $k \in \mathbb{N}^*$, according to the following rules:

R1) For each $t \in [k\tau, (k+1)\tau]$ if

$$V(s) \leq V(k\tau) \exp^{-\sigma(s-k\tau)} + \frac{\epsilon}{\sigma} \quad \forall s \in [k\tau, t],$$

then $\dot{\hat{\theta}}(t) = 0$; otherwise,

$$\hat{\theta}(r) = \hat{\theta}(k\tau) + \Delta \quad \forall r \in [t, (k+1)\tau]. \quad (32)$$

R2) On the interval $[0, \tau)$, we set $\hat{\theta} = \hat{\theta}(0) \geq 0$.

According to Algorithm 3, if $t \mapsto \hat{\theta}(t)$ is bounded, being non-decreasing, we can show that $\hat{\theta}$ becomes constant after some finite time. Hence, according to R1) in Algorithm 3, the function $t \mapsto V(t)$ converges to a ball of radius ϵ/σ . Since such a radius can be made arbitrarily small, we conclude that we can achieve the convergence of V to an arbitrarily small neighborhood of the origin.

Theorem 3 Consider system (1) under (28) such that Assumption 2 holds. Let V be defined in (29), κ be defined in (10), and the parameter $\hat{\theta}$ governed by Algorithm 3. Then, we conclude that (1) and (28) in closed-loop with $u := \kappa(V, u_{xxx}(0), \hat{\theta})$ satisfies the following properties:

- L^2 globally practically attractivity (GpA) of the origin; namely, for any $\eta > 0$, we can find adaptation gains (τ, σ, ϵ) such that, for any $\hat{\theta}(0) \geq 0$ and for any $V(0) \geq 0$, we have $\limsup_{t \rightarrow +\infty} V(t) \leq \eta$.
- For each $\hat{\theta}(0)$, there exists $M > 0$ such that $|\hat{\theta}|_\infty \leq M$ for all $V(0)$.
- The input u_1 remains bounded.

Proof. We first show that $\hat{\theta}$ admits an upperbound that does not depend on $V(0)$. As a result, since $\hat{\theta}$ is non-decreasing and, when it increases, it does so according to (32), we conclude that $\hat{\theta}$ becomes constant after some finite time $T \geq 0$. Next, we analyze the Lyapunov function candidate V in (29) and show that, after a finite time, V starts decaying exponentially towards a neighborhood of the origin, whose size is proportional to ϵ . Hence, we conclude L^2 -global practical attractivity of the origin. Finally, using boundedness of $(V, \hat{\theta})$ and the structure of the feedback law κ , boundedness of the control input u_1 follows. Let us show that

$$\hat{\theta}(t) \leq \max\{\theta + C^2/\epsilon + \sigma + \Delta, \hat{\theta}(0)\} \quad \forall t \geq 0. \quad (33)$$

To conclude that the inequality (33) is verified, we first suppose that $\hat{\theta}(0) \leq \theta + C^2/\epsilon + \sigma$. As a result, either

$$\hat{\theta}(t) \leq \theta + C^2/\epsilon + \sigma \quad \forall t \geq 0.$$

Otherwise, in view of R1) in Algorithm 3, there exist $k^* \geq 1$ and $t \in [(k^* - 1)\tau, k^*\tau]$ such that

$$\theta + C^2/\epsilon + \sigma < \hat{\theta}(t) \leq \theta + C^2/\epsilon + \sigma + \Delta.$$

Using (31), we then conclude that

$$V(s) \leq V(t) \exp^{-\sigma(s-t)} + \frac{\epsilon}{\sigma} \quad \forall s \in [t, k^*\tau].$$

This implies that $\hat{\theta}$ is constant on $[t, k^*\tau]$. Since $\hat{\theta}$ is non-decreasing, it follows that

$$V(s) \leq V(t) \exp^{-\sigma(s-t)} + \frac{\epsilon}{\sigma} \quad \forall s \geq t,$$

which in turn implies that

$$\hat{\theta}(s) = \hat{\theta}(t) \leq \theta + C^2/\epsilon + \sigma + \Delta \quad \forall s \geq t.$$

If $\hat{\theta}(0) > \theta + C^2/\epsilon + \sigma$, we use the fact that $\hat{\theta}$ is non-decreasing to conclude that

$$V(s) \leq V(0) \exp^{-\sigma s} + \frac{\epsilon}{\sigma} \quad \forall s \geq 0,$$

which in turn implies that $\hat{\theta}(s) = \hat{\theta}(0)$ for all $s \geq 0$. To analyze the function V , we let $k \geq 0$ such that $\hat{\theta}$ is constant on $[k\tau, +\infty)$. As a result, for each $t \in [k\tau, +\infty)$, there exists $n \geq 0$ such that $t \in [(k+n)\tau, (k+1+n)\tau)$. Now, according to R1), we have

$$V(t) \leq V((k+n)\tau) \exp^{-\sigma(t-(k+n)\tau)} + \frac{\epsilon}{\sigma} \quad \forall t \in [(k+n)\tau, (k+1+n)\tau]. \quad (34)$$

Moreover, by continuity of V and according to R1), we conclude that, for each $i \in \{1, \dots, n\}$, we have

$$V((k+i)\tau) \leq V((k+i-1)\tau) \exp^{-\sigma\tau} + \frac{\epsilon}{\sigma}.$$

We show next that

$$V((k+n)\tau) \leq V(k\tau) \exp^{-\sigma n\tau} + \frac{\epsilon}{\sigma(1 - \exp^{-\sigma\tau})}. \quad (35)$$

Indeed, the latter inequality combined with (34) allows us to conclude L^2 -global practical attractivity of the origin. To prove (35), we note that

$$\begin{aligned} V((k+n)\tau) &\leq V((k+n-1)\tau) \exp^{-\sigma\tau} + \frac{\epsilon}{\sigma} \\ &\leq V((k+n-2)\tau) \exp^{-2\sigma\tau} + \frac{\epsilon}{\sigma} (1 + \exp^{-\sigma\tau}) \\ &\leq V(k\tau) \exp^{-\sigma n\tau} + \frac{\epsilon}{\sigma} \left(\sum_{j=0}^n \exp^{-j\sigma\tau} \right). \end{aligned}$$

Finally, (35) follows using the fact that

$$\sum_{j=0}^n \exp^{-j\sigma\tau} \leq \sum_{j=0}^{\infty} \exp^{-j\sigma\tau} \leq \frac{1}{1 - \exp^{-\sigma\tau}}.$$

■

Remark 5 Under full sensing and when $f \equiv 0$, we can guarantee L^2 global exponential stability of the origin by following an approach analogous to the one in this section. More specifically, since $f \equiv 0$, then $\dot{V} \leq (\theta - \hat{\theta})V$. The parameter $\hat{\theta}$ is then updated according to Algorithm 3, while setting $\epsilon := 0$ therein. The rest follows using the arguments in the proof of Theorem 3.

6 Simulation Example

The system (1) under (3) is simulated with the controller introduced in Section 4.1. We set $\lambda(x) := 4\pi^2/0.25 + 50 + 2\sin(4x)^2$ for all $x \in (0, Y)$ and $\lambda(x) := \lambda(x - Y)$ for all $x \in (Y, 1)$ and $f(x, t) := 12 \times 10^3 \sin(2 \times 10^4 t) + \xi(x, t)$, where ξ is a white noise of power 80 dBW generated using the Matlab function 'wgn'. We set $I_1 := [0, 1) \cup [2, 2.8) \cup [3.9, 5) \cup [5.5, 6.5) \cup [7, 7.6) \times 10^{-3}$ and $I_2 := [1, 2) \cup [2.8, 3.9) \cup [5, 5.5) \cup [6.5, 7) \cup [7.6, 8) \times 10^{-3}$. Finally, (u_1, u_2) are given by (7), with κ defined in (10) for $(\varepsilon, \delta) := (1, 2)$, and $\hat{\theta}_1$ and $\hat{\theta}_2$ designed according to Algorithm 1 with $\hat{\theta}_1(0) = \hat{\theta}_2(0) = 0$, $\Delta_1 = \Delta_2 = 0.01$, and $\sigma = 100$.

The system is discretized with the mesh-free collocation method in [26], based on radial basis functions (RBFs). We estimate the boundary values $w_{xxx}(0)$ and $w_x(0)$ using the Euler forward scheme and $v_{xxx}(1)$ and $v_x(1)$ using the Euler backward scheme. The Lyapunov function candidates (V_1, V_2) are estimated using Riemannian sums.

We use multiquadric RBFs, which depend on a shape parameter $c := 0.4$. The initial and final simulation times are selected as $t_1 := 0$ and $t_f := 8 \times 10^{-3}$, respectively. We select 10 uniformly separated collocation points on the interval $[0, Y]$ (with $Y := 0.5$), ranging from $x_o := 0$ to $x_9 := Y$, and select the same number of points on the interval $[Y, 1]$. Hence, our spatial step is $\Delta x \approx 0.05$, which makes us choose the time step of $\Delta t := 10^{-7}$ to keep the ratio $\Delta t/\Delta x^4$ small. For comparison, the time step $\Delta t := 10^{-9}$ and the spatial step $\Delta x := 5 \times 10^{-3}$ are used in [27], to simulate the third-order Korteweg-de Vries-Burgers' equation defined on $[0, 1]$. Finally, the initial condition $u_o(x) := -A(\cos(4\pi x) - 1)$ for all $x \in (0, 1)$ is tested for different values of $A > 0$.

Figure 1 shows the closed-loop solutions corresponding to the initial condition u_o under $A = 3$. The corresponding control inputs are shown in Figure 2. As expected, the inputs are bounded, at the price of some chattering due to the discontinuous nature of our controller. Furthermore, the corresponding plots of $t \mapsto (V_1(t), V_2(t))$ and $t \mapsto V_1(t) + V_2(t)$ are depicted in Figure 3. On every interval in I_2 , where $u_1 = 0$, the final value of V_1 is greater than its initial value at the beginning of that time interval. This increase is compensated over the next intervals in I_1 , where $u_1 = \kappa(V_1, w_{xxx}(0), \hat{\theta}_1)$. The same behavior is observed for V_2 , which increases over intervals in I_1 and decreases over intervals in I_2 . In Figure 4, we plot the evolution of the adaptation parameters $t \mapsto (\hat{\theta}_1(t), \hat{\theta}_2(t))$,

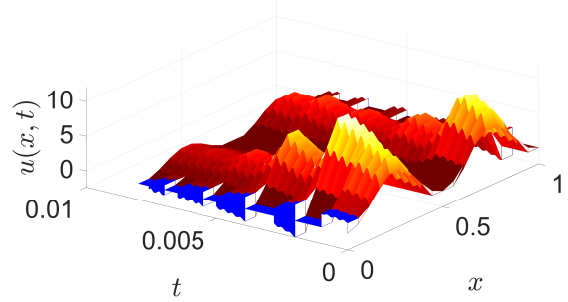


Fig. 1. The KS response to (3), (7), and Algorithm 1.

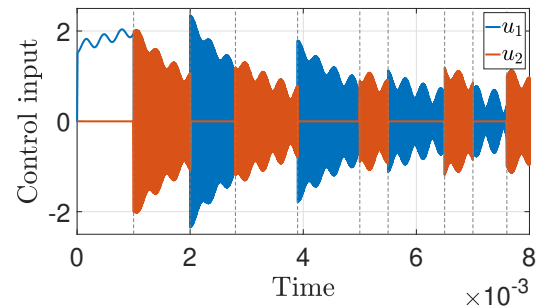


Fig. 2. The inputs (u_1, u_2) in (7) and Algorithm 1.

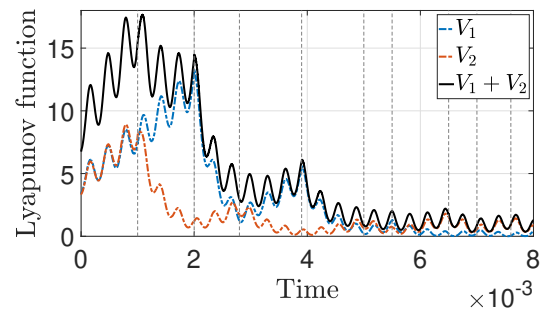


Fig. 3. Lyapunov functions (V_1, V_2) and $V_1 + V_2$ along the KS response under (3), (7), and Algorithm 1.

which shows the existence of an adaptation phase, during which $\hat{\theta}_1$ and $\hat{\theta}_2$ increase before they become, as expected, constant. Finally, Figure 5 illustrate the evolution of $t \mapsto V_1(t) + V_2(t)$ for the initial condition u_o with $A \in \{2, 3, 4, 5, 6, 7\}$. The conclusions of Theorem 1 are in agreement with Figure 5, which shows finite-time convergence of $V_1 + V_2$ to a bound that is independent of the various choices of A .

7 Conclusion

In this paper, we studied the boundary stabilization problem for the perturbed KS equation under an intermittent sensing scenario, and compared the obtained results with those we obtained with full sensing. The intermittent sensing scenario forces us to consider in-domain conditions, that we do not necessarily require under full

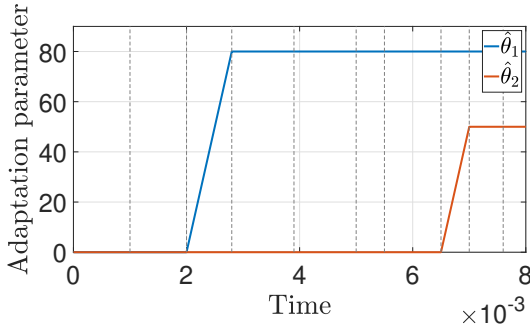


Fig. 4. The adaptation parameters $(\hat{\theta}_1, \hat{\theta}_2)$ constructed according to Algorithm 1.

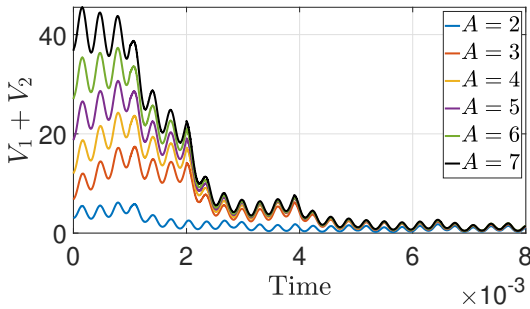


Fig. 5. $V_1 + V_2$ along the KS response to (3), (7), and Algorithm 1, for different sizes of initial condition.

sensing. Our design is Lyapunov-based, tailored to some adaptive design to handle unknown parameters and perturbations. The obtained results are summarized in the following table.

| Sensing Scenario | $f = 0$ | \bar{f} known | \bar{f} unknown |
|----------------------|------------|-----------------|-------------------|
| Full Sensing | L^2 -GES | L^2 -GpA | L^2 -GpA |
| Intermittent Sensing | L^2 -GES | L^2 -ISS | L^2 -GUUB |

Several challenging research directions emerge from our work. A primary extension would involve considering measurements taken intermittently over an arbitrary number of subregions, rather than just two. Another compelling direction concerns uncertainty in the location Y , which becomes particularly relevant when the Lyapunov functions (V_1, V_2) must be approximated numerically using sensor measurements. A related scenario arises when enforcing the in-domain condition at $x = Y$ while gathering measurements over intervals $[0, Y_o]$ and $[Y_o, 1]$ with $Y_o < Y$. The consideration of packet losses under an average dwell-time condition is also an interesting research perspective. Finally, extending our approach to accommodate measurements from scanning and pointwise sensors [21,28] represents a natural progression of our work.

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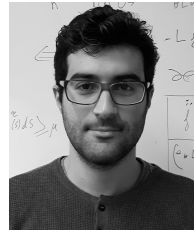
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Appendix

Lemma 3 ([25], page 84) Given $u : (a, b) \rightarrow \mathbb{R}$ be twice continuously differentiable and let $\epsilon > 0$. Then,

$$\int_a^b u_x^2 dx \leq \left[\frac{1}{\epsilon} + \frac{12}{(b-a)^2} \right] \int_a^b u^2 dx + \epsilon \int_a^b u_{xx}^2 dx. \quad (36)$$

Lemma 4 Let $\{t_i\}_{i=1}^\infty \subset \mathbb{R}$, with $t_1 = 0$ and $t_{i+1} > t_i$, $I_1 := \bigcup_{k=1}^\infty [t_{2k-1}, t_{2k}]$, and $I_2 := \bigcup_{k=1}^\infty [t_{2k}, t_{2k+1}]$. Moreover, let $\underline{T}_1 \leq t_{2k} - t_{2k-1} \leq \bar{T}_1$ and $\underline{T}_2 \leq t_{2k+1} - t_{2k} \leq \bar{T}_2$ for all $k \geq 1$, and for some constants $\underline{T}_1, \underline{T}_2, \bar{T}_1, \bar{T}_2 > 0$. Furthermore, let $\theta, C \in \mathbb{R}_{\geq 0}$, and $t \mapsto \hat{\theta}(t) \geq 0$ be a non-decreasing function such that there exists $T \geq 0$, such that, for all $t \geq T$, we have

$$\hat{\theta}(t) \geq \theta + C + \frac{(\theta + 1)\bar{T}_2 + \sigma(\bar{T}_1 + \bar{T}_2)}{\underline{T}_1} + 1, \quad (37)$$

where $\sigma > 0$. We let $V : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be a locally absolutely continuous solution to the switched system of differential inequalities

$$\begin{cases} \dot{V} \leq (\theta - \hat{\theta})V + C\sqrt{V} & \text{a.e. on } I_1, \\ \dot{V} \leq \theta V + C\sqrt{V} & \text{a.e. on } I_2. \end{cases} \quad (38)$$

Then, for all $k > 1$ such that $t_{2k-3} \geq T$, we have

$$V(t_{2k-2}) \leq V(t_{2k-3}) \exp^{-\sigma(t_{2k-2} - t_{2k-3})} + C, \quad (39)$$

$$V(t_{2k-2}) \leq V(t_{2k-3}) \exp^{-\sigma(t_{2k-2} - t_{2k-3})} + \hat{\theta}(t_{2k-3}). \quad (40)$$

Proof. To prove (40), first note that, a.e. on $[t_{2k-3}, t_{2k-2}) \subset I_1$, we have, according to (38),

$$\begin{aligned} \dot{V} &\leq (\theta - \hat{\theta})V + C\sqrt{V} \\ &\leq (\theta + C - \hat{\theta})V + C, \end{aligned} \quad (41)$$

where we have used the fact that $\sqrt{V} \leq V + 1$. Integrating (41), we obtain

$$\begin{aligned} V(t_{2k-2}) &\leq V(t_{2k-3}) \exp^{\int_{t_{2k-3}}^{t_{2k-2}} (\theta + C - \hat{\theta}(t)) dt} \\ &\quad + C \int_{t_{2k-3}}^{t_{2k-2}} \exp^{\int_{\tau}^{t_{2k-2}} (\theta + C - \hat{\theta}(s)) ds} d\tau. \end{aligned}$$

Furthermore, since $t \mapsto \hat{\theta}(t)$ is non-decreasing, we conclude that $t \mapsto \theta + C - \hat{\theta}(t)$ is non-increasing. Therefore,

$$\theta + C - \hat{\theta}(t) \leq \theta + C - \hat{\theta}(t_{2k-3}), \quad \forall t \geq t_{2k-3}.$$

Using the latter inequality, we can write

$$\begin{aligned} V(t_{2k-2}) &\leq V(t_{2k-3}) \exp^{(\theta + C - \hat{\theta}(t_{2k-3}))(t_{2k-2} - t_{2k-3})} \\ &\quad + C \int_{t_{2k-3}}^{t_{2k-2}} \exp^{(\theta + C - \hat{\theta}(t_{2k-3}))(t_{2k-2} - \tau)} d\tau \\ &\leq V(t_{2k-3}) \exp^{(\theta + C - \hat{\theta}(t_{2k-3}))(t_{2k-2} - t_{2k-3})} \\ &\quad + C \left[\frac{\exp^{(\theta + C - \hat{\theta}(t_{2k-3}))(t_{2k-2} - \tau)}}{-(\theta + C - \hat{\theta}(t_{2k-3}))} \right]_{\tau=t_{2k-3}}^{\tau=t_{2k-2}} \\ &\leq V(t_{2k-3}) \exp^{(\theta + C - \hat{\theta}(t_{2k-3}))(t_{2k-2} - t_{2k-3})} \\ &\quad - \left[\frac{C}{(\theta + C - \hat{\theta}(t_{2k-3}))} \right] \\ &\quad + C \left[\frac{\exp^{(\theta + C - \hat{\theta}(t_{2k-3}))(t_{2k-2} - t_{2k-3})}}{(\theta + C - \hat{\theta}(t_{2k-3}))} \right]. \end{aligned}$$

Next, in view of inequality (37), we conclude that

$$\begin{aligned} [\theta + C - \hat{\theta}(t_{2k-3})] \underline{T}_1 &\leq -\underline{T}_1 - (\theta + 1)\bar{T}_2 - \sigma(\bar{T}_1 + \bar{T}_2) \\ &< -\sigma(\bar{T}_1 + \bar{T}_2). \end{aligned} \quad (42)$$

and

$$\theta + C - \hat{\theta}(t_{2k-3}) \leq -1.$$

Now, using the fact that

$$\underline{T}_1 \leq t_{2k-2} - t_{2k-3} \quad \text{and} \quad \theta + C - \hat{\theta}(t_{2k-3}) < 0,$$

we obtain

$$\begin{aligned} \exp^{(\theta + C - \hat{\theta}(t_{2k-3}))(t_{2k-2} - t_{2k-3})} &\leq \exp^{(\theta + C - \hat{\theta}(t_{2k-3}))\underline{T}_1}, \\ C \left[\frac{\exp^{(\theta + C - \hat{\theta}(t_{2k-3}))(t_{2k-2} - t_{2k-3})}}{(\theta + C - \hat{\theta}(t_{2k-3}))} \right] &< 0, \end{aligned}$$

and thus

$$\begin{aligned} V(t_{2k-2}) &\leq V(t_{2k-3}) \exp^{(\theta + C - \hat{\theta}(t_{2k-3}))\underline{T}_1} \\ &\quad - \left[\frac{C}{(\theta + C - \hat{\theta}(t_{2k-3}))} \right] \\ &\quad + C \left[\frac{\exp^{(\theta + C - \hat{\theta}(t_{2k-3}))(t_{2k-2} - t_{2k-3})}}{(\theta + C - \hat{\theta}(t_{2k-3}))} \right] \\ &\leq V(t_{2k-3}) \exp^{(\theta + C - \hat{\theta}(t_{2k-3}))\underline{T}_1} \\ &\quad - \left[\frac{C}{(\theta + C - \hat{\theta}(t_{2k-3}))} \right]. \end{aligned}$$

Furthermore, in view of (42) and, since $t_{2k-2} - t_{2k-3} \leq \bar{T}_1 + \bar{T}_2$, we conclude that

$$\begin{aligned} \exp^{(\theta + C - \hat{\theta}(t_{2k-3}))\underline{T}_1} &\leq \exp^{-\sigma(\bar{T}_1 + \bar{T}_2)} \\ &\leq \exp^{-\sigma(t_{2k-2} - t_{2k-3})}. \end{aligned}$$

Combining the aforementioned inequalities, and the fact that $-\frac{C}{\theta+C-\hat{\theta}(t_{2k-3})} \leq C$, we obtain

$$\begin{aligned} V(t_{2k-2}) &\leq V(t_{2k-3}) \exp^{(\theta+C-\hat{\theta}(t_{2k-3}))\underline{T}_1} + C \\ &\leq V(t_{2k-3}) \exp^{-\sigma(t_{2k-2}-t_{2k-3})} + C. \end{aligned}$$

Finally, (40) is obtained by combining (39) with the fact that, from (37), we have $\hat{\theta}(t_{2k-3}) \geq C$. \blacksquare

Lemma 5 *Let $V : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be a locally absolutely continuous solution to the differential inequality*

$$\dot{V} \leq \theta V + C\sqrt{V} \quad \text{a.e. on } [0, T] \subset \mathbb{R}_{\geq 0}, \quad (43)$$

where $\theta, C \geq 0$ are constants. Then, for any constant $\delta > 0$ we have, for all $t \in [0, T]$,

$$V(t) \leq V(0) \exp^{(\theta+\delta)t} + \frac{C^2/4\delta}{(\theta+\delta)} \left(\exp^{(\theta+\delta)t} - 1 \right). \quad (44)$$

Proof. Let $\delta > 0$ and consider the function $f(V) := \sqrt{V} - \frac{\delta V}{C} - \frac{C}{4\delta}$. By differentiating f , we find for all $V > 0$, $f'(V) = \frac{1}{2\sqrt{V}} - \frac{\delta}{C}$. As a result, the function f is strictly increasing on $[0, C^2/(4\delta^2)]$ and strictly decreasing on $[C^2/(4\delta^2), \infty)$. Moreover, $f(0) = -C/4\delta$, and $f(C^2/(4\delta^2)) = 0$. Therefore, for all $V \geq 0$, we have $f(V) \leq 0$. As a consequence, we can rewrite (43) as $\dot{V} \leq \theta V + C\sqrt{V} \leq (\theta + \delta)V + \frac{C^2}{4\delta}$. By integrating this inequality from 0 to t , (44) follows. \blacksquare

Lemma 6 *Let the function $V : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be locally absolutely continuous, let a sequence $\{T_i\}_{i=0}^{\infty}$ and $\underline{T}, \bar{T} > 0$ such that $T_0 = 0$ and $\bar{T} \geq T_{i+1} - T_i \geq \underline{T} \quad \forall i \in \mathbb{N}$. Let $\{i_1, i_2, \dots, i_{N^*}\} \subset \mathbb{N}$, with $N^* \in \mathbb{N}$, and let (M, ψ, σ) be nonnegative constants. Assume that*

- For each $i \in \{i_1, i_2, \dots, i_{N^*}\}$,

$$V(T_{i+1}) \leq \left(V(T_i) + \frac{M^2}{4} \right) \exp^{\psi(T_{i+1}-T_i)}. \quad (45)$$

- For each $i \in \mathbb{N} \setminus \{i_1, i_2, \dots, i_{N^*}\}$,

$$\begin{aligned} V(T_{i+1}) &\leq V(T_i) \exp^{-\sigma(T_{i+1}-T_i)} \\ &\quad + \left(M + \frac{M^2}{4} \right) \exp^{\psi(T_{i+1}-T_i)}. \end{aligned} \quad (46)$$

Then, we have

$$\begin{aligned} V(T_i) &\leq \exp^{(\sigma+\psi)N^*\bar{T}} V(0) \exp^{-\sigma T_i} \\ &\quad + \left(\frac{\exp^{(\sigma+\psi)N^*\bar{T}}}{1 - \exp^{-\sigma\underline{T}}} \right) \frac{4M + M^2}{4} \exp^{\psi\bar{T}}, \quad \forall i \in \mathbb{N}. \end{aligned}$$

Proof. To prove the Lemma, it is enough to show that

$$V(T_i) \leq \exp^{(\sigma+\psi)N^*\bar{T}} \left(V(0) \exp^{-\sigma T_i} + \eta(i) \right) \quad \forall i \in \mathbb{N}, \quad (47)$$

where, for each $i \in \mathbb{N}$,

$$\begin{aligned} \eta(i) &:= \left(\sum_{k=0}^i \exp^{-k\sigma\underline{T}} \right) \frac{4M + M^2}{4} \exp^{\psi\bar{T}} \\ &\leq \left(\frac{1}{1 - \exp^{-\sigma\underline{T}}} \right) \frac{4M + M^2}{4} \exp^{\psi\bar{T}}. \end{aligned}$$

To prove (47), it is sufficient to show that

$$V(T_i) \leq \exp^{(\sigma+\psi)N(i)\bar{T}} \left(V(0) \exp^{-\sigma T_i} + \eta(i) \right) \quad \forall i \in \mathbb{N}, \quad (48)$$

where $N(i) := \text{card}\{[T_j, T_{j+1}] : j+1 \leq i, j \in \{i_1, i_2, \dots, i_{N^*}\}\}$ is the number of time intervals $[T_j, T_{j+1}]$, $j \in \{i_1, i_2, \dots, i_{N^*}\}$, prior to T_i , which satisfies $N(i) \leq N^*$ for all $i \in \mathbb{N}$. To show (48), we proceed by recurrence. Indeed, for $i = 0$, the inequality in (48) is trivially satisfied. Suppose now that the inequality in (48) is verified for $i \in \mathbb{N}$ and let us show that it is also verified for $i+1$.

Note that either $N(i+1) = N(i)$ or $N(i+1) = N(i) + 1$. If $N(i+1) = N(i)$ then, using (46), we obtain

$$\begin{aligned} V(T_{i+1}) &\leq V(T_i) \exp^{-\sigma(T_{i+1}-T_i)} \\ &\quad + \left(M + \frac{M^2}{4} \right) \exp^{\psi(T_{i+1}-T_i)} \\ &\leq \left(\exp^{(\sigma+\psi)N(i)\bar{T}} \left(V(0) \exp^{-\sigma T_i} \right. \right. \\ &\quad \left. \left. + \eta(i) \right) \right) \exp^{-\sigma(T_{i+1}-T_i)} \\ &\quad + \left(M + \frac{M^2}{4} \right) \exp^{\psi(T_{i+1}-T_i)} \\ &\leq \exp^{(\sigma+\psi)N(i)\bar{T}} V(0) \exp^{-\sigma(T_{i+1}-T_i+T_i)} \\ &\quad + \eta(i) \exp^{(\sigma+\psi)N(i)\bar{T}} \exp^{-\sigma(T_{i+1}-T_i)} \\ &\quad + \left(M + \frac{M^2}{4} \right) \exp^{\psi(T_{i+1}-T_i)} \\ &\leq \exp^{(\sigma+\psi)N(i)\bar{T}} V(0) \exp^{-\sigma T_{i+1}} \\ &\quad + \left(\eta(i) \exp^{-\sigma(T_{i+1}-T_i)} \right) \exp^{(\sigma+\psi)N(i)\bar{T}} \\ &\quad + \left(M + \frac{M^2}{4} \right) \exp^{\psi(T_{i+1}-T_i)}. \end{aligned}$$

Using the fact that

$$\eta(i) \exp^{-\sigma(T_{i+1}-T_i)} \leq \left[\sum_{k=1}^{i+1} \exp^{-k\sigma\underline{T}} \right] \left(M + \frac{M^2}{4} \right) \exp^{\psi\bar{T}},$$

we obtain

$$\begin{aligned} V(T_{i+1}) &\leq \exp^{(\sigma+\psi)N(i)\bar{T}} V(0) \exp^{-\sigma T_{i+1}} \\ &+ \left[\sum_{k=1}^{i+1} \exp^{-k\sigma\bar{T}} \right] \left(M + \frac{M^2}{4} \right) \exp^{\psi\bar{T}} \exp^{(\sigma+\psi)N(i)\bar{T}} \\ &+ \left(M + \frac{M^2}{4} \right) \exp^{\psi(T_{i+1}-T_i)}. \end{aligned}$$

It implies that

$$\begin{aligned} V(T_{i+1}) &\leq \exp^{(\sigma+\psi)N(i)\bar{T}} V(0) \exp^{-\sigma T_{i+1}} \\ &+ \left[\sum_{k=1}^{i+1} \exp^{-k\sigma\bar{T}} \right] \left(M + \frac{M^2}{4} \right) \exp^{\psi\bar{T}} \exp^{(\sigma+\psi)N(i)\bar{T}} \\ &+ \left(M + \frac{M^2}{4} \right) \exp^{\psi\bar{T}} \exp^{(\sigma+\psi)N(i)\bar{T}}. \end{aligned}$$

Combining the latter two terms, we obtain

$$\begin{aligned} V(T_{i+1}) &\leq \exp^{(\sigma+\psi)N(i)\bar{T}} V(0) \exp^{-\sigma T_{i+1}} \\ &+ \left[\sum_{k=0}^{i+1} \exp^{-k\sigma\bar{T}} \right] \left(M + \frac{M^2}{4} \right) \exp^{\psi\bar{T}} \exp^{(\sigma+\psi)N(i)\bar{T}}. \end{aligned}$$

Finally, since $N(i) = N(i+1) \leq N^*$ and

$$\eta(i+1) = \left[\sum_{k=0}^{i+1} \exp^{-k\sigma\bar{T}} \right] \left(M + \frac{M^2}{4} \right) \exp^{\psi\bar{T}},$$

we obtain

$$\begin{aligned} V(T_{i+1}) &\leq \exp^{(\sigma+\psi)N(i+1)\bar{T}} V(0) \exp^{-\sigma T_{i+1}} \\ &+ \eta(i+1) \exp^{(\sigma+\psi)N(i)\bar{T}}. \end{aligned}$$

If $N(i+1) = N(i) + 1$, we use (45) to conclude that

$$\begin{aligned} V(T_{i+1}) &\leq \left(V(T_i) + \frac{M^2}{4} \right) \exp^{\psi\bar{T}} \\ &\leq \exp^{(\sigma+\psi)N(i)\bar{T}} \exp^{\psi\bar{T}} \\ &\times \left(V(0) \exp^{-\sigma T_i} + \eta(i) + \exp^{-(\sigma+\psi)N(i)\bar{T}} \frac{M^2}{4} \right). \end{aligned}$$

Now, using the fact that $N(i+1) = N(i) + 1$, we obtain

$$\begin{aligned} V(T_{i+1}) &\leq \exp^{(\sigma+\psi)N(i+1)\bar{T}} \exp^{\psi\bar{T}} \exp^{-(\sigma+\psi)\bar{T}} \\ &\times \left(V(0) \exp^{-\sigma T_i} + \eta(i) + \exp^{-(\sigma+\psi)N(i)\bar{T}} \frac{M^2}{4} \right) \\ &\leq \exp^{(\sigma+\psi)N(i+1)\bar{T}} \exp^{\psi\bar{T}} \\ &\times \left(V(0) \exp^{-\sigma T_i} + \eta(i) \exp^{-(\sigma+\psi)\bar{T}} + \frac{M^2}{4} \right). \end{aligned}$$

The proof is completed by showing that

$$\eta(i) \exp^{-(\sigma+\psi)\bar{T}} + M^2/4 \leq \eta(i+1).$$

Indeed, we note that

$$\begin{aligned} &\eta(i) \exp^{-(\sigma+\psi)\bar{T}} + \frac{M^2}{4} \\ &\leq \eta(i) \exp^{-\sigma\bar{T}} \exp^{-\psi\bar{T}} + \left(M + \frac{M^2}{4} \right) \exp^{\psi\bar{T}} \\ &\leq \left(\sum_{k=1}^{i+1} \exp^{-k\sigma\bar{T}} \right) \left(M + \frac{M^2}{4} \right) \exp^{\psi\bar{T}} \\ &+ \left(M + \frac{M^2}{4} \right) \exp^{\psi\bar{T}} = \eta(i+1). \end{aligned}$$

■

Proof of Lemma 1

By differentiating V_1 along (2), we obtain

$$\begin{aligned} \dot{V}_1 &= \int_0^Y w(x) w_t(x) dx \\ &= \int_0^Y w(x) [-w(x) w_x(x) - \lambda(x) w_{xx}(x) \\ &\quad - w_{xxx}(x) + f(x)] dx. \end{aligned}$$

Note that $-3 \int_0^Y w(x)^2 w_x(x) dx = -w(Y)^3 + w(0)^3$. Using integration by part, we obtain

$$\begin{aligned} &-\int_0^Y w(x) w_{xxx}(x) dx \\ &= -[w(x) w_{xxx}(x)]_0^Y + \int_0^Y w_x(x) w_{xxx}(x) dx \\ &= -[w(x) w_{xxx}(x)]_0^Y + [w_x(x) w_{xx}(x)]_0^Y - \int_0^Y w_{xx}(x)^2 dx. \end{aligned}$$

Using the boundary conditions $w_x(0) = w_x(Y) = 0$, we obtain

$$\begin{aligned} \int_0^Y w(x) w_{xxx}(x) dx &= [w(x) w_{xxx}(x)]_0^Y - \int_0^Y w_{xx}(x)^2 dx \\ &= -u_1 w_{xxx}(0) + \int_0^Y w_{xx}(x)^2 dx. \end{aligned}$$

Similarly, note that

$$\begin{aligned} -\int_0^Y \lambda(x) w(x) w_{xx}(x) dx &= \int_0^Y \lambda(x) w_x(x)^2 dx \\ &+ \int_0^Y \lambda'(x) w(x) w_x(x) dx. \end{aligned}$$

Using Young inequality, we obtain

$$|\lambda'(x)w(x)w_x(x)| \leq \frac{1}{2}(\bar{\lambda}_l'^2 w(x)^2 + w_x(x)^2).$$

This allows us to conclude that

$$-\int_0^Y \lambda(x)w(x)w_{xx}(x)dx \leq \left(\bar{\lambda}_l + \frac{1}{2}\right) \int_0^Y w_x(x)^2 dx + \bar{\lambda}_l'^2 V_1.$$

Finally, using Cauchy-Schwarz inequality, we obtain $\int_0^Y w(x)f(x)dx \leq \bar{f} \int_0^Y |w(x)|dx \leq C_1 \sqrt{V_1}$. As a consequence, we have

$$\begin{aligned} \dot{V}_1 &\leq \bar{\lambda}_l'^2 V_1 + \left(\bar{\lambda}_l + \frac{1}{2}\right) \int_0^Y w_x(x)^2 dx \\ &\quad - \int_0^Y w_{xx}(x)^2 dx + C_1 \sqrt{V_1} + \frac{u_1^3}{3} + u_1 w_{xxx}(0). \end{aligned}$$

Invoking Lemma 3 with $\epsilon := 1/(\bar{\lambda}_l + \frac{1}{2})$, we find

$$\begin{aligned} \left(\bar{\lambda}_l + \frac{1}{2}\right) \int_0^Y w_x(x)^2 dx - \int_0^Y w_{xx}(x)^2 dx \\ \leq (\theta_1 - \bar{\lambda}_l'^2) V_1, \end{aligned}$$

which proves inequality (5). We show inequality (6) in a similar way. ■

Proof of Lemma 2

Consider first the case when $|\omega| \geq l(V, \hat{\theta})$. It implies that $\kappa^3 + 3\kappa\omega \leq V - 3\sqrt[3]{V}l(V, \hat{\theta}) \leq -3\hat{\theta}V$. On the other hand, $|\omega| < l(V, \hat{\theta})$ implies that

$$\begin{aligned} \kappa^3 + 3\kappa\omega + 3\hat{\theta}V &\leq \kappa^3 + 3|\kappa|l(V, \hat{\theta}) + 3\hat{\theta}V \\ &\leq [-\epsilon^3(\hat{\theta} + \delta)^3 + \epsilon(\hat{\theta} + \delta)(1 + 3\hat{\theta}) + 3\hat{\theta}]V \\ &\leq [-\epsilon^3\hat{\theta}^3 - 3\epsilon(\epsilon\delta) - 1]\hat{\theta}^2 \\ &\quad - 3(\epsilon^3\delta^2 - (\epsilon/3) - \epsilon\delta - 1)\hat{\theta} - (\epsilon\delta)^3 + \epsilon\delta]V. \end{aligned}$$

Since $\hat{\theta} \geq 0$, we conclude that $\kappa^3 + 3\kappa\omega + 3\hat{\theta}V \leq 0$. ■