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Boussad Hamour, François Murat

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Quasilinear problems involving a perturbation with quadratic growth in the gradient and a non coercive zeroth order term

Boussad Hamour ¹ & François Murat ²

Abstract

In this paper we consider the problem

$$\begin{cases} u \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Du) = H(x, u, Du) + f(x) + a_0(x)u \quad \text{in } \mathcal{D}'(\Omega), \end{cases}$$

where Ω is an open bounded set of \mathbb{R}^N , $N \geq 3$, $A(x)$ is a coercive matrix with coefficients in $L^\infty(\Omega)$, $H(x, s, \xi)$ is a Carathéodory function which satisfies

$$-c_0 A(x) \xi \xi \leq H(x, s, \xi) \operatorname{sign}(s) \leq \gamma A(x) \xi \xi, \quad \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N,$$

$f \in L^{N/2}(\Omega)$, and $a_0 \in L^q(\Omega)$, $q > N/2$, $a_0 \geq 0$. For f and a_0 sufficiently small, we prove the existence of at least one solution u of this problem which is moreover such that $e^{\delta_0|u|} - 1 \in H_0^1(\Omega)$ for some $\delta_0 \geq \gamma$.

Résumé

Dans cet article nous étudions le problème

$$\begin{cases} u \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Du) = H(x, u, Du) + f(x) + a_0(x)u \quad \text{dans } \mathcal{D}'(\Omega), \end{cases}$$

où Ω est un ouvert borné de \mathbb{R}^N , $N \geq 3$, $A(x)$ est une matrice coercive à coefficients $L^\infty(\Omega)$, $H(x, s, \xi)$ est une fonction de Carathéodory qui satisfait

$$-c_0 A(x) \xi \xi \leq H(x, s, \xi) \operatorname{sign}(s) \leq \gamma A(x) \xi \xi, \quad \text{p.p. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N,$$

$f \in L^{N/2}(\Omega)$, et $a_0 \in L^q(\Omega)$, $q > N/2$, $a_0 \geq 0$. Pour f et a_0 suffisamment petits, nous démontrons qu'il existe au moins une solution u de ce problème qui est de plus telle que $e^{\delta_0|u|} - 1 \in H_0^1(\Omega)$ pour un certain $\delta_0 \geq \gamma$.

¹Département de Mathématiques, Ecole Normale Supérieure, Boîte postale 92, Vieux Kouba, 16050 Alger, Algérie; e-mail: hamour_b2000@yahoo.fr

²Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie (Paris VI), Boîte courrier 187, 75252 Paris Cedex 05, France; e-mail: murat@ann.jussieu.fr

1 Introduction

In this paper, we consider the quasilinear problem

$$\begin{cases} u \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Du) = H(x, u, Du) + f(x) + a_0(x)u \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \quad (1.1)$$

where Ω is a bounded open set of \mathbb{R}^N , $N \geq 3$, where A is a coercive matrix with bounded measurable coefficients, where $H(x, s, \xi)$ is a Carathéodory function which has quadratic growth in ξ , and more precisely which satisfies for some $\gamma > 0$ and $c_0 \geq 0$

$$-c_0 A(x) \xi \xi \leq H(x, s, \xi) \operatorname{sign}(s) \leq \gamma A(x) \xi \xi, \quad \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N, \quad (1.2)$$

where $f \in L^{N/2}(\Omega)$, and where $a_0 \in L^q(\Omega)$, $q > \frac{N}{2}$, with

$$a_0 \geq 0, \quad \text{with } a_0 \neq 0. \quad (1.3)$$

When f and a_0 are sufficiently small (and more precisely when f and a_0 satisfy the two smallness conditions (2.14) and (2.15)), we prove in the present paper that problem (1.1) has a least one solution, which is moreover such that

$$e^{\delta_0|u|} - 1 \in H_0^1(\Omega), \quad (1.4)$$

with

$$\left\| \frac{e^{\delta_0|u|} - 1}{\delta_0} \right\|_{H_0^1(\Omega)} \leq Z_{\delta_0}, \quad (1.5)$$

where $\delta_0 \geq \gamma$ and Z_{δ_0} are two constants which depend only on the data of the problem (see (6.15), (6.16), (6.17)) for the definitions of δ_0 and Z_{δ_0}).

The main originality of our result is the fact that we assume that a_0 satisfies (1.3), namely that a_0 is a non negative function.

Let us begin with some review of the literature.

Problem (1.1) has been studied in many papers in the case where $a_0 \leq 0$. Among these papers is a series of papers [8], [9], [10] and [11] by L. Boccardo, F. Murat and J.-P. Puel (see also the paper [23] by J.-M. Rokotoson), which are concerned with the case where

$$a_0(x) \leq -\alpha_0 < 0. \quad (1.6)$$

In these papers (which also consider nonlinear monotone operators and not only the linear operator $-\operatorname{div}A((x)Du)$), the authors prove that when a_0 satisfies (1.6) and when f belongs to $L^q(\Omega)$, $q > \frac{N}{2}$, then there exists at least one solution of (1.1) which moreover belongs to $L^\infty(\Omega)$ and which satisfies some a priori estimates. The uniqueness of such a solution has been proved, under some further structure assumptions, by G. Barles and F. Murat in [4], by G. Barles, A.-P. Blanc, C. Georgelin and M. Kobylanski in [3] and by G. Barles and A. Porretta in [5].

The case where

$$a_0 = 0 \quad (1.7)$$

was considered, among others, by A. Alvino, P.-L. Lions and G. Trombetti in [1], by C. Maderna, C. Pagani and S. Salsa in [21], by V. Ferone and M.-R. Posteraro in [16], and by N. Grenon-Isselkou and J. Mossino in [17]. In these papers (which also consider nonlinear monotone operators), the authors prove that when a_0 satisfies (1.7) and when f belongs to $L^q(\Omega)$, $q > \frac{N}{2}$, with $\|f\|_{L^q(\Omega)}$ sufficiently small, then there exists at least one solution of (1.1) which moreover belongs to $L^\infty(\Omega)$ and which satisfies some a priori estimates.

The case where a_0 satisfies (1.7) but where f only belongs to $L^{N/2}(\Omega)$ for $N \geq 3$ (and no more to $L^q(\Omega)$ with $q > \frac{N}{2}$) was considered by V. Ferone and F. Murat in [13] (and in [14] in the nonlinear monotone case). These authors proved that when $\|f\|_{L^{N/2}(\Omega)}$ is sufficiently small, there exists at least one solution of (1.1) which is moreover such that $e^{\delta|u|} - 1 \in H_0^1(\Omega)$ for some $\delta > \gamma$ and which satisfies some a priori estimates. Similar results were obtained in the case where $f \in L^{N/2}(\Omega)$ by A. Dall'Aglio, D. Giachetti and J.-P. Puel in [12] when a_0 satisfies (1.6) and in the recent paper [15] by V. Ferone and F. Murat when a_0 satisfies $a_0 \leq 0$.

To finish with the case where a_0 satisfies $a_0 \leq 0$, let us quote the paper [22] by A. Porretta, where the author studies the asymptotic behaviour of the solution u of (1.1) when a_0 is a strictly positive constant which tends to zero, and proves that an ergodic constant appears at the limit $a_0 = 0$. Let us also mention the case where the nonlinearity $H(x, s, \xi)$ has the ‘‘good sign property’’, namely satisfies

$$-H(x, s, \xi) \operatorname{sign}(s) \geq 0. \tag{1.8}$$

In this case, when $a_0 \leq 0$ and when f just belongs to $H^{-1}(\Omega)$, L. Boccardo, F. Murat and J.-P. Puel in [7] and A. Bensoussan, L. Boccardo and F. Murat in [6] proved the existence of least one solution of (1.1) which belongs to $H_0^1(\Omega)$.

In contrast with the cases (1.6) and (1.7), the present paper is concerned with the case (1.3) where $a_0 \geq 0$.

In this setting we are only aware of three papers. In [20], L. Jeanjean and B. Sirakov proved the existence of at least two solutions of (1.1) (which moreover belong to $L^\infty(\Omega)$) when $A(x) = Id$, $H(x, s, \xi) = \mu|\xi|^2$, $\mu > 0$, $f \in L^q(\Omega)$, $q > \frac{N}{2}$, $f \geq 0$, and $a_0 \in L^q(\Omega)$, $a_0 \geq 0$, with $\|f\|_{L^q(\Omega)}$ and $\|a_0\|_{L^q(\Omega)}$ sufficiently small. In the recent preprint [2], D. Arcoya, C. De Coster, L. Jeanjean and K. Tanaka proved the existence of a continuum (u, λ) of solutions (with u which moreover belong to $L^\infty(\Omega)$) when $A(x) = Id$, $H(x, s, \xi) = \mu(x)|\xi|^2$, with $\mu \in L^\infty(\Omega)$, $\mu(x) \geq \mu > 0$, $f \in L^q(\Omega)$, $q > \frac{N}{2}$, $f \geq 0$, $f \neq 0$ and $a_0(x) = \lambda a_0^*(x)$ with $a_0^* \in L^q(\Omega)$, $a_0^* \geq 0$ and $a_0^* \neq 0$; moreover, under some further conditions on f , these authors proved that this continuum is defined for $\lambda \in]-\infty, \lambda_0]$ with $\lambda_0 > 0$ and that there are at least two non negative solutions of (1.1) when $\lambda > 0$ is sufficiently small. Finally, in the recent preprint [19], L. Jeanjean and H. Ramos Quoirin proved the existence of two positive solutions (which moreover belong to $L^\infty(\Omega)$) when $A(x) = Id$, $H(x, s, \xi) = \mu|\xi|^2$, $\mu > 0$, $f \in L^q(\Omega)$, $q > \frac{N}{2}$, $f \geq 0$, $f \neq 0$, and $a_0 \in C(\bar{\Omega})$ which can change sign with $a_0^+ \neq 0$ and which satisfies the so called ‘‘thick zero set condition’’, and when the first eigenvalue of the operator $-\Delta - (a_0 + \mu f)$ in $H_0^1(\Omega)$ is positive.

With respect to the results obtained in the three latest papers, we prove in the present paper, as said above, the existence of (only) one solution of (1.1) in the case (1.3) ($a_0 \geq 0$) when a_0 and f satisfy the two smallness conditions (2.14) and (2.15), but our result is obtained in the general case of a nonlinearity $H(x, s, \xi)$ which satisfies only (1.2), for $f \in L^{N/2}(\Omega)$ and for $a_0 \in L^q(\Omega)$, $q > \frac{N}{2}$. Moreover, the method which allows us to prove this result continues *mutatis mutandis* to

work in the nonlinear monotone case where the linear operator $-\operatorname{div}((A(x)Du))$ is replaced by a Leray-Lions operator $-\operatorname{div}(a(x, u, Du))$ working in $W_0^{1,p}(\Omega)$, for some $1 < p < N$, and where the quasilinear term $H(x, u, Du)$ has p -growth in $|Du|$.

Let us now describe the contents of the present paper.

The precise statement of our result is given in Section 2 (Theorem 2.1), as well as the precise assumptions under which we are able to prove it. These conditions include the two smallness conditions (2.14) and (2.15).

Our method for proving Theorem 2.1 is based on an equivalence result (see Theorem 3.5) that we state in Section 3 once we have introduced the functions $K_\delta(x, s, \zeta)$ and $g_\delta(s)$ (see (3.6) and (3.7)) and made some technical remarks on them. This result is very close to the equivalence result given in the paper [14] by V. Ferone and F. Murat.

This equivalence result implies that in order to prove the existence of a solution u of (1.1) which satisfies (1.4) and (1.5), it is equivalent to prove (see Theorem 3.8) the existence of a function w with (see (3.31))

$$w = \frac{1}{\delta_0}(e^{\delta_0|u|} - 1)\operatorname{sign}(u) \quad (1.9)$$

which satisfies (see (3.33))

$$\begin{cases} w \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Dw) + K_{\delta_0}(x, w, Dw) \operatorname{sign}(w) = \\ = (1 + \delta_0|w|)f(x) + a_0(x)w + a_0(x)g_{\delta_0}(w) \operatorname{sign}(w) \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \quad (1.10)$$

and the estimate (see (3.34))

$$\|w\|_{H_0^1(\Omega)} \leq Z_{\delta_0}, \quad (1.11)$$

which is nothing but (1.5).

Our goal thus becomes to prove Theorem 3.8, namely to prove the existence of a solution w which satisfies (1.10) and (1.11).

Problem (1.10) is very similar to problem (1.1), since it involves a term $-K_{\delta_0}(x, w, Dw) \operatorname{sign}(w)$ which has quadratic growth in Dw , as well as a zeroth order term $\delta_0|w|f(x) + a_0(x)w + a_0(x)g_{\delta_0}(w) \operatorname{sign}(w)$. But this problem is also very different from (1.1), since the term $-K_{\delta_0}(x, w, Dw) \operatorname{sign}(w)$ with quadratic growth has now the ‘‘good sign property’’ (see (1.8)), since $K_{\delta_0}(x, s, \xi)$ satisfies

$$K_{\delta_0}(x, s, \xi) \geq 0,$$

while the zeroth order term is now no more a linear one but a semilinear term with $|s|^{1+\theta}$ growth (see (6.19)) due to presence of the term $a_0(x)g_{\delta_0}(w) \operatorname{sign}(w)$.

We will prove Theorem 3.8 essentially by applying Schauder’s fixed point theorem. But there are some difficulties to do it directly, since the term with quadratic growth $K_{\delta_0}(x, w, Dw) \operatorname{sign}(w)$ only belongs to $L^1(\Omega)$ in general. We therefore begin by defining an approximate problem (see (4.1)) where $K_\delta(x, w, Dw)$ is replaced by its truncation at height k , namely $T_k(K_\delta(x, w, Dw))$,

and we prove (see Theorem 4.1) that if f and a_0 satisfy the two smallness conditions (2.14) and (2.15), this approximate problem has at least one solution w_k which satisfies the a priori estimate

$$\|w_k\|_{H_0^1(\Omega)} \leq Z_{\delta_0}. \quad (1.12)$$

This result, which is proved in Section 4, is obtained by applying Schauder's fixed point theorem in a classical way.

We then pass to the limit as k tends to infinity and we prove in Section 5 that (for a subsequence of k) w_k tend to some w^* strongly in $H_0^1(\Omega)$ (see Lemma 5.2) and that this w^* is a solution of (1.10) which satisfies (1.11) (see End of the proof of Theorem 3.8).

This completes the proof of Theorem 3.8, and therefore proves Theorem 2.1, as announced.

This proof follows along the lines of the proof used by V. Ferone and F. Murat in [13] in the case where $a_0 = 0$. As mentioned above, this method can be applied *mutatis mutandis* to the nonlinear case where the linear operator $-\operatorname{div}(A(x)Du)$ is replaced by a Leray-Lions operator $-\operatorname{div}(a(x, u, Du))$ working in $W_0^1(\Omega)$ for some $1 < p < N$, and where the quasilinear term $H(x, u, Du)$ has p -growth in $|Du|$, as it was done in [14] by V. Ferone and F. Murat in this non linear case when $a_0 = 0$. This will be the goal of our next paper [18].

2 Main result

In this paper we consider the following quasilinear problem

$$\begin{cases} u \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Du) = H(x, u, Du) + a_0(x)u + f(x) \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \quad (2.1)$$

where the set Ω satisfies (note that no regularity is assumed on the boundary of Ω)

$$\Omega \quad \text{is a bounded open subset of } \mathbb{R}^N, \quad N \geq 3, \quad (2.2)$$

where the matrix A is a coercive matrix with bounded measurable coefficients, i.e.

$$\begin{cases} A \in (L^\infty(\Omega))^{N \times N}, \\ \exists \alpha > 0, \quad A(x)\xi\xi \geq \alpha|\xi|^2 \quad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^N, \end{cases} \quad (2.3)$$

where the function $H(x, s, \xi)$ is a Carathéodory function with quadratic growth in ξ , and more precisely satisfies

$$\begin{cases} H : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \text{ is a Carathéodory function} \\ -c_0 A(x) \xi\xi \leq H(x, s, \xi) \operatorname{sign}(s) \leq \gamma A(x) \xi\xi, \quad \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N, \\ \text{where } \gamma > 0 \text{ and } c_0 \geq 0, \end{cases} \quad (2.4)$$

where $\text{sign} : \mathbb{R} \rightarrow \mathbb{R}$ denotes the function defined by

$$\text{sign}(s) = \begin{cases} +1 & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ -1 & \text{if } s < 0, \end{cases} \quad (2.5)$$

where the coefficient a_0 satisfies

$$a_0 \in L^q(\Omega) \quad \text{for some } q > \frac{N}{2}, \quad a_0 \geq 0, \quad a_0 \neq 0, \quad (2.6)$$

as well as the technical assumption (which can be made without loss generality with respect to (2.6), since $\frac{N}{2} < \frac{2N}{6-N}$ when $3 \leq N \leq 6$)

$$\frac{N}{2} < q < \frac{2N}{6-N} \quad \text{when } 3 \leq N \leq 6, \quad (2.7)$$

and finally where

$$f \in L^{N/2}(\Omega), \quad f \neq 0, \quad (2.8)$$

(note that $L^{N/2}(\Omega) \subset H^{-1}(\Omega)$ since $N \geq 3$ and since Ω is bounded; note also that if $f = 0$, then $u = 0$ is a solution of (2.1) so that the results of the present paper become trivial).

Let 2^* be the Sobolev's exponent defined by

$$\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N},$$

and let C_N be the Sobolev's constant defined as the best constant such that

$$\|\varphi\|_{2^*} \leq C_N \|D\varphi\|_2, \quad \forall \varphi \in H_0^1(\Omega). \quad (2.9)$$

We claim that one has

$$0 < \frac{2^*}{q'} - 2 < 1, \quad (2.10)$$

where q' the Hölder's conjugate of the exponent q which appears in (2.6), (2.7), i.e.

$$\frac{1}{q'} + \frac{1}{q} = 1;$$

indeed easy computations show that

$$0 < \frac{2^*}{q'} - 2 \iff q > \frac{N}{2},$$

$$\frac{2^*}{q'} - 2 < 1 \iff \frac{1}{q} > \frac{6-N}{2N};$$

the latest inequality is satisfied when $N \geq 6$ and is equivalent to $q < \frac{2N}{6-N}$ when $N \leq 6$.

We now define the number θ by

$$\theta = \frac{2^*}{q'} - 2. \quad (2.11)$$

In view of (2.10) we have

$$0 < \theta < 1. \quad (2.12)$$

Since Ω is bounded, we equip the space $H_0^1(\Omega)$ with the norm

$$\|u\|_{H_0^1(\Omega)} = \|Du\|_{L^2(\Omega)}. \quad (2.13)$$

We finally assume that f and a_0 are sufficiently small (see Remark 2.2), and more precisely that

$$\alpha - C_N^2 \|a_0\|_{N/2} - \gamma C_N^2 \|f\|_{N/2} > 0, \quad (2.14)$$

$$\|f\|_{H^{-1}(\Omega)} \leq \frac{\theta}{1+\theta} \frac{(\alpha - C_N^2 \|a_0\|_{N/2} - \gamma C_N^2 \|f\|_{N/2})^{(1+\theta)/\theta}}{((1+\theta)GC_N^{2+\theta} \|a_0\|_q)^{1/\theta}}, \quad (2.15)$$

where the constant G is defined by (6.13). Observe that in place of (2.14) we could as well have assumed that

$$\alpha - C_N^2 \|a_0\|_{N/2} - \gamma C_N^2 \|f\|_{N/2} \geq 0,$$

but that when equality takes places in the latest inequality, inequality (2.15) implies that $f = 0$, and then $u = 0$ is a solution of (2.1), so that the result of Theorem 2.1 becomes trivial.

Our main result is the following Theorem.

Theorem 2.1 *Assume that (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8) hold true. Assume moreover that (2.14) and (2.15) hold true.*

Then there exist a constant δ_0 with $\delta_0 \geq \gamma$, and a constant Z_{δ_0} , which are defined in Lemma 6.2 (see (6.15), (6.16)) and (6.17)), such that there exists at least one solution u of (2.1) which further satisfies

$$(e^{\delta_0|u|} - 1) \in H_0^1(\Omega), \quad (2.16)$$

with

$$\|e^{\delta_0|u|} Du\|_{L^2(\Omega)} = \left\| \frac{e^{\delta_0|u|} - 1}{\delta_0} \right\|_{H_0^1(\Omega)} \leq Z_{\delta_0}. \quad (2.17)$$

Our proof of Theorem 2.1 is based on an equivalence result (Theorem 3.5) which will be stated and proved in Section 3. This equivalence Theorem will allow us to replace proving Theorem 2.1 by proving Theorem 3.8 which is equivalent to Theorem 2.1.

Remark 2.2 In this Remark, we consider that the open set Ω , the matrix A and the function H are fixed (and therefore in particular that the constants $\alpha > 0$ and $\gamma > 0$ are fixed), and we consider the functions a_0 and f as parameters.

Our first assumption on these parameters (see assumptions (2.6) and (2.7)) is that a_0 belongs to $L^q(\Omega)$ with $q > \frac{N}{2}$ and $q < \frac{2N}{N-6}$ when $N \geq 6$; this is essential to ensure (see (2.10)) that

the exponent θ defined by (2.11) satisfies $0 < \theta < 1$ (see (2.12)). We also assume $a_0 \geq 0$ and $a_0 \neq 0$.

Our second set of assumptions on these parameters are the two smallness assumptions (2.14) and (2.15).

Indeed, if, for example, a_0 is sufficiently small and satisfies

$$\alpha - C_N^2 \|a_0\|_{N/2} > 0,$$

then (2.14) and (2.15) are satisfied if $\|f\|_{N/2}$ (and therefore $\|f\|_{H^{-1}(\Omega)}$, since $L^{N/2}(\Omega) \subset H^{-1}(\Omega)$) is sufficiently small.

Similarly, if, for example, f is sufficiently small and satisfies

$$\alpha - \gamma C_N^2 \|f\|_{N/2} > 0,$$

then (2.14) and (2.15) are satisfied if $\|a_0\|_q$ (and therefore $\|a_0\|_{N/2}$, since $L^q(\Omega) \subset L^{N/2}(\Omega)$) is sufficiently small, since $\|a_0\|_q$ appears in the denominator of the right-hand side of (2.15). \square

Remark 2.3 The definitions of the two constants δ_0 and Z_{δ_0} which appear in Theorem 2.1 are given in (the technical) Appendix 6 (see Lemma 6.2): these definitions are based on the properties of the function Φ_δ (see (6.12)) which looks like a parabola (see Figure 2): the constant δ_0 is the value of the parameter δ such that the function Φ_{δ_0} has a double zero, and Z_{δ_0} is the value of this double zero. The two smallness conditions (2.14) and (2.15) ensure that δ_0 satisfies $\delta_0 \leq \gamma$, a condition which is essential in our proof.

In Remark 3.9 we try to explain where the two smallness conditions (2.14) and (2.15) come from.

In Remark 3.10, we explain why we have chosen to state Theorem 3.8 with $\delta = \delta_0$ rather than with a fixed δ with $\gamma \leq \delta \leq \delta_0$. \square

3 An equivalence result

The main results of this Section are Theorems 3.5 and 3.8. Remarks 3.1, 3.2 and 3.4 and Lemma 3.3 can be considered as technical results.

Indeed, as said before, the proof of Theorem 2.1 is based on the equivalence result of Theorem 3.5 that we state and prove in this Section. This equivalence Theorem in particular implies that Theorem 3.8, which we state at the end of this Section, is equivalent to Theorem 2.1. Theorem 3.8 will be proved in Sections 4 and 5. This Section also includes Remark 3.9 in which we try to explain where the two smallness conditions (2.14) and (2.15) come from, and Remark 3.10 where we explain why we have chosen to state Theorem 3.8 for $\delta = \delta_0$.

In this Section we always assume that

$$\delta > 0. \tag{3.1}$$

Let us first proceed with a formal computation. If u is a solution of

$$\begin{cases} -\operatorname{div}(A(x)Du) = H(x, u, Du) + f(x) + a_0(x)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

and if we formally define the function w_δ by

$$w_\delta = \frac{1}{\delta} \left(e^{\delta|u|} - 1 \right) \text{sign}(u), \quad (3.3)$$

where the function sign is defined by (2.5), we have, at least formally,

$$\begin{cases} e^{\delta|u|} = 1 + \delta|w_\delta|, & |u| = \frac{1}{\delta} \log(1 + \delta|w_\delta|), & \text{sign}(u) = \text{sign}(w_\delta), \\ Dw_\delta = e^{\delta|u|} Du, & ADw_\delta = e^{\delta|u|} ADu, \\ -\text{div}(A(x)Dw_\delta) = -\delta e^{\delta|u|} A(x)DuDu \text{sign}(u) - e^{\delta|u|} (\text{div}(A(x)Du)), \end{cases} \quad (3.4)$$

and therefore w_δ is, at least formally, a solution of

$$\begin{cases} -\text{div}(A(x)Dw_\delta) = \\ = -\delta e^{\delta|u|} A(x)DuDu \text{sign}(u) + e^{\delta|u|} H(x, u, Du) + e^{\delta|u|} f(x) + e^{\delta|u|} a_0(x)u = \\ = -K_\delta(x, w_\delta, Dw_\delta) \text{sign}(w_\delta) + \\ + (1 + \delta|w_\delta|) f(x) + a_0(x)w_\delta + a_0(x)g_\delta(w_\delta) \text{sign}(w_\delta) \quad \text{in } \Omega, \\ w_\delta = 0 \quad \text{on } \partial\Omega, \end{cases} \quad (3.5)$$

where the functions $K_\delta : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ and $g_\delta : \mathbb{R} \rightarrow \mathbb{R}$ are defined by the formulas

$$\begin{cases} K_\delta(x, s, \zeta) = \\ = \frac{\delta}{1 + \delta|s|} A(x)\zeta\zeta - (1 + \delta|s|)H(x, \frac{1}{\delta} \log(1 + \delta|s|) \text{sign}(s), \frac{\zeta}{1 + \delta|s|}) \text{sign}(s), \\ \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \zeta \in \mathbb{R}^N, \end{cases} \quad (3.6)$$

and

$$g_\delta(s) = -|s| + \frac{1}{\delta}(1 + \delta|s|) \log(1 + \delta|s|), \quad \forall s \in \mathbb{R}, \quad (3.7)$$

which is nothing but

$$s + g_\delta(s) \text{sign}(s) = (1 + \delta|s|) \frac{1}{\delta} \log(1 + \delta|s|) \text{sign}(s), \quad \forall s \in \mathbb{R}. \quad (3.8)$$

Conversely, if w_δ is a solution of (3.5), and if u is formally defined by

$$u = \frac{1}{\delta} \log(1 + \delta|w_\delta|) \text{sign}(w_\delta), \quad (3.9)$$

the same formal computation easily shows that u is a solution of (3.2).

The first goal of this Section is to transform this formal equivalence into a mathematical result, namely Theorem 3.5. We begin with three remarks on the functions K_δ and g_δ , namely Remarks 3.1, 3.2 and 3.4.

Remark 3.1 Observe that, because of inequality (2.4) on the function H , and because of the coercivity (2.3) of the matrix A , one has

$$\left\{ \begin{array}{l} (c_0 + \delta)A(x)\zeta\zeta \geq \\ \frac{c_0 + \delta}{(1 + \delta|s|)}A(x)\zeta\zeta = \frac{\delta}{(1 + \delta|s|)}A(x)\zeta\zeta + (1 + \delta|s|)\frac{c_0}{(1 + \delta|s|)^2}A(x)\zeta\zeta \geq \\ \geq K_\delta(x, s, \zeta) \geq \\ \geq \frac{\delta}{(1 + \delta|s|)}A(x)\zeta\zeta - (1 + \delta|s|)\frac{\gamma}{(1 + \delta|s|)^2}A(x)\zeta\zeta = \frac{(\delta - \gamma)}{(1 + \delta|s|)}A(x)\zeta\zeta \geq \\ \geq -|\delta - \gamma|A(x)\zeta\zeta, \\ \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \zeta \in \mathbb{R}^N, \quad \forall \delta > 0, \end{array} \right. \quad (3.10)$$

and therefore in particular, when $\delta \geq \gamma$,

$$(c_0 + \delta)A(x)\zeta\zeta \geq K_\delta(x, s, \zeta) \geq 0 \quad \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \zeta \in \mathbb{R}^N \quad \text{if } \delta \geq \gamma. \quad (3.11)$$

□

Remark 3.2 In this technical Remark we prove that the functions $K_\delta(x, w, Dw)$ and $K_\delta(x, w, Dw) \text{sign}(w)$ are correctly defined measurable functions when $w \in H^1(\Omega)$, and we prove their continuity with respect to the almost everywhere convergence of w and Dw (see Lemma 3.3).

Note that the function $K_\delta(x, s, \zeta)$ defined by (3.6) and the function $K_\delta(x, s, \zeta) \text{sign}(s)$ are not Carathéodory functions, because their definitions involve the function $\text{sign}(s)$, which it is not a Carathéodory function since it is not continuous at $s = 0$. This lack of continuity in $s = 0$ is however the only obstruction for the functions $K_\delta(x, s, \zeta)$ and $K_\delta(x, s, \zeta) \text{sign}(s)$ to be Carathéodory functions, and

$$\left\{ \begin{array}{l} \text{for every } w \in H^1(\Omega), \\ \text{the functions } K_\delta(x, w, Dw) \text{ and } K_\delta(x, w, Dw) \text{sign}(w) \text{ are well defined} \\ \text{and are measurable functions,} \end{array} \right. \quad (3.12)$$

as it immediately results from the two formulas

$$\left\{ \begin{array}{l}
 K_\delta(x, s, \zeta) = \\
 = \frac{\delta}{1 + \delta|s|} A(x)\zeta\zeta - (1 + \delta|s|) H(x, \frac{1}{\delta} \log(1 + \delta|s|) \operatorname{sign}(s), \frac{\zeta}{1 + \delta|s|}) \operatorname{sign}(s) = \\
 = \frac{\delta}{1 + \delta|s|} A(x)\zeta\zeta + \\
 + \chi_{\{s < 0\}}(s) (1 + \delta|s|) H(x, -\frac{1}{\delta} \log(1 + \delta|s|), \frac{\zeta}{1 + \delta|s|}) - \chi_{\{s=0\}}(s) 0 + \\
 - \chi_{\{s > 0\}}(s) (1 + \delta|s|) H(x, \frac{1}{\delta} \log(1 + \delta|s|), \frac{\zeta}{1 + \delta|s|}), \\
 \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \zeta \in \mathbb{R}^N,
 \end{array} \right. \quad (3.13)$$

$$\left\{ \begin{array}{l}
 K_\delta(x, s, \zeta) \operatorname{sign}(s) = \\
 = \operatorname{sign}(s) \frac{\delta}{1 + \delta|s|} A(x)\zeta\zeta - (1 + \delta|s|) H(x, \frac{1}{\delta} \log(1 + \delta|s|) \operatorname{sign}(s), \frac{\zeta}{1 + \delta|s|}) = \\
 = - \chi_{\{s < 0\}}(s) \frac{\delta}{1 + \delta|s|} A(x)\zeta\zeta + \chi_{\{s=0\}}(s) 0 + \chi_{\{s > 0\}}(s) \frac{\delta}{1 + \delta|s|} A(x)\zeta\zeta + \\
 - \chi_{\{s < 0\}}(s) (1 + \delta|s|) H(x, -\frac{1}{\delta} \log(1 + \delta|s|), \frac{\zeta}{1 + \delta|s|}) - \chi_{\{s=0\}}(s) 0 + \\
 - \chi_{\{s > 0\}}(s) (1 + \delta|s|) H(x, \frac{1}{\delta} \log(1 + \delta|s|), \frac{\zeta}{1 + \delta|s|}), \\
 \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \zeta \in \mathbb{R}^N.
 \end{array} \right. \quad (3.14)$$

Moreover, in view of (3.10) one has

$$K_\delta(x, w, Dw) \in L^1(\Omega), \quad K_\delta(x, w, Dw) \operatorname{sign}(w) \in L^1(\Omega), \quad \forall w \in H^1(\Omega), \quad \forall \delta > 0. \quad (3.15)$$

One also has the following convergence result.

Lemma 3.3 *Consider a sequence w_n such that*

$$w_n \in H^1(\Omega), \quad w \in H^1(\Omega), \quad w_n \rightarrow w \quad \text{a.e. in } \Omega, \quad Dw_n \rightarrow Dw \quad \text{a.e. in } \Omega. \quad (3.16)$$

Then

$$\left\{ \begin{array}{l}
 K_\delta(x, w_n, Dw_n) \rightarrow K_\delta(x, w, Dw) \quad \text{a.e. in } \Omega, \\
 K_\delta(x, w_n, Dw_n) \operatorname{sign}(w_n) \rightarrow K_\delta(x, w, Dw) \operatorname{sign}(w) \quad \text{a.e. in } \Omega.
 \end{array} \right. \quad (3.17)$$

Proof On the first hand, we have the following almost everywhere convergences

$$\left\{ \begin{array}{l} \frac{\delta}{1 + \delta|w_n|} A(x) Dw_n Dw_n \rightarrow \frac{\delta}{1 + \delta|w|} A(x) Dw Dw \quad \text{a.e. in } \Omega, \\ (1 + \delta|w_n|) H(x, -\frac{1}{\delta} \log(1 + \delta|w_n|), \frac{Dw_n}{1 + \delta|w_n|}) \rightarrow \\ \rightarrow (1 + \delta|w|) H(x, -\frac{1}{\delta} \log(1 + \delta|w|), \frac{Dw}{1 + \delta|w|}) \quad \text{a.e. in } \Omega, \\ (1 + \delta|w_n|) H(x, \frac{1}{\delta} \log(1 + \delta|w_n|), \frac{Dw_n}{1 + \delta|w_n|}) \rightarrow \\ \rightarrow (1 + \delta|w|) H(x, \frac{1}{\delta} \log(1 + \delta|w|), \frac{Dw}{1 + \delta|w|}) \quad \text{a.e. in } \Omega. \end{array} \right. \quad (3.18)$$

On the other hand, for almost every x fixed in the set $\{y \in \Omega : w(y) > 0\}$, the assertion $w_n(x) \rightarrow w(x)$ implies that, since $w(x) > 0$, one has $w_n(x) > 0$ for $n > \nu(x)$, i.e. for n sufficiently large (depending on x); therefore

$$\left\{ \begin{array}{l} \chi_{\{w_n < 0\}}(w_n(x)) = 0 = \chi_{\{w < 0\}}(w(x)), \quad \text{for } n > \nu(x), \\ \chi_{\{w_n = 0\}}(w_n(x)) = 0 = \chi_{\{w = 0\}}(w(x)), \quad \text{for } n > \nu(x), \\ \chi_{\{w_n > 0\}}(w_n(x)) = 1 = \chi_{\{w > 0\}}(w(x)), \quad \text{for } n > \nu(x). \end{array} \right. \quad (3.19)$$

These convergences and (3.18) imply that

$$\left\{ \begin{array}{l} K_\delta(x, w_n, Dw_n) \rightarrow K_\delta(x, w, Dw) \quad \text{a.e. in } \{y \in \Omega : w(y) > 0\}, \\ K_\delta(x, w_n, Dw_n) \text{sign}(w_n) \rightarrow K_\delta(x, w, Dw) \text{sign}(w) \quad \text{a.e. in } \{y \in \Omega : w(y) > 0\}. \end{array} \right.$$

The same proof gives the similar result in the set $\{y \in \Omega : w(y) < 0\}$.

The proof of the similar result in the set $\{y \in \Omega : w(y) = 0\}$ is a little bit more delicate. Since for $w \in H^1(\Omega)$ one has

$$Dw = 0 \quad \text{a.e. in } \{y \in \Omega : w(y) = 0\}, \quad (3.20)$$

and since inequality (2.4) on the function H implies that

$$H(x, s, 0) = 0 \quad \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, s \neq 0, \quad (3.21)$$

and therefore by continuity in s that

$$H(x, 0, 0) = 0 \quad \text{a.e. } x \in \Omega, \quad (3.22)$$

formulas (3.13) and (3.14) imply on the first hand that

$$K_\delta(x, w, Dw) = K_\delta(x, w, Dw) \text{sign}(w) = 0 \quad \text{a.e. in } \{y \in \Omega : w(y) = 0\}. \quad (3.23)$$

On the other hand, in view of (3.20) and (3.22), the three functions which appear in the limits in (3.18) vanish almost everywhere in the set $\{y \in \Omega : w(y) = 0\}$. Even if we do not know anything about the (almost everywhere) convergence of the functions $\chi_{\{w_n < 0\}}(w_n(x))$, and $\chi_{\{w_n > 0\}}(w_n(x))$ in the set $\{y \in \Omega : w(y) = 0\}$, this fact and formulas (3.13) and (3.14) prove that

$$\begin{cases} K_\delta(x, w_n, Dw_n) \rightarrow 0 & \text{a.e. in } \{y \in \Omega : w(y) = 0\}, \\ K_\delta(x, w_n, Dw_n) \text{sign}(w_n) \rightarrow 0 & \text{a.e. in } \{y \in \Omega : w(y) = 0\}. \end{cases} \quad (3.24)$$

From (3.23) and (3.24) we deduce that

$$\begin{cases} K_\delta(x, w_n, Dw_n) \rightarrow K_\delta(x, w, Dw) & \text{a.e. in } \{y \in \Omega : w(y) = 0\}, \\ K_\delta(x, w_n, Dw_n) \text{sign}(w_n) \rightarrow K_\delta(x, w, Dw) \text{sign}(w) & \text{a.e. in } \{y \in \Omega : w(y) = 0\}. \end{cases} \quad (3.25)$$

This completes the proof of (3.17). \square

Remark 3.4 Observe that the function $g_\delta(s) \text{sign}(s)$ is a Carathéodory function since in view of (6.4) this function is continuous at $s = 0$. This allows one to define $g_\delta(w) \text{sign}(w)$ as a measurable function for every $w \in H^1(\Omega)$. \square

The main result of this Section is the following equivalence Theorem.

Theorem 3.5 *Assume that (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8) hold true, and let $\delta > 0$ be fixed. Let the functions K_δ and g_δ be defined by (3.6) and (3.7).*

If u is any solution of (2.1) which satisfies

$$(e^{\delta|u|} - 1) \in H_0^1(\Omega), \quad (3.26)$$

then the function w_δ defined by (3.3), namely by

$$w_\delta = \frac{1}{\delta} (e^{\delta|u|} - 1) \text{sign}(u),$$

satisfies

$$\begin{cases} w_\delta \in H_0^1(\Omega), \\ -\text{div}(A(x)Dw_\delta) + K_\delta(x, w_\delta, Dw_\delta) \text{sign}(w_\delta) = \\ = (1 + \delta|w_\delta|)f(x) + a_0(x)w_\delta + a_0(x)g_\delta(w_\delta) \text{sign}(w_\delta) & \text{in } \mathcal{D}'(\Omega). \end{cases} \quad (3.27)$$

Conversely, if w_δ is any solution of (3.27), then the function u defined by (3.4), namely by

$$u = \frac{1}{\delta} \log(1 + \delta|w_\delta|) \text{sign}(w_\delta),$$

is a solution of (2.1) which satisfies (3.26).

Remark 3.6 Every term of the equation in (3.27) makes sense as a measurable function (see Remarks 3.2 and 3.4): actually the second term of the left-hand side of the equation in (3.27) belongs to $L^1(\Omega)$ in view of (3.10), while the three terms of the right-hand side of (3.27) can be proved to belong to $(L^{2^*}(\Omega))'$ (and therefore to $H^{-1}(\Omega)$, see e.g. the proof of (3.40) in Remark 3.9 and the proof of (4.8) in the proof of Lemma 4.2). \square

Remark 3.7 Observe that the equivalence Theorem 3.5 holds true without assuming the two smallness conditions (2.14) and (2.15); moreover one could even have removed in (2.3) the hypothesis that the matrix A is coercive, and still obtain this equivalence result.

Note however that Theorem 3.5 is an equivalence result which does not prove neither the existence of a solution of (2.1) nor the existence of a solution of (3.27), but which assumes as an hypothesis either the existence of a solution of (2.1) which also satisfies (3.26), or the existence of a solution of (3.27). \square

Proof of Theorem 3.5 Define the function \hat{f} by

$$\hat{f}(x) = f(x) + a_0(x) u(x), \quad (3.28)$$

In view of (3.3) and of the definition (3.7) of $g_\delta(s)$, one has (see (3.8) and (3.4))

$$\left\{ \begin{array}{l} (1 + \delta|w_\delta|) f(x) + a_0(x) w_\delta + a_0(x) g_\delta(w_\delta) \operatorname{sign}(w_\delta) = \\ = (1 + \delta|w_\delta|) \left(f(x) + a_0(x) \frac{1}{\delta} \log(1 + \delta|w_\delta|) \operatorname{sign}(w_\delta) \right) = \\ = (1 + \delta|w_\delta|) (f(x) + a_0(x) u(x)) = (1 + \delta|w_\delta|) \hat{f}(x). \end{array} \right. \quad (3.29)$$

Then Theorem 3.5 becomes an immediate application of Proposition 1.8 of [13], once one observes that

$$\hat{f} \in L^{N/2}(\Omega); \quad (3.30)$$

such is the case in the setting of Theorem 3.5: indeed $\hat{f} = f + a_0 u$, where f belongs to $L^{N/2}(\Omega)$ by hypothesis (2.8), and where $a_0 u$ also belongs to $L^{N/2}(\Omega)$, since a_0 is assumed to belong to $L^q(\Omega)$, $q > \frac{N}{2}$ (see hypothesis (2.6)), while u belongs to $L^r(\Omega)$ for every $r < +\infty$, since by (3.26) $(e^{\delta|u|} - 1)$ is assumed to belong to $H_0^1(\Omega)$, hence in particular to $L^1(\Omega)$, which implies that $e^{\delta|u|}$ belongs to $L^1(\Omega)$. The first part of Theorem 3.5 is therefore proved.

To prove the converse, observe that when \hat{f} is defined by (3.29), then \hat{f} is also given by (3.28), where u belongs to $L^r(\Omega)$ for every $r < +\infty$. Theorem 3.5 is proved. \square

From the equivalence Theorem 3.5 one immediately deduces, setting

$$w = \frac{1}{\delta_0} (e^{\delta_0|u|} - 1) \operatorname{sign}(u) \quad (3.31)$$

and equivalently

$$u = \frac{1}{\delta_0} \log(1 + \delta_0|w|) \operatorname{sign}(w), \quad (3.32)$$

that Theorem 2.1 is equivalent to the following Theorem.

Theorem 3.8 *Assume that (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8) hold true. Assume moreover that (2.14) and (2.15) hold true.*

Then there exist a constant δ_0 with $\delta_0 \geq \gamma$, and a constant Z_{δ_0} , which are defined in Lemma 6.2 (see (6.15), (6.16) and (6.17)), such that there exists at least one solution w of

$$\begin{cases} w \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Dw) + K_{\delta_0}(x, w, Dw) \operatorname{sign}(w) = \\ = (1 + \delta_0|w|)f(x) + a_0(x)w + a_0(x)g_{\delta_0}(w) \operatorname{sign}(w) \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \quad (3.33)$$

which satisfies

$$\|w\|_{H_0^1(\Omega)} \leq Z_{\delta_0}. \quad (3.34)$$

The rest of this paper will therefore be devoted to prove Theorem 3.8. This will be done in Section 4 (where the existence of a solution satisfying (3.34) of a problem which approximates (3.33) will be proved, see Theorem 4.1), and in Section 5 (where we will pass to the limit in this approximate problem and prove that the limit satisfies (3.33) and (3.34)).

Remark 3.9 In this Remark we assume that (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8) hold true. We also assume that the two smallness conditions (2.14) and (2.15) hold true, and we try to explain how these two conditions come from an “a priori estimate” that one can obtain on the solutions of (3.27).

If w_δ is any solution of (3.27), using $T_k(w_\delta) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function, where $T_k : \mathbb{R} \rightarrow \mathbb{R}$ is the usual truncation at height k defined by

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k, \\ k \frac{s}{|s|}, & \text{if } |s| \geq k, \end{cases} \quad (3.35)$$

one has

$$\begin{cases} \int_{\Omega} A(x)Dw_\delta DT_k(w_\delta) + \int_{\Omega} K_\delta(x, w_\delta, Dw_\delta)|T_k(w_\delta)| = \\ = \int_{\Omega} f(x)T_k(w_\delta) + \int_{\Omega} \delta f(x)|w_\delta|T_k(w_\delta) + \int_{\Omega} a_0(x)w_\delta T_k(w_\delta) + \\ + \int_{\Omega} a_0(x)g_\delta(w_\delta)|T_k(w_\delta)|. \end{cases} \quad (3.36)$$

From now we assume in this Remark that δ satisfies

$$\gamma \leq \delta \leq \delta_1, \quad (3.37)$$

where δ_1 is defined by (6.11) (note that $\gamma < \delta_1$ results from (2.14), see (6.22)). Since $\delta \geq \gamma$ by (3.37), we deduce from (3.11) that $K_\delta(x, s, \zeta) \geq 0$, and therefore that the second term of the left-hand side of (3.36) is nonnegative. We then use the fact that the matrix A is coercive (see (2.3)) in the first term of the left-hand side of (3.36), the fact that $f \in L^{N/2}(\Omega)$ (see (2.8)), which implies that $f \in H^{-1}(\Omega)$, in the first term of the right-hand of (3.36), Hölder’s inequality with

$$\frac{1}{\frac{N}{2}} + \frac{2}{2^*} = 1$$

in the second and in the third terms of the right-hand side of (3.36), and finally the fact that using (6.18), namely

$$0 \leq g_\delta(s) < G|s|^{1+\theta}, \quad \forall s \in \mathbb{R}, \quad s \neq 0, \quad \forall \delta, \quad 0 < \delta \leq \delta_1, \quad (3.38)$$

(note that here we use $\delta \leq \delta_1$, see (3.37)), and Hölder's inequality with

$$\frac{1}{q} + \frac{2+\theta}{2^*} = 1$$

(which results from the definition (2.11) of θ) in the fourth term of the right-hand side of (3.36). This allows us to pass to the limit in k in (3.36) and to obtain that

$$\begin{cases} \alpha \|Dw_\delta\|_2^2 < \|f\|_{H^{-1}(\Omega)} \|Dw_\delta\|_2 + \delta \|f\|_{N/2} \|w_\delta\|_{2^*}^2 + |a_0|_{N/2} \|w_\delta\|_{2^*}^2 + G \|a_0\|_q \|w_\delta\|_{2^*}^{2+\theta}, \\ \text{if } w_\delta \neq 0, \end{cases} \quad (3.39)$$

(note that in view of (3.38), inequality (3.39) is a strict inequality). Using Sobolev's inequality (2.9) and dividing by $\|Dw_\delta\|_2$ this implies that (even in the case where $w_\delta = 0$)

$$\alpha \|Dw_\delta\|_2 < \|f\|_{H^{-1}(\Omega)} + \delta C_N^2 \|f\|_{N/2} + C_N^2 \|a_0\|_{N/2} + G C_N^{2+\theta} \|a_0\|_q \|Dw_\delta\|_2^{1+\theta}. \quad (3.40)$$

In view of the definition (6.12) of the function Φ_δ (see also Figure 2), we have proved that if w_δ is any solution of (3.27), one has

$$\Phi_\delta(\|Dw_\delta\|_2) > 0 \quad \text{if } \gamma \leq \delta \leq \delta_1. \quad (3.41)$$

But by the definition of δ_0 , one has

$$\Phi_\delta(X) > 0, \quad \forall X \geq 0, \quad \forall \delta, \quad \delta_0 < \delta \leq \delta_1,$$

and therefore inequality (3.41) does not imply anything on $\|Dw_\delta\|_2$ when δ satisfies $\delta_0 < \delta \leq \delta_1$. In contrast, when $\delta < \delta_0$, the strict inequality (3.41) implies that

$$\text{either } \|Dw_\delta\|_2 < Y_\delta^- \quad \text{or} \quad \|Dw_\delta\|_2 > Y_\delta^+ \quad \text{if } \delta < \delta_0, \quad (3.42)$$

where $Y_\delta^- < Y_\delta^+$ are the two zeros of the function Φ_δ (see Remark 2.4 and Figure 2), and when $\delta = \delta_0$, the strict inequality (3.41) implies that

$$\text{either } \|Dw_{\delta_0}\|_2 < Z_{\delta_0} \quad \text{or} \quad \|Dw_{\delta_0}\|_2 > Z_{\delta_0} \quad \text{if } \delta = \delta_0. \quad (3.43)$$

Inequalities (3.42) and (3.43) are not really a priori estimates, since they do not imply any bound on $\|Dw_\delta\|_2$. Nevertheless these inequalities exclude the closed interval $[Y_\delta^-, Y_\delta^+]$ or the point Z_{δ_0} for $\|Dw_\delta\|_2$, and they give the hope to prove the existence of a fixed point in the set $\|Dw_\delta\|_2 \leq Y_\delta^-$, when $\delta < \delta_0$, or in the set $\|Dw_{\delta_0}\|_2 \leq Z_{\delta_0}$, when $\delta = \delta_0$.

These inequalities also explain where the two smallness conditions (2.14) and (2.15) come from, since we claim that these two conditions imply that the value δ_0 of δ for which Φ_δ has a double zero satisfies $\delta_0 \geq \gamma$, which is the case where hope is allowed: indeed (2.14) ensures that (and actually is equivalent to the fact that) the minimum on $X \geq 0$ of $\Phi_\gamma(X)$ is attained for some $X = Z_\gamma > 0$, and (2.15) then ensures that (and actually is equivalent to the fact that) $\Phi_\gamma(Z_\gamma) \leq 0$. \square

Remark 3.10 In the present paper, we have chosen to prove the existence of a function w which is solution of (3.33) (or equivalently of a function $w = w_{\delta_0}$ which is solution of (3.27) with $\delta = \delta_0$) which satisfies $\|w\|_{H_0^1(\Omega)} \leq Z_{\delta_0}$ (see (3.34)). We could as well have chosen to prove that for any fixed δ with $\gamma \leq \delta \leq \delta_0$, there exists a function \hat{w}_δ which is solution of (3.27) and which satisfies $\|\hat{w}_\delta\|_{H_0^1(\Omega)} \leq Y_\delta^-$, where Y_δ^- is the smallest zero of the function Φ_δ (see Remark 6.4 and Figure 2): indeed the proofs made in Sections 4 and 5 continue to work in this framework.

In this framework, defining \hat{u}_δ by

$$\hat{u}_\delta = \frac{1}{\delta} \log(1 + \delta|\hat{w}_\delta|) \text{sign}(\hat{w}_\delta) \quad (3.44)$$

(compare with the definition (3.32) of u where $\delta = \delta_0$), the existence of \hat{w}_δ which is solution of (3.27) and which satisfies $\|\hat{w}_\delta\|_{H_0^1(\Omega)} \leq Y_\delta^-$ proves (see the equivalence Theorem 3.5) that \hat{u}_δ defined by (3.44) is solution of (2.1) and satisfies (see (3.4))

$$\|e^{\delta|\hat{u}_\delta}| D\hat{u}_\delta\|_2 = \|D\hat{w}_\delta\|_2 \leq Y_\delta^-.$$

But the function u which is defined by (3.32) from the function w given by Theorem 3.8 satisfies (2.1) (by the equivalence Theorem 3.5 with $\delta = \delta_0$) and (see (3.4) again)

$$\|e^{\delta_0|u}| Du\|_2 = \|Dw\|_2 \leq Z_{\delta_0}.$$

When $\delta < \delta_0$, we therefore have

$$\|e^{\delta|u}| Du\|_2 \leq \|e^{\delta_0|u}| Du\|_2 \leq Z_{\delta_0}, \quad \text{if } \delta < \delta_0,$$

and therefore $e^{\delta|u} - 1 \in H_0^1(\Omega)$. By the equivalence Theorem 3.5, the function \bar{w}_δ defined from u by

$$\bar{w}_\delta = \frac{1}{\delta} (e^{\delta|u} - 1) \text{sign}(u) \quad (3.45)$$

is solution of (3.27). Moreover, since $D\bar{w}_\delta = e^{\delta|u}| Du$ and since $Z_{\delta_0} < Y_\delta^+$ for $\delta < \delta_0$ (see (6.26)), one has in particular

$$\|D\bar{w}_\delta\|_2 < Y_\delta^+, \quad \text{if } \delta < \delta_0,$$

which implies by (3.42) that \bar{w}_δ satisfies

$$\|D\bar{w}_\delta\|_2 < Y_\delta^-. \quad (3.46)$$

Therefore the result of Theorem 3.8 (which is concerned with the case $\delta = \delta_0$) provides us with a function w , and then with a function u defined by (3.32), and finally with a function \bar{w}_δ defined by (3.45) which is solution of (3.27) and which satisfies (3.46). So this function \bar{w}_δ is a solution \hat{w}_δ of (3.27) which satisfies $\|\hat{w}_\delta\|_{H_0^1(\Omega)} \leq Y_\delta^-$. Therefore the result of Theorem 3.8 (where $\delta = \delta_0$) provides us with a solution for every δ with $\gamma \leq \delta \leq \delta_0$ of the problem posed in the second paragraph of this Remark. \square

4 Existence of a solution for an approximate problem

In this Section we introduce an approximation of problem (3.33) (see (4.1)). Under the two smallness assumptions (2.14) and (2.15), we prove by applying Schauder's fixed point theorem that this approximate problem has a solution which satisfies the estimate (3.34).

Let δ_0 be defined by (6.15), (6.16) and (6.17). For any $k > 0$, we consider the approximate problem of finding a solution w_k of (compare with (3.33))

$$\begin{cases} w_k \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Dw_k) + T_k(K_{\delta_0}(x, w_k, Dw_k)) \operatorname{sign}_k(w_k) = \\ = (1 + \delta_0|w_k|)f(x) + a_0(x)w_k + a_0(x)g_{\delta_0}(w_k) \operatorname{sign}(w_k) \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \quad (4.1)$$

where T_k is the usual truncation at height k defined by (3.35) and where $\operatorname{sign}_k : \mathbb{R} \rightarrow \mathbb{R}$ is the approximation of the function sign which is defined by

$$\operatorname{sign}_k(s) = \begin{cases} ks, & \text{if } |s| \leq \frac{1}{k}, \\ \operatorname{sign}(s), & \text{if } |s| \geq \frac{1}{k}. \end{cases} \quad (4.2)$$

Theorem 4.1 *Assume that (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8) hold true. Assume moreover that (2.14) and (2.15) hold true. Let δ_0 be defined in Lemma 6.2 (see (6.15) and (6.16)), and let $k > 0$ be fixed.*

Then there exists at least one solution of (4.1) such that

$$\|w_k\|_{H_0^1(\Omega)} \leq Z_{\delta_0}, \quad (4.3)$$

where Z_{δ_0} is defined in Lemma 6.2 (see (6.15), (6.16) and (6.17)).

The proof of Theorem 4.1 consists in applying Schauder's fixed point theorem. First we prove the two following lemmas.

Lemma 4.2 *Assume that (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8) hold true. Let $k > 0$ be fixed.*

Then, for any $w \in H_0^1(\Omega)$, there exists a unique solution W of the following semilinear problem

$$\begin{cases} W \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)DW) + T_k(K_{\delta_0}(x, w, Dw)) \operatorname{sign}_k(W) = \\ = (1 + \delta_0|w|)f(x) + a_0(x)w + a_0(x)g_{\delta_0}(w) \operatorname{sign}(w) \quad \text{in } \mathcal{D}'(\Omega). \end{cases} \quad (4.4)$$

Moreover W satisfies

$$\alpha \|DW\|_2 \leq \|f\|_{H^{-1}(\Omega)} + \delta_0 C_N^2 \|f\|_{N/2} \|Dw\|_2 + C_N^2 \|a_0\|_{N/2} \|Dw\|_2 + G C_N^{2+\theta} \|a_0\|_q \|Dw\|_2^{1+\theta}, \quad (4.5)$$

where C_N and G are the constants given by (2.8) and (6.13).

Proof Problem (4.4) is of the form

$$\begin{cases} W \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)DW) + \hat{b}(x)\operatorname{sign}_k(W) = \hat{F}(x) \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \quad (4.6)$$

where $\hat{b}(x)$ and $\hat{F}(x)$ are given. Since $\hat{b}(x) = T_k(K_{\delta_0}(x, w, Dw))$ belongs to $L^\infty(\Omega)$ and is nonnegative in view of (3.11) and of $\delta_0 \geq \gamma$, since the function sign_k is continuous and non decreasing, and since \hat{F} belong to $H^{-1}(\Omega)$ (see e.g. the computation which allows one to obtain (4.8) below), this problem has a unique solution.

Since $W \in H_0^1(\Omega)$, the use of W as a test function in (4.4) is licit. Since $T_k(K_\delta(x, s, \zeta))$ is nonnegative, this gives

$$\begin{cases} \int_{\Omega} A(x)DW DW \, dx \leq \\ \leq \int_{\Omega} (1 + \delta_0|w|) f(x) W \, dx + \int_{\Omega} a_0(x) w W \, dx + \int_{\Omega} a_0(x) g_{\delta_0}(w) \operatorname{sign}(w) W \, dx. \end{cases} \quad (4.7)$$

As in the computation made in Remark 3.9 to obtain the ‘‘a priori estimate’’ (3.39), we use in (4.7) the coercivity (2.3) of the matrix A , Hölder’s inequality with $\frac{1}{\frac{N}{2}} + \frac{2}{2^*} = 1$, inequality (6.19) on g_{δ_0} , $\frac{1}{q} + \frac{1+\theta}{2^*} + \frac{1}{2^*} = 1$ (which results from the definition (2.11) of θ), and finally Sobolev’s inequality (2.9). We obtain

$$\begin{cases} \alpha \|DW\|_2^2 \leq \|f\|_{H^{-1}(\Omega)} \|DW\|_2 + \delta_0 \|f\|_{N/2} \|w\|_{2^*} \|W\|_{2^*} + \|a_0\|_{N/2} \|w\|_{2^*} \|W\|_{2^*} + \\ + G \|a_0\|_q \|w\|_{2^*}^{1+\theta} \|W\|_{2^*} \leq \\ \leq \|f\|_{H^{-1}(\Omega)} \|DW\|_2 + \delta_0 C_N^2 \|f\|_{N/2} \|Dw\|_2 \|DW\|_2 + C_N^2 \|a_0\|_{N/2} \|Dw\|_2 \|DW\|_2 + \\ + GC_N^{2+\theta} \|a_0\|_q \|Dw\|_2^{1+\theta} \|DW\|_2, \end{cases} \quad (4.8)$$

which immediately implies (4.5). \square

Lemma 4.3 *Assume that (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8) hold true. Let $k > 0$ be fixed.*

Let w_n be a sequence such that

$$w_n \rightharpoonup w \quad \text{in } H_0^1(\Omega) \quad \text{weakly and a.e. in } \Omega. \quad (4.9)$$

Define W_n as the unique solution of (4.4) for $w = w_n$, i.e.

$$\begin{cases} W_n \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)DW_n) + T_k(K_{\delta_0}(x, w_n, Dw_n)) \operatorname{sign}_k(W_n) = \\ = (1 + \delta_0|w_n|) f(x) + a_0(x) w_n + a_0(x) g_{\delta_0}(w_n) \operatorname{sign}(w_n) \quad \text{in } \mathcal{D}'(\Omega). \end{cases} \quad (4.10)$$

Assume moreover that for some $W^* \in H_0^1(\Omega)$, W_n satisfies

$$W_n \rightharpoonup W^* \text{ in } H_0^1(\Omega) \text{ weakly and a.e. in } \Omega. \quad (4.11)$$

Then

$$W_n \rightarrow W^* \text{ in } H_0^1(\Omega) \text{ strongly.} \quad (4.12)$$

Proof Since $W_n - W^* \in H_0^1(\Omega)$, the use of $(W_n - W^*)$ as test function in (4.10) is licit. This gives

$$\left\{ \begin{array}{l} \int_{\Omega} A(x)D(W_n - W^*)D(W_n - W^*) dx = \\ = - \int_{\Omega} A(x)DW^*D(W_n - W^*) dx + \\ - \int_{\Omega} T_k(K_{\delta_0}(x, w_n, Dw_n)) \text{sign}_k(W_n) (W_n - W^*) dx + \\ + \int_{\Omega} (1 + \delta_0|w_n|)f(x) (W_n - W^*) dx + \int_{\Omega} a_0(x) w_n(W_n - W^*) dx + \\ + \int_{\Omega} a_0(x)g_{\delta_0}(w_n) \text{sign}(w_n) (W_n - W^*) dx. \end{array} \right. \quad (4.13)$$

We claim that every term of the right-hand of (4.13) tends to zero as n tends to infinity.

For the first term, we just use the fact that $W_n - W^*$ tends to zero in $H_0^1(\Omega)$ weakly.

For the second term, we use the fact $T_k(K_{\delta_0}(x, w_n, Dw_n)) \text{sign}_k(W_n)$ is bounded in $L^\infty(\Omega)$, since k is fixed, while $W_n - W^*$ tends to zero in $L^1(\Omega)$ strongly.

For the last three terms we observe that, in view of (4.9) and (4.11), we have

$$\left\{ \begin{array}{ll} (1 + \delta_0|w_n|) f(x) (W_n - W^*) \rightarrow 0 & \text{a.e. in } \Omega, \\ a_0(x) w_n (W_n - W^*) \rightarrow 0 & \text{a.e. in } \Omega, \\ a_0(x) g_{\delta_0}(w_n) \text{sign}(w_n) (W_n - W^*) \rightarrow 0 & \text{a.e. in } \Omega. \end{array} \right. \quad (4.14)$$

We will now prove that each of the three sequences which appear in (4.14) are equiintegrable. Together with (4.14), this will imply that these sequences converge to zero in $L^1(\Omega)$ strongly, and this will prove that the three last terms of the right-hand side of (4.13) tend to zero as n tends to infinity.

In order to prove that the sequence $(1 + \delta_0|w_n|) f(x) (W_n - W^*)$ is uniformly equiintegrable, we use Hölder's inequality with $\frac{1}{N} + \frac{2}{2^*} = 1$. For any measurable set E , $E \subset \Omega$, we have

$$\left\{ \begin{array}{l} \int_E |(1 + \delta|w_n|) f(x) (W_n - W^*)| dx \leq \\ \leq \left(\int_E |f(x)|^{N/2} dx \right)^{2/N} \|(1 + \delta_0|w_n|)\|_{2^*} \|W_n - W^*\|_{2^*} \leq c \left(\int_E |f(x)|^{N/2} dx \right)^{2/N}, \end{array} \right.$$

where c denotes a constant which is independent of n .

Proving that the sequence $a_0(x) w_n (W_n - W^*)$ is uniformly equiintegrable is similar, since for any measurable set E , $E \subset \Omega$, we have

$$\begin{cases} \int_E |a_0(x) w_n (W_n - W^*)| dx \leq \\ \leq \left(\int_E |a_0(x)|^{N/2} dx \right)^{2/N} \|w_n\|_{2^*} \|W_n - W^*\|_{2^*} \leq c \left(\int_E |a_0(x)|^{N/2} dx \right)^{2/N}. \end{cases}$$

Finally, in order to prove that the sequence $a_0(x) g_{\delta_0}(w_n) \text{sign}(w_n) (W_n - W^*)$ is uniformly equiintegrable, we use as in (4.8) inequality (6.19) and Hölder's inequality with $\frac{1}{q} + \frac{1+\theta}{2^*} + \frac{1}{2^*} = 1$; for any measurable set E , $E \subset \Omega$, we have

$$\begin{cases} \int_E |a_0(x) g_{\delta_0}(w_n) \text{sign}(w_n) (W_n - W^*)| dx \leq \int_E |a_0(x)| G |w_n|^{1+\theta} |W_n - W^*| dx \leq \\ \leq \left(\int_E |a_0(x)|^q dx \right)^{1/q} G \|w_n\|_{2^*}^{1+\theta} \|W_n - W^*\|_{2^*} \leq c \left(\int_E |a_0(x)|^q dx \right)^{1/q}. \end{cases}$$

We have proved that the right-hand side of (4.13) tends to zero. Since the matrix A is coercive (see (2.3)), this proves that W_n tends to W^* in $H_0^1(\Omega)$ strongly. Lemma 4.3 is proved. \square

Proof of Theorem 4.1 Recall that $k > 0$ is fixed.

Consider the ball B of $H_0^1(\Omega)$ defined by

$$B = \{w \in H_0^1(\Omega) : \|Dw\|_2 \leq Z_{\delta_0}\}, \quad (4.15)$$

where Z_{δ_0} is defined from δ_0 by (6.17).

Consider also the mapping $S : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ defined by

$$S(w) = W, \quad (4.16)$$

where for every $w \in H_0^1(\Omega)$, W is the unique solution of (4.4) (see Lemma 4.2).

We will apply Schauder's fixed point theorem in the Hilbert space $H_0^1(\Omega)$ to the mapping S and to the ball B .

First step In this step we prove that S maps B into itself.

Indeed by Lemma 4.2, $W = S(w)$ satisfies (4.5); therefore, when $\|Dw\|_2 \leq Z_{\delta_0}$, one has, in view of the definition (6.12) the function Φ_δ and of the property (6.16) of Z_{δ_0} ,

$$\begin{cases} \alpha \|DW\|_2 \leq \\ \leq \|f\|_{H^{-1}(\Omega)} + \delta_0 C_N^2 \|f\|_{N/2} \|Dw\|_2 + C_N^2 \|a_0\|_{N/2} \|Dw\|_2 + GC_N^{2+\theta} \|a_0\|_q \|Dw\|_2^{1+\theta} \leq \\ \leq \|f\|_{H^{-1}(\Omega)} + \delta_0 C_N^2 \|f\|_{N/2} Z_{\delta_0} + C_N^2 \|a_0\|_{N/2} Z_{\delta_0} + GC_N^{2+\theta} \|a_0\|_q Z_{\delta_0}^{1+\theta} = \\ = \alpha Z_{\delta_0} + \Phi_{\delta_0}(Z_{\delta_0}) = \alpha Z_{\delta_0}, \end{cases} \quad (4.17)$$

i.e. $\|DW\|_2 \leq Z_{\delta_0}$, or in other terms $W \in B$, which proves that $S(B) \subset B$.

Second step In this step we prove that S is continuous from $H_0^1(\Omega)$ strongly into $H_0^1(\Omega)$ strongly. For this we consider a sequence such that

$$w_n \in B, \quad w_n \rightarrow w \quad \text{in } H_0^1(\Omega) \quad \text{strongly,} \quad (4.18)$$

and define W_n as $W_n = S(w_n)$, i.e. as the solution of (4.10).

The functions w_n belong to B , and therefore the functions W_n belong to B in view of the First step. We can therefore extract a subsequence, still denoted by n , such that for some $W^* \in H_0^1(\Omega)$,

$$W_n \rightharpoonup W^* \quad \text{in } H_0^1(\Omega) \quad \text{weakly and a.e. in } \Omega. \quad (4.19)$$

We can moreover assume that for the same subsequence

$$w_n \rightharpoonup w \quad \text{a.e. in } \Omega \quad \text{and} \quad Dw_n \rightharpoonup Dw \quad \text{a.e. in } \Omega.$$

Since the assumptions of Lemma 4.3 are satisfied by the subsequences w_n and W_n , the subsequence W_n converges to W^* in $H_0^1(\Omega)$ strongly.

We now pass to the limit in equation (4.10) as n tends to infinity by using the fact that $\text{sign}_k(s)$ and $g_{\delta_0}(s) \text{sign}(s)$ are Carathéodory functions, and the first result of (3.17) as far as $T_k(K_{\delta_0}(x, w_n, Dw_n))$ is concerned (this point is the only part of the proof where the hypothesis of strong $H_0^1(\Omega)$ convergence in (4.18) is used). This proves that W^* is a solution of (4.4). Since the solution of (4.4) is unique, one has $W^* = S(w)$.

In view of the fact that W^* is uniquely determined, we conclude that it was not necessary to extract a subsequence in (4.19) and that the whole sequence $W_n = S(w_n)$ converges in $H_0^1(\Omega)$ strongly to $W^* = S(w)$. This proves the continuity of the application S .

Third step In this step we prove that $S(B)$ is precompact in $H_0^1(\Omega)$.

For this we consider a sequence $w_n \in B$ and we define W_n as $W_n = S(w_n)$; in other terms W_n is the solution of (4.10). Since w_n and W_n belong to B , they are bounded in $H_0^1(\Omega)$, and we can extract a subsequence, still denoted by n , such that

$$w_n \rightharpoonup w \quad \text{in } H_0^1(\Omega) \quad \text{weakly and a.e. in } \Omega,$$

$$W_n \rightharpoonup W^* \quad \text{in } H_0^1(\Omega) \quad \text{weakly and a.e. in } \Omega.$$

Since w_n and W_n satisfies the assumptions of Lemma 4.3, we have

$$W_n \rightarrow W^* \quad \text{in } H_0^1(\Omega) \quad \text{strongly.}$$

This proves that $S(B)$ is precompact in $H_0^1(\Omega)$ (note that in contrast with the second step, we do not need here to prove that $W^* = S(w)$).

End of the proof of Theorem 4.1 We have proved that the application S and the ball B satisfy the assumptions of Schauder's fixed point theorem. Therefore there exists at least one $w_k \in B$ such that $S(w_k) = w_k$. This proves Theorem 4.1. \square

5 Proof of Theorem 3.8

Theorem 4.1 asserts that for every $k > 0$ fixed there exists at least a solution w_k of (4.1) which satisfies (4.3). We can therefore extract a subsequence, still denoted by k , such that for some $w^* \in H_0^1(\Omega)$

$$w_k \rightharpoonup w^* \quad \text{in } H_0^1(\Omega) \text{ weakly and a.e. in } \Omega, \quad (5.1)$$

where w^* satisfies

$$\|w^*\|_{H_0^1(\Omega)} \leq Z_{\delta_0},$$

i.e. (3.34).

In this Section we will first prove that

$$w_k \rightarrow w^* \quad \text{in } H_0^1(\Omega) \text{ strongly,} \quad (5.2)$$

and then that w^* is a solution of (3.33) and (3.34). This will prove Theorem 3.8.

To prove (5.2), we use a technique which traces back to [6] (see also [13]).

For $n > 0$, we define $G_n : \mathbb{R} \rightarrow \mathbb{R}$ as the remainder of the truncation at height n , namely

$$G_n(s) = s - T_n(s), \quad \forall s \in \mathbb{R} \quad (5.3)$$

where T_n is the truncation at height n defined by (3.35), or in other terms

$$G_n(s) = \begin{cases} s + n, & \text{if } s \leq -n, \\ 0, & \text{if } -n \leq s \leq n, \\ s - n, & \text{if } s \geq n. \end{cases} \quad (5.4)$$

First we prove the two following Lemmas.

Lemma 5.1 *Assume that (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8) hold true. Assume moreover that (2.14) and (2.15) hold true. Let w_k be a solution of (4.1). Assume that the sequence w_k satisfies (5.1).*

Then we have

$$\limsup_{k \rightarrow +\infty} \int_{\Omega} |DG_n(w_k)|^2 dx \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (5.5)$$

Proof Since $G_n(w_k) \in H_0^1(\Omega)$, the use of $G_n(w_k)$ as test function in (4.1) is licit. This gives

$$\begin{cases} \int_{\Omega} A(x) Dw_k DG_n(w_k) dx + \int_{\Omega} T_k(K_{\delta_0}(x, w_k, Dw_k)) \text{sign}_k(w_k) G_n(w_k) dx = \\ = \int_{\Omega} \left((1 + \delta_0 |w_k|) f(x) + a_0(x) w_k + a_0(x) g_{\delta_0}(w_k) \text{sign}(w_k) \right) G_n(w_k) dx. \end{cases} \quad (5.6)$$

Using the coercivity (2.3) of the matrix A , we have for the first term of (5.6)

$$\int_{\Omega} A(x) Dw_k DG_n(w_k) dx = \int_{\Omega} A(x) DG_n(w_k) DG_n(w_k) dx \geq \alpha \int_{\Omega} |DG_n(w_k)|^2 dx. \quad (5.7)$$

On the other hand, since

$$\text{sign}_k(s) G_n(s) = |\text{sign}_k(s)| |G_n(s)| \geq 0, \quad \forall s \in \mathbb{R},$$

and since $T_k(K_{\delta_0}(x, w_k, Dw_k)) \geq 0$ in view of (3.11) and of $\delta_0 \geq \gamma$, we have

$$\int_{\Omega} T_k(K_{\delta_0}(x, w_k, Dw_k)) \text{sign}_k(w_k) G_n(w_k) dx \geq 0. \quad (5.8)$$

Finally, we observe that, by a proof which is similar to the one that we used in the proof of Lemma 4.3, we have

$$\left\{ \begin{array}{l} \left((1 + \delta_0 |w_k|) f(x) + a_0(x) w_k + a_0(x) g_{\delta_0}(w_k) \text{sign}(w_k) \right) G_n(w_k) \rightarrow \\ \rightarrow \left((1 + \delta_0 |w^*|) f(x) + a_0(x) w^* + a_0(x) g_{\delta_0}(w^*) \text{sign}(w^*) \right) G_n(w^*) \\ \text{in } L^1(\Omega) \text{ strongly,} \end{array} \right. \quad (5.9)$$

since the functions in the left-hand side of (5.9) converge almost everywhere in Ω and are equi-integrable.

Together with (5.6), the three results (5.7), (5.8) and (5.9) imply that

$$\left\{ \begin{array}{l} \limsup_{k \rightarrow +\infty} \alpha \int_{\Omega} |DG_n(w_k)|^2 dx \leq \\ \leq \int_{\Omega} \left((1 + \delta_0 |w^*|) |f(x)| + a_0(x) |w^*| + a_0(x) |g_{\delta_0}(w^*)| \right) |w^*| dx. \end{array} \right. \quad (5.10)$$

But since $|G_n(w^*)| \leq |w^*|$ and since $G_n(w^*) = 0$ in the set $\{|w^*| \leq n\}$, the right-hand side of (5.10) is bounded from above by

$$\int_{\{|w^*| \geq n\}} \left((1 + \delta_0 |w^*|) |f(x)| + a_0(x) |w^*| + a_0(x) |g_{\delta_0}(w^*)| \right) |w^*| dx, \quad (5.11)$$

which tends to zero when n tends to infinity because the integrand belongs to $L^1(\Omega)$.

This prove (5.5). \square

Lemma 5.2 *Assume that (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8) hold true. Assume moreover that (2.14) and (2.15) hold true. Let w_k be a solution of (4.1). Assume that the sequence w_k satisfies (5.1).*

Then for every $n > 0$ fixed we have

$$T_n(w_k) \rightarrow T_n(w^*) \text{ in } H_0^1(\Omega) \text{ strongly as } k \rightarrow +\infty. \quad (5.12)$$

Proof In this proof n is fixed. We define

$$z_k = T_n(w_k) - T_n(w^*), \quad (5.13)$$

and we fix a C^1 function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\psi(0) = 0, \quad \psi'(s) - (c_0 + \delta_0) |\psi(s)| \geq 1/2, \quad \forall s \in \mathbb{R}, \quad (5.14)$$

where c_0 is the constant which appears in the hypothesis (2.4) on H . Such that ψ exists; indeed an example is

$$\psi(s) = s \exp\left(\frac{(c_0 + \delta_0)^2}{4} s^2\right).$$

First step Since $z_k \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and since $\psi(0) = 0$, the function $\psi(z_k)$ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$. The use of $\psi(z_k)$ as test function in (4.1) is therefore licit. This gives

$$\left\{ \begin{aligned} & \int_{\Omega} A(x) Dw_k Dz_k \psi'(z_k) dx + \int_{\Omega} T_k(K_{\delta_0}(x, w_k, Dw_k)) \text{sign}_k(w_k) \psi(z_k) dx = \\ & = \int_{\Omega} \left((1 + \delta_0 |w_k|) f(x) + a_0(x) w_k + a_0(x) g_{\delta_0}(w_k) \text{sign}(w_k) \right) \psi(z_k) dx. \end{aligned} \right. \quad (5.15)$$

Since

$$Dw_k = DT_n(w_k) + DG_n(w_k) = Dz_k + DT_n(w^*) + DG_n(w_k), \quad (5.16)$$

the first term of the left-hand side of (5.15) reads as

$$\left\{ \begin{aligned} & \int_{\Omega} A(x) Dw_k Dz_k \psi'(z_k) dx = \int_{\Omega} A(x) Dz_k Dz_k \psi'(z_k) dx + \\ & + \int_{\Omega} A(x) DT_n(w^*) Dz_k \psi'(z_k) dx + \int_{\Omega} A(x) DG_n(w_k) Dz_k \psi'(z_k) dx. \end{aligned} \right. \quad (5.17)$$

On the other hand, splitting Ω into $\Omega = \{|w_k| > n\} \cup \{|w_k| \leq n\}$, the second term of the left-hand side of (5.15) reads as

$$\left\{ \begin{aligned} & \int_{\Omega} T_k(K_{\delta_0}(x, w_k, Dw_k)) \text{sign}_k(w_k) \psi(z_k) dx = \\ & = \int_{\{|w_k| > n\}} T_k(K_{\delta_0}(x, w_k, Dw_k)) \text{sign}_k(w_k) \psi(z_k) dx + \\ & + \int_{\{|w_k| \leq n\}} T_k(K_{\delta_0}(x, w_k, Dw_k)) \text{sign}_k(w_k) \psi(z_k) dx. \end{aligned} \right. \quad (5.18)$$

For what concerns the first term of the right-hand of (5.18), we claim that

$$\int_{\{|w_k| > n\}} T_k(K_{\delta_0}(x, w_k, Dw_k)) \text{sign}_k(w_k) \psi(z_k) dx \geq 0. \quad (5.19)$$

Indeed in $\{|w_k| > n\}$, the integrand is non negative since on the first hand $T_k(K_{\delta_0}(x, w_k, Dw_k)) \geq 0$ in view of (3.11) and of $\delta_0 \geq \gamma$, and since on the other hand one has

$$\text{sign}_k(w_k) \psi(z_k) \geq 0 \quad \text{in } \{|w_k| > n\}; \quad (5.20)$$

indeed since $\text{sign}(s)$ and $\text{sign}_k(s)$ have the same sign, it is equivalent to prove (5.20) or to prove that

$$\text{sign}(w_k) \psi(z_k) \geq 0 \quad \text{in } \{|w_k| > n\}; \quad (5.21)$$

but in $\{|w_k| > n\}$ one has $z_k = n \text{sign}(w_k) - T_n(w^*)$, and therefore $\text{sign}(z_k) = \text{sign}(w_k)$; this implies that

$$\text{sign}(w_k) \psi(z_k) = \text{sign}(z_k) \psi(z_k) = |\psi(z_k)| \quad \text{in } \{|w_k| > n\},$$

which proves (5.21).

For what concerns the second term of the right-hand side of (5.18), we observe that, in view of (3.11) and of $\delta_0 \geq \gamma$, we have

$$\begin{cases} |T_k(K_{\delta_0}(x, w_k, Dw_k)) \text{sign}_k(w_k) \psi(z_k)| \leq |K_{\delta_0}(x, w_k, Dw_k)| |\psi(z_k)| \leq \\ \leq (c_0 + \delta_0) |\psi(z_k)| A(x) Dw_k Dw_k. \end{cases} \quad (5.22)$$

Since in view of (5.16) one has

$$Dw_k = Dz_k + DT_n(w^*) \quad \text{in } \{|w_k| \leq n\},$$

we obtain

$$\left\{ \begin{aligned} & \int_{\{|w_k| \leq n\}} T_k(K_{\delta_0}(x, w_k, Dw_k)) \text{sign}_k(w_k) \psi(z_k) dx \geq \\ & \geq - \int_{\{|w_k| \leq n\}} (c_0 + \delta_0) |\psi(z_k)| A(x) Dw_k Dw_k dx = \\ & = - \int_{\{|w_k| \leq n\}} (c_0 + \delta_0) |\psi(z_k)| A(x) (Dz_k + DT_n(w^*)) (Dz_k + DT_n(w^*)) dx \geq \\ & = - \int_{\Omega} (c_0 + \delta_0) |\psi(z_k)| A(x) (Dz_k + DT_n(w^*)) (Dz_k + DT_n(w^*)) dx \geq \\ & \geq - \int_{\Omega} (c_0 + \delta_0) |\psi(z_k)| A(x) Dz_k Dz_k dx + \\ & - \int_{\Omega} (c_0 + \delta_0) |\psi(z_k)| \\ & \quad \left(A(x) DT_n(w^*) Dz_k + A(x) Dz_k DT_n(w^*) + A(x) DT_n(w^*) DT_n(w^*) \right) dx. \end{aligned} \right. \quad (5.23)$$

From (5.15), (5.17), (5.18), (5.19) and (5.23) we deduce that

$$\left\{ \begin{aligned}
 & \int_{\Omega} A(x) Dz_k Dz_k (\psi'(z_k) - (c_0 + \delta_0)|\psi(z_k)|) dx \leq \\
 & \leq - \int_{\Omega} A(x) DT_n(w^*) Dz_k \psi'(z_k) dx - \int_{\Omega} A(x) DG_n(w_k) Dz_k \psi'(z_k) dx + \\
 & + \int_{\Omega} (c_0 + \delta_0)|\psi(z_k)| \\
 & \quad \left(A(x) DT_n(w^*) Dz_k + A(x) Dz_k DT_n(w^*) + A(x) DT_n(w^*) DT_n(w^*) \right) dx + \\
 & + \int_{\Omega} \left((1 + \delta_0|w_k|) f(x) + a_0(x) w_k + a_0(x) g_{\delta_0}(w_k) \text{sign}(w_k) \right) \psi(z_k) dx.
 \end{aligned} \right. \quad (5.24)$$

Second step We claim that each term of the right-hand side of (5.24) tends to zero as k tends to infinity. Since $\psi'(z_k) - (c_0 + \delta)|\psi(z_k)| \geq 1/2$ by (5.14), and since the matrix A is coercive (see (2.3)), this will imply that

$$z_k \rightarrow 0 \text{ in } H_0^1(\Omega) \text{ strongly,}$$

or in other terms (see the definition (5.13) of z_k) that

$$T_n(w_k) \rightarrow T_n(w^*) \text{ in } H_0^1(\Omega) \text{ strongly as } k \rightarrow +\infty,$$

which is nothing but (5.12). Lemma 5.2 will therefore be proved.

In order to prove the claim let us recall that in view of (5.1) and of the definition (5.13) of z_k we have

$$z_k \rightharpoonup 0 \text{ in } H_0^1(\Omega) \text{ weakly, } L^\infty(\Omega) \text{ weakly } \star \text{ and a.e. in } \Omega \text{ as } k \rightarrow +\infty.$$

Since $\psi(0) = 0$, this implies that $\psi(z_k)$ tends to zero almost everywhere in Ω and in $L^\infty(\Omega)$ weakly \star as k tends to infinity, which in turn implies that

$$Dz_k \psi'(z_k) = D\psi(z_k) \rightharpoonup 0 \text{ in } L^2(\Omega)^N \text{ weakly as } k \rightarrow +\infty.$$

This implies that the first term of the right-hand side of (5.24) tends to zero as k tends to infinity.

For the second term of the right-hand side of (5.24) we observe that

$$A(x) DG_n(w_k) Dz_k = A(x) DG_n(w_k) (DT_n(w_k) - DT_n(w^*)) = -A(x) DG_n(w_k) DT_n(w^*),$$

and that by Lebesgue's dominated convergence theorem

$$DT_n(w^*) \psi'(z_k) \rightarrow DT_n(w^*) \psi'(0) \text{ in } L^2(\Omega)^N \text{ strongly as } k \rightarrow +\infty,$$

while $DG_n(w_k)$ tends to $DG_n(w^*)$ weakly in $L^2(\Omega)^N$. Since $A(x) DG_n(w^*) DT_n(w^*) = 0$ almost everywhere, the second term of the right-hand side of (5.24) tends to zero.

For the third term of the right-hand side of (5.24), we observe that

$$(c_0 + \delta_0)|\psi(z_k)|A(x)DT_n(w^*) \rightarrow 0 \text{ in } L^2(\Omega)^N \text{ strongly as } k \rightarrow +\infty$$

by Lebesgue's dominated convergence theorem, since $\psi(z_k)$ is bounded in $L^\infty(\Omega)$ and since $\psi(z_k)$ tends almost everywhere to zero because $\psi(0) = 0$. Since Dz_k is bounded in $(L^2(\Omega))^N$, this implies that the first part of this third term tends to zero. A similar proof holds true for the two others parts of this third term.

Finally the fourth term of the right-hand side of (5.24) tends to zero by a proof which is similar to the one that we used in the proof of Lemma 4.3, since the integrand converges almost everywhere to zero and is equiintegrable.

The claim made at the beginning of the second term is proved. This completes the proof of Lemma 5.2. \square

End of the proof of Theorem 3.8

First step Since we have

$$w_k - w^* = T_n(w_k) + G_n(w_k) - T_n(w^*) - G_n(w^*),$$

and since by Lemma 5.2 (see (5.12)) we have

$$\|T_n(w_k) - T_n(w^*)\|_{H_0^1(\Omega)} \rightarrow 0 \text{ as } k \rightarrow +\infty \text{ for every } n > 0 \text{ fixed,}$$

while by Lemma 5.1 (see (5.5)) we have

$$\limsup_{n \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \|G_n(w_k)\|_{H_0^1(\Omega)} = 0,$$

and while we have

$$\limsup_{n \rightarrow +\infty} \|G_n(w^*)\|_{H_0^1(\Omega)} = 0,$$

since $w^* \in H_0^1(\Omega)$, we conclude that

$$w_k \rightarrow w^* \text{ in } H_0^1(\Omega) \text{ strongly as } k \rightarrow +\infty.$$

This proves (5.2).

Second step Let us now pass to the limit in (4.1) as k tends to infinity. This is easy for the first term of the left-hand side of (4.1) as well as for the three terms of the right-hand side of (4.1), which pass to the limit in $(L^{2^*}(\Omega))'$ strongly by a proof which is similar to the one that we used in the proof of Lemma 4.3.

It remains to pass to the limit in the second term of the left-hand side of (4.1), namely in

$$T_k(K_{\delta_0}(x, w_k, Dw_k)) \text{ sign}_k(w_k).$$

We first observe that in view of (3.11) and of $\delta_0 \geq \gamma$, we have

$$|T_k(K_{\delta_0}(x, w_k, Dw_k)) \operatorname{sign}_k(w_k)| \leq |K_{\delta_0}(x, w_k, Dw_k)| \leq (c_0 + \delta_0) \|A\|_\infty |Dw_k|^2 \quad \text{a.e. in } \Omega,$$

which implies that the functions $T_k(K_{\delta_0}(x, w_k, Dw_k)) \operatorname{sign}_k(w_k)$ are equiintegrable since Dw_k converges strongly to Dw^* in $L^2(\Omega)^N$.

Extracting if necessary a subsequence, still denoted by k , such that

$$Dw_k \rightarrow Dw^* \quad \text{a.e. in } \Omega,$$

we claim that

$$T_k(K_{\delta_0}(x, w_k, Dw_k)) \operatorname{sign}_k(w_k) \rightarrow K_{\delta_0}(x, w^*, Dw^*) \operatorname{sign}(w^*) \quad \text{a.e. in } \Omega. \quad (5.25)$$

On the first hand we use the first part of (3.17), which asserts that

$$K_{\delta_0}(x, w_k, Dw_k) \rightarrow K_{\delta_0}(x, w^*, Dw^*) \quad \text{a.e. in } \Omega,$$

and the fact that for every $s \in \mathbb{R}$

$$T_k(s_k) \rightarrow s \quad \text{if } k \rightarrow +\infty \quad \text{when } s_k \rightarrow s,$$

to deduce that

$$T_k(K_{\delta_0}(x, w_k, Dw_k)) \rightarrow K_{\delta_0}(x, w^*, Dw^*) \quad \text{a.e. in } \Omega. \quad (5.26)$$

On the other hand we use the fact that

$$\operatorname{sign}_k(w_k) \rightarrow \operatorname{sign}(w^*) \quad \text{a.e. in } \{y \in \Omega : w^*(y) \neq 0\},$$

which together with (5.26) proves the convergence (5.25) in the set $\{y \in \Omega : w^*(y) \neq 0\}$.

Finally, as far as the convergence in the set $\{y \in \Omega : w^*(y) = 0\}$ is concerned, convergence (5.26) and the fact that (see (3.23))

$$K_{\delta_0}(x, w^*, Dw^*) = 0 \quad \text{a.e. in } \{y \in \Omega : w^*(y) = 0\},$$

prove that

$$T_k(K_{\delta_0}(x, w_k, Dw_k)) \operatorname{sign}_k(w_k) \rightarrow 0 = K_{\delta_0}(x, w^*, Dw^*) \operatorname{sign}(w^*) \quad \text{a.e. in } \{y \in \Omega : w^*(y) = 0\}.$$

This completes the proof of (5.25).

The equiintegrability and the almost everywhere convergence of $T_k(K_{\delta_0}(x, w_k, Dw_k)) \operatorname{sign}_k(w_k)$ then imply that

$$T_k(K_{\delta_0}(x, w_k, Dw_k)) \operatorname{sign}_k(w_k) \rightarrow K_{\delta_0}(x, w^*, Dw^*) \operatorname{sign}(w^*) \quad \text{in } L^1(\Omega) \quad \text{strongly.}$$

This proves that w^* satisfies (3.33). Since w^* also satisfies (3.34), Theorem 3.8 is proved. \square

6 Appendix

In this Appendix, we give an estimate on the function g_δ defined by (3.7) (see Lemma 6.1), and the definitions of the constants δ_0 and Z_{δ_0} which appear in Theorem 2.1 (see Lemma 6.2).

6.1 An estimate for the function g_δ

Lemma 6.1 *For $\delta > 0$, let $g_\delta : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by (3.7), i.e. by*

$$g_\delta(s) = -|s| + \frac{1}{\delta}(1 + \delta|s|) \log(1 + \delta|s|), \quad \forall s \in \mathbb{R}. \quad (6.1)$$

Then, for every λ and δ_ with*

$$0 < \lambda < 1, \quad 0 < \delta_* < +\infty, \quad (6.2)$$

there exists a constant $C(\lambda)$ which depends only on λ , with

$$0 < C(\lambda) \leq \sup \left\{ 1, \frac{2^{1+\lambda}}{\lambda e} \right\}, \quad (6.3)$$

such that

$$0 \leq g_\delta(s) \leq \delta_*^\lambda C(\lambda) |s|^{1+\lambda}, \quad \forall s \in \mathbb{R}, \quad \forall \delta, \quad 0 < \delta \leq \delta_*. \quad (6.4)$$

Moreover

$$0 \leq g_\delta(s) < \delta_*^\lambda C(\lambda) |s|^{1+\lambda}, \quad \forall s \in \mathbb{R}, \quad s \neq 0, \quad \forall \delta, \quad 0 < \delta \leq \delta_*. \quad (6.5)$$

Proof Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ be the function defined by

$$g(t) = -t + (1+t) \log(1+t), \quad \forall t \geq 0.$$

Since $g(0) = 0$ and $g'(t) \geq 0$, one has

$$g(t) \geq 0, \quad \forall t \geq 0. \quad (6.6)$$

On the other hand, since $\log(1+t) < t$ for $t > 0$, one has $g(t) < t^2 \leq t^{1-\lambda} t^{1+\lambda}$ for $t > 0$, and therefore for $0 < \lambda < 1$ and for every $m > 0$

$$g(t) < m^{1-\lambda} t^{1+\lambda}, \quad \forall t, \quad 0 < t \leq m. \quad (6.7)$$

One has also

$$\frac{g(t)}{t^{1+\lambda}} < \frac{(1+t) \log(1+t)}{t^{1+\lambda}} = \left(\frac{1+t}{t} \right)^{1+\lambda} \frac{\log(1+t)}{(1+t)^\lambda}, \quad \forall t > 0,$$

and therefore

$$\frac{g(t)}{t^{1+\lambda}} < \left(\frac{1+m}{m} \right)^{1+\lambda} \frac{\log(1+t)}{(1+t)^\lambda} \quad \forall t \geq m > 0.$$

But the function $\frac{\log(1+t)}{(1+t)^\lambda}$ reaches its maximum for t_0 defined by $(1+t_0) = e^{1/\lambda}$, hence

$$\frac{\log(1+t)}{(1+t)^\lambda} \leq \frac{1}{\lambda e}, \quad \forall t \geq 0.$$

This implies that for $0 < \lambda < 1$ and for every $m > 0$

$$g(t) < \left(\frac{1+m}{m}\right)^{1+\lambda} \frac{1}{\lambda e} t^{1+\lambda}, \quad \forall t \geq m, \quad (6.8)$$

which, with (6.6) and (6.7), implies that for $0 < \lambda < 1$ and for every $m > 0$

$$0 \leq g(t) < \sup \left\{ m^{1-\lambda}, \left(\frac{1+m}{m}\right)^{1+\lambda} \frac{1}{\lambda e} \right\} t^{1+\lambda}, \quad \forall t > 0, \quad (6.9)$$

or in other terms that for every λ , $0 < \lambda < 1$,

$$0 \leq g(t) < C(\lambda) t^{1+\lambda}, \quad \forall t > 0, \quad (6.10)$$

for some $C(\lambda)$, with (take $m = 1$)

$$0 < C(\lambda) \leq \sup \left\{ 1, \frac{2^{1+\lambda}}{\lambda e} \right\},$$

which is nothing but (6.3).

Since

$$g_\delta(s) = \frac{1}{\delta} g(\delta|s|), \quad \forall s \in \mathbb{R},$$

one deduces from (6.10) that g_δ satisfies

$$\begin{cases} 0 \leq g_\delta(s) \leq \delta^\lambda C(\lambda) |s|^{1+\lambda}, & \forall s \in \mathbb{R}, \quad \forall \delta > 0, \\ 0 \leq g_\delta(s) < \delta^\lambda C(\lambda) |s|^{1+\lambda}, & \forall s \in \mathbb{R}, \quad s \neq 0, \quad \forall \delta > 0, \end{cases}$$

which proves (6.4) and (6.5) with a constant $C(\lambda)$ which satisfies (6.3).

□

6.2 Definition of δ_0 and Z_{δ_0}

The goal of this Subsection is to define the constants δ_0 and Z_{δ_0} which appear in Theorem 2.1. We will prove the following result.

Lemma 6.2 *Assume that (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8) hold true. Assume moreover that (2.14) and (2.15) hold true.*

Let δ_1 be the number defined by

$$\delta_1 = \frac{\alpha - C_N^2 \|a_0\|_{N/2}}{C_N^2 \|f\|_{N/2}}. \quad (6.11)$$

Note that $\delta_1 > 0$ since $\alpha - C_N^2 \|a_0\|_{N/2} > 0$ in view of (2.14).

For $\delta \geq 0$, let $\Phi_\delta : \mathbb{R}^+ \rightarrow \mathbb{R}$ (see Figure 2) be the function defined by

$$\Phi_\delta(X) = GC_N^{2+\theta} \|a_0\|_q X^{1+\theta} - (\alpha - C_N^2 \|a_0\|_{N/2} - \delta C_N^2 \|f\|_{N/2})X + \|f\|_{H^{-1}(\Omega)}, \quad (6.12)$$

where θ is defined by (2.11) and where G is the constant defined by

$$G = \left(\frac{\alpha - C_N^2 \|a_0\|_{N/2}}{C_N^2 \|f\|_{N/2}} \right)^\theta C(\theta), \quad (6.13)$$

with C_N the best constant in the Sobolev's inequality (2.9) and $C(\theta)$ the constant which appears in (6.4) (see also (6.3)).

Then, for $0 \leq \delta \leq \delta_1$, the function Φ_δ has a unique minimizing point Z_δ on \mathbb{R}^+ , which is given by

$$Z_\delta = \left(\frac{\alpha - C_N^2 \|a_0\|_{N/2} - \delta C_N^2 \|f\|_{N/2}}{(1 + \theta)GC_N^{2+\theta} \|a_0\|_q} \right)^{1/\theta}, \quad \text{for } 0 \leq \delta \leq \delta_1. \quad (6.14)$$

Moreover, there exists a unique number δ_0 such that

$$\gamma \leq \delta_0 < \delta_1, \quad (6.15)$$

and

$$\Phi_{\delta_0}(Z_{\delta_0}) = 0. \quad (6.16)$$

This number is the number δ_0 which appear in Theorem 2.1, and Z_{δ_0} is then defined from δ_0 through formula (6.14), namely by

$$Z_{\delta_0} = \left(\frac{\alpha - C_N^2 \|a_0\|_{N/2} - \delta_0 C_N^2 \|f\|_{N/2}}{(1 + \theta)GC_N^{2+\theta} \|a_0\|_q} \right)^{1/\theta}. \quad (6.17)$$

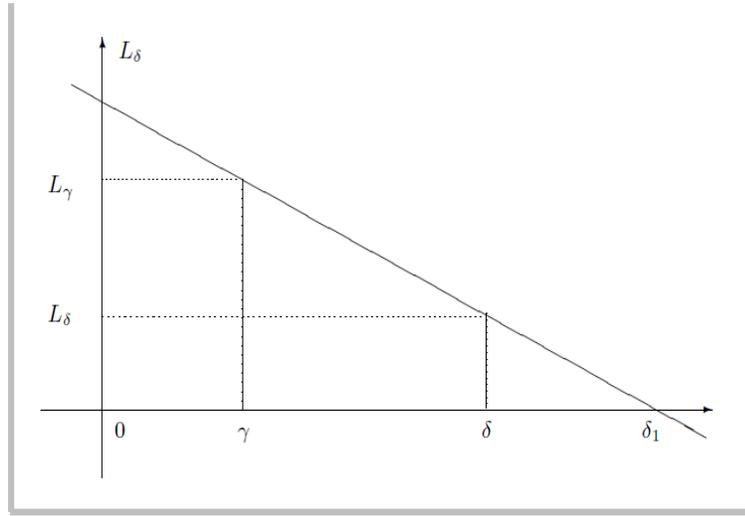
Remark 6.3 In the present paper we use Lemma 6.1 with $\lambda = \theta$ defined by (2.11) (note that $0 < \theta < 1$ in view of (2.12)) and with $\delta_* = \delta_1$ defined by (6.11). Using the fact that G defined by (6.13) is nothing but $G = \delta_1^\theta C(\theta)$, inequalities (6.4) and (6.5) imply that

$$\begin{cases} 0 \leq g_\delta(s) \leq \delta_1^\theta C(\theta) |s|^{1+\theta} = G |s|^{1+\theta}, & \forall s \in \mathbb{R}, \quad \forall \delta, 0 < \delta \leq \delta_1, \\ 0 \leq g_\delta(s) < \delta_1^\theta C(\theta) |s|^{1+\theta} = G |s|^{1+\theta}, & \forall s \in \mathbb{R}, \quad s \neq 0, \quad \forall \delta, 0 < \delta \leq \delta_1. \end{cases} \quad (6.18)$$

In particular for $\delta = \delta_0$ defined by (6.15) and (6.16) we have

$$0 \leq g_{\delta_0}(s) \leq G |s|^{1+\theta}, \quad \forall s \in \mathbb{R}. \quad (6.19)$$

□


 Figure 1: The graph of the straight line L_δ

Proof of Lemma 6.2 For $\delta \geq 0$, let L_δ be the constant defined by (see Figure 1)

$$L_\delta = \alpha - C_N^2 \|a_0\|_{N/2} - \delta C_N^2 \|f\|_{N/2}, \quad (6.20)$$

where C_N is the best constant in the Sobolev's inequality (2.9). Note that L_δ is decreasing with respect to δ .

Since δ_1 is defined by (6.11), one has

$$L_{\delta_1} = 0. \quad (6.21)$$

On the other hand, hypothesis (2.14) is nothing but $L_\gamma > 0$. Since L_δ is decreasing in δ , we have

$$\gamma < \delta_1. \quad (6.22)$$

Let us now study the function $\Phi_\delta : \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by (6.12), i.e., in view of the definition (6.20) of L_δ , by

$$\Phi_\delta(X) = GC_N^{2+\theta} \|a_0\|_q X^{1+\theta} - L_\delta X + \|f\|_{H^{-1}(\Omega)}, \quad \forall X \geq 0, \quad (6.23)$$

(see Figure 2).

Since $a_0 \neq 0$ (see (2.6)), the function Φ_δ looks like a parabola. Observe that the function Φ_δ is a decreasing family of functions for $0 \leq \delta \leq \delta_1$.

When $0 \leq \delta \leq \delta_1$, one has $L_\delta \geq 0$, and the function Φ_δ has a unique minimizing point Z_δ . A simple computation shows that Z_δ is given by

$$Z_\delta = \left(\frac{L_\delta}{(1+\theta)GC_N^{2+\theta} \|a_0\|_q} \right)^{1/\theta} = \left(\frac{\alpha - C_N^2 \|a_0\|_{N/2} - \delta C_N^2 \|f\|_{N/2}}{(1+\theta)GC_N^{2+\theta} \|a_0\|_q} \right)^{1/\theta}, \quad (6.24)$$

i.e (6.14), and that the minimum value $\Phi_\delta(Z_\delta)$ of Φ_δ is given by

$$\begin{cases} \Phi_\delta(Z_\delta) = \|f\|_{H^{-1}(\Omega)} - \frac{\theta}{1+\theta} \frac{L_\delta^{(1+\theta)/\theta}}{((1+\theta)GC_N^{2+\theta}\|a_0\|_q)^{1/\theta}} = \\ = \|f\|_{H^{-1}(\Omega)} - \frac{\theta}{1+\theta} \frac{(\alpha - C_N^2\|a_0\|_{N/2} - \delta C_N^2\|f\|_{N/2})^{(1+\theta)/\theta}}{((1+\theta)GC_N^{2+\theta}\|a_0\|_q)^{1/\theta}}. \end{cases} \quad (6.25)$$

When $0 \leq \delta \leq \delta_1$, the function L_δ is nonnegative, continuous and decreasing with respect to δ . Therefore Z_δ is continuous and decreasing with respect to δ , while $\Phi_\delta(Z_\delta)$ is continuous and increasing with respect to δ .

When $\delta = \delta_1$, one has $L_{\delta_1} = 0$, the function Φ_{δ_1} attains its minimum in $Z_{\delta_1} = 0$ and $\Phi_{\delta_1}(Z_{\delta_1}) = \|f\|_{H^{-1}(\Omega)} > 0$, while hypothesis (2.15) is nothing but $\Phi_\gamma(Z_\gamma) \leq 0$. Therefore there exists a unique δ_0 with $\gamma \leq \delta_0 < \delta_1$ such that $\Phi_{\delta_0}(Z_{\delta_0}) = 0$. This is the definition of δ_0 given by (6.15) and (6.16) in Lemma 6.2.

Lemma 6.2 is proved. \square

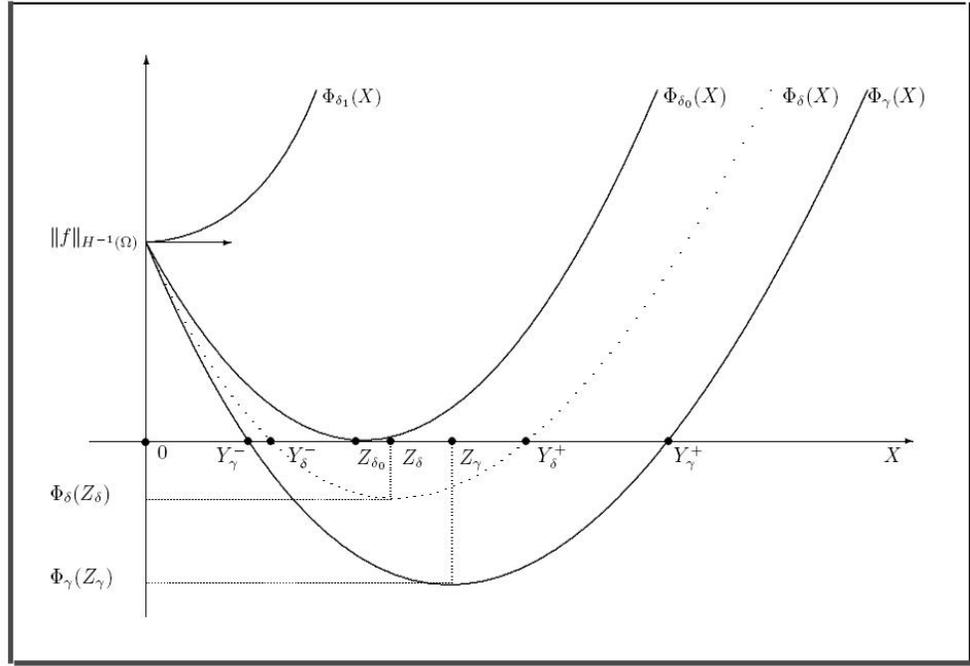


Figure 2: The graphs of the functions $\Phi_\delta(X)$ for $\delta = \gamma$, $\gamma < \delta < \delta_0$, $\delta = \delta_0$ and $\delta = \delta_1$

Remark 6.4 The case where equality takes places in inequality (2.15) corresponds to the case where $\delta_0 = \gamma$.

On the other hand, we have $\delta_0 < \gamma$ when (2.15) is a strict inequality, and for δ with $\delta_0 < \delta \leq \gamma$, the function Φ_δ has two zeros Y_δ^- and Y_δ^+ with $0 < Y_\delta^- < Y_\delta^+$. Since the family of functions Φ_δ is a decreasing family of functions, one has

$$0 < Y_\delta^- < Z_{\delta_0} < Y_\delta^+ \quad \text{if } \delta_0 < \delta \leq \gamma. \quad (6.26)$$

□

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