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| Jules Chenal. On the First Page of the Renaudineau-Shaw Spectral Sequence. 2024. hal-04520683

HAL Id: hal-04520683

<https://hal.science/hal-04520683v1>

Preprint submitted on 25 Mar 2024

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ON THE FIRST PAGE OF THE RENAUDINEAU-SHAW SPECTRAL SEQUENCE

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March 2024

Abstract

A T-hypersurface is a combinatorial hypersurface of the real locus of a projective toric variety Y . It is constructed from a primitive triangulation K of a moment polytope P of Y and a 0-cochain ε on K with coefficients in the field with two elements \mathbb{F}_2 , called a *sign distribution*. O. Viro showed that when K is convex the T-hypersurface is ambiently isotopic to a real algebraic hypersurface of Y . A. Renaudineau and K. Shaw gave upper bounds on the Betti numbers of T-hypersurfaces in terms of the Hodge numbers of a generic section of the ample line bundle L associated with the moment polytope. In particular, the number of connected components of a T-hypersurface cannot exceed the geometric genus of a generic section of L plus one. In this article we investigate whether this upper bound is attainable. We are able to characterise the couples $(K; \varepsilon)$ leading to T-hypersurfaces realising the Renaudineau-Shaw upper bound on the number of connected components. This theorem generalises B. Haas' theorem for T-curves. In contrast with this results we find that the upper bound is not always attainable on every primitive triangulations. For some of those on which it is not attainable we provide a sharper upper bound. Finally we use our characterisation to show that there always exist a triangulation and a sign distribution on the standard simplex that reach the Renaudineau-Shaw upper bound. We also study the growth of the expected number of connected components of a T-hypersurface as we dilate the moment polytope by d (i.e. we tensorise the line bundle d -times with itself) and show that it is always of the order of d^n where n is the dimension of P .

Acknowledgements

This research was funded, in whole or in part, by l'Agence Nationale de la Recherche (ANR), project ANR-22-CE40-0014. For the purpose of open access, the author has applied a CC-BY public copyright licence to any Author Accepted Manuscript (AAM) version arising from this submission.

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Introduction

This article is about the first page of the Renaudineau-Shaw spectral sequence of T -hypersurfaces. In particular we compute explicitly its boundary operators. These computations lead us to a generalisation of B. Haas' theorem¹. This result is a characterisation of the triangulations and sign distributions that produce T -hypersurfaces having a maximal number of connected components with respect to the Renaudineau-Shaw inequality². Recall that for all T -hypersurfaces $\mathbb{R}X_\varepsilon$, A. Renaudineau and K. Shaw showed the following upper bound:

$$\dim H_0(\mathbb{R}X_\varepsilon; \mathbb{F}_2) \leq 1 + h^{n-1,0}(X), \quad (1)$$

where $h^{n-1,0}(X)$ is the number of interior lattice points of P the moment polytope of the ambient toric variety. In the following we will say that $\mathbb{R}X_\varepsilon$ has a maximal number of connected components when the upper bound (1) is achieved. In order to state our result we define the notion of ρ -uniformity for primitive triangulations. It is a technical property involving the relative parities of the coordinates of the vertices of two maximal simplices of the triangulation sharing a face of codimension 1.

Theorem 4.15 (Generalised Haas Theorem). *Let P be a non-singular polytope of $\mathfrak{t}^*(\mathbb{R})$ endowed with a primitive triangulation K . For all sign distributions ε , the T -hypersurface $\mathbb{R}X_\varepsilon$ has a maximal number of connected components if and only if:*

1. *The triangulation K is ρ -uniform;*
2. *The homological “inclusion” $H_1(\mathbb{R}X_\varepsilon; \mathbb{F}_2) \rightarrow H_1(\mathbb{R}P; \mathbb{F}_2)$ is onto whenever $n > 2$;*
3. *The sign distribution ε satisfies:*

$$D^2\varepsilon = \rho \text{ in } H_{n-1}(K; \mathcal{B}_1K; N). \quad (2)$$

4. *For all simplices σ^{n-2} of \mathcal{B}_1K not contained in \mathcal{B}_0K , $(\text{ap}(\sigma^{n-2}); \sigma^{n-2})_\varepsilon$ vanishes.*

A 2-dimensional triangulation is always ρ -uniform, hence this new hypothesis (relatively to Haas' statement) is automatically satisfied for T -curves. However it is not always the case in dimension bigger than 3. We can also note that since all T -surfaces on the same triangulation share the same Euler characteristic³ a T -surface is maximal as a real variety if and only if it has a maximal number of connected components with respect to the Renaudineau-Shaw inequality. In addition we define a number $\kappa(K)$ which measure the defect of ρ -uniformity of the triangulation K and use this quantity to refine the Renaudineau-Shaw inequality on the number of connected components of $\mathbb{R}X_\varepsilon$.

Proposition 4.22. *Let K be a primitive triangulation of a non-singular polytope. For all sign distributions $\varepsilon \in C^0(K; \mathbb{F}_2)$ the rank of $M_{0,n-2}(\varepsilon)$ is always at least equal to $\kappa(K)$. As a consequence we have the following upper bound:*

$$\dim H_0(\mathbb{R}X_\varepsilon; \mathbb{F}_2) \leq 1 + \dim H_{0,n-1}(X; \mathbb{F}_2) - \kappa(K). \quad (3)$$

¹c.f. [Haa97] or [RS23] Theorem 7.5 p.977.

²c.f. [RS23] Theorem 1.4 p.947.

³c.f. [Ber10].

Moreover we use our computations to determine the asymptotic growth of the expected number of connected components of $\mathbb{R}X_\varepsilon$ when one choses the sign distribution uniformly at random and considers dilatations of the moment polytope. More precisely we consider P a non-singular lattice polytope and $(K_d)_{d \geq 1}$ a sequence of primitive triangulations of the dilates dP . Then we show that if ε is chosen uniformly at random in $C^0(K_d; \mathbb{F}_2)$ we have the following property.

Proposition 4.25. *For all sequences $(K_d)_{d \geq 1}$ of primitive triangulations of $(dP)_{d \geq 1}$ we have:*

$$1 \geq \limsup_{d \rightarrow +\infty} \mathbb{E} \left[\frac{b_0(\mathbb{R}X_\varepsilon(d))}{1 + h^{n-1,0}(X(d))} \right] \geq \liminf_{d \rightarrow +\infty} \mathbb{E} \left[\frac{b_0(\mathbb{R}X_\varepsilon(d))}{1 + h^{n-1,0}(X(d))} \right] > 0.$$

Recall that $h^{n-1,0}(X(d)) = \text{vol}_{\mathbb{Z}}(P)d^n + o(d^n)$.

To conclude we provide examples of triangulations of the standard simplices and sign distributions for which the Renaudineau-Shaw upper bound is achieved. To do so we found inspiration in the construction of maximal T-surfaces in \mathbb{P}^3 by I. Itenberg, [Ite97].

Theorem 0.1. *For all $n \geq 2$, and all $d \geq 1$, the Itenberg-Viro triangulation IV_d^n is simply integrable. In particular, the Harnack distribution:*

$$h^n(x_1, \dots, x_n) := \sum_{i < j} x_i x_j,$$

produces a projective T-hypersurfaces made of $h^{n-1,0}(X)$ spheres bounding disjoint balls and another connected component that intersects every toric divisors of \mathbb{P}^n .

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1 T-hypersurfaces and Tools to Study Them

In this first section we set some notations, remind some definitions and introduce the construction of T-hypersurfaces and their Renaudineau-Shaw spectral sequence.

Simplicial Complexes and Cellular Homology We recall that a finite simplicial complex K is the data of a set of vertices V and a set of simplices S of subsets of V that contains every

singletons of V and is closed under the operation of taking non-empty⁴ subsets. The latter condition means that if σ belongs to S and τ is a non-empty subset of σ then it also belongs to S . The *dimension* of a simplex σ of K is by definition one less than its number of elements. We will never mention the sets V or S but, for all non-negative integer p , we will denote $\sigma^p \in K$ to indicate that σ^p is a p -dimensional simplex of K (also called a p -*simplex* of K). Also we will make no difference between a vertex v and the 0-simplex $\{v\}$ it defines.

Every finite simplicial complex defines a regular CW-complex⁵ called is *geometric realisation*. Let C be the convex hull of the vertices of K in the real vector space freely spanned by said vertices. The geometric realisation of K is the union of all the faces of K generated by the vertices of the simplices of K . This is an embedding of the set of simplices of K into the lattice of faces of C , we will not really make a difference between the abstract complex K and its geometric realisation. We indicate by $\sigma^p \leq \sigma^q$ that σ^p is a face of σ^q , i.e. that the former is included in the latter when seen as a set. The cells of the geometric realisation of K are thought to be open hence the simplex $\sigma^p = \{v_0; \dots; v_p\}$ correspond to:

$$\left\{ \sum_{k=0}^p t_k v_k \mid \forall 0 \leq k \leq p, t_k > 0, \text{ and } \sum_{k=0}^p t_k = 1 \right\}.$$

The *star* of a simplex σ^p is the reunion of all the simplices of which it is a face. Its *link* (denoted by $\text{lk}(\sigma^p; K)$) is the reunion of all the faces σ^q of the members of its star whose closure $\bar{\sigma}^q$ has empty intersection with $\bar{\sigma}^p$. These concepts extends to regular CW-complexes.

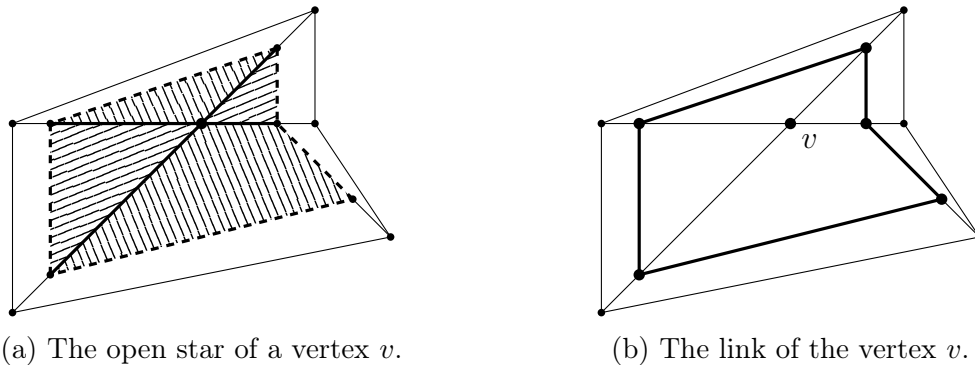


Figure 1: An open star and a link.

Definition 1.1. A Δ -complex is a regular CW-complex in which every closed cell is isomorphic (as a CW-complex) to a simplex.

Definition 1.2 (Barycentric and Cubical Subdivision). Let K be a finite Δ -complex. Its *barycentric subdivision*, sometimes denoted $\text{Sd } K$ is the simplicial complex:

- whose vertices are the simplices of K ;
- whose simplices are the set of simplices $\{\sigma^{p_0} < \dots < \sigma^{p_k}\}$ of K that can be totally ordered by adjacency.

⁴We adopt this convention as we will never have to consider the empty simplex.

⁵A CW-complex in which the pair made of a cell and its closure is homeomorphic to an open ball in its closure.

The geometric realisation of $\text{Sd } K$ is a regular subdivision of K . The *cubical subdivision* of K is a regular CW-complex subdivision of K that is coarser than its barycentric subdivision. An open cell of this subdivision is indexed by a pair of simplices $[\sigma^p; \sigma^q]$ where σ^p is a face of σ^q . This particular cell has dimension $q - p$ and is the reunion of the barycentric simplices $\{\sigma^{p_0} < \dots < \sigma^{p_k}\}$ for which σ^{p_0} is σ^p and σ^{p_k} is σ^q . Every closed k -dimensional cubical cell has the combinatorics of a k -dimensional cube triangulated into $k!$ simplices of dimension k .



(a) The barycentric subdivision of a triangle. (b) The cubical subdivision of a triangle.

Figure 2: Subdivisions of a triangle.

Definition 1.3 (Cellular Cosheaves and Cellular Sheaves). Let E be a regular CW-complex, and denote by $\mathbf{Cell } E$ the category whose objects are the cells of E and whose morphisms corresponds to adjacency, i.e. there is exactly one morphism $e^p \rightarrow e^q$ if e^p is a face of e^q and none otherwise. A *cellular cosheaf* (resp. *sheaf*) on E is a contravariant (resp. covariant) functor:

$$F : \mathbf{Cell } E \rightarrow \mathbf{Vect}_{\mathbb{F}_2}^{\mathbf{f}},$$

where $\mathbf{Vect}_{\mathbb{F}_2}^{\mathbf{f}}$ is the category of finite dimensional vector spaces over \mathbb{F}_2 . If $e^p \leq e^q$ are two cells of E and v is a vector of $F(e^q)$ we will denote by $v|_{e^p}^{e^q}$ its image in $F(e^p)$ by the morphism $F(e^p \rightarrow e^q)$ called the extension morphism (we use a similar notation in the case of sheaves). A morphism f between two cosheaves (or two sheaves) on E is a natural transformation. The kernel, image, and cokernel of f are formed “cell-wise”.

When E is the cubical subdivision of a Δ -complex K we might call a cellular cosheaf on E a *cubical cosheaf* on K . Such an object is an assignment of a \mathbb{F}_2 -vector space to every pair of adjacent simplices $\sigma^p \leq \sigma^q$ that is covariant in the first coordinate and contravariant in the second.

Definition 1.4. Let E be a regular CW-complex, F be a cellular cosheaf on E and k be a non-negative integer. The space of cellular k -chains with value in F is the vector space:

$$C_k(E; F) := \bigoplus_{e^k \in E} F(e^k).$$

A typical vector of this space is written $c = \sum_{e^k \in E} v_{e^k} \otimes e^k$ where v_{e^k} belongs to $F(e^k)$ for all cells e^k . We might denote the coefficient v_{e^k} by $\langle c; e^k \rangle$. When F is a cubical cosheaf on the Δ -complex K we denote $C_k(E; F)$, where E is the cubical subdivision of K , by $\Omega_k(K; F)$. This vector space has a canonical decomposition:

$$\Omega_k(K; F) := \bigoplus_{q-p=k} \Omega_{p,q}(K; F), \quad (4)$$

where $\Omega_{p,q}(K; F)$ is the direct sum over all adjacent pairs of simplices $\sigma^p \leq \sigma^q$ of the vector spaces $F(\sigma^p; \sigma^q)$. The vector spaces $(C_k(E; F))_{k \geq 0}$ form a chain complex for the following boundary operator ∂ :

$$\forall e^{k+1} \in E, \forall v \in F(e^{k+1}), \quad \partial(v \otimes e^{k+1}) := \sum_{e^k \leq e^{k+1}} v|_{e^k} \otimes e^k.$$

The homology of F is then defined as the homology $(H_k(E; F))_{k \geq 0}$ of this complex. We note that a morphism of cosheaves induces a morphism of chain complexes.

Notations. Let σ^p be a simplex of a Δ -complex K , we denote by χ_{σ^p} the characteristic cochain of σ^p . This is the p -cochain with value in \mathbb{F}_2 whose value is 1 on σ^p and 0 elsewhere.

Remark 1.5. For a more thorough exposition of the theory of cellular sheaves and cosheaves one can look at [KS10] (under the name *constructible sheaves*), [She85], or even [Che23a] where we discuss *dihomologic cosheaves*, a generalisation of cubical cosheaves. However we can still note that:

1. Whenever E' is a subcomplex of E we can define relative homology with coefficients in F by the usual quotient of chain complexes;
2. Definition 1.4 can be dualised for a cellular sheaf. This defines cellular cohomology with coefficients in a cellular sheaf;
3. The decomposition (4) makes $(\Omega_k(K; F); \partial)_{k \geq 0}$ into the total complex of a bicomplex. For all p, q , the boundary ∂ maps $\Omega_{p,q}$ in $\Omega_{p,q-1} \oplus \Omega_{p+1,q}$;
4. Assuming that E' is a regular subdivision of the regular CW-complex E there is a natural subdivision functor from the category of cosheaves on E to the cosheaves on E' . It associates to F the precomposition $F \circ S$ where $S : \mathbf{Cell} E' \rightarrow \mathbf{Cell} E$ is the increasing map that takes a cell of E' to the unique cell of E that contains it. This procedure always induces a quasi-isomorphism:

$$\begin{aligned} (C_k(E; F))_{k \geq 0} &\longrightarrow (C_k(E'; F \circ S))_{k \geq 0} \\ v \otimes e^k &\longmapsto \sum_{(e')^k \subset e^k} v \otimes (e')^k. \end{aligned}$$

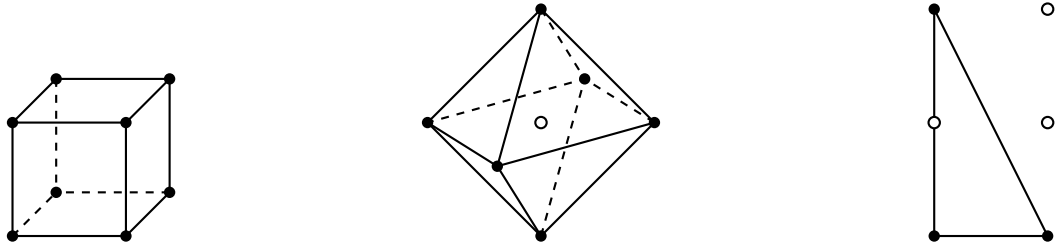
Non-Singular Polytopes, Primitive Triangulations, and Tropical Cosheaves. In the whole article $\mathfrak{t}^*(\mathbb{Z})$ will denote a free Abelian group of rank $n \geq 1$. For every commutative ring R , $\mathfrak{t}^*(R)$ denotes the associated free module $\mathfrak{t}^*(\mathbb{Z}) \otimes_{\mathbb{Z}} R$. $\mathfrak{t}(\mathbb{Z})$ denotes the dual lattice of the dual vector space $\mathfrak{t}(\mathbb{R})$. We say that a sub-vector space $W \subset \mathfrak{t}^*(\mathbb{R})$ is rational if there exists a sub-module $M \subset \mathfrak{t}^*(\mathbb{Z})$ satisfying $M \otimes_{\mathbb{Z}} \mathbb{R} = W$. In this case we denote by $W(\mathbb{Z})$ the lattice $W \cap \mathfrak{t}^*(\mathbb{Z})$, and for every commutative ring R , by $W(R)$ the free module $W(\mathbb{Z}) \otimes_{\mathbb{Z}} R$. We say that an affine sub-space A of $\mathfrak{t}^*(\mathbb{R})$ is rational if its tangent space TA is.

Definition 1.6. A non-singular polytope of $\mathfrak{t}^*(\mathbb{R})$ is a simple⁶ convex polytope P whose vertices lie in the lattice⁷ $\mathfrak{t}^*(\mathbb{Z})$ and such that for every vertex V of P the group $\sum_{E \geq V} TE(\mathbb{Z})$,

⁶A polytope P is simple if every face of codimension k of P is the intersection of k faces of codimension 1.

⁷We say that P is integral.

where the E 's are the edges of P containing V , is the full lattice $\mathfrak{t}^*(\mathbb{Z})$. A primitive triangulation K of P is a decomposition of P into integral simplices of minimal integral volume⁸, i.e. $\frac{1}{n!}$.



(a) The cube is non-singular. (b) The octahedron is not simple. (c) This triangle is simple but singular.

Figure 3: Examples of integer polytopes.

Remark 1.7. With this definition a non-singular polytope P has necessarily dimension n . Also a polytope P is non-singular if and only if its associated projective toric variety is smooth⁹ and the action of the torus $\text{Hom}(\mathfrak{t}^*(\mathbb{Z}); \mathbb{C}^\times)$ is effective. The toric variety of a polytope is smooth if and only if the latter is non-singular in the affine space it spans.

Definition 1.8. Let P be a non-singular polytope of $\mathfrak{t}^*(\mathbb{R})$ endowed with a primitive triangulation K . The dual hypersurface X of K is the sub-complex of the cubical subdivision of K made of all the cubes $[\sigma^p; \sigma^q]$ for which p is at least 1. See Figure 4 for instance.

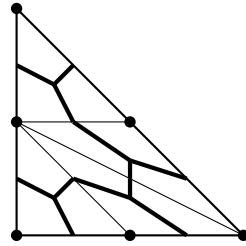


Figure 4: A dual hypersurface.

Definition 1.9 (Contraction). Let V be a finite dimensional vector space over a field \mathbb{F} , and $k, l \in \mathbb{N}$ be integers. For all $\alpha \in \bigwedge^l V^*$ and $v \in \bigwedge^{l+k} V$ the contraction $\alpha \cdot v$ is the only element of $\bigwedge^k V$ satisfying:

$$\beta(\alpha \cdot v) = (\beta \wedge \alpha)(v),$$

for all $\beta \in \bigwedge^k V^*$.

Definition 1.10. Let V be a finite dimensional \mathbb{F}_2 -vector space. Let k be the dimension of V . We denote by $[V]$ the unique generator of the line $\bigwedge^k V$. Whenever V is a subspace of W we see $[V]$ as a k -vector of W . In particular, if K is a primitive subdivision of a non-singular polytope P of $\mathfrak{t}^*(\mathbb{R})$, we denote, for all simplices σ^k of K , by $\omega(\sigma^k)$ the element of $\bigwedge^k \mathfrak{t}^*(\mathbb{F}_2)$ given by $[T\sigma^k(\mathbb{F}_2)]$.

Definition 1.11 (Tropical Cosheaves). Let P be a non-singular polytope of $\mathfrak{t}^*(\mathbb{R})$ endowed with a primitive triangulation K . We give some of the cosheaves of tropical homology given by I. Itenberg, L. Katzarkov, G. Mikhalkin, and I. Zharkov¹⁰ :

⁸The Lebesgue measure normalised so that a parallelogram on a basis of the lattice $\mathfrak{t}^*(\mathbb{Z})$ has volume 1.

⁹c.f. [Ful93], §2.1 Local Properties of Toric Varieties pp.28-31.

¹⁰c.f. [IKMZ19].

1. The *sedentarity* cosheaf Sed : for a simplex σ^p of K the group $\text{Sed}(\sigma^p)$ is given by:

$$\text{Sed}(\sigma^p) := \left\{ v \in \mathfrak{t}(\mathbb{F}_2) \mid \alpha(v) = 0, \forall \alpha \in (TQ \cap \mathfrak{t}^*(\mathbb{Z})) \right\},$$

where Q is the smallest face of P containing the relative interior¹¹ of σ^p . If $\sigma^p \leq \sigma^q$ the extension morphism $\text{Sed}(\sigma^q) \rightarrow \text{Sed}(\sigma^p)$ is the inclusion. It is defined on K , we denote its cubical subdivision by the same symbol, hence for all $\sigma^p \leq \sigma^q$, $\text{Sed}(\sigma^p; \sigma^q)$ is by definition $\text{Sed}(\sigma^q)$;

2. The cosheaf F_1^P , defined as the quotient: $F_1^P := \mathfrak{t}(\mathbb{F}_2) / \text{Sed}$, where $\mathfrak{t}(\mathbb{F}_2)$ is understood as the constant cosheaf. We also denote its subdivisions by the same symbol;
3. The cosheaves F_p^P , for all $p \in \mathbb{N}$, are the exterior powers of F_1^P : $F_p^P := \bigwedge^p F_1^P$. In particular F_0^P is the constant cosheaf \mathbb{F}_2 ;
4. The cosheaves¹² F_p^X , for all $p \in \mathbb{N}$, defined as the sub-cosheaf of F_p^P whose groups are, for all $\sigma^k \leq \sigma^l$:

$$F_p^X(\sigma^k; \sigma^k) := \ker \left(f : F_p^P(\sigma^k; \sigma^l) \rightarrow F_{p-k}^P(\sigma^k; \sigma^l) \right),$$

where f is the contraction against $\omega(\sigma^k)$: the generator of $\bigwedge^k T\sigma^k(\mathbb{F}_2)$. We have the relation : $F_k^X(\sigma^p; \sigma^q) = \sum_{\sigma^1 \leq \sigma^p} F_k^X(\sigma^1; \sigma^q) = \sum_{\sigma^1 \leq \sigma^p} \bigwedge^k F_1^X(\sigma^1; \sigma^q)$. See Figure 5 for an example. The morphisms between these groups are given by quotient maps and inclusions. Note that even though F_k^X is defined on the whole cubical subdivision of K , its support¹³ is contained in X . We denote its inclusion in F_p^P by $i_p : F_p^X \subset F_p^P$.

For all integers $p, q \geq 0$ we will denote by $H_{p,q}(P; \mathbb{F}_2)$ (resp. $H_{p,q}(X; \mathbb{F}_2)$) the q -th homology group of F_p^P (resp. F_p^X) and by $i_{p,q} : H_{p,q}(X; \mathbb{F}_2) \rightarrow H_{p,q}(P; \mathbb{F}_2)$ the morphism induced by the inclusion i_p . We will denote their dual vector spaces by lifting the indices.

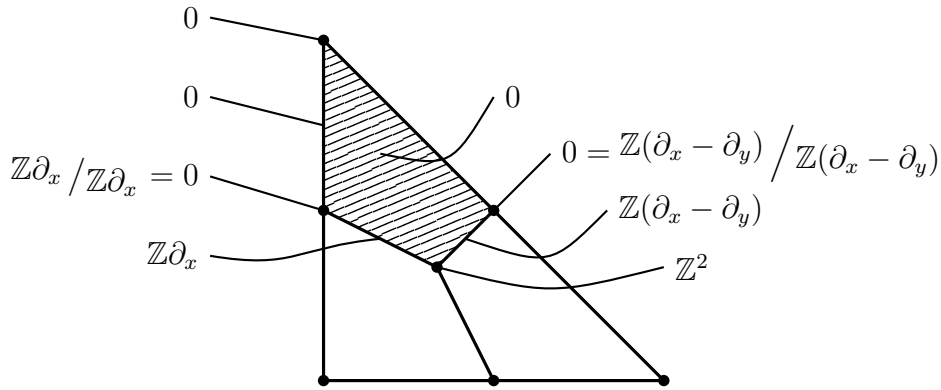


Figure 5: A triangle and the groups associated by F_1^X to some of its cubical cells.

¹¹i.e. the interior of σ^p in the affine space it spans.

¹²This definition is different from the usual one but is equivalent in the case of primitive triangulations. Such a triangulation K has $\theta(K) = 1$, c.f. [Che23a] Definition 3.5.

¹³i.e. the union of the closed cubes carrying non-trivial groups.

T-Hypersurfaces. These objects are combinatorial hypersurfaces of the real loci of smooth projective toric varieties. They naturally arise as topological description of the hypersurfaces obtained by primitive patchworks of Viro¹⁴. They are *dual complexes* of a particular kind of closed 1-cochains on a Δ -complex structure of the real loci of smooth projective toric varieties.

Definition 1.12. Let K be a Δ -complex of pure dimension $n \geq 1$ and ζ be a closed 1-cochain of K with values in \mathbb{F}_2 . The *dual complex* of ζ is the sub-complex of the cubical subdivision of K made of all the cubes $[\sigma^p; \sigma^q]$ whose first simplex σ^p is contains an edge σ^1 on which the value of ζ is 1.

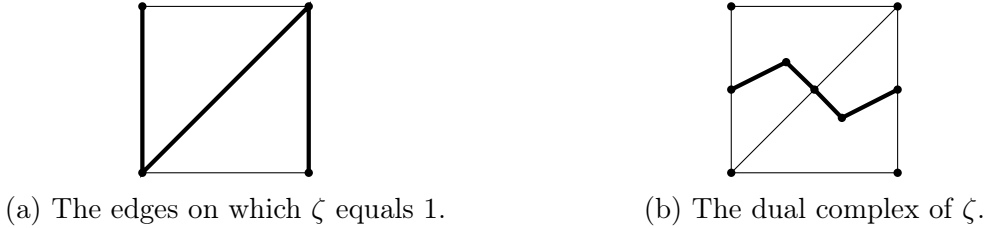


Figure 6: Examples of dual complex.

Definition 1.13. Let P be a non-singular polytope of $\mathfrak{t}^*(\mathbb{R})$ and Y denote its toric variety. The moment map¹⁵ $\mu : Y(\mathbb{R}) \rightarrow P$ induces an homeomorphism between P and the quotient of $Y(\mathbb{R})$ by the action of the unitary real torus $\text{Hom}(\mathfrak{t}^*(\mathbb{Z}); \{\pm 1\}) \cong \mathfrak{t}(\mathbb{F}_2)$. The moment map is split and admits 2^n sections called the quadrants. One of these quadrants is distinguished. It is made of all the points with non-negative coordinates¹⁶, we call it the positive quadrant. As a results $Y(\mathbb{R})$ is homeomorphic to the quotient space of $P \times \mathfrak{t}(\mathbb{F}_2)$ by the equivalence relation: $(x; v)$ is equivalent to $(y; w)$ if and only if x equals y and $v - w$ belongs to the stabilisator of any point in the orbit of x . The stabilisator of a point in the $\mathfrak{t}(\mathbb{F}_2)$ -orbit $x \in P$ is given by $\text{Sed}(x)$ ¹⁷. This description endows $Y(\mathbb{R})$ with a structure of regular CW-complex that we denote by $\mathbb{R}P$. Any closed cell of this cellular structure is a lift of a face of P . If Q is a face of P , the set of cells of $\mathbb{R}P$ lifting Q is canonically in bijection with $\mathfrak{t}(\mathbb{F}_2) / \text{Sed}(Q)$. We denote by $|\cdot| : \mathbb{R}P \rightarrow P$ the cellular map induced the moment map.

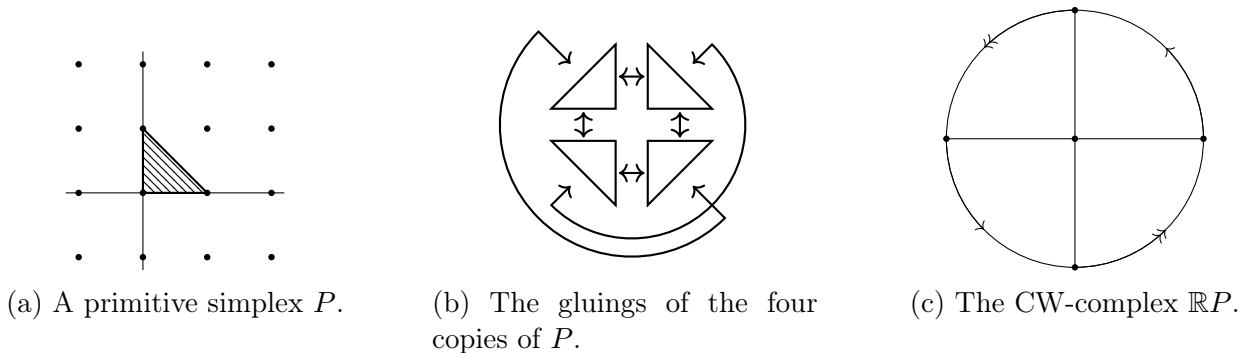


Figure 7: The CW-complex structure of the real projective plane induced by the moment map.

¹⁴c.f. [Vir06] or [Ris93].

¹⁵c.f. [Ful93], §4.2 Moment map pp.81-83.

¹⁶c.f. *ibid.*, §The manifold with corners pp.78-80.

¹⁷The orthogonal of $TQ(\mathbb{F}_2)$ with Q the face of P that contains x in its relative interior.

Definition 1.14. Let P be a non-singular polytope of $\mathfrak{t}^*(\mathbb{R})$ and K be a primitive triangulation of P . We denote by $\mathbb{R}K$ the subdivision of $\mathbb{R}P$ induced by K , and $|\cdot| : \mathbb{R}K \rightarrow K$ the induced cellular map. Let σ^p be a simplex of K . Each lift¹⁸ $\sigma_{\mathbb{R}}^p$ of σ^p in $\mathbb{R}K$ is uniquely represented by an element of $\mathfrak{t}(\mathbb{F}_2) / \text{Sed}(\sigma^p) = F_1^P(\sigma)$ that we call the *argument* of σ^p , and denote by $\arg(\sigma_{\mathbb{R}}^p)$. More generally the procedure $K \mapsto \mathbb{R}K$ applies to any regular subdivision of P . In particular if E is the cubical subdivision of K then $\mathbb{R}E$ is the cubical subdivision of $\mathbb{R}K$ and $[[\sigma_{\mathbb{R}}^p; \sigma_{\mathbb{R}}^q]]$ is $[[\sigma_{\mathbb{R}}^p]; |\sigma_{\mathbb{R}}^q|]$ and $\arg[\sigma_{\mathbb{R}}^p; \sigma_{\mathbb{R}}^q]$ equals $\arg(\sigma_{\mathbb{R}}^q)$.

Definition 1.15. Let P be a non-singular polytope of $\mathfrak{t}^*(\mathbb{R})$ and K be a primitive triangulation of P with dual hypersurface X . We denote by $\omega_{\mathbb{R}X}$ the closed 1-cochain defined by the formula:

$$\forall \sigma_{\mathbb{R}}^1 \in \mathbb{R}K, \quad \omega_{\mathbb{R}X}(\sigma_{\mathbb{R}}^1) := \omega(|\sigma_{\mathbb{R}}^1|) \left(\arg(\sigma_{\mathbb{R}}^1) \right).$$

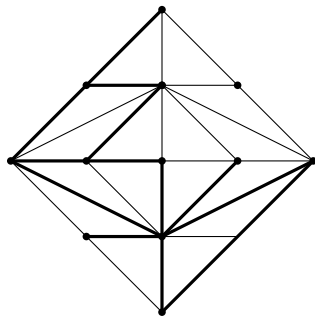
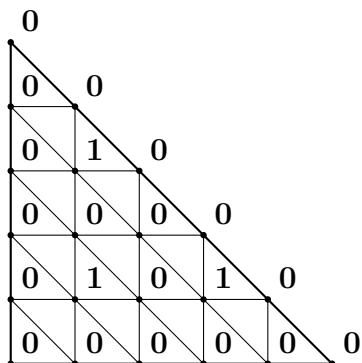
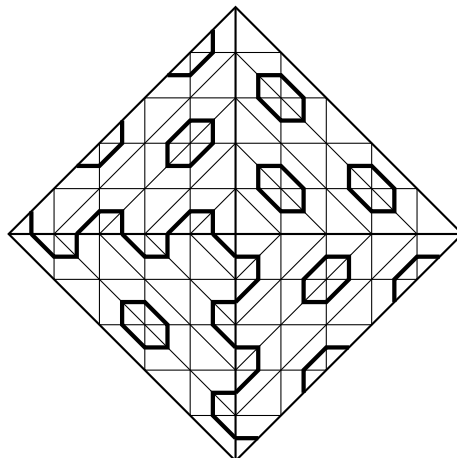


Figure 8: The edges of $\mathbb{R}K$ on which $\omega_{\mathbb{R}X}$ is 1.

Definition 1.16. Let P be a non-singular polytope of $\mathfrak{t}^*(\mathbb{R})$, K be a primitive triangulation of P with dual hypersurface X , and ε be a 0-cochain of K . We denote by $|\varepsilon|^*$ the pullback of ε by the cellular map $|\cdot|$. The *T-hypersurface* defined by ε is the dual complex of $d|\varepsilon|^* + \omega_{\mathbb{R}X}$ in $\mathbb{R}K$. We denote it by $\mathbb{R}X_\varepsilon$. We note that $|\mathbb{R}X_\varepsilon|$ is precisely X .



(a) Une distribution de signe sur une subdivision convexe du triangle de taille 5.



(b) Le patchwork combinatoire associé.

Figure 9: Le patchwork combinatoire d'une quintique plane maximale.

¹⁸i.e. $|\sigma_{\mathbb{R}}^p| = \sigma^p$.

Proposition 1.17 (E. Brugallé, L. López de Medrano, and J. Rau, Proposition 4.11 p.26, [BLdMR22]). *For all non-singular polytopes P , all primitive triangulations K of P and all sign distributions ε , the T -hypersurface $\mathbb{R}X_\varepsilon$ is PL-smooth.*

The Renaudineau-Shaw Spectral Sequence. Since $\mathbb{R}P$ and $\mathbb{R}X_\varepsilon$ can be respectively folded on P and X by $|\cdot|$ whose fibers are discrete, we can compute the cellular homology of $\mathbb{R}P$ and $\mathbb{R}X_\varepsilon$ with coefficients in \mathbb{F}_2 as the homology of cosheaves on P and X respectively. Using inclusion and subdivision we can see these cosheaves as cubical cosheaves on K .

Definition 1.18. Let P be a non-singular polytope of $\mathfrak{t}^*(\mathbb{R})$, K be a primitive triangulation of P with dual hypersurface X , and ε be a 0-cochain of K . Let $\mathcal{K}^{\mathbb{R}P}$ be the cosheaf that associates to every cube of K the vector space freely generated by its lifts in $\mathbb{R}K$. This cosheaf is canonically isomorphic to the cosheaf of group algebras $\mathbb{F}_2[F_1^P]$ and hence as canonical structure of cosheaf of algebras. Now we denote by Arg_ε the cosheaf of sets that associates to a cube of K its set of lifts in $\mathbb{R}K$ that belong to $\mathbb{R}X_\varepsilon$. We denote by $\mathcal{K}^{\mathbb{R}X_\varepsilon}$ the sub-cosheaf of $\mathcal{K}^{\mathbb{R}P}$ spanned by Arg_ε .

Remark 1.19. In [RS23], A. Renaudineau and K. Shaw called $\mathcal{K}^{\mathbb{R}X_\varepsilon}$ the sign cosheaf of $\mathbb{R}X_\varepsilon$ and denoted it \mathcal{S}_ε . They also called Arg_ε the real phase structure of $\mathbb{R}X_\varepsilon$ and denoted it \mathcal{E} .

Notations. Let G be an Abelian group, in the group algebra $\mathbb{F}_2[G]$ we denote the generator associated to the element g by x^g .

Remark 1.20. We can note that if V is a finite dimensional vector space over \mathbb{F}_2 then $\mathbb{F}_2[V]$ is a local ring with maximal ideal the kernel of the augmentation map $\mathbb{F}_2[V] \rightarrow \mathbb{F}_2$ that sends all elements x^v to 1. Every element not in the kernel of the augmentation is its own inverse.

Proposition 1.21 (A. Renaudineau and K. Shaw, Proposition 3.17 of p.964, [RS23]). *The chain complexes $(\Omega_k(\mathbb{R}K; \mathbb{F}_2))_{k \geq 0}$ and $(\Omega_k(K; \mathcal{K}^{\mathbb{R}P}))_{k \geq 0}$ are canonically isomorphic. Moreover this isomorphism sends the sub-complex $(C_k(\mathbb{R}X_\varepsilon; \mathbb{F}_2))_{k \geq 0}$ to $(\Omega_k(K; \mathcal{K}^{\mathbb{R}X_\varepsilon}))_{k \geq 0}$.*

This isomorphism sends $1 \otimes [\sigma_{\mathbb{R}}^p; \sigma_{\mathbb{R}}^q]$ to $[\sigma_{\mathbb{R}}^p; \sigma_{\mathbb{R}}^q] \otimes [|\sigma_{\mathbb{R}}^p|; |\sigma_{\mathbb{R}}^q|]$ or equivalently to $x^{\text{arg}(\sigma_{\mathbb{R}}^q)} \otimes [|\sigma_{\mathbb{R}}^p|; |\sigma_{\mathbb{R}}^q|]$.

Definition 1.22. Let P be a non-singular polytope of $\mathfrak{t}^*(\mathbb{R})$, K be a primitive triangulation of P with dual hypersurface X , and ε be a 0-cochain of K . The decreasing filtration $(\mathcal{K}_{(p)}^{\mathbb{R}P})_{0 \leq p \leq n}$ is defined by the iterated powers of its maximal ideal. We denote by $(\mathcal{K}_{(p)}^{\mathbb{R}X_\varepsilon})_{0 \leq p \leq n}$ the induced filtration of $\mathcal{K}^{\mathbb{R}X_\varepsilon}$. We denote by $(E_{p,q}^r(\mathbb{R}P))_{p,q,r \geq 0}$ and $(E_{p,q}^r(\mathbb{R}X_\varepsilon))_{p,q,r \geq 0}$ the respective spectral sequences. The inclusion $i : \mathcal{K}^{\mathbb{R}X_\varepsilon} \rightarrow \mathcal{K}^{\mathbb{R}P}$ induces a morphism of spectral sequences $(i_{p,q}^r : E_{p,q}^r(\mathbb{R}X_\varepsilon) \rightarrow E_{p,q}^r(\mathbb{R}P))_{p,q,r \geq 0}$.

Proposition 1.23 (Lemma 4.8 and Proposition 4.10, [RS23] and Proposition 4.22, [Che23b]). *Let $p \geq 0$ be an integer. There is a unique surjective morphism of cosheaves $\text{bv}_p : \mathcal{K}_{(p)}^{\mathbb{R}P} \rightarrow F_p^P$ that sends an element $\sum_{w \in W} x^w$ to $[W]$, for every p -dimensional subspace W of F_1^P . Its kernel is $\mathcal{K}_{(p+1)}^{\mathbb{R}P}$ and it sends $\mathcal{K}_{(p)}^{\mathbb{R}X_\varepsilon}$ onto F_p^X . We call them the Borel-Viro morphisms. For all $p, q \geq 0$*

they induce the following commutative square:

$$\begin{array}{ccc}
E_{p,q}^1(\mathbb{R}P) & \xrightarrow[\cong]{\text{bv}_{p,q}} & H_{p,q}(P; \mathbb{F}_2) \\
i_{p,q}^1 \uparrow & & \uparrow i_{p,q} \\
E_{p,q}^1(\mathbb{R}X_\varepsilon) & \xrightarrow[\cong]{\text{bv}_{p,q}} & H_{p,q}(X; \mathbb{F}_2)
\end{array} \tag{5}$$

Hence in what follows we shall always identify $(E_{p,q}^1(\mathbb{R}P))_{p,q \geq 0}$ and $(E_{p,q}^1(\mathbb{R}X_\varepsilon))_{p,q \geq 0}$ with $(H_{p,q}(P; \mathbb{F}_2))_{p,q \geq 0}$ and $(H_{p,q}(X; \mathbb{F}_2))_{p,q \geq 0}$ respectively, using the Borel-Viro isomorphisms.

Remark 1.24. Let $\sigma^a \leq \sigma^b$ be two simplices of K , and $v_0, \dots, v_p \in F_1^P(\sigma^a; \sigma^b)$, we have:

$$\text{bv}_p \left(x^{v_0} \prod_{i=1}^p (1 + x^{v_i}) \right) = \bigwedge_{i=1}^p v_i.$$

2 Differential Geometry of Primitive Complexes

Shellings. In this paragraph we provide the definitions and basic properties of shellings of simplicial complexes.

Definition 2.1. Let K be a finite simplicial complex of dimension $n \in \mathbb{N}$. The polynomial \mathbf{f}_K is given by the formula:

$$\mathbf{f}_K(X) = \sum_{i=0}^{n+1} f_{i-1}(K) X^{n+1-i},$$

where $f_i(K)$ is the number of i -simplices of K .

Definition 2.2. A *shelling* of a finite simplicial complex K of pure dimension $n \in \mathbb{N}$, is a numbering of its maximal simplices $\sigma_0^n, \dots, \sigma_k^n$ such that, for all integers $0 \leq i \leq k-1$, the intersection of σ_{i+1}^n with the previous simplices $\cup_{j \leq i} \sigma_j^n$ is a non-empty union of faces of codimension 1 of σ_{i+1}^n . Let $0 \leq i \leq k$, we call the *order* of σ_i^n , the number of its faces of codimension 1 that belong to the previous simplices. For all $0 \leq j \leq n+1$, we denote by $h_j(K)$ the number of simplices of order j of the shelling. Moreover, we define the polynomial \mathbf{h}_K by the formula:

$$\mathbf{h}_K(X) = \sum_{j=0}^{n+1} h_j(K) X^{n+1-j}.$$

A simplicial complex that admits a shelling is called *shellable*.

The polynomial \mathbf{h}_K of a shellable simplicial complex K does not depend on the particular shelling used to compute it. This property is a corollary of the following formula.

Proposition 2.3 ([Zie95], p.249). *Let K be a shellable simplicial complex. We have the following identity:*

$$\mathbf{h}_K(X) = \mathbf{f}_K(X-1). \tag{6}$$

The relation (6) implies that $h_0(K)$ equals $f_{-1}(K)$, i.e. 1, and also that $h_{n+1}(K)$ equals $\mathbf{f}_K(-1)$, i.e. $(-1)^{n+1} + (-1)^n \chi(|K|)$ (the Euler characteristic of $|K|$).

Proposition 2.4 ([Bjö84], Proposition 4.3 p.12). *Let K be a shellable simplicial complex of dimension n in which every $(n - 1)$ -simplex is contained in at most two n -simplices. Then:*

1. *If every $(n - 1)$ -simplex is contained in exactly two n -simplices, the geometric realisation of K is homeomorphic to a sphere;*
2. *If there is a $(n - 1)$ -simplex contained in exactly one n -simplex, the geometric realisation of K is homeomorphic to a ball.*

In particular, in the first case $h_{n+1}(K)$ equals 1 and in the second 0.

Lemma 2.5. *Let K be an n -dimensional shellable simplicial complex in which every $(n - 1)$ -simplex is contained in at most two n -simplices. Let $\sigma_0^n, \dots, \sigma_k^n$ be a shelling of K . For all $0 \leq i \leq k$, we denote by τ_i the intersection of all the codimension 1 faces of σ_i^n along which it intersects the previous simplices. For all $0 \leq i \leq k$, the star of τ_i in K is contained in $\cup_{j \leq i} \sigma_j^n$. In other words, when we add the simplex σ_i^n we complete the star of τ_i .*

Proof. Let $0 \leq i \leq k$. The link of τ_i is a pure $(n - 1 - \dim(\tau_i))$ -dimensional simplicial complex in which every simplex of codimension 1 is contained in at most two maximal simplices. The shelling of K induces a shelling of the link $\text{lk}(\tau_i; K)$. The proof of Lemma 8.7 p.237 of [Zie95] is directly adaptable from the case of a vertex to the case of a simplex of higher dimension. By definition of τ_i , this shelling has a simplex of order $n - \dim(\tau_i)$, i.e. $\dim \text{lk}(\tau_i; K) + 1$. This is $\text{lk}(\tau_i; \sigma_i^n)$. Hence, by Proposition 2.4, this link must be a sphere. Thus, every simplex of codimension one of this link belongs to exactly two maximal simplices. The shelling of the link can be reversed to find another shelling of the link. This would be Lemma 8.10 p.240 of [Zie95] if we were considering boundary of simplicial polytopes. However, its proof only uses the fact that every simplex of codimension one belongs to two maximal simplices, hence it is directly transposable to our case. Reversing the order changes the order of every maximal simplex by the involution $x \mapsto \dim \text{lk}(\tau_i; K) + 1 - x$. Therefore, the simplex of order $\dim \text{lk}(\tau_i; K) + 1$ has to be the last for the simplex of order 0 is always the first. Therefore, when we add σ_i^n we complete the link of τ_i , hence its star. \square

Differential Geometry. Some concepts of differential geometry are adaptable to the discrete context of simplicial complexes primitively embedded in affine spaces. We give the definitions of this framework.

Definition 2.6. We call a *primitive complex* in $\mathfrak{t}^*(\mathbb{R})$, a purely n -dimensional piecewise-affinely embedded simplicial complex $K \subset \mathfrak{t}^*(\mathbb{R})$ whose simplices are primitive, i.e. of minimal integral volume. The *boundary* of K is the sub-complex of simplices contained in exactly one maximal simplex of K . We denote it by ∂K .

Definition 2.7. Let K be primitive complex in $\mathfrak{t}^*(\mathbb{R})$. The \mathbb{F}_2 -*normal cosheaf* N of K is defined for all simplices σ^p by the formula:

$$N(\sigma^p) := (T\sigma^p)^\perp (\mathbb{F}_2),$$

with extension morphisms given by inclusions. Likewise, the \mathbb{F}_2 -cotangent cosheaf T^* of K is defined for all simplices σ^p by the formula:

$$T^*(\sigma^p) := T^*\sigma^p(\mathbb{F}_2),$$

with extension morphisms given by the quotient morphisms. We have the following exact sequence of cosheaves:

$$0 \rightarrow N \rightarrow [K; \mathfrak{t}(\mathbb{F}_2)] \rightarrow T^* \rightarrow 0,$$

where $[K; \mathfrak{t}(\mathbb{F}_2)]$ denotes the constant cosheaf on K with value $\mathfrak{t}(\mathbb{F}_2)$.

Definition 2.8. Let K be primitive complex in $\mathfrak{t}^*(\mathbb{R})$ and σ^p be a simplex of K . We denote by $\omega(\sigma^p) \in \bigwedge^p \mathfrak{t}^*(\mathbb{F}_2)$ the generator of the line $\bigwedge^p T\sigma^p(\mathbb{F}_2)$. If σ^1 is an edge of K with vertices α and β then $\omega(\sigma^1)$ is the reduction modulo 2 of $\beta - \alpha$.

Proposition 2.9. *The cochain $[\sigma^p \mapsto \omega(\sigma^p)] \in C^p(K; \bigwedge^p \mathfrak{t}^*(\mathbb{F}_2))$ is closed.*

Proof. If p equals 0 then by definition $[\sigma^0 \mapsto \omega(\sigma^0)]$ is constant to 1. It is closed. Now let $p \geq 1$ be an integer, σ^p be a simplex of K , and β be a vertex of σ^p . Since K is primitive, $\omega(\sigma^p)$ is given by the following formula:

$$\begin{aligned} \omega(\sigma^p) &= \bigwedge_{\alpha \in \text{lk}(\beta; \sigma^p)^{(0)}} (\alpha - \beta) = \bigwedge_{\alpha \in \text{lk}(\beta; \sigma^p)^{(0)}} \alpha + \sum_{\alpha \in \text{lk}(\beta; \sigma^p)^{(0)}} \beta \wedge \bigwedge_{\substack{\alpha' \in \text{lk}(\beta; \sigma^p)^{(0)} \\ \alpha' \neq \alpha}} \alpha' \\ &= \sum_{\sigma^{p-1} \leq \sigma^p} \bigwedge_{\alpha \in (\sigma^{p-1})^{(0)}} \alpha. \end{aligned}$$

If we denote by λ the cochain $[\sigma^{p-1} \mapsto \bigwedge_{\alpha \in (\sigma^{p-1})^{(0)}} \alpha]$, the cochain ω is the differential of λ . Hence, it is closed. \square

Proposition 2.10. *Let K be a primitive complex in $\mathfrak{t}^*(\mathbb{R})$. For all simplices $\sigma^p \in K$ and all vertices α of σ^p the set $\{\beta - \alpha : \beta \text{ vertex of } \text{lk}(\alpha; \sigma^p)\}$ is, up to ordering, a basis of $T\sigma^p(\mathbb{Z})$. Moreover for all $q \in \mathbb{N}$, the morphism:*

$$\begin{aligned} z_{\sigma^p}^q : \bigwedge^q \mathfrak{t}(\mathbb{F}_2) &\longrightarrow Z^q(\sigma^p; \mathbb{F}_2) \\ v &\longmapsto \left[\sigma^q \mapsto (\omega(\sigma^q))(v) \right], \end{aligned}$$

fits in the short exact sequence:

$$0 \rightarrow N(\sigma^p) \wedge \bigwedge^{q-1} \mathfrak{t}(\mathbb{F}_2) \rightarrow \bigwedge^q \mathfrak{t}(\mathbb{F}_2) \xrightarrow{z_{\sigma^p}^q} Z^q(\sigma^p; \mathbb{F}_2) \rightarrow 0.$$

Moreover, for all pairs $\sigma^p \leq \sigma^r$ and all vectors $v \in \bigwedge^q \mathfrak{t}(\mathbb{F}_2)$ the restriction of $z_{\sigma^r}^q(v)$ to σ^p is $z_{\sigma^p}^q(v)$.

Proof. The first statement is a direct consequence of the primitivity of the simplices of K . The morphism $z_{\sigma^0}^0$ is an isomorphism, $z_{\sigma^0}^q$ is null for all $q > 0$, and $N(\sigma^p)$ equals $\mathfrak{t}(\mathbb{F}_2)$. The proposition is true for $p = 0$. Let σ^p be a simplex of K of positive dimension, and $q \geq 0$ be an integer. Proposition 2.9 ensures us that $z_{\sigma^p}^q$ is well defined with $Z^q(\sigma^p; \mathbb{F}_2)$ as target space.

Let β be a vertex of σ^p and let us label $\sigma_1^q, \dots, \sigma_k^q$, with k being $\binom{p}{q}$, the q -simplices of σ^p that contain β . The first statement implies that $\{\omega(\sigma_1^q), \dots, \omega(\sigma_k^q)\}$ is a basis of $\bigwedge^q T\sigma^p(\mathbb{F}_2)$. Let ζ be a q -cocycle of σ^p . We define v to be the unique vector of $\bigwedge^q \mathfrak{t}(\mathbb{F}_2)/(\bigwedge^q T\sigma^p(\mathbb{F}_2))^\perp$ satisfying:

$$\omega(\sigma_i^q)(v) = \zeta(\sigma_i^q),$$

for all $1 \leq i \leq k$. Let σ^q be a simplex of σ^p that does not contain β . The boundary of $\beta * \sigma^q$ is made of σ^q and $\sigma_{i_1}^q, \dots, \sigma_{i_{q+1}}^q$. Since both ω and ζ are closed we have:

$$\omega(\sigma^q)(v) = \sum_{j=1}^{q+1} \omega(\sigma_{i_j}^q)(v) = \sum_{j=1}^{q+1} \zeta(\sigma_{i_j}^q) = \zeta(\sigma^q).$$

Hence, for any lift \bar{v} of v in $\bigwedge^q \mathfrak{t}(\mathbb{F}_2)$, $z_{\sigma^p}^q(\bar{v})$ equals ζ , and $z_{\sigma^p}^q$ is surjective. Moreover, the kernel of $z_{\sigma^p}^q$ is $(\bigwedge^q T\sigma^p(\mathbb{F}_2))^\perp$. This vector space has codimension $\binom{p}{q}$ and contains $N(\sigma^p) \wedge \bigwedge^{q-1} \mathfrak{t}(\mathbb{F}_2)$. Using a basis of $\mathfrak{t}(\mathbb{F}_2)$ whose $(n-p)$ first vectors belong to $N(\sigma^p)$, we find that the latter space has dimension $\binom{n}{q} - \binom{p}{q}$. Therefore, they must be equal. The last statement of the proposition is a direct consequence of the definition of $z_{\sigma^p}^q$. \square

Proposition 2.11. *Let K be primitive complex in $\mathfrak{t}^*(\mathbb{R})$ and σ^n be a maximal simplex of K . For all vertices α of σ^n , we denote by $e_{\sigma^n}(\alpha)$ the generator of the line $N(\text{lk}(\alpha; \sigma^n))$. For all vertices α of σ^n the collection $\{e_{\sigma^n}(\beta) : \beta \in \text{lk}(\alpha; \sigma^p)^{(0)}\}$ is the dual basis of the reduction modulo 2 of $\{\beta - \alpha : \beta \in \text{lk}(\alpha; \sigma^p)^{(0)}\}$. Moreover, we have the relation:*

$$\sum_{\alpha \in (\sigma^n)^{(0)}} e_{\sigma^n}(\alpha) = 0.$$

In addition, for all simplices $\sigma^p \leq \sigma^n$, the set $\{e_{\sigma^n}(\alpha) : \alpha \in \text{lk}(\sigma^p; \sigma^n)^{(0)}\}$ is a basis of the vector space $N(\sigma^p)$. Also, for all simplices $\sigma^p \leq \sigma^n$, with $p \geq 1$, and all vertices $\alpha \in \sigma^n$, the interior product $\iota_{e_{\sigma^n}(\alpha)}\omega(\sigma^p)$ vanishes if α is not a vertex of σ^p and equals $\omega(\text{lk}(\alpha; \sigma^p))$ when α belongs to σ^p .

Proof. Let σ^n be a simplex of K and α, β be distinct vertices of σ^n . If $\beta' \in \sigma^n$ is different from α and β , then, by definition, $(\beta' - \alpha)(e_{\sigma^n}(\beta))$ must vanish for the edge $[\beta'; \alpha]$ belongs to $\text{lk}(\beta; \sigma^n)$. The number $(\beta - \alpha)(e_{\sigma^n}(\beta))$ must be 1, for otherwise $e_{\sigma^n}(\beta)$ would vanish. Indeed, $\{\beta - \alpha : \beta \in \text{lk}(\alpha; \sigma^p)^{(0)}\}$ is a basis of $\mathfrak{t}^*(\mathbb{Z})$, hence its reduction modulo 2 is a basis of $\mathfrak{t}^*(\mathbb{F}_2)$. It follows that $\{e_{\sigma^n}(\beta) : \beta \in \text{lk}(\alpha; \sigma^p)^{(0)}\}$ is the dual basis of $\{\beta - \alpha \pmod{2} : \beta \in \text{lk}(\alpha; \sigma^p)^{(0)}\}$. Moreover, it implies that for any two vertices β, β' of σ^n , we have:

$$(\beta - \beta') \left(\sum_{\alpha \in (\sigma^n)^{(0)}} e_{\sigma^n}(\alpha) \right) = (\beta - \beta')(e_{\sigma^n}(\beta) + e_{\sigma^n}(\beta')) = 1 + 1 = 0.$$

Therefore, the vector $\sum_{\alpha \in (\sigma^n)^{(0)}} e_{\sigma^n}(\alpha)$ is 0. We also deduce that, for all pairs $\sigma^p \leq \sigma^n$, the set $\{e_{\sigma^n}(\alpha) : \alpha \in \text{lk}(\sigma^p; \sigma^n)^{(0)}\}$ is a linearly independent family of $N(\sigma^p)$. Since its cardinality is $(n-p)$ it must be a basis. As a consequence if $\alpha \in \sigma^n$ is not a vertex of σ^p , with $p \geq 1$, then $\iota_{e_{\sigma^n}(\alpha)}\omega(\sigma^p)$ vanishes. If on the contrary α is a vertex of σ^p , then $\omega(\sigma^p)$

equals $\omega(\text{lk}(\alpha; \sigma^p)) \wedge (\alpha - \beta)$ for any other vertex β of σ^p . It follows that $\iota_{e_{\sigma^n}(\alpha)}\omega(\sigma^p)$ equals $(\alpha - \beta)(e_{\sigma^n}(\alpha))\omega(\text{lk}(\alpha; \sigma^p))$, i.e. $\omega(\text{lk}(\alpha; \sigma^p))$. \square

Remark 2.12. Using Proposition 2.10 and Proposition 2.11, we find, for all simplices σ^n of K and all vertices β of σ^n , that the image of $e_{\sigma^n}(\beta)$ by $z_{\sigma^n}^1$ is the exact cocycle $d\chi_\beta$ ¹⁹. Moreover, every form $\alpha \in \mathfrak{t}^*(\mathbb{F}_2)$ equals $\sum_{\beta \in \sigma^n} \alpha(e_{\sigma^n}(\beta))\beta$.

Proposition 2.13. *Let K be primitive complex in $\mathfrak{t}^*(\mathbb{R})$ and σ^{n-1} be an interior simplex of K , i.e. bounding two maximal simplices σ_+^n and σ_-^n . For all vertices $\alpha \in \sigma^{n-1}$ we denote by $\rho_{\sigma^{n-1}}(\alpha)$ the difference $e_{\sigma_+^n}(\alpha) - e_{\sigma_-^n}(\alpha)$. This vector lies on the line $N(\sigma^{n-1})$ and we denote by $|\rho_{\sigma^{n-1}}(\alpha)| \in \mathbb{F}_2$ the associated number. Moreover, if α_\pm denotes the vertex of σ_\pm^n opposite to σ^{n-1} then:*

$$\alpha_+ - \alpha_- = \sum_{\beta \in \sigma^{n-1}} |\rho_{\sigma^{n-1}}(\beta)|\beta \pmod{2}.$$

Proof. Let α be a vertex of σ^{n-1} and σ^1 be an edge of σ^{n-1} . Following Remark 2.12, we have:

$$\omega(\sigma^1)(\rho_{\sigma^{n-1}}(\alpha)) = \omega(\sigma^1)(e_{\sigma_+^n}(\alpha) - e_{\sigma_-^n}(\alpha)) = 2d\chi_\alpha(\sigma^1) = 0.$$

Since $\{\omega(\sigma^1): \sigma^1 \leq \sigma^{n-1}\}$ spans $T\sigma^{n-1}(\mathbb{F}_2)$, the vector $\rho_{\sigma^{n-1}}(\alpha)$ belongs to the normal line $N(\sigma^{n-1})$. By Proposition 2.11, $e_{\sigma_+^n}(\alpha_+)$ equals $e_{\sigma_-^n}(\alpha_-)$. They are the generator $[N(\sigma^{n-1})]$ of $N(\sigma^{n-1})$. Hence, for any vertex $\beta \in \sigma^{n-1}$, we have:

$$\begin{aligned} (\alpha_+ - \alpha_-)[N(\sigma^{n-1})] &= (\alpha_+ - \beta)[N(\sigma^{n-1})] + (\alpha_- - \beta)[N(\sigma^{n-1})] \\ &= (\alpha_+ - \beta)(e_{\sigma_+^n}(\alpha_+)) + (\alpha_- - \beta)(e_{\sigma_-^n}(\alpha_-)) \\ &= 1 + 1 = 0. \end{aligned}$$

And the form $(\alpha_+ - \alpha_-)$ is tangent to σ^{n-1} modulo 2. Therefore, by Proposition 2.11 and Remark 2.12, we have the following formula:

$$\alpha_+ - \alpha_- = \sum_{\beta \in \sigma^{n-1}} (\alpha_+ - \alpha_-)(e_{\sigma_+^n}(\beta))\beta.$$

For all $\beta \in \sigma^{n-1}$, we have:

$$\begin{aligned} (\alpha_+ - \alpha_-)(e_{\sigma_+^n}(\beta)) &= (\alpha_+ - \beta)(e_{\sigma_+^n}(\beta)) + (\beta - \alpha_-)(e_{\sigma_+^n}(\beta)) \\ &= 1 + (\beta - \alpha_-)(e_{\sigma_-^n}(\beta)) + (\beta - \alpha_-)(\rho_{\sigma^{n-1}}(\beta)) \\ &= 1 + 1 + |\rho_{\sigma^{n-1}}(\beta)|(\beta - \alpha_-)[N(\sigma^{n-1})] \\ &= |\rho_{\sigma^{n-1}}(\beta)|, \end{aligned}$$

since $(\beta - \alpha_-)[N(\sigma^{n-1})]$ equals 1. Indeed, the opposite would imply that $[N(\sigma^{n-1})]$ is zero. \square

Proposition 2.14. *For all interior simplices $\sigma^{n-1} \in K$ we have:*

$$\sum_{\alpha \in (\sigma^{n-1})^{(0)}} \rho_{\sigma^{n-1}}(\alpha) = 0.$$

¹⁹Recall that χ_β is the characteristic cochain of the vertex β . Its values are 1 on β and 0 elsewhere.

Proof. Let σ_+^n and σ_-^n be the two maximal simplices bounded by σ^{n-1} . By Proposition 2.11, we have:

$$\sum_{\alpha \in (\sigma^{n-1})^{(0)}} \rho_{\sigma^{n-1}}(\alpha) = e_{\sigma_+^n}(\alpha_+) + e_{\sigma_-^n}(\alpha_-).$$

However, $e_{\sigma_+^n}(\alpha_+)$ and $e_{\sigma_-^n}(\alpha_-)$ both equal the generator of $N(\sigma^{n-1})$. Hence, the proposition follows. \square

Definition 2.15. Let $\varepsilon \in C^0(K; \mathbb{F}_2)$. The *first derivative* $D\varepsilon \in H_n(K; \partial K; T^*)$ is defined for all $\sigma^n \in K$ by the formula:

$$D\varepsilon_{\sigma^n} := (z_{\sigma^n}^1)^{-1}(d\varepsilon|_{\sigma^n}).$$

The following proposition, which is a direct consequence of Definition 2.15, justifies the choice of terminology.

Proposition 2.16. *For all $\varepsilon \in C^0(K; \mathbb{F}_2)$, all $\sigma^n \in K$ and all vertices $\alpha, \beta \in \sigma^n$ we have:*

$$\varepsilon(\beta) = \varepsilon(\alpha) + D\varepsilon_{\sigma^n}(\beta - \alpha).$$

Therefore, $D\varepsilon$ vanishes if and only if ε is constant.

Definition 2.17. Let $\varepsilon \in C^0(K; \mathbb{F}_2)$. The *second derivative* $D^2\varepsilon \in H_{n-1}(K; \partial K; N)$ is defined by the formula:

$$D^2\varepsilon := \partial(D\varepsilon),$$

where ∂ denotes the connecting morphism $H_n(K; \partial K; T^*) \rightarrow H_{n-1}(K; \partial K; N)$ of the homological long exact sequence associated with the short exact sequence of cosheaves of Definition 2.7.

Remark 2.18. When the support of K is a closed ball the homological long exact sequence associated with the short exact sequence of cosheaves of Definition 2.7 splits around the dimension n . It induces the following short exact sequence:

$$0 \longrightarrow \underbrace{H_n(K; \partial K; \mathfrak{t}(\mathbb{F}_2))}_{\cong \mathfrak{t}(\mathbb{F}_2)} \longrightarrow H_n(K; \partial K; T^*) \xrightarrow{\partial} H_{n-1}(K; \partial K; N) \longrightarrow 0.$$

Because N vanishes on maximal simplices. We note that $H_{n-1}(K; \partial K; N)$ is a group of cycles.

Proposition 2.19. *Let σ^{n-1} be an interior simplex bounding two maximal simplices σ_{\pm}^n , and α_{\pm} be the vertex of σ_{\pm}^n opposite to σ^{n-1} . The number $D^2\varepsilon_{\sigma^{n-1}}(\alpha_- - \beta)$ does not depend on the choice of vertex $\beta \in \sigma^{n-1}$. It satisfies the following property:*

$$\varepsilon(\alpha_+) = \varepsilon(\alpha_-) + D\varepsilon_{\sigma_+^n}(\alpha_+ - \alpha_-) + D^2\varepsilon_{\sigma^{n-1}}(\alpha_- - \beta).$$

Moreover, if $|K|$ is homeomorphic to a ball, $D^2\varepsilon$ vanishes if and only if ε is affine.

Proof. Let β be a vertex of σ^{n-1} . We have:

$$\begin{aligned} \varepsilon(\alpha_+) + \varepsilon(\alpha_-) &= D\varepsilon_{\sigma_+^n}(\alpha_+ + \beta) + D\varepsilon_{\sigma_-^n}(\alpha_- + \beta) \\ &= D\varepsilon_{\sigma_+^n}(\alpha_+ + \alpha_-) + D^2\varepsilon_{\sigma^{n-1}}(\alpha_- + \beta). \end{aligned}$$

Hence $\varepsilon(\alpha_+)$ equals $\varepsilon(\alpha_-) + D\varepsilon_{\sigma_+^n}(\alpha_+ - \alpha_-) + D^2\varepsilon_{\sigma^{n-1}}(\alpha_- - \beta)$ and $D^2\varepsilon_{\sigma^{n-1}}(\alpha_- - \beta)$ does not depend on β . The cochain ε is represented in each n -simplex σ^n of K by an affine function of $\mathfrak{t}^*(\mathbb{F}_2)$ whose linear part is given by $D\varepsilon_{\sigma^n}$. Using the exact sequence of Remark 2.18, we see that if $D^2\varepsilon$ vanishes then this linear part is the same on every maximal simplices of K since K is connected. Hence ε is globally given by an affine map. \square

Lemma 2.20. *If the primitive complex K is shellable then both D and D^2 are surjective. Therefore, we have the exact sequences:*

$$0 \longrightarrow \mathbb{F}_2 \longrightarrow C^0(K; \mathbb{F}_2) \xrightarrow{D} H_n(K; \partial K; T^*) \longrightarrow 0,$$

and:

$$0 \longrightarrow \text{Aff}(\mathfrak{t}^*(\mathbb{F}_2)) \longrightarrow C^0(K; \mathbb{F}_2) \xrightarrow{D^2} H_{n-1}(K; \partial K; N) \longrightarrow 0.$$

Proof. Let $\varepsilon \in C^0(K; \mathbb{F}_2)$ and σ^n be a maximal simplex of K . By Definition 2.15, we have the following formula:

$$D\varepsilon_{\sigma^n} = \sum_{\alpha \in \sigma^n} \varepsilon(\alpha) e_{\sigma^n}(\alpha).$$

Now, let σ^{n-1} be an interior simplex of K . By Definition 2.17, we have:

$$D^2\varepsilon_{\sigma^{n-1}} = (\varepsilon(\alpha_+) + \varepsilon(\alpha_-)) [N(\sigma^{n-1})] + \sum_{\alpha \in \sigma^{n-1}} \varepsilon(\alpha) \rho_{\sigma^{n-1}}(\alpha),$$

where $\alpha_{\pm} * \sigma^{n-1}$ are the two n -simplices bounded by σ^{n-1} , and $[N(\sigma^{n-1})]$ is the generator of $N(\sigma^{n-1})$. We can also write it as follows:

$$D^2\varepsilon_{\sigma^{n-1}} = \left(\varepsilon(\alpha_+) + \varepsilon(\alpha_-) + \sum_{\alpha \in \sigma^{n-1}} \varepsilon(\alpha) |\rho_{\sigma^{n-1}}(\alpha)| \right) [N(\sigma^{n-1})].$$

We will use this formula to show that when K is shellable, D^2 is surjective. Let $(\sigma_i^n)_{0 \leq i \leq k}$ be a shelling of K , and $c \in H_{n-1}(K; \partial K; N)$. We will construct a ε whose image by D^2 is c . For all $0 \leq i \leq k$, we denote by K_i the union $\cup_{0 \leq j \leq i} K(\sigma_j^n)$. For all interior simplices σ^{n-1} , we denote by $|\langle c; \sigma^{n-1} \rangle| \in \mathbb{F}_2$ the unique number for which $|\langle c; \sigma^{n-1} \rangle| [N(\sigma^{n-1})]$ equals $\langle c; \sigma^{n-1} \rangle$. We note that $K_{i+1}^{(0)}$ equals $K_i^{(0)}$ if the order of σ_{i+1}^n is bigger than 1. Otherwise, $K_{i+1}^{(0)} \setminus K_i^{(0)}$ is reduced to the vertex $\text{lk}(K_i; K_{i+1})$. It induces an order $(\alpha_j)_{1 \leq j \leq h_1(K)}$ on the vertices of K that do not belong to σ_0^n . For all $1 \leq j \leq h_1(K)$, we denote by σ_j^{n-1} the simplex for which $\sigma_j^{n-1} * \alpha_j$ is the simplex of order 1 at which α_j appears. We also denote by α_j^- the other vertex of the link of σ_j^{n-1} in K . Now we inductively define ε by the following rule:

$$\begin{cases} \varepsilon|_{\sigma_0^n} = 0 \\ \varepsilon(\alpha_j) = |\langle c; \sigma_j^{n-1} \rangle| + \varepsilon(\alpha_j^-) + \sum_{\alpha \in \sigma_j^{n-1}} \varepsilon(\alpha) |\rho_{\sigma_j^{n-1}}(\alpha)|, \quad \forall 1 \leq j \leq h_1(K). \end{cases}$$

By construction, $D^2\varepsilon_{\sigma_j^{n-1}}$ equals $\langle c; \sigma_j^{n-1} \rangle$ for all $1 \leq j \leq h_1(K)$. We claim this is also true on the other interior $(n-1)$ -simplices of K . We will show by induction on $0 \leq i \leq k$, that $D^2\varepsilon_{\sigma^{n-1}}$ equals $\langle c; \sigma^{n-1} \rangle$ for all interior $(n-1)$ -simplices of K_i . The assertion is true on K_1 for σ_1^n has

order 1. Let us assume it true on K_i . The only interior $(n-1)$ -simplices of K_{i+1} that were not already interior in K_i are those contained in σ_{i+1}^n that contain τ_{i+1} (the intersection of the contact faces of σ_{i+1}^n with K_i). If $\text{ord}(\sigma_{i+1}^n)$ is 1, then by construction the property is true on K_{i+1} . If $\text{ord}(\sigma_{i+1}^n)$ is bigger than 1, let us consider two $(n-1)$ -simplices σ_{\pm}^{n-1} that contain τ_i . Since $D^2\varepsilon$ and c are cycles we have:

$$D^2\varepsilon_{\sigma_+^{n-1}} + D^2\varepsilon_{\sigma_-^{n-1}} = \sum_{\substack{\sigma^{n-1} \geq \sigma_+^{n-1} \cap \sigma_-^{n-1} \\ \sigma^{n-1} \neq \sigma_{\pm}^{n-1}}} D^2\varepsilon_{\sigma^{n-1}}, \quad (7)$$

and:

$$\langle c; \sigma_+^{n-1} \rangle + \langle c; \sigma_-^{n-1} \rangle = \sum_{\substack{\sigma^{n-1} \geq \sigma_+^{n-1} \cap \sigma_-^{n-1} \\ \sigma^{n-1} \neq \sigma_{\pm}^{n-1}}} \langle c; \sigma^{n-1} \rangle. \quad (8)$$

If $\sigma^{n-1} \in K$ contains $\sigma_+^{n-1} \cap \sigma_-^{n-1}$ and is different from σ_+^{n-1} and σ_-^{n-1} it cannot belong to σ_{i+1}^n for $\sigma_+^{n-1} \cap \sigma_-^{n-1}$ has codimension 2. Hence, by Lemma 2.5, it belongs to K_i . Thus, the right hand sides of (7) and (8) are equal by recursion hypothesis. Therefore, we have the following relation:

$$D^2\varepsilon_{\sigma_+^{n-1}} + D^2\varepsilon_{\sigma_-^{n-1}} = \langle c; \sigma_+^{n-1} \rangle + \langle c; \sigma_-^{n-1} \rangle. \quad (9)$$

The vector $[N(\sigma_+^{n-1})]$ is linearly independent from $[N(\sigma_-^{n-1})]$. Thus (9) implies that $D^2\varepsilon_{\sigma_{\pm}^{n-1}}$ equals $\langle c; \sigma_{\pm}^{n-1} \rangle$ for $D^2\varepsilon_{\sigma_+^{n-1}}$ and $\langle c; \sigma_{\pm}^{n-1} \rangle$ belong to the line $N(\sigma_{\pm}^{n-1})$.

Since K is embedded in a real vector space of the same dimension, its support can only be homeomorphic²⁰ to a ball. Hence, we have the exact sequence of Remark 2.18. From Proposition 2.16, Proposition 2.19, and the surjectivity of D^2 , we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \uparrow & & & & \\ 0 & \longrightarrow & H_n(K; \partial K; \mathfrak{t}(\mathbb{F}_2)) & \longrightarrow & H_n(K; \partial K; T^*) & \xrightarrow{\partial} & H_{n-1}(K; \partial K; N) \longrightarrow 0 \\ & & \uparrow \text{D} & & \uparrow \text{D} & & \uparrow \text{id} \\ 0 & \longrightarrow & \text{Aff}(\mathfrak{t}^*(\mathbb{F}_2)) & \longrightarrow & C^0(K; \mathbb{F}_2) & \xrightarrow{D^2} & H_{n-1}(K; \partial K; N) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \\ & & \mathbb{F}_2 & & \mathbb{F}_2 & & \\ & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & \end{array}$$

from which derive the exact sequences of the lemma. □

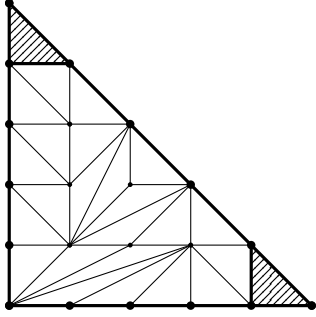
Definition 2.21. We define two enlargement of the boundary of K :

1. the sub-complex \mathcal{B}_0K of K made of all the simplices of K having all of their vertices in ∂K ;

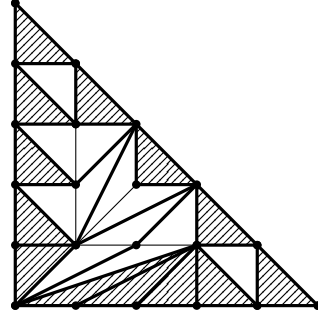
²⁰c.f. Proposition 2.4.

2. the sub-complex \mathcal{B}_1K of K made of the union of the closed simplices $K(\sigma^p)$ of K for which all of their vertices $v \in K(\sigma^p)$ satisfy $v \in \partial K$ or $\text{lk}(v; \sigma^p) \in \partial K$. See Figure 10 for an example.

We note that \mathcal{B}_1K contains \mathcal{B}_0K and all the vertices of K . The simplices of \mathcal{B}_1K that do not belong to \mathcal{B}_0K are joins of an interior vertex of K and a simplex of ∂K . For such a simplex σ we denote by $\text{ap}(\sigma)$ its interior vertex.



(a) The sub-complex \mathcal{B}_0K (in bold and hatched).



(b) The sub-complex \mathcal{B}_1K (in bold and hatched).

Figure 10: Examples of \mathcal{B}_0K and \mathcal{B}_1K .

Definition 2.22. Let σ^{n-1} be an interior simplex of K . we denote by $I(\sigma^{n-1})$ the set of vertices α of σ^{n-1} that do not belong to ∂K and whose link $\text{lk}(\alpha; \sigma^{n-1})$ is not contained in ∂K either. We say that K is ρ -uniform if for all interior simplices $\sigma^{n-1} \in K$ the restriction of $\rho_{\sigma^{n-1}}$ to $I(\sigma^{n-1})$ is constant. In particular, because of Proposition 2.14, if $\dim K \leq 2$ then K is ρ -uniform. If K is not ρ -uniform we say that K fails to be ρ -uniform at σ^{n-1} if $\rho_{\sigma^{n-1}}$ is not constant on $I(\sigma^{n-1})$. For all ρ -uniform complexes K , we denote by $\rho \in C_{n-1}(K; \mathcal{B}_0K; N)$ the chain whose value on $\sigma^{n-1} \in K \setminus \mathcal{B}_1K$ is $\rho_{\sigma^{n-1}}(\alpha)$ for any $\alpha \in I(\sigma^{n-1})$, and whose value on $\sigma^{n-1} \in \mathcal{B}_1K \setminus \mathcal{B}_0K$ is $\rho_{\sigma^{n-1}}(\text{ap}(\sigma^{n-1}))$.

Proposition 2.23. *If K is ρ -uniform, the chain ρ is a cycle. It induces a homology class denoted by the same symbol.*

Proof. Let σ^{n-2} be a simplex of K that does not belong to ∂K . If n equals 2 then $\langle \partial\rho; \sigma^0 \rangle$ is $\sum_{\sigma^1 \geq \sigma^0} \rho_{\sigma^1}(\sigma^0)$ since ρ is constant on σ^1 . This vector is zero by definition of $\rho_{\sigma^1}(\sigma^0)$ as the difference $\sum_{\sigma^2 \geq \sigma^1} e_{\sigma^2}(\sigma^0)$. Now if n is bigger than 2, then either σ^{n-2} belongs to \mathcal{B}_1K or not but either way there is an interior vertex α of σ^{n-2} for which $\langle \partial\rho; \sigma^{n-2} \rangle$ equals $\sum_{\sigma^{n-1} \geq \sigma^{n-2}} \rho_{\sigma^{n-1}}(\alpha)$. This vector vanishes for the same reason as in the case $n = 2$. \square

3 Computing the Renaudineau-Shaw Spectral Sequence

The aim of this section is to give an effective computation of the first page of the Renaudineau-Shaw spectral sequence for T-hypersurfaces. This spectral sequence is derived from a filtration of the cellular chain complex of the T-hypersurface $\mathbb{R}X_\varepsilon$. A. Renaudineau and K. Shaw showed that the groups of its first page are isomorphic to the tropical homology groups of the tropical

counterpart X of $\mathbb{R}X_\varepsilon$. We will compute here the boundary operators of this first page. To do so, we will follow a usual path: in a first time we will lift a tropical homology class of X as a relative cycle of $\mathbb{R}X_\varepsilon$, then we will compute its boundary and reproject it as a tropical homology class of X . To simplify this task we will reduce ourself to subspaces of tropical homology groups made of classes having a suitable representing cycle. Following the computation of the boundary of these lifts, we will finally display the obtained information in matrices and use their ranks to express the dimensions of the groups of the second page of the sequence.

Lifting Tropical Cycles. Let P be a smooth n -dimensional polytope of $\mathfrak{t}^*(\mathbb{R})$ endowed with a primitive triangulation K . For all $\sigma^p \in K$, we choose $w(\sigma^p) = w_1(\sigma^p) \wedge \dots \wedge w_p(\sigma^p)$ a totally decomposed p -vector of $\mathfrak{t}(\mathbb{F}_2)$ satisfying $\omega(\sigma^p)(w(\sigma^p)) = 1$.

Definition 3.1. Let $0 \leq p \leq n-1$ be an integer. We define $g^p: C^p(K; \partial K; \mathbb{F}_2) \rightarrow H_{p, n-1-p}(X; \mathbb{F}_2)$ by the following formula. For all p -cochain γ we set:

$$g^p(\gamma) = \left[\sum_{\sigma^{p+1} \leq \sigma^n} d(\gamma w)(\sigma^{p+1}) \otimes [\sigma^{p+1}; \sigma^n] \right]. \quad (10)$$

We can note that g^0 is known to be an isomorphism. Indeed, since F_0^X is the constant cosheaf \mathbb{F}_2 , the group $H_{0, n-1}(X; \mathbb{F}_2)$ is the (usual) homology group of X with coefficients in \mathbb{F}_2 . The space X is homotopic to a wedge of $(n-1)$ -spheres, one for each interior vertex of K . Let p be an interior vertex of K , the class $g^0(\chi_p)$ is the fundamental class of the corresponding sphere, which is the link of p in the barycentric subdivision of K .

Lemma 3.2. For all integers $0 \leq p \leq n-1$, the morphism g^p does not depend on the choice of w . Moreover its image is the kernel of morphism $i_{p, n-1-p}: H_{p, n-1-p}(X; \mathbb{F}_2) \rightarrow H_{p, n-1-p}(P; \mathbb{F}_2)$.

Proof. In a first time we can notice the cycle :

$$z = \sum_{\sigma^{p+1} \leq \sigma^n} d(\gamma w)(\sigma^{p+1}) \otimes [\sigma^{p+1}; \sigma^n] \in \Omega_{n-1-p}(K; F_p^X),$$

is the boundary of the chain :

$$b = \sum_{\sigma^p \leq \sigma^n} \gamma(\sigma^p) w(\sigma^p) \otimes [\sigma^p; \sigma^n] \in \Omega_{n-p}(K; F_p^P).$$

Hence it shows that the image of $g^p(\gamma)$ in the tropical homology of the toric variety P vanishes. Also, should we choose another w , say w' , the chain :

$$b - b' = \sum_{\sigma^p \leq \sigma^n} \gamma(\sigma^p) (w - w')(\sigma^p) \otimes [\sigma^p; \sigma^n],$$

belongs to $\Omega_{n-p}(K; F_p^X)$ since $\omega(\sigma^p)(w(\sigma^p) - w'(\sigma^p))$ vanishes by hypothesis on w and w' . Now the boundary of $b - b'$ is :

$$\partial(b - b') = z - \sum_{\sigma^{p+1} \leq \sigma^n} d(\gamma w')(\sigma^{p+1}) \otimes [\sigma^{p+1}; \sigma^n].$$

And the class $g^p(\gamma)$ does not depend on w . The fact that the image of g^p is the kernel of $i_{p,n-1-p}$ comes from Theorem 1 p.20 and Theorem 2 p.32 of [Che23a]. They assert that both the tropical homology of P and X can be computed as the cohomology of cellular sheaves on K . Let Q denote either P or X and define, as in the proof of aforementioned Theorem 2, the cellular sheaves of local tropical homology:

$$G_p^Q : \sigma^q \mapsto H_n(K; K - \sigma^q; F_p^Q).$$

Then the inclusion $i_p : F_p^X \rightarrow F_p^P$ induces an inclusion $j_p : G_p^X \rightarrow G_p^P$ and we have a commutative diagram:

$$\begin{array}{ccc} H_{p,n-1-p}(X; \mathbb{F}_2) & \xrightarrow{i_{p,n-1-p}} & H_{p,n-1-p}(P; \mathbb{F}_2) \\ \downarrow \cong & & \downarrow \cong \\ H^{p+1}(K; G_p^X) & \xrightarrow{j_p^{p+1}} & H^{p+1}(K; G_p^P) \end{array}$$

The cokernel sheaf of j_p is supported²¹ on the p -th skeleton of K and for all simplex σ^p , $\text{coker}(j_p)(\sigma^p)$ is either \mathbb{F}_2 or 0, the latter occurring if and only if σ^p lies on the boundary of P . Therefore the group of p -cocycles with coefficients in $\text{coker}(j_p)$ is precisely $C^p(K; \partial K; \mathbb{F}_2)$ and the exactness of the sequence:

$$C^p(K; \partial K; \mathbb{F}_2) \xrightarrow{g^p} H_{p,n-1-p}(X; \mathbb{F}_2) \longrightarrow H_{p,n-1-p}(P; \mathbb{F}_2) \longrightarrow 0,$$

comes from the exactness of the sequence:

$$H^p(K; \text{coker}(j_p)) \xrightarrow{d} H^{p+1}(K; L_p^X) \longrightarrow H^{p+1}(K; L_p^P) \longrightarrow 0.$$

□

Remark 3.3. The morphism $i_{p,n-1-p} : H_{p,n-1-p}(X; \mathbb{F}_2) \rightarrow H_{p,n-1-p}(P; \mathbb{F}_2)$ is surjective. This is a consequence of the Tropical Lefschetz Theorem²².

Lemma 3.4 (Origin). *Let $\varepsilon \in C^0(K; \mathbb{F}_2)$ be a sign distribution, $\sigma^p \leq \sigma^{p+1} \leq \sigma^n$ be three simplices of K , and W be a supplementary sub-space of $N(e^p)$ in $\mathfrak{t}(\mathbb{F}_2)$. The point:*

$$\mathbf{O}_{\sigma^n}(e^p) := D\varepsilon_{\sigma^n} + \sum_{\sigma^0 \leq \sigma^p} e_{\sigma^n}(\sigma^0) \in \mathfrak{t}(\mathbb{F}_2),$$

satisfies $\mathbf{O}_{\sigma^n}(e^p) + W \subset \text{Arg}_\varepsilon(\sigma^{p+1}; \sigma^n)$.

Proof. Following Lemma 3.7 p.18 in [Che23b], the set $\text{Arg}_\varepsilon(\sigma^{p+1}; \sigma^n)$ is described as the following set $\{v \in \mathfrak{t}(\mathbb{F}_2) \mid z_{\sigma^{p+1}}^1(v) \neq d\varepsilon|_{\sigma^{p+1}}\}$. By Definition 2.15, the image of the vector $D\varepsilon_{\sigma^n}$ by $z_{\sigma^n}^1$ is $d\varepsilon|_{\sigma^n}$. Hence, by Proposition 2.10, $z_{\sigma^{p+1}}^1(D\varepsilon_{\sigma^n})$ equals $d\varepsilon|_{\sigma^{p+1}}$. Let us denote by v the vector:

$$v := \sum_{\beta \text{ vertex of } \sigma^p} e_{\sigma^n}(\beta).$$

²¹c.f.[Che23a] Theorem 2 p.32, especially the end of the proof.

²²c.f. [ARS21] Theorem 1.2 p.1349, [BLdMR22] Proposition 3.2 p.15, or [Che23a] Corollary 3 p.34.

Proposition 2.11 ensures that $z_{\sigma^{p+1}}^1(v)$ equals $d\chi_\alpha$ where α is the vertex of σ^{p+1} opposite to σ^p . If σ^1 is an edge of σ^{p+1} that contains α then $z_{\sigma^{p+1}}^1(D\varepsilon_{\sigma^n} + v)(\sigma^1)$ equals $d\varepsilon(\sigma^1) + 1$, and $D\varepsilon_{\sigma^n} + v$ belongs to $\text{Arg}_\varepsilon(\sigma^{p+1}; \sigma^n)$. Let $w \in \mathfrak{t}(\mathbb{F}_2)$ be a vector that does not belong to $N(e^p)$. The restriction of $z_{\sigma^{p+1}}^1(D\varepsilon_{\sigma^n} + v + w)$ to σ^p is $d\varepsilon|_{\sigma^p} + z_{\sigma^p}^1(w)$. Since w does not belong to $N(e^p)$, $z_{\sigma^p}^1(w)$ is different from zero and, consequently, $D\varepsilon_{\sigma^n} + v + w$ belongs to $\text{Arg}_\varepsilon(\sigma^{p+1}; \sigma^n)$. \square

Lemma 3.5 (Lift). *Let $\varepsilon \in C^0(K; \mathbb{F}_2)$ be a sign distribution, $0 \leq p \leq n - 1$ be an integer, and γ be a cochain of $C^p(K; \partial K; \mathbb{F}_2)$. The chain:*

$$c := \sum_{\sigma^{p+1} \leq \sigma^n} d \left(\gamma x^{\mathbf{O}_{\sigma^n}} \prod_{i=1}^p (1 + x^{w_i}) \right) (\sigma^{p+1}) \otimes [\sigma^{p+1}; \sigma^n] \in \Omega_{n-1-p} \left(K; \mathcal{K}_{(p)}^{\mathbb{R}X_\varepsilon} \right),$$

has its boundary equal to:

$$\sum_{\sigma^{p+1} \leq \sigma^{n-1}} d \left(\gamma G_{\sigma^{n-1}} \prod_{i=1}^p (1 + x^{w_i}) \right) (\sigma^{p+1}) \otimes [\sigma^{p+1}; \sigma^{n-1}],$$

where $G_{\sigma^{n-1}}(\sigma^p)$ is null on the p -simplices of the boundary and otherwise equals $\sum_{\sigma^n \geq \sigma^{n-1}} x^{\mathbf{O}_{\sigma^n}(\sigma^p)}$. This boundary belongs to $\Omega_{n-2-p}(K; \mathcal{K}_{(p+1)}^{\mathbb{R}X_\varepsilon})$ and c represents the homology class $g^p(\gamma)$.

Proof. Let $0 \leq p \leq n - 1$ be an integer. By definition, $(E_{p,q}^1(\mathbb{R}X_\varepsilon))_{q \geq 0}$ is the homology of the p -th graded piece of the filtered cosheaf $\mathcal{K}^{\mathbb{R}X_\varepsilon}$. By Proposition 4.22 p.29 in [Che23b], this cosheaf is isomorphic to F_p^X through bv_p^X . Therefore, any homology class $[c] \in H_{p,q}(X; \mathbb{F}_2)$ can be represented by a relative cycle, i.e. a chain $c \in \Omega_q(K; \mathcal{K}_{(p)}^{\mathbb{R}X_\varepsilon})$ whose boundary belongs to $\Omega_{q-1}(K; \mathcal{K}_{(p+1)}^{\mathbb{R}X_\varepsilon})$. Let $\sigma^{p+1} \leq \sigma^n$ be a pair of simplices of K , and $\mathbf{O} + \langle v_1, \dots, v_p \rangle$ be a p -affine subspace of $\mathfrak{t}(\mathbb{F}_2)$ contained in $\text{Arg}_\varepsilon(\sigma^{p+1}; \sigma^n)$. From Remark 1.24, we have:

$$\text{bv}_p^X(\sigma^{p+1}; \sigma^n) \left(x^{\mathbf{O}} \prod_{i=1}^p (1 + x^{v_i}) \right) = \bigwedge_{i=1}^p v_i.$$

Since $\omega(\sigma^p)(w(\sigma^p))$ equals 1, for all $\sigma^p \leq \sigma^{p+1}$, the vector space $\langle w_1(\sigma^p), \dots, w_p(\sigma^p) \rangle$ is a supplementary sub-space of $N(\sigma^p)$ in $\mathfrak{t}(\mathbb{F}_2)$. Using Lemma 3.4, it follows that the chain:

$$c := \sum_{\sigma^{p+1} \leq \sigma^n} d \left(\gamma x^{\mathbf{O}_{\sigma^n}} \prod_{i=1}^p (1 + x^{w_i}) \right) (\sigma^{p+1}) \otimes [\sigma^{p+1}; \sigma^n],$$

lifts the chain $\sum_{\sigma^{p+1} \leq \sigma^n} d(\gamma w) (\sigma^{p+1}) \otimes [\sigma^{p+1}; \sigma^n]$. Let $\sigma^q \leq \sigma^{q+n-p-2}$ be a pair of simplices of K . We have:

$$\langle \partial c; [\sigma^q; \sigma^{q+n-p-2}] \rangle = \sum_{\sigma^{q-1} \leq \sigma^q} \langle c; [\sigma^{q-1}; \sigma^{q+n-p-2}] \rangle + \sum_{\sigma^{q+n-p-2}; \sigma^{q+n-p-1}} \langle c; [\sigma^q; \sigma^{q+n-p-1}] \rangle.$$

If q is different from $p + 1$ and $p + 2$, $\langle \partial c; [\sigma^q \leq \sigma^{q+n-p-2}] \rangle$ vanishes. For the remaining pieces we have:

$$\langle \partial c; [\sigma^{p+2}; \sigma^n] \rangle = \sum_{\sigma^{p+1} \leq \sigma^{p+2}} d \left(\gamma x^{\mathbf{O}_{\sigma^n}} \prod_{i=1}^p (1 + x^{w_i}) \right) (\sigma^{p+1}) = 0,$$

and:

$$\begin{aligned}
\langle \partial c; [\sigma^{p+1}; \sigma^{n-1}] \rangle &= \sum_{\sigma^{n-1} \leq \sigma^n} d \left(\gamma x^{\mathbf{O}_{\sigma^n}} \prod_{i=1}^p (1 + x^{w_i}) \right) (\sigma^{p+1}) \\
&= \sum_{\sigma^p \leq \sigma^{p+1}} \gamma(\sigma^p) \sum_{\sigma^{n-1} \leq \sigma^n} (x^{\mathbf{O}_{\sigma^n}(\sigma^p)}) \prod_{i=1}^p (1 + x^{w_i(\sigma^p)}) \\
&= \sum_{\sigma^p \leq \sigma^{p+1}} \gamma(\sigma^p) G_{\sigma^{n-1}}(\sigma^p) \prod_{i=1}^p (1 + x^{w_i(\sigma^p)}) \\
&= d \left(\gamma G_{\sigma^{n-1}} \prod_{i=1}^p (1 + x^{w_i}) \right) (\sigma^{p+1}),
\end{aligned}$$

since γ vanishes on the boundary of K . Whenever we have a triple $\sigma^p \leq \sigma^{p+1} \leq \sigma^{n-1}$ with σ^p not contained in ∂K , there are two n -simplices σ_{\pm}^n adjacent to σ^{n-1} . In this case $\text{Arg}_{\varepsilon}(\sigma^{p+1}; \sigma^{n-1})$ equals $\text{Arg}_{\varepsilon}(\sigma^{p+1}; \sigma_{\pm}^n)$, and $\mathbf{O}_{\sigma_{\pm}^n} + \langle w_1(\sigma^p), \dots, w_p(\sigma^p) \rangle$ are two parallel affine sub-spaces contained in this set. Therefore $G_{\sigma^{n-1}}(\sigma^p) \prod_{i=1}^p (1 + x^{w_i(\sigma^p)})$ belongs to $\mathcal{K}_{(p+1)}^{\mathbb{R}X_{\varepsilon}}(\sigma^{p+1}; \sigma^{n-1})$. Thus ∂c belongs to $\Omega_{n-2-p}(K; \mathcal{K}_{(p+1)}^{\mathbb{R}X_{\varepsilon}})$. \square

Proposition 3.6. *Let $\varepsilon \in C^0(K; \mathbb{F}_2)$ be a sign distribution on K , $0 \leq p \leq n-1$ be an integer, and γ be a cochain of $C^p(K; \partial K; \mathbb{F}_2)$. We have:*

$$\partial_{p, n-1-p}^1 g^p(\gamma) = \left[\sum_{\sigma^{p+1} \leq \sigma^{n-1}} d(\gamma w \wedge (D^2 \varepsilon_{\sigma^{n-1}} + \rho_{\sigma^{n-1}}^p)) (\sigma^{p+1}) \otimes [\sigma^{p+1}; \sigma^{n-1}] \right],$$

where $\rho_{\sigma^{n-1}}^p \in C^p(\sigma^{n-1}; N(\sigma^{n-1}))$ is defined, for all $\sigma^p \leq \sigma^{n-1}$, by the formula:

$$\rho_{\sigma^{n-1}}^p(\sigma^p) := \sum_{\alpha \text{ vertex of } \sigma^p} \rho_{\sigma^{n-1}}(\alpha).$$

Proof. By Lemma 3.5, $\partial_{p, n-1-p}^1 g^p(\gamma)$ is given by the formula:

$$\left[\sum_{\sigma^{p+1} \leq \sigma^{n-1}} \text{bv}_{p+1}^X(\sigma^{p+1}; \sigma^{n-1}) \left(d \left(\gamma G_{\sigma^{n-1}} \prod_{i=1}^p (1 + x^{w_i}) \right) (\sigma^{p+1}) \right) \otimes [\sigma^{p+1}; \sigma^n] \right].$$

Let $\sigma^p \leq \sigma^{p+1} \leq \sigma^{n-1}$ be a triple of simplices with σ^p not contained in ∂K . From Remark 1.24, we have:

$$\begin{aligned}
\text{bv}_{p+1}^X(\sigma^{p+1}; \sigma^{n-1}) \left(G_{\sigma^{n-1}}(\sigma^p) \prod_{i=1}^p (1 + x^{w_i(\sigma^p)}) \right) &= \left(\sum_{\sigma^n \geq \sigma^{n-1}} \mathbf{O}_{\sigma^n}(\sigma^p) \right) \wedge w(\sigma^p) \\
&= (D^2 \varepsilon_{\sigma^{n-1}} + \rho_{\sigma^{n-1}}^p(\sigma^p)) \wedge w(\sigma^p).
\end{aligned}$$

Therefore we find that:

$$\partial_{p,n-1-p}^1 g^p(\gamma) = \left[\sum_{\sigma^{p+1} \leq \sigma^{n-1}} d(\gamma w \wedge (D^2 \varepsilon_{\sigma^{n-1}} + \rho_{\sigma^{n-1}}^p)) (\sigma^{p+1}) \otimes [\sigma^{p+1}; \sigma^{n-1}] \right].$$

□

Intersection Matrices. We use the tropical intersection product to represent the boundary operators of the first page as matrices and provide formulæ for the dimensions of the groups of the second page that take into account that the morphisms g^p are not surjective in general.

Definition 3.7. Let $\varepsilon \in C^0(K; \mathbb{F}_2)$ be a sign distribution, and σ^p and σ^q be two interior simplices of K with $p + q = n - 2$. We denote by $(\sigma^p; \sigma^q)_\varepsilon$ the number:

$$(\sigma^p; \sigma^q)_\varepsilon := g^p(\chi_{\sigma^p}) \cdot \partial_{q,p+1}^1 g^q(\chi_{\sigma^q}) \in H_{0,0}(X; \mathbb{F}_2) = \mathbb{F}_2,$$

where $-\cdot-$ denotes the tropical intersection product. For all integers $p, q \geq 0$ for which $p + q = n - 2$, let $M_{p,q}(\varepsilon)$ denote the matrix whose entries are given by the numbers $((\sigma^p; \sigma^q)_\varepsilon)_{\sigma^p, \sigma^q \in K \setminus \partial K}$.

Definition 3.8. We assume that $n = 2k + 1$ is odd. We define the *radical* of X , denoted by $\text{Rad}(X)$, to be the radical of the intersection form restricted to the kernel of the morphism $i_{k,k} : H_{k,k}(X; \mathbb{F}_2) \rightarrow H_{k,k}(P; \mathbb{F}_2)$. We denote by $\text{rad}(X)$ the dimension of $\text{Rad}(X)$.

Proposition 3.9. *We assume that $n = 2k + 1$ is odd. Let ω_X denote the Poincaré dual in P of the fundamental class of X . The radical $\text{Rad}(X)$ is generated by the classes $i^{k,k}(\gamma) \cap [X] \in H_{k,k}(X; \mathbb{F}_2)$ for all $\gamma \in H^{k,k}(P; \mathbb{F}_2)$ for which $\gamma \cup \omega_X$ vanishes.*

Proof. The Poincaré duality in X provides us with an isomorphism between the group $H_{k,k}(X; \mathbb{F}_2)$ and its dual $H^{k,k}(X; \mathbb{F}_2)$. The orthogonal of $\ker(i_{k,k})$ is the image under this isomorphism of $\text{im}(i^{k,k})$ i.e. is spanned by the element of the form $i^{k,k}(\gamma) \cap [X]$ where γ belongs to $H^{k,k}(P; \mathbb{F}_2)$. If we denote by ω_X the Poincaré dual in P of $[X]$, we have:

$$i_{k,k}(i^{k,k}(\gamma) \cap [X]) = (\gamma \cup \omega_X) \cap [P].$$

Therefore:

$$\text{Rad}(X) = \left\langle i^{k,k}(\gamma) \cap [X] \in H_{k,k}(X; \mathbb{F}_2) : \gamma \in H^{k,k}(P; \mathbb{F}_2) \text{ s.t. } \gamma \cup \omega_X = 0 \right\rangle.$$

□

Proposition 3.10. *Let $\varepsilon \in C^0(K; \mathbb{F}_2)$ be a sign distribution. For all $p + q = n - 2$, $M_{p,q}(\varepsilon)$ is the transpose of $M_{q,p}(\varepsilon)$. Moreover:*

1. *If n is even, the rank of $M_{p,q}(\varepsilon)$ equals the rank of $\partial_{q,n-1-q}^1$;*
2. *If $n = 2k + 1$ is odd, it is also the case except when (p, q) equals $(k - 1, k)$ or $(k, k - 1)$. In the first case the rank of $M_{k-1,k}(\varepsilon)$ equals the rank of the restriction of $\partial_{k,k}^1$ to the kernel of $i_{k,k}^1 : E_{k,k}^1(\mathbb{R}X_\varepsilon) \rightarrow E_{k,k}^1(\mathbb{R}P)$. In the second one, $M_{k,k-1}(\varepsilon)$ equals the rank of the composition of $\partial_{k+1,k-1}^1$ with the projection of $\ker(i_{k,k}^1)$ to its quotient by $\text{Rad}(X)$.*

Proof. Let σ^p, σ^q be two simplices of K satisfying $p + q = n - 2$, i.e. $q = n - 1 - (p + 1)$. By Definition 3.7 we have:

$$(\sigma^p; \sigma^q)_\varepsilon = g^p(\chi_{\sigma^p}) \cdot \partial_{n-1-(p+1), p+1}^1 g^q(\chi_{\sigma^q}).$$

Let $\gamma \in H^{n-1-p, p}(X; \mathbb{F}_2)$ denote the Poincaré dual²³ of $g^p(\chi_{\sigma^p})$. By definition of the tropical intersection product, we have:

$$(\sigma^p; \sigma^q)_\varepsilon = \gamma(\partial_{n-1-(p+1), p+1}^1 g^q(\chi_{\sigma^q})) = (d_1^{n-1-p, p} \gamma)(g^q(\chi_{\sigma^q})),$$

since $(\partial_{a,b}^r)^*$ equals $d_r^{a+r, b-1}$ by duality of the spectral sequences²⁴. Following Theorem 4.27 p.32 and Proposition 1.6 p.9 of [Che23b], we find that $d_1^{n-1-p, p} \gamma$ is the Poincaré dual of $\partial_{p, n-1-p}^1 g^p(\chi_{\sigma^p})$. There is a sign in the aforementioned Proposition 1.6, however since we consider \mathbb{F}_2 -vector spaces we can discard it. Finally we have:

$$(\sigma^p; \sigma^q)_\varepsilon = \partial_{p, n-1-p}^1 g^p(\chi_{\sigma^p}) \cdot g^q(\chi_{\sigma^q}),$$

which is equal to $(\sigma^q; \sigma^p)_\varepsilon$ since the intersection product is symmetric over \mathbb{F}_2 . Now we assume n even in a first time. The family $\{g^q(\chi_{\sigma^q}) : \sigma^q \in K \setminus \partial K\}$ spans $E_{q, n-1-q}^1(\mathbb{R}X_\varepsilon)$ and the family $\{(g^p(\chi_{\sigma^p}) \cdot -) : \sigma^p \in K \setminus \partial K\}$ spans $E_1^{q+1, n-2-q}(\mathbb{R}X_\varepsilon)$. As such the rank of $M_{p,q}(\varepsilon)$ equals the rank of $\partial_{q, n-1-q}^1$. If now we assume $n = 2k + 1$ odd and that (p, q) is neither $(k-1, k)$ nor $(k, k-1)$, the family:

$$\{g^q(\chi_{\sigma^q}) : \sigma^q \in K \setminus \partial K\},$$

still spans $E_{q, n-1-q}^1(\mathbb{R}X_\varepsilon)$ and the family $\{(g^p(\chi_{\sigma^p}) \cdot -) : \sigma^p \in K \setminus \partial K\}$ still spans $E_1^{q+1, n-2-q}(\mathbb{R}X_\varepsilon)$. The rank of $M_{p,q}(\varepsilon)$ is still equal to the rank of $\partial_{q, n-1-q}^1$. For the two remaining cases we consider the following commutative diagram with exact columns:

$$\begin{array}{ccccc}
& & & 0 & \\
& & & \uparrow & \\
& & & 0 & \\
& & & \uparrow & \\
0 & \longrightarrow & E_{k,k}^1(\mathbb{R}P) & \longrightarrow & 0 \\
& \uparrow i_{k-1, k+1}^1 & \uparrow i_{k,k}^1 & & \uparrow i_{k+1, k-1}^1 \\
E_{k-1, k+1}^1(\mathbb{R}X_\varepsilon) & \xrightarrow{\partial_{k-1, k+1}^1} & E_{k,k}^1(\mathbb{R}X_\varepsilon) & \xrightarrow{\partial_{k,k}^1} & E_{k+1, k-1}^1(\mathbb{R}X_\varepsilon) \\
& \uparrow & \uparrow & & \uparrow \\
\ker(i_{k-1, k+1}^1) & \longrightarrow & \ker(i_{k,k}^1) & \longrightarrow & \ker(i_{k+1, k-1}^1) \\
& \uparrow & \uparrow & & \uparrow \\
0 & & 0 & & 0
\end{array}$$

The vector family $\{g^k(\chi_{\sigma^k}) : \sigma^k \in K \setminus \partial K\}$ spans the kernel $\ker(i_{k,k}^1)$ and the vector family $\{(g^{k-1}(\chi_{\sigma^{k-1}}) \cdot -) : \sigma^{k-1} \in K \setminus \partial K\}$ spans $E_1^{k+1, k-1}(\mathbb{R}X_\varepsilon)$. Therefore the rank of $M_{k-1, k}(\varepsilon)$ equals the rank of the restriction of $\partial_{k,k}^1$ to the kernel $\ker(i_{k,k}^1)$. The family $\{(g^k(\chi_{\sigma^k}) \cdot -) : \sigma^k \in$

²³c.f. [JRS18] Theorem 5.3 p.16.

²⁴c.f. [Che23b] Definition 4.19 p.28, for the dual spectral sequence.

$K \setminus \partial K$ spans the cokernel of $i_1^{k,k}$ and the family $\{g^{k-1}(\chi_{\sigma^{k-1}}): \sigma^{k-1} \in K \setminus \partial K\}$ spans $E_{k-1,k+1}^1(\mathbb{R}X_\varepsilon)$. The cokernel of $i_1^{k,k}$ is the quotient of $E_1^{k,k}(\mathbb{R}X_\varepsilon)$ by the orthogonal of $\ker(i_{k,k}^1)$. Using the Poincaré duality, the latter quotient is isomorphic to the quotient of $E_{k,k}^1(\mathbb{R}X_\varepsilon)$ by the orthogonal of $\ker(i_{k,k}^1)$ with respect to the intersection product. Since $\partial_{k,k}^1$ takes values in $\ker(i_{k,k}^1)$ the rank of $M_{k-1,k}(\varepsilon)$ equals the rank of the composition of $\partial_{k,k}^1$ with the quotient projection $\ker(i_{k,k}^1) \rightarrow \ker(i_{k,k}^1) / \text{Rad}(X)$. \square

Proposition 3.11. *Let $\varepsilon \in C^0(K; \mathbb{F}_2)$ be a sign distribution.*

1. *If n is even, then for all integers $0 \leq p \leq n-1$, we have:*

$$\dim E_{p,n-1-p}^2(\mathbb{R}X_\varepsilon) = \dim H_{p,n-1-p}(X; \mathbb{F}_2) - \text{rk } M_{n-2-p,p}(\varepsilon) - \text{rk } M_{n-1-p,p-1}(\varepsilon).$$

2. *If $n = 2k + 1$ is odd, then for all integers $0 \leq p \leq n-1$, different from $k-1$, k , and $k+1$, we have:*

$$\dim E_{p,n-1-p}^2(\mathbb{R}X_\varepsilon) = \dim H_{p,n-1-p}(X; \mathbb{F}_2) - \text{rk } M_{n-2-p,p}(\varepsilon) - \text{rk } M_{n-1-p,p-1}(\varepsilon).$$

In the other cases, we have:

$$\dim E_{k-1,k+1}^2(\mathbb{R}X_\varepsilon) = \dim H_{k-1,k+1}(X; \mathbb{F}_2) - \text{rk } M_{k+1,k-2}(\varepsilon) - \text{rk } M_{k,k-1}(\varepsilon) - \delta,$$

$$\dim E_{k+1,k-1}^2(\mathbb{R}X_\varepsilon) = \dim H_{k+1,k-1}(X; \mathbb{F}_2) - \text{rk } M_{k-2,k+1}(\varepsilon) - \text{rk } M_{k-1,k}(\varepsilon) - \delta,$$

and:

$$\dim E_{k,k}^2(\mathbb{R}X_\varepsilon) = \dim H_{k,k}(X; \mathbb{F}_2) - \text{rk } M_{k-1,k}(\varepsilon) - \text{rk } M_{k,k-1}(\varepsilon) - 2\delta.$$

where δ is the dimension of the cokernel of $i_k : H_k(\mathbb{R}X_\varepsilon; \mathbb{F}_2) \rightarrow H_k(\mathbb{R}P; \mathbb{F}_2)$. Moreover, δ cannot exceed $\text{rad}(X)$. We can also note that $H_k(\mathbb{R}X_\varepsilon; \mathbb{F}_2)$ is isomorphic to $E_{k,k}^2(\mathbb{R}X_\varepsilon)$.

Proof. The first point is a consequence of the Rank Formula and Proposition 3.10. We assume now that $n = 2k + 1$ is an odd integer. On the line $\{p + q = n - 1\}$ we have the short exact sequence of complexes:

$$0 \rightarrow (\ker(i_{p,q}^1); \partial_{p,q}^1) \rightarrow (E_{p,q}^1(\mathbb{R}X_\varepsilon); \partial_{p,q}^1) \rightarrow (E_{p,q}^1(\mathbb{R}P); \partial_{p,q}^1) \rightarrow 0.$$

The complex $(E_{p,q}^1(\mathbb{R}P); \partial_{p,q}^1)_{p+q=n-1}$ has only one non-trivial group: $E_{k,k}^1(\mathbb{R}P)$. Therefore, the associated long exact sequence splits into the following exact sequences:

$$0 \rightarrow H_{p,q}(\ker(i^1)) \rightarrow E_{p,q}^2(\mathbb{R}X_\varepsilon) \rightarrow 0,$$

for all $p + q = n - 1$ different from (k, k) and $(k + 1, k - 1)$. And the exact sequence:

$$0 \rightarrow H_{k,k}(\ker(i^1)) \longrightarrow E_{k,k}^2(\mathbb{R}X_\varepsilon) \xrightarrow{i_{k,k}^2} E_{k,k}^1(P) \longrightarrow \dots \tag{11}$$

$$\dots \longrightarrow H_{k+1,k-1}(\ker(i^1)) \rightarrow E_{k+1,k-1}^2(\mathbb{R}X_\varepsilon) \rightarrow 0.$$

The Rank Formula and Proposition 3.10 imply that, for all $p + q = n - 1$ except (k, k) and $(k - 1, k + 1)$, the dimension of $H_{p,q}(\ker(i^1))$ is equal to:

$$\dim E_{p,q}^1(\mathbb{R}X_\varepsilon) - \text{rk } M_{q-1,p}(\varepsilon) - \text{rk } M_{q,p-1}(\varepsilon).$$

Moreover, if we write $\text{rk } \partial_{k-1,k+1}^1 = \text{rk } M_{k,k-1}(\varepsilon) + \delta$, δ cannot exceed $\text{rad}(X)$, and:

$$\dim H_{k-1,k+1}(\ker(i^1)) = \dim E_{k-1,k+1}^1(\mathbb{R}X_\varepsilon) - \text{rk } M_{k,k-1}(\varepsilon) - \delta - \text{rk } M_{k+1,k-2}(\varepsilon),$$

and:

$$\dim H_{k,k}(\ker(i^1)) = \dim E_{k,k}^1(\mathbb{R}X_\varepsilon) - \dim E_{k,k}^1(\mathbb{R}P) - \text{rk } M_{k-1,k}(\varepsilon) - \text{rk } M_{k,k-1}(\varepsilon) - \delta.$$

It follows that if $p + q = n - 1$ is different from $(k + 1, k - 1)$, (k, k) , and $(k - 1, k + 1)$ we have:

$$\begin{aligned} \dim E_{p,q}^2(\mathbb{R}X_\varepsilon) &= \dim E_{p,q}^1(\mathbb{R}X_\varepsilon) - \text{rk } M_{q-1,p}(\varepsilon) - \text{rk } M_{q,p-1}(\varepsilon) \\ &= \dim H_{p,q}(X; \mathbb{F}_2) - \text{rk } M_{q-1,p}(\varepsilon) - \text{rk } M_{q,p-1}(\varepsilon). \end{aligned}$$

Using Theorem 4.27 p.32 of [Che23b] and Proposition 3.10 we find that:

$$\begin{aligned} \dim E_{k-1,k+1}^2(\mathbb{R}X_\varepsilon) &= \dim E_{k-1,k+1}^1(\mathbb{R}X_\varepsilon) - \text{rk } M_{k,k-1}(\varepsilon) - \text{rk } M_{k+1,k-2}(\varepsilon) - \delta \\ &= \dim H_{k-1,k+1}(X; \mathbb{F}_2) - \text{rk } M_{k,k-1}(\varepsilon) - \text{rk } M_{k+1,k-2}(\varepsilon) - \delta, \end{aligned}$$

and:

$$\begin{aligned} \dim E_{k+1,k-1}^2(\mathbb{R}X_\varepsilon) &= \dim E_{k+1,k-1}^1(\mathbb{R}X_\varepsilon) - \text{rk } M_{k-1,k}(\varepsilon) - \text{rk } M_{k-2,k+1}(\varepsilon) - \delta \\ &= \dim H_{k+1,k-1}(X; \mathbb{F}_2) - \text{rk } M_{k-1,k}(\varepsilon) - \text{rk } M_{k-2,k+1}(\varepsilon) - \delta. \end{aligned}$$

From (11) we find the last formula:

$$\dim E_{k,k}^2(\mathbb{R}X_\varepsilon) = \dim H_{k,k}(X; \mathbb{F}_2) - \text{rk } M_{k-1,k}(\varepsilon) - \text{rk } M_{k,k-1}(\varepsilon) - 2\delta.$$

By Theorem 4.26 p.31 of [Che23b] the group $E_{k,k}^2(\mathbb{R}X_\varepsilon)$ is isomorphic to $H_k(\mathbb{R}X_\varepsilon; \mathbb{F}_2)$. The map $i_{k,k}^2$ is conjugated to $i_k : H_k(\mathbb{R}X_\varepsilon; \mathbb{F}_2) \rightarrow H_k(\mathbb{R}P; \mathbb{F}_2)$. From (11) we find that the dimension of the cokernel of $i_{k,k}^2$ is δ . \square

4 The First and Last Boundaries

Twists of $\mathbb{R}X_\varepsilon$ In this paragraph we provide a geometric interpretation of the coefficients of $M_{0,n-1}(\varepsilon)$.

Proposition 4.1. *Let σ^n be a simplex of K . The lift $\sigma_{\mathbb{R}}^n$ of σ^n in $\mathbb{R}K$, indexed by the vector $D\varepsilon_{\sigma^n}$ is the only one avoided by $\mathbb{R}X_\varepsilon$, i.e. the only lift for which $\sigma_{\mathbb{R}}^n \cap \mathbb{R}X_\varepsilon$ is empty. More generally, if H is the hypersurface of σ^n dual to the cocycle $\zeta \in Z^1(\sigma^n; \mathbb{F}_2)$, the only lift $\sigma_{\mathbb{R}}^n$ of σ^n for which $\sigma_{\mathbb{R}}^n \cap \mathbb{R}X_\varepsilon$ corresponds to H is the one indexed by the vector $D\varepsilon_{\sigma^n} + v$ where $z_{\sigma^n}^1(v)$ equals ζ .*

Proof. Let $\sigma_{\mathbb{R}}^n$ be the lift of σ^n indexed by the vector $D\varepsilon_{\sigma^n} + v$. The hypersurface $\sigma_{\mathbb{R}}^n \cap \mathbb{R}X_\varepsilon$ is the dual hypersurface of the cocycle $\zeta : \sigma^1 \mapsto d\varepsilon(\sigma^1) + \omega(\sigma^1)(D\varepsilon_{\sigma^n} + v)$. By Definition 2.15, $\omega(\sigma^1)(D\varepsilon_{\sigma^n})$ equals $d\varepsilon(\sigma^1)$ for all edges σ^1 of σ^n . Therefore, ζ equals $z_{\sigma^n}^1(v)$. \square

Definition 4.2 (Twists). Let ε be a sign distribution on K , and σ^{n-1} be an interior simplex of K . For all vertices β of σ^{n-1} , we say that $\mathbb{R}X_\varepsilon$ is *twisted along σ^{n-1} in the direction of β* if $D^2\varepsilon_{\sigma^{n-1}}$ is different from $\rho_{\sigma^{n-1}}(\beta)$. The vector $D^2\varepsilon_{\sigma^{n-1}} + \rho_{\sigma^{n-1}}(\beta)$ belongs to the line $N(\sigma^{n-1})$, we denote by $|D^2\varepsilon_{\sigma^{n-1}} + \rho_{\sigma^{n-1}}(\beta)| \in \mathbb{F}_2$ the corresponding number.

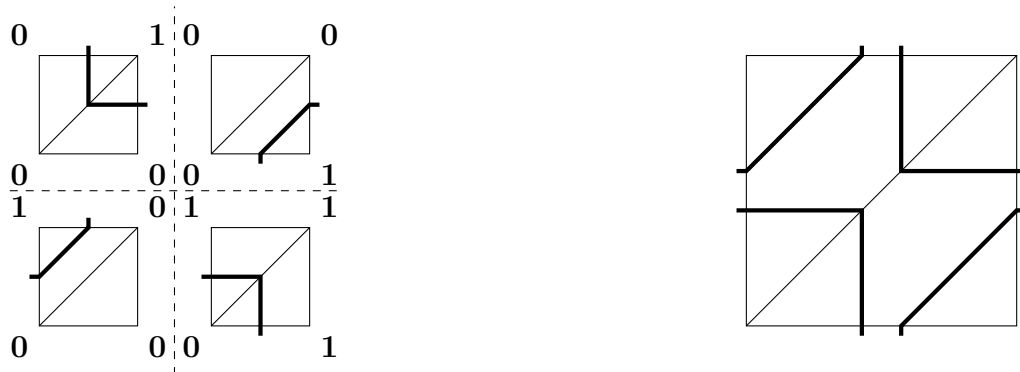
Remark 4.3. Since both $D^2\varepsilon_{\sigma^{n-1}}$ and $\rho_{\sigma^{n-1}}(\beta)$ belong to the line $N(\sigma^{n-1})$, knowing if $\mathbb{R}X_\varepsilon$ is twisted or not along a simplex in a given direction is a numerical question. The hypersurface $\mathbb{R}X_\varepsilon$ is twisted along σ^{n-1} in the direction of β if and only if $\varepsilon(\alpha_+) + \varepsilon(\alpha_-) + \sum_{\alpha \in \sigma^{n-1}} \varepsilon(\alpha)|\rho_{\sigma^{n-1}}(\alpha)| + |\rho_{\sigma^{n-1}}(\beta)| = 1$, where $\sigma^{n-1} * \alpha_\pm$ are the two n -simplices of K adjacent to σ^{n-1} .

Let σ^{n-1} be an interior simplex of K . The set of directions in which $\mathbb{R}X_\varepsilon$ can be twisted along σ^{n-1} is strongly influenced by the geometry of K . The function $\rho_{\sigma^{n-1}}$ partitions the vertices into two sets. The set of those on which the function vanishes and the set of the others. For all sign distributions ε , the hypersurface $\mathbb{R}X_\varepsilon$ is then either twisted along σ^{n-1} in all the directions of the former set or in all the directions of the latter. Let S denote the closed star of σ^{n-1} . Since σ^{n-1} is interior it has exactly 2^n different lifts in $\mathbb{R}K$ and so does S . A lift of S is indexed by a vector $v \in \mathfrak{t}(\mathbb{F}_2)$ and we denote it by S_v . The intersection of $\mathbb{R}X_\varepsilon$ with S_v is the dual hypersurface of $[\sigma^1 \mapsto d\varepsilon(\sigma^1) + \omega(\sigma^1)(v)]$. Using Proposition 2.10, we see that the restrictions of these cocycles to σ^{n-1} describe the whole set of cocycle $Z^1(\sigma^{n-1}; \mathbb{F}_2)$. Therefore $\mathbb{R}X_\varepsilon$ intersects σ^{n-1} in every possible way. By reflecting back all of the intersections $\mathbb{R}X_\varepsilon \cap S_v$ into S , we get a collection of 2^n hypersurfaces of S (some could be empty). Exactly $2^n - 2$ of them intersect σ^{n-1} and are connected. For the remaining two we have the following alternative: one is empty and the other consists of two connected components that are “parallel” to σ^{n-1} , or none of them is empty and each is a “parallel” copy of σ^{n-1} . Some examples are depicted in Examples 4.5. Either way we have a collection of 2^n connected hypersurfaces of S . These hypersurfaces are naturally paired by their intersection with σ^{n-1} . We call the pair of hypersurfaces that does not intersect σ^{n-1} the parallel pair. Beside the parallel pair there are also n distinguished pairs of hypersurfaces. For all vertices β of σ^{n-1} , there is exactly one pair $\{H_1(\beta); H_2(\beta)\}$ whose intersection with σ^{n-1} separates β from the other vertices of σ^{n-1} .

Proposition 4.4. *The hypersurface $\mathbb{R}X_\varepsilon$ is not twisted along σ^{n-1} in the direction of β if and only if one of the two hypersurfaces $\{H_1(\beta); H_2(\beta)\}$ separates β from all the other vertices of S .*

Proof. Let σ_{\pm}^n denote the two n -simplices of S . Following Proposition 4.1, the lift of σ_{\pm}^n in which the lift of β is separated from all the other vertices by $\mathbb{R}X_\varepsilon$, is indexed by the vector $v_{\pm} = D\varepsilon_{\sigma_{\pm}^n} + e_{\sigma_{\pm}^n}(\beta)$. Indeed, $z_{\sigma_{\pm}^n}^1(e_{\sigma_{\pm}^n}(\beta))$ equals $d\chi_\beta$. Let us denote by H_{\pm} the hypersurface of S corresponding to $S_{v_{\pm}} \cap \mathbb{R}X_\varepsilon$. By construction H_{\pm} belongs to $\{H_1(\beta); H_2(\beta)\}$. We have two cases: either H_+ equals H_- , i.e. v_+ equals v_- , or H_+ is different from H_- and so are v_+ and v_- . In the first case H_+ separates β from all the other vertices of S and $v_+ + v_-$ vanishes, i.e. $D^2\varepsilon_{\sigma^{n-1}}$ equals $\rho_{\sigma^{n-1}}(\beta)$. Hence $\mathbb{R}X_\varepsilon$ is not twisted along σ^{n-1} in the direction of β . In the

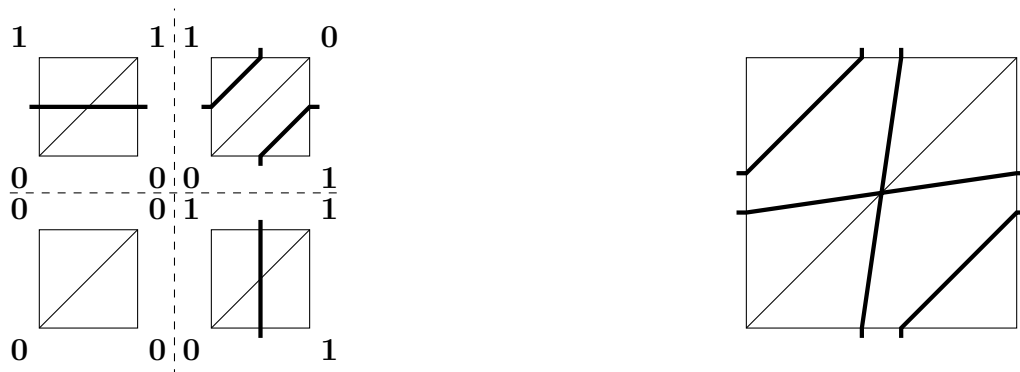
other case, $\mathbb{R}X_\varepsilon$ is twisted along σ^{n-1} in the direction of β , and H_\pm does not separate β from $\text{lk}(\sigma^{n-1}; \sigma_\mp^n)$. \square



(a) The four different intersections of $\mathbb{R}X_\varepsilon$ with the lifts of S .

(b) The four different intersections of $\mathbb{R}X_\varepsilon$ with the lifts of S inside S .

Figure 11: The behavior of a curve $\mathbb{R}X_\varepsilon$ in the star of an edge along which it is untwisted.



(a) The four different intersections of $\mathbb{R}X_\varepsilon$ with the lifts of S .

(b) The four different intersections of $\mathbb{R}X_\varepsilon$ with the lifts of S inside S .

Figure 12: The behavior of a curve $\mathbb{R}X_\varepsilon$ in the star of an edge along which it is twisted.

Examples 4.5. 1. We assume that n equals 2, so that $\mathbb{R}X_\varepsilon$ is a curve. We consider an interior edge σ^1 of K . Following Proposition 2.14, the function ρ_{σ^1} is constant. Hence $\mathbb{R}X_\varepsilon$ can either be twisted along σ^1 in all the directions or in none. This corresponds to the usual notion of twist²⁵.

- (a) In Figure 11a we depicted all the intersections of $\mathbb{R}X_\varepsilon$ with the star S of σ^1 . In this case the curve $\mathbb{R}X_\varepsilon$ is not twisted along σ^1 . In Figure 11b, we represented all these intersections in S . The two curves intersecting σ^1 are represented disjoint, but in reality they would meet “tangentially” at the middle of σ^1 .
- (b) Figure 12 is the analogous of Figure 11 for a curve $\mathbb{R}X_\varepsilon$, twisted along σ^1 . In Figure 12b, the two curves intersecting σ^1 meet “transversally” at the middle point of σ^1 .

²⁵c.f. [RS23], §7. Case of plane curves p.973.

2. We assume now that n equals 3, so that $\mathbb{R}X_\epsilon$ is a surface. Let σ^2 be an interior triangle of K . As we discussed earlier, the set of directions in which $\mathbb{R}X_\epsilon$ can be twisted along σ^2 is determined by the geometry of K . If we assume that the function ρ_{σ^2} vanishes, then either $\mathbb{R}X_\epsilon$ is twisted along σ^2 in every direction or in none.

- (a) In Figure 13, We depicted the intersections of a surface $\mathbb{R}X_\epsilon$, that is untwisted along σ^2 , with all the lifts of the star S of σ^2 . Figure 14 represents the four pairs of surfaces of S obtained by intersecting its lifts with $\mathbb{R}X_\epsilon$. In Figure 14b, Figure 14c, and Figure 14d, the pairs of surfaces are disjoint. In reality they would meet “tangentially” along their common intersection with σ^2 .
- (b) Now we consider a surface $\mathbb{R}X_\epsilon$ that is twisted along σ^2 in every direction. In this example Figure 15 is the analogous of Figure 13. It represents the different intersections of $\mathbb{R}X_\epsilon$ with the lifts of S . Figure 16 represents the four pairs of surfaces of S obtained by intersecting its lifts with $\mathbb{R}X_\epsilon$. In Figure 16b, Figure 16c, and Figure 16d, the pairs of surfaces meet “transversally” along their common intersection with σ^2 .

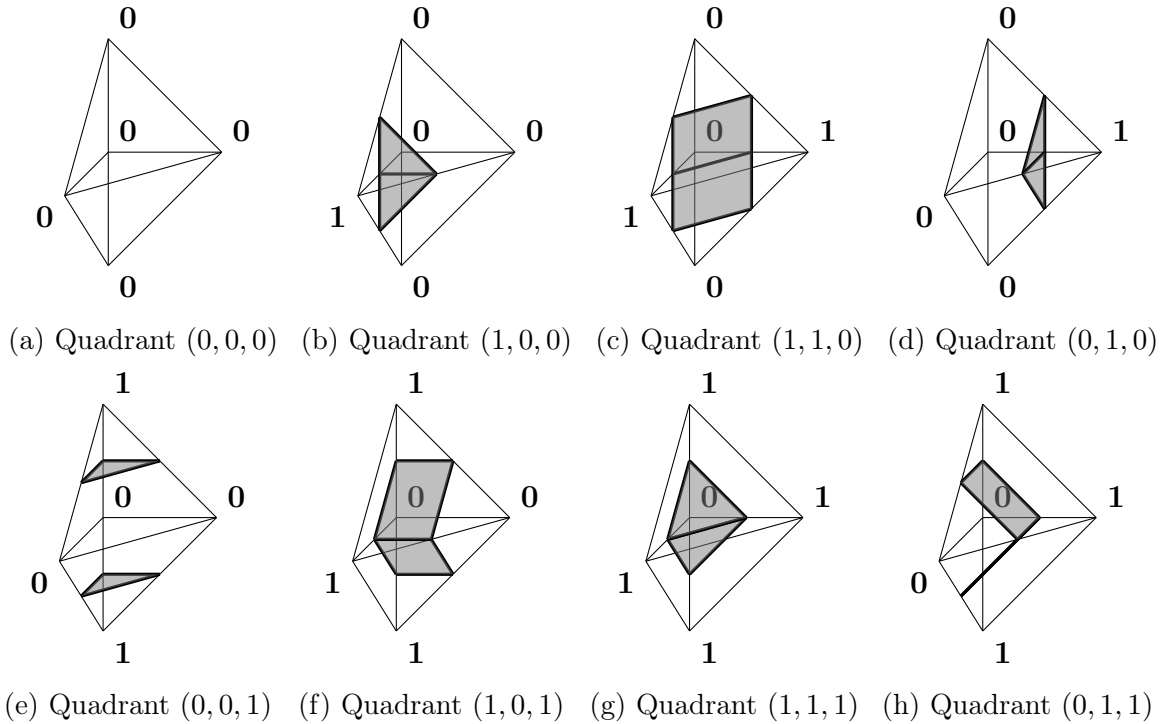


Figure 13: The eight intersections of $\mathbb{R}X_\epsilon$ with the different lifts of S . In this example $\mathfrak{t}(\mathbb{F}_2)$ is \mathbb{F}_2^3 .

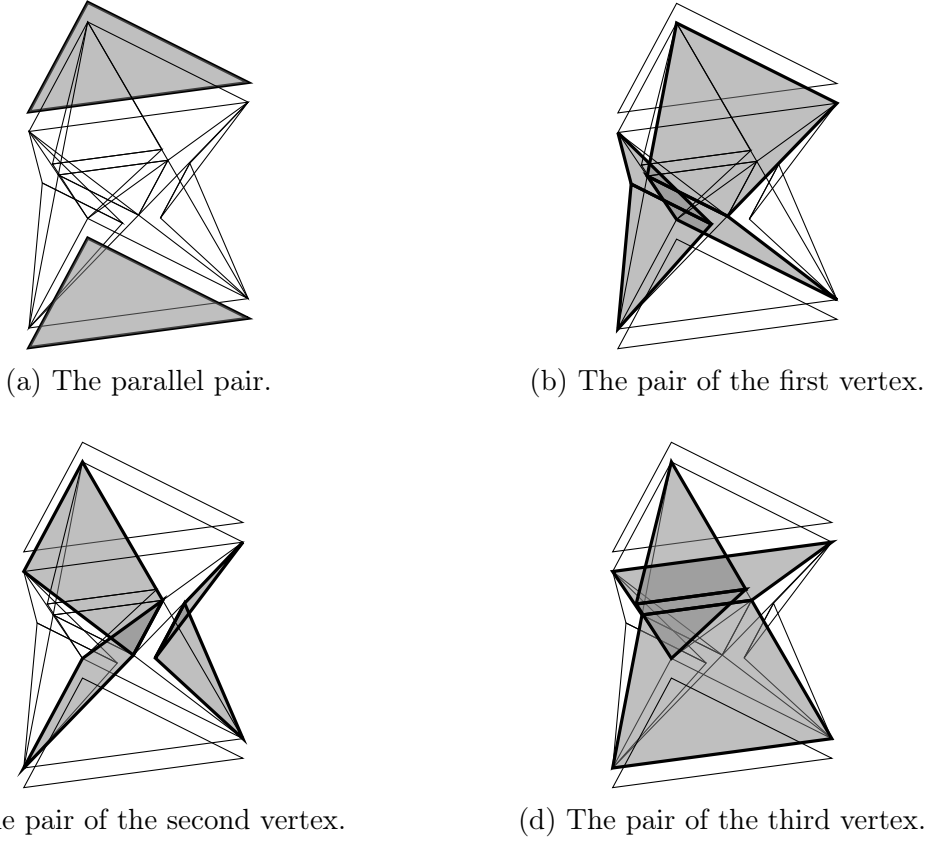


Figure 14: A representation of the four pairs of intersections of lifts of S with $\mathbb{R}X_\epsilon$.

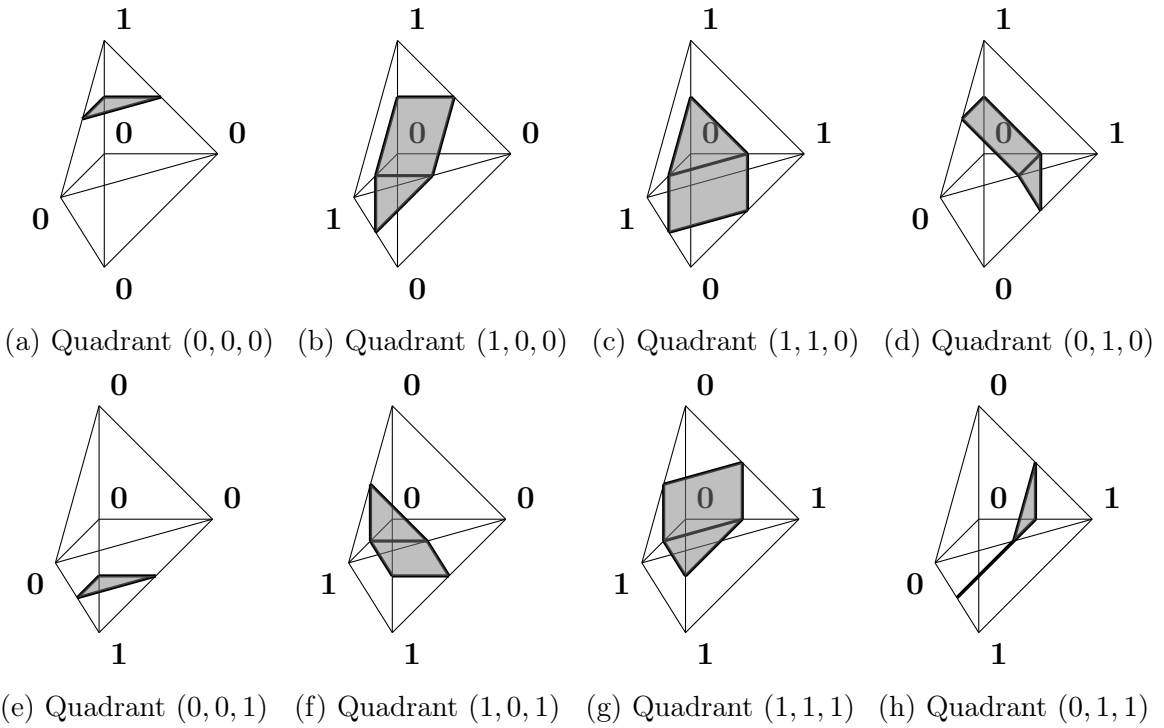
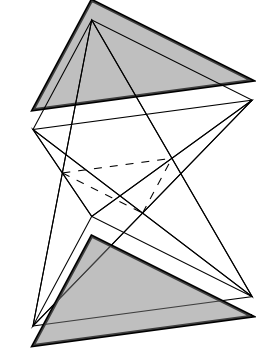
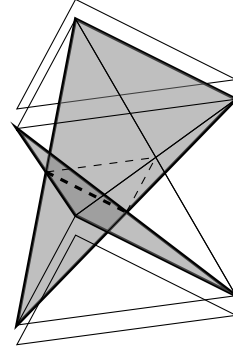


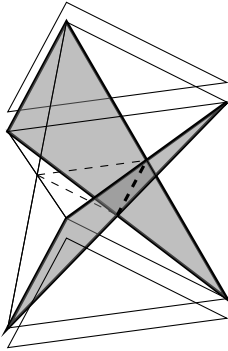
Figure 15: The eight intersections of $\mathbb{R}X_\epsilon$ with the different lifts of S .



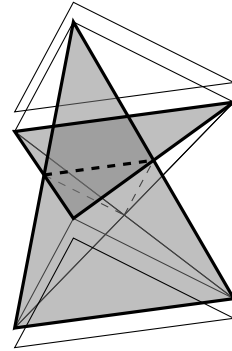
(a) The Parallel Pair.



(b) The pair of the first vertex.



(c) The pair of the second vertex.



(d) The pair of the third vertex.

Figure 16: A representation of the four pairs of intersections of lifts of S with $\mathbb{R}X_\varepsilon$.

Maximality of the number of connected component. We recall from Theorem 1.4 p.947 of [RS23] that the number of connected components of $\mathbb{R}X_\varepsilon$ cannot exceed the number $1 + h^{n-1,0}(X)$. When a T-hypersurface achieves this upper bound we will say that it has a maximal number of connected components. In this paragraph we compute the matrix $M_{0,n-2}(\varepsilon)$ which leads us to a characterisation of T-hypersurfaces achieving this upper bound.

Notations. Let W be a p -dimensional sub-space of $\mathfrak{t}(\mathbb{F}_2)$. We denote by $[W] \in \bigwedge^p \mathfrak{t}(\mathbb{F}_2)$ the generator of the line $\bigwedge^p W$.

Lemma 4.6. *Let $\sigma^p \leq \sigma^q$ be a pair of simplices of K with p positive. We have the following alternative:*

- Either σ^q belongs to the boundary ∂K and $F_{n-1}^X(\sigma^p; \sigma^q)$ vanishes;
- Or σ^q is interior and we have the exact sequence:

$$C_2(\sigma^p; \mathbb{F}_2) \xrightarrow{\partial} C_1(\sigma^p; \mathbb{F}_2) \xrightarrow{\Upsilon_{\sigma^p, \sigma^q}} F_{n-1}^X(\sigma^p; \sigma^q) \longrightarrow 0$$

$$c \longmapsto \sum_{\sigma^1 \leq \sigma^p} \langle c; \sigma^1 \rangle [N(\sigma^1)]$$

Moreover, if we have $\sigma^p \leq \sigma^r \leq \sigma^q$ with σ^q interior, and $c \in C_1(\sigma^p; \mathbb{F}_2)$ then $\Upsilon_{\sigma^r, \sigma^q}(c)$ equals $\Upsilon_{\sigma^p, \sigma^q}(c)$.

Proof. If σ^q belongs to ∂K , then $s := \dim \text{Sed}(\sigma^q)$ is positive. If s is greater than 1, the vector space $\bigwedge^{n-1} \mathfrak{t}(\mathbb{F}_2) / \text{Sed}(\sigma^q)$ vanishes. This latter vector space equals $F_{n-1}^P(\sigma^p; \sigma^q)$. It contains $F_{n-1}^X(\sigma^p; \sigma^q)$ which must vanish as well. If s equals 1, $\bigwedge^{n-1} \mathfrak{t}(\mathbb{F}_2) / \text{Sed}(\sigma^q)$ is a line. Since $\omega(\sigma^p)$ divides the generator of the dual line, the contraction against it is injective. Thus $F_{n-1}^X(\sigma^p; \sigma^q)$ is also trivial. We assume now that σ^q is interior. For all $\sigma^1 \leq \sigma^p$, the $(n-1)$ -vector $[N(\sigma^1)]$ is the generator of $F_{n-1}^X(\sigma^1; \sigma^q)$. Hence by the fourth point of Definition 1.11, the morphism $\Upsilon_{\sigma^p, \sigma^q}$ is surjective. Let σ^1 be an edge of σ^p . The vector $[N(\sigma^1)]$ equals $\omega(\sigma^1) \cdot [\mathfrak{t}(\mathbb{F}_2)]$. However, the cochain $[\sigma^1 \mapsto \omega(\sigma^1)]$ is closed, c.f. Proposition 2.9. Therefore, $[\sigma^1 \mapsto [N(\sigma^1)]]$ is also closed, and the image of ∂ is contained in the kernel of $\Upsilon_{\sigma^p, \sigma^q}$. It follows from Proposition 2.11, that for all vertices β of σ^p , the set $\{[N(\sigma^1)]: \sigma^p \geq \sigma^1 \geq \beta\}$ is linearly independent. Hence, it is a basis of $F_{n-1}^X(\sigma^p; \sigma^q)$, and this space has dimension p . The rank of the morphism ∂ equals the codimension of the kernel of $d : C^1(\sigma^p; \mathbb{F}_2) \rightarrow C^2(\sigma^p; \mathbb{F}_2)$. However $\ker(d)$ has dimension $(p+1) - 1 = p$. Therefore, the image of ∂ has codimension p and the sequence is exact. If we have $\sigma^p \leq \sigma^r \leq \sigma^q$ with σ^q interior, $F_{n-1}^X(\sigma^p; \sigma^q)$ is included in $F_{n-1}^X(\sigma^r; \sigma^q)$. The last statement is then a direct consequence of the definition. \square

Lemma 4.7. *Let γ be a cochain in $C^0(K; \partial K; \mathbb{F}_2)$ and $[\sum_{\sigma^n \in K} w(\sigma^n) \otimes [\sigma^n; \sigma^n]]$ be a homology class in $H_{n-1,0}(X; \mathbb{F}_2)$. We have the formula:*

$$g^0(\gamma) \cdot \left[\sum_{\sigma^n} w(\sigma^n) \otimes [\sigma^n; \sigma^n] \right] = \sum_{\sigma^n \in K} \left(\sum_{\beta * \sigma^{n-1} = \sigma^n} \gamma(\beta) \omega(\sigma^{n-1}) \right) (w(\sigma^n)).$$

Proof. Let $[w]$ denote the homology class $[\sum_{\sigma^n} w(\sigma^n) \otimes [\sigma^n; \sigma^n]]$. The intersection product $g^0(\gamma) \cdot [w]$ equals $\zeta[w]$ where $\zeta \in H^{n-1,0}(X; \mathbb{F}_2)$ is the Poincaré dual of $g^0(\gamma)$, i.e. $\zeta \cap [X]$ equals $g^0(\gamma)$. Therefore, we will begin by finding a cycle $\zeta \in Z^0\Omega(K; F_X^{n-1})$ whose cohomology class is Poincaré dual to $g^0(\gamma)$. Let σ^p be an interior simplex of K . We denote by $\zeta(\sigma^p)$ the unique form in $F_X^{n-1}(\sigma^p; \sigma^p)$ satisfying:

$$\Upsilon_{\sigma^p, \sigma^p}^*(\zeta(\sigma^p)) = d\gamma|_{\sigma^p},$$

where $\Upsilon_{\sigma^p, \sigma^p}^*$ is the adjoint morphism of $\Upsilon_{\sigma^p, \sigma^p}$ from Lemma 4.6. For all simplices σ^p of the boundary ∂K and all vertices of K , we set $\zeta(\sigma^p)$ to be 0. Then we have a well defined 0-cochain $\zeta \in \Omega^0(K; F_X^{n-1})$. Let us show that it is closed. Let $\sigma^p \leq \sigma^{p+1}$ be an adjacent pair of simplices of K . If σ^p is a vertex then $F_X^{n-1}(\sigma^p; \sigma^{p+1})$ vanishes and so does $d\zeta(\sigma^p; \sigma^{p+1})$. We assume now that p is positive. If σ^{p+1} belongs to ∂K , so does σ^p , and $d\zeta(\sigma^p; \sigma^{p+1})$ vanishes. If σ^p belongs to ∂K but σ^{p+1} does not, $d\zeta(\sigma^p; \sigma^{p+1})$ equals the image $\zeta(\sigma^{p+1})'$ of $\zeta(\sigma^{p+1})$ by the restriction morphism $F_X^{n-1}(\sigma^{p+1}; \sigma^{p+1}) \rightarrow F_X^{n-1}(\sigma^p; \sigma^{p+1})$. By Lemma 4.6, $\Upsilon_{\sigma^p, \sigma^{p+1}}^*(\zeta(\sigma^{p+1})')$ equals $d\gamma|_{\sigma^p}$. However, the latter cocycle vanishes for γ is identically zero on ∂K . Since $\Upsilon_{\sigma^p, \sigma^{p+1}}^*$ is injective, it means that $d\zeta(\sigma^p; \sigma^{p+1})$ vanishes. If now both σ^p and σ^{p+1} are interior simplices, then:

$$\Upsilon_{\sigma^p, \sigma^{p+1}}^*(d\zeta(\sigma^p; \sigma^{p+1})) = d\gamma|_{\sigma^p} + d\gamma|_{\sigma^p} = 0.$$

The cochain ζ is closed. We have:

$$\begin{aligned} \zeta \cap \sum_{\sigma^1 \leq \sigma^n} [N(\sigma^1)] \otimes [\sigma^1; \sigma^n] &= \sum_{\sigma^1 \leq \sigma^n} \zeta(\sigma^1)[N(\sigma^1)] \otimes [\sigma^1; \sigma^n] \\ &= \sum_{\sigma^1 \leq \sigma^n} \Upsilon_{\sigma^1, \sigma^n}^*(\zeta(\sigma^1))(\sigma^1) \otimes [\sigma^1; \sigma^n] \\ &= \sum_{\sigma^1 \leq \sigma^n} d\gamma(\sigma^1) \otimes [\sigma^1; \sigma^n]. \end{aligned}$$

The cohomology class of ζ is the Poincaré dual of $g^0(\gamma)$ for the fundamental class $[X]$ is represented by the cycle $\sum_{\sigma^1 \leq \sigma^n} [N(\sigma^1)] \otimes [\sigma^1; \sigma^n]$. Therefore, we have:

$$g^0(\gamma) \cdot [w] = \sum_{\sigma^n \in K} \zeta(\sigma^n)(w(\sigma^n)).$$

To complete our proof we need to compute $\zeta(\sigma^n)$ for all simplices $\sigma^n \in K$. Let σ^1 and σ^{n-1} be two simplices of σ^n . The number $\omega(\sigma^{n-1})[N(\sigma^1)]$ equals the value of $(\omega(\sigma^{n-1}) \wedge \omega(\sigma^1))$ on $[\mathfrak{t}(\mathbb{F}_2)]$ i.e. 1 if and only if σ^1 is not contained in σ^{n-1} . Therefore, for all $\sigma^1 \leq \sigma^n$, we have:

$$\left(\sum_{\beta * \sigma^{n-1} = \sigma^n} \gamma(\beta) \omega(\sigma^{n-1}) \right) [N(\sigma^1)] = d\gamma(\sigma^1),$$

and the lemma follows. \square

Definition 4.8. Let σ^{n-1} be an interior simplex of K , and $\sigma^n \in K$ be adjacent to σ^{n-1} . For all vertices α of σ^{n-1} the vector $e_{\sigma^n}(\alpha) \pmod{N(\sigma^{n-1})}$ does not depend on the choice of σ^n and is denoted by $e_{\sigma^{n-1}}(\alpha)$.

Proposition 4.9. Let ε be a sign distribution on K . Let β be an interior vertex of K and σ^{n-2} be an interior simplex of K . We have the following alternative:

1. The intersection of the open stars of β and σ^{n-2} is empty and $(\beta; \sigma^{n-2})_\varepsilon$ vanishes;
2. The join of β and σ^{n-2} is an $(n-1)$ -simplex σ^{n-1} of K and:

$$(\beta; \sigma^{n-2})_\varepsilon [N(\sigma^{n-1})] = D^2 \varepsilon_{\sigma^{n-1}} + \rho_{\sigma^{n-1}}(\beta),$$

where $[N(\sigma^{n-1})]$ denotes the generator of the line $N(\sigma^{n-1})$;

3. The vertex β belongs to σ^{n-2} and:

$$(\beta; \sigma^{n-2})_\varepsilon [N(\sigma^{n-2})] = \sum_{\sigma^{n-1} \geq \sigma^{n-2}} e_{\sigma^{n-1}}(\beta) \wedge \left(D^2 \varepsilon_{\sigma^{n-1}} + \rho_{\sigma^{n-1}}(\text{lk}(\sigma^{n-2}; \sigma^{n-1})) \right),$$

where $[N(\sigma^{n-2})]$ denotes the generator of the line $\wedge^2 N(\sigma^{n-2})$.

Proof. Let σ^{n-2} be an interior simplex of K . Following Proposition 3.6, the homology class

$\partial_{n-2,1}^1 g^{n-2}(\chi_\sigma^{n-2})$ is represented by the cycle:

$$c := \sum_{\sigma^{n-1} \geq \sigma^{n-2}} w(\sigma^{n-2}) \wedge (D^2 \varepsilon_{\sigma^{n-1}} + \rho_{\sigma^{n-1}}^{n-2}(\sigma^{n-2})) \otimes [\sigma^{n-1}; \sigma^{n-1}].$$

Since σ^{n-2} is interior, its link L in K is homeomorphic to a circle. We denote the vertices of L by α_i , for all $i \in \mathbb{Z}/k\mathbb{Z}$, in such a way that α_i and α_{i+1} are the vertices of an edge of L . For all $i \in \mathbb{Z}/k\mathbb{Z}$, we denote $\sigma^{n-2} * \alpha_i$ by σ_i^{n-1} , and $\sigma^{n-2} * \alpha_i * \alpha_{i+1}$ by $\sigma_{i,i+1}^n$. Following Proposition 2.14, $\rho_{\sigma_i^{n-1}}^{n-2}(\sigma^{n-2})$ equals $\rho_{\sigma_i^{n-1}}(\alpha_i)$ for all $i \in \mathbb{Z}/k\mathbb{Z}$. Then we can write:

$$c = \sum_{i \in \mathbb{Z}/k\mathbb{Z}} w(\sigma^{n-2}) \wedge (D^2 \varepsilon_{\sigma_i^{n-1}} + \rho_{\sigma_i^{n-1}}(\alpha_i)) \otimes [\sigma_i^{n-1}; \sigma_i^{n-1}].$$

Let us consider the chain of $\Omega_{n-1,n}(K; F_{n-1}^X)$ given by the formula:

$$b := \sum_{i \in \mathbb{Z}/k\mathbb{Z}} u(\sigma^{n-2}) \wedge (D^2 \varepsilon_{\sigma_i^{n-1}} + \rho_{\sigma_i^{n-1}}(\alpha_i)) \otimes [\sigma_i^{n-1}; \sigma_{i,i+1}^n].$$

By construction, if σ^p is a simplex of K that does not contain σ^{n-2} , then $\langle \partial b; [\sigma^p; \sigma^p] \rangle$ vanishes. For all $i \in \mathbb{Z}/k\mathbb{Z}$, both $\langle \partial b; [\sigma_i^{n-1}; \sigma_i^{n-1}] \rangle$ and $\langle \partial b; [\sigma_{i,i+1}^n; \sigma_{i,i+1}^n] \rangle$ equal $\langle c; [\sigma_i^{n-1}; \sigma_i^{n-1}] \rangle$. Therefore, the cycle c is homologous to the cycle:

$$c' := \sum_{i \in \mathbb{Z}/k\mathbb{Z}} w(\sigma^{n-2}) \wedge (D^2 \varepsilon_{\sigma_i^{n-1}} + \rho_{\sigma_i^{n-1}}(\alpha_i)) \otimes [\sigma_{i,i+1}^n; \sigma_{i,i+1}^n].$$

Let β be an interior vertex of K . Following Lemma 4.7, we find that:

$$\begin{aligned} (\beta; \sigma^{n-2})_\varepsilon &= g^0(\chi_\beta) \cdot [c'] \\ &= \sum_{\substack{i \in \mathbb{Z}/k\mathbb{Z} \\ \beta \in \sigma_{i,i+1}^n}} \omega(\text{lk}(\beta; \sigma_{i,i+1}^n)) \left(w(\sigma^{n-2}) \wedge (D^2 \varepsilon_{\sigma_i^{n-1}} + \rho_{\sigma_i^{n-1}}(\alpha_i)) \right). \end{aligned} \quad (12)$$

It implies that $(\beta; \sigma^{n-2})_\varepsilon$ vanishes whenever the open stars of β and σ^{n-2} are disjoint. If now the join of β and σ^{n-2} is an $(n-1)$ -simplex of K , then β is α_i for a certain $i \in \mathbb{Z}/k\mathbb{Z}$. In this case we have:

$$\begin{aligned} (\beta; \sigma^{n-2})_\varepsilon &= \omega(\text{lk}(\alpha_i; \sigma_{i,i+1}^n)) \left(w(\sigma^{n-2}) \wedge (D^2 \varepsilon_{\sigma_i^{n-1}} + \rho_{\sigma_i^{n-1}}(\alpha_i)) \right) \\ &\quad + \omega(\text{lk}(\alpha_i; \sigma_{i-1,i}^n)) \left(w(\sigma^{n-2}) \wedge (D^2 \varepsilon_{\sigma_{i-1}^{n-1}} + \rho_{\sigma_{i-1}^{n-1}}(\alpha_{i-1})) \right) \\ &= \omega(\sigma_{i+1}^{n-1}) \left(w(\sigma^{n-2}) \wedge (D^2 \varepsilon_{\sigma_i^{n-1}} + \rho_{\sigma_i^{n-1}}(\alpha_i)) \right) \\ &\quad + \omega(\sigma_{i-1}^{n-1}) \left(w(\sigma^{n-2}) \wedge (D^2 \varepsilon_{\sigma_{i-1}^{n-1}} + \rho_{\sigma_{i-1}^{n-1}}(\alpha_{i-1})) \right) \\ &= |D^2 \varepsilon_{\sigma_i^{n-1}} + \rho_{\sigma_i^{n-1}}(\alpha_i)|, \end{aligned}$$

Because $D^2\varepsilon_{\sigma_{i-1}^{n-1}} + \rho_{\sigma_{i-1}^{n-1}}(\alpha_{i-1})$ belongs to $N(\sigma_{i-1}^{n-1})$, and $\omega(\sigma_{i+1}^{n-1})\left(w(\sigma^{n-2}) \wedge [N(\sigma_i^{n-1})]\right)$ equals 1. Indeed, if α is a vertex of σ^{n-2} , we have:

$$\begin{aligned}\omega(\sigma_{i+1}^{n-1})\left(w(\sigma^{n-2}) \wedge [N(\sigma_i^{n-1})]\right) &= \omega(\sigma^{n-2}) \wedge (\alpha_{i+1} - \alpha)\left(w(\sigma^{n-2}) \wedge [N(\sigma_i^{n-1})]\right) \\ &= (\alpha_{i+1} - \alpha)([N(\sigma_i^{n-1})]) = 1,\end{aligned}$$

since the vanishing of $(\alpha_{i+1} - \alpha)([N(\sigma_i^{n-1})])$ would imply the nullity of $[N(\sigma_i^{n-1})]$. If we assume now β to be a vertex of σ^{n-2} , the formula (12) implies that:

$$(\beta; \sigma^{n-2})_\varepsilon = \sum_{i \in \mathbb{Z}/k\mathbb{Z}} \omega(\text{lk}(\beta; \sigma_{i,i+1}^n))\left(w(\sigma^{n-2}) \wedge (D^2\varepsilon_{\sigma_i^{n-1}} + \rho_{\sigma_i^{n-1}}(\alpha_i))\right).$$

The contraction $\omega(\text{lk}(\beta; \sigma_{i,i+1}^n)) \cdot [\mathbf{t}(\mathbb{F}_2)]$ equals the vector $e_{\sigma_{i,i+1}^n}(\beta)$. Therefore, we have:

$$(\beta; \sigma^{n-2})_\varepsilon[\mathbf{t}(\mathbb{F}_2)] = w(\sigma^{n-2}) \wedge \sum_{i \in \mathbb{Z}/k\mathbb{Z}} e_{\sigma_{i,i+1}^n}(\beta) \wedge (D^2\varepsilon_{\sigma_i^{n-1}} + \rho_{\sigma_i^{n-1}}(\alpha_i)).$$

If n equals 2, then $w(\sigma^{n-2})$ equals 1 by definition and we have established the last formula. Now we assume that n is at least equal to 3. Let σ^1 be an edge of σ^{n-2} . The number $\omega(\sigma^1)(e_{\sigma_{i,i+1}^n}(\beta))$ equals 1 if and only if β belongs to σ^1 , c.f. Proposition 2.11. In particular it does not depend on i . The number $\omega(\sigma^1)(D^2\varepsilon_{\sigma_i^{n-1}} + \rho_{\sigma_i^{n-1}}(\alpha_i))$ is 0 for all $i \in \mathbb{Z}/k\mathbb{Z}$ for σ^1 is contained in σ_i^{n-1} and $D^2\varepsilon_{\sigma_i^{n-1}} + \rho_{\sigma_i^{n-1}}(\alpha_i)$ belongs to $N(\sigma_i^{n-1})$. Using these computations we find that:

$$\omega(\sigma^1) \cdot \sum_{i \in \mathbb{Z}/k\mathbb{Z}} e_{\sigma_{i,i+1}^n}(\beta) \wedge (D^2\varepsilon_{\sigma_i^{n-1}} + \rho_{\sigma_i^{n-1}}(\alpha_i)) = d\chi_\beta(\sigma^1) \sum_{i \in \mathbb{Z}/k\mathbb{Z}} D^2\varepsilon_{\sigma_i^{n-1}} + \rho_{\sigma_i^{n-1}}(\alpha_i) = 0,$$

and that the vector $\sum_{i \in \mathbb{Z}/k\mathbb{Z}} e_{\sigma_{i,i+1}^n}(\beta) \wedge (D^2\varepsilon_{\sigma_i^{n-1}} + \rho_{\sigma_i^{n-1}}(\alpha_i))$ belongs to the line $\wedge^2 N(\sigma^{n-2})$. Since $u(\sigma^{n-2})$ is the volume element of a supplementary sub-space of $N(\sigma^{n-2})$ we finally find the formula:

$$(\beta; \sigma^{n-2})_\varepsilon[N(\sigma^{n-2})] = \sum_{i \in \mathbb{Z}/k\mathbb{Z}} e_{\sigma_{i,i+1}^n}(\beta) \wedge (D^2\varepsilon_{\sigma_i^{n-1}} + \rho_{\sigma_i^{n-1}}(\alpha_i)).$$

□

Proposition 4.10. *For all interior vertices β of K and all simplices $\sigma^{n-2} \geq \beta$ we have:*

$$\sum_{\sigma^{n-1} \geq \sigma^{n-2}} e_{\sigma^{n-1}}(\beta) \wedge \rho_{\sigma^{n-1}}(\beta) = \sum_{\sigma^{n-1} \geq \sigma^{n-2}} e_{\sigma^{n-1}}(\beta) \wedge \rho_{\sigma^{n-1}}(\text{lk}(\sigma^{n-2}; \sigma^{n-1})).$$

Proof. We will establish this formula by using a particular sign distribution: the characteristic cochain χ_β of the vertex β . On one hand we have:

$$D^2\chi_\beta = \sum_{\beta \in \sigma^{n-1}} \rho_{\sigma^{n-1}}(\beta) \otimes \sigma^{n-1}.$$

Using the formulæ of Proposition 4.9, we have for all σ^{n-2} containing β :

$$(\beta; \sigma^{n-2})_{\chi_\beta} [N(\sigma^{n-2})] = \sum_{\sigma^{n-1} \geq \sigma^{n-2}} e_{\sigma^{n-1}}(\beta) \wedge \left(\rho_{\sigma^{n-1}}(\beta) + \rho_{\sigma^{n-1}}(\text{lk}(\sigma^{n-2}; \sigma^{n-1})) \right).$$

On the other hand, we have:

$$(\beta; \sigma^{n-2})_{\chi_\beta} = g^{n-2}(\chi_{\sigma^{n-2}}) \cdot \partial_{0,n-1}^1 g^0(\chi_\beta),$$

However the tropical homology class $g^0(\chi_\beta)$ can be lifted as a $(n-1)$ -homology class of $\mathbb{R}X_{\chi_\beta}$: the fundamental class of the small sphere surrounding β in the positive quadrant of $\mathbb{R}K$. This small sphere is isotopic to the link of β in K . As a consequence $\partial_{0,n-1}^1 g^0(\chi_\beta)$ vanishes. Therefore, we have:

$$\sum_{\sigma^{n-1} \geq \sigma^{n-2}} e_{\sigma^{n-1}}(\beta) \wedge \left(\rho_{\sigma^{n-1}}(\beta) + \rho_{\sigma^{n-1}}(\text{lk}(\sigma^{n-2}; \sigma^{n-1})) \right) = 0.$$

□

Proposition 4.11. *For all sign distributions ε on K , all interior vertices β of K , and all simplices $\sigma^{n-2} \geq \beta$ we have:*

$$(\beta; \sigma^{n-2})_\varepsilon [N(\sigma^{n-2})] = \sum_{\sigma^{n-1} \geq \sigma^{n-2}} e_{\sigma^{n-1}}(\beta) \wedge (D^2 \varepsilon_{\sigma^{n-1}} + \rho_{\sigma^{n-1}}(\beta)).$$

Proof. This is a direct consequence of Proposition 4.9 and Proposition 4.10. □

Proposition 4.12. *We assume n to be equal to 2. For all interior vertices β of K , the number $(\beta; \beta)_\varepsilon$ equals the parity of the number of edges of K , adjacent to β , along which $\mathbb{R}X_\varepsilon$ is twisted. In other words, we have:*

$$(\beta; \beta)_\varepsilon = \sum_{\sigma^1 \geq \beta} |D^2 \varepsilon_{\sigma^1} + \rho_{\sigma^1}|.$$

Proof. Let σ^2 be a triangle of K and σ^1 be an edge of σ^2 . For all vertices β of σ^2 , the set $\{e_{\sigma^2}(\beta); [N(\sigma^1)]\}$ forms a basis of $\mathfrak{t}(\mathbb{F}_2)$. Hence $e_{\sigma^2}(\beta) \wedge [N(\sigma^1)]$ equals $[\mathfrak{t}(\mathbb{F}_2)]$. Moreover for all interior vertex β of K , $[N(\beta)]$ equals $[\mathfrak{t}(\mathbb{F}_2)]$. Therefore, Proposition 4.11 implies the formula of the proposition. □

Remark 4.13. When n is greater than 3 the formula of Proposition 4.11, can also be interpreted as a number of twists in the direction of β along the $(n-1)$ -simplices of its star. However, this time some twists must be counted and some must not. For instance, if β is an interior vertex and σ^{n-2} contains β , $(\beta; \sigma^{n-2})_\varepsilon$ equals the number of twists of $\mathbb{R}X_\varepsilon$ in the direction of β along the $(n-1)$ -simplices of K for which $w(\sigma^{n-2}) \wedge e_{\sigma^{n-1}}(\beta) \wedge [N(\sigma^{n-1})]$ equals $[\mathfrak{t}(\mathbb{F}_2)]$. The set of twists we are counting may depend on the choice of the decomposed vectors $w(\sigma^{n-2})$. However, the count itself, will not.

Definition 4.14. Let $\mathbb{R}X_\varepsilon$ be a T-hypersurface of $\mathbb{R}P$. The rank of $\mathbb{R}X_\varepsilon$, denoted by $\ell(\mathbb{R}X_\varepsilon)$, is the biggest integer $\ell \geq 0$ for which the homological ‘‘inclusions’’ $H_k(\mathbb{R}X_\varepsilon; \mathbb{F}_2) \rightarrow H_k(\mathbb{R}P; \mathbb{F}_2)$ are onto for all $0 \leq k \leq \ell$.

Theorem 4.15 (Generalised Haas Theorem). *Let P be a non-singular polytope of $\mathfrak{t}^*(\mathbb{R})$ endowed with a primitive triangulation K . For all sign distributions ε , the T -hypersurface $\mathbb{R}X_\varepsilon$ has a maximal number of connected components if and only if:*

1. *The triangulation K is ρ -uniform;*
2. *The number $\ell(\mathbb{R}X_\varepsilon)$ is at least equal to $\min(n - 2; 1)$;*
3. *The sign distribution ε satisfies:*

$$D^2\varepsilon = \rho \text{ in } H_{n-1}(K; \mathcal{B}_1K; N). \quad (13)$$

4. *For all simplices σ^{n-2} of \mathcal{B}_1K not contained in \mathcal{B}_0K , $(\text{ap}(\sigma^{n-2}); \sigma^{n-2})_\varepsilon$ vanishes.*

Proof. Using Theorem 4.26 p.31 in [Che23b], we have the short exact sequence:

$$0 \longrightarrow E_{0,0}^\infty(\mathbb{R}X_\varepsilon) \longrightarrow H_0(\mathbb{R}X_\varepsilon; \mathbb{F}_2) \longrightarrow E_{n-1,0}^\infty(\mathbb{R}X_\varepsilon) \longrightarrow 0.$$

Since $E_{0,0}^\infty(\mathbb{R}X_\varepsilon)$ is isomorphic to $E_{0,0}^1(\mathbb{R}X_\varepsilon)$, itself isomorphic to $H_{0,0}(X; \mathbb{F}_2)$ it has dimension 1. Hence $\mathbb{R}X_\varepsilon$ has a maximal number of connected components if and only if $E_{n-1,0}^\infty(\mathbb{R}X_\varepsilon)$ is isomorphic to $E_{n-1,0}^1(\mathbb{R}X_\varepsilon)$. We are now going to distinguish three cases: n equals 2, n equals 3, and n is greater than 3. In a first time we assume that n is greater than 3. $E_{n-1,0}^\infty(\mathbb{R}X_\varepsilon)$ is isomorphic to $E_{n-1,0}^1(\mathbb{R}X_\varepsilon)$ if and only if the two following morphisms vanish:

$$\begin{cases} \partial_{n-2,1}^1 : E_{n-2,1}^1(\mathbb{R}X_\varepsilon) \longrightarrow E_{n-1,0}^1(\mathbb{R}X_\varepsilon) \\ \partial_{1,1}^{n-2} : E_{1,1}^{n-2}(\mathbb{R}X_\varepsilon) \longrightarrow E_{n-1,0}^{n-2}(\mathbb{R}X_\varepsilon). \end{cases}$$

By Theorem 4.30 p.34 in [Che23b] and duality $\partial_{1,1}^{n-2}$ vanishes if and only if the morphism:

$$i^1 : H^1(\mathbb{R}P; \mathbb{F}_2) \rightarrow H^1(\mathbb{R}X_\varepsilon; \mathbb{F}_2),$$

is injective. Since $i^0 : H^0(\mathbb{R}P; \mathbb{F}_2) \rightarrow H^0(\mathbb{R}X_\varepsilon; \mathbb{F}_2)$ is always injective, $\partial_{1,1}^{n-2}$ vanishes if and only if $\ell(\mathbb{R}X_\varepsilon)$ is at least equal to 1. Since n is greater than 3, Proposition 3.10 ensures us that $\partial_{n-2,1}^1$ vanishes if and only if $M_{0,n-2}(\varepsilon)$ does. The second point of Proposition 4.9 implies that for $M_{0,n-2}(\varepsilon)$ to vanish we must have:

$$\rho_{\sigma^{n-1}}(\beta) = D^2\varepsilon_{\sigma^{n-1}},$$

for all interior simplices σ^{n-1} of K , and all vertices β of σ^{n-1} whose link $\text{lk}(\beta; \sigma^{n-1})$ is not contained in the boundary. In particular, K must be ρ -uniform. Reciprocally, if K is ρ -uniform, then we must have:

$$D^2\varepsilon = \rho \text{ in } H_{n-1}(K; \mathcal{B}_1K; N).$$

Now if we suppose that K is ρ -uniform and ε is a solution of (13), $M_{0,n-2}(\varepsilon)$ vanishes if and only if $(\beta; \sigma^{n-2})_\varepsilon$ vanishes for all pairs of interior simplices $\beta \leq \sigma^{n-2}$. We assume that we have such a pair $\beta \leq \sigma^{n-2}$ with σ^{n-2} containing at least 2 interior vertices, one of them being β . If σ^{n-1} contains σ^{n-2} then, by hypothesis, the link of β in σ^{n-1} contains at least one interior vertex of

K . As a consequence of (13), $D^2\varepsilon_{\sigma^{n-1}}$ equals $\rho_{\sigma^{n-1}}(\beta)$. Hence $(\beta; \sigma^{n-2})_\varepsilon$ vanishes. Therefore, $M_{0,n-2}(\varepsilon)$ vanishes if and only if K is ρ -uniform, ε is a solution of (13), and for all simplices σ^{n-2} of $\mathcal{B}_1K \setminus \mathcal{B}_0K$, $(\text{ap}(\sigma^{n-2}); \sigma^{n-2})_\varepsilon$ vanishes. When n is at most equal to 3, Theorem 4.26 p.31 of [Che23b] implies that the Renaudineau-Shaw spectral sequence of $\mathbb{R}X_\varepsilon$ degenerates at the second page. Hence, in these cases $\mathbb{R}X_\varepsilon$ has a maximal number of connected components if and only if $E_{n-1,0}^2(\mathbb{R}X_\varepsilon)$ is isomorphic to $E_{n-1,0}^1(\mathbb{R}X_\varepsilon)$. This latter condition is equivalent²⁶ to the vanishing of $\partial_{n-1,0}^1$. Let n be equal to 2. Following Proposition 3.10, the vanishing of $\partial_{1,0}^1$ is equivalent to the vanishing of $M_{0,0}(\varepsilon)$. We recall that every primitive triangulation of a two dimensional polytope is ρ -uniform²⁷. As in the preceding case, $M_{0,0}(\varepsilon)$ vanishes if and only if:

$$D^2\varepsilon = \rho \text{ in } H_1(K; \mathcal{B}_1K; N),$$

and $(\beta; \beta)_\varepsilon$ vanishes for all $\beta \in \mathcal{B}_1K \setminus \mathcal{B}_0K$. Finally, the rank $\ell(\mathbb{R}X_\varepsilon)$ being always at least 0, the condition $\ell(\mathbb{R}X_\varepsilon) \geq \min(0, 1)$ is always satisfied. Only remains the case n equals 3.

Proposition 3.11 provides us with the formula:

$$\dim E_{2,0}^2(\mathbb{R}X_\varepsilon) = \dim E_{2,0}^1(\mathbb{R}X_\varepsilon) - \text{rk } M_{0,1}(\varepsilon) - \delta,$$

where δ denotes the dimension of the cokernel of $i_1 : H_1(\mathbb{R}X_\varepsilon; \mathbb{F}_2) \rightarrow H_1(\mathbb{R}P; \mathbb{F}_2)$. Hence, $E_{2,0}^2(\mathbb{R}X_\varepsilon)$ is isomorphic to $E_{2,0}^1(\mathbb{R}X_\varepsilon)$ if and only if both $M_{0,1}(\varepsilon)$ and δ vanish. The latter condition is equivalent to $\ell(\mathbb{R}X_\varepsilon)$ being at least equal to 1. The former, as in the case $n > 3$ is equivalent to K being ρ -uniform and ε satisfying:

$$D^2\varepsilon = \rho \text{ in } H_2(K; \mathcal{B}_1K; N),$$

and $(\text{ap}(\sigma^1); \sigma^1)_\varepsilon = 0$ for all $\sigma^1 \in \mathcal{B}_1K \setminus \mathcal{B}_0K$. □

Remark 4.16. The case n equals 2 of Theorem 4.15 was already known and is due to B. Haas, c.f. [Haa97], Theorem 7.3.0.10 p.70. Our generalisation can also be formulated in terms of twists. Equation (13) forbids $\mathbb{R}X_\varepsilon$ to be twisted in certain directions along the $(n-1)$ -simplices of $K \setminus \mathcal{B}_1K$ and the fourth condition of Theorem 4.15 imposes that the weighted count of the twists of $\mathbb{R}X_\varepsilon$ around the $(n-2)$ -simplices of $\mathcal{B}_1K \setminus \mathcal{B}_0K$ has to be even. The two first conditions of the theorem are automatically satisfied for T-curves. Because of Proposition 4.39 p.38 of [Che23b], the second condition is automatically satisfied when $\iota[\omega_X] \geq 1$. The latter condition only depends on the ‘‘degree’’ of X . For instance odd degree hypersurfaces in projective spaces satisfy $\iota[\omega_X] = n - 1$.

In the remaining of this paragraph we tackle the question of knowing if every ρ -uniform triangulation admits a sign distribution ε whose T-hypersurface $\mathbb{R}X_\varepsilon$ has a maximal number of connected components.

Proposition 4.17. *Let K be a ρ -uniform triangulation. Any solution ε of the following equation:*

$$D^2\varepsilon = \rho \text{ in } H_{n-1}(K; \mathcal{B}_0K; N), \tag{14}$$

²⁶c.f. [Che23b] Theorem 4.27 p.32.

²⁷c.f. Proposition 2.14.

yields a T -hypersurface $\mathbb{R}X_\varepsilon$ with the maximal number of connected components. Moreover, the T -hypersurfaces constructed from the solutions of (14) are made of:

1. Exactly $h^{0,n-1}(X)$ spheres, bounding disjoint balls. None of them intersect the toric divisors of $\mathbb{R}P$;
2. Another connected component that intersects all of the toric divisors of $\mathbb{R}P$.

When (14) has a solution we say that K is simply integrable.

Proof. Let ε be a solution of (14) and β be an interior vertex of K . For all $(n-1)$ -simplices adjacent to β , $D^2\varepsilon_{\sigma^{n-1}} + \rho_{\sigma^{n-1}}(\beta)$ vanishes. It implies that the vector $v := D\varepsilon_{\sigma^n} + e_{\sigma^n}(\beta)$ does not depend on the choice of the n -simplex $\sigma^n \geq \beta$ we used to compute it. For all $\sigma^n \geq \beta$, the cocycle $d\varepsilon|_{\sigma^n} + z_{\sigma^n}^1(v)$ equals $d\chi_\beta|_{\sigma^n}$. Therefore, it follows from Proposition 4.1 that the intersection of $\mathbb{R}X_\varepsilon$ with the lift of the closed star of β corresponding to v consists of a sphere confining the lift of β in its interior. Since $\mathbb{R}X_\varepsilon$ must intersect all the toric divisors of $\mathbb{R}P$ in a non-empty set, $\mathbb{R}X_\varepsilon$ has at least $1 + \text{card}(K^{(0)} \setminus \partial K^{(0)})$ connected component. However, $\text{card}(K^{(0)} \setminus \partial K^{(0)})$ equals $h^{n-1,0}(X)$. Hence, $\mathbb{R}X_\varepsilon$ has a maximal number of connected components and satisfies the description of the proposition. \square

Proposition 4.18. *A shellable, ρ -uniform triangulation K of a non-singular polytope P is simply integrable if and only if $\partial\rho = 0$ where ∂ denotes the connecting morphism:*

$$\partial : H_{n-1}(K; \mathcal{B}_0K; N) \rightarrow H_{n-2}(\mathcal{B}_0K; \partial K; N),$$

of the long exact sequence associated with the triple $\partial K \subset \mathcal{B}_0K \subset K$.

Proof. Let K be a ρ -uniform triangulation. If K is simply integrable, we consider ε a solution of (14). In this case, the second derivative of ε belongs to $H_{n-1}(K; \partial K; N)$ and equals ρ modulo \mathcal{B}_0K . Since the kernel of ∂ is the image of $H_{n-1}(K; \partial K; N) \rightarrow H_{n-1}(K; \mathcal{B}_0K; N)$, $\partial\rho$ has to vanish. Reciprocally, if we assume K to be a ρ -uniform, shellable triangulation, then $\partial\rho = 0$ implies that ρ has a lift $\bar{\rho}$ in $H_{n-1}(K; \partial K; N)$. Since K is shellable, Lemma 2.20 ensures that we can find a sign distribution ε , whose second derivative equals $\bar{\rho}$. By Proposition 4.17, any such sign distribution yields a solution of (14). \square

T-curves. In what follows we assume n equal to 2. In this special case, Proposition 2.14 implies that there is a canonical class $\bar{\rho} \in H_1(K; \partial K; N)$ that lifts the class $\rho \in H_1(K; \mathcal{B}_0K; N)$. It is defined as follows:

$$\bar{\rho} := \sum_{\sigma^1 \notin \partial K} \rho_{\sigma^1} \otimes \sigma^1.$$

Theorem 4.19. *The solutions of the equation:*

$$D^2\varepsilon = \bar{\rho} \text{ in } H_1(K; \partial K; N), \tag{15}$$

consist of the images of the polynomial functions of degree 2 in $C^0(K; \mathbb{F}_2)$. As such, every 2-dimensional triangulation, even a non-shellable one, is simply integrable.

Proof. Let h be a quadratic sign distribution. Now let β_0 and β_1 be the two vertices of an interior edge of K , and $[\beta_0; \beta_1; \alpha_{\pm}]$ the two triangles of K adjacent to $[\beta_0; \beta_1]$. Following Proposition 2.19, affine maps are in the kernel of D^2 . Since h is quadratic, we can assume, to compute $D^2 h_{[\beta_0; \beta_1]}$, that h is given by $[(x; y) \mapsto xy]$ where $(x; y)$ are the affine coordinates in the basis $(\beta_1 + \beta_0; \alpha_+ + \beta_0)$ with origin at β_0 . If $(x_-; y_-)$ are the coordinates of the vector $\alpha_- + \beta_0$, then y_- equals 1 for $(\beta_1 + \beta_0; \alpha_- + \beta_0)$ is also a basis of $\mathfrak{t}^*(\mathbb{F}_2)$. Hence, we have:

$$\alpha_+ + \alpha_- = x_-(\beta_1 + \beta_0),$$

and $\rho_{[\beta_0; \beta_1]}$ equals $x_-[N([\beta_0; \beta_1])]$ by Proposition 2.13. Let us write:

$$D^2 h_{[\beta_0; \beta_1]} = |D^2 h_{[\beta_0; \beta_1]}| \left[N([\beta_0; \beta_1]) \right].$$

Then, we have:

$$\begin{aligned} |D^2 h_{[\beta_0; \beta_1]}| &= x(\alpha_+)y(\alpha_+) + x_-y_- + y_-(x(\beta_0)y(\beta_0) + x(\beta_1)y(\beta_1)) \\ &= 0 \times 1 + x_- \times 1 + x_-(0 \times 0 + 1 \times 0) \\ &= x_-. \end{aligned}$$

And $D^2 h_{[\beta_0; \beta_1]}$ equals $\rho_{[\beta_0; \beta_1]}$. Now by Proposition 2.19, the kernel of D^2 is exactly the space of affine sign distributions, hence the theorem follows. \square

Remarks 4.20. 1. Since $\mathfrak{t}^*(\mathbb{F}_2)$ has dimension 2 there is essentially one quadratic polynomial function $f : \mathfrak{t}^*(\mathbb{F}_2) \rightarrow \mathbb{F}_2$. It is a consequence of Fermat's little theorem. Hence, all the T-curves $\mathbb{R}X_\varepsilon$ obtained from a solution ε of (15) are reflections of one another;

2. I. Itenberg and O. Viro proved²⁸ that a particular sign distribution always produces a maximal curve of a given topological type for all convex triangulations of the standard triangle of size $d \geq 1$: $\{(x; y) \in \mathbb{R}_+^2 \mid x + y \leq d\}$. They named their sign distribution the Harnack distribution. It is given by the formula $\varepsilon(x; y) := 1 + (1 + x)(1 + y) \pmod{2}$, hence by a quadratic polynomial.

A sharper bound for the number of connected components. In this paragraph, we provide an estimation of the maximal number of connected components of a T-hypersurface constructed on a non- ρ -uniform triangulation.

Definition 4.21. Let K be a primitive triangulation of a non-singular polytope. The distance between two simplices is the minimum of the graph length of their vertices in the 1-skeleton of K . We denote by $\kappa(K)$ the maximal number of $(n - 1)$ -simplices of K at which it fails to be ρ -uniform and that are all at distance at least 2 of each others.

Proposition 4.22. *Let K be a primitive triangulation of a non-singular polytope. For all sign distributions $\varepsilon \in C^0(K; \mathbb{F}_2)$ the rank of $M_{0, n-2}(\varepsilon)$ is always at least equal to $\kappa(K)$. As a consequence we have the following upper bound:*

$$\dim H_0(\mathbb{R}X_\varepsilon; \mathbb{F}_2) \leq 1 + \dim H_{0, n-1}(X; \mathbb{F}_2) - \kappa(K). \quad (16)$$

²⁸c.f. [IV96], Proposition p.24.

Proof. Let $\sigma_1^{n-1}, \dots, \sigma_k^{n-1}$ be simplices of K at which it fails to be ρ -uniform and that are all at distance at least 2 of each others. Let ε be a sign distribution on K . Since K fails to be ρ -uniform at the simplices $\sigma_1^{n-1}, \dots, \sigma_k^{n-1}$, we can find, for all $1 \leq i \leq k$, two different interior vertices α_i, β_i of σ_i^{n-1} for which $\rho_{\sigma_i^{n-1}}(\alpha_i)$ and $\rho_{\sigma_i^{n-1}}(\beta_i)$ are different. Using Proposition 4.9, it follows that either $(\alpha_i; \text{lk}(\alpha_i; \sigma_i^{n-1}))_\varepsilon$ or $(\beta_i; \text{lk}(\beta_i; \sigma_i^{n-1}))_\varepsilon$ equals 1. We assume that it happens on α_i , for all $1 \leq i \leq k$. The matrix:

$$\left((\alpha_i; \text{lk}(\alpha_j; \sigma_j^{n-1}))_\varepsilon \right)_{1 \leq i, j \leq k},$$

is a sub-matrix of $M_{0, n-2}(\varepsilon)$ equal to the identity matrix of size k . Indeed for all i different from j , α_i and $\text{lk}(\alpha_j; \sigma_j^{n-1})$ are at distance at least 2. Therefore, they cannot be both included in the same simplex. Hence, the rank of $M_{0, n-2}(\varepsilon)$ is at least equal to k . Consequently, the rank of $M_{0, n-2}(\varepsilon)$ is at least equal to $\kappa(K)$, and the upper bound (16) follows from Proposition 3.11. \square

Asymptotic growth of the expected number of connected components. We assume that we are given a smooth polytope P of $\mathfrak{t}^*(\mathbb{R})$, and for every integer $d \geq 1$, a primitive triangulation K_d of its dilate dP . We denote by $X(d)$ the dual hypersurface of K_d . By Theorem 1.4 p.947 of [RS23], we know that for all $\varepsilon \in C^0(K(d); \mathbb{F}_2)$, the number of connected components $b_0(\mathbb{R}X_\varepsilon(d))$ is less than $1 + h^{n-1,0}(X(d))$, where $h^{n-1,0}(X(d))$ is the dimension of $H_{n-1,0}(X(d); \mathbb{F}_2)$. In this paragraph we give an idea of the growth of the number connected components of a typical T-hypersurface as d grows to $+\infty$. If we choose ε uniformly at random in $C^0(K(d); \mathbb{F}_2)$, the number $b_0(\mathbb{R}X_\varepsilon(d))$ becomes a random variable. We denote by $\mathbb{E}[\xi]$ the expected value of a random variable ξ .

Proposition 4.23. *The asymptotic of $h^{n-1,0}(X(d))$ is the following:*

$$h^{n-1,0}(X(d)) = \text{vol}_{\mathbb{Z}}(P)d^n + o(d^{n-1}),$$

where $\text{vol}_{\mathbb{Z}}(P)$ is the integral volume of P . That is to say the Lebesgue measure of P , normalised so that a parallelogram on a basis of $\mathfrak{t}^*(\mathbb{Z})$ has volume 1.

Proof. The number $h^{n-1,0}(X(d))$ equals the number of lattice points in the relative interior of P . By Ehrhart-Macdonald reciprocity²⁹ it equals $(-1)^n L(P; -d)$, where $L(P; X)$ is the Ehrhart polynomial³⁰ of P . The polynomial $L(P; X)$ has degree n and its leading coefficient is $\text{vol}_{\mathbb{Z}}(P)$. \square

Proposition 4.24. *Let P be a n -dimensional integral polytope of $\mathfrak{t}^*(\mathbb{R})$ and $(K_d)_{d \geq 1}$ be a sequence of primitive triangulations of $(dP)_{d \geq 1}$. The function $d \mapsto f_1(K_d)$ is a polynomial of degree n .*

Proof. Let $d \geq 1$ be an integer. The number $f_1(K_d)$ is a linear combination of the h^* -vector of dP . In particular there is a $\mathfrak{t}^*(\mathbb{Z})$ -translation invariant valuation φ on the integral polytopes of $\mathfrak{t}^*(\mathbb{R})$ such that $f_1(K_d)$ equals $\varphi(dP)$ for all $d \geq 1$. Theorem 5 p.121 of [McM77] implies that $d \mapsto f_1(K_d)$ is given by a polynomial of degree at most n . However, the number $f_1(K_d)$

²⁹c.f. [Mac71], Theorem 4.6 p.192.

³⁰c.f. [Ehr62], pp.616-618.

is not smaller than $\frac{1}{2}f_0(K_d)$ since no vertex is isolated. Hence, $d \mapsto f_1(K_d)$ is of degree n for $d \mapsto f_0(K_d) = \text{card}(dP \cap \mathfrak{t}^*(\mathbb{Z}))$ is polynomial of degree n . \square

Proposition 4.25. *For all sequences $(K_d)_{d \geq 1}$ of primitive triangulations of $(dP)_{d \geq 1}$ we have:*

$$1 \geq \limsup_{d \rightarrow +\infty} \mathbb{E} \left[\frac{b_0(\mathbb{R}X_\varepsilon(d))}{1 + h^{n-1,0}(X(d))} \right] \geq \liminf_{d \rightarrow +\infty} \mathbb{E} \left[\frac{b_0(\mathbb{R}X_\varepsilon(d))}{1 + h^{n-1,0}(X(d))} \right] > 0.$$

Proof. Let $d \geq 1$ be an integer and $\beta \in K_d$ be an interior vertex. We denote by S_β the subcomplex of cubical subdivision of K_d made of all the cubes indexed by the pairs of simplices $\sigma^p \leq \sigma^q$ with $p \geq 1$ and σ^p containing β . We also denote by St_β the open star of β , and ξ_β the Bernoulli random variable whose value is 1 if and only if $\mathbb{R}X_\varepsilon(d)$ contains a lift of S_β . By Proposition 4.1 and Proposition 4.17 $\xi_\beta(\varepsilon)$ equals 1 if and only if the image of $D^2\varepsilon$ in $H_{n-1}(K_d(\text{St}_\beta); \partial K_d(\text{St}_\beta); N)$ equals $\rho(\beta) = \sum_{\sigma^{n-1} \geq \beta} \rho_{\sigma^{n-1}}(\beta) \otimes \sigma^{n-1}$. We have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ & & & C^0(K_d; K_d(\text{St}_\beta); \mathbb{F}_2) & & & \\ & & & \downarrow & & & \\ 0 & \longrightarrow & \text{Aff}(\mathfrak{t}^*(\mathbb{F}_2)) & \longrightarrow & C^0(K_d; \mathbb{F}_2) & \xrightarrow{D^2} & H_{n-1}(K_d; \partial K_d; N) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Aff}(\mathfrak{t}^*(\mathbb{F}_2)) & \longrightarrow & C^0(K_d(\text{St}_\beta); \mathbb{F}_2) & \xrightarrow{D^2} & H_{n-1}(K_d(\text{St}_\beta); \partial K_d(\text{St}_\beta); N) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Hence, we have the following exact sequence:

$$0 \longrightarrow \begin{array}{c} C^0(K_d; K_d(\text{St}_\beta); \mathbb{F}_2) \\ \oplus \text{Aff}(\mathfrak{t}^*(\mathbb{F}_2)) \end{array} \longrightarrow C^0(K_d; \mathbb{F}_2) \xrightarrow{D_\beta^2} H_{n-1}(K_d(\text{St}_\beta); \partial K_d(\text{St}_\beta); N),$$

where D_β^2 is the composition $D^2 \circ |_{K_d(\text{St}_\beta)}$. Since $D_\beta^2 \chi_\beta$ equals $\rho(\beta)$ there is exactly $2^{n+1+f_0(K_d)-f_0(K_d(\text{St}_\beta))}$ solutions to the equation $D_\beta^2 \varepsilon = \rho(\beta)$. Therefore the probability that ξ_β equals 1 is $2^{n+1-f_0(K_d(\text{St}_\beta))}$.

We have the following inequality:

$$b_0(\mathbb{R}X_\varepsilon(d)) \geq 1 + \sum_{\beta \in K \setminus \partial K} \xi_\beta(\varepsilon).$$

Which implies the following one:

$$\mathbb{E} \left[\frac{b_0(\mathbb{R}X_\varepsilon(d))}{1 + h^{n-1,0}(X(d))} \right] \geq \frac{1 + \sum_{\beta \in K \setminus \partial K} 2^{n+1-f_0(K_d(\text{St}_\beta))}}{1 + h^{n-1,0}(X(d))}.$$

The number $f_0(K_d(\text{St}_\beta))$ equals the degree³¹ of the vertex β plus 1. Hence, the number $\sum_{\beta \in K \setminus \partial K} 2^{n+1-f_0(K_d(\text{St}_\beta))}$ equals $2^n \sum_{\beta \in K \setminus \partial K} 2^{-\deg \beta}$. By convexity of the function $x \mapsto 2^{-x}$, we have:

$$2^n \sum_{\beta \in K \setminus \partial K} 2^{-\deg \beta} \geq 2^n h^{n-1,0}(X(d)) 2^{-\frac{\sum_{\beta \in K \setminus \partial K} \deg \beta}{h^{n-1,0}(X(d))}} \geq h^{n-1,0}(X(d)) 2^{n-2 \frac{f_1(K_d)}{h^{n-1,0}(X(d))}}.$$

The sequence $\left(\frac{f_1(K_d)}{h^{n-1,0}(X(d))} \right)_{d \geq 1}$ converges toward a number $c > 0$ as d grows to $+\infty$. Thus, the $\lim. \inf.$ is at least equal to $2^{n-2c} > 0$. Since $\mathbb{E} \left[\frac{b_0(\mathbb{R}X_\varepsilon(d))}{1+h^{n-1,0}(X(d))} \right]$ is at most equal to 1 by A. Renaudineau and K. Shaw's inequality, the proposition follows. \square

A Discrete Dirichlet Problem for T-Curves. We finally remark that in the special case of curves one can use the symmetry of $M_{0,0}(\varepsilon)$ to define a Laplacian type operator on the space of 0-cochains of K . Doing so we can link the number of connected components of $\mathbb{R}X_\varepsilon$ to the dimension of the space of harmonic cochains.

Definition 4.26. Let K be a primitive triangulation of the 2-dimensional polytope P and ε be a sign distribution on K . We define the *Laplacian* Δ_ε to be the following endomorphism of $C^0(K; \partial K; \mathbb{F}_2)$:

$$\begin{aligned} \Delta_\varepsilon : C^0(K; \partial K; \mathbb{F}_2) &\longrightarrow C^0(K; \partial K; \mathbb{F}_2) \\ f &\longmapsto \left[\beta \mapsto \sum_\alpha (\beta; \alpha)_\varepsilon f(\alpha) \right]. \end{aligned}$$

This operator is symmetric with respect to the canonical scalar product of $C^0(K; \partial K; \mathbb{F}_2)$ since the matrix $M_{0,0}(\varepsilon)$ is symmetric. Also, using Proposition 3.11, the number of connected components of $\mathbb{R}X_\varepsilon$ is equal to 1 plus the dimension of the space of harmonic cochains $\ker(\Delta_\varepsilon)$ that vanish on the boundary.

5 Examples and Non-Examples

Let n and d be two positive integers. We recall that we denote the standard n -simplex of size d by the symbol P_d^n . In this context $\mathfrak{t}^*(\mathbb{R})$ is \mathbb{R}^n , $\mathfrak{t}^*(\mathbb{Z})$ is \mathbb{Z}^n , e_1, \dots, e_n is the canonical basis of \mathbb{R}^n and P_d^n is the convex hull of $0, de_1, \dots, de_n$.

Regular Triangulations of F. Knudsen. In the third chapter of [KKMSD73], F. Knudsen introduces a family of primitive convex triangulations of $(P_d^n)_{n,d \geq 1}$ called *regular triangulations*. For all $n, d \geq 1$, the regular triangulation R_d^n of P_d^n is obtained by cutting P_d^n by the following hyperplanes:

$$H_k^{i,j} := \left\{ \sum_{l=i}^j x_l = k \right\},$$

for all $1 \leq i \leq j \leq n$, and all $1 \leq k \leq d-1$. The Lemma 2.12 p.126 of [KKMSD73] implies that R_d^n is never ρ -uniform when n is bigger than 2 and d is big enough.

³¹The number of edges adjacent to the vertex.

Viro Triangulations. In Definition 5.3 p.43 in [Che23b] we described a method of construction of triangulations of P_{d+1}^{n+1} from a triangulation of P_d^{n+1} and of P_{d+1}^n : If K is a triangulation of P_d^{n+1} and L is a triangulation of P_{d+1}^n , $K + L$ is the unique triangulation of P_{d+1}^{n+1} for which:

1. The restriction of $K + L$ to $P_{d+1}^{n+1} \cap \{x_{n+1} \geq 1\}$ is the translation of K ;
2. For all integers $0 \leq i \leq n$, the restriction of the triangulation $K + L$ to the join of simplices $[e_{n+1}, e_{n+1} + de_1, \dots, e_{n+1} + de_i] * [(d+1)e_i, \dots, (d+1)e_n]$, is the join of the restriction of the translation of K to $[e_{n+1}, e_{n+1} + de_1, \dots, e_{n+1} + de_i]$ with the restriction of L to $[(d+1)e_i, \dots, (d+1)e_n]$.

If both K and L are primitive then so is $K + L$. By Proposition 5.5 p.44 of [Che23b] the same holds for the convexity. Then we recursively defined the Viro triangulations $(V_d^n)_{n,d \geq 1}$ by the formulæ:

$$\begin{cases} V_1^n = P_1^n & \forall n \geq 1 \\ V_{d+1}^{n+1} = V_d^{n+1} + V_{d+1}^n & \forall n, d \geq 1. \end{cases}$$

This process can be generalised. Let $L = (L_d)_{d \geq 1}$ be a sequence of primitive triangulations of $(P_d^n)_{d \geq 1}$ we denote by $(V_d(L))_{d \geq 1}$ the sequence of triangulations of $(P_d^{n+1})_{d \geq 1}$ defined by the formulæ:

$$\begin{cases} V_1(L) = P_1^n \\ V_{d+1}(L) = V_d(L) + L_{d+1} \quad \forall d \geq 1. \end{cases}$$

Proposition 5.1. *Let $L = (L_d)_{d \geq 1}$ be a sequence of primitive triangulations of $(P_d^{2n})_{d \geq 1}$. Let d be an even integer. If L_d has a $2n$ -simplex that contains only interior vertices, then $V_D(L)$ is not ρ -uniform for all $D \geq d + 1$.*

Proof. Let d be such an integer and σ^{2n} be a $2n$ -simplex of L_d whose vertices are all interior. Using Proposition 2.13, we see that $V_{d+1}(L)$ does not fail to be ρ -uniform at σ^{2n} if and only if the two apices of the two $(2n+1)$ -simplices adjacent to σ^{2n} are congruent modulo 2. Indeed, since σ^{2n} has an odd number of vertices, and $\sum_{\beta \in \sigma^{2n}} \rho_{\sigma^{2n}}(\beta)$ vanishes, $\rho_{\sigma^{2n}}$ can only be constant to 0. Now the two apices of the two $(2n+1)$ -simplices adjacent to σ^{2n} are $2e_{2n+1}$ and $(d+1)e_{2n}$. Since d is even they are not congruent and $V_{d+1}(L)$ fails to be ρ -uniform at σ^{2n} . Since all the triangulations $(V_D(L))_{D \geq d+1}$ contain $V_{d+1}(L)$, they all fail to be ρ -uniform at σ^{2n} . \square

Itenberg-Viro Triangulations In the last example we always perform the first subdivision of the prism $P_{d+1}^{n+1} \cap \{0 \leq x_{n+1} \leq 1\}$ in the same order. However, we can also reverse the order of the joins.

Definition 5.2. If K is a triangulation of P_{d+1}^{n+1} and L is a triangulation of P_{d+1}^n we define $K + (-1)L$ as the unique triangulation of P_{d+1}^{n+1} for which:

1. The restriction of $K + (-1)L$ to $P_{d+1}^{n+1} \cap \{x_{n+1} \geq 1\}$ is the translation of K ;
2. For all integers $0 \leq i \leq n$, the restriction of the triangulation $K + (-1)L$ to the join of simplices $[e_{n+1} + de_i, \dots, e_{n+1} + de_n] * [0, (d+1)e_1, \dots, (d+1)e_i]$ is the join of the restriction of the translation of K to $[e_{n+1} + de_i, \dots, e_{n+1} + de_n]$ with the restriction of L to $[0, (d+1)e_1, \dots, (d+1)e_i]$.

As a direct consequence of the definition if both K and L are primitive then so is $K + (-1)L$. The same holds for the convexity by a small adaptation of Proposition 5.5 p.44 in [Che23b]. For all $n, d \geq 1$, let Tr_d^n denote the set of primitive triangulations of P_d^n . For all subset I of $\{0, \dots, n\}$ of cardinality $(k + 1)$, there is a map $\pi_I : \text{Tr}_d^n \rightarrow \text{Tr}_d^k$ that sends a triangulation K to its restriction on the convex hull of $\{de_i : i \in I\}$, where $e_0 = 0$ and the order of the vertices is preserved. Another direct consequence of the definition is that if $K \in \text{Tr}_d^{n+1}$, $L \in \text{Tr}_{d+1}^n$, and $I \subset \{0, \dots, n\}$ then $\pi_{I \cup \{n+1\}}(K + (-1)^\eta L)$ equals $\pi_{I \cup \{n+1\}}(K) + (-1)^\eta \pi_I(L)$ for all $\eta \in \mathbb{F}_2$. Following [Ite97], we define the *Itenberg-Viro triangulations* $(IV_d^n)_{n,d \geq 1}$ by the formulæ:

$$\begin{cases} IV_d^1 \text{ is the only primitive triangulation of } P_d^1 \\ IV_1^n = P_1^n & \forall n \geq 1 \\ IV_{d+1}^{n+1} = IV_d^{n+1} + (-1)^d IV_{d+1}^n & \forall n, d \geq 1. \end{cases}$$

An example of such triangulation is illustrated in Figure 17. As in the case of Viro triangulations this process can be generalised from a sequence of primitive triangulations $(L_d)_{d \geq 1}$ of $(P_d^n)_{d \geq 1}$ to a sequence $(IV_d(L))_{d \geq 1}$ of triangulations of $(P_d^{n+1})_{d \geq 1}$ by the formulæ:

$$\begin{cases} IV_1(L) = P_1^{n+1} \\ IV_{d+1}(L) = V_d(L) + (-1)^d L_{d+1} \quad \forall d \geq 1. \end{cases}$$

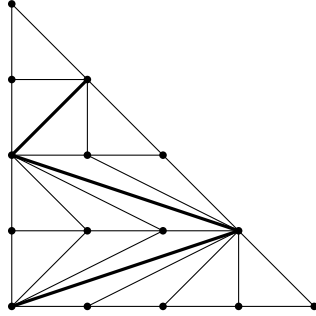


Figure 17: The triangulation IV_4^2 .

Proposition 5.3. *The triangulations L_d are ρ -uniform for all $d \geq 1$ if and only if the triangulations $IV_d(L)$ are ρ -uniform for all $d \geq 1$.*

Proof. Let K be a triangulation of P_d^{n+1} . We denote, for all $0 \leq q \leq n + 1$, the set of q -simplices of K having at least one vertex in the interior $K - \partial K$ by $A_q(K)$. If L is a triangulation of P_{d+1}^n , we denote, for all $0 \leq q \leq n + 1$, by $A_q^\pm(K; L)$ the set of q -simplices of $K + e_{n+1}$ that belong to $A_q(K \pm L)$ but not to $A_q(K) + e_{n+1}$. Let us denote by \underline{K} the restriction of K to the bottom face $[0; de_1; \dots; de_n]$ of P_d^{n+1} . We have:

$$A_n(K + L) = (A_n(K) + e_{n+1}) \amalg A_n^+(K; L) \amalg ([A_{n-1}(\underline{K}) + e_{n+1}] * (d + 1)e_n).$$

Indeed, the n -simplices of $A_n(K + L)$ that are neither translates of simplices of $A_n(K)$ nor in $A_n^+(K; L)$ come from joins of simplices of \underline{K} with simplices of L . By construction the only such simplices not having all of their vertices in $\partial(K + L)$ belong to the subdivision of the

join $(e_{n+1} + [0; de_1; \dots; de_n]) * (d+1)e_n$. Since $(d+1)e_n$ belongs to $\partial(K+L)$, the remaining n -simplices are in the set $([A_{n-1}(\underline{K}) + e_{n+1}] * (d+1)e_n)$. Likewise we have:

$$A_n(K-L) = (A_n(K) + e_{n+1}) \amalg A_n^-(K; L) \amalg ([A_{n-1}(\underline{K}) + e_{n+1}] * 0).$$

Now let L' be a triangulation of P_{d+2}^n . We have:

$$A_n^-(K+L; L') = (A_n(L) + e_{n+1}) \amalg ([A_{n-1}(L) + e_{n+1}] * 2e_{n+1}),$$

and:

$$A_n^+(K-L; L') = (A_n(L) + e_{n+1}) \amalg ([A_{n-1}(L) + e_{n+1}] * (2e_{n+1} + de_n)).$$

Therefore, we have:

$$\begin{aligned} A_n(K+L-L') &= (A_n(K+L) + e_{n+1}) \amalg A_n^-(K+L; L') \amalg ([A_{n-1}(L) + e_{n+1}] * 0) \\ &= (A_n(K+L) + e_{n+1}) \amalg (A_n(L) + e_{n+1}) \\ &\quad \amalg ([A_{n-1}(L) + e_{n+1}] * 2e_{n+1}) \amalg ([A_{n-1}(L) + e_{n+1}] * 0), \end{aligned}$$

and:

$$\begin{aligned} A_n(K-L+L') &= (A_n(K-L) + e_{n+1}) \amalg A_n^+(K-L; L') \\ &\quad \amalg ([A_{n-1}(L) + e_{n+1}] * (d+2)e_n) \\ &= (A_n(K-L) + e_{n+1}) \amalg (A_n(L) + e_{n+1}) \\ &\quad \amalg ([A_{n-1}(L) + e_{n+1}] * (2e_{n+1} + de_n)) \\ &\quad \amalg ([A_{n-1}(L) + e_{n+1}] * (d+2)e_n), \end{aligned}$$

for $\underline{K} \pm \underline{L}$ equals L . If we assume $K+L$ to be ρ -uniform, $K+L-L'$ is ρ -uniform as well if and only if the simplices of $(A_n(L) + e_{n+1})$, $([A_{n-1}(L) + e_{n+1}] * 2e_{n+1})$, and $([A_{n-1}(L) + e_{n+1}] * 0)$ satisfy the requirement of Definition 2.22. Let σ^n be a simplex of the first set. It bounds the two $(n+1)$ -simplices $\sigma^n * 2e_{n+1}$ and $\sigma^n * 0$. By Proposition 2.13 we have:

$$\sum_{\alpha \in \sigma^n} |\rho_{\sigma^n}(\alpha)|\alpha = 2e_{n+1} \pmod{2},$$

and therefore, $\rho_{\sigma^n} = 0$. Now let σ^{n-1} be a simplex of $A_{n-1}(L) + e_{n+1}$. It bounds two n -simplices of $L + e_{n+1}$, σ_+^n and σ_-^n , so $\sigma^{n-1} * 2e_{n+1}$ bounds the two $(n+1)$ -simplices of $K+L-L'$, $\sigma_+^n * 2e_{n+1}$ and $\sigma_-^n * 2e_{n+1}$. Let α_{\pm} denote the vertex of σ_{\pm}^n opposite to σ^{n-1} . From Proposition 2.13, we have:

$$\sum_{\beta \in \sigma^n} |\rho_{\sigma^{n-1} * 2e_{n+1}}(\beta)|\beta = \alpha_+ - \alpha_- \pmod{2}.$$

Since the difference $\alpha_+ - \alpha_-$ belongs to the hyperplane $\{x_{n+1} = 0\}$, the number $|\rho_{\sigma^{n-1} * 2e_{n+1}}(2e_{n+1})|$ vanishes and the restriction of $|\rho_{\sigma^{n-1} * 2e_{n+1}}|$ to σ^{n-1} equals $|\rho_{\sigma^{n-1}}^L|$ (the function $|\rho|$ of the triangulation L). The vertex $2e_{n+1}$ belongs to the boundary $\partial(K+L-L')$. It implies that $\rho_{\sigma^{n-1} * 2e_{n+1}}$ is constant on the interior vertices $I(\sigma^{n-1} * 2e_{n+1})$ and meets the requirements of

Definition 2.22 if and only if $\rho_{\sigma^{n-1}}^L$ is constant on the interior³² vertices $I(\sigma^{n-1})$. The exact same reasoning applies to the simplices of the third set $[A_{n-1}(L) + e_{n+1}] * 0$ and we find that $K + L - L'$ is ρ -uniform if and only if K and L are. We can also prove that $K - L + L'$ is ρ -uniform if and only if K and L are by the same arguments. The only difference in this case is the two apices with which we suspend the horizontal n -simplices σ^n of L . They are $2e_{n+1} + de_n$ and $(d+2)e_n$. Their difference is also 0 modulo 2 so ρ_{σ^n} vanishes as in the case $K + L - L'$. The proposition follows from the construction of the triangulations $(IV_d(L))_{d \geq 1}$. \square

Remark 5.4. In the last proof we showed that if σ^n is a n -simplex of the triangulation $IV_d(L)$ of \mathbb{P}_d^{n+1} written as a link $\alpha * \sigma^{n-1}$ with σ^{n-1} in a floor L_k then the numerical value $|\rho_{\alpha * \sigma^{n-1}}(\beta)|$ is zero for $\beta = \alpha$ and equals $\rho_{\sigma^{n-1}}^{L_k}(\beta)$ otherwise.

Proposition 5.5. *For all $n, d \geq 1$, for all $\varepsilon \in C^0(IV_d^n; \mathbb{F}_2)$, the number $\ell(\mathbb{R}X_\varepsilon)$ is at least equal to $\lfloor \frac{n-1}{2} \rfloor$.*

Proof. This is Theorem 5.9 p.45 of [Che23b] for Itenberg-Viro triangulations. The crucial point in the proof of this theorem was Proposition 5.8 of the same article: the fact that the restriction of the Viro triangulation to the face opposite to 0 is still a Viro triangulation. However, the same is true for the Itenberg-Viro triangulation, hence the proposition holds. \square

Theorem 5.6. *For all $n \geq 2$, and all $d \geq 1$, the Itenberg-Viro triangulation IV_d^n is simply integrable. In particular, the Harnack distribution:*

$$h^n(x_1, \dots, x_n) := \sum_{i < j} x_i x_j,$$

is a solution of (14).

Proof. We will prove, by recursion on n , that h^n is a solution of (14) for all $d \geq 1$. Theorem 4.19 ensures that the property is true for $n = 2$. Let us now assume that the property is true for some $n \geq 2$. Let $1 \leq d \leq D$ be two integers. The n -simplices of $IV_D^{n+1} \setminus \mathcal{B}_0 IV_D^{n+1}$ that have non-empty intersection with the horizontal hyperplane $\{x_{n+1} = D - d\}$ are n -simplices of IV_d^n and cones over $(n-1)$ -simplices of $IV_d^n \setminus \mathcal{B}_0 IV_d^n$. If d is even the apices of the cones are $(D - d - 1)e_{n+1}$ and $(D - d + 1)e_{n+1}$. And if d is odd, they are $(D - d - 1)e_{n+1} + (d + 1)e_n$ and $(D - d + 1)e_{n+1} + (d - 1)e_n$. Either way we showed³³ that $\rho_{\alpha * \sigma^{n-1}}$, with $\sigma^{n-1} \in IV_d^n$, is zero on the apex α of the cone and that the numerical value of $|\rho_{\alpha * \sigma^{n-1}}|$ equals the numerical value $|\rho_{\sigma^{n-1}}^{IV_d^n}|$ of IV_d^n . As a consequence, $|D^2 h_{\alpha * \sigma^{n-1}}^{n+1}|$ equals $|D^2(h^{n+1}|_{IV_d^n})_{\sigma^{n-1}}|$. However, $h^{n+1}|_{IV_d^n}$ only differs from h^n by an affine function. Hence, by recursion hypothesis $|D^2(h^{n+1}|_{IV_d^n})_{\sigma^{n-1}}|$ equals $|\rho_{\sigma^{n-1}}^{IV_d^n}|$, thus $|D^2 h_{\alpha * \sigma^{n-1}}^{n+1}|$ equals $|\rho_{\alpha * \sigma^{n-1}}|$. Now, if σ^n belongs to IV_d^n , the two $(n+1)$ -simplices of IV_D^{n+1} that are adjacent to σ^n have their apices congruent modulo 2. It implies that ρ_{σ^n} vanishes. These two $(n+1)$ -simplices are identical modulo 2, and the value of h^{n+1} is the same on the two apices since they have congruent coordinates. Hence, $D^2 h_{\sigma^n}^{n+1}$ vanishes as well. Therefore, we proved that h^{n+1} is a solution of (14). \square

Remark 5.7. The hypersurface $\mathbb{R}X_{h^n}$ of $\mathbb{R}\mathbb{P}_d^n$ has a maximal number of connected components but also intersects all the toric divisors of $\mathbb{R}\mathbb{P}_d^n$ along T-hypersurfaces that have a maximal number of connected components.

³²interior inside L i.e. in $L - \partial L$.

³³c.f. Remark 5.4.

References

- [ARS21] Charles Arnal, Arthur Renaudineau, and Kris Shaw. Lefschetz Section Theorems for Tropical Hypersurfaces. *Annales Henri Lebesgue*, 4:1347–1387, 2021.
- [Ber10] Benoit Bertrand. Euler Characteristic of Primitive T-Hypersurfaces and Maximal Surfaces. *Journal of the Institute of Mathematics of Jussieu*, 9(1):1–27, 2010.
- [Bjö84] Anders Björner. Posets, Regular CW Complexes and Bruhat Order. *European Journal of Combinatorics*, 5(1):7–16, 1984.
- [BLdMR22] Erwan Brugallé, Lucía López de Medrano, and Johannes Rau. Combinatorial Patchworking: Back from Tropical Geometry. To appear in *Transaction of the AMS*, arXiv:2209.14043, 2022.
- [Che23a] Jules Chenal. A Poincaré-Lefschetz Theorem for Cellular Cosheaves and an Application to the Tropical Homology of Orbifold Toric Varieties. hal-04164269, jul. 2023.
- [Che23b] Jules Chenal. Poincaré Duality, Degeneracy, and Real Lefschetz Property for T-Hypersurfaces. hal-04349997, dec. 2023.
- [Ehr62] Eugène Ehrhart. Sur les polyèdres rationnels homothétiques à n dimensions. *Comptes rendus de l'académie des Sciences*, 254(1):616–618, january 1962.
- [Ful93] William Fulton. *Introduction to Toric Varieties*. Annals of mathematics studies, no. 131.; William H. Roever lectures in geometry. Princeton University Press, 1993.
- [Haa97] Bertrand Haas. *Real Algebraic Curves and Combinatorial Constructions*. PhD thesis, Universität Basel, 1997.
- [IKMZ19] Ilia Itenberg, Ludmil Katzarkov, Grigory Mikhalkin, and Ilia Zharkov. Tropical Homology. *Mathematische Annalen*, 374(1):963–1006, 2019.
- [Ite97] Ilia Itenberg. Topology of Real Algebraic T-surfaces. *Revista Matemática de la Universidad Complutense de Madrid*, 10(Supl.):131–152, 1997.
- [IV96] Ilia Itenberg and Oleg Viro. Patchworking Algebraic Curves Disproves the Ragsdale Conjecture. *The Mathematical Intelligencer*, 18(4):19–28, 1996.
- [JRS18] Philipp Jell, Johannes Rau, and Kristin Shaw. Lefschetz (1,1)-theorem in tropical geometry. *Épjournal de Géométrie Algébrique*, Volume 2, 2018.
- [KKMSD73] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal Embeddings 1*. Lecture Notes in Mathematics. Springer Berlin, Heidelberg, 1 edition, 1973.
- [KS10] Masaki Kashiwara and Pierre Schapira. *Sheaves on Manifolds*, chapter Constructible sheaves, pages 320–359. Grundlehren der mathematischen Wissenschaften. Springer Berlin, Heidelberg, 1 edition, 12 2010.

- [Mac71] I. G. MacDonald. Polynomials Associated with Finite Gell-Complexes. *Journal of the London Mathematical Society*, s2-4(1):181–192, 2023/09/16 1971.
- [McM77] P. McMullen. Valuations and Euler-Type Relations on Certain Classes of Convex Polytopes. *Proceedings of the London Mathematical Society*, s3-35(1):113–135, 2023/10/05 1977.
- [Ris93] Jean-Jacques Risler. Construction d’hypersurfaces réelles [d’après Viro]. In *Séminaire Bourbaki : volume 1992/93, exposés 760-774*, number 216 in Astérisque. Société mathématique de France, 1993. talk:763.
- [RS23] Arthur Renaudineau and Kris Shaw. Bounding the Betti Numbers of Real Hypersurfaces Near the Tropical Limit. *Annales Scientifiques de l’École Normale Supérieure*, 56(3):945–980, 2023.
- [She85] Allan D. Shepard. *A Cellular Description of the Derived Category of a Stratified Space*. PhD thesis, Brown University, 05 1985.
- [Vir06] Oleg Viro. Patchworking Real Algebraic Varieties. arXiv:math/0611382, 2006.
- [Zie95] Günter M. Ziegler. *Lectures on Polytopes*. Graduate Texts in Mathematics. Springer New York, NY, 1 edition, 1995.