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Local stabilization conditions for discrete-time switched affine systems

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Abstract

This paper deals with the switching law design problem for discrete-time switched affine systems. While the existing literature addresses the case of globally stabilizing switching laws, in discrete time, many switched affine systems can only be stabilized locally. In this work, methods are proposed for the design of switching control laws ensuring the local uniform ultimate boundedness of the equilibrium. The approach is based on the existence of a stabilizing continuous state feedback for a bilinear system. Estimates of the domain of attraction and of the invariant attractive neighborhood of the equilibrium are given. Numerical methods that allows a simple implementation of the proposed results are given.

Key words: Discrete-time switched affine systems; local stability; uniform ultimate boundedness; bilinear systems.

1 Introduction

Switched systems [17] represent a class of hybrid systems [11]. They consist of a family of subsystems and a switching law that selects the subsystem to be activated. A very common problem when studying switching systems is the switching control law design. This problem is particularly challenging for the case of switched affine systems [18], [4], [13] that have many practical applications, for example in the domain of power converters [7], [5], [1], [21]. This paper deals with the problem of the switching control law design for the local stabilization of discrete-time switched affine systems. State dependent switching laws are considered. This means finding conditions to select the subsystem to be activated according to the system's state in order to steer the trajectories towards a neighborhood of an equilibrium. An important challenge related to this stabilization objective is that the equilibrium is generally different from the ones of the subsystems. In addition, in discrete-time [6], [2], for switched affine systems generally only practical stabilization is possible: the stabilization cannot be addressed to the equilibrium point but only to a neighborhood containing it or to a limit cycle [9] (this is usually called chattering). Results concerning the practical stabilization of discrete-time switched affine systems can be found in [8], [10], [19], [20]. Various design methods and characterizations of the practical stability domain are proposed. In [8], a stabilizing min-type switching state feedback function and an ellipsoidal characterization of the attractive invariant domain are provided based on general quadratic Lyapunov functions. An improvement of the estimate of the attractive invariant domain can be found in [10], where switching control law design conditions are given based on Lyapunov-Metzler inequalities. In [20], the positive invariant set is characterized by the union of several potentially disjoint ellipsoids. The estimate of the invariant set and the stabilizing switching law are given based on the existence of multiple shifted Lyapunov functions. In [9], conditions for stabilization towards a limit cycle are given. The results are based on the existence of a Schur product of system matrices. The extension to the case of uncertain switched affine systems has been recently proposed in [19]. All the existing results provide global stabilization conditions. However, there exist classes of switched affine systems that

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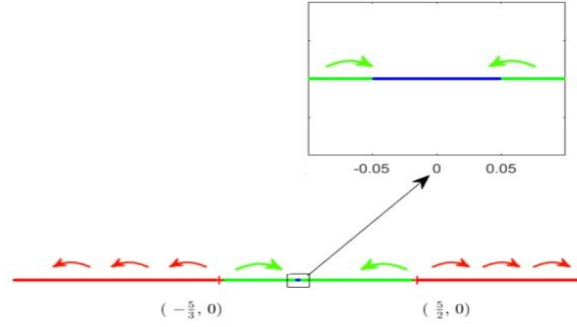


Fig. 1. Behavior of the system trajectories

can be only locally stabilized. Let us recall the following motivating example that we have presented in a preliminary version of this paper in [16]. Consider the scalar system

$$x_{k+1} = a_{\sigma(x_k)}x_k + b_{\sigma(x_k)}, \quad \sigma(x_k) \in \{1, 2\}.$$

where $a_1 = 1.02$, $b_1 = -0.05$, $a_2 = 1.03$, $b_2 = 0.05$. Clearly, since all the values of the state coefficient are greater than 1, the system cannot be globally stabilized. No matter what the switching law is, if $|x_k| > \max\{\frac{5}{2}, \frac{5}{3}\}$ then $|x_{k+1}| > |x_k|$. Now, the following switching law is considered: $\sigma(x_k) = 1$ when $x_k \geq 0$ and $\sigma(x_k) = 2$, otherwise.

One may easily check in simulations that when $|x_0| < \min\{\frac{5}{2}, \frac{5}{3}\}$ the solutions of the system are bounded (see Figure 1 for an illustration). By simple computations, one may show that if $|x_k| < 1.666$ and $|x_k| > 0.024$ then $|x_{k+1}| < |x_k|$, that is the system can be locally practically stabilized. The existing literature cannot address the example under study. For this reason, in this work, we are interested in the local practical stabilization problem.

A switching control law design that ensures the local practical stability is provided based on the existence of a stabilizing state feedback controller for a bilinear system. Estimates of the domain of attraction and of the attractive invariant set are given as ellipsoids. The approach is based on tools related to the study of systems with input constraints [14], [22],[3]. With respect to the preliminary version of the article in [16], the approach is generalized in several directions. First, in this article we take into account switched systems with state matrices and affine terms that switch in the same way, while in the conference version independent switching laws have been considered. Second, the assumptions considered here are less restrictive. Third, numerically tractable methods are presented here.

The paper is organized as follows: In Section 2, the system under study is presented. In Section 3, preliminaries are given. Then, the main results are presented in Section 4, followed by numerically implementable conditions in Section 5. In Section 6, numerical examples that illustrate the results are provided. The paper ends with a conclusion in Section 7.

Notations: For a given matrix $P \in \mathbb{R}^{n \times n}$, $P \succ 0$ (resp. $P \prec 0$, $P \succeq 0$, $P \preceq 0$) indicates it is positive definite (resp. negative definite, positive semi-definite, negative semi-definite). For a symmetric positive definite matrix P and a positive scalar γ , $\mathcal{E}(P, \gamma)$ denotes the ellipsoid $\mathcal{E}(P, \gamma) = \{x \in \mathbb{R}^n : x^T P x \leq \gamma\}$. $[u]_i$ denotes the i -th element of the vector u . $\text{conv}(S)$ denotes the convex hull of a set S and $\text{int}(S)$ its interior. For $N \in \mathbb{N}$, Δ_N is the set $\Delta_N = \left\{ \delta \in \mathbb{R}^N : \delta_i \geq 0, \forall i \in \{1, \dots, N\}, \sum_{j=1}^N \delta_j = 1 \right\}$. For a scalar x , $\lceil x \rceil$ denotes the smallest integer larger than or equal to x .

2 System description

Consider the following switched affine system

$$x^+ = \bar{A}(u)x + \bar{b}(u). \quad (1)$$

where $u \in \mathbb{R}^m$ is a switching input taking values in the finite set of vectors $\mathcal{V} = \{v_1, \dots, v_N\} \subset \mathbb{R}^m$. The matrix $\bar{A} : \mathcal{V} \rightarrow \mathbb{R}^{n \times n}$ and the affine term $\bar{b} : \mathcal{V} \rightarrow \mathbb{R}^n$ are switching according to the values of the input u . Here, they take the following form

$$\bar{A}(u) = A_0 + \sum_{j=1}^m N_j [u]_j, \quad \bar{b}(u) = \sum_{j=1}^m b_j [u]_j, \quad (2)$$

where $A_0 \in \mathbb{R}^{n \times n}$, and for $j \in \{1, \dots, m\}$, $N_j \in \mathbb{R}^{n \times n}$, $b_j \in \mathbb{R}^n$, are known matrices. Moreover, we assume that $0 \in \text{int}(\text{conv}(\mathcal{V}))$.

Objective: find a switching control $\kappa_s : \mathbb{R}^n \rightarrow \mathcal{V}$ such that system (1),(2) with

$$u = \kappa_s(x) \quad (3)$$

is locally uniformly ultimately bounded in a neighborhood of the origin. The precise mathematical notion of local uniform ultimate boundedness that we consider will be formalized later in the paper.

System description (1),(2) is very general in the sense that, even though in the existing literature switched affine systems are formulated differently, they can be transformed to the form (1),(2) by an adequate change of coordinates.

Consider for example the case of systems [12] of the form

$$z^+ = \bar{A}(\theta)z + \bar{b}(\theta), \quad (4)$$

with $\bar{A}(\theta) = \bar{A}_0 + \sum_{i=1}^m \theta_i \bar{A}_i$, $\bar{b}(\theta) = \sum_{i=1}^m \theta_i \bar{b}_i$. and $\theta_i \in \{0, 1\}$, $i \in \{1, 2, \dots, m\}$. Assume that there exists $(z^*, \theta^*) \in \mathbb{R}^n \times (0, 1)^m$ such that $(I - \bar{A}(\theta^*)) z^* = \bar{b}(\theta^*)$. Then, by performing the change of variable, $x = z - z^*$, $v = \theta - \theta^*$, system (4) can be transformed to a system of the form (1) with

$$\mathcal{V} = \left\{ v \in \mathbb{R}^m : [v]_i \in \{-\theta_i^*, 1 - \theta_i^*\} \right\}.$$

The problem of stabilizing (4) to a neighborhood of z^* becomes a problem of stabilization of system (1) to a neighborhood of the origin. The approach presented in this paper can also be used for switched affine systems in the classical form

$$z^+ = \sum_{i=1}^N \theta_i (A_i z + b_i), \quad (5)$$

with

$$\theta \in \left\{ \delta \in \mathbb{R}^N : \delta_i \in \{0, 1\}, i \in \{1, \dots, N\}, \sum_{j=1}^N \delta_j = 1 \right\}.$$

Following similar arguments to the ones used for the continuous-time case in [13], system (5) can be written in the form (1).

3 Preliminaries

In this section, we present some preliminary definitions and basic assumptions.

Definition 1 A sequence $\{\phi_k\}_{k \in \mathbb{N}}$ is called a solution to (1), (3) originating from $\phi_0 \in \mathbb{R}^n$ if

$$\phi_{k+1} = \bar{A}(\kappa_s(\phi_k))\phi_k + \bar{b}(\kappa_s(\phi_k)), \quad \forall k \in \mathbb{N}. \quad (6)$$

In order to study the local practical stabilization properties of the switched affine system (1),(2), the notion of local uniform ultimate boundedness is considered.

Definition 2 (adapted from [3]) Let \mathcal{S} and \mathcal{D} be bounded neighborhoods of the origin such that $\mathcal{S} \subset \mathcal{D}$. System (1),(3) is said to be locally uniformly ultimately bounded in \mathcal{S} with a domain of attraction \mathcal{D} if there exists $k > 0$ such that the solutions $\{\phi_k\}_{k \in \mathbb{N}}$ of (1),(3), with initial conditions $\phi_0 \in \mathcal{D}$ satisfy $\phi_k \in \mathcal{S}$ for all $k \geq k$.

Consider the notation

$$G(x) = \begin{bmatrix} N_1 x + b_1 & \cdots & N_m x + b_m \end{bmatrix}. \quad (7)$$

The results presented in this work are based on the following reformulation of system (1),(2)

$$x^+ = A_0 x + G(x)u. \quad (8)$$

This is a bilinear system with a control taking values in a finite set \mathcal{V} . The contributions in this paper are based on the following assumption

Assumption 3 Consider system (1), (2) and the notation $G(x)$ given in (7). We assume that there exists two scalars $\beta \in (0, 1)$ and $\gamma > 0$, a matrix $P \in \mathbb{R}^{n \times n}$ such that $P = P^T \succ 0$, and a continuous function $\kappa_c : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $\kappa_c(0) = 0$, such that the quadratic function $V(x) = x^T P x$ satisfies, for all $x \in \mathcal{E}(P, \gamma)$

$$\kappa_c(x) \in \text{conv}(\mathcal{V}),$$

$$V(A_0 x + G(x)\kappa_c(x)) \leq (1 - \beta)V(x). \quad (9)$$

Remark 4 Assumption 3 implies the existence of a continuous stabilizing controller κ_c ensuring the decay of a quadratic Lyapunov function along the solutions of a bilinear system of the form (8), where the input u is unconstrained (u may take any values in \mathbb{R}^n , it is not restricted to \mathcal{V}). In what follows, we will show how the existence of the continuous stabilizer κ_c can be used in order to derive a switching controller of the form (3).

4 Main Result

Based on Assumption 3, the following theorem provides a switching control (3) that ensures the local uniform ultimate boundedness of system (1),(2). Ellipsoidal estimates of the domain of attraction and of the uniform ultimate boundedness set are also given.

Theorem 5 Consider system (1),(2) and the notation $G(x)$ given in (7). Let there exist scalars $\gamma > 0$, $0 < \beta < 1$, a matrix $P = P^T \succ 0$ and a function $\kappa_c : \mathbb{R}^n \rightarrow \mathbb{R}^m$, such that Assumption 3 holds. Given α such that $0 < \alpha < \beta$, and $c > 0$ such that

$\max_{v \in \mathcal{V}, x \in \mathcal{E}(P, \gamma)} v^T G^T(x) P G(x) v \leq c$, (10) denote $\rho = \frac{c}{\beta - \alpha}$. If $\rho < \gamma$, then the closed loop system (1),(2) with the switching control

$$\kappa_s(x) \in \arg \min_{v \in \mathcal{V}} x^T A_0^T P G(x) v \quad (11)$$

is locally uniformly ultimately bounded in $\mathcal{E}(P, \rho)$ with a domain of attraction $\mathcal{E}(P, \gamma)$. Moreover, any solution $\{\phi_k\}_{k \in \mathbb{N}}$ of (1),(2) with $\phi_0 \in \mathcal{E}(P, \gamma)$ with the control (11) reaches $\mathcal{E}(P, \rho)$ in at most τ steps with

$$\tau = \left\lceil \frac{\ln\left(\frac{\rho}{\gamma}\right)}{\ln(1 - \alpha)} \right\rceil.$$

Proof of Theorem 5. Theorem 5 states that every trajectory of system (1) starting in the ellipsoid $\mathcal{E}(P, \gamma)$ converges to the ellipsoid $\mathcal{E}(P, \rho)$ with a decay rate $1 - \alpha$ when the control law (11) is applied. The proof of this theorem is constructed step by step and is divided into several intermediate results (Propositions 6, 7, 8 and 9 below). First, based on the existence of the stabilizing continuous controller κ_c that ensures the decay of $V(x)$ for the bilinear system (8), Proposition 6 proposes a control law, of the form (11) that ensures the decay of the Lyapunov function

$V(x) = x^T P x$ in a set contained in $\mathcal{E}(P, \gamma)$. Next, Proposition 7 presents an estimate of the ultimate boundedness set, which is always reached in an interval of time less than a bound that is given by Proposition 9. Proposition 8 shows the invariance of the proposed estimate of the uniform ultimate bounded set.

Proposition 6 *Suppose Assumption 3 holds for some matrix P , some scalars γ, β and a function κ_c . Let α such that $0 < \alpha < \beta$. Denote*

$$\Gamma = \{x \in \mathbb{R}^n : \kappa_c^T(x) G^T(x) P G(x) \kappa_c(x) + (\beta - \alpha) x^T P x - v^T G^T(x) P G(x) v < 0, \forall v \in \mathcal{V}\}. \quad (12)$$

System (1),(2) with the switching control law given by (11)

satisfies

$$V(A_0 x + G(x) \kappa_s(x)) - V(x) \leq -\alpha V(x), \quad (13)$$

whenever $x \in \mathcal{E}(P, \gamma)$ and $x \notin \Gamma$.

Proof: See Appendix A. □

The previous Proposition shows that the switching control (11) ensures the decay of the function V inside $\mathcal{E}(P, \gamma)$ when x is outside the region Γ . An estimate of the uniform ultimate boundedness set containing Γ is given as an ellipsoid in the following proposition.

Proposition 7 *Let Assumption 3 hold for some matrix P , some scalars γ, β and a function κ_c . Let α such that $0 < \alpha < \beta < 1$. Let $c > 0$ such that*

$$\max_{v \in \mathcal{V}, x \in \mathcal{E}(P, \gamma)} v^T G^T(x) P G(x) v \leq c.$$

If $\frac{c}{\beta - \alpha} < \gamma$, then, system (1),(2) with the control law (11)

satisfies

$$V(A_0 x + G(x) \kappa_s(x)) - V(x) \leq -\alpha V(x), \quad (14)$$

for all x such that $x \in \mathcal{E}(P, \gamma)$ and $x \notin \mathcal{E}(P, \frac{c}{\beta - \alpha})$.

Proof: See Appendix A. □

It follows from the previous propositions that trajectories starting in $\mathcal{E}(P, \gamma)$ are converging to $\mathcal{E}(P, \rho)$, with $\rho = \frac{c}{\beta - \alpha}$, if $\rho = \frac{c}{\beta - \alpha} < \gamma$. However, Proposition 7 does not say anything about the behavior of the system's solutions inside $\mathcal{E}(P, \rho)$. One may think that trajectories in $\mathcal{E}(P, \rho)$ can leave it. The following Proposition shows that the ellipsoid $\mathcal{E}(P, \rho)$ is positive-invariant for (1),(2) with the switching control (11).

Proposition 8 *Consider system (1),(2) such that Assumption 3 holds for some matrix P , some scalars γ, β and a function κ_c . Given α such that $0 < \alpha < \beta < 1$, let $c > 0$ such that*

$$\max_{v \in \mathcal{V}, x \in \mathcal{E}(P, \gamma)} v^T G^T(x) P G(x) v \leq c.$$

and $\rho = \frac{c}{\beta - \alpha}$. If $\rho \leq \gamma$, the ellipsoid $\mathcal{E}(P, \rho)$ is positive-invariant for the trajectories of system (1) with the control law (11).

Proof: See Appendix A. □

The following proposition shows that $\mathcal{E}(P, \rho)$ is always reached in finite time by all the trajectories of (1),(2) starting in $\mathcal{E}(P, \gamma)$ with a switching control (11). An upper bound of the convergence time is given.

Proposition 9 Suppose Assumption 3 holds for some matrix P , some scalars γ, β and a function κ_c . Given α such that $0 < \alpha < \beta < 1$. Consider $c > 0$ such that

$$\max_{v \in \mathcal{V}, x \in \mathcal{E}(P, \gamma)} v^T G^T(x) P G(x) v \leq c.$$

Let $\rho = \frac{c}{\beta - \alpha}$. If $\rho < \gamma$, let $\tau = \left\lceil \frac{\ln(\frac{\rho}{\gamma})}{\ln(1 - \alpha)} \right\rceil$. Then, every solution $\{\phi_k\}_{k \in \mathbb{N}}$ of system (1),(2) with $\phi_0 \in \mathcal{E}(P, \gamma)$ with the switching law κ_s as in (11), satisfy $\phi_k \in \mathcal{E}(P, \rho)$ for all $k \geq \tau$.

Proof: See Appendix A. □

Remark 10 Theorem 5 provides a switching control that ensures the local uniform ultimate boundedness of system (1),(2) with a guaranteed domain of attraction as the ellipsoid $\mathcal{E}(P, \gamma)$. Every trajectory starting in $\mathcal{E}(P, \gamma)$ is converging, with a known decay rate, to the uniform ultimate boundedness set, which results in a finite time convergence of this set (an upper bound of this convergence time is given). For the ellipsoidal estimation of the uniform ultimate boundedness set $\mathcal{E}(P, \rho)$ (estimate of the chattering set), the theorem assumes the existence of a scalar c such that (5) holds. Note that such a scalar always exists. This is because the set $\{x \in \mathbb{R}^n : x^T P x \leq \gamma\}$ is compact and $G(x)$ is continuous with respect to x . Then, the level set ρ (related to the ellipsoidal estimate of the uniform ultimate boundedness set) can be computed, provided that $\frac{c}{\beta} < \gamma$. Note that since the approach is based on a quadratic Lyapunov function, the estimate of the domain of attraction might be conservative.

5 Numerical Implementation

In this section, we provide numerically tractable results for control design.

Proposition 11 Consider system (1),(2) and vectors h_1, \dots, h_{n_h} such that

$$\text{conv}(\mathcal{V}) = \{y \in \mathbb{R}^m : h_i^T y \leq 1, \forall i \in \{1, \dots, n_h\}\}. \quad (15)$$

Denote $\Theta = [b_1 \ \dots \ b_m]$, $A_i = A_0 + \sum_{j=1}^m N_j [v_i]_j$, $\forall i \in \{1, \dots, N\}$, and

$$M_v = \left[\sum_{j=1}^m N_j [v]_j \quad \sum_{j=1}^m b_j [v]_j \right], \quad \forall v \in \mathcal{V}.$$

Given $\beta > 0$ and $R = R^T \in \mathbb{R}^{n \times n} \succ 0$, assume there exists $Q \in \mathbb{R}^{n \times n}$, $Q = Q^T \succ 0$, $\Upsilon \in \mathbb{R}^{m \times n}$ and $\xi > 0$ that satisfy

$$\begin{bmatrix} \xi R & I \\ I & Q \end{bmatrix} \succeq 0, \quad (16a)$$

$$\begin{bmatrix} Q & \Upsilon^T h_i \\ h_i^T \Upsilon & I \end{bmatrix} \succeq 0, \quad \forall i \in \{1, \dots, n_h\}, \quad (16b)$$

$$\begin{bmatrix} (1 - \beta)Q & (A_j Q + \Theta \Upsilon)^T \\ A_j Q + \Theta \Upsilon & Q \end{bmatrix} \succeq 0, \quad \forall j \in \{1, \dots, N\}. \quad (16c)$$

For Q satisfying (16), denote $P = Q^{-1}$. For $i \in \{1, \dots, N\}$, consider positive scalars μ_{v_i} and c_{v_i} solutions to

$$M_{v_i}^T P M_{v_i} + \begin{bmatrix} -\mu_{v_i} P & 0 \\ 0 & -c_{v_i} + \mu_{v_i} \end{bmatrix} \preceq 0. \quad (17)$$

For a scalar α such that $0 < \alpha < \beta$, if $\frac{c}{\beta-\alpha} < 1$, with

$$c = \max\{c_v : v \in \mathcal{V}\},$$

then system (1),(2) with the control

$$\kappa_s(x) \in \arg \min_{v \in \mathcal{V}} x^T A_0^T Q^{-1} G(x) v \quad (18)$$

is locally uniformly ultimately bounded in $\mathcal{E}(Q^{-1}, \frac{c}{\beta-\alpha})$ with a domain of attraction $\mathcal{E}(Q^{-1}, 1)$. In addition, $\mathcal{E}(R, \frac{1}{\xi}) \subseteq \mathcal{E}(Q^{-1}, 1)$. Moreover, any solution $\{\phi_k\}_{k \in \mathbb{N}}$ of (1),(2) with the control (18), when $\phi_0 \in \mathcal{E}(Q^{-1}, 1)$, reaches the set $\mathcal{E}(Q^{-1}, \frac{c}{\beta-\alpha})$ in at most τ steps with

$$\tau = \left\lceil \frac{\ln\left(\frac{c}{\beta-\alpha}\right)}{\ln(1-\alpha)} \right\rceil. \quad (19)$$

Proof: The proof of this Proposition follows from the results of Theorem 5. It is constructed in four steps detailed below. Assume that conditions (16) hold true.

Step 1: We show that condition (16a) guarantees that $\mathcal{E}(R, \frac{1}{\xi}) \subseteq \mathcal{E}(Q^{-1}, 1)$. If for some $\xi > 0$ condition (16a) holds then $\xi R - Q^{-1} \geq 0 \iff \xi R \geq Q^{-1}$. This means that $x^T \xi R x \geq x^T Q^{-1} x, \forall x \in \mathbb{R}^n$, thus $x^T R x < \frac{1}{\xi} \implies x^T Q^{-1} x < 1$, which means $\mathcal{E}(R, \frac{1}{\xi}) \subseteq \mathcal{E}(Q^{-1}, 1)$.

Step 2: For matrices Υ and Q satisfying (16), let $P = Q^{-1}$ and $\Pi = \Upsilon Q^{-1}$. This step consists in showing that conditions (16b) imply that

$$\Pi x \in \text{conv}(\mathcal{V}) \quad \text{when} \quad x \in \mathcal{E}(P, 1).$$

Let us define the following set:

$$\mathcal{C}_v(\Pi) = \{x \in \mathbb{R}^n : \Pi x \in \text{conv}(\mathcal{V})\}. \quad (20)$$

Showing that $\Pi x \in \text{conv}(\mathcal{V})$ when $x \in \mathcal{E}(P, 1)$ is the same as proving that $\mathcal{E}(P, 1) \subset \mathcal{C}_v(\Pi)$. We proceed, by using the polytopic representation of $\text{conv}(\mathcal{V})$ given in (15), $\Pi x \in \text{conv}(\mathcal{V})$ is then satisfied when

$$h_i^T \Pi x \leq 1, \quad \forall i \in \{1, \dots, n_h\}.$$

It is sufficient for $\mathcal{E}(P, 1) \subset \mathcal{C}_v(\Pi)$, that $\forall i \in \{1, \dots, n_h\}$ the following is satisfied

$$x^T \Pi^T h_i h_i^T \Pi x \leq 1 \quad \text{whenever} \quad x^T P x \leq 1. \quad (21)$$

The latter is satisfied if:

$$\Pi^T h_i h_i^T \Pi - P \preceq 0, \quad \forall i \in \{1, \dots, n_h\}. \quad (22)$$

Using the Schur complement

$$\begin{bmatrix} P & \Pi^T h_i \\ h_i^T \Pi & I \end{bmatrix} \succeq 0, \quad \forall i \in \{1, \dots, n_h\}. \quad (23)$$

Pre- and post-multiplying by $\begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix}$ gives

$$\begin{bmatrix} P^{-1} & P^{-1} \Pi^T h_i \\ h_i^T \Pi P^{-1} & 1 \end{bmatrix} \succeq 0, \quad \forall i \in \{1, \dots, n_h\}.$$

which is the same as (16b). Therefore, when (16b) holds true, $\Pi x \in \text{conv}(\mathcal{V})$ for all $x \in \mathcal{E}(P, 1)$.

Step 3: This step consists in showing that conditions (16c), guarantee that the function $V(x) = x^T P x$, with $P = Q^{-1}$, satisfies

$$V(A_0 x + G(x) \Pi x) \leq (1 - \beta) V(x), \quad \forall x \in \mathcal{E}(P, 1).$$

Pre- and post-multiplying (16c) by $\begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}$ gives

$$\begin{bmatrix} (1-\beta)P & (A_j + \Theta\Pi)^T \\ A_j + \Theta\Pi & P^{-1} \end{bmatrix} \succeq 0, \quad \forall j \in \{1, \dots, N\}.$$

Let $\delta \in \Delta_N$. Since $[\delta]_i \geq 0, \forall i \in \{1, \dots, N\}$,

$$\begin{bmatrix} (1-\beta)P & (\sum_{j=1}^N [\delta]_j A_j + \Theta\Pi)^T \\ \sum_{j=1}^N [\delta]_j A_j + \Theta\Pi & P^{-1} \end{bmatrix} \succeq 0. \quad (24)$$

Applying Schur complement, the latter implies that

$$(1-\beta)P - \left(\sum_{j=1}^N [\delta]_j A_j + \Theta\Pi\right)^T P \left(\sum_{j=1}^N [\delta]_j A_j + \Theta\Pi\right) \succeq 0. \quad (25)$$

Note that

$$\begin{aligned} \sum_{j=1}^N [\delta]_j A_j &= A_0 + \sum_{j=1}^N [\delta]_j \sum_{i=1}^m N_i [v_j]_i \\ &= A_0 + \sum_{i=1}^m N_i \sum_{j=1}^N [\delta]_j [v_j]_i. \end{aligned}$$

Let $x \in \mathcal{E}(P, 1)$. Based on Step 2 of this proof, Πx is in $\text{conv}(\mathcal{V})$. Then there exists $\alpha(x) \in \Delta_N$ such that $\Pi x = \sum_{j=1}^N [\alpha(x)]_j v_j$. For $x \in \mathcal{E}(P, 1)$, we multiply (25) from left and right by x^T and x respectively. Note that, (25) holds also for $\delta = \alpha(x)$. We obtain the following

$$(1-\beta)x^T P x - \left(A_0 x + \sum_{i=1}^m N_i x \sum_{j=1}^N [\alpha(x)]_j [v_j]_i + \Theta \Pi x\right)^T P \left(A_0 x + \sum_{i=1}^m N_i x \sum_{j=1}^N [\alpha(x)]_j [v_j]_i + \Theta \Pi x\right) > 0. \quad (26)$$

Consider the following partition of Π :

$$\Pi = \left[\Pi_1^T \ \dots \ \Pi_m^T \right]^T, \quad \text{with } \Pi_i \in \mathbb{R}^{1 \times n}.$$

$\Pi x = \sum_{j=1}^N [\alpha(x)]_j v_j$ means that

$$\Pi_i x = \sum_{j=1}^N [\alpha(x)]_j [v_j]_i.$$

Then, we can write (26) as

$$(1-\beta)x^T P x - \left(A_0 x + \sum_{i=1}^m N_i x \Pi_i x + \Theta \Pi x\right)^T P \left(A_0 x + \sum_{i=1}^m N_i x \Pi_i x + \Theta \Pi x\right) \geq 0. \quad (27)$$

Note that

$$\begin{aligned} \sum_{i=1}^m N_i x \Pi_i x + \Theta \Pi x &= \sum_{i=1}^m (N_i x + b_i) \Pi_i x \\ &= G(x) \Pi x. \end{aligned}$$

Thus (27) is the same as

$$\left(A_0 x + G(x) \Pi x\right)^T P \left(A_0 x + G(x) \Pi x\right) - (1-\beta)x^T P x \leq 0. \quad (28)$$

The latter is the same as

$$V(A_0x + G(x)\Pi x) \leq (1 - \beta)V(x), \quad \text{for } x \in \mathcal{E}(P, 1).$$

To sum up, the conditions (16) imply that, in addition to $\mathcal{E}(R, \frac{1}{\xi}) \subseteq \mathcal{E}(P, 1)$, whenever $x \in \mathcal{E}(P, 1)$, $\Pi x \in \text{conv}(\mathcal{V})$ and $V(A_0x + G(x)\Pi x) \leq (1 - \beta)V(x)$. Therefore, Assumption 3 holds with $\gamma = 1$, $P = Q^{-1}$, and $\kappa_c(x) = \Pi x$ with $\Pi = \Upsilon Q^{-1}$.

Step 4: In what follows we show that for every $v \in \mathcal{V}$, when conditions (17) are satisfied, then

$$v^T G^T(x) P G(x) v \leq c_v, \quad \forall x \in \mathcal{E}(P, 1).$$

Let $v \in \mathcal{V}$, suppose (17) is satisfied for $c_v > 0$, $\mu_v > 0$. Then, the following holds for all $x \in \mathcal{E}(P, 1)$,

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T M_v^T P M_v \begin{bmatrix} x \\ 1 \end{bmatrix} + \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} -\mu_v P & 0 \\ 0 & -c_v + \mu_v \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0.$$

The last inequality can be written as

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T M_v^T P M_v \begin{bmatrix} x \\ 1 \end{bmatrix} - c_v + \mu_v(1 - x^T P x) \leq 0. \quad (29)$$

Note that, for all $x \in \mathcal{E}(P, 1)$, $x^T P x \leq 1$ which means that $1 - x^T P x \geq 0$. Since, $\mu_v > 0$, by S-procedure (29) gives

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T M_v^T P M_v \begin{bmatrix} x \\ 1 \end{bmatrix} \leq c_v, \quad \text{for all } x \in \mathcal{E}(P, 1). \quad (30)$$

From the definition of M_v , for all $v \in \mathcal{V}$

$$G(x)v = M_v \begin{bmatrix} x \\ 1 \end{bmatrix}^T. \quad (31)$$

Then, for all $v \in \mathcal{V}$, one can write

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T M_v^T P M_v \begin{bmatrix} x \\ 1 \end{bmatrix} = v^T G^T(x) P G(x) v.$$

Then (30) gives that

$$v^T G^T(x) P G(x) v \leq c_v, \quad \forall x \in \mathcal{E}(P, 1).$$

(17) is satisfied for all $v \in \mathcal{V}$. Let $c = \max\{c_v : \forall v \in \mathcal{V}\}$, then it follows that

$$v^T G^T(x) P G(x) v \leq c, \quad \forall v \in \mathcal{V}, \quad \forall x \in \mathcal{E}(P, 1). \quad (32)$$

Assumption 3 is satisfied by conditions (16), as mentioned above, and the scalar c satisfies (32), then following Theorem 5 with some α such that $0 < \alpha < \beta$, the control (18) ensures the local uniform ultimate boundedness of system (1),(2) in $\mathcal{E}(P, \frac{c}{\beta - \alpha})$, with $\mathcal{E}(P, 1)$ the domain of attraction, provided that $\frac{c}{\beta - \alpha} < 1$. Moreover, following Proposition 9, $\mathcal{E}(P, \frac{c}{\beta - \alpha})$ is reached by all the trajectories in $\mathcal{E}(P, 1)$ in at most τ steps with τ given in (19). \square

Proposition 11 provides simple conditions that allow to find a switching law ensuring the local uniform ultimate boundedness of system (1),(2) with ellipsoidal characterizations of the domain of attraction and of the uniform ultimate boundedness set. First, based on the Fundamental Theorem of Polytopes, the existence of vectors h_1, \dots, h_{n_h} such that (15) holds is guaranteed. They describe the polytope whose vertices are the control vectors \mathcal{V} . These vectors can either be computed analytically or using existing numerical routines. The matrix R , which is called a shape reference set, is fixed by the designer. It is chosen in a manner to prioritize some specified directions of the domain of attraction. For example (from details about the shape reference set choice from [14]), when R is a diagonal matrix, the state corresponding to a smaller diagonal element is desired to have a larger range of variation in the domain

of attraction. The relative weightings at the diagonals of R can be determined from the physical operating range of the states. Also a value of β , which is related to the decay rate of the bilinear system with the continuous controller is chosen. By having these parameters fixed, conditions (16) are Linear Matrix Inequalities with variables Q , Υ and ξ . Therefore, the provided conditions can be verified numerically with powerful convex optimization tools. Some elements concerning the limitations in terms of applicability of the existing solvers can be found in [23]. These LMIs guarantee that β is a decay rate of the quadratic Lyapunov function $V(x) = x^T Q^{-1} x$ that satisfies Assumption 3. The level set of the quadratic Lyapunov function $\mathcal{E}(P, 1)$ contains the ellipsoid $\mathcal{E}(R, \frac{1}{\xi})$. In the next step, conditions (17) are solved. If $\frac{c}{\beta} < 1$, with c the maximum of c_{v_1}, \dots, c_{v_N} solutions of (17), then $\mathcal{E}(P, 1)$ is invariant and there exists an α (at least a sufficiently small one) such that $\frac{c}{\beta-\alpha} < 1$. We can conclude that system (1),(2) is locally uniformly ultimately bounded in $\mathcal{E}(P, \frac{c}{\beta-\alpha})$ with a domain of attraction $\mathcal{E}(P, 1)$ and that trajectories are converging to $\mathcal{E}(P, \frac{c}{\beta-\alpha})$ with a decay rate α .

Remark 12 *When solving LMIs (16) by minimizing ξ , the estimate of the domain of attraction $\mathcal{E}(P, 1)$ is maximized. Since minimizing ξ maximizes $\frac{1}{\xi}$, this allows to maximize the volume of the ellipsoid $\mathcal{E}(R, \frac{1}{\xi})$ that is contained in $\mathcal{E}(P, 1)$. In addition, different scenarios can be executed when solving LMIs (16) in order to find an adequate trade-off between a bigger estimate of the domain of attraction or a smaller estimate of the uniform ultimate boundedness set. This can be shown for example by looking at the influence of the parameter β on both sets. Larger values of β allow to have better (smaller) estimates of the positive-invariant ellipsoid $\mathcal{E}(P, \frac{c}{\beta-\alpha})$ (uniform ultimate boundedness set); however, the estimate of the domain of attraction is smaller. Moreover, $\frac{c}{\beta-\alpha} < \gamma$ cannot be guaranteed for any β . For this reason, a value of β can be found using a line search algorithm in order to find $\frac{c}{\beta-\alpha} < \gamma$ and to, deal with the trade-off related to the sets' volume. The parameter α on the other hand, influences the convergence and the volume of the uniform ultimate boundedness set. A larger value of α means a faster convergence of the uniform ultimate boundedness set (the attractive invariant ellipsoid $\mathcal{E}(P, \frac{c}{\beta-\alpha})$) but its volume is bigger.*

6 Numerical examples

Example 1. Consider the motivating example presented in the introduction. This example cannot be treated using the tools in the literature, since global stabilization is not possible (as mentioned above). The system can be written in the form (5), as

$$\begin{aligned} x^+ &= \theta_1(a_1x + b_1) + (1 - \theta_1)(a_2x + b_2), \\ &= ((a_1 - a_2)x + (b_1 - b_2))\theta_1 + a_2x + b_2. \end{aligned}$$

with $\theta_1 \in \{0, 1\}$. For $\theta_1^* = 0.5$, we have $b_1\theta_1^* + (1 - \theta_1^*)b_2 = 0$. By taking $u = \theta_1 - \theta_1^*$, $u \in \{v_1, v_2\}$ with $v_1 = -\theta_1^*$, $v_2 = 1 - \theta_1^*$. The system can be transformed to the form of system (1) with $A_0 = 1.025$, $N_1 = -0.01$, $b_1 = -0.10$, $\mathcal{V} = \{-0.5, 0.5\}$. The vectors h_1, h_2 that describe the polytope $\text{conv}(\mathcal{V})$ as in (15) are $h_1 = 2$, and $h_2 = -2$. Since the domain of attraction and the uniform ultimate bounded set depend on β , a line search is used in order to find a value of β that satisfies the conditions of the Proposition 11 while ensuring a trade-off between the sets' volume (maximization of the ratio between the two volumes). For $\beta = 0.02$, $R = 1$, and $\alpha = 0.001$, solving conditions (16) gives $P = 0.641$. In addition, from conditions (17), we find $c = 0.002$, which gives $\frac{c}{\beta-\alpha} = 0.107$. These results mean that the closed loop system is locally uniformly ultimately bounded in $\mathcal{E}(0.641, 0.107)$ with a domain of attraction $\mathcal{E}(0.641, 1)$. In other words, whenever $|x_0| \leq 1.248$, the trajectory $\{\phi_k\}_{k \in \mathbb{N}}$ is converging to the chattering ball $(-0.408, 0.408)$ (to be compared with 1.666 and 0.05 given in the introduction). On the other hand, for the same system with $\beta = 0.50$, the value $P = 41.707$ and $c = 0.107$ are found. This gives $\frac{c}{\beta-\alpha} = 0.215$. These results guarantee that the system is locally uniformly ultimately bounded in $\mathcal{E}(41.707, 0.215)$ with a domain of attraction $\mathcal{E}(41.707, 1)$. This means that, the trajectories starting in $(-0.154, 0.154)$ are converging to $(-0.071, 0.071)$. While a larger domain of attraction can be guaranteed when using a smaller value of β , the estimate of the ultimate boundedness set in this case, may be more conservative.

Example 2. In order to provide a graphical illustration of the method here presented, consider a two dimensional system of the form (1) obtained using an Euler discretization of the continuous-time switched affine system in [15]. Consider the system

$$\dot{x} = \bar{A}^c(u)x + \bar{b}^c(u)$$

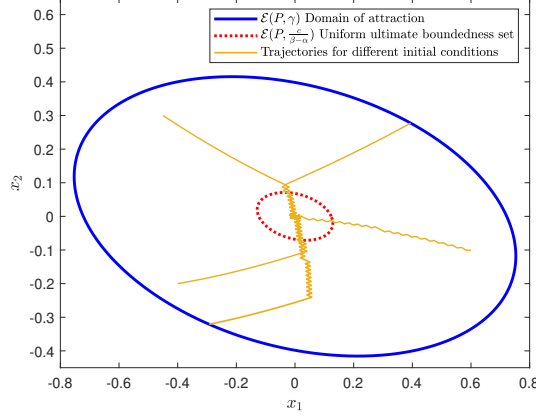


Fig. 2. Illustration of the domain of attraction ($\mathcal{E}(P, \gamma)$, blue bold line) and of the uniform ultimate boundedness set ($\mathcal{E}(P, \frac{c}{\beta-\alpha})$, estimate of the chattering set, dotted red line) for the system in *Example 2* with $\beta = 0.03$.

where $\bar{A}^c(u) = A_0^c + \sum_{j=1}^2 N_j^c[u]_j$, $\bar{b}^c(u) = \sum_{j=1}^2 b_j^c[u]_j$,

$$A_0^c = \begin{bmatrix} 1 & 2 \\ 4 & -5 \end{bmatrix}, N_1^c = \begin{bmatrix} 1 & -5 \\ 0.5 & 2 \end{bmatrix}, N_2^c = \begin{bmatrix} -1 & 5 \\ -0.5 & -2 \end{bmatrix},$$

$b_1^c = [15 \ -1]^T$, $b_2^c = [1 \ -5]^T$. The set of control vectors is given by

$$\mathcal{V} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}.$$

In continuous-time, each system mode is unstable (the maximum real part of the state matrix eigenvalues is always strictly greater than 0). We use an Euler discretization with a sampling period $T = 10^{-3}$. The discrete-time matrices are, then, given by

$$A_0 = I + TA_0^c, N_i = TN_i^c, b_i = Tb_i^c, i \in \{1, 2\}.$$

The eigenvalues of the discrete-time system matrices $A_i = A_0 + \sum_{j=1}^m N_j[v_i]_j$, $i \in \{1, \dots, 4\}$ are outside the unit circle. The polytope $\text{conv}(\mathcal{V})$ in (15) is described by the following vectors: $h_1 = [1 \ 0]^T$, $h_2 = [-1 \ 0]^T$, $h_3 = [0 \ 1]^T$, $h_4 = [0 \ -1]^T$. Proposition 11 is used in order to design a control with guaranteed domain of attraction and uniform ultimate boundedness set. For $\beta = 0.03$ and R as the identity matrix, LMIs (16) are solved in combination with an optimization constraint aiming to minimize ξ , such that the ball $\mathcal{E}(R, \frac{1}{\xi})$, which is contained in the domain of attraction, is maximized. The solution

$$P = \begin{bmatrix} 1.919 & 0.991 \\ 0.991 & 6.306 \end{bmatrix}.$$

is found. LMIs (17) can then be solved and $c = 8.4916 \times 10^{-4}$ is found. By setting $\alpha = 0.001$, the corresponding estimate of the domain of attraction is $\mathcal{E}(P, \gamma)$ with $\gamma = 1$ and the attractive invariant set is $\mathcal{E}(P, \frac{c}{\beta-\alpha})$ with $\frac{c}{\beta-\alpha} = 0.029 < 1$ (see Figure 2 for an illustration). In addition, the maximal \mathbb{R}^2 ball that is contained in the domain of attraction has radius 0.391. We use a line search on β in order to find a good trade-off between the sets' estimates since they are both influenced by the value of β . For this example a larger value of β , for instance $\beta = 0.20$, provides

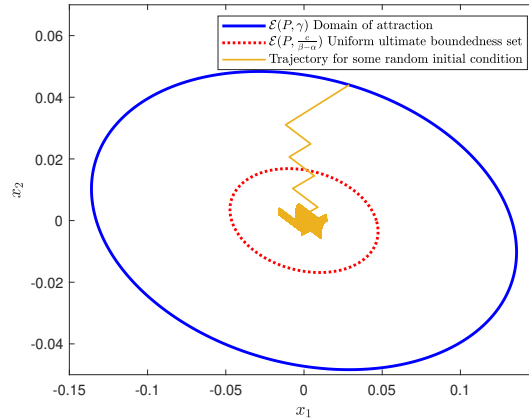


Fig. 3. Domain of attraction ($\mathcal{E}(P, \gamma)$, blue bold line) and uniform ultimate boundedness set ($\mathcal{E}(P, \frac{c}{\beta-\alpha})$, estimate of the chattering set, dotted red line) for the system in *Example 2* with $\beta = 0.2$.

a smaller domain of attraction (the maximal ball contained in it has radius 0.047) with

$$P = \begin{bmatrix} 56.720 & 33.784 \\ 33.784 & 447.636 \end{bmatrix}.$$

Also, $\frac{c}{\beta-\alpha} = 0.121$ is found. While the domain of attraction is smaller in this case, the estimate of the uniform ultimate boundedness set is better (it is closer to the domain observed in simulations - see Figure 3).

7 Conclusion

In this paper, stabilization conditions ensuring the local uniform ultimate boundedness of discrete-time switched affine systems are provided. A switching control design ensuring the attractive invariance of a set containing the equilibrium is proposed. In addition, ellipsoidal estimates of the domain of attraction and of the attractive invariant set are given. Moreover, Linear Matrix Inequalities that allow to find the conditions of the proposed switching control are given. In the future, better estimates of the domain of attraction and of the ultimately bounded set can be looked for, based on the existence of more complex classes of Lyapunov functions.

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A Proofs of Propositions

A.1 Proof of Proposition 6

When Assumption 3 is satisfied we have:

$$V(A_0x + G(x)\kappa_c(x)) - (1 - \beta)V(x) \leq 0, \quad \forall x \in \mathcal{E}(P, \gamma).$$

This is the same as

$$x^T A_0^T P A_0 x + 2x^T A_0^T P G(x)\kappa_c(x) + (\kappa_c(x))^T G^T(x) P G(x)\kappa_c - (1 - \beta)x^T P x \leq 0.$$

When $x^T P x \leq \gamma$, $\kappa_c(x) \in \text{conv}(\mathcal{V})$, which means that there exists $\alpha_i(x)$, $i \in \{1, \dots, N\}$, such that $\sum_{i=1}^N \alpha_i(x) = 1$ and $\kappa_c(x) = \sum_{i=1}^N \alpha_i(x)v_i$. Therefore, for all $x \in \mathcal{E}(P, \gamma)$,

$$\sum_{i=1}^N \alpha_i(x)(x^T A_0^T P A_0 x + 2x^T A_0^T P G(x)v_i - (1 - \beta)x^T P x + (\kappa_c(x))^T G^T(x) P G(x)\kappa_c(x)) \leq 0. \quad (\text{A.1})$$

Then, the control $\kappa_s(x) = \arg \min_{v \in \mathcal{V}} x^T A_0^T P G(x)v$ satisfies the following relation for all $x \in \mathcal{E}(P, \gamma)$

$$x^T A_0^T P A_0 x + 2x^T A_0^T P G(x)\kappa_s(x) + (\kappa_c(x))^T G^T(x) P G(x)\kappa_c(x) - (1 - \beta)x^T P x \leq 0. \quad (\text{A.2})$$

By adding and subtracting

$$((\kappa_s(x))^T G^T(x) P G(x)\kappa_s(x) + \alpha x^T P x)$$

to (A.2) we have the following, for all $x \in \mathcal{E}(P, \gamma)$,

$$\begin{aligned} & x^T A_0^T P A_0 x + 2x^T A_0^T P G(x)\kappa_s(x) + (\kappa_s(x))^T G^T(x) P G(x)\kappa_s(x) + \alpha x^T P x - x^T P x \\ & + (\kappa_c(x))^T G^T(x) P G(x)\kappa_c(x) + (\beta - \alpha)x^T P x - (\kappa_s(x))^T G^T(x) P G(x)\kappa_s(x) \leq 0. \end{aligned} \quad (\text{A.3})$$

This is the same as

$$V(A_0x + G(x)\kappa_s(x)) - V(x) + \alpha V(x) + (\beta - \alpha)V(x) + (\kappa_c(x))^T G^T(x)PG(x)\kappa_c(x) - (\kappa_s(x))^T G^T(x)PG(x)\kappa_s(x) \leq 0, \quad \forall x \in \mathcal{E}(P, \gamma).$$

The latter means that

$$V(A_0x + G(x)\kappa_s(x)) - V(x) \leq -\alpha V(x)$$

whenever

$$(\beta - \alpha)V(x) + (\kappa_c(x))^T G^T(x)PG(x)\kappa_c(x) - (\kappa_s(x))^T G^T(x)PG(x)\kappa_s(x) \geq 0.$$

and $x \in \mathcal{E}(P, \gamma)$. Since $\kappa_s(x) \in \mathcal{V}$, $\forall x \in \mathcal{E}(P, \gamma)$, and by the definition of Γ given (12) in Proposition 6, it follows from the two last conditions that

$$V(A_0x + G(x)\kappa_s(x)) - V(x) \leq -\alpha V(x)$$

when $x \in \mathcal{E}(P, \gamma)$ and $x \notin \Gamma$.

A.2 Proof of Proposition 7

From Proposition 6, system (1),(2) with the control (11) ensures the decay of the function $V(x) = x^T Px$ with a decay rate of $(1 - \alpha)$ for x satisfying $x^T Px < \gamma$ and

$$(\kappa_c(x))^T G^T(x)PG(x)\kappa_c(x) + (\beta - \alpha)x^T Px - v^T G^T(x)PG(x)v \geq 0, \quad \forall v \in \mathcal{V}.$$

The latter holds when $(\beta - \alpha)x^T Px \geq v^T G^T(x)PG(x)v$, $\forall v \in \mathcal{V}$. Since c satisfies

$$v^T G^T(x)PG(x)v \leq c, \quad \forall x \in \mathcal{E}(P, \gamma) \quad \text{and} \quad \forall v \in \mathcal{V},$$

then a sufficient condition for

$$V(A_0x + G(x)\kappa_s(x)) \leq (1 - \alpha)V(x)$$

to hold is $(\beta - \alpha)x^T Px \geq c$.

A.3 Proof of Proposition 8

In order to prove the invariance of the ellipsoid $\mathcal{E}(P, \rho)$ for system (1),(2) with the control (11), we show that for all x such that $x \in \mathcal{E}(P, \rho)$, we have $A_0x + G(x)\kappa_s \in \mathcal{E}(P, \rho)$ with κ_s as in (11).

Let $x \in \mathcal{E}(P, \rho)$, $x^+ = A_0x + G(x)\kappa_s(x)$ satisfies

$$\begin{aligned} (x^+)^T Px^+ &= (A_0x + G(x)\kappa_s(x))^T P(A_0x + G(x)\kappa_s(x)) \\ &= x^T A_0^T P A_0 x + 2x^T A_0^T P G(x)\kappa_s(x) + (\kappa_s(x))^T G^T(x)PG(x)\kappa_s(x). \end{aligned}$$

Denote

$$M(x) = x^T A_0^T P A_0 x + 2x^T A_0^T P G(x)\kappa_s(x) + (\kappa_c(x))^T G^T(x)PG(x)\kappa_c(x) - (1 - \beta)x^T Px.$$

Then, we can write

$$(x^+)^T Px^+ = M(x) - (\kappa_c(x))^T G^T(x)PG(x)\kappa_c(x) + (\kappa_s(x))^T G^T(x)PG(x)\kappa_s(x) + (1 - \beta)x^T Px.$$

From Assumption 3 and following the same steps that led to (A.2), we have $M(x) \leq 0$, $\forall x \in \mathcal{E}(P, \rho)$. Note that

$$(\kappa_c(x))^T G^T(x)PG(x)\kappa_c(x) \geq 0, \quad \forall x \in \mathcal{E}(P, \rho).$$

Thus, we have

$$(x^+)^T Px^+ \leq (\kappa_s(x))^T G^T(x)PG(x)\kappa_s(x) - c + x^T Px - \beta x^T Px + c, \quad \forall x \in \mathcal{E}(P, \rho).$$

Since $\max_{v \in \mathcal{V}, x \in \mathcal{E}(P, \gamma)} v^T G^T(x) P G(x) v \leq c$ and $\kappa_s(x) \in \mathcal{V}, \forall x \in \mathcal{E}(P, \rho)$, we have that

$$(\kappa_s(x))^T G^T(x) P G(x) \kappa_s(x) - c \leq 0.$$

Therefore $(x^+)^T P x^+ \leq x^T P x - \beta x^T P x + c, \forall x \in \mathcal{E}(P, \rho)$. Note that $\rho = \frac{c}{\beta - \alpha}$ implies $c < \rho\beta$ since $\alpha < \beta$. Therefore the following inequality holds

$$(x^+)^T P x^+ \leq (1 - \beta)x^T P x + \rho\beta.$$

Furthermore, $x \in \mathcal{E}(P, \rho)$ means that $x^T P x \leq \rho$, and this gives $(x^+)^T P x^+ \leq (1 - \beta)\rho + \rho\beta$, which leads to $(x^+)^T P x^+ \leq \rho$. From the last inequality, we have that $x^+ \in \mathcal{E}(P, \rho)$ whenever $x \in \mathcal{E}(P, \rho)$. Thus, the positive-invariance of the ellipsoid $\mathcal{E}(P, \rho)$ for the trajectories of system (1),(2) with the control law (11) is proven.

A.4 Proof of Proposition 9

Let $\{\phi_k\}_{k \in \mathbb{N}}$ a solution of (1),(2) with $\phi_0 \in \mathcal{E}(P, \gamma)$ subject to the control (11). Applying Proposition 7, the decrease of the Lyapunov Function in $\mathcal{E}(P, \gamma) \setminus \mathcal{E}(P, \rho)$ is guaranteed with a decay rate $(1 - \alpha)$. This means that $V(x^+) < (1 - \alpha)V(x)$ when $x \in \mathcal{E}(P, \gamma) \setminus \mathcal{E}(P, \rho)$. Then

$$V(\phi_1) < (1 - \alpha)V(\phi_0).$$

It follows that $V(\phi_2) < (1 - \alpha)^2 V(\phi_0)$, which gives by iteration (as long as $\phi_k \in \mathcal{E}(P, \gamma) \setminus \mathcal{E}(P, \rho)$)

$$V(\phi_k) < (1 - \alpha)^k V(\phi_0).$$

Since $\phi_0 \in \mathcal{E}(P, \gamma)$, we obtain $V(\phi_k) < (1 - \alpha)^k \gamma$. Since $(1 - \alpha)^k V(\phi_0)$ is exponentially decreasing to 0, there exists $\tau > 0$ such that $V(\phi_\tau) \in \mathcal{E}(P, \rho)$. Consider τ that satisfies $\tau \leq \min\{k \in \mathbb{N}^* : (1 - \alpha)^k \gamma \leq \rho\}$, that is $\tau \leq \left\lceil \frac{\ln(\frac{\rho}{\gamma})}{\ln(1 - \alpha)} \right\rceil$. Then since $\mathcal{E}(P, \rho)$ is positive-invariant according to Proposition 8, $\phi_k \in \mathcal{E}(P, \rho), \forall k \geq \tau$.