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# Minimizing Movements for Nonlinear Mean Curvature Flows

Daniele De Gennaro

## Abstract

In this note we prove existence for level-set solutions to a nonlinear variant of the mean curvature flow via the minimizing movements scheme à la Almgren-Taylor-Wang.

## 1 Introduction

In this note we study a nonlinear version of the anisotropic mean curvature flow (MCF in short) with forcing and mobility. In particular, given a continuous non decreasing function  $G : \mathbb{R} \rightarrow \mathbb{R}$ , we consider the flow of sets  $t \mapsto E_t$  formally governed by the evolution law

$$V(x, t) = \psi(\nu_{E_t}(x))G\left(-\kappa_{E_t}^\phi(x) + \mathbf{f}(t)\right), \quad \text{for all } x \in \partial^* E_t, t \geq 0, \quad (1.1)$$

law evol

where  $\psi, \phi$  are two anisotropies (with  $\psi$  usually called the mobility),  $\kappa_{E_t}^\phi$  denotes the  $\phi$ -curvature of the set of finite perimeter  $E_t$ ,  $\nu_{E_t}$  denotes the outer normal vector and  $\mathbf{f}$  is a forcing term constant in space. We are interested in showing that the mimizing movement approximation scheme produces discrete-in-time solutions that converge to the unique viscosity solution to (1.1) as the time-step parameter tends to 0.

The evolution law (1.1) is relevant from a numerical point of view, as suggested e.g. in [10, Remark 3.5]. For example, a truncation of the evolution speed is usually encoded in algorithms for the MCF, which would correspond to choosing  $G(s) = (-M) \vee s \wedge M$  in (1.1). Another interesting choice could be  $G(s) = -s^-$ , which amounts to consider a purely shrinking evolution. Moreover, evolution by powers of the mean curvature have been previously studied in the smooth or convex setting [11, 19, 3] and have been used to prove isoperimetric inequalities [20], or considered in the setting of image processing algorithms [2, 18]. In particular, in [2, Section 4.5] it is remarked that the evolution law (1.1) with  $G(s) = s^{\frac{1}{3}}$  and  $\phi = \psi = |\cdot|$  is particularly interesting as it is invariant under affine motions (isometries and rescalings).

In the present nonlinear setting, only two concept of solutions are currently available: smooth solutions (starting from smooth sets, in general existing only in a finite time span) and viscosity/level-set solutions, which are weak solutions of (1.1) defined globally in time and starting from any initial compact set. On the other hand, it is not clear whether a notion of “BV-solutions” in the spirit of [1, 16] can be properly defined.

Inspired by the techniques developed in our recent paper [5], we will show that the mimizing movement scheme à la Almgren-Taylor-Wang or Luckhaus-Sturzenhecker [1, 16] provides existence by approximation of a level set solution to the nonlinear MCF, under suitable smoothness assumptions on the quantities involved. From a technical point of view, the main difficulties arise in the case where  $G$  is bounded from above or below, as some tools heavily employed in the linear setting

are no longer available (see e.g. the commonly used reformulation (2.12)). Anyhow, by an approximation approach we can recover all the necessary results, which we then pair with the variational approach used in [5] in order to prove our main result.

To conclude, it would be interesting to study the much more challenging case where  $\phi$  is non smooth, i.e. the so-called crystalline case. In this setting the availability of the viscosity solutions of [13, 14] and the development of distribution solutions of [8, 7, 6] may suggest the possibility of a future investigation in this direction.

We have been informed that a work in progress on the same topic is being prepared in [4]. We want to point out that our result is more general as the authors work in the isotropic setting without mobility ( $\psi = \phi = |\cdot|$ ), require further regularity on the function  $G$  and assume that

$$\lim_{s \rightarrow \pm\infty} G(s) = \pm\infty,$$

which simplifies many arguments.

## 2 The minimizing movements scheme

### 2.1 Preliminaries

We start introducing some notations. We will use both  $B_r(x)$  and  $B(x, r)$  to denote the Euclidean ball in  $\mathbb{R}^N$  centered in  $x$  and of radius  $r$ ; with  $S^N$  we denote the sphere  $\partial B_1(0) \subseteq \mathbb{R}^N$ . If the ball is centered in zero, we simply write  $B_r$ . In the following, we will always speak about measurable sets and refer to a set as the union of all the points of density 1 of that set i.e.  $E = E^{(1)}$ . Moreover, if not otherwise stated, we implicitly assume that the function spaces considered are defined on  $\mathbb{R}^N$ , e.g.  $L^\infty = L^\infty(\mathbb{R}^N)$ . Moreover, we often drop the measure with respect to which we are integrating, if clear from the context. For  $\delta \in \mathbb{R}$  we denote

$$E_\delta = \{x \in \mathbb{R}^N : \text{sd}_E^\psi(x) \leq \delta\},$$

and use the notation  $E_{-\infty} := \emptyset, E_{+\infty} := \mathbb{R}^N$ .

**Definition 2.1.** We define anisotropy a function  $\psi : \mathbb{R}^N \rightarrow [0, +\infty)$  which is continuous, convex, even and positively 1-homogeneous. Moreover, there exists  $c_\psi > 0$  such that  $\forall p \in \mathbb{R}^N$  it holds

$$\frac{1}{c_\psi} |p| \leq \psi(p) \leq c_\psi |p|. \quad (2.1)$$

**Definition 2.2.** Given an anisotropy  $\phi$ , we define the  $\phi$ -perimeter ( $P_\phi$  in short) as follows

$$P_\phi(E) = \int_{\partial^* E} \phi(\nu_E(x)) \, d\mathcal{H}^{N-1}(x),$$

where  $\partial^* E$  denotes the reduced boundary of  $E$ .

We refer to [17] for all the basic references on sets of finite perimeter. By the bounds of (2.1) we immediately see that  $P(E)/c_\phi \leq P_\phi(E) \leq c_\phi P(E)$ . Furthermore, one can see that the following variational characterization of the  $\phi$ -perimeter holds

$$P_\phi(E) = \sup \left\{ \int_E \text{div } v : v \in C_c^1(\mathbb{R}^N; \mathbb{R}^N), \phi^\circ(v) \leq 1 \right\}, \quad (2.2)$$

where the polar function  $\phi^\circ$  of an anisotropy  $\phi$  is defined by

$$\phi^\circ(v) := \sup_{\phi(\xi) \leq 1} \xi \cdot v.$$

We recall the following identities holding for smooth anisotropies:  $\forall v, \xi \in \mathbb{R}^N$  it holds

$$\phi(v)\phi^\circ(\xi) \geq v \cdot \xi, \quad \phi^\circ(\nabla\phi(v)) = v, \quad \nabla\phi(v) \cdot v = \phi(v).$$

**Definition 2.3.** Given an anisotropy  $\psi$  and a set  $E$ , we define the  $\psi$ -distance from  $E$  as

$$\text{dist}_E^\psi(x) = \inf_{y \in E} \psi^\circ(x - y),$$

and the signed  $\psi$ -distance from  $E$  as

$$\text{sd}_E^\psi(x) = \text{dist}_E^\psi(x) - \text{dist}_{E^c}^\psi(x).$$

Note that (2.1) implies that

$$\frac{1}{c_\psi} \text{dist}_E(x) \leq \text{dist}_E^\psi(x) \leq c_\psi \text{dist}_E(x), \quad (2.3)$$

where  $\text{dist}_E$  denotes the Euclidean distance from the set  $E$ .

We focus on the following nonlinear evolution by anisotropic mean curvature with mobility of a family of sets  $\{E_t\}_{t \geq 0}$  starting from a set  $E_0 \subseteq \mathbb{R}^N$  which is either bounded or has bounded complement

$$V(x, t) = \psi(\nu_{E_t}(x))G\left(-\kappa_{E_t}^\phi(x) + \mathbf{f}(t)\right) \quad x \in \partial^* E_t, t \geq 0, \quad (2.4) \quad \boxed{\text{smooth law}}$$

where  $\nu_{E_t}(x)$  is the outer normal vector to  $E_t$  at  $x$  and  $\phi, \psi$  are two anisotropies, and under the following hypotheses on the quantity involved:

- $G : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous, non-decreasing function, with  $G(0) = 0$ ;
- $\mathbf{f} \in C_b^0(\mathbb{R})$ ;
- $\phi \in C^3$  and it is strictly convex.

We then set

$$\lim_{s \rightarrow -\infty} G(s) = -a \in [-\infty, 0], \quad \lim_{s \rightarrow +\infty} G(s) = b \in [0, +\infty].$$

Consider a function  $u : \mathbb{R}^N \times [0, +\infty) \rightarrow \mathbb{R}$  whose superlevel sets  $E_s := \{u(\cdot, t) \geq s\}$  evolve according to the nonlinear mean curvature equation (2.4). By classical computations (see e.g. [12]), the function  $u$  satisfies the equation

$$\partial_t u = |\nabla u|V(x, t) = \psi(\nabla u)G\left(-\nabla^2\phi(\nabla u) : \nabla^2 u + \mathbf{f}\right) =: -H(t, \nabla u, \nabla^2 u)$$

where we defined the Hamiltonian  $H : [0, +\infty) \times \mathbb{R}^N \setminus \{0\} \times S^N \rightarrow \mathbb{R}$  as

$$H(t, p, X) := -\psi(p)G\left(-\kappa^\phi(p, X) + \mathbf{f}(t)\right), \quad (2.5) \quad \boxed{\text{def hamiltonian}}$$

and  $\kappa^\phi(p, X)$  is defined as  $\kappa^\phi(p, X) = \nabla^2 \phi(p) : X$ . Therefore, one is let to solve the parabolic Cauchy problem in bounded time intervals  $[0, T]$ , for  $T > 0$ , given by

$$\begin{cases} \partial_t u + H(t, \nabla u, \nabla^2 u) = 0 & \text{on } \mathbb{R}^N \times [0, T] \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R}^N. \end{cases} \quad (2.6)$$

cauchy problem

Existence and uniqueness to (2.6) can be proved in the framework of viscosity solutions as done in [15]. Let us recall the notion of viscosity solution used in [15], starting by a family of auxiliary functions.

family F

**Definition 2.4.** The family  $\mathcal{F}$  is composed of smooth functions  $\ell \in C_c^\infty([0, +\infty))$  satisfying  $\ell(0) = \ell'(0) = \ell''(0) = 0$ ,  $\ell''(r) > 0$  in a neighborhood of 0,  $\ell$  constant in  $(0, M)^c$  for some  $M > 0$  (depending on  $\ell$ ), and such that

$$\lim_{p \rightarrow 0} \frac{\ell'(|p|)}{|p|} H(t, p, \pm I) = 0$$

holds uniformly in time.

**Remark.** We note that  $\mathcal{F} \neq \emptyset$ . Since for all  $t \in [0, +\infty)$ ,  $p \in \mathbb{R}^N$  it holds

$$G(-c/|p| - \|\mathbf{f}\|_\infty) \leq H(t, p, I) \leq H(t, p, -I) \leq G(c/|p| + \|\mathbf{f}\|_\infty),$$

for a suitable positive constant  $c = c(\phi)$ , one can repeat the construction used in [15, page 229] to show that  $\mathcal{F} \neq \emptyset$ .

With a slight abuse of notation, in the following we will say that a function is spatially constant outside a compact set even if the value of such constant is time-dependent.

def visco sol

**Definition 2.5.** Let  $\hat{z} = (\hat{x}, \hat{t}) \in \mathbb{R}^N \times (0, T)$  and let  $A \subseteq (0, T)$  be any open interval containing  $\hat{t}$ . We will say that  $\eta \in C^0(\mathbb{R}^N \times \bar{A})$  is admissible at the point  $\hat{z}$  if it is of class  $C^2$  in a neighborhood of  $\hat{z}$ , it is spatially constant outside a compact set, and, in case  $\nabla \eta(\hat{z}) = 0$ , the following holds: there exists  $\ell \in \mathcal{F}$  and  $\omega \in C^\infty([0, \infty))$  with  $\omega'(0) = 0$ ,  $\omega(r) > 0$  for  $r \neq 0$  and such that

$$|\eta(x, t) - \eta(\hat{z}) - \eta_t(\hat{z})(t - \hat{t})| \leq \ell(|x - \hat{x}|) + \omega(|t - \hat{t}|),$$

for all  $(x, t) \in \mathbb{R}^N \times A$ .

The notion of viscosity solution used in [15] is the following one.

**Definition 2.6.** An upper semicontinuous function  $u : \mathbb{R}^N \times [0, T] \rightarrow \mathbb{R}$ , constant outside a compact set, is a viscosity subsolution of the Cauchy problem (2.6) if  $u(\cdot, 0) \leq u_0$  and, for all  $z := (x, t) \in \mathbb{R}^N \times [0, T]$  and all  $C^\infty$ -test functions  $\eta$  such that  $\eta$  is admissible at  $z$  and  $u - \eta$  has a maximum at  $z$ , the following holds:

i) If  $\nabla \eta(z) = 0$ , then

$$\eta_t(z) \leq 0 \quad (2.7)$$

eq viscosa degen

ii) If  $\nabla \eta(z) \neq 0$ , then

$$\partial_t \eta(z) + H(\nabla \eta(z), \nabla^2 \eta(z)) \leq 0. \quad (2.8)$$

eq viscosa

A lower semicontinuous function  $u : \mathbb{R}^N \times [0, T] \rightarrow \mathbb{R}$ , constant outside a compact set, is a viscosity supersolution of the Cauchy problem (2.6) if  $-u$  is a viscosity supersolution to (2.6), with  $-u_0$  replacing  $u_0$ . Finally, a function  $u$  is a viscosity solution for the Cauchy problem (2.6) if it is both a subsolution and a supersolution of (2.6).

**Remark.** By classical arguments, one could assume that the maximum of  $u - \eta$  is strict in the definition of subsolution above (an analogous remark holds for supersolutions).

We thus recall the existence and uniqueness result proved in [15].

**Theorem 2.7.** *Given an initial datum  $u_0$  uniformly continuous and constant outside a compact set, the Cauchy problem (2.6) admits a unique viscosity solution. Moreover, if  $u, v$  are, respectively, a super- and subsolution to (2.6) satisfying  $u(\cdot, 0) \geq v(\cdot, 0)$ , then  $u(\cdot, t) \geq v(\cdot, t)$  for every  $t \in [0, T]$ .*

## 2.2 The minimizing movements scheme

In order to give the definition of the discrete scheme we introduce some notations. We set  $g$  as a selection of the set-valued inverse of  $G$ , that is  $g(x) \in G^{-1}(x)$  for every  $x \in (-a, b)$  and extend it setting  $g = -\infty$  for every  $x \leq -a$ ,  $g = +\infty$  for every  $x \geq b$ . Here, we extended  $G$  to  $[-\infty, +\infty]$  setting  $G(\pm\infty) = \lim_{x \rightarrow \pm\infty} G(x)$ . We assume also that  $g(0) = 0$ . Note that these definitions imply  $G \circ g = id$  in  $[-a, b]$ . Moreover,  $g$  is strictly increasing. In the following we will denote for  $k \in \mathbb{N}, h > 0$

$$f(kh) = \int_{kh}^{(k+1)h} \mathbf{f}(s) \, ds.$$

Given a bounded set of finite perimeter  $E$  and  $h > 0, t \in (0, +\infty)$  we define a functional on the measurable sets as

$$\mathcal{F}_{h,t}^E(F) = P_\phi(F) + \int_{E \Delta F} \left| g \left( \frac{\text{sd}_E^\psi}{h} \right) \right| - f([t/h]h) |F|, \quad (2.9) \quad \boxed{\text{def mathscr}}$$

where  $[\cdot]$  denotes the integer part.

**lemma existence** **Lemma 2.8.** *Let  $E$  be a bounded set of finite perimeter and  $h > 0, t \in [0, +\infty)$ . Then, there exist minimizers of  $\mathcal{F}_{h,t}^E$  and, denoting  $E'$  one such minimizer, it has the following properties: it is a bounded set of finite  $\phi$ -perimeter such that (up to negligible sets)*

$$E_{-ah} \subseteq E' \subseteq E_{bh}.$$

Moreover, there exist a maximal and a minimal minimizer (with respect to inclusion) of  $\mathcal{F}_{h,t}^E$ .

*Proof.* Since  $\mathcal{F}_{h,t}^E(E) < +\infty$  and  $\mathcal{F}_{h,t}^E \geq -\|f\|_\infty$ , by the standard method of calculus of variations we have the existence of a minimizer of  $\mathcal{F}_{h,t}^E$ . Since it has finite energy, it is straightforward to check that  $\text{sd}_E^\psi \in [-ah, bh]$  a.e. on  $E' \Delta E$ . If  $b < +\infty$  this clearly implies that  $E'$  is bounded; if  $b = +\infty$  a classic contradiction argument yields the same result. Finally, by classical arguments one shows that, if  $E'_1, E'_2$  are minimizers of  $\mathcal{F}_{h,t}^E$ , then so are  $E'_1 \cap E'_2, E'_1 \cup E'_2$ , implying the existence of a minimal and a maximal solution (see e.g. [9, Proposition 6.1]).  $\square$

For a given bounded set  $E$  and  $t \in (0, +\infty)$ , we thus denote

$$T_{h,t}^- E = \min \operatorname{argmin} \mathcal{F}_{h,t}^E, \quad T_{h,t}^+ E = \max \operatorname{argmin} \mathcal{F}_{h,t}^E, \quad (2.10)$$

incremental pb

where the minimum and maximum above are made with respect to inclusion. We will often denote  $T_{h,t} := T_{h,t}^-$ . We now prove some classical results following the lines of [16].

comparison principle

**Lemma 2.9** (Weak comparison principle). *Fix  $h > 0, t \in (0, +\infty)$  and assume that  $F_1, F_2$  are bounded sets with  $F_1 \subset\subset F_2$ . Then, for any two minimizers  $E_i$  of  $\mathcal{F}_{h,t}^{F_i}$  for  $i = 1, 2$ , we have  $E_1 \subseteq E_2$ . If, instead,  $F_1 \subseteq F_2$ , then we have that the minimal (respectively maximal) minimizer of  $\mathcal{F}_{h,t}^{F_1}$  is contained in the minimal (respectively maximal) minimizer of  $\mathcal{F}_{h,t}^{F_2}$ .*

*Proof.* Firstly, we assume  $F_1 \subset\subset F_2$ , Testing the minimality of  $E_1, E_2$  with their intersection and union, respectively, we obtain

$$\begin{aligned} P_\phi(E_1) + \int_{(E_1 \setminus E_2) \setminus F_1} g\left(\frac{\operatorname{sd}_{F_1}^\psi}{h}\right) + \int_{(E_1 \setminus E_2) \cap F_1} g\left(\frac{\operatorname{sd}_{F_1}^\psi}{h}\right) &\leq P_\phi(E_1 \cap E_2) + f([t/h]h) |E_1 \setminus E_2| \\ P_\phi(E_2) &\leq P_\phi(E_1 \cup E_2) + \int_{(E_1 \setminus E_2) \setminus F_2} g\left(\frac{\operatorname{sd}_{F_2}^\psi}{h}\right) + \int_{(E_1 \setminus E_2) \cap F_2} g\left(\frac{\operatorname{sd}_{F_2}^\psi}{h}\right) - f([t/h]h) |E_1 \setminus E_2|. \end{aligned}$$

Summing the two inequalities above and using the submodularity of the perimeter we get

$$\int_{(E_1 \setminus E_2) \setminus F_1} g\left(\frac{\operatorname{sd}_{F_1}^\psi}{h}\right) + \int_{(E_1 \setminus E_2) \cap F_1} g\left(\frac{\operatorname{sd}_{F_1}^\psi}{h}\right) \leq \int_{(E_1 \setminus E_2) \cap F_2} g\left(\frac{\operatorname{sd}_{F_2}^\psi}{h}\right) + \int_{(E_1 \setminus E_2) \setminus F_2} g\left(\frac{\operatorname{sd}_{F_2}^\psi}{h}\right). \quad (2.11)$$

ineq comp

Assume by contradiction that  $|E_1 \setminus E_2| > 0$ . Since  $\operatorname{sd}_{F_2}^\psi < \operatorname{sd}_{F_1}^\psi$  and by the strict monotonicity of  $g$ , we estimate the *rhs* of (2.11) by

$$\int_{(E_1 \setminus E_2) \setminus F_2} g\left(\frac{\operatorname{sd}_{F_2}^\psi}{h}\right) + \int_{(E_1 \setminus E_2) \cap F_2} g\left(\frac{\operatorname{sd}_{F_2}^\psi}{h}\right) < \int_{(E_1 \setminus E_2) \setminus F_2} g\left(\frac{\operatorname{sd}_{F_1}^\psi}{h}\right) + \int_{(E_1 \setminus E_2) \cap F_1} g\left(\frac{\operatorname{sd}_{F_1}^\psi}{h}\right)$$

and plug it in (2.11) to reach the desired contradiction. The other cases follow analogously, reasoning by approximation if  $F_1 \subseteq F_2$ .  $\square$

comparison function

**Lemma 2.10.** *Let  $c \in \mathbb{R}$ . Consider  $E$  a bounded set of finite perimeter and non-decreasing functions  $g_1, g_2 : \mathbb{R} \rightarrow \mathbb{R}$  such that  $g_1 < g_2$  in  $\mathbb{R} \setminus \{0\}$  and  $g_1(0) = g_2(0) = 0$ . Then, if  $E_i$  solves*

$$\min_F \left\{ P_\phi(F) + \int_{E \triangle F} |g_i(\operatorname{sd}_E^\psi(x))| dx + c|F| \right\}$$

for  $i = 1, 2$ , we have that  $E_2 \subseteq E_1$ . If  $g_1 \leq g_2$  instead, an analogous statement holds for the maximal and minimal solutions.

*Proof.* Denote  $g_i = g_i \circ \operatorname{sd}_E^\psi$  for  $i = 1, 2$  and assume by contradiction that  $|E_2 \setminus E_1| > 0$ . Reasoning as in Lemma 2.9, one gets

$$\int_{E_1 \triangle E} |g_1| + \int_{E_2 \triangle E} |g_2| \leq \int_{(E_1 \cup E_2) \triangle E} |g_1| + \int_{(E_1 \cap E_2) \triangle E} |g_2|.$$

Simplifying<sup>1</sup> the above expression and recalling that  $g_i \geq 0$  on  $E^c$ ,  $g_i \leq 0$  on  $E$ , we reach

$$0 \leq \int_{(E_2 \setminus E_1) \setminus E} (g_1 - g_2) + \int_{(E_2 \setminus E_1) \cap E} (g_1 - g_2) = \int_{E_2 \setminus E_1} (g_1 - g_2),$$

which implies the contradiction. The case  $g_1 \leq g_2$  follows by approximation.  $\square$

In the linear case ( $g = id$ ), minimizers of  $\mathcal{F}_{h,t}^E$  minimize also the functional

$$F \mapsto P_\phi(F) + \int_F \text{sd}_E^\psi / h - f([t/h]h)|F|. \quad (2.12) \quad \boxed{\text{pb equi inf}}$$

In the present setting, since  $\int_E g(\text{sd}_E^\psi)$  may be infinite in the case  $a < +\infty$ , we can not draw this conclusion straightforwardly. We can nonetheless recover the minimal and the maximal solution to (2.10) by means of a sequence of minimizers of a functional similar to (2.12).

**Corollary 2.11.** *Let  $E$  be a bounded set of finite perimeter and  $t \in (0, +\infty)$ ,  $h > 0$ . Then, there exists a sequence of uniformly bounded sets  $(E_n)_{n \in \mathbb{N}}$  such that  $E_n \nearrow T_{h,t}^- E$  and for any  $n \in \mathbb{N}$ ,  $E_n$  is a minimizer of*

$$F \mapsto P_\phi(F) + \int_F g\left(\frac{\text{sd}_E^\psi}{h}\right) \vee (-n) - f([t/h]h)|F| =: \mathcal{F}_{h,t}^{E,n}(F). \quad (2.13) \quad \boxed{\text{funct coroll}}$$

Analogously, there exists a sequence of uniformly bounded sets  $(E_n)_{n \in \mathbb{N}}$  such that  $E_n \searrow T_{h,t}^+ E$  in  $L^1$  and for any  $n \in \mathbb{N}$ ,  $E_n$  is a solution to

$$\min \left\{ P_\phi(F) + \int_{B_R \setminus F} g\left(\frac{\text{sd}_E^\psi}{h}\right) \wedge n - f([t/h]h)|F| : F \subseteq B_R \right\}, \quad (2.14) \quad \boxed{\text{funct coroll2}}$$

where  $T_{h,t}^\pm E \subseteq B_R$ .

*Proof.* We prove the statement for  $T_{h,t}^- E$ , the other case being analogous. Assume  $a < +\infty$  (otherwise the result follows by the boundedness of  $T_{h,t}^- E$ ). We set  $c = f([t/h]h)$ ,  $g_n := g(\text{sd}_E^\psi/h) \vee (-n)$ , and  $E' = T_{h,t}^- E$ . Consider the sequence of sets  $(E_n)_{n \in \mathbb{N}}$ , each being the minimal minimizer of  $\mathcal{F}_{h,t}^{E,n}$ . By the same arguments recalled above, note that there exists a constant  $R > 0$  such that  $E_n \subseteq B_R$  for all  $n \in \mathbb{N}$ . By Lemma 2.10, the sequence  $E_n$  is increasing as  $g_n \geq g_{n+1}$  and moreover  $E' \supseteq E_n$  as  $g \leq g_n$ . Therefore, one has that  $E_n \nearrow \tilde{E} := \bigcup_n E_n \subseteq E'$  and also  $\chi_{E_n \Delta E'} = |\chi_{E_n} - \chi_{E'}| \rightarrow \chi_{\tilde{E} \Delta E'}$  a.e. as  $n \rightarrow \infty$ . By lower semicontinuity of the perimeter and Fatou's lemma we get

$$\mathcal{F}_{h,t}^E(\tilde{E}) = P_\phi(\tilde{E}) - c|\tilde{E}| + \int_{\tilde{E} \Delta E'} |g(\text{sd}_E^\psi/h)| = P_\phi(\tilde{E}) - c|\tilde{E}| + \int_{\mathbb{R}^N} \liminf_{n \rightarrow \infty} (|g_n| \chi_{E_n \Delta E'})$$

<sup>1</sup>Noting that

$$\begin{aligned} E_1 \Delta E &= ((E_1 \setminus E_2) \setminus E) \cup ((E_1 \cap E_2) \setminus E) \cup ((E \setminus E_1) \setminus E_2) \cup ((E \cap E_2) \setminus E_1) \\ (E_1 \cup E_2) \Delta E &= (E_2 \setminus E_1 \setminus E) \cup ((E_1 \cap E_2) \setminus E) \cup ((E_1 \setminus E_2) \setminus E) \cup ((E \setminus E_1) \setminus E_2) \\ (E_1 \cap E_2) \Delta E &= ((E_2 \cap E_1) \setminus E) \cup ((E \setminus E_1) \setminus E_2) \cup ((E \cap E_1) \setminus E_2) \cup ((E \cap E_2) \setminus E_1). \end{aligned}$$



$$\leq \liminf_{n \rightarrow \infty} \left( P_\phi(E_n) - c|E_n| + \int_{E_n \Delta E} |g_n| \right).$$

Since  $E_n$  minimizes  $\mathcal{F}_{h,t}^{E,n}$  we get

$$\mathcal{F}_{h,t}^E(\tilde{E}) \leq \liminf_n \left( P_\phi(E') + \int_{E' \Delta E} |g_n| - c|E'| \right) \leq \mathcal{F}_{h,t}^E(E'), \quad (2.15)$$

final eq

where in the last inequality we used that  $|g_n| \leq |g|$ . Since  $E'$  is the minimal minimizer of  $\mathcal{F}_{h,t}^E$  we conclude  $\tilde{E} = E'$ . The functional (2.13) is obtained from (2.9) adding  $\int_E g_n(\text{sd}_E^\psi/h)$ . Finally, the functional in (2.14) is obtained from functional (2.9) adding the (finite) term  $-\int_{B_R \setminus E} g(\text{sd}_E^\psi/h) \wedge n$  and restricting the family of competitors.  $\square$

We define the discrete flow starting from the initial set  $E_0$  by setting  $E_t^{(h)} = E_0$  for  $t \in [0, h)$  and iteratively

$$E_t^{(h)} = T_{h,t-h} E_{t-h}^{(h)}, \quad t \in [h, +\infty). \quad (2.16)$$

def discrete flow

We now provide an estimate on the evolution speed of balls. It is interesting to note that, in the isotropic setting ( $\psi = \phi = |\cdot|$ ) and under the hypothesis of strict monotonicity of  $G$ , an explicit evolution law for the radii of evolving balls can be obtained. In our more general case we need to employ the variational proofs of [9, 5].

tion bounded sets

**Lemma 2.12.** *For every  $R > 0$  and every  $t \in (0, +\infty)$ ,  $h > 0$  it holds*

$$T_{h,t}^\pm B_R \subseteq B_{R + \frac{h}{c_\psi} G(\|f\|_\infty)}.$$

*Proof.* We fix  $h > 0$  and set  $c := f([t/h]h)$  and  $E' = T_{h,t}^\pm B_R$ . Let  $\varepsilon > 0$  and set  $H \subseteq \mathbb{R}^N$  as an half-space containing the ball centered at 0 of radius  $R + \frac{h}{c_\psi} G(c + \varepsilon)$ . By the minimality of  $E'$  we get

$$\int_{E' \Delta B_R} |g(\text{sd}_{B_R}^\psi/h)| - \int_{(E' \cap H) \Delta B_R} |g(\text{sd}_{B_R}^\psi/h)| \leq P_\phi(E' \cap H) - P_\phi(E') + c|E' \setminus H|.$$

By a simple computation, since  $B_R \subset H$  we find

$$\int_{E' \setminus H} g(\text{sd}_{B_R}^\psi/h) \leq P_\phi(E' \cap H) - P_\phi(E') + c|E' \setminus H| \leq c|E' \setminus H| \quad (2.17)$$

ineq boundedness

where in the last inequality we used that cutting sets of finite perimeter by half-spaces decreases  $P_\phi$ . Therefore, since  $\text{sd}_{B_R}^\psi \geq hG(c + \varepsilon)$  on  $E' \setminus H$ , one concludes  $|E' \setminus H| = 0$ . Thus the result follows sending  $\varepsilon \rightarrow 0$ .  $\square$

We then provide an upper bound on the evolution speed of balls in the spirit of [9, 5]. We remark that the significant case is  $a = +\infty$  as otherwise Lemma 2.8 yields

$$T_{h,t}^\pm B_R \supseteq B_{R-ah}.$$

estimates on balls

**Lemma 2.13.** *Let  $R_0 > 0$  and  $\sigma > 1$  be fixed. Assume  $a = +\infty$ . Then, there exist a positive constant  $c$  such that, if  $h > 0$  is small enough, for all  $R \geq R_0$  and  $t \in (0, +\infty)$  it holds*

$$T_{h,t}^\pm B_R \supseteq B_{R + \frac{h}{c_\psi} G(-\sigma \frac{c}{R} - \|f\|_\infty)}. \quad (2.18)$$

decay speed ball

*Proof.* We prove the result for  $E := T_{h,t}B_R$ . Take  $h$  small enough so that  $T_{h,t}B_{\frac{1}{4}R_0} \neq \emptyset$ . By Lemma 2.12, translation invariance and taking  $h$  small, one can see that<sup>2</sup>  $B_{\frac{R}{4}} \subseteq E$ . We set

$$\bar{\rho} = \sup\{r \in [0, R] : |B_r \setminus E| = 0\} \in \left[\frac{R}{4}, R + \frac{h}{c_\psi} G(\|f\|_\infty)\right]. \quad (2.19) \quad \boxed{\text{def bar rho}}$$

Assume *wlog*  $\bar{\rho} < R$ . Let  $\bar{x} \in \partial B_{\bar{\rho}}$  be such that  $|B(\bar{x}, \varepsilon) \setminus E| > 0$  for any  $\varepsilon > 0$ . Set  $\rho \in (0, \bar{\rho})$  and  $\tau = (1 - \rho/\bar{\rho})\bar{x}$  such that  $\partial B(\tau, \rho) \cap \partial B_{\bar{\rho}} = \{\bar{x}\}$ . Setting  $B^\varepsilon := ((1 + \varepsilon)\tau, \rho)$ , consider the sets

$$W^\varepsilon := B^\varepsilon \setminus E.$$

Notice that by construction, for  $\varepsilon$  small,  $W^\varepsilon$  has positive measure and it converges to  $\{x\}$  as  $\varepsilon \rightarrow 0$ . By (2.2) with  $v = \nabla\phi(x/|x|)$  and by submodularity, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla\phi\left(\frac{x}{|x|}\right) \cdot D\chi_{W^\varepsilon} &= \int_{\mathbb{R}^N} \nabla\phi\left(\frac{x}{|x|}\right) \cdot \left(D\chi_{B^\varepsilon} - D\chi_{B^\varepsilon \cap T_{h,t}^\pm B_R}\right) \\ &\leq P_\phi(B^\varepsilon \cap T_{h,t}^\pm B_R) - P_\phi(B^\varepsilon) \leq P_\phi(T_{h,t}^\pm B_R) - P_\phi(B^\varepsilon \cup T_{h,t}^\pm B_R). \end{aligned} \quad (2.20) \quad \boxed{\text{split}}$$

Since  $E$  minimizes (2.12) (as  $a = +\infty$ ), we use its minimality on the *rhs* of (2.20) and the divergence theorem on the *lhs* of (2.20) to arrive at

$$- \int_{W^\varepsilon} \operatorname{div} \nabla\phi\left(\frac{x}{|x|}\right) \leq f([t/h]h)|W^\varepsilon| + \int_{W^\varepsilon} g\left(\frac{\operatorname{sd}_{B_R}^\psi}{h}\right). \quad (2.21) \quad \boxed{\text{eq 1}}$$

By the regularity assumptions on  $\phi$  we remark that it holds

$$|\operatorname{div} \nabla\phi(p)| = |\operatorname{tr}(\nabla^2\phi(p))| \leq \frac{c}{|p|}.$$

We plug the estimate above in (2.21), divide by  $|W^\varepsilon|$  and send  $\varepsilon \rightarrow 0$  to conclude

$$-\frac{c}{\rho} - \|f\|_\infty \leq \limsup_{s \rightarrow c_\psi(\bar{\rho}-R)/h} g(s).$$

Applying  $G$  to both sides and letting  $\rho \rightarrow \bar{\rho}$ , we conclude

$$\bar{\rho} \geq R + \frac{h}{c_\psi} G\left(-\frac{c}{\bar{\rho}} - \|f\|_\infty\right) \geq R + \frac{h}{c_\psi} G\left(-\frac{4c}{R} - \|f\|_\infty\right), \quad (2.22) \quad \boxed{\text{incremental eq}}$$

where in the last inequality we recalled that  $\bar{\rho} \geq R/4$ . Using again the previous analysis with the bound (2.22), we show (2.18) by taking  $h$  small enough.  $\square$

<sup>2</sup>Indeed, by translation invariance and Lemma 2.12 it holds

$$T_{h,t}B_{\frac{R}{4}} + B_{\frac{3}{4}R} \subseteq T_{h,t}B_R \subseteq B_{R+ch}.$$

### 2.3 The scheme for unbounded sets

We now define the discrete evolution scheme for unbounded sets having compact boundary. For every compact set  $K$  and  $h > 0, t \geq 0$ , we will denote by  $\tilde{T}_{h,t}^\pm K$  the maximal and the minimal minimizer of  $\tilde{\mathcal{F}}_{h,t}^K$ , which corresponds to (2.9) with  $\tilde{g}(s) := -g(-s)$  instead of  $g(s)$  and  $-f$  instead of  $f$ . By changing variable  $\tilde{F} := F^c$  in (2.9), we see that  $(\tilde{T}_{h,t}^- K)^c$  is the maximal solution to

$$\min \left\{ P_\phi(\tilde{F}) + \int_{\tilde{F} \Delta K^c} \left| g \left( \text{sd}_{K^c}^\psi / h \right) \right| + f([t/h]h) |\tilde{F}^c| \right\}. \quad (2.23)$$

equiv pb unbounded

Therefore, for every unbounded set  $E$  with compact boundary we define<sup>3</sup>

$$T_{h,t}^\pm E := \left( \tilde{T}_{h,t}^\mp E^c \right)^c. \quad (2.24)$$

def unbounded

As in the case of compact sets, we set  $T_{h,t} E := T_{h,t}^- E$ . Given an unbounded set  $E_0$  having compact boundary, we define the discrete flow  $\{E_t^{(h)}\}_{t \geq 0}$  as follows:  $E_t^{(h)} := E_0$  for  $t \in [0, h)$  and

$$E_t^{(h)} = T_{h,t-h} E_{t-h}^{(h)}, \quad \forall t \in [h, +\infty).$$

Since  $\tilde{g}$  has the same properties of  $g$ , one easily checks that analogous results to Lemmas 2.12, 2.9 and 2.13 hold also for (2.24).

principle, unbounded

**Lemma 2.14.** *Let  $t, h > 0$ . The following statements hold.*

- Let  $F_1 \subseteq F_2$  be unbounded sets with compact boundary. Then,  $T_{h,t} F_1 \subseteq T_{h,t} F_2$ .
- There exists  $c > 0$  such that for every  $R > 0, h > 0$  it holds  $T_{h,t}^\pm B_R^c \supseteq B_{R+ch}^c$ .
- Let  $R_0 > 0$  and  $\sigma > 1$  be fixed. Then, if  $a = +\infty$  there exist  $c > 0$  such that for  $h > 0$  small enough and for all  $R \geq R_0$ , it holds

$$T_{h,t}^\pm B_R^c \subseteq B_{R+\frac{h}{c\psi} G(-\sigma \frac{c}{R} - \|f\|_\infty)}^c. \quad (2.25)$$

If instead  $a < +\infty$  it holds

$$T_{h,t}^\pm B_R^c \subseteq B_{R-ah}^c. \quad (2.26)$$

Furthermore, Corollary 2.11 implies straightforwardly the following approximation result.

approx unbounded

**Corollary 2.15.** *Set  $t, h > 0$  and let  $E$  be an unbounded set of finite perimeter with bounded complement. Then, there exists two sequences of sets  $(E_n)_{n \in \mathbb{N}}, (E'_n)_{n \in \mathbb{N}}$  with uniformly bounded complement with the following property. Each  $(E_n)^c$  is a minimizer of (2.23) with  $g \vee (-n)$  substituting  $g$ , and  $(E'_n)^c$  is a minimizer of (2.23) with  $g \wedge n$  substituting  $g$ . Moreover  $E_n \nearrow T_{h,t}^- E$  and  $E'_n \searrow T_{h,t}^+ E$ .*

<sup>3</sup>To justify this, one can check that if a set  $E$  is moving according to (2.4), its complement moves according to

$$V(x, t) = -\psi(\nu_{E^c}(x))G(\kappa_{E^c}(x) + \mathbf{f}) \quad \text{in the direction } \nu_{E^c},$$

from which the incremental problem follows.

We now deduce an equivalent version of (2.23), which will be used in the final proof, following [9]. Let us consider  $E$  such that  $E^c \subseteq B_R$  and assume  $a = +\infty$ . Recall that  $T_{h,t}^\pm E \supseteq B_{R+ch}^c$  for some  $c > 0$  by Lemma 2.14. Adding to the functional in (2.23) the term  $\int_{B_{R+ch} \setminus (T_{h,t}^- E)^c} g(\text{sd}_E^\psi/h)$  and restricting the family of competitors, we note that  $T_{h,t}^- E$  is the minimal solution to

$$\min \left\{ P_\phi(\tilde{F}) + \int_{\tilde{F} \cap B_{R+ch}} g\left(\text{sd}_E^\psi/h\right) + f([t/h]h)|\tilde{F}^c| : \tilde{F}^c \subseteq B_{R+ch} \right\}. \quad (2.27)$$

pb equi unbounded

The case  $a < +\infty$  needs to be treated by approximation using Corollary 2.15. Lastly, we state a comparison principle between bounded and unbounded sets. Its proof follows the one of [9, Lemma 6.10], up to employing Corollary 2.15.

**Lemma 2.16.** *Let  $E_1$  be a compact set and let  $E_2$  be an open, unbounded set with compact boundary, and such that  $E_1 \subseteq E_2$ . Then, for every  $h \in (0, 1), t \geq 0$  it holds  $T_{h,t}^\pm E_1 \subseteq T_{h,t}^\pm E_2$ .*

bounded-unbounded

### 3 Main result

We now describe the discrete-in-time approximation of the viscosity solution based on the operators  $T_{h,t}^\pm$  previously defined. Given a continuous function  $v : \mathbb{R}^N \rightarrow \mathbb{R}$  which is constant outside a compact set, we define the transformation

$$T_{h,t}v(x) = \sup \{s \in \mathbb{R} : x \in T_{h,t}\{v \geq s\}\}, \quad (3.1)$$

def operator

which defines a new function on  $\mathbb{R}^N \times [0, +\infty)$  by setting  $v_h(x, t) = v(x)$  for  $t \in [0, h)$  and

$$v_h(x, t) := (T_{h,t-h}v_h(\cdot, t-h))(x). \quad (3.2)$$

def increm supers

By lemmas 2.9 and 2.14, one can see that the operator  $T_{h,t}$  maps functions into functions. Moreover, the following holds.

**Lemma 3.1.** *Given  $t, h > 0$ , the operator  $T_{h,t}$  defined in (3.1) satisfies the following properties:*

- $T_{h,t}$  is monotone, meaning that  $u_0 \leq v_0$  implies  $T_{h,t}u_0 \leq T_{h,t}v_0$ ;
- $T_{h,t}$  is translation invariant, as for any  $z \in \mathbb{R}^N$ , setting  $\tau_z u_0(x) := u_0(x - z)$ , it holds  $T_{h,t}(\tau_z u_0) = \tau_z(T_{h,t}u_0)$ ;
- $T_{h,t}$  commutes with constants, meaning  $T_{h,t}(u + c) = (T_{h,t}u) + c$  for every  $c \in \mathbb{R}$ .

*Proof.* The first assertion follows from Lemma 2.9 and 2.14. The second one follows easily employing the definition (3.1), recalling the fact that the functional defined in (2.9) is invariant under translations and that  $\{\tau_z u_0 \geq \lambda\} = \{u_0 \geq \lambda\} + z$  for all  $\lambda \in \mathbb{R}$ . The last result follows analogously.  $\square$

The previous properties satisfied by the operator, in turn, preserve the continuity in space of the initial function. Indeed, assume  $u_0$  is uniformly continuous and let  $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be an increasing, continuous modulus of continuity for  $u_0$ . Then, for any  $s > s'$  we have

$$\{u > s\} + B_{\omega^{-1}(s-s')} \subseteq \{u > s'\},$$

thus, by translation invariance we deduce

$$T_{h,t}\{u > s\} + B_{\omega^{-1}(s-s')} \subseteq T_{h,t}\{u > s'\}.$$

This inclusion implies that the function  $T_{h,t}u_0$  is uniformly continuous in space, with the same modulus of continuity  $\omega$  of  $u_0$ .

In order to study the continuity in time of the functions  $u_h$ , we state the following lemma, where equality between sets must be understood up to negligible sets.

mma 6.12 Nonlocal

**Lemma 3.2.** *Fix  $t, h > 0$  and  $u_0$  a uniformly continuous function. For all  $\lambda \in \mathbb{R}$  it holds*

$$T_{h,t}\{u_h(\cdot, t) > \lambda\} = \{u_h(\cdot, t+h) > \lambda\}, \quad T_{h,t}^+\{u_h(\cdot, t) \geq \lambda\} = \{u_h(\cdot, t+h) \geq \lambda\}.$$

*Proof.* Given  $\varepsilon > 0$ , by definition it is easy to see that

$$\{T_{h,0}u_0 > \lambda + \varepsilon\} \subseteq T_{h,0}^\pm\{u_0 > \lambda\} \subseteq \{T_{h,0}u_0 > \lambda - \varepsilon\}.$$

Passing to the limit  $\varepsilon \rightarrow 0$ , we deduce

$$\{u_h(\cdot, h) \geq \lambda\} \subseteq T_{h,0}^\pm\{u_0 > \lambda\} \subseteq \{u_h(\cdot, h) \geq \lambda\}.$$

Finally, since  $u_h(\cdot, h)$  is a continuous function, the equalities  $\{u_h(\cdot, h) > \lambda\} = \text{int}\{u_h(\cdot, h) \geq \lambda\}$  and  $\{u_h(\cdot, h) \geq \lambda\} = \overline{\{u_h(\cdot, h) > \lambda\}}$  holds and we prove the result for  $t = h$ . The other cases follow by iteration.  $\square$

With the previous results and reasoning exactly as in [9, Lemma 6.13], we can prove that the functions  $u_h$  are uniformly continuous in time.

**Lemma 3.3.** *For any  $\varepsilon > 0$ , there exists  $\tau > 0$  and  $h_0 = h_0(\varepsilon) > 0$  such that for all  $|t - t'| \leq \tau$  and  $h \leq h_0$  we have  $|u_h(\cdot, t) - u_h(\cdot, t')| \leq \varepsilon$ .*

Thus, the family  $\{u_h\}_{h>0}$  is equicontinuous and uniformly bounded as implied by Lemma 2.12. By the Ascoli-Arzelà theorem we can pass to the limit  $h \rightarrow 0$  (up to subsequences) to conclude that  $u_h \rightarrow u$  uniformly in any compact in time subset of  $\mathbb{R}^N \times [0, +\infty)$ , with  $u$  being a uniformly continuous function. Moreover, the function  $u$  is bounded and constant outside a compact set.

istence by approx

**Proposition 3.4.** *Let  $T > 0$ . Up to a subsequence, the family  $\{u_h\}_{h>0}$  converges uniformly on  $\mathbb{R}^N \times [0, T]$  to a uniformly continuous function  $u$ , which is bounded and constant out of a compact set.*

We can thus state our main result.

teo sol viscosa

**Theorem 3.5.** *The function  $u$  defined in Proposition 3.4 coincides with the unique continuous viscosity solution of the Cauchy problem (2.6).*

We finally recall the notion of a level-set solution to the evolution equation (2.4) (cp. e.g. [12]).

**Definition 3.6.** Given an initial bounded set  $E_0$  (or unbounded set with bounded complement) define an uniformly continuous function  $u_0 : \mathbb{R}^N \rightarrow \mathbb{R}$  such that  $\{u_0 > 0\} = E_0$ . Then, setting  $u$  as the solution to (2.6) with initial datum  $u_0$  given by Theorem 3.5, we define the level-set solution to the nonlinear mean curvature evolution (2.4) of  $E_0$  as

$$E_t := \{u(\cdot, t) > 0\}.$$

Our main result, Theorem 3.5 amounts thus in showing that the discrete flow converges to the unique level set solution to equation (1.1).

### 3.1 Proof of the main result

We start by an estimate on the evolution speed. For every  $r > 0$ , using the notation of Lemma 2.13, we set

$$\hat{\kappa}(r) = \min \left\{ -1, \frac{1}{c_\psi} G \left( -\frac{c}{r} - \|f\|_\infty \right) \right\}$$

and, given  $r_0 > 0$ , we set  $r(t)$  as the unique solution to

$$\begin{cases} \dot{r}(t) = \hat{\kappa}(r(t)) \\ r(0) = r_0. \end{cases} \quad (3.3) \quad \boxed{\text{ODE}}$$

Note that, in general, the solution  $r(t)$  will exist in a finite time interval  $[0, T^*(r_0)]$ , where  $T^*(r_0)$  denotes the extinction time of the solution starting from  $r_0$  i.e. the first time  $t$  such that  $r(t) = 0$ .

alls in levelsets

**Lemma 3.7.** *Let  $u$  be the function given by Proposition 3.4 and assume that there exists  $\lambda \in \mathbb{R}$  such that  $B(x_0, r_0) \subseteq \{u(\cdot, t_0) > \lambda\}$ . Then, if  $a = +\infty$ , it holds*

$$B(x_0, r(t - t_0)) \subseteq \{u(\cdot, t) > \lambda\}$$

for every  $t \leq T^*(r_0) + t_0$ , where  $r(t)$  is the solution to (3.3) with extinction time  $T^*(r_0)$ . If instead  $a < +\infty$  it holds

$$B(x_0, r_0 - a(t - t_0)) \subseteq \{u(\cdot, t) > \lambda\}$$

for all  $t$  such that  $r_0 - a(t - t_0) \geq 0$ . The same result holds for sublevels substituting superlevel sets.

*Proof.* The result in the case  $a < +\infty$  follows directly by Lemma 2.8, so we assume  $a = +\infty$ . We consider wlog  $\{u(\cdot, t_0) > \lambda\}$  bounded, as the other case is analogous. For a fixed  $R_0 < r_0$ , taking  $h(R_0)$  small enough, we can ensure that  $B(x_0, R_0) \subseteq \{u_h(\cdot, t_0) > \lambda\}$ . We then fix  $\sigma > 1$  and define recursively the radii  $R_n$  by

$$R_{n+1} = R_n + \frac{h}{c_\psi} G \left( -\sigma \frac{C_\phi}{R_n} - \|f\|_\infty \right).$$

By Lemmas 2.9, 2.13 and 3.2, we see that  $B(x_0, R_{\lfloor (t-t_0)/h \rfloor + 1}) \subseteq \{u(\cdot, t) > \lambda\}$  for every  $t \geq t_0$  such that  $R_{\lfloor (t-t_0)/h \rfloor + 1} > 0$ . Let then  $r_\sigma$  be the unique solution to the ODE

$$\begin{cases} \dot{r}_\sigma(t) = \hat{\kappa}(r_\sigma(t)/\sigma) \\ r_\sigma(0) = R_0. \end{cases} \quad (3.4) \quad \boxed{\text{ODE modif}}$$

Employing the monotonicity of  $\hat{\kappa}$ , if  $r_\sigma(t) \leq R_n$ , then

$$\begin{aligned} r_\sigma((n+1)h) &\leq R_n + \int_{nh}^{(n-1)h} \hat{\kappa} \left( \frac{r_\sigma(s)}{\sigma} \right) ds \leq R_n + \int_{nh}^{(n-1)h} \hat{\kappa} \left( \frac{R_n}{\sigma} \right) ds \\ &\leq R_n + \int_{nh}^{(n-1)h} \frac{1}{c_\psi} G \left( -\sigma \frac{C_\phi}{R_n} - \|f\|_\infty \right) ds = R_{n+1}. \end{aligned}$$

Therefore,  $B(x_0, r_\sigma(h \lfloor (t-t_0)/h \rfloor + h)) \subseteq \{u_h(\cdot, t) > \lambda\}$  for  $t \geq t_0$  as long as the radius is positive. We conclude sending  $h \rightarrow 0$ , then  $R_0 \rightarrow r_0$  and  $\sigma \rightarrow 1$ .  $\square$

We are now in the position to prove our main result.

*Proof of Theorem 3.5.* Consider  $u$  as defined in (3.4): we show that  $u$  is a subsolution, as proving that it is a supersolution is analogous. Let  $\eta(x, t)$  be an admissible test function in  $\bar{z} := (\bar{x}, \bar{t})$  and assume that  $(\bar{x}, \bar{t})$  is a strict maximum point for  $u - \eta$ . Assume furthermore that  $u - \eta = 0$  in such point.

**Case 1:** We assume that  $\nabla\eta(\bar{z}) \neq 0$ . Firstly, in the case  $a < +\infty$  we remark that if  $\partial_t\eta/\psi(\nabla\eta(\hat{z})) \leq -a$ , then (2.8) is trivially satisfied, thus we can assume *wlog* that

$$\frac{\partial_t\eta(\bar{z})}{\psi(\nabla\eta(\hat{z}))} > -a. \quad (3.5)$$

rmk test

By classical arguments (recalled in [5]) we can assume that each function  $u_{h_k} - \eta$  assumes a local supremum in  $B_\rho(\bar{z})$  at a point  $z_{h_k} =: (x_k, t_k)$  and that  $u_{h_k}(z_{h_k}) \rightarrow u(\bar{z})$  as  $k \rightarrow \infty$ . Moreover, we can assume that  $\nabla\eta(z_k) \neq 0$  for  $k$  large enough.

**Step 1:** We define a suitable competitor for the minimality of the level sets of  $u_h$ . By the previous remarks we have that

$$u_h(x, t) \leq \eta(x, t) + c_k \quad (3.6)$$

eq 6.19 Nonlocal

where  $c_k := u_{h_k}(x_k, t_k) - \eta(x_k, t_k)$ , with equality if  $(x, t) = (x_k, t_k)$ . Let  $\sigma > 0$  and set

$$\eta_{h_k}^\sigma(x) := \eta(x, t_k) + c_k + \frac{\sigma}{2}|x - x_k|^2.$$

Then, for all  $x \in \mathbb{R}^N$ ,

$$u_{h_k}(x, t_k) \leq \eta_{h_k}^\sigma(x)$$

with equality if and only if  $x = x_k$ . We set  $l_k = u_{h_k}(x_k, t_k) = \eta_{h_k}^\sigma(x_k)$ . We fix  $\varepsilon > 0$ , to be chosen later, and define  $E_\varepsilon^k := \{u_{h_k}(\cdot, t_k) > l_k - \varepsilon\} = T_{h_k, t_k - h_k} \{u_{h_k}(\cdot, t_k - h_k) > l_k - \varepsilon\}$ <sup>4</sup> and

$$W_\varepsilon^k := E_\varepsilon^k \setminus \{\eta_{h_k}^\sigma(\cdot) > l_k + \varepsilon\}. \quad (3.7)$$

Assume that  $E_\varepsilon^k$  is bounded and let us define  $E_{\varepsilon, n}^k$  as the sets constructed by Corollary 2.11 where  $\{u_{h_k}(\cdot, t_k - h_k) > l_k - \varepsilon\}$ ,  $E_\varepsilon^k$  substitute  $E, T_{h, t}^- E$  respectively. We thus have that  $E_{\varepsilon, n}^k \nearrow E_\varepsilon^k$  as  $n \rightarrow \infty$  and that each  $E_{\varepsilon, n}^k$  is the minimal minimizer of a problem in the form (2.12). We define

$$W_{\varepsilon, n}^k := E_{\varepsilon, n}^k \setminus \{\eta_{h_k}^\sigma(\cdot) > l_k + \varepsilon, \}. \quad (3.8)$$

It is easy to see that, along any subsequence  $n(\varepsilon) \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ , it holds  $W_{\varepsilon, n(\varepsilon)}^k \rightarrow \{x\}$  as  $\varepsilon \rightarrow 0$ . Furthermore, we check that for every  $\varepsilon, k > 0$  there exists  $n(\varepsilon, k)$  large enough such that  $|W_{\varepsilon, n}^k| > 0$  for all  $n \geq n(\varepsilon, k)$ . Indeed, by the continuity of  $\eta^\sigma$  and since  $|\nabla\eta(\bar{z})| \neq 0$  there exists a positive radius  $r$  such that

$$(B(x_k, r) \cap E_\varepsilon^k) \subseteq W_\varepsilon^k.$$

Since  $x_k \in E_\varepsilon^k$  and it is an open set, it holds  $|W_\varepsilon^k| > 0$ . Recalling that  $E_{\varepsilon, n}^k \rightarrow E_\varepsilon^k$  in  $L^1$ , we conclude that  $|W_{\varepsilon, n}^k| > 0$  for all  $n = n(\varepsilon, k)$  large enough. Note also that, for every fixed  $k$ ,  $n(\varepsilon, k) \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ .

<sup>4</sup>The choice of working with the open superlevel sets is motivated by our need to employ (2.13)

By minimality of  $E_{\varepsilon,n}^k$  we have

$$\begin{aligned} & P_\phi(E_{\varepsilon,n}^k) + \int_{E_{\varepsilon,n}^k} g\left(\text{sd}_{\{u_{h_k}(\cdot, t_k - h_k) > l_k - \varepsilon\}}^\psi(x)/h_k\right) \vee (-n) \, dx - f\left(\left[\frac{t}{h_k}\right] h_k\right) |W_{\varepsilon,n}^k| \\ & \leq P_\phi(E_{\varepsilon,n}^k \cap \{\eta_{h_k}^\sigma > l_k\}) + \int_{E_{\varepsilon,n}^k \cap \{\eta_{h_k}^\sigma > l_k\}} g\left(\text{sd}_{\{u_{h_k}(\cdot, t_k - h_k) > l_k - \varepsilon\}}^\psi(x)/h_k\right) \vee (-n) \, dx. \end{aligned} \quad (3.9)$$

eq 6.21 Nonlocal

Adding to both sides  $P_\phi(\{\eta_{h_k}^\sigma > l_k\} \cup E_{\varepsilon,n}^k)$  and using the submodularity of the perimeter, we obtain

$$\begin{aligned} & P_\phi(\{\eta_{h_k}^\sigma > l_k + \varepsilon\} \cup W_{\varepsilon,n}^k) - P_\phi(\{\eta_{h_k}^\sigma > l_k + \varepsilon\}) - f\left(\left[\frac{t}{h_k}\right] h_k\right) |W_{\varepsilon,n}^k| \\ & \quad + \int_{W_{\varepsilon,n}^k} g\left(\text{sd}_{\{u_{h_k}(\cdot, t_k - h_k) > l_k - \varepsilon\}}^\psi(x)/h_k\right) \vee (-n) \, dx \leq 0. \end{aligned}$$

Equation (3.6) implies  $\{u_{h_k}(\cdot, t_k - h_k) > l_k - \varepsilon\} \subseteq \{\eta(\cdot, t_k - h_k) > l_k - c_k - \varepsilon\}$ , therefore by monotonicity we get

$$\begin{aligned} & P_\phi(\{\eta_{h_k}^\sigma > l_k + \varepsilon\} \cup W_{\varepsilon,n}^k) - P_\phi(\{\eta_{h_k}^\sigma > l_k + \varepsilon\}) - f\left(\left[\frac{t}{h_k}\right] h_k\right) |W_{\varepsilon,n}^k| \\ & \quad + \int_{W_{\varepsilon,n}^k} g\left(\text{sd}_{\{\eta(\cdot, t_k - h_k) > l_k - c_k - \varepsilon\}}^\psi(x)/h_k\right) \vee (-n) \, dx \leq 0. \end{aligned} \quad (3.10)$$

eq 6.22 Nonlocal

If instead  $E_\varepsilon^k$  is an unbounded set with compact boundary, we employ (2.27) instead of (3.9) to obtain (3.10) in the computations above. See [9, 5] for details.

**Step 2:** We now estimate the terms appearing in (3.10). We start with the first two perimeter terms  $P_\phi(\{\eta_{h_k}^\sigma > l_k + \varepsilon\} \cup W_{\varepsilon,n}^k) - P_\phi(\{\eta_{h_k}^\sigma > l_k + \varepsilon\})$ . Reasoning as in Lemma 2.13, we use the divergence theorem and (2.2) with the vector field  $v := \nabla\phi(\nabla\eta^\sigma/|\nabla\eta^\sigma|)$  to obtain

$$\begin{aligned} & P_\phi(\{\eta_{h_k}^\sigma > l_k + \varepsilon\} \cup W_{\varepsilon,n}^k) - P_\phi(\{\eta_{h_k}^\sigma \geq l_k + \varepsilon\}) \\ & \geq \int_{\partial(\{\eta_{h_k}^\sigma > l_k + \varepsilon\} \cup W_{\varepsilon,n}^k)} v \cdot \nu - \int_{\partial\{\eta_{h_k}^\sigma > l_k + \varepsilon\}} v \cdot \nu = \int_{W_{\varepsilon,n}^k} \text{div } v, \end{aligned} \quad (3.11)$$

eq 6.30 Nonlocal

where  $\nu$  denotes the unit outer vector to the set we are integrating on.

The last term in (3.10) can be treated as follows. For any  $z \in W_\varepsilon$ , we have

$$\eta(z, t_k) + c_k + \frac{\sigma}{2}|z - x_k|^2 \leq l_k + \varepsilon. \quad (3.12)$$

eq 6.23 Nonlocal

Since, in turn,  $\eta(z, t_k) + c_k > l_k - \varepsilon$  it follows that  $\sigma|z - x_k|^2 < 4\varepsilon$  and thus, for  $\varepsilon$  small enough,

$$W_\varepsilon^k \subseteq B_{c\sqrt{\varepsilon}}(x_k). \quad (3.13)$$

Therefore, by Hausdorff convergence it holds that for every  $\varepsilon, k > 0$  there exists  $n = n(\varepsilon, k)$  large enough such that

$$W_{\varepsilon,n}^k \subseteq B_{2c\sqrt{\varepsilon}}(x_k). \quad (3.14)$$

eq 6.24 Nonlocal



On the other hand, by a Taylor expansion, for every  $z \in W_{\varepsilon, n}^k$  we have

$$\eta(z, t_k - h_k) = \eta(z, t_k) - h_k \partial_t \eta(z, t_k) + h_k^2 \int_0^1 (1-s) \partial_{tt}^2 \eta(z, t_k - sh_k) ds. \quad (3.15) \quad \boxed{\text{eq 6.25 Nonlocal}}$$

Then, we consider  $y \in \{\eta(\cdot, t_k - h_k)(y) = l_k - c_k - \varepsilon\}$  being a point of minimal  $\psi$ -distance from  $z$ , that is,  $\psi(z - y) = \text{sd}_{\{\eta(\cdot, t_k - h_k)(y) > l_k - c_k - \varepsilon\}}^\psi(z)$ . One can prove (see [5, eq. (4.26)] for details) that

$$|z - y| = O(h_k). \quad (3.16) \quad \boxed{\text{decay y}}$$

Moreover, it holds (see [9, eq (6.26)] for details)

$$(z - y) \cdot \frac{\nabla \eta(y, t_k - h_k)}{|\nabla \eta(y, t_k - h_k)|} = \pm \psi \left( \frac{\nabla \eta(y, t_k - h_k)}{|\nabla \eta(y, t_k - h_k)|} \right) \text{dist}_{\{\eta(\cdot, t_k - h_k)(y) = l_k - c_k - \varepsilon\}}^\psi(z),$$

with a “+” if  $z \in \{\eta(\cdot, t_k - h_k)(y) \leq l_k - c_k - \varepsilon\}$  and a “-” otherwise. We get

$$\begin{aligned} \eta(z, t_k - h_k) &= \eta(y, t_k - h_k) + (z - y) \cdot \nabla \eta(y, t_k - h_k) \\ &\quad + \int_0^1 (1-s) (\nabla^2 \eta(y + s(z-y), t_k - h_k)(z-y)) \cdot (z-y) ds \\ &= l_k - c_k - \varepsilon - \text{sd}_{\{\eta(\cdot, t_k - h_k)(y) = l_k - c_k - \varepsilon\}}^\psi(z) \psi(\nabla \eta(y, t_k - h_k)) \\ &\quad + \int_0^1 (1-s) (\nabla^2 \eta(y + s(z-y), t_k - h_k)(z-y)) \cdot (z-y) ds. \end{aligned} \quad (3.17) \quad \boxed{\text{eq 6.27 Nonlocal}}$$

Note that, in view of (3.12) it holds  $|\eta(z, t_k) - \eta(y, t_k)| \leq c\varepsilon + ch_k = O(h_k)$ , provided  $\varepsilon \ll h_k$  and small enough. Thus, using also (3.14), (3.16) we deduce

$$\begin{aligned} \frac{1}{h_k} \text{sd}_{\{\eta(\cdot, t_k - h_k) > l_k - c_k - \varepsilon\}}^\psi(z) &\geq \frac{\partial_t \eta(z, t_k) - \frac{2\varepsilon}{h_k} - O(h_k) - O_{h_k}(1)}{\psi(\nabla \eta(y, t_k - h_k))} \\ &= \frac{\partial_t \eta(x_k, t_k) + O(\sqrt{\varepsilon}) - \frac{2\varepsilon}{h_k} - O(h_k) - O_{h_k}(1)}{\psi(\nabla \eta(x_k, t_k - h_k)) + O(\sqrt{\varepsilon}) + O(h_k)}, \end{aligned}$$

and we apply  $g$  to both sides to conclude

$$g \left( \text{sd}_{\{\eta(\cdot, t_k - h_k) > l_k - c_k - \varepsilon\}}^\psi(z) / h_k \right) \geq g \left( \frac{\partial_t \eta(x_k, t_k) - O_{h_k}(1)}{\psi(\nabla \eta(x_k, t_k - h_k)) + O(h_k)} \right) \quad (3.18) \quad \boxed{\text{eq 6.29 Nonlocal}}$$

**Step 4:** We conclude the proof. Combining (3.10), (3.11) and (3.18), we arrive at

$$0 \geq \int_{W_{\varepsilon, n}^k} \text{div } v + |W_{\varepsilon, n}^k| \left( -f \left( \left[ \frac{t}{h_k} \right] h_k \right) + g \left( \frac{\partial_t \eta(x_k, t_k) - O_{h_k}(1)}{\psi(\nabla \eta(x_k, t_k - h_k)) + O(h_k)} \right) \vee (-n) \right). \quad (3.19) \quad \boxed{\text{end eq}}$$

Choosing  $n = n(\varepsilon, k)$ , we can divide by  $|W_{\varepsilon, n(\varepsilon, k)}^k| > 0$  and apply  $G$  to both sides to get

$$G \left( - \int_{W_{\varepsilon, n(\varepsilon, k)}^k} \text{div } v + f \left( \left[ \frac{t}{h_k} \right] h_k \right) \right) \geq G \left( g \left( \frac{\partial_t \eta(x_k, t_k) - O_{h_k}(1)}{\psi(\nabla \eta(x_k, t_k - h_k)) + O(h_k)} \right) \vee (-n(\varepsilon, k)) \right).$$

Let us fix  $k > 0$  and send  $\varepsilon \rightarrow 0$  (thus also  $n(\varepsilon, k) \rightarrow 0$ ). Thanks to the continuity of  $G$  and recalling also that  $W_{\varepsilon, n(\varepsilon, k)}^k \rightarrow \{x\}$  as  $\varepsilon \rightarrow 0$ , we arrive at

$$G\left(-\kappa_{\{\eta_{h_k}^\sigma \geq \eta_{h_k}^\sigma(x_k)\}}^\phi(x_k) + f\left(\left[\frac{t}{h_k}\right]h_k\right)\right) \geq \frac{\partial_t \eta(x_k, t_k) - O_{h_k}(1)}{\psi(\nabla \eta(x_k, t_k)) + O(h_k)},$$

which finally implies the thesis by letting simultaneously  $\sigma \rightarrow 0$  and  $k \rightarrow +\infty$ .

**Case 2:** We assume  $\nabla \eta(\bar{x}, \bar{t}) = 0$  and prove that  $\partial_t \eta(\bar{x}, \bar{t}) \leq 0$ . The proof follows the line of the one in [9]. We focus on the case  $a = +\infty$ , the other being simpler.

Since  $\nabla \eta(\bar{z}) = 0$ , there exist  $\ell \in \mathcal{F}$  and  $\omega \in C^\infty(\mathbb{R})$  with  $\omega'(0) = 0$  such that

$$|\eta(x, t) - \eta(\bar{z}) - \partial_t \eta(\bar{z})(t - \bar{t})| \leq \ell(|x - \bar{x}|) + \omega(|t - \bar{t}|)$$

thus, we can define

$$\begin{aligned} \tilde{\eta}(x, t) &= \partial_t \eta(\bar{z})(t - \bar{t}) + 2\ell(|x - \bar{x}|) + 2\omega(|t - \bar{t}|) \\ \tilde{\eta}_k(x, t) &= \tilde{\eta}(x, t) + \frac{1}{k(\bar{t} - t)}. \end{aligned}$$

We remark that  $u - \tilde{\eta}$  achieves a strict maximum in  $\bar{z}$  and the local maxima of  $u - \tilde{\eta}_k$  in  $\mathbb{R}^N \times [0, \bar{t}]$  are in points  $(x_k, t_k) \rightarrow \bar{z}$  as  $k \rightarrow \infty$ , with  $t_n \leq \bar{t}$ . From now on, the only difference from [9] is in the case  $x_k = \bar{x}$  for an (unrelabeled) subsequence. We thus assume  $x_k = \bar{x}$  for all  $k > 0$  and define  $b_k = \bar{t} - t_k > 0$  and the radii

$$r_k := \ell^{-1}(a_k b_k),$$

where  $a_k \rightarrow 0$  must be chosen such that the extinction time for the solution of (3.3) satisfies  $T^*(r_k) \geq \bar{t} - t_k$ , for  $k$  large enough. To show that such a choice for  $a_k$  is possible, we set

$$\beta(t) = \sup_{0 \leq s \leq t} \hat{\kappa}(\ell^{-1}(s))\ell'(\ell^{-1}(s)), \quad (3.20)$$

where  $\hat{\kappa}$  is as in (3.3). Note that by Definition 2.4 it holds  $\beta(t) \leq \hat{\kappa}(t)$  for  $t$  small,  $\beta$  is non decreasing in  $t$  and  $g(t) \rightarrow 0$  as  $t \rightarrow 0$ . We then have

$$\begin{aligned} \frac{T^*(r_k)}{b_k} &\geq \frac{1}{b_k} \int_{r_k/2}^{r_k} \frac{1}{\hat{\kappa}(s)} ds = \frac{1}{b_k} \int_{\ell^{-1}(a_k b_k/2)}^{\ell^{-1}(a_k b_k)} \frac{1}{\hat{\kappa}(s)} ds \\ &= \frac{a_k}{2} \int_{a_k b_k/2}^{a_k b_k} \frac{1}{\hat{\kappa}(\ell^{-1}(r))\ell'(\ell^{-1}(r))} dr \geq \frac{a_k}{2} \frac{1}{\beta(b_k)} = 2, \end{aligned} \quad (3.21) \quad \boxed{\text{eq 6.32 Nonlocal}}$$

where in the last equality we chose  $a_k := 4\beta(b_k)$  which tends to 0 as  $k \rightarrow \infty$ .

By definition of  $\tilde{\eta}_k$  it holds

$$\begin{aligned} B(\bar{x}, r_k) &\subseteq \{\tilde{\eta}_k(\cdot, t_k) \leq \tilde{\eta}_k(\bar{x}, t_k) + 2\ell(r_k)\} \\ &\subseteq \{u(\cdot, t_k) \leq u(\bar{x}, t_k) + 2\ell(r_k)\}, \end{aligned}$$

by maximality of  $u - \tilde{\eta}_k$  at  $z_k$  and since  $u(z_k) = \tilde{\eta}_k(z_k)$ . Since the balls  $B(\cdot, r_k)$  are not vanishing, by Lemma 3.7 we have

$$\bar{x} \in \{u(\cdot, \bar{t}) \leq u(\bar{x}, t_k) + 2\ell(r_k)\}. \quad (3.22) \quad \boxed{\text{eq bar x}}$$

Finally, using again the maximality of  $u - \eta$  at  $\bar{z}$ , the choice of  $r_k$  and (3.22), we obtain

$$\frac{\eta(\bar{z}) - \eta(\bar{x}, t_k)}{\bar{t} - t_k} = \frac{\eta(\bar{z}) - \eta(\bar{x}, t_k)}{b_k} \leq \frac{u(\bar{z}) - u(\bar{x}, t_k)}{b_k} \leq \frac{2\ell(r_k)}{b_k} = 2a_k.$$

Passing to the limit  $k \rightarrow \infty$ , we conclude that  $\partial_t \eta(\bar{z}) \leq 0$ .  $\square$

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