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MINIMIZING MOVEMENTS FOR NONLINEAR AND NONLOCAL CURVATURE FLOWS

DANIELE DE GENNARO

ABSTRACT. We prove that the minimizing movements scheme à la Almgren-Taylor-Wang converges towards level-set solutions to a nonlinear and nonlocal variant of the mean curvature flow, in a rather general framework of possibly nonlocal motions.

1. INTRODUCTION

In this paper we study a nonlinear version of the mean curvature flow (MCF in short) with forcing and mobility. In particular, given a continuous non decreasing function $G : \mathbb{R} \rightarrow \mathbb{R}$, we consider the flow of sets $t \mapsto E_t$ formally governed by the evolution law

$$(1.1) \quad V(x, t) = \psi(\nu_{E_t}(x))G(-\kappa(x, E_t) + \mathbf{f}(t)), \quad \text{for all } x \in \partial E_t, t \geq 0,$$

where ψ is an anisotropy (usually called the mobility), ν_{E_t} denotes the outer normal vector and \mathbf{f} is a forcing term constant in space. In (1.1) the curvature $\kappa(\cdot, E)$ denotes a variational curvature, belonging to a class of (nonlocal) generalized mean curvatures introduced in [13].

Generalised curvatures are functions defined on pairs (x, E) , where E is a set of class C^2 with compact boundary and $x \in \partial E$, and such that they are non-decreasing with respect to inclusion of sets touching at x , continuous w.r.t. C^2 -convergence of sets, and translation invariant (see conditions (A)-(C) below). We will focus on a particular instance of generalized curvatures, namely variational curvature. These curvatures arise as the first variation (in a suitable sense) of perimeter-like functionals, which are called generalized perimeters. A generalized perimeter $J : \mathcal{M} \rightarrow [0, +\infty]$ is a translation invariant functional on the class of measurable sets \mathcal{M} , which is insensitive to modifications on negligible sets, finite on C^2 -sets with compact boundary, lower semicontinuous w.r.t. the L^1_{loc} -convergence, and satisfies a submodularity condition: $J(E \cap F) + J(E \cup F) \leq J(E) + J(F)$ for every $E, F \in \mathcal{M}$.

The evolution law (1.1) is relevant even in the specific instance where κ is the classical mean curvature, arising as first variation of the perimeter. From a numerical point of view, as suggested e.g. in [14, Remark 3.5], a truncation of the classical evolution speed

$$V(x, t) = -\kappa(x, E_t)$$

is usually encoded in algorithms for the MCF, which corresponds to choosing $G(s) = (-M) \vee s \wedge M$ in (1.1), for $M > 0$ large. Another interesting choice could be $G(s) = -s^-$, which amounts to consider a purely shrinking evolution. Moreover, evolution by powers of the mean curvature have been previously studied in the smooth or convex setting [15, 24, 3] and have been used to prove isoperimetric inequalities [25], or considered in the setting of image processing algorithms [2, 23]. In particular, in [2, Section 4.5] it is remarked that the evolution law (1.1) with $G(s) = s^{\frac{1}{3}}$ and $\psi = |\cdot|$, $\mathbf{f} = 0$ is particularly interesting as it is invariant under affine motions (isometries and

rescalings). We refer also to [16] for interesting links between a time-fractional Allen-Cahn equation and motion by powers of the mean curvature.

On the other hand, being able to address this study in the framework of generalized curvature allows us to prove new results for different geometric flows. As detailed in [13, Section 5], some instances of geometric flows driven by variational curvatures are the following: classical *anisotropic MCF* (driven by a suitably smooth and translation invariant anisotropy), *fractional MCF*, *capacity flows*, and flows driven by the curvature associated to the *regularized pre-Minkowski content*. See also [4] for some extensions.

Given the definition of variational curvature, and the formal gradient flow structure of the MCF, one is naturally led to consider the minimizing movements approach, in the spirit of [1, 21], as a way to prove existence for (1.1). This scheme provides a discrete-in-time approximation of the evolution law (1.1) by iteratively solving a variational problem, where the energy to minimize consists of the sum of J and a suitable dissipation term that penalizes the L^2 -distance between sets. In our setting, we will modify the iterative scheme defined in [13] (reminiscent of [1, 21]) by taking into account the nonlinearity in the dissipation term. The main result of the paper, Theorem 3.5, asserts that such an approximation scheme produces discrete-in-time functions that converge, as the time-step parameter tends to 0, towards a viscosity solution to (1.1).

The proof is inspired by the techniques developed in [13] (see also [6]), and the recent study [8]. In [13], the authors prove existence and uniqueness of viscosity solutions to curvature flows of the form $V = -\kappa$, with κ being a generalized curvature (the uniqueness result requires additional hypotheses on κ beyond those previously mentioned). In the specific case of variational curvatures, existence can be also proved by using the minimizing movements scheme, similar to the one sketched above.

The starting observation is that, under our standing assumptions on G , if κ is a generalized curvature, then $-G(-\kappa)$ is still a generalized curvature. Therefore, the same viscosity theory of [13] applies to evolution laws of the form $V = G(-\kappa)$, providing existence of viscosity solutions, and uniqueness under further hypotheses on G and κ . Anyhow, when dealing with (1.1) two problems arise. Firstly, it is no longer true in general that if κ is a variational curvature, then so is $-G(-\kappa)$. Thus, it is interesting to modify the minimizing movements scheme to account for the nonlinear term, even in the simplified case of $V = G(-\kappa)$. Furthermore, the introduction of a time-dependent forcing term and a mobility requires some care. Indeed, the level set formulation for the evolution law (1.1) with time-dependent forcing and mobility does not fall in the framework of [13]. We provide here the first existence result for such curvature flows in the presence of continuous time-dependent forcing and mobility.

From a technical point of view, we will also use some refinements of the techniques of [13] that we developed in [8], where we focused on similar evolutions driven by inhomogeneous curvatures (i.e. non translation invariant). Moreover, this work can be seen as an extension and an improvement of the unpublished (and unfinished) preprint [7]. We want to point out that our results are more general than those of [7], where the authors consider as κ the classical mean curvature with no mobility, require further regularity on the function G , and furthermore assume that

$$\lim_{s \rightarrow \pm\infty} G(s) = \pm\infty,$$

which simplifies many arguments. Indeed, the main difficulties arise in the case where G is bounded from above or below, as some tools heavily employed in the linear setting are no longer available (see e.g. the commonly used reformulation (2.17)). This issue will be circumvented by an approximation

procedure. Other nontrivial difficulties arise when dealing with generalized perimeters, as many tools from the theory of sets of finite perimeter are no longer available.

To conclude, it would be interesting to study the much more challenging case where the subjacent perimeter is of crystalline type. In this setting the availability of the viscosity solutions of [18, 19] and the development of distribution solutions of [12, 11, 10] may suggest the possibility of a future investigation in this direction. Another interesting instance is the non translation invariant case, and a first step could be considering the same setting of [8]. Concerning uniqueness of viscosity solutions to (1.1), some remarks can be made in the case $f = 0$, see Remark 3.8, but clearly the true question is whether a general uniqueness theory may be developed for generalized curvatures depending on time $\kappa(x, t, E)$, at least in the easier setting of the present paper.

Shortly after a first version of this paper was uploaded, the work [5] appeared in preprint. Here, the authors address a different problem, showing *consistency* of the minimizing movement scheme with smooth solutions, and continuity in L^1 of the limiting flat flows, in the particular case where G is a power and κ is the classical anisotropic mean curvature.

2. THE MINIMIZING MOVEMENTS SCHEME

2.1. Preliminaries. We start introducing some notations. We will use both $B_r(x)$ and $B(x, r)$ to denote the Euclidean ball in \mathbb{R}^N centered in x and of radius r ; with S^N we denote the sphere $\partial B_1(0) \subseteq \mathbb{R}^N$. If the ball is centered in zero, we simply write B_r . We let \mathcal{M} denote the family of the measurable sets in \mathbb{R}^N , and $E \in C^2$ to say that the set E is of class C^2 . In the following, we will always speak about measurable sets and refer to a set as the union of all the points of density 1 of that set i.e. $E = E^{(1)}$. Moreover, if not otherwise stated, we implicitly assume that the function spaces considered are defined on \mathbb{R}^N , e.g. $L^\infty = L^\infty(\mathbb{R}^N)$. Moreover, we often drop the measure with respect to which we are integrating, if clear from the context.

Definition 2.1. We define anisotropy a function $\psi : \mathbb{R}^N \rightarrow [0, +\infty)$ which is continuous, convex, even and positively 1-homogeneous. Moreover, there exists $c_\psi > 0$ such that $\forall p \in \mathbb{R}^N$ it holds

$$(2.1) \quad \frac{1}{c_\psi} |p| \leq \psi(p) \leq c_\psi |p|.$$

We recall that the polar function ψ° of an anisotropy ψ is defined by

$$\psi^\circ(v) := \sup_{\psi(\xi) \leq 1} \xi \cdot v.$$

The following identities hold for smooth anisotropies: $\forall v, \xi \in \mathbb{R}^N$

$$\psi(v)\psi^\circ(\xi) \geq v \cdot \xi, \quad \psi^\circ(\nabla\psi(v)) = v, \quad \nabla\psi(v) \cdot v = \psi(v).$$

Definition 2.2. Given an anisotropy ψ and a set E , we define the ψ -distance from E as

$$\text{dist}_E^\psi(x) = \inf_{y \in E} \psi^\circ(x - y),$$

and the signed ψ -distance from E as

$$\text{sd}_E^\psi(x) = \text{dist}_E^\psi(x) - \text{dist}_{E^c}^\psi(x).$$

For $\delta \in \mathbb{R}$ and $E \in \mathcal{M}$, we denote

$$E_\delta = \{x \in \mathbb{R}^N : \text{sd}_E^\psi(x) \leq \delta\},$$

and use the notation $E_{-\infty} := \emptyset, E_{+\infty} := \mathbb{R}^N$.

Note that (2.1) implies that

$$(2.2) \quad \frac{1}{c_\psi} \text{dist}_E(x) \leq \text{dist}_E^\psi(x) \leq c_\psi \text{dist}_E(x),$$

where dist_E denotes the Euclidean distance from the set E .

In this section we extend the previous study to nonlocal instances, in the spirit of [13]. We recall some notation. For any given $E \in C^2$, we consider¹ a function $x \mapsto \kappa(x, E)$, defined for $x \in \partial E$, and that we will call (generalized) curvature of E at x . This function must satisfy the following axioms:

- (A) Monotonicity: If $E, F \in C^2$ and $x \in \partial E \cap \partial F$ with $E \subseteq F$, then $\kappa(x, E) \geq \kappa(x, F)$;
- (B) translation invariance: For every $E \in C^2$, $x \in \partial E$ and $y \in \mathbb{R}^N$, it holds $\kappa(x, E) = \kappa(x + y, E + y)$;
- (C) Continuity: If $E_n \rightarrow E$ in C^2 and $x_n \in \partial E_n \rightarrow x \in \partial E$, then $\kappa(x_n, E_n) \rightarrow \kappa(x, E)$.

Defining for $x \in \mathbb{R}^N$ and $\rho > 0$

$$(2.3) \quad \begin{aligned} \bar{c}(\rho) &= \max_{x \in \partial B_\rho} \max \{ \kappa(x, B_\rho), -\kappa(x, B_\rho^c) \}, \\ \underline{c}(\rho) &= \min_{x \in \partial B_\rho} \min \{ \kappa(x, B_\rho), -\kappa(x, B_\rho^c) \}, \end{aligned}$$

we note that by (C) these functions are continuous in ρ . We further require

- (D) Curvature of balls: There exists $K > 0$ such that $\underline{c}(\rho) \geq -K > -\infty$.

In the following we will focus on the study of the geometric evolution equation

$$(2.4) \quad V(x, t) = \psi(\nu_{E_t})(x)G(-\kappa(x, E_t) + \mathbf{f}(t)), \quad \text{for } x \in \partial E_t \text{ and } t > 0,$$

starting from an initial bounded set E_0 (or an unbounded set with bounded complement), where ψ is an anisotropy, $\kappa(\cdot, E_t)$ is a variational curvature in the sense above, and \mathbf{f} is a bounded forcing term. Here and in the following, we fix $T > 0$ and consider the evolution for $t \in (0, T)$. The functions G, \mathbf{f} are required to satisfy the following conditions:

- $G : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous, non-decreasing function, with $G(0) = 0$;
- $\mathbf{f} \in C_b^0(\mathbb{R})$;

We then set

$$\lim_{s \rightarrow -\infty} G(s) = -a \in [-\infty, 0], \quad \lim_{s \rightarrow +\infty} G(s) = b \in [0, +\infty].$$

Consider a function $u : \mathbb{R}^N \times [0, +\infty) \rightarrow \mathbb{R}$ whose superlevel sets $E_s := \{u(\cdot, t) \geq s\}$ evolve according to the nonlinear mean curvature equation (2.4). By classical computations (see e.g. [17]), the function u satisfies

$$(2.5) \quad \begin{cases} \partial_t u(x, t) - \psi(\nabla u(x, t))G(-\kappa(x, \{u(\cdot, t) \geq u(x, t)\}) + \mathbf{f}(t)) = 0 \\ u(\cdot, 0) = u_0. \end{cases}$$

Let us recall the notion of viscosity solutions employed in [13]. One first introduces a family of auxiliary functions.

¹One can slightly generalize this definition by considering sets in $C^{\ell, \beta}$ with $\ell \geq 2$, $\beta \in [0, 1]$, but for simplicity we consider the C^2 case only.

Definition 2.3. Given a curvature κ defined as above, we consider a family \mathcal{F}_κ of functions $\ell \in C^\infty([0, +\infty))$, such that $\ell(0) = \ell'(0) = \ell''(0) = 0$, $\ell(\rho) > 0$ for all ρ in a neighborhood of 0, ℓ is constant in $[M, +\infty)$ for some $M > 0$ (depending on ℓ), and

$$(2.6) \quad \lim_{\rho \rightarrow 0^+} \ell'(\rho) \bar{c}(\rho) = 0,$$

where \bar{c} is as in (2.3).

One refers to [20, page 229] for a proof that the family \mathcal{F}_κ is not empty. The notion of admissible test function is the following. With a slight abuse of notation, in the following we will say that a function is spatially constant outside a compact set even if the value of such constant is time-dependent.

Definition 2.4. Let $\hat{z} = (\hat{x}, \hat{t}) \in \mathbb{R}^N \times (0, T)$ and let $A \subseteq (0, T)$ be any open interval containing \hat{t} . We say that $\eta \in C^0(\mathbb{R}^N \times \bar{A})$ is admissible at the point \hat{z} if it is of class C^2 in a neighborhood of \hat{z} , if it is constant out of a compact set, and, in case $\nabla \eta(\hat{z}) = 0$, the following holds: there exists $\ell \in \mathcal{F}_\kappa$ and $\omega \in C^\infty([0, +\infty))$ with $\omega'(0) = 0$, $\omega(\rho) > 0$ for $\rho > 0$ such that

$$|\eta(x, t) - \eta(\hat{z}) - \varphi_t(\hat{z})(t - \hat{t})| \leq \ell(|x - \hat{x}|) + \omega(|t - \hat{t}|)$$

for all (x, t) in $\mathbb{R}^N \times A$.

Then, the notion of viscosity solutions employed in [13] is the following.

Definition 2.5. An upper semicontinuous function $u : \mathbb{R}^N \times [0, T] \rightarrow \mathbb{R}$, constant outside a compact set, is a viscosity subsolution of the Cauchy problem (2.5) if $u(\cdot, 0) \leq u_0$ and, for all $z := (x, t) \in \mathbb{R}^N \times [0, T]$ and all C^∞ -test functions η such that η is admissible at z and $u - \eta$ has a maximum at z , the following holds:

i) If $\nabla \eta(z) = 0$, then

$$(2.7) \quad \eta_t(z) \leq 0;$$

ii) If $\nabla \eta(z) \neq 0$, then

$$(2.8) \quad \partial_t \eta(z) + \psi(\nabla \eta(x, t))G(-\kappa(x, \{\eta(\cdot, t) \geq \eta(z)\}) + \mathbf{f}(t)) \leq 0.$$

A lower semicontinuous function $u : \mathbb{R}^N \times [0, T] \rightarrow \mathbb{R}$, constant outside a compact set, is a viscosity supersolution of the Cauchy problem (2.5) if $u(\cdot, 0) \geq u_0$ and, for all $z := (x, t) \in \mathbb{R}^N \times [0, T]$ and all C^∞ -test functions η such that η is admissible at z and $u - \eta$ has a minimum at z , the following holds:

i) If $\nabla \eta(z) = 0$, then $\eta_t(z) \geq 0$,

ii) If $\nabla \eta(z) \neq 0$, then $\partial_t \eta(z) + \psi(\nabla \eta(x, t))G(-\kappa(x, \{\eta(\cdot, t) \geq \eta(x, t)\}) + \mathbf{f}(t)) \geq 0$.

Finally, a function u is a viscosity solution for the Cauchy problem (2.5) if it is both a subsolution and a supersolution of (2.5).

Remark. By classical arguments, one could assume that the maximum of $u - \eta$ is strict in the definition of subsolution above (an analogous remark holds for supersolutions).

In the rest of the section we will consider a particular instance of generalized curvatures, namely the variational curvatures introduced in [13]. We start by recalling the notion of generalized perimeters.

Definition 2.6. We will say that a functional $J : \mathcal{M} \rightarrow [0, +\infty]$ is a generalized perimeter if it satisfies the following properties: for every E, E' measurable sets and $x \in \mathbb{R}^N$

- (i) $J(E) < +\infty$ for every bounded C^2 -set E ;
- (ii) $J(\emptyset) = J(\mathbb{R}^N) = 0$;
- (iii) $J(E) = J(E')$ if $|E \Delta E'| = 0$;
- (iv) J is lower semicontinuous in L^1_{loc} ;
- (v) J is submodular, that is

$$(2.9) \quad J(E \cap E') + J(E \cup E') \leq J(E) + J(E');$$

- (vi) J is translation invariant: for every $E \in C^2$ and $x \in \mathbb{R}^N$ it holds $J(x + E) = J(E)$.

A generalized perimeter J can be extended to a functional on $L^1_{loc}(\mathbb{R}^N)$ enforcing a generalized co-area formula:

$$(2.10) \quad J(u) = \int_{-\infty}^{+\infty} J(\{u \geq s\}) ds \quad \text{for every } u \in L^1_{loc}(\mathbb{R}^N).$$

It turns out that the functional above is a convex *lsc* functional on $L^1_{loc}(\mathbb{R}^N)$ see [9].

Definition 2.7. Given a bounded C^2 -set E and $x \in \partial E$, we define

$$(2.11) \quad \kappa^+(x, E) = \inf \left\{ \liminf_m \frac{J(E \cup W_\varepsilon) - J(E)}{|W_\varepsilon \setminus E|} : \overline{W_\varepsilon} \xrightarrow{\mathcal{H}} \{x\}, |W_\varepsilon \setminus E| > 0 \right\},$$

and

$$\kappa^-(x, E) = \inf \left\{ \liminf_m \frac{J(E) - J(E \setminus W_\varepsilon)}{|W_\varepsilon \cap E|} : \overline{W_\varepsilon} \xrightarrow{\mathcal{H}} \{x\}, |W_\varepsilon \cap E| > 0 \right\},$$

where $\xrightarrow{\mathcal{H}}$ denotes Hausdorff convergence. We say that $\kappa(x, E)$ is the curvature of E at x if $\kappa^+(x, E) = \kappa^-(x, E) =: \kappa(x, E)$.

In the rest of the section we will assume that κ exists for all sets of class C^2 , and furthermore that it satisfies assumption (C) and (D). Assumptions (A) and (B) follow from the assumptions on J , furthermore one can prove that the weak notion of curvature of Definition 2.7 coincides with the more standard one based on the first variation of the functional J , whenever the latter exists (see [13, Section 4] for details).

2.2. The minimizing movements scheme. We set g as a selection of the set-valued inverse of G , that is $g(x) \in G^{-1}(x)$ for every $x \in (-a, b)$ and extend it setting $g = -\infty$ for every $x \leq -a$, $g = +\infty$ for every $x \geq b$. Here, we extended G to $[-\infty, +\infty]$ setting $G(\pm\infty) = \lim_{x \rightarrow \pm\infty} G(x)$. We assume also that $g(0) = 0$. Note that these definitions imply $G \circ g = id$ in $[-a, b]$. Moreover, g is strictly increasing. In the following we will denote for $k \in \mathbb{N}, h > 0$

$$f(kh) = \int_{kh}^{(k+1)h} \mathbf{f}(s) ds.$$

Given a bounded set $E \in \mathcal{M}$ and $h > 0, t \in (0, +\infty)$ we define a functional on the measurable sets as

$$(2.12) \quad \mathcal{F}_{h,t}^E(F) = J(F) + \int_{E \Delta F} \left| g \left(\frac{\text{sd}_E^\psi}{h} \right) \right| - f(\lfloor t/h \rfloor h) |F|,$$

where $\lfloor \cdot \rfloor$ denotes the integer part. Before proving existence for the functional 2.12 we recall the following existence result for a related problem, see [13, Proposition 6.1].

Lemma 2.8. *Assume that η is a measurable function satisfying $(-\eta) \vee 0 = \eta^- \in L^1(\mathbb{R}^N)$. Then, the problem*

$$(2.13) \quad \min \left\{ J(F) + \int_F \eta(x) \, dx \right\}$$

admits a minimal and a maximal solution (with respect to inclusion). Moreover, if $\eta_1 \leq \eta_2$ then the minimal (resp. maximal) solution to (2.13) with η_1 replacing η contains the minimal (resp. maximal) solution to (2.13) with η_2 replacing η .

We then prove existence of minimizers to $\mathcal{F}_{h,t}^E$. The proof of the boundedness of minimizers has been taken from [22].

Lemma 2.9. *Let $E \in \mathcal{M}$ be a bounded set and $h > 0, t \in [0, +\infty)$. Then, there exist minimizers of $\mathcal{F}_{h,t}^E$ and, denoting E' one such minimizer, it has the following properties: it is a bounded set such that (up to negligible sets)*

$$E_{-ah} \subseteq E' \subseteq E_{bh}.$$

Moreover, there exist a maximal and a minimal minimizer (with respect to inclusion) of $\mathcal{F}_{h,t}^E$.

Proof. We fix $h > 0$ and $t \in (0, T)$, and $c = f([t/h]h)$. Let $n \in \mathbb{N}$ and denote $g_n := g(\frac{\text{sd}_E^\psi}{h}) \vee -n$ and $\tilde{g} := g(\frac{\text{sd}_E^\psi}{h})$. We note that $g_n^- \in L_{loc}^1$, thus Lemma 2.8 implies that the functional

$$J(F) + \int_F (g_n - c)$$

admits a minimal minimizer E_n . Since $\int_E g_n$ is finite, one can check that E_n minimizes also

$$(2.14) \quad J(F) + \int_{E \Delta F} |g_n| - c |F|.$$

Note that $E_n \subseteq E_{n+1}$ by Lemma 2.8, therefore $E_n \rightarrow E' = \bigcup_{n \in \mathbb{N}} E_n$ in L_{loc}^1 . Since $|\tilde{g}|$ is coercive, there exists $R > 0$ such that $|\tilde{g}| \geq 2\|f\|_{L^\infty(\mathbb{R})}$ in B_R^c and $E \subseteq B_R$. Testing (2.14) with \emptyset , we deduce

$$0 \geq J(E_n) + \int_{E_n} (g_n - c) \geq \|f\|_{L^\infty(\mathbb{R})} |E_n \setminus B_R|,$$

that implies $E_n \subseteq B_R$ for every $n \in \mathbb{N}$. By semicontinuity and Fatou's lemma we get

$$\mathcal{F}_{h,t}^E(E') \leq \lim_{n \rightarrow \infty} J(E_n) + \int_{E_n \Delta E} |g_n| - c |E_n|.$$

Since $|g_n| \leq |\tilde{g}|$, we conclude that E' is a minimizer of $\mathcal{F}_{h,t}^E$. By classical arguments, one can check that if E'_1, E'_2 are minimizers of $\mathcal{F}_{h,t}^E$, then so are $E'_1 \cap E'_2, E'_1 \cup E'_2$, implying the existence of a minimal and a maximal solution (see e.g. [13, Proposition 6.1]).

Let now \tilde{E} denote a minimizer of $\mathcal{F}_{h,t}^E$. Since \tilde{E} has finite energy, it is straightforward to check that $|\tilde{E}| < +\infty$ and $\text{sd}_E^\psi \in [-ah, bh]$ a.e. on $\tilde{E} \Delta E$. If $b < +\infty$ this clearly implies that \tilde{E} is bounded; if $b = +\infty$ we use a different argument. We first prove some preliminary results. \square

For a given bounded set $E \in \mathcal{M}$ and $t \in (0, +\infty)$, we denote

$$(2.15) \quad T_{h,t}^- E = \min \operatorname{argmin} \mathcal{F}_{h,t}^E, \quad T_{h,t}^+ E = \max \operatorname{argmin} \mathcal{F}_{h,t}^E,$$

where the minimum and maximum above are made with respect to inclusion. We will often denote $T_{h,t} := T_{h,t}^-$. We now prove some classical results following the lines of [21].

Lemma 2.10 (Weak comparison principle). *Fix $h > 0, t \in (0, +\infty)$ and assume that F_1, F_2 are bounded sets with $F_1 \subset\subset F_2$. Then, for any two minimizers E_i of $\mathcal{F}_{h,t}^{F_i}$ for $i = 1, 2$, we have $E_1 \subseteq E_2$. If, instead, $F_1 \subseteq F_2$, then we have that the minimal (respectively maximal) minimizer of $\mathcal{F}_{h,t}^{F_1}$ is contained in the minimal (respectively maximal) minimizer of $\mathcal{F}_{h,t}^{F_2}$.*

Proof. Firstly, we assume $F_1 \subset\subset F_2$, Testing the minimality of E_1, E_2 with their intersection and union, respectively, we obtain

$$\begin{aligned} J(E_1) + \int_{(E_1 \setminus E_2) \setminus F_1} g\left(\frac{\text{sd}_{F_1}^\psi}{h}\right) + \int_{(E_1 \setminus E_2) \cap F_1} g\left(\frac{\text{sd}_{F_1}^\psi}{h}\right) &\leq J(E_1 \cap E_2) + f([t/h]h)|E_1 \setminus E_2| \\ J(E_2) &\leq J(E_1 \cup E_2) + \int_{(E_1 \setminus E_2) \setminus F_2} g\left(\frac{\text{sd}_{F_2}^\psi}{h}\right) + \int_{(E_1 \setminus E_2) \cap F_2} g\left(\frac{\text{sd}_{F_2}^\psi}{h}\right) - f([t/h]h)|E_1 \setminus E_2|. \end{aligned}$$

Summing the two inequalities above and using the submodularity of J we get

$$(2.16) \quad \int_{(E_1 \setminus E_2) \setminus F_1} g\left(\frac{\text{sd}_{F_1}^\psi}{h}\right) + \int_{(E_1 \setminus E_2) \cap F_1} g\left(\frac{\text{sd}_{F_1}^\psi}{h}\right) \leq \int_{(E_1 \setminus E_2) \cap F_2} g\left(\frac{\text{sd}_{F_2}^\psi}{h}\right) + \int_{(E_1 \setminus E_2) \setminus F_2} g\left(\frac{\text{sd}_{F_2}^\psi}{h}\right).$$

Assume by contradiction that $|E_1 \setminus E_2| > 0$. Since $\text{sd}_{F_2}^\psi < \text{sd}_{F_1}^\psi$ and by the strict monotonicity of g , we estimate the *rhs* of (2.16) by

$$\int_{(E_1 \setminus E_2) \setminus F_2} g\left(\frac{\text{sd}_{F_2}^\psi}{h}\right) + \int_{(E_1 \setminus E_2) \cap F_2} g\left(\frac{\text{sd}_{F_2}^\psi}{h}\right) < \int_{(E_1 \setminus E_2) \setminus F_2} g\left(\frac{\text{sd}_{F_1}^\psi}{h}\right) + \int_{(E_1 \setminus E_2) \cap F_1} g\left(\frac{\text{sd}_{F_1}^\psi}{h}\right)$$

and plug it in (2.16) to reach the desired contradiction. The other cases follow analogously, reasoning by approximation if $F_1 \subseteq F_2$. \square

Lemma 2.11. *Let $c \in \mathbb{R}$. Consider a bounded set $E \in \mathcal{M}$ and non-decreasing functions $g_1, g_2 : \mathbb{R} \rightarrow \mathbb{R}$ such that $g_1 < g_2$ in $\mathbb{R} \setminus \{0\}$ and $g_1(0) = g_2(0) = 0$. Then, if E_i solves*

$$\min_F \left\{ J(F) + \int_{E \Delta F} |g_i(\text{sd}_E^\psi(x))| dx + c|F| \right\}$$

for $i = 1, 2$, we have that $E_2 \subseteq E_1$. If $g_1 \leq g_2$ instead, an analogous statement holds for the maximal and minimal solutions.

Proof. Denote $g_i = g_i \circ \text{sd}_E^\psi$ for $i = 1, 2$ and assume by contradiction that $|E_2 \setminus E_1| > 0$. Reasoning as in Lemma 2.10, one gets

$$\int_{E_1 \Delta E} |g_1| + \int_{E_2 \Delta E} |g_2| \leq \int_{(E_1 \cup E_2) \Delta E} |g_1| + \int_{(E_1 \cap E_2) \Delta E} |g_2|.$$

Simplifying² the above expression and recalling that $g_i \geq 0$ on E^c , $g_i \leq 0$ on E , we reach

$$0 \leq \int_{(E_2 \setminus E_1) \setminus E} (g_1 - g_2) + \int_{(E_2 \setminus E_1) \cap E} (g_1 - g_2) = \int_{E_2 \setminus E_1} (g_1 - g_2),$$

which implies the contradiction. The case $g_1 \leq g_2$ follows by approximation. \square

We can then conclude the proof of the boundedness of minimizers to $\mathcal{F}_{h,t}^E$.

End of proof of Lemma 2.9. We prove that any minimizer \tilde{E} of $\mathcal{F}_{h,t}^E$ is bounded. Recall that $|\tilde{E}| < +\infty$. We assume by contradiction the existence of points $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}^N$ of density one for \tilde{E} , with $|x_n| \rightarrow +\infty$ as $n \rightarrow +\infty$. Fixed $M > 0$, since $|g|$ is coercive there exists $R > 0$ such that $|\tilde{g}| \geq M$ in B_R^c . We can assume that $E \subseteq B_R$, and, up to extracting an unrelabelled subsequence, that $|x_n - x_m| > 2R$ for $n \neq m$ and $|x_n| > 3R$ for all $n \in \mathbb{N}$. We note that

$$M\chi_{B_{2R}^c} < |\tilde{g}|(\cdot + \tau) \quad \text{for all } |\tau| \leq R.$$

Let us denote by E_M a minimizer of

$$J(F) + \int_{F \Delta E} M\chi_{B_{2R}^c} = J(F) + M|F \setminus B_{2R}|.$$

By translation invariance $\tilde{E} + \tau$ minimizes (2.12) with $|\tilde{g}|(\cdot + \tau)$ substituting $|\tilde{g}|$, thus by comparison

$$\tilde{E} + \tau \subseteq E_M \quad \text{for all } |\tau| \leq R.$$

In particular, the disjoint balls $B_R(x_n)$ are all contained (up to negligible sets) in E_M . This implies

$$J(E_M) + M|E_M \setminus B_{2R}| \geq M \left| \bigcup_{n \in \mathbb{N}} B_R(x_n) \right| = +\infty,$$

a contradiction. \square

If $\int_E g(\text{sd}_E^\psi) < +\infty$, minimizers of $\mathcal{F}_{h,t}^E$ minimize also the functional

$$(2.17) \quad F \mapsto J(F) + \int_F g \left(\frac{1}{h} \text{sd}_E^\psi \right) - f([t/h]h)|F|,$$

as can be see adding the (constant term) $\int_E g(\text{sd}_E^\psi)$ to the functional $\mathcal{F}_{h,t}^E$. In the present setting, since $\int_E g(\text{sd}_E^\psi)$ may be infinite in the case $a < +\infty$, we can not draw this conclusion straightforwardly. We can nonetheless recover the minimal and the maximal solution to (2.15) by means of a sequence of minimizers of a functional similar to (2.17), essentially as in the proof of Lemma 2.9.

Corollary 2.12. *Assume $a < +\infty$. Let $E \in \mathcal{M}$ be a bounded set and $t \in (0, +\infty)$, $h > 0$. Then, there exists a sequence of uniformly bounded sets $(E_n)_{n \in \mathbb{N}}$ such that $E_n \nearrow T_{h,t}^- E$ and for any $n \in \mathbb{N}$, E_n is a minimizer of*

$$(2.18) \quad F \mapsto J(F) + \int_F g \left(\frac{\text{sd}_E^\psi}{h} \right) \vee (-n) - f([t/h]h)|F| =: \mathcal{F}_{h,t}^{E,n}(F).$$

²Noting that

$$\begin{aligned} E_1 \Delta E &= ((E_1 \setminus E_2) \setminus E) \cup ((E_1 \cap E_2) \setminus E) \cup ((E \setminus E_1) \setminus E_2) \cup ((E \cap E_2) \setminus E_1) \\ (E_1 \cup E_2) \Delta E &= (E_2 \setminus E_1 \setminus E) \cup ((E_1 \cap E_2) \setminus E) \cup ((E_1 \setminus E_2) \setminus E) \cup ((E \setminus E_1) \setminus E_2) \\ (E_1 \cap E_2) \Delta E &= ((E_2 \cap E_1) \setminus E) \cup ((E \setminus E_1) \setminus E_2) \cup ((E \cap E_1) \setminus E_2) \cup ((E \cap E_2) \setminus E_1). \end{aligned}$$

Analogously, there exists a sequence of uniformly bounded sets $(E_n)_{n \in \mathbb{N}}$ such that $E_n \searrow T_{h,t}^+ E$ in L^1 and for any $n \in \mathbb{N}$, E_n is a solution to

$$(2.19) \quad \min \left\{ J(F) + \int_{B_R \setminus F} g \left(\frac{\text{sd}_E^\psi}{h} \right) \wedge n - f([t/h]h)|F| : F \subseteq B_R \right\},$$

where $T_{h,t}^\pm E \subseteq B_R$.

Proof. We prove the statement for $T_{h,t}^- E$, the other case being analogous. We set $c = f([t/h]h)$, $g_n := g(\text{sd}_E^\psi/h) \vee (-n)$, and $E' = T_{h,t}^- E$. Arguing as in the proof of Lemma 2.9, one builds a sequence of sets $(E_n)_{n \in \mathbb{N}}$, each being the minimal minimizer of $\mathcal{F}_{h,t}^{E,n}$, $E_n \subseteq B_R$ for all $n \in \mathbb{N}$ and $E_n \nearrow \bigcup_{n \in \mathbb{N}} E_n =: \tilde{E}$. Note that $E' \supseteq E_n$ as $g \leq g_n$, therefore $\tilde{E} \subseteq E'$ and also $\chi_{E_n \Delta E'} = |\chi_{E_n} - \chi_{E'}| \rightarrow \chi_{\tilde{E} \Delta E'}$ a.e. as $n \rightarrow \infty$. By lower semicontinuity of J and Fatou's lemma we get

$$\begin{aligned} \mathcal{F}_{h,t}^E(\tilde{E}) &= J(\tilde{E}) - c|\tilde{E}| + \int_{\tilde{E} \Delta E'} |g(\text{sd}_E^\psi/h)| = J(\tilde{E}) - c|\tilde{E}| + \int_{\mathbb{R}^N} \liminf_{n \rightarrow \infty} (|g_n| \chi_{E_n \Delta E}) \\ &\leq \liminf_{n \rightarrow \infty} \left(J(E_n) - c|E_n| + \int_{E_n \Delta E} |g_n| \right). \end{aligned}$$

Since E_n minimizes $\mathcal{F}_{h,t}^{E,n}$ we get

$$(2.20) \quad \mathcal{F}_{h,t}^E(\tilde{E}) \leq \liminf_n \left(J(E') + \int_{E' \Delta E} |g_n| - c|E'| \right) \leq \mathcal{F}_{h,t}^E(E'),$$

where in the last inequality we used that $|g_n| \leq |g|$. Since E' is the minimal minimizer of $\mathcal{F}_{h,t}^E$ we conclude $\tilde{E} = E'$. The functional (2.18) is obtained from (2.12) adding $\int_E g_n(\text{sd}_E^\psi/h)$. Finally, the functional in (2.19) is obtained from functional (2.12) adding the (finite) term $-\int_{B_R \setminus E} g(\text{sd}_E^\psi/h) \wedge n$ and restricting the family of competitors. \square

We now provide an estimate on the evolution speed of balls. It is interesting to note that, in the isotropic setting ($\psi = \phi = |\cdot|$) and under the hypothesis of strict monotonicity of G , an explicit evolution law for the radii of evolving balls can be obtained. In our more general case we need to employ the variational proofs of [13, 8]. By Lemma 2.9, the interesting case is when $b = +\infty$.

Lemma 2.13. *Assume $b = +\infty$. There exists a positive constant C such that, for every $R > 0$ and every $t \in (0, +\infty)$, $h > 0$ it holds*

$$T_{h,t}^\pm B_R \subseteq B_{R+Ch}.$$

Proof. It is sufficient to prove the claim for $T_{h,t}^+ B_R$. We fix $h > 0$ and set $E = T_{h,t}^+ B_R$ and $\tilde{g} = g(\text{sd}_{B_R}^\psi/h)$. We define

$$\bar{\rho} = \sup\{\rho \in (0, +\infty) : |E \setminus B_\rho| = 0\},$$

and note that $\bar{\rho} < +\infty$ since E is bounded. We can assume wlog $\bar{\rho} > R$. Let $\bar{x} \in \partial B_{\bar{\rho}}$ such that $|E \cap B(\bar{x}, \varepsilon)| > 0$ for any $\varepsilon > 0$, and let $\rho > \bar{\rho}$. Let $\tau = (\frac{\rho}{\bar{\rho}} - 1)\bar{x}$ and note that $B(-\tau, \rho) \supseteq B_{\bar{\rho}}$ and $\partial B(-\tau, \rho)$ is tangent to $\partial B_{\bar{\rho}}$ at \bar{x} .

We let for $\varepsilon > 0$ small $B^\varepsilon = B(-(1+\varepsilon)\tau, \rho)$ and $W^\varepsilon = E \setminus B^\varepsilon$. We note that by construction $|W^\varepsilon| > 0$ and it converges to \bar{x} in the Hausdorff sense as $\varepsilon \rightarrow 0$.

Testing the minimality of E against $E \cap B^\varepsilon$, we find

$$(2.21) \quad J(E) - J(B^\varepsilon \cap E) \leq f([t/h]h)|W_\varepsilon| + \int_{B^\varepsilon \cap E \Delta B_R} |\tilde{g}| - \int_{E \Delta B_R} |\tilde{g}|.$$

We remark that, by the choice of $\bar{\rho}$ and τ , taking ε small it holds $B_R \subseteq B^\varepsilon \cap E$. Therefore, (2.21) reads

$$J(E) - J(B^\varepsilon \cap E) \leq f([t/h]h)|W_\varepsilon| + \int_{B^\varepsilon \cap E \setminus B_R} |\tilde{g}| - \int_{E \setminus B_R} |\tilde{g}|,$$

implying

$$(2.22) \quad J(E) - J(B^\varepsilon \cap E) \leq f([t/h]h)|W_\varepsilon| - \int_{(E \setminus B^\varepsilon) \setminus B_R} |\tilde{g}| = f([t/h]h)|W_\varepsilon| - \int_{W^\varepsilon} |\tilde{g}|.$$

By submodularity (2.9), using the definition of \underline{c} and hypothesis (D) we conclude

$$-K + o_\varepsilon(1) \leq \|f\|_\infty - \int_{W^\varepsilon} |\tilde{g}| \leq \|f\|_\infty - \int_{W^\varepsilon} g(c_\psi(|x| - R)/h).$$

Passing to the limit $\varepsilon \rightarrow 0$ we get

$$K + \|f\|_\infty \geq \liminf_{s \rightarrow c_\psi(\bar{\rho} - R)h} g(s),$$

from which the thesis follows applying G on both sides. \square

Note that the previous result implies, in particular, that the discrete evolution starting from an initial bounded set remains bounded in every bounded time interval $(0, T)$.

We then provide an upper bound on the evolution speed of balls in the spirit of [13, 8]. We remark that the significant case is $a = +\infty$ as otherwise Lemma 2.9 yields

$$T_{h,t}^\pm B_R \supseteq B_{R-ah}.$$

Lemma 2.14. *Let $R_0 > 0$ and $\sigma > 1$ be fixed. Assume $a = +\infty$. Then, there exist a positive constant c such that, if $h > 0$ is small enough, for all $R \geq R_0$ and $t \in (0, +\infty)$ it holds*

$$(2.23) \quad T_{h,t}^\pm B_R \supseteq B_{R + \frac{h}{c_\psi} G(-\bar{c}(R/\sigma) - \|f\|_\infty)}.$$

Proof. We prove the result for $E := T_{h,t}^- B_R$. Take h small enough so that $T_{h,t} B_{\frac{3}{4}R_0} \neq \emptyset$. By translation invariance and taking h small, one can see that³ $B_{\frac{R}{4}} \subseteq E$. We set

$$(2.24) \quad \bar{\rho} = \sup\{\rho \in [0, +\infty) : |B_\rho \setminus E| = 0\} \in [\frac{R}{4}, +\infty),$$

and note that $\bar{\rho} < +\infty$ by the boundedness of E . Assume wlog $\bar{\rho} < R$. Let $\bar{x} \in \partial B_{\bar{\rho}}$ be such that $|B(\bar{x}, \varepsilon) \setminus E| > 0$ for any $\varepsilon > 0$. Set $\rho \in (0, \bar{\rho})$ and $\tau = (1 - \rho/\bar{\rho})\bar{x}$ such that $\partial B(\tau, \rho) \cap \partial B_{\bar{\rho}} = \{\bar{x}\}$. Setting $B^\varepsilon := ((1 + \varepsilon)\tau, \rho)$, consider the sets

$$W^\varepsilon := B^\varepsilon \setminus E.$$

³Indeed, by translation invariance it holds

$$T_{h,t} B_{\frac{R}{4}} + B_{\frac{3}{4}R} \subseteq T_{h,t} B_R,$$

and for h small (depending on R) the set $T_{h,t}^\pm B_{R/4}$ is not empty.

Notice that by construction, for ε small, W^ε has positive measure and it converges to $\{x\}$ as $\varepsilon \rightarrow 0$ in the Hausdorff sense. Since E minimizes (2.17) (as $a = +\infty$), we use its minimality to get

$$J(T_{h,t}^\pm B_R) - J(B^\varepsilon \cup T_{h,t}^\pm B_R) \leq f([t/h]h)|W_\varepsilon| + \int_{W^\varepsilon} g\left(\frac{\text{sd}_{B_R}^\psi}{h}\right).$$

Dividing by $|W_\varepsilon| > 0$ the equation above reads

$$(2.25) \quad \frac{J(T_{h,t}^\pm B_R) - J(B^\varepsilon \cup T_{h,t}^\pm B_R)}{|W_\varepsilon|} \leq f([t/h]h) + \int_{W^\varepsilon} g\left(\frac{\text{sd}_{B_R}^\psi}{h}\right).$$

By submodularity and the definition of variational curvature (2.11) we see that

$$J(T_{h,t}^\pm B_R) - J(B^\varepsilon \cup T_{h,t}^\pm B_R) \geq J(B^\varepsilon \setminus W_\varepsilon) - J(B^\varepsilon) \geq |W_\varepsilon|(-\kappa(\bar{x}, B^\varepsilon) + o_\varepsilon(1)),$$

where $o_\varepsilon(1) \rightarrow 0$ as $\varepsilon \rightarrow 0$. We plug the estimate above in (2.25) and send $\varepsilon \rightarrow 0$ to conclude

$$-\bar{c}(\bar{\rho}) - \|f\|_\infty \leq \limsup_{s \rightarrow c_\psi(\bar{\rho}-R)/h} g(s).$$

Applying G to both sides of (2.25), we conclude

$$(2.26) \quad \bar{\rho} \geq R + \frac{h}{c_\psi} G(-\bar{c}(\bar{\rho}) - \|f\|_\infty) \geq R + \frac{h}{c_\psi} G(-\bar{c}(R/4) - \|f\|_\infty),$$

where in the last inequality we recalled that $\bar{\rho} \geq R/4$. Using again the previous analysis with the bound (2.26), we show (2.23) by taking h small enough. \square

2.3. The scheme for unbounded sets. We now define the discrete evolution scheme for unbounded sets having compact boundary. Let us introduce the generalized perimeter

$$\tilde{J}(E) := J(E^c) \quad \text{for all } E \in \mathcal{M}.$$

It is easily checked that \tilde{J} satisfies all the assumptions of Definition 2.6, and, denoting $\tilde{\kappa}$ the corresponding curvature, that

$$\tilde{\kappa}(x, E) = -\kappa(x, E^c).$$

Therefore, one has the bounds

$$\begin{aligned} \bar{c}(\rho) &= \max_{x \in \partial B_\rho} \max \{ \tilde{\kappa}(x, B_\rho), -\tilde{\kappa}(x, B_\rho^c) \}, \\ \underline{c}(\rho) &= \min_{x \in \partial B_\rho} \min \{ \tilde{\kappa}(x, B_\rho), -\tilde{\kappa}(x, B_\rho^c) \}, \end{aligned}$$

where the functions \bar{c}, \underline{c} are defined in (2.3). For every compact set K and $h, t > 0$, we let $\tilde{T}_{h,t}^\pm K$ denote the maximal and the minimal minimizer of $\tilde{\mathcal{F}}_{h,t}^K$, which corresponds to (2.12) with $\tilde{g}(s) := -g(-s)$ instead of $g(s)$ and $-f$ instead of f . By changing variable $\tilde{F} := F^c$ in (2.12), we see that $(\tilde{T}_{h,t}^- K)^c$ is the maximal solution to

$$(2.27) \quad \min \left\{ J(\tilde{F}) + \int_{\tilde{F} \Delta K^c} \left| g\left(\frac{\text{sd}_{K^c}^\psi}{h}\right) \right| + f([t/h]h) \right\}.$$

Therefore, for every unbounded set E with compact boundary we define⁴

$$(2.28) \quad T_{h,t}^{\pm} E := \left(\tilde{T}_{h,t}^{\mp} E^c \right)^c.$$

As in the case of compact sets, we set $T_{h,t} E := T_{h,t}^- E$. Since \tilde{g} has the same properties of g , one easily checks that analogous results to Lemmas 2.13, 2.10 and 2.14 hold also for (2.28).

Lemma 2.15. *Let $t, h > 0$. The following statements hold.*

- Let $F_1 \subseteq F_2$ be unbounded sets with compact boundary. Then, $T_{h,t}^{\pm} F_1 \subseteq T_{h,t}^{\pm} F_2$.
- There exists $C > 0$ such that for every $R > 0, h > 0$ it holds $T_{h,t}^{\pm} B_R^c \supseteq B_{R+Ch}^c$.
- Let $R_0 > 0$ and $\sigma > 1$ be fixed. Then, if $a = +\infty$ there exist $c > 0$ such that for $h > 0$ small enough and for all $R \geq R_0$, it holds

$$T_{h,t}^{\pm} B_R^c \subseteq B_{R+\frac{h}{c\psi}G(-\sigma\frac{c}{R}-\|f\|_{\infty})}^c.$$

If instead $a < +\infty$ it holds

$$T_{h,t}^{\pm} B_R^c \subseteq B_{R-ah}^c.$$

Furthermore, Corollary 2.12 implies straightforwardly the following approximation result.

Corollary 2.16. *Set $t, h > 0$ and let $E \in \mathcal{M}$ be an unbounded set with bounded complement. Then, there exists two sequences of sets $(E_n)_{n \in \mathbb{N}}, (E'_n)_{n \in \mathbb{N}}$ with uniformly bounded complement with the following property. Each $(E_n)^c$ is a minimizer of (2.27) with $g \vee (-n)$ substituting g , and $(E'_n)^c$ is a minimizer of (2.27) with $g \wedge n$ substituting g . Moreover $E_n \nearrow T_{h,t}^- E$ and $E'_n \searrow T_{h,t}^+ E$.*

We now deduce an equivalent version of (2.27), which will be used in the final proof. Let us consider E such that $E^c \subseteq B_R$ and assume $a = +\infty$. Recall that $T_{h,t}^{\pm} E \supseteq B_{R+Ch}^c$ for some $C > 0$ by Lemma 2.15. Adding to the functional in (2.27) the term $\int_{B_{R+Ch} \setminus (T_{h,t} E)^c} g(\text{sd}_E^{\psi}/h)$ and restricting the family of competitors, we note that $T_{h,t}^- E$ is the minimal solution to

$$(2.29) \quad \min \left\{ J(\tilde{F}) + \int_{\tilde{F} \cap B_{R+Ch}} g(\text{sd}_E^{\psi}/h) + f([t/h]h)|\tilde{F}^c| : \tilde{F}^c \subseteq B_{R+Ch} \right\}.$$

The case $a < +\infty$ needs to be treated by approximation using Corollary 2.16. Lastly, we state a comparison principle between bounded and unbounded sets. Its proof follows the one of [13, Lemma 6.10], up to employing Corollary 2.16.

Lemma 2.17. *Let E_1 be a compact set and let E_2 be an open, unbounded set with compact boundary, and such that $E_1 \subseteq E_2$. Then, for every $h, t > 0$ it holds $T_{h,t}^{\pm} E_1 \subseteq T_{h,t}^{\pm} E_2$.*

3. MAIN RESULT

We start by introducing the discrete approximation scheme. Given a continuous function $u_0 : \mathbb{R}^N \rightarrow \mathbb{R}$ which is constant outside a compact set, we define the transformation

$$(3.1) \quad T_{h,t} u(x) = \sup \{ s \in \mathbb{R} : x \in T_{h,t} \{ u_0 \geq s \} \},$$

⁴To justify this, one can check that if a set E is moving according to (1.1), its complement moves according to

$$V(x, t) = -\psi(\nu_{E^c}(x))G(\kappa(x, E^c) + \mathbf{f}) \quad \text{in the direction } \nu_{E^c},$$

from which the incremental problem follows.

and set $u_h(x, t) = u_0(x)$ for $t \in [0, h)$ and

$$(3.2) \quad u_h(x, t) := (T_{h, t-h} u_h(\cdot, t-h))(x).$$

By lemmas 2.10 and 2.15, one can see that the operator $T_{h,t}$ maps functions into functions. The following properties of the operator $T_{h,t}$ hold.

Lemma 3.1. *Given $t, h > 0$, the operator $T_{h,t}$ defined in (3.1) satisfies the following properties:*

- $T_{h,t}$ is monotone, meaning that $u_0 \leq v_0$ implies $T_{h,t}u_0 \leq T_{h,t}v_0$;
- $T_{h,t}$ is translation invariant, as for any $z \in \mathbb{R}^N$, setting $\tau_z u_0(x) := u_0(x - z)$, it holds $T_{h,t}(\tau_z u_0) = \tau_z(T_{h,t}u_0)$;
- $T_{h,t}$ commutes with constants, meaning $T_{h,t}(u + c) = (T_{h,t}u) + c$ for every $c \in \mathbb{R}$.

Proof. The first assertion follows from Lemma 2.10 and 2.15. The second one follows easily employing the definition (3.1), recalling the fact that the functional defined in (2.12) is invariant under translations and that $\{\tau_z u_0 \geq \lambda\} = \{u_0 \geq \lambda\} + z$ for all $\lambda \in \mathbb{R}$. The last result follows analogously. \square

The previous properties satisfied by the operator, in turn, preserve the continuity in space of the initial function. Indeed, assume u_0 is uniformly continuous and let $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an increasing, continuous modulus of continuity for u_0 . Then, for any $s > s'$ we have

$$\{u > s\} + B_{\omega^{-1}(s-s')} \subseteq \{u > s'\},$$

thus, by translation invariance we deduce

$$T_{h,t}\{u > s\} + B_{\omega^{-1}(s-s')} \subseteq T_{h,t}\{u > s'\}.$$

This inclusion implies that the function $T_{h,t}u_0$ is uniformly continuous in space, with the same modulus of continuity ω of u_0 .

The following lemma provides an estimate on the continuity in time of u_h . Here, equality between sets must be understood up to negligible sets.

Lemma 3.2. *Fix $t, h > 0$ and u_0 a uniformly continuous function. For all $\lambda \in \mathbb{R}$ it holds*

$$T_{h,t}\{u_h(\cdot, t) > \lambda\} = \{u_h(\cdot, t+h) > \lambda\}, \quad T_{h,t}^+\{u_h(\cdot, t) \geq \lambda\} = \{u_h(\cdot, t+h) \geq \lambda\}.$$

Proof. Given $\varepsilon > 0$, by definition it is easy to see that

$$\{T_{h,0}u_0 > \lambda + \varepsilon\} \subseteq T_{h,0}^\pm\{u_0 > \lambda\} \subseteq \{T_{h,0}u_0 > \lambda - \varepsilon\}.$$

Passing to the limit $\varepsilon \rightarrow 0$, we deduce

$$\{u_h(\cdot, h) \geq \lambda\} \subseteq T_{h,0}^\pm\{u_0 > \lambda\} \subseteq \{u_h(\cdot, h) \geq \lambda\}.$$

Finally, since $u_h(\cdot, h)$ is a continuous function, the equalities $\{u_h(\cdot, h) > \lambda\} = \text{int}\{u_h(\cdot, h) \geq \lambda\}$ and $\{u_h(\cdot, h) \geq \lambda\} = \overline{\{u_h(\cdot, h) > \lambda\}}$ holds and we prove the result for $t = h$. The other cases follow by iteration. \square

With the previous results and reasoning exactly as in [13, Lemma 6.13], we can prove that the functions u_h are uniformly continuous in time.

Lemma 3.3. *For any $\varepsilon > 0$, there exists $\tau > 0$ and $h_0 = h_0(\varepsilon) > 0$ such that for all $|t - t'| \leq \tau$ and $h \leq h_0$ we have $|u_h(\cdot, t) - u_h(\cdot, t')| \leq \varepsilon$.*

Thus, the family $\{u_h\}_{h>0}$ is equicontinuous and uniformly bounded as implied by Lemma 2.13. By the Ascoli-Arzelà theorem we can pass to the limit $h \rightarrow 0$ (up to subsequences) to conclude that $u_h \rightarrow u$ uniformly in any compact in time subset of $\mathbb{R}^N \times [0, +\infty)$, with u being a uniformly continuous function. Moreover, the function u is bounded and constant outside a compact set (as implied by Lemma 2.13).

Proposition 3.4. *Let $T > 0$. Up to a subsequence, the family $\{u_h\}_{h>0}$ defined in (3.2) converges uniformly on $\mathbb{R}^N \times [0, T]$ to a uniformly continuous function u , which is bounded and constant out of a compact set.*

We can thus state our main result.

Theorem 3.5. *The function u defined in Proposition 3.4 is a continuous viscosity solution to the Cauchy problem (2.5).*

We finally recall the notion of a level-set solution to the evolution equation (1.1) (see e.g. [17]).

Definition 3.6. Given an initial bounded set E_0 (or unbounded set with bounded complement) define an uniformly continuous function $u_0 : \mathbb{R}^N \rightarrow \mathbb{R}$ such that $\{u_0 > 0\} = E_0$. Then, setting u as the solution to (2.5) with initial datum u_0 given by Theorem 3.5, we define the level-set solution to the nonlinear mean curvature evolution (1.1) of E_0 as

$$E_t := \{u(\cdot, t) > 0\}.$$

3.1. Proof of the main result. We start by an estimate on the evolution speed. For every $r > 0$, using the notation of Lemma 2.14, we set

$$\hat{\kappa}(r) = \min \left\{ -1, \frac{1}{c_\psi} G(-\bar{c}(r) - \|f\|_\infty) \right\}$$

and, given $r_0 > 0$, we set $r(t)$ as the unique solution to

$$(3.3) \quad \begin{cases} \dot{r}(t) = \hat{\kappa}(r(t)) \\ r(0) = r_0. \end{cases}$$

Note that, in general, the solution $r(t)$ will exist in a finite time interval $[0, T^*(r_0)]$, where $T^*(r_0)$ denotes the extinction time of the solution starting from r_0 i.e. the first time t such that $r(t) = 0$.

Lemma 3.7. *Let u be the function given by Proposition 3.4 and assume that there exists $\lambda \in \mathbb{R}$ such that $B(x_0, r_0) \subseteq \{u(\cdot, t_0) > \lambda\}$. Then, if $a = +\infty$, it holds*

$$B(x_0, r(t - t_0)) \subseteq \{u(\cdot, t) > \lambda\}$$

for every $t \leq T^*(r_0) + t_0$, where $r(t)$ is the solution to (3.3) with extinction time $T^*(r_0)$. If instead $a < +\infty$ it holds

$$B(x_0, r_0 - a(t - t_0)) \subseteq \{u(\cdot, t) > \lambda\}$$

for all t such that $r_0 - a(t - t_0) \geq 0$. The same result holds for sublevels substituting superlevel sets.

Proof. The result in the case $a < +\infty$ follows directly by Lemma 2.9, so we assume $a = +\infty$. We consider wlog $\{u(\cdot, t_0) > \lambda\}$ bounded, as the other case is analogous. For a fixed $R_0 < r_0$, taking $h(R_0)$ small enough, we can ensure that $B(x_0, R_0) \subseteq \{u_h(\cdot, t_0) > \lambda\}$. We then fix $\sigma > 1$ and define recursively the radii R_n by

$$R_{n+1} = R_n + \frac{h}{c_\psi} G(-\bar{c}(R_n/\sigma) - \|f\|_\infty).$$

By Lemmas 2.10, 2.14 and 3.2, we see that $B(x_0, R_{[(t-t_0)/h]+1}) \subseteq \{u(\cdot, t) > \lambda\}$ for every $t \geq t_0$ such that $R_{[(t-t_0)/h]+1} > 0$. Let then r_σ be the unique solution to the ODE

$$(3.4) \quad \begin{cases} \dot{r}_\sigma(t) = \hat{\kappa}(r_\sigma(t)/\sigma) \\ r_\sigma(0) = R_0. \end{cases}$$

Employing the monotonicity of $\hat{\kappa}$, if $r_\sigma(t) \leq R_n$, then

$$\begin{aligned} r_\sigma((n+1)h) &\leq R_n + \int_{nh}^{(n-1)h} \hat{\kappa}\left(\frac{r_\sigma(s)}{\sigma}\right) ds \leq R_n + \int_{nh}^{(n-1)h} \hat{\kappa}\left(\frac{R_n}{\sigma}\right) ds \\ &\leq R_n + \int_{nh}^{(n-1)h} \frac{1}{c_\psi} G(-\bar{c}(R_n/\sigma) - \|f\|_\infty) ds = R_{n+1}. \end{aligned}$$

Therefore, $B(x_0, r_\sigma(h[(t-t_0)/h] + h)) \subseteq \{u_h(\cdot, t) > \lambda\}$ for $t \geq t_0$ as long as the radius is positive. We conclude sending $h \rightarrow 0$, then $R_0 \rightarrow r_0$ and $\sigma \rightarrow 1$. \square

We are now in the position to prove our main result.

Proof of Theorem 3.5. Consider u as defined in (3.4): we show that u is a subsolution, as proving that it is a supersolution is analogous. Let $\eta(x, t)$ be an admissible test function in $\bar{z} := (\bar{x}, \bar{t})$ and assume that (\bar{x}, \bar{t}) is a strict maximum point for $u - \eta$. Assume furthermore that $u - \eta = 0$ in such point.

Case 1: We assume that $\nabla\eta(\bar{z}) \neq 0$. Firstly, in the case $a < +\infty$ we remark that if $\partial_t\eta/\psi(\nabla\eta(\bar{z})) \leq -a$, then (2.8) is trivially satisfied, thus we can assume wlog that

$$(3.5) \quad \frac{\partial_t\eta(\bar{z})}{\psi(\nabla\eta(\bar{z}))} > -a.$$

By classical arguments (recalled in [8]) we can assume that each function $u_{h_k} - \eta$ assumes a local supremum in $B_\rho(\bar{z})$ at a point $z_{h_k} := (x_k, t_k)$ and that $u_{h_k}(z_{h_k}) \rightarrow u(\bar{z})$ as $k \rightarrow \infty$. Moreover, we can assume that $\nabla\eta(z_k) \neq 0$ for k large enough.

Step 1: We define a suitable competitor for the minimality of the level sets of u_h . By the previous remarks we have that

$$(3.6) \quad u_h(x, t) \leq \eta(x, t) + c_k$$

where $c_k := u_{h_k}(x_k, t_k) - \eta(x_k, t_k)$, with equality if $(x, t) = (x_k, t_k)$. Let $\sigma > 0$ and set

$$\eta_{h_k}^\sigma(x) := \eta(x, t_k) + c_k + \frac{\sigma}{2}|x - x_k|^2.$$

Then, for all $x \in \mathbb{R}^N$,

$$u_{h_k}(x, t_k) \leq \eta_{h_k}^\sigma(x)$$

with equality if and only if $x = x_k$. We set $l_k = u_{h_k}(x_k, t_k) = \eta_{h_k}^\sigma(x_k)$. We fix $\varepsilon > 0$, to be chosen later, and define $E_\varepsilon^k := \{u_{h_k}(\cdot, t_k) > l_k - \varepsilon\} = T_{h_k, t_k - h_k} \{u_{h_k}(\cdot, t_k - h_k) > l_k - \varepsilon\}$ ⁵ and

$$(3.7) \quad W_\varepsilon^k := E_\varepsilon^k \setminus \{\eta_{h_k}^\sigma(\cdot) > l_k + \varepsilon\}.$$

⁵The choice of working with the open superlevel sets is motivated by our need to employ (2.18)

Assume that E_ε^k is bounded and let us define $E_{\varepsilon,n}^k$ as the sets constructed by Corollary 2.12 where $\{u_{h_k}(\cdot, t_k - h_k) > l_k - \varepsilon\}$, E_ε^k substitute $E, T_{h,t}^- E$ respectively. We thus have that $E_{\varepsilon,n}^k \nearrow E_\varepsilon^k$ as $n \rightarrow \infty$ and that each $E_{\varepsilon,n}^k$ is the minimal minimizer of a problem in the form (2.17). We define

$$(3.8) \quad W_{\varepsilon,n}^k := E_{\varepsilon,n}^k \setminus \{\eta_{h_k}^\sigma(\cdot) > l_k + \varepsilon, \}$$

It is easy to see that, along any subsequence $n(\varepsilon) \rightarrow \infty$ as $\varepsilon \rightarrow 0$, it holds $W_{\varepsilon,n(\varepsilon)}^k \rightarrow \{x\}$ as $\varepsilon \rightarrow 0$ in the Hausdorff sense. Furthermore, we check that for every $\varepsilon, k > 0$ there exists $n(\varepsilon, k)$ large enough such that $|W_{\varepsilon,n}^k| > 0$ for all $n \geq n(\varepsilon, k)$. Indeed, by the continuity of η^σ and since $|\nabla\eta(\bar{z})| \neq 0$ there exists a positive radius r such that

$$(B(x_k, r) \cap E_\varepsilon^k) \subseteq W_\varepsilon^k.$$

Since $x_k \in E_\varepsilon^k$ and it is an open set, it holds $|W_\varepsilon^k| > 0$. Recalling that $E_{\varepsilon,n}^k \rightarrow E_\varepsilon^k$ in L^1 , we conclude that $|W_{\varepsilon,n}^k| > 0$ for all $n = n(\varepsilon, k)$ large enough. Note also that, for every fixed k , $n(\varepsilon, k) \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

By minimality of $E_{\varepsilon,n}^k$ we have

$$(3.9) \quad \begin{aligned} & J(E_{\varepsilon,n}^k) + \int_{E_{\varepsilon,n}^k} g\left(\frac{1}{h_k} \text{sd}_{\{u_{h_k}(\cdot, t_k - h_k) > l_k - \varepsilon\}}^\psi(x)\right) \vee (-n) \, dx - f\left(\left[\frac{t}{h_k}\right] h_k\right) |W_{\varepsilon,n}^k| \\ & \leq J(E_{\varepsilon,n}^k \cap \{\eta_{h_k}^\sigma > l_k\}) + \int_{E_{\varepsilon,n}^k \cap \{\eta_{h_k}^\sigma > l_k\}} g\left(\frac{1}{h_k} \text{sd}_{\{u_{h_k}(\cdot, t_k - h_k) > l_k - \varepsilon\}}^\psi(x)\right) \vee (-n) \, dx. \end{aligned}$$

Adding to both sides $J(\{\eta_{h_k}^\sigma > l_k\} \cup E_{\varepsilon,n}^k)$ and using the submodularity of J , we obtain

$$\begin{aligned} & J(\{\eta_{h_k}^\sigma > l_k + \varepsilon\} \cup W_{\varepsilon,n}^k) - J(\{\eta_{h_k}^\sigma > l_k + \varepsilon\}) - f\left(\left[\frac{t}{h_k}\right] h_k\right) |W_{\varepsilon,n}^k| \\ & \quad + \int_{W_{\varepsilon,n}^k} g\left(\frac{1}{h_k} \text{sd}_{\{u_{h_k}(\cdot, t_k - h_k) > l_k - \varepsilon\}}^\psi(x)\right) \vee (-n) \, dx \leq 0. \end{aligned}$$

Equation (3.6) implies $\{u_{h_k}(\cdot, t_k - h_k) > l_k - \varepsilon\} \subseteq \{\eta(\cdot, t_k - h_k) > l_k - c_k - \varepsilon\}$, therefore by monotonicity we get

$$(3.10) \quad \begin{aligned} & J(\{\eta_{h_k}^\sigma > l_k + \varepsilon\} \cup W_{\varepsilon,n}^k) - J(\{\eta_{h_k}^\sigma > l_k + \varepsilon\}) - f\left(\left[\frac{t}{h_k}\right] h_k\right) |W_{\varepsilon,n}^k| \\ & \quad + \int_{W_{\varepsilon,n}^k} g\left(\frac{1}{h_k} \text{sd}_{\{\eta(\cdot, t_k - h_k) > l_k - c_k - \varepsilon\}}^\psi(x)\right) \vee (-n) \, dx \leq 0. \end{aligned}$$

If instead E_ε^k is an unbounded set with compact boundary, we employ (2.29) instead of (3.9) to obtain (3.10) in the computations above. See [13, 8] for details.

Step 2: We now estimate the terms appearing in (3.10). We start with the first two terms $J(\{\eta_{h_k}^\sigma > l_k + \varepsilon\} \cup W_{\varepsilon,n}^k) - J(\{\eta_{h_k}^\sigma > l_k + \varepsilon\})$. By definition of variational curvature, we get

$$(3.11) \quad J(\{\eta_{h_k}^\sigma > l_k + \varepsilon\} \cup W_{\varepsilon,n}^k) - J(\{\eta_{h_k}^\sigma > l_k + \varepsilon\}) \geq |W_{\varepsilon,n}^k| (\kappa(x_k, \{\eta_{h_k}^\sigma > l_k + \varepsilon\}) + o_\varepsilon(1)),$$

The last term in (3.10) can be treated as follows. For any $z \in W_\varepsilon$, we have

$$(3.12) \quad \eta(z, t_k) + c_k + \frac{\sigma}{2} |z - x_k|^2 \leq l_k + \varepsilon.$$

Since, in turn, $\eta(z, t_k) + c_k > l_k - \varepsilon$ it follows that $\sigma |z - x_k|^2 < 4\varepsilon$ and thus, for ε small enough,

$$(3.13) \quad W_\varepsilon^k \subseteq B_{c\sqrt{\varepsilon}}(x_k).$$

Therefore, by Hausdorff convergence it holds that for every $\varepsilon, k > 0$ there exists $n = n(\varepsilon, k)$ large enough such that

$$(3.14) \quad W_{\varepsilon, n}^k \subseteq B_{2c\sqrt{\varepsilon}}(x_k).$$

On the other hand, by a Taylor expansion, for every $z \in W_{\varepsilon, n}^k$ we have

$$(3.15) \quad \eta(z, t_k - h_k) = \eta(z, t_k) - h_k \partial_t \eta(z, t_k) + h_k^2 \int_0^1 (1-s) \partial_{tt}^2 \eta(z, t_k - sh_k) ds.$$

Then, we consider $y \in \{\eta(\cdot, t_k - h_k)(y) = l_k - c_k - \varepsilon\}$ being a point of minimal ψ -distance from z , that is, $\psi(z - y) = |\text{sd}_{\{\eta(\cdot, t_k - h_k)(y) > l_k - c_k - \varepsilon\}}^\psi(z)|$. One can prove (see [8, eq. (4.26)] for details) that

$$(3.16) \quad |z - y| = O(h_k).$$

Moreover, it holds (see [13, eq (6.26)] for details)

$$(z - y) \cdot \frac{\nabla \eta(y, t_k - h_k)}{|\nabla \eta(y, t_k - h_k)|} = \pm \psi \left(\frac{\nabla \eta(y, t_k - h_k)}{|\nabla \eta(y, t_k - h_k)|} \right) \text{dist}_{\{\eta(\cdot, t_k - h_k)(y) = l_k - c_k - \varepsilon\}}^\psi(z),$$

with a “+” if $z \in \{\eta(\cdot, t_k - h_k)(y) \leq l_k - c_k - \varepsilon\}$ and a “-” otherwise. We get

$$(3.17) \quad \begin{aligned} \eta(z, t_k - h_k) &= \eta(y, t_k - h_k) + (z - y) \cdot \nabla \eta(y, t_k - h_k) \\ &\quad + \int_0^1 (1-s) (\nabla^2 \eta(y + s(z-y), t_k - h_k)(z-y)) \cdot (z-y) ds \\ &= l_k - c_k - \varepsilon - \text{sd}_{\{\eta(\cdot, t_k - h_k)(y) = l_k - c_k - \varepsilon\}}^\psi(z) \psi(\nabla \eta(y, t_k - h_k)) \\ &\quad + \int_0^1 (1-s) (\nabla^2 \eta(y + s(z-y), t_k - h_k)(z-y)) \cdot (z-y) ds. \end{aligned}$$

Note that, in view of (3.12) it holds $|\eta(z, t_k) - \eta(y, t_k)| \leq c\varepsilon + ch_k = O(h_k)$, provided $\varepsilon \ll h_k$ and small enough. Thus, using also (3.14), (3.16) we deduce

$$\begin{aligned} \frac{1}{h_k} \text{sd}_{\{\eta(\cdot, t_k - h_k) > l_k - c_k - \varepsilon\}}^\psi(z) &\geq \frac{\partial_t \eta(z, t_k) - \frac{2\varepsilon}{h_k} - O(h_k) - O_{h_k}(1)}{\psi(\nabla \eta(y, t_k - h_k))} \\ &= \frac{\partial_t \eta(x_k, t_k) + O(\sqrt{\varepsilon}) - \frac{2\varepsilon}{h_k} - O(h_k) - O_{h_k}(1)}{\psi(\nabla \eta(x_k, t_k - h_k)) + O(\sqrt{\varepsilon}) + O(h_k)}, \end{aligned}$$

and we apply g to both sides to conclude

$$(3.18) \quad g \left(\frac{1}{h_k} \text{sd}_{\{\eta(\cdot, t_k - h_k) > l_k - c_k - \varepsilon\}}^\psi(z) \right) \geq g \left(\frac{\partial_t \eta(x_k, t_k) - O_{h_k}(1)}{\psi(\nabla \eta(x_k, t_k - h_k)) + O(h_k)} \right)$$

Step 4: We conclude the proof. Combining (3.10), (3.11) and (3.18), we arrive at

$$(3.19) \quad 0 \geq |W_{\varepsilon, n}^k| \left(\kappa(x_k, \{\eta_{h_k}^\sigma > l_k + \varepsilon\}) + o_\varepsilon(1) - f \left(\left[\frac{t}{h_k} \right] h_k \right) + g \left(\frac{\partial_t \eta(x_k, t_k) - O_{h_k}(1)}{\psi(\nabla \eta(x_k, t_k - h_k)) + O(h_k)} \right) \vee (-n) \right).$$

Choosing $n = n(\varepsilon, k)$, we can divide by $|W_{\varepsilon, n(\varepsilon, k)}^k| > 0$ and apply G to both sides to get

$$G \left(-\kappa(x_k, \{\eta_{h_k}^\sigma > l_k + \varepsilon\}) + o_\varepsilon(1) + f \left(\left[\frac{t}{h_k} \right] h_k \right) \right) \geq$$

$$G\left(g\left(\frac{\partial_t \eta(x_k, t_k) - O_{h_k}(1)}{\psi(\nabla \eta(x_k, t_k - h_k)) + O(h_k)}\right) \vee (-n(\varepsilon, k))\right).$$

Let us fix $k > 0$ and send $\varepsilon \rightarrow 0$ (thus also $n(\varepsilon, k) \rightarrow 0$). Thanks to the continuity of G and recalling also that $W_{\varepsilon, n(\varepsilon, k)}^k \rightarrow \{x\}$ as $\varepsilon \rightarrow 0$, we let $\varepsilon \rightarrow 0$ and arrive at

$$G\left(-\kappa(x_k, \{\eta_{h_k}^\sigma > l_k + \varepsilon\}) + f\left(\left[\frac{t}{h_k}\right] h_k\right)\right) \geq \frac{\partial_t \eta(x_k, t_k) - O_{h_k}(1)}{\psi(\nabla \eta(x_k, t_k)) + O(h_k)},$$

which finally implies the thesis by letting simultaneously $\sigma \rightarrow 0$ and $k \rightarrow +\infty$.

Case 2: We assume $\nabla \eta(\bar{x}, \bar{t}) = 0$ and prove that $\partial_t \eta(\bar{x}, \bar{t}) \leq 0$. The proof follows the line of the one in [13]. We focus on the case $a = +\infty$, the other being simpler.

Since $\nabla \eta(\bar{z}) = 0$, there exist $\ell \in \mathcal{F}_\kappa$ and $\omega \in C^\infty(\mathbb{R})$ with $\omega'(0) = 0$ such that

$$|\eta(x, t) - \eta(\bar{z}) - \partial_t \eta(\bar{z})(t - \bar{t})| \leq \ell(|x - \bar{x}|) + \omega(|t - \bar{t}|)$$

thus, we can define

$$\begin{aligned} \tilde{\eta}(x, t) &= \partial_t \eta(\bar{z})(t - \bar{t}) + 2\ell(|x - \bar{x}|) + 2\omega(|t - \bar{t}|) \\ \tilde{\eta}_k(x, t) &= \tilde{\eta}(x, t) + \frac{1}{k(\bar{t} - t)}. \end{aligned}$$

We remark that $u - \tilde{\eta}$ achieves a strict maximum in \bar{z} and the local maxima of $u - \tilde{\eta}_k$ in $\mathbb{R}^N \times [0, \bar{t}]$ are in points $(x_k, t_k) \rightarrow \bar{z}$ as $k \rightarrow \infty$, with $t_n \leq \bar{t}$. From now on, the only difference from [13] is in the case $x_k = \bar{x}$ for an (unrelabeled) subsequence. We thus assume $x_k = \bar{x}$ for all $k > 0$ and define $b_k = \bar{t} - t_k > 0$ and the radii

$$r_k := \ell^{-1}(a_k b_k),$$

where $a_k \rightarrow 0$ must be chosen such that the extinction time for the solution of (3.3) satisfies $T^*(r_k) \geq \bar{t} - t_k$, for k large enough. To show that such a choice for a_k is possible, we set

$$(3.20) \quad \beta(t) = \sup_{0 \leq s \leq t} \hat{\kappa}(\ell^{-1}(s)) \ell'(\ell^{-1}(s)),$$

where $\hat{\kappa}$ is as in (3.3). Note that by Definition 2.3 it holds $\beta(t) \leq \hat{\kappa}(t)$ for t small, β is non decreasing in t and $\beta(t) \rightarrow 0$ as $t \rightarrow 0$. We then have

$$(3.21) \quad \begin{aligned} \frac{T^*(r_k)}{b_k} &\geq \frac{1}{b_k} \int_{r_k/2}^{r_k} \frac{1}{\hat{\kappa}(s)} ds = \frac{1}{b_k} \int_{\ell^{-1}(a_k b_k/2)}^{\ell^{-1}(a_k b_k)} \frac{1}{\hat{\kappa}(s)} ds \\ &= \frac{a_k}{2} \int_{a_k b_k/2}^{a_k b_k} \frac{1}{\hat{\kappa}(\ell^{-1}(r)) \ell'(\ell^{-1}(r))} dr \geq \frac{a_k}{2} \frac{1}{\beta(b_k)} = 2, \end{aligned}$$

where in the last equality we chose $a_k := 4\beta(b_k)$ which tends to 0 as $k \rightarrow \infty$.

By definition of $\tilde{\eta}_k$ it holds

$$\begin{aligned} B(\bar{x}, r_k) &\subseteq \{\tilde{\eta}_k(\cdot, t_k) \leq \tilde{\eta}_k(\bar{x}, t_k) + 2\ell(r_k)\} \\ &\subseteq \{u(\cdot, t_k) \leq u(\bar{x}, t_k) + 2\ell(r_k)\}, \end{aligned}$$

by maximality of $u - \tilde{\eta}_k$ at z_k and since $u(z_k) = \tilde{\eta}_k(z_k)$. Since the balls $B(\cdot, r_k)$ are not vanishing, by Lemma 3.7 we have

$$(3.22) \quad \bar{x} \in \{u(\cdot, \bar{t}) \leq u(\bar{x}, t_k) + 2\ell(r_k)\}.$$

Finally, using again the maximality of $u - \eta$ at \bar{z} , the choice of r_k and (3.22), we obtain

$$\frac{\eta(\bar{z}) - \eta(\bar{x}, t_k)}{\bar{t} - t_k} = \frac{\eta(\bar{z}) - \eta(\bar{x}, t_k)}{b_k} \leq \frac{u(\bar{z}) - u(\bar{x}, t_k)}{b_k} \leq \frac{2\ell(r_k)}{b_k} = 2a_k.$$

Passing to the limit $k \rightarrow \infty$, we conclude that $\partial_t \eta(\bar{z}) \leq 0$. \square

Remark 3.8. We conclude with the following remark. The theory of [13] allows to show *uniqueness* for the Cauchy problem

$$\begin{cases} \partial_t u(x, t) + |\nabla u(x, t)| \kappa(x, \{u(\cdot, t) \geq u(x, t)\}) = 0 \\ u(\cdot, 0) = u_0, \end{cases}$$

which corresponds to (2.5) for $G = id$, $\psi = |\cdot|$ and $\mathbf{f} = 0$, under some more hypotheses on the curvature considered. In particular, the curvature κ is required to be either of “first order” (that is, its semicontinuous extensions do not depend on the second derivative variable), or satisfy a uniform continuity property (see conditions (FO) and (C’), respectively, in [13] for the precise definitions). Since our nonlinearity G is continuous, it is easy to see that if the first order condition is satisfied by κ , then also $-G(-\kappa)$ satisfies the same condition. On the other hand, requiring that

G is uniformly continuous

we can deduce that if κ satisfies the uniform continuity condition, so does $-G(-\kappa)$. In particular, our result Theorem 3.5 shows convergence of the minimizing movements scheme towards the *unique* continuous viscosity solution to

$$\begin{cases} \partial_t u(x, t) - |\nabla u(x, t)| G(-\kappa(x, \{u(\cdot, t) \geq u(x, t)\})) = 0 \\ u(\cdot, 0) = u_0. \end{cases}$$

Note that this applies to some relevant examples. For instance, the fractional mean curvature satisfies the first order condition. On the other hand, other instances of (variational) curvatures are of second order, as the classical mean curvature. In this case, uniqueness is ensured, for instance, for choices of G as $G(s) = (-M) \vee s \wedge M$ for $M > 0$, or $G(s) = s^{\frac{1}{3}}$.

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