

WEAK SOLUTIONS TO KOLMOGOROV-FOKKER-PLANCK EQUATIONS: REGULARITY, EXISTENCE AND UNIQUENESS

PASCAL AUSCHER, CYRIL IMBERT, AND LUKAS NIEBEL

ABSTRACT. In this article, we establish embeddings à la Lions and transfer of regularity à la Bouchut for a large scale of kinetic spaces. We use them to identify a notion of weak solutions to Kolmogorov-Fokker-Planck equations with (local or integral) diffusion and rough (measurable) coefficients under minimal requirements. We prove their existence and uniqueness for a large class of source terms, first in full space for the time, position and velocity variables and then for the kinetic Cauchy problem on infinite and finite time intervals.

CONTENTS

1. Introduction	1
2. Homogeneous kinetic spaces and Kolmogorov operators	9
3. Uniqueness and kinetic embeddings	20
4. Construction of weak solutions to rough equations	24
5. Kinetic Cauchy problems	27
6. The special case of local diffusion operators	32
7. Extensions and link with other kinetic embeddings	36
References	40

1. INTRODUCTION

1.1. **Kolmogorov-Fokker-Planck equations.** This work is concerned with the construction of weak solutions for kinetic equations of Kolmogorov-Fokker-Planck type. The prototypical example is given by the equation considered by A. Kolmogorov in [22],

$$(1.1) \quad (\partial_t + v \cdot \nabla_x)f - \Delta_v f = S, \quad (t, x, v) \in \Omega.$$

The unknown function $f = f(t, x, v)$ of the time t , position x and velocity v variables is defined in (a subset of) $\Omega = \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ and the source term S is given. We are

Date: March 25, 2024.

2010 *Mathematics Subject Classification.* Primary: 35K65, 35R05, 35D30, 35Q84, 35R09 Secondary: 35K70, 35B65.

Key words and phrases. Kolmogorov-Fokker-Planck equations, kinetic spaces, transfer of regularity, weak solutions, kinetic Cauchy problems.

The third author is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044 –390685587, Mathematics Münster: Dynamics–Geometry–Structure.

also interested in the fractional counterpart of the previous equation [19, 27, 28],

$$(1.2) \quad (\partial_t + v \cdot \nabla_x) f + (-\Delta_v)^\beta f = S$$

where $\beta \in (0, 1)$ – see below for a definition of the fractional Laplacian $(-\Delta_v)^\beta$.

Motivated by the study of non-linear equations, we choose a framework that encompasses equations with coefficients. For this reason, we consider equations under the following general form,

$$(1.3) \quad (\partial_t + v \cdot \nabla_x) f + \mathcal{A}f = S$$

where \mathcal{A} is a linear operator acting on the velocity variable v (in a uniform fashion with respect to the other variables t, x). Kinetic Fokker-Planck equations with rough coefficients correspond to

$$(1.4) \quad \mathcal{A}f(t, x, v) = -\nabla_v \cdot (\mathbf{A} \nabla_v f)(t, x, v)$$

where $\mathbf{A} = \mathbf{A}(t, x, v)$ is a real symmetric matrix satisfying for a.e. $(t, x, v) \in \Omega$,

$$\text{eigenvalues of } \mathbf{A}(t, x, v) \text{ lie in } [\lambda, \Lambda]$$

for some $0 < \lambda \leq \Lambda < \infty$. Another important class of examples are kinetic equations with integral diffusion. They correspond to

$$(1.5) \quad \mathcal{A}f(t, x, v) = \text{PV} \int_{\mathbb{R}^d} (f(t, x, v) - f(t, x, v')) k(t, x, v, v') dv'$$

where the function $k: \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ is non-negative and satisfies appropriate ellipticity conditions. The main example is $\mathcal{A}f(t, x, v) = (-\Delta_v)^\beta f(t, x, v)$ for which $k(t, x, v, v') = c_{d,\beta} |v' - v|^{-d-2\beta}$ for some explicit constant $c_{d,\beta} > 0$. In the general integro-differential case, since the kernel k can be singular, the meaning of $\mathcal{A}f$ has to be understood in the principal value (PV) sense. The simplest form of the ellipticity conditions for the kernel k are the following ones,

$$\begin{cases} k(t, x, v, v') = k(t, x, v', v) \geq 0, \\ \frac{\lambda}{|v' - v|^{d+2\beta}} \leq k(t, x, v, v') \leq \frac{\Lambda}{|v' - v|^{d+2\beta}}. \end{cases}$$

These conditions are considered in many articles concerned with the regularity of parabolic equations with integral diffusion, see for instance [9, 10, 21]. Our approach allows us to deal with a larger class of kernels, satisfying weaker ellipticity conditions. More precisely, our results apply to the class of kinetic equations with integral diffusion introduced in [19]. They naturally appear in the context of the study of the Boltzmann equation without Grad's cut-off assumption. See Remark 1.8 below.

1.2. Kinetic embedding and transfer of regularity. A question one may ask is: what is the correct definition of a weak solution? By correct, we mean a definition that allows one to prove existence and uniqueness even for rough coefficients or kernels. Recall that the linear operator \mathcal{A} is diffusive in the velocity variable so one necessarily requires regularity of order β in the v -variable. Do we need more *a priori* regularity conditions in the definition? The answer is no. Indeed, it is known since the discovery of hypoellipticity and averaging lemmas that the transport operator transfers v -regularity to the other variables. The expected amount of transferred regularity in the x variable with respect to the diffusion parameter $\beta > 0$ can be computed by the underlying scaling invariance $(t, x, v) \mapsto (r^{2\beta}t, r^{2\beta+1}x, rv)$ of the

fractional Kolmogorov equation. Our first main result quantifies this in a way that is both reminiscent of Bouchut's transfer of regularity result in [8] and Lions' embedding theorem in [24].

Theorem 1.1 (Kinetic embedding and transfer of regularity). *Let $\beta \in (0, d/2)$. For any $f \in \mathcal{D}'(\Omega)$ such that $f \in L_{t,x}^2 \dot{H}_v^\beta$ and $(\partial_t + v \cdot \nabla_x)f = S_1 + S_2 + S_3$, with $S_1 \in L_{t,x}^2 \dot{H}_v^{-\beta}$, $S_2 \in L_{t,v}^2 \dot{H}_x^{-\frac{\beta}{2\beta+1}}$ and $S_3 \in L_t^1 L_{x,v}^2$, we have,*

(i) $f \in L_{t,v}^2 \dot{H}_x^{\frac{\beta}{2\beta+1}} \cap C_0(\mathbb{R}_t; L_{x,v}^2)$ with

$$\|f\|_{L_{t,v}^2 \dot{H}_x^{\frac{\beta}{2\beta+1}}} + \sup_{t \in \mathbb{R}} \|f(t)\|_{L_{x,v}^2} \lesssim_{d,\beta} \|f\|_{L_{t,x}^2 \dot{H}_v^\beta} + \|S_1\|_{L_{t,x}^2 \dot{H}_v^{-\beta}} + \|S_2\|_{L_{t,v}^2 \dot{H}_x^{-\frac{\beta}{2\beta+1}}} + \|S_3\|_{L_t^1 L_{x,v}^2}.$$

(ii) The map $t \mapsto \|f(t)\|_{L_{x,v}^2}^2$ is absolutely continuous on \mathbb{R} with

$$(1.6) \quad \frac{d}{dt} \|f(t)\|_{L_{x,v}^2}^2 = 2 \operatorname{Re} \left(\int_{\mathbb{R}^d} \langle S_1, f \rangle dx + \int_{\mathbb{R}^d} \langle S_2, f \rangle dv + \iint_{\mathbb{R}^{2d}} S_3 \bar{f} dx dv \right).$$

In (1.6), the first bracket denotes the sesquilinear duality between $\dot{H}_v^{-\beta}$ and \dot{H}_v^β , giving an integrable function of (t, x) , and the second one refers to the duality between $\dot{H}_x^{-\frac{\beta}{2\beta+1}}$ and $\dot{H}_x^{\frac{\beta}{2\beta+1}}$, giving an integrable function of (t, v) . Integrals are in the sense of Lebesgue.

Remark 1.2. We use homogeneous spaces and obtain scale invariant estimates. The homogeneous Sobolev spaces are defined in Section 2 where the limitation on β will be explained.

Remark 1.3. In Section 6 we also consider a different definition in the local case $\beta = 1$ that eliminates the constraint on β .

Remark 1.4. We shed light on the fact that we do not assume *a priori* control on $f(t) \in L_{x,v}^2$ at any time, which could look surprising. In Section 7, we will see that assuming it, but only qualitatively, allows to deal with $\beta \geq d/2$.

Remark 1.5. We allow the free transport of f to belong to a larger space than just the dual of $L_{t,x}^2 \dot{H}_v^\beta$.

Remark 1.6. Not only do we obtain continuity of $t \mapsto f(t) \in L_{x,v}^2$ but we also get absolute continuity of $t \mapsto \|f(t)\|_{L_{x,v}^2}^2$ and zero limit at infinity, both being crucial when coming to weak solutions.

Remark 1.7. Theorem 1.1 is a special case of a more general result, see Theorem 3.6, which itself is a consequence of embeddings obtained for a scale of kinetic spaces, see Theorem 2.2. Such generalizations could be useful for further developments in the field for weak solutions and we shall prove them in this article.

1.3. Weak solutions to the Kolmogorov-Fokker-Planck equation. We explained above that we are interested in constructing solutions of the Kolmogorov-Fokker-Planck equation (1.3) for some general operator \mathcal{A} . Keeping in mind examples (1.4) and (1.5), we now make precise the definition of this operator and the assumptions that we need. As mentioned above, we want that for all $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, \mathcal{A} maps \dot{H}_v^β to $\dot{H}_v^{-\beta}$ in a uniform fashion. To make it rigorous, we assume that $\mathbb{R} \times \mathbb{R}^d \ni (t, x) \mapsto a_{t,x}$ is a family of continuous sesquilinear forms $a_{t,x}: \dot{H}_v^\beta \times \dot{H}_v^\beta \rightarrow \mathbb{C}$ with

$$(1.7) \quad |a_{t,x}(f, g)| \leq \Lambda \|f\|_{\dot{H}_v^\beta} \|g\|_{\dot{H}_v^\beta}$$

for some $\Lambda < \infty$, uniformly for $(f, g) \in \dot{H}_v^\beta \times \dot{H}_v^\beta$, $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, and $(t, x) \mapsto a_{t,x}(f, g)$ measurable. We also assume that there exists $\lambda > 0$ such that for all $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ and all $f \in \dot{H}_v^\beta$,

$$(1.8) \quad \operatorname{Re} a_{t,x}(f, f) \geq \lambda \|f\|_{\dot{H}_v^\beta}^2.$$

The space

$$\dot{F}^\beta = L_{t,x}^2 \dot{H}_v^\beta$$

plays a central role and we set

$$(1.9) \quad \forall f, g \in \dot{F}^\beta, \quad a(f, g) = \iint_{\mathbb{R} \times \mathbb{R}^d} a_{t,x}(f(t, x), g(t, x)) \, dt \, dx.$$

For short, we write $a_{t,x}(f(t, x), g(t, x))$ as $a_{t,x}(f, g)$. Then, a is a continuous and coercive sesquilinear form on $\dot{F}^\beta \times \dot{F}^\beta$. Denote by $(\dot{F}^\beta)'$ the dual of \dot{F}^β realized in the duality extending the $L_{t,x,v}^2$ inner product and $\mathcal{A}: \dot{F}^\beta \rightarrow (\dot{F}^\beta)'$ the isomorphism associated to a by

$$\langle \mathcal{A}f, g \rangle = a(f, g), \quad f, g \in \dot{F}^\beta.$$

Remark 1.8. We can work with weaker forms of boundedness and ellipticity conditions using inhomogeneous spaces, see (5.3) and (5.4). The situation is well-understood when \mathcal{A} is given by (1.4). When \mathcal{A} is given by a diffusion kernel (1.5), [19] gives sufficient conditions on the kernel for obtaining the weaker forms: see Condition (1.3) with $\bar{R} = +\infty$ for ellipticity and Conditions (1.4), (1.5), (1.6), (1.7) with $\bar{R} = +\infty$ for boundedness. The interested reader is referred to [19, Theorem 4.1] and [26, Theorem 2.1].

With the kinetic embedding and the transfer of regularity at hand, and the general form of the elliptic operator $\mathcal{A}: L_{t,x}^2 \dot{H}_v^\beta \rightarrow L_{t,x}^2 \dot{H}_v^{-\beta}$ with rough coefficients, we define weak solutions of $(\partial_t + v \cdot \nabla_x) + \mathcal{A}$ and state our second main result concerned with their construction for fairly general source terms S in appropriate spaces.

Definition 1.9 (Weak solutions). Let $\beta \in (0, d/2)$ and $S \in \mathcal{D}'(\Omega)$. A distribution $f \in \mathcal{D}'(\Omega)$ is said to be a *weak solution* to (1.3) if $f \in L_{t,x}^2 \dot{H}_v^\beta$ and for all $h \in \mathcal{D}(\Omega)$,

$$-\iiint_{\Omega} f(\partial_t + v \cdot \nabla_x) \bar{h} \, dt \, dx \, dv + a(f, h) = \langle S, h \rangle.$$

Remark 1.10. We may also consider the adjoint equation

$$(1.10) \quad -(\partial_t + v \cdot \nabla_x) f + \mathcal{A}^* f = S,$$

where \mathcal{A}^* is the adjoint of \mathcal{A} . We refer to it as the backward kinetic equation and we define weak solutions similarly. The theory we develop will need to consider both equations (1.3) and (1.10).

Remark 1.11. We do not assume more defining conditions for weak solutions than being in $L_{t,x}^2 \dot{H}_v^\beta$ and in particular no *a priori* $L_{x,v}^2$ condition like the common assumption $L_{t,\text{loc}}^\infty L_{x,v}^2$ (see for example [12, 15, 16, 19]). Our definition of \dot{H}_v^β and the restriction $0 < \beta < d/2$ will guarantee that a weak solution belongs to L_{loc}^2 in all variables and is a tempered distribution, that is $f \in \mathcal{S}'(\Omega)$. Thus the source S must also be in $\mathcal{S}'(\Omega)$. The sesquilinear duality $\langle S, h \rangle$ between the distribution S and test functions will be specified depending on the assumptions on S . In Section 7, we explain how to treat the case $\beta \geq d/2$.

Theorem 1.12 (Existence and uniqueness of weak solutions). *Let $\beta \in (0, d/2)$. Assume that the family a of sesquilinear forms satisfies (1.7) and (1.8). Let $S = S_1 + S_2 + S_3$ with $S_1 \in L_{t,x}^2 \dot{H}_v^{-\beta}$, $S_2 \in L_{t,v}^2 \dot{H}_x^{-\frac{\beta}{2\beta+1}}$ and $S_3 \in L_t^1 L_{x,v}^2$. There exists a unique weak solution $f \in L_{t,x}^2 \dot{H}_v^\beta$ to (1.3) in the sense of Definition 1.9. Moreover, this solution belongs to $L_{t,v}^2 \dot{H}_x^{\frac{\beta}{2\beta+1}} \cap C_0(\mathbb{R}_t; L_{x,v}^2)$ and*

$$\begin{aligned} \sup_{t \in \mathbb{R}} \|f(t)\|_{L_{x,v}^2} + \|D_x^{\frac{\beta}{2\beta+1}} f\|_{L_{t,x,v}^2} + \|D_v^\beta f\|_{L_{t,x,v}^2} \\ \lesssim \|D_x^{-\frac{\beta}{2\beta+1}} S_1\|_{L_{t,x,v}^2} + \|D_v^{-\beta} S_2\|_{L_{t,x,v}^2} + \|S_3\|_{L_t^1 L_{x,v}^2} \end{aligned}$$

for an implicit constant depending only on $d, \beta, \lambda, \Lambda$, where D_x and D_v denote $(-\Delta_x)^{\frac{1}{2}}$ and $(-\Delta_v)^{\frac{1}{2}}$, respectively. In addition, the solution satisfies the energy equality: for any $s, t \in \mathbb{R}$ with $s < t$,

$$\begin{aligned} \|f(t)\|_{L_{x,v}^2}^2 + 2 \operatorname{Re} \int_s^t \int_{\mathbb{R}^d} a_{\tau,x}(f, f) \, dx \, d\tau = \\ (1.11) \quad \|f(s)\|_{L_{x,v}^2}^2 + 2 \operatorname{Re} \int_s^t \left(\int_{\mathbb{R}^d} \langle S_1, f \rangle \, dx + \int_{\mathbb{R}^d} \langle S_2, f \rangle \, dv + \iint_{\mathbb{R}^{2d}} S_3 \bar{f} \, dx \, dv \right) \, d\tau. \end{aligned}$$

Remark 1.13. As the problem is linear, all sources that belong to any interpolation space that can be obtained from the above three ones give rise to a weak solution. We shall not attempt to write such statements.

Remark 1.14. The time variable describes \mathbb{R} and we obtain homogeneous estimates. We also construct solutions for the kinetic Cauchy problem on half-infinite time intervals with homogeneous estimates, see Theorem 5.2.

Remark 1.15. We extend our approach to inhomogeneous spaces and construct solutions for the kinetic Cauchy problem in the corresponding inhomogeneous framework on finite time intervals, under weaker ellipticity conditions (5.3) and (5.4), see Theorem 5.7.

Remark 1.16. The same statements and remarks hold for the backward equations corresponding to the operator in (1.10).

1.4. Highlights and observations. The theory of kinetic equations related to this work is much developed and we shall provide a review. We wish to highlight some points in our contribution and make a few observations on tools that we use or not.

The Hilbertian framework proposed by J.-L. Lions to construct weak solutions had already been used for some particular kinetic equations. We show it applies in full generality, thanks to the complete description of the kinetic embedding and transfer of regularity for scales of adapted kinetic spaces: it allows us to obtain uniqueness and also to obtain existence with more source terms.

A second highlight concerns the strategy of proof of the kinetic embedding and transfer of regularity: it boils down to estimates and appropriate uniqueness of distributional solutions of the constant coefficients Kolmogorov equation $(\partial_t + v \cdot \nabla_x)f + (-\Delta_v)^\beta f = S$ and/or its adjoint. This also shows isomorphism properties of the integral Kolmogorov operators which is interesting in its own right.

As for the techniques of proofs, they only use partial Fourier transformation for the position and velocity variables and the Galilean change of variables. Using the homogeneous norms also provides a very convenient setup in which estimates are quite elementary and do not require subtle decompositions in Fourier space.

We also mention that there are common tools in the field that we do not need. We neither use the Galilean group law associated with the fields $(\partial_t + v \cdot \nabla_x), \nabla_v$ nor rely on commutator techniques à la Hörmander in the context of hypoelliptic equations.

Our final observation is that we do not use the local regularity theory of weak solutions, such as the De Giorgi-Nash-Moser estimates. In other words, the construction of weak solutions does not require knowing that weak solutions have local regularity properties.

1.5. From Lions embedding to kinetic embeddings. We think it would be useful for the reader to focus on the genesis of our kinetic embedding and the subtlety of the homogeneous situation by describing the parabolic case where there is no transport and no position variable x .

The starting point of J.-L. Lions in his 1957 article [24] is his famous and well-known abstract embedding. Three Hilbert spaces (V, H, V') form a *Gelfand triple* if there are continuous and dense inclusions $V \hookrightarrow H \hookrightarrow V'$. With such a triple, Lions established that if $f \in L^2(0, T; V)$ and $f \in H^1(0, T; V')$ then $f \in C([0, T]; H)$ and $t \mapsto \|f(t)\|_H^2$ is absolutely continuous. His proof uses the *a priori* knowledge that $u(t)$ exists in H almost everywhere, approximation procedures that are compatible in the three spaces V, H, V' and integration theory for vector-valued functions.

For example, with $V = H^1(\mathbb{R}^d)$ the classical Sobolev space, this shows that if $f \in L^2(0, T; H^1(\mathbb{R}^d))$ and $f \in H^1(0, T; H^{-1}(\mathbb{R}^d))$ then $f \in C([0, T]; L^2(\mathbb{R}^d))$. Given the natural scaling, the embedding could be envisioned with H^1, H^{-1} replaced by their homogeneous versions \dot{H}^1, \dot{H}^{-1} : however, it is not true when $T < \infty$, for the simple reason that both embeddings $\dot{H}^1(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow \dot{H}^{-1}(\mathbb{R}^d)$ fail. Indeed, take any $g \in \dot{H}^1(\mathbb{R}^d)$ which is neither an element of $L^2(\mathbb{R}^d)$ nor $\dot{H}^{-1}(\mathbb{R}^d)$ and consider $f(t) \equiv g \in L^2(0, 1; \dot{H}^1(\mathbb{R}^d)) \cap \dot{H}^1(0, 1; \dot{H}^{-1}(\mathbb{R}^d))$ which cannot be an element of $C([0, 1]; L^2(\mathbb{R}^d))$. In other words, there is no possible temporal trace theory with homogeneous spaces on finite time intervals.

This failure is not an obstacle when $T = \infty$. Indeed, it is proved by the first author with S. Monniaux and P. Portal in [5] that the embedding is true provided one works on an infinite interval. The strategy is to prove first a uniqueness result for the backward heat equation so that any distribution f that belongs to $L^2(0, \infty; \dot{H}^1(\mathbb{R}^d)) \cap \dot{H}^1(0, \infty; \dot{H}^{-1}(\mathbb{R}^d))$ is represented as a Duhamel integral for the backward heat equation (up to a constant, depending on how homogeneous Sobolev spaces are defined) with source term $\partial_t f + \Delta f$. As this integral belongs to $C_0([0, \infty); L^2(\mathbb{R}^d))$ by construction, this yields the desired conclusion for the embedding. This can also be done on \mathbb{R} .

With another strategy using half-time derivatives, it was shown by the first author and M. Egert in [3] that the condition $f \in \dot{H}^1(\mathbb{R}; \dot{H}^{-1}(\mathbb{R}^d))$, that is, $\partial_t f \in L^2(\mathbb{R}; \dot{H}^{-1}(\mathbb{R}^d))$, can even be relaxed for the embedding, so as to allow more source terms in the equation.

Our proof of continuity in time in Theorem 1.1 will rely on using the strategy of [5] in a kinetic context: working on infinite intervals is required to obtain the scale-invariant estimates; and we allow as general source terms as possible, in the spirit of [3]. We also make it work for fractional diffusion.

1.6. Review of literature. This work is concerned with the study of kinetic equations with both local (typically $\beta = 1$) and integral diffusion. It is related to the existing literature stemming from various trends of research.

1.6.1. Kolmogorov equations. The Kolmogorov equation was first exhibited by A. Kolmogorov in 1934 in [22]. He derived explicitly the fundamental solution of (1.1), which allows us to observe the regularizing effect of the, at first glance, degenerate diffusion equation. This was the starting point of Hörmander's hypoellipticity theory [17]. One of the examples in his paper gave birth to a trend of research devoted to a class of equations now referred to as ultra-parabolic equations or equations of Kolmogorov type. It was launched by E. Lanconelli and S. Polidoro [23]. In this article, they first considered such equations with constant coefficients, exhibited a Lie group structure associated to the equation (extending the Galilean invariance of the original Kolmogorov equation) and established a Harnack inequality.

1.6.2. Construction of weak solutions. Weak solutions to the kinetic Fokker-Planck equation ($\beta = 1$) for constant diffusion coefficients in bounded domains were constructed in inhomogeneous function spaces using Lions existence theorem by J. A. Carrillo in [11]. More recently, D. Albritton, S. Armstrong, J. C. Mourrat and M. Novack [1] defined a kinetic space associated to a class of Fokker-Planck equations ($\beta = 1$) and apply again Lions's variational approach for well-posedness. This space is made of functions $f \in L^2_{t,x} H^1_v(\gamma dv)$ such that $(\partial_t + v \cdot \nabla_x) f \in L^2_{t,x} H^{-1}_v(\gamma dv)$. Because a term associated with confinement appears in the equation they are interested in, they work with a Gaussian weight γ in the velocity variable. Solutions are proven to be continuous in time with values in $L^2_{x,v}(\gamma dv)$.

K. Nyström and M. Litsgård considered the Cauchy-Dirichlet Problem for the Fokker-Planck equation ($\beta = 1$) in possibly unbounded domains in [25]. Weak solutions are constructed on the full space in the case where coefficients are allowed to be rough using a minimization procedure of a convex functional. For the uniqueness result, coefficients are allowed to be rough only in a compact set; they are assumed to

be constant in the rest of the domain. See also the work of F. Anceschi and A. Rebusci in [2] for a follow-up on an obstacle problem.

In [28], the third author together with R. Zacher introduced and developed a notion called *kinetic maximal L^2 -regularity*. This concept is applied to distributional solutions of the fractional Kolmogorov ($0 < \beta \leq 1$) with constant diffusion coefficients. The Cauchy problem is studied on $[0, T]$ and in inhomogeneous spaces in (x, v) -variables on the full space. Continuity in time is proven for the space of solutions and the temporal trace space is characterized in terms of an anisotropic Sobolev space. There are similarities and differences with our approach of kinetic embedding that we shall explain in the last section.

1.6.3. *Transfer of regularity.* The regularizing effect of equations combining free transport ($\partial_t + v \cdot \nabla_x$) and diffusion in the velocity variable v was first exhibited by A. Kolmogorov in [22]. It can be seen from his explicit computation of the fundamental solution of (1.1). This transfer of regularity has been quantified in Hörmander's work, although only in a crude way for the regularity exponents.

An important step in the study of the regularity of kinetic equations is the observation that the means of solutions in the velocity variable v are more regular than the solution itself. This was observed in particular by F. Golse, B. Perthame and R. Sentis [14] (see also [13]). Results of this type are nowadays called *averaging lemmas*. The theory was further developed by the study of the regularity of the solution itself, exhibiting the *transfer of regularity* property of the free transport. In this perspective, F. Bouchut's article [8] was influential as it was among the first ones to provide optimal and global estimates for this phenomenon, which is sometimes also called *kinetic regularisation*. We emphasize that the results of [8] apply to weak solutions only when $\beta = 1$ and in inhomogeneous spaces.

This regularising effect can also be seen for the weak solutions constructed in [1] by working with a commutator method inspired by the work of Hörmander. Sharp estimates are proven in adapted Besov spaces.

1.6.4. *Regularity of weak solutions.* Although we do not use the local regularity of weak solutions at all, recent developments in such topics have been crucial to our understanding. In the case of rough and real coefficients satisfying an ellipticity condition but without any smoothness assumptions on coefficients, a local L^∞ bound for weak solutions was obtained through Moser's iterative method by A. Pascucci and S. Polidoro [31]. Then W. Wang and L. Zhang [33] proved a local Hölder regularity thanks to a weak Poincaré inequality satisfied by weak subsolutions. In both works, weak solutions f together with $\nabla_v f$ and $(\partial_t + v \cdot \nabla_x)f$ are assumed to be locally square integrable, [which has no reason to hold for rough coefficients](#). Another proof of the local Hölder estimate was proposed in [12] by F. Golse, the second author, C. Mouhot and A. Vasseur, where a Harnack inequality is established. This latter work focuses on (1.3) with $\mathcal{A}f = -\nabla_v \cdot (\mathbf{A}\nabla_v f)$ and weak solutions are assumed to be locally in $L_t^\infty L_{x,v}^2 \cap L_{t,x}^2 H_v^1$ and such that $(\partial_t + v \cdot \nabla_x)f \in L_{t,x}^2 H_v^{-1}$. See also the works of J. Guerand with the second author and with C. Mouhot, and also the work of the third author with R. Zacher [15, 16, 30] for some further developments.

We conclude this review of literature by mentioning that we focused mainly on literature concerning weak solutions. For an overview of the literature on strong L^p

solutions or strong Hölder solutions we refer to [27, 29] and [18, 20] and the references therein. Here, the pioneering work of L. Rothschild and E. Stein [32] was influential.

1.7. Future research. We announce some subsequent works in upcoming articles. We shall use the material of this article to provide a construction of the fundamental solution. In the local case, we also obtain Gaussian estimates, see [4]. We shall also establish self-improvement of the regularity properties of local weak solutions with improved source terms, both in the local and non-local cases. The authors also plan to treat L^p estimates in the spirit of kinetic maximal L^p -regularity as in [29] for weak solutions of the (fractional) Kolmogorov equation with constant (and even possibly rough) coefficients. We will also develop further regularity theory along the characteristics with some new variational methods, with consequences for weak solutions.

Among possible other developments is the construction of weak solutions on bounded domains.

It is an interesting question how far one can stretch the methods of the article. While it is natural to apply them to operators of Kolmogorov type as considered in [23], another problem is to tackle Hörmander sum of squares with a drift term.

1.8. Organisation of the article. In Section 2, homogeneous kinetic spaces are introduced and estimates for Kolmogorov operators are obtained. Section 3 concerns kinetic embeddings. They rely on an appropriate uniqueness result for the equation associated with Kolmogorov operators. Weak solutions for equations with rough coefficients are constructed in Section 4. The kinetic Cauchy problem is treated in Section 5. In particular, inhomogeneous kinetic spaces are introduced and they are used to solve the kinetic Cauchy problem on finite time intervals. In Section 6, we make a special focus on the case of local diffusion operators ($\beta = 1$). Section 7 contains an alternative definition of homogeneous kinetic spaces that permits us to relax the constraint $\gamma \leq 2\beta$ and allows us to compare our findings with previously known kinetic embeddings. We also explain in this final section how to lift the limitation $\beta < d/2$.

1.9. Notation. The full space $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ is denoted by Ω while Ω_+ denotes $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. The set of distributions in Ω (resp. tempered distributions in Ω) is denoted by $\mathcal{D}'(\Omega)$ (resp. $\mathcal{S}'(\Omega)$). The operators $(-\Delta_v)^{1/2}$ and $(-\Delta_x)^{1/2}$ are denoted by D_v and D_x respectively. We work with functions of (t, x, v) . As we use mixed spaces, we shall put variables in the index, e.g. $L^2_{t,x} \dot{H}^\beta_v$. The Galilean change of variables is denoted by $\Gamma f(t, x, v) = f(t, x + tv, v)$. Given a Banach space B , $C_0(\mathbb{R}; B)$ denotes the space of continuous functions valued into B , with limit zero at $\pm\infty$. Given two Banach spaces X and Y , $X \hookrightarrow Y$ means that X is continuously embedded in Y . We use the same bracket notation for the duality pairing in different contexts with subsequent explanations, hoping this does not create any confusion.

2. HOMOGENEOUS KINETIC SPACES AND KOLMOGOROV OPERATORS

2.1. Main functional spaces. Before defining rigorously the homogeneous kinetic spaces that we will work with, we give here a short list and state the main result for the reader's convenience. The ranges of γ, β will be specified below.

- The space \dot{H}^α is the homogeneous Sobolev space on \mathbb{R}^d .
- The space \dot{F}^β is $L^2_{t,x} \dot{H}_v^\beta$.
- The space \dot{X}_β^γ equals $L^2_x \dot{H}_v^\gamma \cap L^2_v \dot{H}_x^{\frac{\gamma}{2\beta+1}}$ when $\gamma \geq 0$ and $\dot{X}_\beta^\gamma = L^2_x \dot{H}_v^\gamma + L^2_v \dot{H}_x^{\frac{\gamma}{2\beta+1}}$ if $\gamma \leq 0$.
- The space $\dot{\mathcal{F}}_\beta^\gamma$ is made of $f \in L^2_{t,x} \dot{H}_v^\gamma$ such that $(\partial_t + v \cdot \nabla_x)f \in L^2_{t,x} \dot{H}_v^{\gamma-2\beta}$.
- The space $\dot{\mathcal{G}}_\beta^\gamma$ is made of $f \in L^2_{t,x} \dot{H}_v^\gamma$ such that $(\partial_t + v \cdot \nabla_x)f \in L^2_t \dot{X}_\beta^{\gamma-2\beta}$.
- The space $\dot{\mathcal{L}}_\beta^\gamma$ is made of $f \in L^2_{t,x} \dot{H}_v^\gamma$ such that $(\partial_t + v \cdot \nabla_x)f \in L^2_t \dot{X}_\beta^{\gamma-2\beta} + L^1_t \dot{X}_\beta^{\gamma-\beta}$.

We use calligraphic letters whenever we are concerned with kinetic spaces, that is spaces which describe also the regularity in terms of the kinetic operator $\partial_t + v \cdot \nabla_x$.

Remark 2.1. The calligraphic scales are adapted to the fractional Laplacian of order β : formally, $(-\Delta_v)^\beta$ maps $\dot{\mathcal{L}}_\beta^\gamma$ into $L^2_{t,x} \dot{H}_v^{\gamma-2\beta} \subset L^2_t \dot{X}_\beta^{\gamma-2\beta}$ when $\gamma-2\beta \leq 0$. The spaces $\dot{\mathcal{F}}_\beta^\beta$ and $\dot{\mathcal{L}}_\beta^\beta$ are the most important ones here. Having full scales is important for the potential extension of the theory (and proofs are not much different anyway). The spaces $\dot{\mathcal{G}}_\beta^\gamma$ are also related to other kinetic spaces defined in Section 7.

Theorem 2.2 (Kinetic embeddings and transfers of regularity: homogeneous case). *Assume $\gamma < d/2$ and $0 \leq \gamma \leq 2\beta$.*

- $\dot{\mathcal{L}}_\beta^\gamma \hookrightarrow L^2_{t,v} \dot{H}_x^{\frac{\gamma}{2\beta+1}}$ with $\|D_x^{\frac{\gamma}{2\beta+1}} f\|_{L^2_{t,x,v}} \lesssim_{d,\beta,\gamma} \|f\|_{\dot{\mathcal{L}}_\beta^\gamma}$.
- $\dot{\mathcal{L}}_\beta^\gamma \hookrightarrow C_0(\mathbb{R}_t; \dot{X}_\beta^{\gamma-\beta})$ with $\sup_{t \in \mathbb{R}} \|f(t)\|_{\dot{X}_\beta^{\gamma-\beta}} \lesssim_{d,\beta,\gamma} \|f\|_{\dot{\mathcal{L}}_\beta^\gamma}$.
- There is a subspace of $\mathcal{S}(\Omega)$ which for all γ is dense in $\dot{\mathcal{F}}_\beta^\gamma$, $\dot{\mathcal{G}}_\beta^\gamma$ and $\dot{\mathcal{L}}_\beta^\gamma$.
- The families of spaces $(\dot{\mathcal{F}}_\beta^\gamma)_\gamma$ and $(\dot{\mathcal{G}}_\beta^\gamma)_\gamma$ have the complex interpolation property.

Remark 2.3. It is not clear whether the complex interpolation property is satisfied by $(\dot{\mathcal{L}}_\beta^\gamma)_\gamma$ or not.

2.2. Homogeneous kinetic spaces. We come to precise definitions.

2.2.1. Homogeneous Sobolev spaces $\dot{H}^\alpha(\mathbb{R}^d)$. To define homogeneous Sobolev spaces, we use the Fourier transform on tempered distributions in \mathbb{R}^d , equipped with Euclidean norm $|x|$ and inner product $x \cdot y$.

For $0 \leq \alpha < d/2$, we define the homogeneous Sobolev space $\dot{H}^\alpha(\mathbb{R}^d)$ as

$$(2.1) \quad \dot{H}^\alpha(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d); \exists g \in L^2(\mathbb{R}^d), \hat{f} = |\xi|^{-\alpha} \hat{g}\}$$

and we set $(-\Delta)^{\alpha/2} f = g$ and $\|f\|_{\dot{H}^\alpha} = \|g\|_2$. Note that when $g \in L^2(\mathbb{R}^d)$ and $0 \leq \alpha < d/2$, $|\xi|^{-\alpha} \hat{g} \in \mathcal{S}'(\mathbb{R}^d)$ and also $|\xi|^{-\alpha} \hat{g} \in L^1_{\text{loc}}(\mathbb{R}^d)$, so that the elements of $\dot{H}^\alpha(\mathbb{R}^d)$ have locally integrable Fourier transform.

With the restriction $0 \leq \alpha < d/2$, this is a Hilbert space and $\mathcal{S}(\mathbb{R}^d) \subset \dot{H}^\alpha(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ with continuous and dense inclusions. Elements of $\dot{H}^\alpha(\mathbb{R}^d)$ are in fact L^q -functions for $q = q_\alpha = \frac{2d}{d-2\alpha}$ the Sobolev conjugate exponent for the Sobolev inequality $\|f\|_q \leq C(d, \alpha) \|f\|_{\dot{H}^\alpha}$ and $\dot{H}^\alpha(\mathbb{R}^d)$ agrees with $\{f \in L^q(\mathbb{R}^d); (-\Delta)^{\alpha/2} f \in L^2(\mathbb{R}^d)\}$. For $\alpha = 0$, $\dot{H}^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$.

Remark 2.4. Literature on kinetic equations is mostly concerned with $\alpha \in (0, 1]$. We observe that the requirement $\alpha < d/2$ is only to have a space of functions. When $\alpha \geq d/2$ one would need to use distributions modulo polynomials, and it is not clear what to do unless we add some further qualitative conditions, see Section 7. It forces $\alpha < 1$ when $d = 2$ and $\alpha < 1/2$ when $d = 1$. Such a condition is also removed by extending our approach to inhomogeneous functional spaces, see Section 5.

Remark 2.5. When $0 < \alpha < 1$, an equivalent norm on $\dot{H}^\alpha(\mathbb{R}^d)$ is given by the Gagliardo or Beppo-Levi norm (see [6] for example) but we shall not use it. When $\alpha = 1$, then this definition differs from the space $\dot{W}^{1,2}(\mathbb{R}^d) = \{f \in \mathcal{D}'(\mathbb{R}^d); \nabla f \in L^2(\mathbb{R}^d)\}$. We will indicate how this changes the statement and proof of some of our results for kinetic equations, see Section 6.

Define for $\alpha > 0$,

$$\dot{H}^{-\alpha}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d); \exists g \in L^2(\mathbb{R}^d), \hat{f} = |\xi|^\alpha \hat{g}\}$$

with norm $\|f\|_{\dot{H}^{-\alpha}} = \|g\|_2$. This is also a Hilbert space, contained in $\mathcal{S}'(\mathbb{R}^d)$ and with elements having $L^2_{\text{loc}}(\mathbb{R}^d)$ Fourier transforms. When $\alpha < d/2$, it identifies to the dual of $\dot{H}^\alpha(\mathbb{R}^d)$ for the duality induced by L^2 inner product and also $\mathcal{S}(\mathbb{R}^d) \subset \dot{H}^{-\alpha}(\mathbb{R}^d)$.

If $f \in \dot{H}^\alpha(\mathbb{R}^d)$, $\alpha < d/2$, and $\gamma \in \mathbb{R}$ with $\alpha - 2\gamma < d/2$, then $(-\Delta)^\gamma f$ is the tempered distribution whose Fourier transform agrees with $|\xi|^{2\gamma-\alpha} \hat{f}$ where $\hat{f} = |\xi|^{-\alpha} \hat{g}$ and $g \in L^2(\mathbb{R}^d)$. Thus, $(-\Delta)^\gamma f \in \dot{H}^{\alpha-2\gamma}(\mathbb{R}^d)$ and has Fourier transform $|\xi|^{2\gamma} \hat{f}$.

2.2.2. Homogeneous weak kinetic spaces. We work with time, position and velocity variables $(t, x, v) \in \Omega = \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$. As we use mixed spaces, we shall put the variables in the index, e.g. $L^2_{t,x} \dot{H}_v^{-\beta}$ for $L^2(\mathbb{R}_t \times \mathbb{R}_x^d; \dot{H}_v^{-\beta}(\mathbb{R}_v^d))$. Without indication, Lebesgue spaces are with respect to the Lebesgue measure. We also indicate the variables on differentiation operators. We recall that we use the notation D_v for $(-\Delta_v)^{1/2}$ and D_x for $(-\Delta_x)^{1/2}$. In what follows, $\beta > 0$ is fixed.

We introduce next for $\gamma \in [0, d/2)$,

$$(2.2) \quad \dot{X}_\beta^\gamma = L_x^2 \dot{H}_v^\gamma \cap L_v^2 \dot{H}_x^{\frac{\gamma}{2\beta+1}}$$

with Hilbertian norm $\|f\|_{\dot{X}_\beta^\gamma}$ given by

$$\|f\|_{\dot{X}_\beta^\gamma}^2 = \|f\|_{L_x^2 \dot{H}_v^\gamma}^2 + \|f\|_{L_v^2 \dot{H}_x^{\frac{\gamma}{2\beta+1}}}^2.$$

If $\gamma < 0$, we set

$$(2.3) \quad \dot{X}_\beta^\gamma = L_x^2 \dot{H}_v^\gamma + L_v^2 \dot{H}_x^{\frac{\gamma}{2\beta+1}}$$

with Hilbertian norm $\|f\|_{\dot{X}_\beta^\gamma}$ given by

$$\|f\|_{\dot{X}_\beta^\gamma}^2 = \inf_{f=f_1+f_2} \|f_1\|_{L_x^2 \dot{H}_v^\gamma}^2 + \|f_2\|_{L_v^2 \dot{H}_x^{\frac{\gamma}{2\beta+1}}}^2.$$

Remark 2.6. All these spaces are complete and are subspaces of $\mathcal{S}'_{x,v} = \mathcal{S}'(\mathbb{R}^{2d})$, and of $L^2_{\text{loc},x,v} = L^2_{\text{loc}}(\mathbb{R}^{2d})$ when $\gamma \geq 0$.

Remark 2.7. For $\gamma \in [0, d/2)$, the space $\dot{X}_\beta^{-\gamma}$ identifies with the dual of \dot{X}_β^γ for the duality extending the $L_{x,v}^2$ inner product.

We introduce next for $\gamma < d/2$,

$$(2.4) \quad \dot{\mathcal{F}}_\beta^\gamma = \{f \in \mathcal{D}'(\Omega); f \in L_{t,x}^2 \dot{H}_v^\gamma \ \& \ (\partial_t + v \cdot \nabla_x)f \in L_{t,x}^2 \dot{H}_v^{\gamma-2\beta}\}$$

with Hilbertian norm $\|f\|_{\dot{\mathcal{F}}_\beta^\gamma}$ defined by

$$(2.5) \quad \|f\|_{\dot{\mathcal{F}}_\beta^\gamma}^2 = \|D_v^\gamma f\|_{L_{t,x,v}^2}^2 + \|(\partial_t + v \cdot \nabla_x)f\|_{L_{t,x}^2 \dot{H}_v^{\gamma-2\beta}}^2,$$

$$(2.6) \quad \dot{\mathcal{G}}_\beta^\gamma = \{f \in \mathcal{D}'(\Omega); f \in L_{t,x}^2 \dot{H}_v^\gamma \ \& \ (\partial_t + v \cdot \nabla_x)f \in L_t^2 \dot{X}_\beta^{\gamma-2\beta}\}$$

with Hilbertian norm $\|f\|_{\dot{\mathcal{G}}_\beta^\gamma}$ defined by

$$\|f\|_{\dot{\mathcal{G}}_\beta^\gamma}^2 = \|D_v^\gamma f\|_{L_{t,x,v}^2}^2 + \|(\partial_t + v \cdot \nabla_x)f\|_{L_t^2 \dot{X}_\beta^{\gamma-2\beta}}^2,$$

and

$$(2.7) \quad \dot{\mathcal{L}}_\beta^\gamma = \{f \in \mathcal{D}'(\Omega); f \in L_{t,x}^2 \dot{H}_v^\gamma \ \& \ (\partial_t + v \cdot \nabla_x)f \in L_t^2 \dot{X}_\beta^{\gamma-2\beta} + L_t^1 \dot{X}_\beta^{\gamma-\beta}\}$$

with norm $\|f\|_{\dot{\mathcal{L}}_\beta^\gamma}$ defined by

$$\|f\|_{\dot{\mathcal{L}}_\beta^\gamma} = \|D_v^\gamma f\|_{L_{t,x,v}^2} + \|(\partial_t + v \cdot \nabla_x)f\|_{L_t^2 \dot{X}_\beta^{\gamma-2\beta} + L_t^1 \dot{X}_\beta^{\gamma-\beta}}.$$

The action of the field $\partial_t + v \cdot \nabla_x$ is taken in the sense of distributions on Ω .

Remark 2.8. These spaces are complete and $\dot{\mathcal{F}}_\beta^\gamma \subset \dot{\mathcal{G}}_\beta^\gamma \subset \dot{\mathcal{L}}_\beta^\gamma \subset L_t^2 \mathcal{S}'_{x,v} \subset \mathcal{S}'(\Omega)$.

Eventually, we can repeat *verbatim* the definitions on $\Omega_+ = (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. But only the first three inclusions of Remark 2.8 continue to hold as there is no natural definition of $\mathcal{S}'(\Omega_+)$. Yet, to simplify the presentation, we shall use the same notation and make it clear that we work on Ω_+ .

2.3. Kolmogorov operators.

2.3.1. *Galilean change of variables.* The Galilean change of variables is defined by

$$\Gamma f(t, x, v) = f(t, x + tv, v)$$

and also for fixed t ,

$$\Gamma(t)f(x, v) = f(x + tv, v)$$

which are isometries on $L_{t,x,v}^2$ and $L_{x,v}^2$ respectively. Furthermore, $(\Gamma(t))_{t \in \mathbb{R}}$ is a strongly continuous group on $L_{x,v}^2$ [28, Lemma 5.2]. These two properties imply that the mapping $\Gamma: C_0(\mathbb{R}_t; L_{x,v}^2) \rightarrow C_0(\mathbb{R}_t; L_{x,v}^2)$ is continuous.

2.3.2. *Partial Fourier transform.* We use (partial) Fourier transform in the variables (φ, ξ) dual to (x, v) , defined by

$$\widehat{f}(t, \varphi, \xi) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} e^{-i(x \cdot \varphi + v \cdot \xi)} f(t, x, v) dx dv.$$

This technical lemma relating the Fourier transform and the Galilean change of variables Γ will be used frequently. To simplify the presentation we introduce the weights

$$(2.8) \quad W(t, \varphi, \xi) = \sup(|\xi - t\varphi|, |\varphi|^{\frac{1}{1+2\beta}}),$$

and

$$w(\varphi, \xi) = \sup(|\xi|, |\varphi|^{\frac{1}{1+2\beta}})$$

which depend on the fixed parameter $\beta > 0$.

Lemma 2.9. *Let $\gamma \in \mathbb{R}$ with $\gamma < d/2$. Then the map $f \mapsto \widehat{\Gamma}f$ is an isomorphism*

- (i) $\mathcal{S}(\Omega)$ onto $\mathcal{S}(\Omega)$.
- (ii) $\mathcal{S}'(\Omega)$ onto $\mathcal{S}'(\Omega)$.
- (iii) $L_t^p \dot{X}_\beta^\gamma$ onto $L_t^p L_{\varphi, \xi}^2 (W^{2\gamma} d\varphi d\xi)$, $1 \leq p < \infty$.
- (iv) $C_0(\mathbb{R}_t; \dot{X}_\beta^\gamma)$ onto $C_0(\mathbb{R}_t; L_{\varphi, \xi}^2 (W^{2\gamma} d\varphi d\xi))$.
- (v) $L_t^2 \dot{X}_\beta^\gamma$ onto $L_{t, \varphi, \xi}^2 (W^{2\gamma} dt d\varphi d\xi)$.

Proof. We have $\widehat{\Gamma}f(t, \varphi, \xi) = \widehat{f}(t, \varphi, \xi - t\varphi)$. The first item is rather easy and we skip details. The second item follows by duality using the first one replacing Γ by $\Gamma^* = \Gamma^{-1}$. For the third and fourth items, Plancherel's theorem tells us that $u \in \dot{X}_\beta^\gamma$ if and only if $\widehat{u} \in L_{\varphi, \xi}^2 (w^{2\gamma} d\varphi d\xi)$. Apply this to $u(x, v) = f(t, x, v)$ for fixed t and then use the change of variables induced by $\Gamma(t)$. This proves the third item. For the fifth item apply further Fubini's theorem when $p = 2$. Concerning the fourth item, the right-hand space should be understood as the image under the change of variable $\widehat{\Gamma}$ of $C_0(\mathbb{R}_t; L_{\varphi, \xi}^2 (w^{2\gamma} d\varphi d\xi))$ where $\widehat{\Gamma}$ is the conjugate of the Galilean change of variables through the Fourier transform, that is $\widehat{\Gamma}h(t, \varphi, \xi) = h(t, \varphi, \xi - t\varphi)$. \square

2.3.3. *Forward and backward Kolmogorov operators of order β .* The forward and backward Kolmogorov operators of order β are denoted by \mathcal{K}_β^\pm . They are formally defined as inverse maps of $\pm(\partial_t + v \cdot \nabla_x) + (-\Delta_v)^\beta$. Rigorously, we use Lemma 2.9 and for $S \in \mathcal{S}'(\Omega)$, we set

$$f = \mathcal{K}_\beta^+ S$$

whenever

$$(2.9) \quad \widehat{\Gamma}f = T(\widehat{\Gamma}S) \in \mathcal{S}'(\Omega),$$

with

$$(2.10) \quad Th(t, \varphi, \xi) = \int_{-\infty}^t K(t, s, \varphi, \xi) h(s, \varphi, \xi) ds,$$

where for $s, t \in \mathbb{R}$ and $(\varphi, \xi) \in \mathbb{R}^{2d}$,

$$(2.11) \quad K(t, s, \varphi, \xi) = \exp\left(-\int_s^t |\xi - \tau\varphi|^{2\beta} d\tau\right).$$

The operator \mathcal{K}_β^- is the formal adjoint to \mathcal{K}_β^+ . It is associated in the same fashion to the adjoint of T given by the following formula,

$$(2.12) \quad T^*h(s, \varphi, \xi) = \int_s^{+\infty} K(t, s, \varphi, \xi)h(t, \varphi, \xi) dt.$$

Of course, the integral operator T above makes sense only for certain classes of functions and is thought of as the Duhamel formula for the ordinary differential equation,

$$g'(t) + |\xi - t\varphi|^{2\beta}g(t) = h(t), \quad t \in \mathbb{R}.$$

2.3.4. *Density.* In order to establish for instance the energy equality (see Lemma 4.2), it is useful to have a density result at hand. This will also be needed to prove (iii) in Theorem 2.2.

Lemma 2.10 (Density). *There exists a subset \mathcal{S}_K of $\mathcal{S}(\Omega)$ which is dense in $L_t^p \dot{X}_\beta^\gamma$ and in $C_0(\mathbb{R}_t; \dot{X}_\beta^\gamma)$ for all $1 \leq p < \infty$ and $\gamma < d/2$ and such that $\mathcal{K}_\beta^\pm(\mathcal{S}_K) \subset \mathcal{S}(\Omega)$.*

Proof. We begin with the construction of \mathcal{S}_K . Recall that $\Omega = \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$. Again, we argue on the Fourier side. Let $d(\xi, \mathbb{R}\varphi)$ denote the distance from ξ to the line $\mathbb{R}\varphi$. When $\varphi \neq 0$, we have $d(\xi, \mathbb{R}\varphi) = |\xi - (\xi \cdot \varphi) \frac{\varphi}{|\varphi|^2}|$, so it is a continuous function of (φ, ξ) on $\mathbb{R}^d \setminus \{0\} \times \mathbb{R}^d$. Set

$$\Omega_K = \{(t, \varphi, \xi) \in \Omega; |\varphi| > 0, d(\xi, \mathbb{R}\varphi) > 0\},$$

which is an open set. We say that $S \in \mathcal{S}_K$ if and only if $h = \widehat{\Gamma S} \in \mathcal{S}(\Omega)$ and has compact support in Ω_K .

Let us prove the density of \mathcal{S}_K in $L_t^p \dot{X}_\beta^\gamma$ and $C_0(\mathbb{R}_t; \dot{X}_\beta^\gamma)$. Assume f is in one of these spaces. Thanks to Lemma 2.9, we may argue on the Fourier side. We first approximate \widehat{f} with a function in $\mathcal{S}(\Omega)$ with compact support in Ω , and away from $|\varphi| = 0$ and $|\xi| = 0$. This follows right away from the definition of homogeneous Sobolev spaces and density in Lebesgue spaces or C_0 . Thus, we may assume that $h = \widehat{\Gamma f}$ has support in $\{(t, \varphi, \xi) \in \Omega; |t| \leq M, m \leq |\varphi| \leq M, m \leq |\xi - t\varphi| \leq M\}$ for some $0 < m \leq M < \infty$. On this set, the function $(t, \varphi, \xi) \mapsto d(\xi, \mathbb{R}\varphi)^2$ is C^∞ and if we let $\chi \in C^\infty(0, \infty)$ with $\chi(y) = 1$ if $y \geq 4$ and $\chi(y) = 0$ if $y \leq 1$, then $h_\varepsilon : (t, \varphi, \xi) \mapsto \chi(d(\xi, \mathbb{R}\varphi)^2/\varepsilon^2)h(t, \varphi, \xi)$ is C^∞ with compact support in Ω_K . Moreover, for the convergence, we have that h has support in $[-M, M] \times \{m \leq |\varphi| \leq M\} \times \{|\xi| \leq M + M^2\}$. In addition $h - h_\varepsilon = 0$ when $d(\xi, \mathbb{R}\varphi) \geq 2\varepsilon$. If $d(\xi, \mathbb{R}\varphi) \leq 2\varepsilon$, then ξ belongs to the cylinder $\mathbb{R}\varphi + B(0, 2\varepsilon)$. Thus, using all the support information,

$$\left\| \left(\iint |h - h_\varepsilon|^2 W^{2\gamma} d\xi d\varphi \right)^{1/2} \right\|_{L_t^p} \lesssim_{m, M, \gamma, \beta, p} \|h\|_{L^\infty} \varepsilon^{1/2}.$$

We turn to proving that the Kolmogorov operators map \mathcal{S}_K to $\mathcal{S}(\Omega)$. We do it for \mathcal{K}_β^+ , the proof for \mathcal{K}_β^- being the same. Let $S \in \mathcal{S}_K$ and set $h = \widehat{\Gamma S}$. We have to show that $g = Th \in \mathcal{S}(\Omega)$. Since h has compact support in Ω_K , there exists $0 < m < M < \infty$ such that h is supported in $[-M, M] \times F$, with

$$F = \{(\varphi, \xi) \in \mathbb{R}^{2d}; m \leq |\varphi| \leq M, m \leq d(\xi, \mathbb{R}\varphi), |\xi| \leq M\}.$$

On $\mathbb{R} \times (\mathbb{R}^{2d} \setminus F) \cup (-\infty, -M] \times F$, g is clearly 0, that is, g is supported in $[-M, \infty) \times F$. Let $0 < m' < m$ and $M < M' < \infty$ and $O = \{(\varphi, \xi) \in \mathbb{R}^{2d}; m' < |\varphi| < M', m' <$

$d(\xi, \mathbb{R}\varphi), |\xi| < M'\}$. When $m' \leq d(\xi, \mathbb{R}\varphi)$ and $t \geq s$, $\int_s^t |\xi - \tau\varphi|^{2\beta} d\tau \geq (t-s)(m')^{2\beta}$. In particular $(\varphi, \xi) \mapsto \exp(-\int_s^t |\xi - \tau\varphi|^{2\beta} d\tau)$ is C^∞ on O and g is C^∞ with respect to $(\varphi, \xi) \in O$ for any $t > -M'$, and all its partial derivatives with respect to $(\varphi, \xi) \in O$ have exponential decay in $t \rightarrow \infty$, uniformly. Next, as $\partial_t g = -|\xi - t\varphi|^{2\beta} g + h$, the same is true for $\partial_t g$. By induction, we obtain that g is C^∞ with respect to (t, φ, ξ) . Since g has support in $[-M, \infty) \times F$, with all partial derivatives having exponential decay in time, uniformly in (φ, ξ) , we conclude that $g \in \mathcal{S}(\Omega)$. \square

2.3.5. Estimates. The main technical result related to Kolmogorov operators is the following proposition. It makes precise how Kolmogorov operators act on various functional spaces in (t, x, v) , in particular on $L_t^2 \dot{X}_\beta^{\gamma-2\beta}$.

Proposition 2.11 (Estimates for Kolmogorov operators). *We have the following bounds.*

- (i) For $\gamma < d/2$, the operators \mathcal{K}_β^\pm map boundedly $L_t^2 \dot{X}_\beta^{\gamma-2\beta}$ to $L_t^2 \dot{X}_\beta^\gamma$.
- (ii) For $\gamma - \beta < d/2$, the operators \mathcal{K}_β^\pm map boundedly $L_t^2 \dot{X}_\beta^{\gamma-2\beta}$ to $C_0(\mathbb{R}_t; \dot{X}_\beta^{\gamma-\beta})$.
- (iii) For $\gamma < d/2$, the operators \mathcal{K}_β^\pm map boundedly $L_t^1 \dot{X}_\beta^{\gamma-\beta}$ to $L_t^2 \dot{X}_\beta^\gamma$.
- (iv) For $\gamma - \beta < d/2$, \mathcal{K}_β^\pm map boundedly $L_t^1 \dot{X}_\beta^{\gamma-\beta}$ to $C_0(\mathbb{R}_t; \dot{X}_\beta^{\gamma-\beta})$.

Remark 2.12. The continuity is a key aspect of our approach. The only restriction on γ is the one imposed by our definition of Sobolev spaces. See Proposition 2.13 below where the Fourier estimates are proven without restriction.

Thanks to (2.9) and Lemma 2.9, the proof of Proposition 2.11 is done here on the Fourier side. The method consists of working with fixed Fourier variables (φ, ξ) and proving weighted temporal estimates by integrating with respect to t . It makes explicitly clear two facts: on the one hand, the uniformity (or dependency) of the constants with respect to (φ, ξ) , yielding in the end homogeneous estimates, and on the other hand that time intervals can be infinite.

Making the variables (φ, ξ) implicit, we set

$$K(t, s) = K(t, s, \varphi, \xi), \quad Th(t) = \int_{-\infty}^t K(t, s)h(s) ds, \quad W(t) = W(t, \varphi, \xi)$$

for the Kolmogorov kernel, the associated operator and the weight defined in (2.11), (2.10) and (2.8).

2.3.6. Weighted temporal estimates. In order to prove Proposition 2.11, we fix $(\varphi, \xi) \neq (0, 0)$, define weighted temporal spaces and study how the map T acts on them.

For $\varepsilon \in \mathbb{R}$ and $1 \leq p < \infty$, we use the notation $L_t^p(W^{\varepsilon p})$ for the space $L^p(\mathbb{R}; W^{\varepsilon p} dt)$ of measurable functions g such that $W^\varepsilon g \in L_t^p$ with norm $\|g\|_{L_t^p(W^{\varepsilon p})} = \|W^\varepsilon g\|_{L_t^p}$, that is

$$\|g\|_{L_t^p(W^{\varepsilon p})}^p := \int_{\mathbb{R}} |g(t)|^p \left(\sup(|\xi - t\varphi|, |\varphi|^{\frac{1}{1+2\beta}}) \right)^{\varepsilon p} dt.$$

The following proposition will use elementary estimates on the kernel K . By abuse of notation, we also denote by W^ε the operator of multiplication by $W^\varepsilon(t)$.

Proposition 2.13 (Weighted temporal estimates). *Let $\beta > 0$ and $\gamma \in \mathbb{R}$. Then with bounds for the norms that only depend on β and γ ,*

$$T : L_t^2(W^{2\gamma-4\beta}) \rightarrow L_t^2(W^{2\gamma}) \text{ is bounded.}$$

$$T : L_t^1(W^{\gamma-\beta}) \rightarrow L_t^2(W^{2\gamma}) \text{ is bounded.}$$

$$W^{\gamma-\beta}T : L_t^2(W^{2\gamma-4\beta}) \rightarrow C_0(\mathbb{R}_t) \text{ is bounded.}$$

$$W^{\gamma-\beta}T : L_t^1(W^{\gamma-\beta}) \rightarrow C_0(\mathbb{R}_t) \text{ is bounded.}$$

The same estimates are valid for T^* .

2.3.7. *Elementary estimates on the function K .* To prove Proposition 2.13, we begin with two elementary lemmas about the function K .

Lemma 2.14. *Let $\beta > 0$. There are constants $0 < c_\beta < C_\beta < \infty$ such that for all $s, t \in \mathbb{R}$ with $s < t$ and $(\varphi, \xi) \neq (0, 0)$,*

$$c_\beta \leq \frac{\int_s^t |\xi - \tau\varphi|^{2\beta} d\tau}{(t-s)(|\xi - s\varphi|^{2\beta} + ((t-s)|\varphi|)^{2\beta})} \leq C_\beta$$

and

$$c_\beta \leq \frac{\int_s^t |\xi - \tau\varphi|^{2\beta} d\tau}{(t-s)(|\xi - t\varphi|^{2\beta} + ((t-s)|\varphi|)^{2\beta})} \leq C_\beta.$$

Proof. Set $\xi' = \xi - s\varphi$ and $\varphi' = (t-s)\varphi$, then a change of variable yields

$$\int_s^t |\xi - \tau\varphi|^{2\beta} d\tau = (t-s) \int_0^1 |\xi' - \tau\varphi'|^{2\beta} d\tau.$$

Now the function

$$(\xi', \varphi') \mapsto \int_0^1 |\xi' - \tau\varphi'|^{2\beta} d\tau$$

is continuous on \mathbb{R}^{2d} , hence bounded above and below by positive numbers C_β, c_β on the compact set defined by $|\xi'|^{2\beta} + |\varphi'|^{2\beta} = 1$. One can conclude the first inequality by using that it is also homogeneous of order 2β . One can do similarly with $\xi' = \xi - t\varphi$ writing

$$\int_s^t |\xi - \tau\varphi|^{2\beta} d\tau = (t-s) \int_0^1 |\xi' + \tau\varphi'|^{2\beta} d\tau$$

and conclude as before. □

Lemma 2.15. *Let $\beta > 0$, $\varepsilon \in \mathbb{R}$, $\alpha > 0$ and $(\varphi, \xi) \neq (0, 0)$. One has the following estimates with implicit constants depending on $\beta, \varepsilon, \alpha$ and independent of s, t, φ, ξ .*

$$(2.13) \quad \int_{-\infty}^t K(t, s)^\alpha ds \simeq_{\beta, \alpha} W^{-2\beta}(t),$$

$$(2.14) \quad \int_s^\infty K(t, s)^\alpha dt \simeq_{\beta, \alpha} W^{-2\beta}(s),$$

$$(2.15) \quad \sup (W^\varepsilon(t), W^\varepsilon(s)) K(t, s) \lesssim_{\beta, \varepsilon} \inf (W^\varepsilon(t), W^\varepsilon(s)), \quad s \leq t.$$

$$(2.16) \quad \int_{-\infty}^t K(t, s) W^{2\beta-\varepsilon}(s) ds \lesssim_{\beta, \varepsilon} W^{-\varepsilon}(t),$$

$$(2.17) \quad \int_s^\infty K(t, s) W^{2\beta-\varepsilon}(t) dt \lesssim_{\beta, \varepsilon} W^{-\varepsilon}(s).$$

Proof. Proof of (2.13) and (2.14). It is easily checked for all $A, B \geq 0$, and $c > 0$,

$$(2.18) \quad \int_0^\infty e^{-cu^{1+2\beta}A^{2\beta}} e^{-cuB^{2\beta}} du \simeq_{c, \beta} \inf (B^{-2\beta}, A^{-\frac{2\beta}{1+2\beta}}) = (\sup (B, A^{\frac{1}{1+2\beta}}))^{-2\beta}.$$

Thus, use the comparisons in Lemma 2.14, the change of variable $t-s = u$ and (2.18).

Proof of (2.15). First, there is nothing to do if $s = t$ and we assume $s < t$. Also for $\varepsilon = 0$, we see that $0 < K(t, s) \leq 1$, so we are done. Assume next $\varepsilon \neq 0$. Set $A = (t-s)^{\frac{1}{2\beta}} |\xi - t\varphi|$, $B = (t-s)^{\frac{1}{2\beta}} |\xi - s\varphi|$ and $C = (t-s)^{\frac{2\beta+1}{2\beta}} |\varphi|$. We have $C > 0$, $A, B \geq 0$ and $|A - B| \leq C$. Observe that using both inequalities in Lemma 2.14 yields

$$K(t, s) \leq \exp \left(-\frac{c\beta}{2} (A^{2\beta} + B^{2\beta} + 2C^{2\beta}) \right)$$

so that after multiplication by $(t-s)^{\frac{1}{2\beta}}$, the desired inequality follows from

$$\sup(\rho^\varepsilon, \rho^{-\varepsilon}) \exp \left(-\frac{c\beta}{2} (A^{2\beta} + B^{2\beta} + 2C^{2\beta}) \right) \leq C_{\beta, \varepsilon},$$

where

$$\rho = \frac{\sup (A, C^{\frac{1}{2\beta+1}})}{\sup (B, C^{\frac{1}{2\beta+1}})}.$$

The case $\varepsilon < 0$ reduces to the case $\varepsilon > 0$ by symmetry. Thus, we assume now $\varepsilon > 0$. If $\sup (A, C^{\frac{1}{2\beta+1}}) = C^{\frac{1}{2\beta+1}}$, then $\rho \leq 1$. If $\sup (A, C^{\frac{1}{2\beta+1}}) = A$, then, using $A \leq B + C$,

$$\rho \leq 1 + \frac{C}{\sup (B, C^{\frac{1}{2\beta+1}})} \leq 1 + (\sup (B, C^{\frac{1}{2\beta+1}}))^{2\beta}.$$

Similarly, we have $\rho^{-1} \leq 1 + (\sup (A, C^{\frac{1}{2\beta+1}}))^{2\beta}$. Hence, the term $\sup(\rho^\varepsilon, \rho^{-\varepsilon})$ can be absorbed by the exponential and this finishes the proof of (2.15).

Proof of (2.16). Using the pointwise inequality (2.15) and the integral one (2.14),

$$\begin{aligned} \int_{-\infty}^t K(t, s)W^{2\beta-\varepsilon}(s) ds &= \int_{-\infty}^t K(t, s)^{\frac{1}{2}}(W^{4\beta-2\varepsilon}(s)K(t, s))^{\frac{1}{2}} ds \\ &\lesssim_{\beta, 2\beta-\varepsilon} W^{2\beta-\varepsilon}(t) \int_{-\infty}^t K(t, s)^{\frac{1}{2}} ds \\ &\lesssim_{\beta, \varepsilon} W^{-\varepsilon}(t). \end{aligned}$$

The proof of (2.17) is the same. \square

2.3.8. *An intermediate result.* Before proving Proposition 2.13, we prove estimates for T in L^1 and L^∞ spaces.

Proposition 2.16 (L^1 and L^∞ estimates for T). *Let $\beta > 0$ and $\varepsilon \in \mathbb{R}$. Then with bounds for the norms that only depend on β and ε ,*

$$\begin{aligned} W^\varepsilon T W^{-\varepsilon} : L_t^1 &\rightarrow C_0(\mathbb{R}_t) \text{ is bounded.} \\ W^\varepsilon T W^{2\beta-\varepsilon} : L_t^1 &\rightarrow L_t^1 \text{ is bounded.} \\ W^\varepsilon T W^{2\beta-\varepsilon} : L_t^\infty &\rightarrow L_t^\infty \text{ is bounded.} \end{aligned}$$

The same estimates are valid for T^ .*

Proof. We only treat T as the proofs for T^* are similar. We start with the L_t^∞ bound for $W^\varepsilon T(W^{-\varepsilon} f)$ when $f \in L_t^1$. We have by (2.15),

$$W^\varepsilon(t)|T(W^{-\varepsilon} f)(t)| \leq \int_{-\infty}^t W^\varepsilon(t)K(t, s)W^{-\varepsilon}(s)|f(s)| ds \lesssim_{\beta, \varepsilon} \int_{-\infty}^t |f(s)| ds,$$

which is controlled by $\|f\|_{L_t^1}$. This also proves the fact $W^\varepsilon(t)|T(W^{-\varepsilon} f)(t)| \rightarrow 0$ when $t \rightarrow -\infty$.

By a density argument and closedness of $C_0(\mathbb{R}_t)$ in L_t^∞ , it is now enough to prove the continuity of $W^\varepsilon T(W^{-\varepsilon} f)$ and limit 0 at $+\infty$ assuming f to be compactly supported in $[-R, R]$ for some fixed $R > 0$ and continuous. For the limit, the above estimate rewrites for $t \geq R$,

$$W^\varepsilon(t)|T(W^{-\varepsilon} f)(t)| \leq \int_{-R}^R W^\varepsilon(t)K(t, s)W^{-\varepsilon}(s)|f(s)| ds$$

and clearly the integral converges to 0 by dominated convergence as $t \rightarrow +\infty$.

For the continuity, we use that W^ε and $W^{-\varepsilon}$ are continuous and also that $T(W^{-\varepsilon} f)$ is locally Lipschitz in time since, as f is continuous with compact support, it is easy to see that $g = T(W^{-\varepsilon} f)$ solves the ordinary differential equation $g'(t) + |\xi - t\varphi|^{2\beta} g(t) = (W^{-\varepsilon} f)(t)$ in the classical sense.

Next, the $L_t^1 - L_t^1$ and $L_t^\infty - L_t^\infty$ bounds of $W^\varepsilon T W^{2\beta-\varepsilon}$ follow from (2.16) and (2.17) by Fubini's theorem. \square

2.3.9. *Proof of Proposition 2.13.* We are now ready to prove the temporal weighted estimates for the operator T announced in Proposition 2.11.

Proof of Proposition 2.13. The last boundedness statement is already contained in Proposition 2.16. We prove the other three properties in the order they are stated.

First, interpolating $L_t^1 - L_t^1$ and $L_t^\infty - L_t^\infty$ boundedness of $W^\gamma T W^{2\beta-\gamma}$, we obtain $L_t^2 - L_t^2$ boundedness which is equivalent to the first item.

Secondly, we have $W^\gamma T : L_t^1(W^{\gamma-2\beta}) \rightarrow L_t^1$ and $W^\gamma T : L_t^1(W^\gamma) \rightarrow L_t^\infty$, Using Stein-Weiss interpolation with change of measure (see [7, Theorem 5.4.1]), we obtain $W^\gamma T : L_t^1(W^{\gamma-\beta}) \rightarrow L_t^2$ is bounded, which is equivalent to the second item.

Thirdly, the second case applied to T^* rewrites $W^\gamma T^* W^{\beta-\gamma} : L_t^1 \rightarrow L_t^2$ is bounded, so by duality and changing $\beta - \gamma$ to $\gamma - \beta$, we have $W^{\gamma-\beta} T W^{2\beta-\gamma}$ maps L_t^2 into L_t^∞ . Note that if we apply this to continuous and compactly supported functions f then we also have $W^{\gamma-\beta} T W^{2\beta-\gamma} f = W^{\gamma-\beta} T W^{\beta-\gamma}(W^\beta f) \in C_0(\mathbb{R}_t)$ as $W^\beta f \in L_t^1$. By density of such functions f in L_t^2 and closedness of $C_0(\mathbb{R}_t)$ in L_t^∞ , we conclude that $W^{\gamma-\beta} T W^{2\beta-\gamma}$ maps L_t^2 into $C_0(\mathbb{R}_t)$ which is equivalent to the third item. \square

2.3.10. *Proof of Proposition 2.11.* We can now obtain the desired estimates for the Kolmogorov operators.

Proof of Proposition 2.11. Let us concentrate on \mathcal{K}_β^+ , the proof for \mathcal{K}_β^- being similar. We come back to the definition of T depending on the three variables and recall that the weight W also depends on the three variables. By the Fourier transform correspondence $S \mapsto \widehat{\Gamma S}$, Lemma 2.9 and Fubini's theorem, the first assertion for \mathcal{K}_β^+ is equivalent to the weighted boundedness

$$T : L_{\varphi,\xi}^2 L_t^2(W^{2\gamma-4\beta}) \rightarrow L_{\varphi,\xi}^2 L_t^2(W^{2\gamma}).$$

Using the conjugate $\widehat{\Gamma}$ of the Galilean change of variables through the Fourier transform, and using the fact that this map preserves $C_0(\mathbb{R}_t; L_{\varphi,\xi}^2)$, the second is equivalent to

$$W^{\gamma-\beta} T : L_{\varphi,\xi}^2 L_t^2(W^{2\gamma-4\beta}) \rightarrow C_0(\mathbb{R}_t; L_{\varphi,\xi}^2).$$

The third is equivalent to

$$T : L_t^1 L_{\varphi,\xi}^2(W^{2\gamma-2\beta}) \rightarrow L_{\varphi,\xi}^2 L_t^2(W^{2\gamma}),$$

where $L_{\varphi,\xi}^2(W^{2\gamma-2\beta})$ is here the weighted L^2 space with respect to $W^{2\gamma-2\beta}(t, \varphi, \xi) d\varphi d\xi$, for fixed t . With the same idea as for the second assertion, the fourth one is equivalent to

$$W^{\gamma-\beta} T : L_t^1 L_{\varphi,\xi}^2(W^{2\gamma-2\beta}) \rightarrow C_0(\mathbb{R}_t; L_{\varphi,\xi}^2).$$

The first two assertions on T are direct consequences of Proposition 2.13.

For the third one, we have by Proposition 2.13 and by Minkowski integral inequality

$$\begin{aligned} \|\|W^\gamma Th\|_{L_t^2}\|_{L_{\varphi,\xi}^2} &\lesssim_{\beta,\gamma} \|\|W^{\gamma-\beta} h\|_{L_t^1}\|_{L_{\varphi,\xi}^2} \\ &\leq \|\|W^{\gamma-\beta} h\|_{L_{\varphi,\xi}^2}\|_{L_t^1}. \end{aligned}$$

For the L^∞ bound in the fourth one, write

$$\begin{aligned} \|\|W^{\gamma-\beta}(t) Th(t)\|_{L_{\varphi,\xi}^2}\| &\leq \int_{-\infty}^t \|\|W^{\gamma-\beta}(t) K(t,s) h(s)\|_{L_{\varphi,\xi}^2} ds \\ &\lesssim_{\beta,\gamma} \int_{-\infty}^t \|\|W^{\gamma-\beta}(s) h(s)\|_{L_{\varphi,\xi}^2} ds \end{aligned}$$

using Minkowski integral inequality and (2.15) directly, rather than Proposition 2.13. Continuity and limits follow the argument in Proposition 2.16. We leave details to the reader. \square

3. UNIQUENESS AND KINETIC EMBEDDINGS

3.1. Uniqueness. The kinetic embeddings rely on a crucial uniqueness result.

Lemma 3.1 (Uniqueness). *Let $\gamma < d/2$ and $\beta > 0$. If $f \in L_{t,x}^2 \dot{H}_v^\gamma$ satisfies $\pm(\partial_t + v \cdot \nabla_x)f + (-\Delta_v)^\beta f = 0$ in the sense of distributions, then $f = 0$.*

Proof. We only consider $(\partial_t + v \cdot \nabla_x)f + (-\Delta_v)^\beta f = 0$, while the backward equation can be treated by a similar argument. When $f \in L_{t,x}^2 \dot{H}_v^\gamma$ and $\gamma < d/2$, f and $(-\Delta_v)^\beta f$ are tempered distributions, with Fourier transform in $L_{\text{loc}}^1(\Omega)$, and the equation can be interpreted in $\mathcal{S}'(\Omega)$. Setting $G = \widehat{\Gamma}f \in \mathcal{S}'(\Omega)$, we have $G \in L_{\text{loc}}^1(\Omega)$ and $\partial_t G + |\xi - t\varphi|^{2\beta} G = 0$ in $\mathcal{S}'(\Omega)$. From $\partial_t G = -|\xi - t\varphi|^{2\beta} G \in L_{\text{loc}}^1(\Omega)$, we deduce that G has a representative in $C(\mathbb{R}; L_{\text{loc}}^1(\mathbb{R}^{2d}))$ and then in $C^1(\mathbb{R}; L_{\text{loc}}^1(\mathbb{R}^{2d}))$. Identifying G to this representative, the equation can be interpreted as a first-order differential equation valued in $L_{\text{loc}}^1(\mathbb{R}^{2d})$. Thus, there exists $G_0 \in L_{\text{loc}}^1(\mathbb{R}^{2d})$ such that $G = K(t, 0)G_0$ for all $t \in \mathbb{R}$, where $K(t, 0)$ is the kernel defined in (2.11). As $(-\Delta_v)^{\gamma/2} f \in L_{t,x,v}^2$, we have that $|\xi - t\varphi|^\gamma G \in L_{t,\varphi,\xi}^2$, that is

$$(3.1) \quad \iiint_{\Omega} |\xi - t\varphi|^{2\gamma} \exp\left(-2 \int_0^t |\xi - \tau\varphi|^{2\beta} d\tau\right) |G_0(\varphi, \xi)|^2 dt d\varphi d\xi < \infty.$$

We can use Fubini's theorem and notice that the t -integral on \mathbb{R} is infinite when $(\varphi, \xi) \neq (0, 0)$ because of the exponential growth when $t \rightarrow -\infty$. This proves that $G_0 = 0$ almost everywhere. Hence, $G = 0$ and so is f . \square

Remark 3.2. The argument works for a larger class of f . It suffices that $f, (-\Delta_v)^\beta f \in \mathcal{S}'(\Omega)$ with $\hat{f} \in L_{\text{loc}}^1(\Omega)$ such that there is some measurable function m for which $mG \in L_t^1 L_{\text{loc}}^1(\mathbb{R}^{2d})$ and

$$\int_{\mathbb{R}} |m(t, \varphi, \xi)| \exp\left(-\int_0^t |\xi - \tau\varphi|^{2\beta} d\tau\right) dt = \infty$$

almost everywhere. For example this works for $f \in L_{t,v}^2 \dot{H}_x^{\frac{\gamma}{2\beta+1}}$. This also works for sums of spaces for which these conditions hold.

3.2. Isomorphism properties. A consequence of uniqueness is the isomorphism properties of the Kolmogorov operators for our scales of spaces in an appropriate range of exponents.

Lemma 3.3 (Isomorphisms). *Assume $\gamma < d/2$ with $0 \leq \gamma \leq 2\beta$. The Kolmogorov operators \mathcal{K}_β^\pm are isomorphisms from $L_{t,x}^2 \dot{H}_v^{\gamma-2\beta}$ onto $\dot{\mathcal{F}}_\beta^\gamma$, from $L_t^2 \dot{X}_\beta^{\gamma-2\beta}$ onto $\dot{\mathcal{G}}_\beta^\gamma$ and from $L_t^2 \dot{X}_\beta^{\gamma-2\beta} + L_t^1 \dot{X}_\beta^{\gamma-\beta}$ onto $\dot{\mathcal{L}}_\beta^\gamma$.*

Proof. The proofs are all the same. We do the proof of the last statement.

It follows from Proposition 2.11 that \mathcal{K}_β^+ maps continuously $S \in L_t^2 \dot{X}_\beta^{\gamma-2\beta} + L_t^1 \dot{X}_\beta^{\gamma-\beta}$ into $\dot{\mathcal{L}}_\beta^\gamma$. Indeed, $\mathcal{K}_\beta^+ S \in L_{t,x}^2 \dot{H}_v^\gamma$ as $\gamma \geq 0$, and $(\partial_t + v \cdot \nabla_x)\mathcal{K}_\beta^+ S = S - (-\Delta_v)^\beta \mathcal{K}_\beta^+ S \in L_t^2 \dot{X}_\beta^{\gamma-2\beta} + L_t^1 \dot{X}_\beta^{\gamma-\beta}$ since $(-\Delta_v)^\beta \mathcal{K}_\beta^+ S \in L_{t,x}^2 \dot{H}_v^{\gamma-2\beta} \subset L_t^2 \dot{X}_\beta^{\gamma-2\beta}$ when $\gamma - 2\beta \leq 0$.

Next, we show the ontoneess. Consider $f \in \dot{\mathcal{L}}_\beta^\gamma$ and set $S = (\partial_t + v \cdot \nabla_x)f + (-\Delta_v)^\beta f$. By definitions of the spaces $\dot{\mathcal{L}}_\beta^\gamma$ and $\dot{X}_\beta^{\gamma-2\beta}$ with $\gamma - 2\beta \leq 0$, we have $S \in L_t^2 \dot{X}_\beta^{\gamma-2\beta} + L_t^1 \dot{X}_\beta^{\gamma-\beta}$ with the norm controlled by that of f in $\dot{\mathcal{L}}_\beta^\gamma$. We observe that $\mathcal{K}_\beta^+ S \in L_t^2 \dot{X}_\beta^\gamma \subset L_{t,x}^2 \dot{H}_v^\gamma$ by Proposition 2.11 and $\gamma \geq 0$. As $\mathcal{K}_\beta^+ S$ is also a distributional solution of the equation satisfied by f , it agrees with f from Lemma 3.1.

This shows bijectivity together with the continuity estimates in both directions, hence the isomorphism property is proved.

The proof for \mathcal{K}_β^- is the same. \square

3.2.1. Proof of the embeddings and transfers of regularity.

Proof of Theorem 2.2. The first two items (i) and (ii) follow from Lemma 3.3 and Proposition 2.11.

Proof of (iii): Lemma 2.10 asserts that one can construct a subspace \mathcal{S}_K of $\mathcal{S}(\Omega)$ which is dense in any $L_t^2 \dot{X}_\beta^{\gamma-2\beta} + L_t^1 \dot{X}_\beta^{\gamma-\beta}$, and such that $\mathcal{K}_\beta^\pm(\mathcal{S}_K) \subset \mathcal{S}(\Omega)$. We conclude using the isomorphism property in Lemma 3.3 that (each of) $\mathcal{K}_\beta^\pm(\mathcal{S}_K)$ is dense in each of our kinetic spaces.

Proof of (iv): When $\gamma - 2\beta < d/2$, the Fourier transform in the (x, v) variables provides isomorphisms

$$\begin{aligned} L_{t,x}^2 \dot{H}_v^{\gamma-2\beta} &\rightarrow L_{t,\varphi,\xi}^2 (|\xi|^{2\gamma-4\beta} dt d\varphi d\xi), \\ L_t^2 \dot{X}_\beta^{\gamma-2\beta} &\rightarrow L_{t,\varphi,\xi}^2 (w^{2\gamma-4\beta} dt d\varphi d\xi), \end{aligned}$$

where we recall that $w(\varphi, \xi) = \sup(|\xi|, |\varphi|^{\frac{1}{1+2\beta}})$. Hence, complex interpolation for weighted L^2 spaces, see [7], implies that the right-hand spaces in the first two lines have the complex interpolation property, so the same holds for the corresponding left-hand spaces. We conclude from the isomorphisms property of \mathcal{K}_β^+ (Lemma 3.3) in the allowed range of γ 's that the scales $\dot{\mathcal{F}}_\beta^\gamma$ and $\dot{\mathcal{G}}_\beta^\gamma$ of kinetic spaces have the complex interpolation property, respectively. \square

Remark 3.4. We also have the isomorphism

$$L_t^2 \dot{X}_\beta^{\gamma-2\beta} + L_t^1 \dot{X}_\beta^{\gamma-\beta} \rightarrow L_t^2 L_{\varphi,\xi}^2 (w^{2\gamma-4\beta} d\varphi d\xi) + L_t^1 L_{\varphi,\xi}^2 (w^{2\gamma-2\beta} d\varphi d\xi)$$

but it is not clear whether the scale of spaces on the right-hand side is an interpolation family, whence we cannot deduce the complex interpolation property for $\dot{\mathcal{L}}_\beta^\gamma$.

Remark 3.5. When $\gamma = \beta$, we can obtain multiplicative inequalities by scale invariance. Applying (i) to $f_\lambda(t, x, v) = f(\lambda t, x, \lambda v)$ we obtain the multiplicative inequality

$$\|D_x^{\frac{\beta}{2\beta+1}} f\|_{L_{t,x,v}^2} \lesssim_{d,\beta} \|D_v^\beta f\|_{L_{t,x,v}^2}^{\frac{\beta+1}{2\beta+1}} \|(\partial_t + v \cdot \nabla_x) f\|_{L_{t,x}^2 \dot{H}_v^{-\beta}}^{\frac{\beta}{2\beta+1}}$$

after optimizing the choice of λ . Similarly, applying (ii) to $f_\lambda(t, x, v) = f(t, \lambda x, \lambda v)$ and optimizing we obtain the multiplicative inequality

$$\sup_{t \in \mathbb{R}} \|f(t)\|_{L_{t,x,v}^2} \leq (2 \|D_v^\beta f\|_{L_{t,x,v}^2} \|(\partial_t + v \cdot \nabla_x) f\|_{L_{t,x}^2 \dot{H}_v^{-\beta}})^{\frac{1}{2}}.$$

3.2.2. *Proof of Theorem 1.1.* We turn to the proof of the first main theorem of this work.

Proof of Theorem 1.1. For $S = S_1 + (-\Delta_v)^\beta f + S_2 + S_3 \in \mathbf{L}_{t,x}^2 \dot{\mathbf{H}}_v^{-\beta} + \mathbf{L}_{t,v}^2 \dot{\mathbf{H}}_x^{-\frac{\beta}{2\beta+1}} + \mathbf{L}_t^1 \mathbf{L}_{x,v}^2$, we have $(\partial_t + v \cdot \nabla_x) f + (-\Delta_v)^\beta f = S$. As $\mathcal{K}_\beta^+ S \in \mathbf{L}_{t,x}^2 \dot{\mathbf{H}}_v^\beta$ is a distributional solution of this equation, it is the only one by uniqueness in Lemma 3.1. It follows that $f = \mathcal{K}_\beta^+ S$, which also belongs to $\mathbf{L}_{t,v}^2 \dot{\mathbf{H}}_x^{\frac{\beta}{2\beta+1}} \cap \mathbf{C}_0(\mathbb{R}_t; \mathbf{L}_{x,v}^2)$ by Proposition 2.11 and the estimate follows.

Let us prove the absolute continuity on \mathbb{R} . As above, we write $f = \mathcal{K}_\beta^+ S$ with $S = S_1 + (-\Delta_v)^\beta f + S_2 + S_3$. Define $f_1 = \mathcal{K}_\beta^+(S_1 + (-\Delta_v)^\beta f)$, $f_2 = \mathcal{K}_\beta^+ S_2$, $f_3 = \mathcal{K}_\beta^+ S_3$, so that $f = f_1 + f_2 + f_3$. By Lemma 2.10, we can find approximations $(S_1 + (-\Delta_v)^\beta f)_k$, $(S_2)_k$ and $(S_3)_k$ of $S_1 + (-\Delta_v)^\beta f$, S_2 and S_3 in the respective topologies so that $(f_1)_k = \mathcal{K}_\beta^+(S_1 + (-\Delta_v)^\beta f)_k$, $(f_2)_k = \mathcal{K}_\beta^+(S_2)_k$ and $(f_3)_k = \mathcal{K}_\beta^+(S_3)_k$ belong to $\mathcal{S}(\Omega)$. Altogether, we have, when $k \rightarrow \infty$, and $i = 1, 2, 3$,

$$\begin{aligned} (S_1 + (-\Delta_v)^\beta f)_k &\rightarrow S_1 + (-\Delta_v)^\beta f, & \text{in } \mathbf{L}_{t,x}^2 \dot{\mathbf{H}}_v^{-\beta}, \\ (S_2)_k &\rightarrow S_2, & \text{in } \mathbf{L}_{t,v}^2 \dot{\mathbf{H}}_x^{-\frac{\beta}{2\beta+1}}, \\ (S_3)_k &\rightarrow S_3, & \text{in } \mathbf{L}_t^1 \mathbf{L}_{x,v}^2, \\ (f_i)_k &\rightarrow f_i, & \text{in } \mathbf{L}_{t,x}^2 \dot{\mathbf{H}}_v^\beta \cap \mathbf{L}_{t,v}^2 \dot{\mathbf{H}}_x^{\frac{\beta}{2\beta+1}} \cap \mathbf{C}_0(\mathbb{R}_t; \mathbf{L}_{x,v}^2). \end{aligned}$$

We now derive the absolute continuity using the approximations. Let $g_k = (f_1)_k + (f_2)_k + (f_3)_k$. As $\Gamma((\partial_t + v \cdot \nabla_x) g_k) = \partial_t \Gamma g_k$, for all $s < t$,

$$\begin{aligned} \|g_k(t)\|_{\mathbf{L}_{x,v}^2}^2 - \|g_k(s)\|_{\mathbf{L}_{x,v}^2}^2 &= \iint_{\mathbb{R}^{2d}} |(\Gamma g_k)(t, x, v)|^2 dx dv - \iint_{\mathbb{R}^{2d}} |(\Gamma g_k)(s, x, v)|^2 dx dv \\ &= 2 \operatorname{Re} \iint_{\mathbb{R}^{2d}} \int_s^t \partial_t (\Gamma g_k)(\tau, x, v) \overline{\Gamma g_k}(\tau, x, v) d\tau dx dv \\ &= 2 \operatorname{Re} \int_s^t \iint_{\mathbb{R}^{2d}} \partial_t (\Gamma g_k)(\tau, x, v) \overline{\Gamma g_k}(\tau, x, v) dx dv d\tau \\ &= 2 \operatorname{Re} \int_s^t \iint_{\mathbb{R}^{2d}} ((\partial_t + v \cdot \nabla_x) g_k)(\tau, x, v) \overline{g_k}(\tau, x, v) dx dv d\tau \end{aligned}$$

where we used Fubini's theorem, and change of variables $\Gamma(t), \Gamma(s)$ and $\Gamma(\tau)$ in the (x, v) -integrals. The left hand-side converges to $\|f(t)\|_{\mathbf{L}_{x,v}^2}^2 - \|f(s)\|_{\mathbf{L}_{x,v}^2}^2$. For the right-hand side, we split the integral using the decomposition of $(\partial_t + v \cdot \nabla_x) g_k$ into three terms and argue differently for each. Realizing the v -integral as the duality bracket between $\dot{\mathbf{H}}_v^{-\beta}$ and $\dot{\mathbf{H}}_v^\beta$, we have, using the dominated convergence theorem for the (τ, x) -integral,

$$\begin{aligned} &\int_s^t \iint_{\mathbb{R}^{2d}} ((S_1 + (-\Delta_v)^\beta f)_k - (-\Delta_v)^\beta g_k) \overline{g_k} dx dv d\tau \\ &= \int_s^t \int_{\mathbb{R}^d} \langle (S_1 + (-\Delta_v)^\beta f)_k - (-\Delta_v)^\beta g_k, g_k \rangle dx d\tau \rightarrow \int_s^t \int_{\mathbb{R}^d} \langle S_1, f \rangle dx d\tau. \end{aligned}$$

Similarly, realizing the x -integral as the duality bracket between $\dot{H}_x^{-\frac{\beta}{2\beta+1}}$ and $\dot{H}_x^{\frac{\beta}{2\beta+1}}$ and using dominated convergence in the (t, v) -integral

$$\int_s^t \iint_{\mathbb{R}^{2d}} (S_2)_k \overline{g_k} dx dv d\tau = \int_s^t \int_{\mathbb{R}^d} \langle (S_2)_k, g_k \rangle dv d\tau \rightarrow \int_s^t \int_{\mathbb{R}^d} \langle S_2, f \rangle dv d\tau.$$

Eventually, using merely dominated convergence for the (t, x, v) -integral

$$\int_s^t \iint_{\mathbb{R}^{2d}} (S_3)_k \overline{g_k} dx dv d\tau \rightarrow \int_s^t \iint_{\mathbb{R}^d} S_3 \overline{f} dx dv d\tau. \quad \square$$

The following result is a general version of the Theorem 1.1.

Theorem 3.6 (Kinetic embedding and transfer of regularity - polarized version). *Let $-\beta \leq \varepsilon \leq \beta$ such that $\beta \pm \varepsilon < d/2$. Let $f, \tilde{f} \in \mathcal{D}'(\Omega)$ be such that $f \in \dot{\mathcal{L}}_\beta^{\beta+\varepsilon}$ and $\tilde{f} \in \dot{\mathcal{L}}_\beta^{\beta-\varepsilon}$. Then*

(i) $f \in L_{t,v}^2 \dot{H}_x^{\frac{\beta+\varepsilon}{2\beta+1}} \cap C_0(\mathbb{R}_t; \dot{X}_\beta^\varepsilon)$ with

$$\|D_x^{\frac{\beta+\varepsilon}{2\beta+1}} f\|_{L_{t,x,v}^2} + \sup_{t \in \mathbb{R}} \|f(t)\|_{\dot{X}_\beta^\varepsilon} \lesssim_{d,\beta,\varepsilon} \|f\|_{\dot{\mathcal{L}}_\beta^{\beta+\varepsilon}}$$

(ii) $\tilde{f} \in L_{t,v}^2 \dot{H}_x^{\frac{\beta-\varepsilon}{2\beta+1}} \cap C_0(\mathbb{R}_t; \dot{X}_\beta^{-\varepsilon})$ with

$$\|D_x^{\frac{\beta-\varepsilon}{2\beta+1}} \tilde{f}\|_{L_{t,x,v}^2} + \sup_{t \in \mathbb{R}} \|\tilde{f}(t)\|_{\dot{X}_\beta^{-\varepsilon}} \lesssim_{d,\beta,\varepsilon} \|\tilde{f}\|_{\dot{\mathcal{L}}_\beta^{\beta-\varepsilon}}.$$

(iii) *The map $t \mapsto \langle f(t), \tilde{f}(t) \rangle$, where the bracket is for the duality between $\dot{X}_\beta^\varepsilon, \dot{X}_\beta^{-\varepsilon}$ is absolutely continuous on \mathbb{R} and its almost everywhere derivative can be computed as follows. Write*

$$(\partial_t + v \cdot \nabla_x) f = S_1 + S_2 + S_3$$

with $S_1 \in L_{t,x}^2 \dot{H}_v^{-\beta+\varepsilon}$, $S_2 \in L_{t,v}^2 \dot{H}_x^{\frac{-\beta+\varepsilon}{2\beta+1}}$ and $S_3 \in L_t^1 \dot{X}_\beta^{-\beta+\varepsilon}$, and

$$(\partial_t + v \cdot \nabla_x) \tilde{f} = \tilde{S}_1 + \tilde{S}_2 + \tilde{S}_3$$

with $\tilde{S}_1 \in L_{t,x}^2 \dot{H}_v^{-\beta-\varepsilon}$, $\tilde{S}_2 \in L_{t,v}^2 \dot{H}_x^{\frac{-\beta-\varepsilon}{2\beta+1}}$ and $\tilde{S}_3 \in L_t^1 \dot{X}_\beta^{-\beta-\varepsilon}$. Then

$$\frac{d}{dt} \langle f(t), \tilde{f}(t) \rangle = \int_{\mathbb{R}^d} (\langle S_1, \tilde{f} \rangle + \langle f, \tilde{S}_1 \rangle) dx + \int_{\mathbb{R}^d} (\langle S_2, \tilde{f} \rangle + \langle f, \tilde{S}_2 \rangle) dv + \langle S_3, \tilde{f} \rangle + \langle f, \tilde{S}_3 \rangle$$

where each bracket denotes a different sesquilinear duality extending the L^2 inner product by tracing the spaces involved. Putting absolute values around each bracket gives an integrable function in the missing variables.

Proof. Items (i) and (ii) are already in Theorem 2.2. Item (iii) is the polarized version (1.6) of Theorem 1.1 where the exponents are symmetric with respect to β for duality reasons. Details are left to the reader. \square

4. CONSTRUCTION OF WEAK SOLUTIONS TO ROUGH EQUATIONS

This section is devoted to the construction of weak solutions. It relies on a theorem due to J.-L. Lions [24, Chap. II, Thm. 1.1]. Since this article is written in French, we recall the statement before proving Theorem 1.12.

4.1. Existence and uniqueness of weak solutions. The proof of the existence of weak solutions to Kolmogorov-Fokker-Planck equations with rough coefficients uses the following theorem.

Theorem 4.1 (Lions – [24]). *Let F be a Hilbert space equipped with a scalar product $\langle \cdot, \cdot \rangle_F$ and the associated norm $\|\cdot\|_F$. Let $\mathcal{H} \subset F$ be a prehilbertian space equipped with a scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and the associated norm $\|\cdot\|_{\mathcal{H}}$. Assume that there exists $c_1 > 0$ such that for all $f \in \mathcal{H}$, $\|f\|_F \leq c_1 \|f\|_{\mathcal{H}}$. Let E be a sesquilinear form on $F \times \mathcal{H}$ such that*

(H1) *for all $h \in \mathcal{H}$, the linear form $f \mapsto E(f, h)$ is continuous on F ,*

(H2) *there exists $\alpha > 0$ such that $\operatorname{Re} E(h, h) \geq \alpha \|h\|_{\mathcal{H}}^2$ for all $h \in \mathcal{H}$.*

Then for any semi-linear continuous form L on \mathcal{H} , there exists $f \in F$ such that $E(f, h) = L(h)$ for all $h \in \mathcal{H}$.

The following lemma is a consequence of the transfer of regularity obtained in Theorem 1.1.

Lemma 4.2 (Energy equality). *Let $\beta \in (0, d/2)$ and $S = S_1 + S_2 + S_3$ with $S_1 \in L_{t,x}^2 \dot{H}_v^{-\beta}$ and $S_2 \in L_{t,v}^2 \dot{H}_x^{-\frac{\beta}{2\beta+1}}$ and $S_3 \in L_t^1 L_{x,v}^2$. Any weak solution to $(\partial_t + v \cdot \nabla_x)f + \mathcal{A}f = S$ in the sense of Definition 1.9 satisfies the energy equality: for any $s, t \in \mathbb{R}$ with $s < t$,*

$$(4.1) \quad \|f(t)\|_{L_{x,v}^2}^2 + 2 \operatorname{Re} \int_s^t \int_{\mathbb{R}^d} a_{\tau,x}(f, f) \, dx \, d\tau = \|f(s)\|_{L_{x,v}^2}^2 + 2 \operatorname{Re} \int_s^t \left(\int_{\mathbb{R}^d} \langle S_1, f \rangle \, dx + \int_{\mathbb{R}^d} \langle S_2, f \rangle \, dv + \iint_{\mathbb{R}^{2d}} S_3 \bar{f} \, dx \, dv \right) d\tau.$$

Proof. As $(\partial_t + v \cdot \nabla_x)f = S - \mathcal{A}f$, we know from Theorem 1.1 that $t \mapsto \|f(t)\|_{L_{x,v}^2}^2$ is absolutely continuous, with limit 0 at $\pm\infty$ and, for $s < t$, $\|f(t)\|_{L_{x,v}^2}^2 - \|f(s)\|_{L_{x,v}^2}^2$ is equal to

$$2 \operatorname{Re} \int_s^t \left(\int_{\mathbb{R}^d} \langle S_1 - \mathcal{A}f, f \rangle \, dx + \int_{\mathbb{R}^d} \langle S_2, f \rangle \, dv + \iint_{\mathbb{R}^{2d}} S_3 \bar{f} \, dx \, dv \right) d\tau.$$

The bracket inside the first integral denotes the one expressing the $\dot{H}_v^{-\beta}, \dot{H}_v^{\beta}$ duality as a function of (τ, x) , so by definition and writing the variables for clarity

$$\langle \mathcal{A}f, f \rangle(\tau, x) = a_{\tau,x}(f(\tau, x), f(\tau, x)) = a_{\tau,x}(f, f)$$

by convention in the last equality. □

Before proving Theorem 1.12, we recall that $\dot{F}^{\beta} = L_{t,x}^2 \dot{H}_v^{\beta}$ with norm $\|D_v^{\beta} u\|_{L_{t,x,v}^2}$ for $D_v = (-\Delta_v)^{1/2}$ is a Hilbert space as $0 < \beta < d/2$.

Proof of Theorem 1.12. Lemma 4.2 already proves that any weak solution satisfies the energy equality.

In the second step, we prove uniqueness. Assume that $f \in \dot{F}^\beta$ with $(\partial_t + v \cdot \nabla_x)f + \mathcal{A}f = 0$ (in the sense of Definition 1.9). The energy equality reads when $s < t$,

$$\|f(t)\|_{L_{x,v}^2}^2 - \|f(s)\|_{L_{x,v}^2}^2 = -2 \operatorname{Re} \int_s^t \int_{\mathbb{R}^d} a_{\tau,x}(f, f) dx d\tau.$$

Letting now $s \rightarrow -\infty$ and $t \rightarrow +\infty$ and using that limits of $L_{x,v}^2$ norms are zero, we obtain $\operatorname{Re} \int_{-\infty}^{+\infty} \int_{\mathbb{R}^d} a_{\tau,x}(f, f) dx d\tau = 0$. Ellipticity implies $f = 0$.

In a third step, we prove the existence in the case where $S \in L_{t,x}^2 \dot{H}_v^{-\beta} = (\dot{F}^\beta)'$ by use of Theorem 4.1. We verify the hypotheses there.

We let $\mathcal{H} = \mathcal{D}(\Omega)$ equipped with the prehilbertian norm $\|h\|_{\mathcal{H}} = \|h\|_{\dot{F}^\beta}$. In particular, the inclusion $\mathcal{H} \subset \dot{F}^\beta$ is continuous.

Next, consider the sesquilinear form E on $\dot{F}^\beta \times \mathcal{H}$ defined by

$$E(f, h) = - \iiint_{\Omega} f(\partial_t + v \cdot \nabla_x) \bar{h} dt dx dv + a(f, h),$$

where a is defined in (1.9).

We need to check (H1), that is: for all $h \in \mathcal{H}$, $f \mapsto E(f, h)$ is continuous on \dot{F}^β . To this end, we show that $|E(f, h)| \lesssim \|f\|_{\dot{F}^\beta} \|h\|_{\dot{\mathcal{F}}_\beta^\beta}$, where the $\dot{\mathcal{F}}_\beta^\beta$ norm is defined in (2.5). This implies (H1) as $\|h\|_{\dot{\mathcal{F}}_\beta^\beta} < \infty$ for $h \in \mathcal{H}$. Let $h \in \mathcal{H}$ and $f \in \dot{F}^\beta$. Then, by Hölder inequality and Sobolev embedding, we have

$$\begin{aligned} \left| \iiint_{\Omega} f(\partial_t + v \cdot \nabla_x) \bar{h} dt dx dv \right| &\leq \|f\|_{L_{t,x}^2 L_v^q} \|(\partial_t + v \cdot \nabla_x) h\|_{L_{t,x}^2 L_v^{q'}} \\ &\lesssim \|f\|_{L_{t,x}^2 \dot{H}_v^\beta} \|(\partial_t + v \cdot \nabla_x) h\|_{L_{t,x}^2 \dot{H}_v^{-\beta}}. \end{aligned}$$

For the other term, we use the assumption on a in order to get,

$$|a(f, h)| \leq \Lambda \|f\|_{\dot{F}^\beta} \|h\|_{\dot{F}^\beta}.$$

Gathering the two terms using the definition of the $\dot{\mathcal{F}}_\beta^\beta$ norm in (2.5) gives us the conclusion.

We next check (H2), namely that $\operatorname{Re} E(h, h) \geq \lambda \|h\|_{\mathcal{H}}^2$ for all $h \in \mathcal{H}$. Indeed,

$$(4.2) \quad - \operatorname{Re} \iiint_{\Omega} h(\partial_t + v \cdot \nabla_x) \bar{h} dt dx dv = 0.$$

Thus, $\operatorname{Re} E(h, h) = \operatorname{Re} a(h, h) \geq \lambda \|h\|_{\dot{F}^\beta}^2 = \lambda \|h\|_{\mathcal{H}}^2$.

As $S \in (\dot{F}^\beta)'$, the map $h \mapsto \iint_{\mathbb{R} \times \mathbb{R}^d} \langle S, h \rangle dx dt$, where the brackets inside the integral are the $\dot{H}_v^{-\beta}, \dot{H}_v^\beta$ duality as usual, is a continuous semi-linear functional on \mathcal{H} . Applying the Theorem 4.1, there exists $f \in \dot{F}^\beta$ such that

$$E(f, h) = \iint_{\mathbb{R} \times \mathbb{R}^d} \langle S, h \rangle dx dt \quad \text{for all } h \in \mathcal{H}.$$

In our language, this means that f is a weak solution to $(\partial_t + v \cdot \nabla_x)f + \mathcal{A}f = S$ in the sense of Definition 1.9. And since $(\partial_t + v \cdot \nabla_x)f = S - \mathcal{A}f \in (\dot{F}^\beta)'$, we have that $f \in \dot{\mathcal{F}}_\beta^\beta$.

The second and third steps show the isomorphism property of $(\partial_t + v \cdot \nabla_x) + \mathcal{A}$ from $\dot{\mathcal{F}}_\beta^\beta$ onto $(\dot{F}^\beta)'$.

In the fourth step, we assume $S \in L_{t,v}^2 \dot{H}_x^{-\frac{\beta}{2\beta+1}} + L_t^1 L_{x,v}^2$. We prove the existence of a weak solution by a duality scheme and the use of kinetic embedding and transfer of regularity in Theorem 1.1. The same argument as above applies to $((\partial_t + v \cdot \nabla_x) + \mathcal{A})^* = -(\partial_t + v \cdot \nabla_x) + \mathcal{A}^*$, because (4.2) is not changed in changing the sign of the field and \mathcal{A}^* is also elliptic. It is also an isomorphism from $\dot{\mathcal{F}}_\beta^\beta$ onto $(\dot{F}^\beta)'$. In particular, Theorem 1.1 implies that

$$(-(\partial_t + v \cdot \nabla_x) + \mathcal{A}^*)^{-1} : (\dot{F}^\beta)' \rightarrow \dot{\mathcal{F}}_\beta^\beta \hookrightarrow \dot{\mathcal{L}}_\beta^\beta \hookrightarrow L_{t,v}^2 \dot{H}_x^{\frac{\beta}{2\beta+1}} \cap C_0(\mathbb{R}_t; L_{x,v}^2).$$

Hence, the operator $\mathcal{T} : L_{t,v}^2 \dot{H}_x^{-\frac{\beta}{2\beta+1}} + L_t^1 L_{x,v}^2 \rightarrow \dot{F}^\beta$ defined by

$$\langle \mathcal{T}S, h \rangle = \langle S, (-(\partial_t + v \cdot \nabla_x) + \mathcal{A}^*)^{-1}h \rangle, \quad \text{for } h \in (\dot{F}^\beta)' \text{ and } S \in L_{t,v}^2 \dot{H}_x^{-\frac{\beta}{2\beta+1}} + L_t^1 L_{x,v}^2,$$

is continuous.

If $S \in L_{t,v}^2 \dot{H}_x^{-\frac{\beta}{2\beta+1}} + L_t^1 L_{x,v}^2$ satisfies the extra assumption $S \in (\dot{F}^\beta)'$, then $\mathcal{T}S$ agrees with $((\partial_t + v \cdot \nabla_x) + \mathcal{A})^{-1}S$ as previously defined, so the function $f = \mathcal{T}S$ satisfies $E(f, h) = \langle S, h \rangle$ for all $h \in \mathcal{H} = \mathcal{D}(\Omega)$.

We now remove this extra assumption. Approximate $S \in L_{t,v}^2 \dot{H}_x^{-\frac{\beta}{2\beta+1}} + L_t^1 L_{x,v}^2$ by a sequence $S_k \in \mathcal{S}_K \subset \mathcal{S}(\Omega)$ as in Lemma 2.10, and thus, letting $f_k = \mathcal{T}S_k$, we have $E(f_k, h) = \langle S_k, h \rangle$ when $h \in \mathcal{D}(\Omega)$. For fixed h , and using the proper meaning of $\langle S, h \rangle$ explained in Theorem 1.1, the second term converges to $\langle S, h \rangle$ since $S_k \rightarrow S$ in $L_{t,v}^2 \dot{H}_x^{-\frac{\beta}{2\beta+1}} + L_t^1 L_{x,v}^2$. As $f_k \rightarrow \mathcal{T}S$ in \dot{F}^β , we can pass to the limit in the first one using the estimate proved for (H1) above. We have shown that $\mathcal{T}S$ is a weak solution to (1.3). Eventually, the fact that $\mathcal{T}S \in L_{t,v}^2 \dot{H}_x^{\frac{\beta}{2\beta+1}} \cap C_0(\mathbb{R}_t; L_{x,v}^2)$ follows from Theorem 1.1. \square

4.2. Operators of Kolmogorov type. Let us mention a corollary of Theorem 1.12. We introduce the operator

$$(4.3) \quad \mathcal{K}_\mathcal{A} = ((\partial_t + v \cdot \nabla_x) + \mathcal{A})^{-1}$$

defined as follows: For $S \in \mathcal{S}'(\Omega)$, $\mathcal{K}_\mathcal{A}S$ is the weak solution in \dot{F}^β , whenever it exists, to $(\partial_t + v \cdot \nabla_x)f + \mathcal{A}f = S$ in the sense of Definition 1.9.

Corollary 4.3 (Boundedness and isomorphism). *Assume $0 < \beta < d/2$ and the assumptions of Theorem 1.12 on \mathcal{A} .*

(i) *We have the continuous map*

$$(4.4) \quad \mathcal{K}_\mathcal{A} : L_{t,x}^2 \dot{H}_v^{-\beta} + L_{t,v}^2 \dot{H}_x^{-\frac{\beta}{2\beta+1}} + L_t^1 L_{x,v}^2 \rightarrow L_{t,x}^2 \dot{H}_v^\beta \cap L_{t,v}^2 \dot{H}_x^{\frac{\beta}{2\beta+1}} \cap C_0(\mathbb{R}_t; L_{x,v}^2).$$

- (ii) The operator $(\partial_t + v \cdot \nabla_x) + \mathcal{A}$ is an isomorphism from $\dot{\mathcal{F}}_\beta$ onto $L_{t,x}^2 \dot{H}_v^{-\beta}$, from $\dot{\mathcal{K}}_\beta$ onto $L_{t,x}^2 \dot{X}_\beta^{-\beta}$, and from $\dot{\mathcal{L}}_\beta$ onto $L_t^2 \dot{X}_\beta^{-\beta} + L_t^1 L_{x,v}^2$.

Proof. The first item is a consequence of the proof of Theorem 1.12. The first isomorphism was also established in the course of the argument. The other two follow by combining existence in steps 3 and 4, uniqueness in step 2 and the open mapping theorem. \square

Remark 4.4. The formal adjoint of $\mathcal{K}_\mathcal{A}$, namely $(\mathcal{K}_\mathcal{A})^* = -(\partial_t + v \cdot \nabla_x) + \mathcal{A}^*$, has the same properties. When $\mathcal{A} = (-\Delta_v)^\beta$, strictly speaking we should write $\mathcal{A} = I_{t,x} \otimes (-\Delta_v)^\beta$ with $I_{t,x}$ being the identity on $L_{t,x}^2$, then we recover the Kolmogorov operators as $\mathcal{K}_\mathcal{A} = \mathcal{K}_\beta^+$ and $(\mathcal{K}_\mathcal{A})^* = \mathcal{K}_\beta^-$.

5. KINETIC CAUCHY PROBLEMS

This section is devoted to the construction of weak solutions to kinetic Cauchy problems on strips for time intervals of the form $[0, \infty)$ or $[0, T]$ with T finite. In order to construct solutions on strips, we will introduce inhomogeneous kinetic spaces and prove an embedding theorem. We begin with working on infinite intervals and obtain homogeneous estimates.

5.1. The kinetic Cauchy problem in homogeneous spaces. We turn our attention to the kinetic Cauchy problem

$$(5.1) \quad \begin{cases} (\partial_t + v \cdot \nabla_x) f + \mathcal{A} f = S, \\ f(0) = \psi, \end{cases}$$

For the kinetic Cauchy problem on infinite time interval, we replace Ω by $\Omega_+ = (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and one can again use the homogeneous spaces on Ω_+ with same notation, see Section 2.2.2. We use the same notation as well for the operator \mathcal{A} .

Definition 5.1 (Weak solutions to the kinetic Cauchy problem). Let $\beta \in (0, d/2)$. Let $S \in \mathcal{D}'(\Omega_+)$ and $\psi \in L_{x,v}^2$. The distribution S is assumed to extend to a continuous linear functional on $\mathcal{D}(\overline{\Omega_+})$. A distribution $f \in \mathcal{D}'(\Omega_+)$ is said to be a *weak solution* to (5.1) if $f \in L_{t,x}^2 \dot{H}_v^\beta$ and for all $h \in \mathcal{D}(\overline{\Omega_+})$,

$$- \iiint_{\Omega_+} f (\partial_t + v \cdot \nabla_x) \bar{h} \, dt \, dx \, dv + a(f, h) = \langle S, h \rangle + \iint_{\mathbb{R}^{2d}} \psi(x, v) \bar{h}(0, x, v) \, dx \, dv.$$

Again, $f \in L_{t,x}^2 \dot{H}_v^\beta$ implies $f \in L_{t,x}^2 \mathcal{S}'_v \cap L_{t,x}^2 L_{loc,v}^2$ with our definition. When h is smooth, the expression $(\partial_t + v \cdot \nabla_x) \bar{h}$ is nothing but the pointwise partial derivative calculation. Thus the left-hand side is well-defined; in the right-hand side, $\langle S, h \rangle$ expresses the ad-hoc duality for the extension of S (it will be clear in the following). The function ψ could be also a distribution in \mathbb{R}^{2d} , in which case the corresponding term should be a sesquilinear duality bracket $\langle \psi, h(0) \rangle$. When $h \in \mathcal{D}(\Omega_+)$, the last term disappears and this means in particular that f is a weak solution to the first equation $(\partial_t + v \cdot \nabla_x) f + \mathcal{A} f = S$ in the sense of Definition 1.9 on Ω_+ . The second equation in (5.1) is encoded by allowing test functions that do not vanish at $t = 0$.

Theorem 5.2 (Weak solutions to the kinetic Cauchy problem). *Let $S \in L^2_{t,x} \dot{H}_v^{-\beta} + L^2_{t,v} \dot{H}_x^{-\frac{\beta}{2\beta+1}} + L^1_t L^2_{x,v}$ and $\psi \in L^2_{x,v}$. There exists a unique weak solution $f \in L^2_{t,x} \dot{H}_v^\beta$ to the kinetic Cauchy problem in the sense of Definition 5.1.*

Moreover, $f \in L^2_{t,v} \dot{H}_x^{\frac{\beta}{2\beta+1}} \cap C_0([0, \infty); L^2_{x,v})$, hence there is $L^2_{x,v}$ convergence of $f(t)$ to ψ as $t \rightarrow 0$, and

$$\begin{aligned} \|D_v^\beta f\|_{L^2_{t,x,v}} + \|D_x^{\frac{\beta}{2\beta+1}} f\|_{L^2_{t,x,v}} + \sup_{t \in [0, \infty)} \|f(t)\|_{L^2_{x,v}} \\ \lesssim \|S\|_{L^2_{t,x} \dot{H}_v^{-\beta} + L^2_{t,v} \dot{H}_x^{-\frac{\beta}{2\beta+1}} + L^1_t L^2_{x,v}} + \|\psi\|_{L^2_{x,v}} \end{aligned}$$

for an implicit constant depending only on $d, \beta, \lambda, \Lambda$. In addition, f satisfies the energy equality (4.1) for non-negative times.

Proof. The strategy has several steps.

Step 1: The estimates of the backward Kolmogorov operator in Proposition 2.11 when $\gamma = \beta$ hold on Ω_+ , by restriction.

Step 2: The uniqueness statement in Lemma 3.1 holds for the backward Kolmogorov operator on Ω_+ , in particular when $\gamma = \beta$, with the same proof.

Step 3: Combining Steps 1 and 2, the space $\dot{\mathcal{L}}_\beta^\beta$ on Ω_+ embeds into $C_0([0, \infty); L^2_{x,v})$. In particular, for all $f \in \dot{\mathcal{L}}_\beta^\beta$,

$$\|f\|_{C_0([0, \infty); L^2_{x,v})} \lesssim_{d, \beta} \|f\|_{\dot{\mathcal{L}}_\beta^\beta}.$$

Step 4: More generally, the statement of Theorem 1.1 holds on Ω_+ with $C_0(\mathbb{R}_t; L^2_{x,v})$ replaced by $C_0([0, \infty); L^2_{x,v})$. The absolute continuity on $[0, \infty)$ is proved similarly using the backward kinetic Kolmogorov operator to construct approximations for the proof. In particular, any weak solution to the kinetic Cauchy problem as in the statement satisfies the energy equality (4.1) in Lemma 4.2 for non-negative times and converges in the sense of $L^2_{x,v}$ limit at $t = 0$.

Step 5: The uniqueness in the kinetic Cauchy problem proceeds with the energy equality as in the proof of Theorem 1.12.

Step 6: Existence when $S \in L^2_{t,x} \dot{H}_v^{-\beta} = (\dot{F}^\beta)'$ and $\psi \in L^2_{x,v}$ also proceeds as in the proof of Theorem 1.12 with modifications as follows. We change here \mathcal{H} to $\mathcal{H}_+ = \mathcal{D}(\overline{\Omega_+})$ equipped with prehilbertian norm $\|h\|_{\mathcal{H}_+}^2 = \|h\|_{\dot{F}^\beta}^2 + \|h(0)\|_{L^2_{x,v}}^2$. Clearly \mathcal{H}_+ is continuously included in the Hilbert space \dot{F}^β on Ω_+ .

Setting

$$E_+(f, h) := - \iiint_{\Omega_+} f(\partial_t + v \cdot \nabla_x) \bar{h} \, dt \, dx \, dv + a(f, h),$$

the proof of condition (H1) with $|E_+(f, h)| \lesssim \|f\|_{\dot{F}^\beta} \|h\|_{\dot{F}^\beta}$ for $f \in \dot{F}^\beta$ and $h \in \mathcal{H}_+$ is unchanged. Now, for condition (H2), if $h \in \mathcal{H}_+$ then

$$-\operatorname{Re} \iiint_{\Omega_+} h(\partial_t + v \cdot \nabla_x) \bar{h} \, dt \, dx \, dv = \frac{1}{2} \|h(0)\|_{L^2_{x,v}}^2.$$

Thus,

$$\operatorname{Re} E_+(h, h) = \frac{1}{2} \|h(0)\|_{L_{x,v}^2}^2 + \operatorname{Re} a(h, h) \geq \min(\lambda, \frac{1}{2}) \|h\|_{\mathcal{H}_+}^2.$$

Finally, the map $\mathcal{H}_+ \ni h \mapsto L(h) = \iint_{(0,\infty) \times \mathbb{R}^d} \langle S, h \rangle dx dt + \langle \psi, h(0) \rangle$, where the brackets inside the integral are the $\dot{H}_v^{-\beta}, \dot{H}_v^\beta$ duality as usual, is a continuous semi-linear functional on \mathcal{H}_+ . For the first term, this is as before, and for the second, this is simply Cauchy-Schwarz inequality,

$$|\langle \psi, h(0) \rangle| \leq \|\psi\|_{L_{x,v}^2} \|h(0)\|_{L_{x,v}^2} \leq \|\psi\|_{L_{x,v}^2} \|h\|_{\mathcal{H}_+}.$$

Applying Theorem 4.1 again, there exists $f \in \dot{F}^\beta$ such that for all $h \in \mathcal{D}(\overline{\Omega_+})$, $E_+(f, h) = L(h)$, that is explicitly

$$(5.2) \quad - \iiint_{\Omega_+} f(\partial_t + v \cdot \nabla_x) \bar{h} dt dx dv + \iint_{(0,\infty) \times \mathbb{R}^d} a_{t,x}(f, h) dx dt \\ = \iint_{(0,\infty) \times \mathbb{R}^d} \langle S, h \rangle dx dt + \langle \psi, h(0) \rangle.$$

In particular, f is a weak solution on Ω_+ to $(\partial_t + v \cdot \nabla_x)f + \mathcal{A}f = S$. It remains to show $f(0) = \psi$ as we already know that $t \mapsto f(t)$ is continuous on $[0, \infty)$ in $L_{x,v}^2$. We remark again that $t \mapsto \langle f(t), h(t) \rangle$ is absolutely continuous and

$$\langle f(t), h(t) \rangle - \langle f(0), h(0) \rangle = - \int_s^t \int_{\mathbb{R}^d} a_{\tau,x}(f, h) dx d\tau \\ + \int_0^t \int_{\mathbb{R}^d} \langle S, h \rangle dx d\tau + \int_0^t \iint_{\mathbb{R}^{2d}} f(\partial_t + v \cdot \nabla_x) \bar{h} dx dv d\tau.$$

Letting $t \rightarrow +\infty$, the right hand side tends to $-\langle \psi, h(0) \rangle$ by (5.2) and the left hand side to $-\langle f(0), h(0) \rangle$. As $h(0)$ can be arbitrary in $\mathcal{D}(\mathbb{R}^{2d})$, we obtain $u(0) = \psi$.

Step 7: It remains to prove existence when $S \in L_{t,v}^2 \dot{H}_x^{-\frac{\beta}{2\beta+1}} + L_t^1 L_{x,v}^2$ and $\psi = 0$. In order to do so, we extend S by 0 for $t < 0$ and take the weak solution f to $(\partial_t + v \cdot \nabla_x)f + \mathcal{A}f = S$ on $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ of Theorem 1.12. Using the energy equality, we have when $s < t \leq 0$,

$$\|f(t)\|_{L_{x,v}^2}^2 - \|f(s)\|_{L_{x,v}^2}^2 = -2 \operatorname{Re} \int_s^t \int_{\mathbb{R}^d} a_{t,x}(f, f) dx dt,$$

and letting $s \rightarrow -\infty$, we obtain $\|f(t)\|_{L_{x,v}^2}^2 \leq 0$. Therefore, f restricted to Ω_+ is a weak solution on Ω_+ with $f(0) = 0$. \square

Corollary 5.3 (Causality principle). *Let $S \in L_{t,x}^2 \dot{H}_v^{-\beta} + L_{t,v}^2 \dot{H}_x^{-\frac{\beta}{2\beta+1}} + L_t^1 L_{x,v}^2$ on Ω_+ . If S_0 is the zero extension of S for $t < 0$, then $\mathcal{K}_A S_0$, with \mathcal{K}_A as in (4.3), vanishes for $t \leq 0$ and its restriction to Ω_+ is the solution to the kinetic Cauchy problem (5.1) with zero initial value.*

Proof. This is the argument in Step 7. \square

Remark 5.4. Following the same strategy, one can also solve the backward kinetic Cauchy problem for $-(\partial_t + v \cdot \nabla_x) + \mathcal{A}^*$ on intervals $(-\infty, 0]$. Step 2 uses instead uniqueness for distributional solutions of the forward Kolmogorov operator on $(-\infty, 0) \times \mathbb{R}^{2d}$. As integration is on this set, the change of sign for the kinetic field gives the condition (H2) in Step 6. The other steps are rigorously the same.

5.2. Inhomogeneous kinetic spaces. The theory of homogeneous kinetic spaces adapts to inhomogeneous kinetic spaces using the inhomogeneous Sobolev spaces H_v^γ and $H_x^{\frac{\gamma}{2\beta+1}}$ for all $\beta > 0$ and $\gamma \in \mathbb{R}$ (no restriction $\gamma < d/2$ here). It amounts to adding the $L_{t,x,v}^2$ norm to the homogeneous ones. This leads us to define on Ω the inhomogeneous spaces X_β^γ as in (2.2) and (2.3), \mathcal{F}_β^γ as in (2.4), \mathcal{G}_β^γ as in (2.6), and \mathcal{L}_β^γ as in (2.7) for which the analog of Theorem 2.2 holds.

Theorem 5.5 (Kinetic embeddings and transfer of regularity: inhomogeneous case). *Assume $\gamma \in [0, 2\beta]$.*

- (i) $\mathcal{L}_\beta^\gamma \hookrightarrow L_{t,v}^2 H_x^{\frac{\gamma}{2\beta+1}}$ with $\|f\|_{L_{t,v}^2 H_x^{\frac{\gamma}{2\beta+1}}} \lesssim_{d,\beta,\gamma} \|f\|_{\mathcal{L}_\beta^\gamma}$
- (ii) $\mathcal{L}_\beta^\gamma \hookrightarrow C_0(\mathbb{R}_t; X_\beta^{\gamma-\beta})$ with $\sup_{t \in \mathbb{R}} \|f(t)\|_{X_\beta^{\gamma-\beta}} \lesssim_{d,\beta,\gamma} \|f\|_{\mathcal{L}_\beta^\gamma}$.
- (iii) $\mathcal{S}(\Omega)$ is densely contained in all $\mathcal{F}_\beta^\gamma, \mathcal{G}_\beta^\gamma$ and \mathcal{L}_β^γ .
- (iv) The families of spaces $(\mathcal{F}_\beta^\gamma)_\gamma$ and $(\mathcal{G}_\beta^\gamma)_\gamma$ have the complex interpolation property.

Remark 5.6. To the knowledge of the authors, the first estimate (i) for \mathcal{G}_β^γ in the special case $\gamma = \beta = 1$ appears first in [8].

This is proved using the inhomogeneous Kolmogorov operators $\mathcal{K}_{1,\beta}^+$ corresponding to $(\pm(\partial_t + v \cdot \nabla_x) + (-\Delta_v)^\beta + 1)^{-1}$ as in Section 2.3 with the analogs of Lemma 2.10 and Proposition 2.11 without any restriction on γ . The proofs are similar and we skip the details. Theorem 1.1 and its generalization Theorem 3.6 are also true with inhomogeneous assumptions and conclusions.

5.3. Kinetic Cauchy problems on strips. We now come to equations with relaxed ellipticity conditions, still imposing measurability. We assume that there exist $0 < \Lambda < \infty$ and $0 \leq c^0 < \infty$, such that uniformly for $(t, x) \in \mathbb{R} \times \mathbb{R}^d$,

$$(5.3) \quad \forall f, g \in L_{t,x}^2 H_v^\beta, \quad |a_{t,x}(f, g)| \leq \Lambda(\|f\|_{\dot{H}_v^\beta} + c^0 \|f\|_{L_v^2})(\|g\|_{\dot{H}_v^\beta} + c^0 \|g\|_{L_v^2})$$

and that there exists $\lambda > 0, c_0 \geq 0$ such that for all $f \in H_v^\beta$ and uniformly for $(t, x) \in \mathbb{R} \times \mathbb{R}^d$,

$$(5.4) \quad \operatorname{Re} a_{t,x}(f, f) \geq \lambda \|f\|_{\dot{H}_v^\beta}^2 - c_0 \|f\|_{L_v^2}^2.$$

Note that this allows the possibility to include lower-order operators acting in velocity with suitable coefficients.

The inhomogeneous version of existence and uniqueness of weak solutions defined in $F^\beta = L_{t,x}^2 H_v^\beta$ on $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ of $(\partial_t + v \cdot \nabla_x)f + \mathcal{A}f + cf = S$ is completely similar with $c > c_0$. That is, the source and the solution belong to the respective corresponding inhomogeneous spaces. The inhomogeneous kinetic operator $\mathcal{K}_{c,A} = (\partial_t + v \cdot \nabla_x + \mathcal{A} + c)^{-1}$ has the inhomogeneous boundedness corresponding to (4.4).

For the kinetic Cauchy problem, we may work on a finite or infinite time interval. Define all the inhomogeneous kinetic spaces as before on $\Omega_I = I \times \mathbb{R}^d \times \mathbb{R}^d$ where I is an open interval. The embeddings are proved as follows.

For I infinite, say $(0, \infty)$, use the extension to the inhomogeneous backward Kolmogorov operator of estimates in Proposition 2.11 and uniqueness in Lemma 3.1 on Ω_+ . For $(-\infty, T)$ use the same arguments for the forward Kolmogorov operator.

For I finite, say $I = (0, T)$, use cut-off in time: if χ is Lipschitz and supported in $[0, T)$ and is 1 near 0, and $f \in \mathcal{L}_\beta^\gamma$ on $\Omega_{(0,T)}$, then $\chi f \in \mathcal{L}_\beta^\gamma$ on Ω_+ so that we have continuity of f in $X_\beta^{\gamma-\beta}$ in any subinterval $[0, T']$ of $[0, T)$. Also doing the same thing at the other endpoint of $(0, T)$, we conclude that $f \in C([0, T]; X_\beta^{\gamma-\beta})$. If $\gamma = \beta$, we also obtain the absolute continuity of $t \mapsto \|f(t)\|_{L_{x,v}^2}^2$ on $[0, T]$ and its extension to duality brackets $\langle f(t), \tilde{f}(t) \rangle$ of pairs $(f, \tilde{f}) \in \mathcal{L}_\beta^{\beta+\varepsilon} \times \mathcal{L}_\beta^{\beta-\varepsilon}$, $-\beta \leq \varepsilon \leq \beta$.

Moreover, $L_t^2(I; \mathcal{S}(\mathbb{R}^{2d}))$ is dense in all of them and one can also interpolate the \mathcal{F} and \mathcal{G} scales.

The inhomogeneous version of Theorem 5.2 is the following.

Theorem 5.7 (Weak solutions for the kinetic Cauchy problem in inhomogeneous spaces). *Let $0 < T < \infty$. Assume that the sesquilinear form a satisfies (5.3) and (5.4) on $(0, T) \times \mathbb{R}^d \times \mathbb{R}^d$. Let $S \in L_{t,x}^2 H_v^{-\beta} + L_{t,v}^2 H_x^{-\frac{\beta}{2\beta+1}} + L_t^1 L_{x,v}^2$ and $\psi \in L_{x,v}^2$. There exists a unique $f \in L_{t,x}^2 H_v^\beta$ such that*

$$(5.5) \quad \begin{cases} (\partial_t + v \cdot \nabla_x) f + \mathcal{A}f = S, \\ f(0) = \psi \end{cases}$$

in the sense of Definition 5.1 for test functions in $\mathcal{D}([0, T) \times \mathbb{R}^d \times \mathbb{R}^d)$.

Moreover, $f \in L_{t,v}^2 H_x^{\frac{\beta}{2\beta+1}} \cap C([0, T]; L_{x,v}^2)$, $f(t)$ converges to ψ in $L_{x,v}^2$ as $t \rightarrow 0$, and

$$\begin{aligned} \sup_{t \in [0, T]} \|f(t)\|_{L_{x,v}^2} + \|D_v^\beta f\|_{L_{t,x,v}^2} + \|D_x^{\frac{\beta}{2\beta+1}} f\|_{L_{t,x,v}^2} + \|f\|_{L_{t,x,v}^2} \\ \lesssim \|S\|_{L_{t,x}^2 H_v^{-\beta} + L_{t,v}^2 H_x^{-\frac{\beta}{2\beta+1}} + L_t^1 L_{x,v}^2} + \|\psi\|_{L_{x,v}^2} \end{aligned}$$

for an implicit constant depending on $d, \beta, \lambda, \Lambda, c_0, T$. Furthermore, f satisfies the energy equality (4.1) for all times $0 \leq s < t \leq T$.

Proof. For existence, extend $a_{t,x}$ by the form associated with $\lambda(-\Delta_v)^\beta$. We may call it the canonical extension, and note that it preserves the ellipticity constants λ, Λ . Extend S by 0 for $t \geq T$. In the case $c_0 = 0 = c^0$, we have nothing else to do than restrict the construction on $[0, \infty)$ to $[0, T]$. If $c_0 > 0$, use the exponential change of variable trick $f \rightarrow e^{-ct} f$ to go back to the kinetic Cauchy problem $(\partial_t + v \cdot \nabla_x) f + \mathcal{A}f + cf = e^{-ct} S$ with initial data ψ . Solve this problem with $c > c_0$ on $[0, \infty)$ exactly as in Theorem 5.2 but with inhomogeneous spaces, then restrict to $[0, T]$ and undo the change of variable mentioned above.

For uniqueness, recall that any $f \in \mathcal{L}_\beta^\beta$ on $\Omega_{(0,T)}$ belongs to $C([0, T]; L_{x,v}^2)$ with the absolute continuity of $t \mapsto \|f(t)\|_{L_{x,v}^2}^2$. If $f \in \mathcal{F}_\beta^\beta$ and $(\partial_t + v \cdot \nabla_x) f + \mathcal{A}f = 0$ on $\Omega_{(0,T)}$ with $f(0) = 0$, then $f \in \mathcal{F}_\beta^\beta \subset \mathcal{L}_\beta^\beta$ on $\Omega_{(0,T)}$. Thus f satisfies the energy equality

(4.1): for any $t \leq T$, $\|f(t)\|_{L^2_{x,v}}^2 + 2 \operatorname{Re} \int_0^t \int_{\mathbb{R}^d} a_{\tau,x}(f, f) dx d\tau = \|f(0)\|_{L^2_{x,v}}^2 = 0$, and it follows that $f = 0$. \square

Remark 5.8. The same arguments show that one can also solve the backward kinetic Cauchy problem for $-(\partial_t + v \cdot \nabla_x) + \mathcal{A}^*$ on intervals $[0, T]$ with final value at T , proceeding with Theorem 5.2 on $(-\infty, T)$ for operators $-(\partial_t + v \cdot \nabla_x) + \mathcal{A}^* + c$.

6. THE SPECIAL CASE OF LOCAL DIFFUSION OPERATORS

In this section, we focus on $\beta = 1$ and give different definitions of the homogeneous spaces that are valid in any dimension $d \geq 1$.

Define $\dot{W}^{1,2}(\mathbb{R}^d) = \{f \in \mathcal{D}'(\mathbb{R}^d); \nabla f \in L^2(\mathbb{R}^d)\}$, equipped with the semi-norm $\|\nabla f\|_{L^2}$. Remark that $\dot{H}^1(\mathbb{R}^d) \subset \dot{W}^{1,2}(\mathbb{R}^d)$ when $d \geq 3$. The inhomogeneous version is $W^{1,2}(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d); \nabla f \in L^2(\mathbb{R}^d)\}$ equipped with the norm $(\|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2)^{1/2}$. It agrees with $H^1(\mathbb{R}^d)$ when $d \geq 1$. Hence, our inhomogeneous theory is unchanged and we only have to address the homogeneous one.

Lemma 6.1. *Any element of $\dot{W}^{1,2}(\mathbb{R}^d)$ identifies to an element in $L^2_{\text{loc}}(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$, and $\mathcal{D}(\mathbb{R}^d)$ is dense in $\dot{W}^{1,2}(\mathbb{R}^d)$ for the semi-norm.*

Proof. This is known. Here is a proof for the comfort of the reader. Let $f \in \dot{W}^{1,2}(\mathbb{R}^d)$. As f is a distribution, we may regularize it to f_ε in this space by mollification, and $f_\varepsilon \in C^\infty$. Use Poincaré inequalities for f_ε and limiting arguments to deduce that $f \in L^2_{\text{loc}}(\mathbb{R}^d)$. Poincaré inequalities again give a polynomial growth in R of the L^2 norms of f on balls $B(0, R)$. Thus f is a tempered distribution as well. Lastly, for the density, we claim that we can approximate f by compactly supported functions in the $\|\nabla f\|_2$ semi-norm. Pick $f_k(x) = (f(x) - c_k)\varphi(2^{-k}x)$, $k \geq 1$, with φ a smooth function supported on the ball $B(0, 2)$ and that is 1 on the unit ball and where c_k is the mean of f on the ball $B(0, 2^{k+1})$. It is easy to see that ∇f_k converges weakly to ∇f in L^2 and by Mazur's lemma, there exist convex combinations of subsequences of ∇f_k having strong convergence to ∇f in L^2 . From there, use mollification to obtain an approximation in $\mathcal{D}(\mathbb{R}^d)$. \square

We introduce the kinetic space

$$(6.1) \quad \dot{\mathcal{L}} = \{f \in \mathcal{D}'(\Omega); f \in L^2_{t,x} \dot{W}_v^{1,2} \ \& \ (\partial_t + v \cdot \nabla_x)f \in L^2_t \dot{X}_1^{-1} + L^1_t L^2_{x,v}\}$$

with semi-norm $\|f\|_{\dot{\mathcal{L}}}$ defined by

$$\|f\|_{\dot{\mathcal{L}}} = \|\nabla_v f\|_{L^2_{t,x,v}} + \|(\partial_t + v \cdot \nabla_x)f\|_{L^2_t \dot{X}_1^{-1} + L^1_t L^2_{x,v}}.$$

The spaces with negative indices are defined as before and \dot{H}_v^{-1} identifies to the dual of $\dot{W}_v^{1,2}$, so that we also write $\dot{H}_v^{-1} = \dot{W}_v^{-1,2}$ and $\dot{X}_1^{-1} = L^2_x \dot{W}_v^{-1,2} + L^2_v \dot{H}_x^{-\frac{1}{3}}$. Note that constants belong to $\dot{\mathcal{L}}$ (contrarily to $\dot{\mathcal{L}}_1$ to which we naturally compare $\dot{\mathcal{L}}$).

Specializing the results obtained for the Fokker-Planck operators $\mathcal{K}^\pm := \mathcal{K}_1^\pm$ in Proposition 2.11 (recall that the limitation on γ there was only due to definition), we can state the following boundedness result.

Proposition 6.2 (Boundedness of the Fokker-Planck operators). *We have the bounded map*

$$\mathcal{K}^\pm: L^2_{t,x} \dot{W}_v^{-1,2} + L^2_{t,v} \dot{H}_x^{-\frac{1}{3}} + L^1_t L^2_{x,v} \rightarrow L^2_{t,x} \dot{W}_v^{1,2} \cap L^2_{t,v} \dot{H}_x^{\frac{1}{3}} \cap C_0(\mathbb{R}_t; L^2_{x,v}).$$

Corollary 6.3 (Uniqueness up to constant). *Let $f \in \mathcal{D}'(\Omega)$ for which $\nabla_v f \in L^2_{t,x,v}$. Then $\pm(\partial_t + v \cdot \nabla_x)f - \Delta_v f = 0$ in the sense of distributions implies that f is constant.*

Proof. We follow the strategy of Lemma 3.1 and notation there. By Lemma 6.1 $f \in \mathcal{S}'(\Omega)$ but \hat{f} may not be locally integrable on Ω . Still, one can solve the ordinary differential equation in a distributional sense and obtain that $G = K(t, 0)G_0$ with $G_0 \in \mathcal{D}'(\mathbb{R}^{2d})$. Next the condition $(\xi - t\varphi)G \in L^2_{t,\varphi,\xi}$ implies that the restriction of G to the complement of $\{\xi - t\varphi = 0\}$ is locally square integrable. This forces G_0 to be supported at the origin $(\varphi, \xi) = (0, 0)$. Indeed, pick a compact set C in $\mathbb{R}^{2d} \setminus \{(0, 0)\}$. Then one can find a compact interval $I \subset \mathbb{R}$ such that $\min |\xi - t\varphi|^2 > 0$ on $I \times C$. Thus $G \in L^2(I \times C)$ and, using the equation, we see that $G \in C(I; L^2(C))$. This implies that $G_0 \in L^2(C)$. Using again $(\xi - t\varphi)G \in L^2_{t,\varphi,\xi}$, this gives us

$$\iiint_{\mathbb{R} \times C} |\xi - t\varphi|^2 \exp\left(-2 \int_0^t |\xi - \tau\varphi|^2 d\tau\right) |G_0(\varphi, \xi)|^2 dt d\varphi d\xi < \infty,$$

which yields $G_0 = 0$ on C . Hence G is supported in $\mathbb{R} \times \{(0, 0)\}$. Hence $(\xi - t\varphi)G = 0$ as a distribution (because it is locally integrable and supported on a null Lebesgue set). Thus $\partial_t G = 0$ from the equation. Differentiating $\xi G = t\varphi G$ with respect to t leads to $\varphi G = 0$, and consequently $\xi G = 0$. This implies that $G = T_0 \otimes \delta$ where δ is the Dirac mass at the origin in \mathbb{R}^{2d} and $T_0 \in \mathcal{D}'(\mathbb{R})$. One uses again $\partial_t G = 0$ to conclude that T_0 is constant. Going back to f , we conclude that f is constant. \square

Proposition 6.4. *The Kolmogorov operators \mathcal{K}^\pm are isomorphisms from $L^2_{t,x} \dot{W}_v^{-1,2}$ onto $\dot{\mathcal{F}}/\mathbb{C}$, from $L^2_t \dot{X}_1^{-1}$ onto $\dot{\mathcal{G}}/\mathbb{C}$ and from $L^2_t \dot{X}_1^{-1} + L^1_t L^2_{x,v}$ onto $\dot{\mathcal{L}}/\mathbb{C}$, using quotient norms, where the spaces $\dot{\mathcal{F}}$ and $\dot{\mathcal{G}}$ are defined similarly to $\dot{\mathcal{L}}$ replacing $L^2_{t,x} \dot{H}_v^1$ by $L^2_{t,x} \dot{W}_v^{1,2}$.*

Proof. This follows from the estimates and uniqueness, as in Lemma 3.3. \square

Corollary 6.5. *Let $f \in \dot{\mathcal{L}}$. Then there exists a constant $c \in \mathbb{C}$ such that $f - c \in C_0(\mathbb{R}_t; L^2_{x,v}) \cap L^2_{t,v} \dot{H}_x^{\frac{1}{3}}$ with estimate*

$$\|f - c\|_{L^2_{t,v} \dot{H}_x^{\frac{1}{3}}} + \sup_{t \in \mathbb{R}} \|f(t) - c\|_{L^2_{x,v}} \lesssim_d \|f\|_{\dot{\mathcal{L}}}.$$

The space in (iii) of Theorem 2.2 is dense in $\dot{\mathcal{L}}$. Moreover, for $f \in \dot{\mathcal{L}}$, the absolute continuity stated in Theorem 1.1 (and its generalization Theorem 3.6) holds with $f - c$.

Proof. Follow the strategy of proof of Theorem 2.2. Details are left to the reader. \square

Remark 6.6. One can now see that $\dot{\mathcal{L}} = \dot{\mathcal{L}}_1^1 + \mathbb{C}$ when $d \geq 3$.

Let us turn to the construction of weak solutions. The difficulty with this homogeneous space here is that $L^2_{t,x} \dot{W}_v^{1,2}$ is not a Hilbert space so that we cannot apply the Lions theorem directly. We use an approximation procedure starting from the inhomogeneous case. We let $\mathbf{A} = \mathbf{A}(t, x, v)$ be a bounded and coercive matrix with measurable coefficients (real and symmetric is not necessary) in such a way that

$$a_{t,x}(f, g) = \int_{\mathbb{R}^d} \langle \mathbf{A}(t, x, v) \nabla_v f(v), \nabla_v g(v) \rangle dv, \quad f, g \in \dot{W}_v^{1,2},$$

satisfies (1.7) and (1.8) on $\dot{W}_v^{1,2}$. We define \mathcal{A} as in (1.4) with the form a defined by (1.9) on $L^2_{t,x} \dot{W}_v^{1,2}$. Here we assume either $t \in \mathbb{R}$ or $t \in (0, \infty)$.

- Theorem 6.7.** (i) On \mathbb{R} , if $S \in L_t^2 \dot{X}_1^{-1} + L_t^1 L_{x,v}^2$, then there is a weak solution $f \in L_{t,x}^2 \dot{W}_v^{1,2}$, unique up to a constant, to $(\partial_t + v \cdot \nabla_x)f + \mathcal{A}f = S$. For the appropriate constant, $f - c \in C_0(\mathbb{R}_t; L_{x,v}^2) \cap L_{t,v}^2 \dot{H}_x^{\frac{1}{3}}$ and satisfies the energy equality.
- (ii) On $[0, \infty)$, if $S \in L_t^2 \dot{X}_1^{-1} + L_t^1 L_{x,v}^2$ and $\psi \in L_{x,v}^2$, then there is a unique weak solution in $L_{t,x}^2 \dot{W}_v^{1,2}$ to the kinetic Cauchy problem $(\partial_t + v \cdot \nabla_x)f + \mathcal{A}f = S$ with $f(0) = \psi$. We have $f \in C_0([0, \infty); L_{x,v}^2) \cap L_{t,v}^2 \dot{H}_x^{\frac{1}{3}}$ and satisfies the energy equality.

Proof. We start with uniqueness. In case (i), a weak solution $f \in L_{t,x}^2 \dot{W}_v^{1,2}$ to $(\partial_t + v \cdot \nabla_x)f + \mathcal{A}f = S$ belongs to $\dot{\mathcal{L}}$. Thus, there is a constant c such that $\tilde{f} = f - c \in C_0(\mathbb{R}_t; L_{x,v}^2)$ and satisfies the energy equality. If we assume $S = 0$, then for all $s < t$,

$$\|\tilde{f}(t)\|_{L_{x,v}^2}^2 - \|\tilde{f}(s)\|_{L_{x,v}^2}^2 = -2 \operatorname{Re} \int_s^t \int_{\mathbb{R}^d} a_{\tau,x}(\tilde{f}, \tilde{f}) dx d\tau.$$

In particular, this yields after taking limits at $\pm\infty$, $\operatorname{Re} \int_{-\infty}^{+\infty} \int_{\mathbb{R}^d} a_{\tau,x}(\tilde{f}, \tilde{f}) dx d\tau = 0$, hence $\operatorname{Re} a_{\tau,x}(\tilde{f}, \tilde{f}) = 0$ for almost all (τ, x) (because it is non negative). This yields $\|\tilde{f}(t)\|_{L_{x,v}^2}^2 - \|\tilde{f}(s)\|_{L_{x,v}^2}^2 = 0$ for all $s < t$, that is $t \mapsto \|\tilde{f}(t)\|_{L_{x,v}^2}^2$ is constant. As it has zero limits at $\pm\infty$, $\tilde{f} = 0$ and f is constant.

In case (ii), we can produce the same argument following Steps 1 and 2 in the proof of Theorem 5.2 to show that on Ω_+ , for all $f \in \dot{\mathcal{L}}$, there is a constant c such that

$$\|f - c\|_{C_0([0, \infty); L_{x,v}^2)} \lesssim_d \|f\|_{\dot{\mathcal{L}}}.$$

Now consider a weak solution f with $S = 0$ and $\psi = 0$. As we assume that $f(0) = 0$ at time 0, $f \in C_0([0, \infty); L_{x,v}^2)$ and satisfies the energy equality. Therefore, for $t > 0$, $\|f(t)\|_{L_{x,v}^2}^2 = -2 \operatorname{Re} \int_0^t \int_{\mathbb{R}^d} a_{\tau,x}(f, f) dx d\tau \leq 0$, thus $f(t) = 0$.

Now we turn to existence. We begin with (ii) in the case where $S \in L_{t,x}^2 \dot{W}_v^{-1,2}$ and $\psi \in L_{x,v}^2$. We prove existence on any finite strip $\Omega_{(0,T)} = (0, T) \times \mathbb{R}^d \times \mathbb{R}^d$ of a weak solution f^T with $\nabla_v f^T \in L^2(\Omega_{(0,T)})$ and $f^T \in C([0, T]; L_{x,v}^2)$. Assume this is done. Such solutions also verify the energy equality. If $T < T'$ and $g = f^{T'} - f^T$, then g is a weak solution in the same class on $[0, T]$ with zero source and zero initial data. The energy equality yields when $t \in [0, T]$, $\|g(t)\|_{L_{x,v}^2}^2 = -2 \operatorname{Re} \int_0^t \int_{\mathbb{R}^d} a_{\tau,x}(g, g) dx d\tau \leq 0$, thus $g(t) = 0$. Since f^T and $f^{T'}$ agree on the smaller strip, this allows us to define f on Ω_+ and f is a weak solution to the kinetic Cauchy problem in (ii) with the desired properties.

To construct f^T , we argue using the inhomogeneous setup on $[0, T]$ that is now fixed and we work on this interval. For $\varepsilon > 0$, using Theorem 5.7 with $\mathcal{A} + \varepsilon$, we solve in $L_{t,x}^2 \dot{W}_v^{1,2}$ the kinetic Cauchy problem $(\partial_t + v \cdot \nabla_x)f_\varepsilon + \mathcal{A}f_\varepsilon + \varepsilon f_\varepsilon = e^{-\varepsilon t} S$ with $f_\varepsilon(0) = \psi$. Note that $(\partial_t + v \cdot \nabla_x)(e^{\varepsilon t} f_\varepsilon) + \mathcal{A}(e^{\varepsilon t} f_\varepsilon) = S$ and $(e^{\varepsilon t} f_\varepsilon)(0) = \psi$. Using the energy equality for $e^{\varepsilon t} f_\varepsilon$ (absolute continuity follows from that of f_ε) and the fact that $S \in L_x^2 \dot{W}_v^{-1,2}$, we can see that $\|\nabla_v(e^{\varepsilon t} f_\varepsilon)\|_{L_{t,x,v}^2}$ and $\sup_{t \in [0, T]} \|e^{\varepsilon t} f_\varepsilon(t)\|_{L_{x,v}^2}$ are uniformly bounded with respect to ε . Hence, a classical weak*-limit argument furnishes a weak

solution $f^T \in L_t^\infty L_{x,v}^2 \cap L_{t,x}^2 W_v^{1,2}$. As f^T belongs to the inhomogeneous version \mathcal{L} of $\dot{\mathcal{L}}$ on $\Omega_{(0,T)}$, continuity in time follows.

Now, we turn to existence in (i) when $S \in L_{t,x}^2 \dot{W}_v^{-1,2}$ on Ω this time. By applying (ii) with time interval $[-n, \infty)$, $n \in \mathbb{N}$, instead on $[0, \infty)$, consider the weak solution f_n on $[-n, \infty)$ to the kinetic Cauchy problem $(\partial_t + v \cdot \nabla_x) f_n + \mathcal{A} f_n = S$ and $f_n(-n) = 0$. The extension of this solution by 0 when $t \leq -n$ belongs to $L_{t,x}^2 \dot{W}_v^{1,2} \cap C_0(\mathbb{R}_t; L_{x,v}^2) \cap L_{t,v}^2 \dot{H}_x^{\frac{1}{3}}$ on Ω , uniformly in n . Extract a weak*-limit in $L_{t,x}^2 \dot{W}_v^{1,2} \cap L_t^\infty L_{x,v}^2 \cap L_{t,v}^2 \dot{H}_x^{\frac{1}{3}}$. This weak*-limit is a weak solution on \mathbb{R} and as it is already a bounded function of time valued in $L_{x,v}^2$, it further belongs to $C_0(\mathbb{R}_t; L_{x,v}^2)$ using the embedding in Corollary 6.5.

It remains to solve the problems for source terms in $L_{t,v}^2 \dot{H}_x^{-\frac{1}{3}} \cap L_t^1 L_{x,v}^2$. For this, we use the duality scheme in Step 4 of the proof of Theorem 1.12 first on \mathbb{R} . Next, on $(0, \infty)$ with zero initial data, we use this previous case and the causality principle as in Step 7 of the proof of Theorem 5.2. \square

Remark 6.8. When $d \geq 3$, as $L_{t,x}^2 \dot{H}_x^1 \subset L_{t,x}^2 \dot{W}_v^{1,2}$, uniqueness implies that the weak solutions in Theorem 6.7, (ii), and in Theorem 5.2 when $\beta = 1$ are the same. Similarly, the weak solution $f - c$ in Theorem 6.7, (i), is the same as the one in Theorem 1.12.

Let us illustrate our results by a consequence for optimal regularity, to be compared with [8, Theorem 1.5] where f and $\nabla_v f$ were both supposed to belong to $L_{t,x,v}^2$ for the conclusion $\Delta_v f \in L_{t,x,v}^2$.

Corollary 6.9 (Optimal regularity). *Let $f \in \mathcal{D}'(\Omega)$ with $\sup_{\sigma \in \mathbb{R}} \|\nabla_v f\|_{L^2(\Omega_{(\sigma, \sigma+1)})} < \infty$ (in particular, if $\nabla_v f \in L_{t,x,v}^2$) be such that $S = (\partial_t + v \cdot \nabla_x) f - \Delta_v f \in L_{t,x,v}^2$. Then, $(\partial_t + v \cdot \nabla_x) f, \Delta_v f \in L_{t,x,v}^2$ and $\nabla_v f \in C_0(\mathbb{R}_t; L_{x,v}^2)$.*

Proof. Again the estimates obtained using Fourier methods in Proposition 2.13 imply in particular the boundedness

$$\mathcal{K}^\pm: L_{t,x,v}^2 \rightarrow L_{t,x}^2 \dot{W}_v^{2,2} \cap C_0(\mathbb{R}_t; L_x^2 \dot{W}_v^{1,2})$$

where $\dot{W}^{2,2}(\mathbb{R}^d) = \{f \in \mathcal{D}'(\mathbb{R}^d); \nabla^2 f \in L^2(\mathbb{R}^d)\}$. Thus, $\mathcal{K}^+ S$ is a tempered distribution solution of the same equation as f . We show that $f - \mathcal{K}^+ S$ is constant, which concludes the proof from the properties of $\mathcal{K}^+ S$.

Remark that from the above bounds $\nabla_v \mathcal{K}^+ S$ satisfies the same condition as $\nabla_v f$ in the hypothesis. Applying the method of proof of Corollary 6.3 to $f - \mathcal{K}^+ S$ and calling G the Fourier transform of $\Gamma(f - \mathcal{K}^+ S)$, the only thing that changes is the condition $(\xi - t\varphi)G \in L_{t,\varphi,\xi}^2$ replaced by $(\xi - t\varphi)G \in L^2(\sigma, \sigma + 1; L_{\varphi,\xi}^2)$ uniformly in $\sigma \in \mathbb{R}$. This is enough to conclude that the restriction of G_0 to any compact set $C \subset \mathbb{R}^{2d} \setminus \{(0, 0)\}$ is square integrable and that

$$\sup_{\sigma \in \mathbb{R}} \iiint_{(\sigma, \sigma+1) \times C} |\xi - t\varphi|^2 \exp\left(-2 \int_0^t |\xi - \tau\varphi|^2 d\tau\right) |G_0(\varphi, \xi)|^2 dt d\varphi d\xi < \infty.$$

By taking $\sigma \rightarrow -\infty$, this is only possible if $G_0 = 0$ almost everywhere on C . The rest of the proof is verbatim the one of Corollary 6.3. \square

Remark 6.10. At the expense of more technicalities, one can make the same discussion with the higher-order homogeneous space $\dot{W}^{m,2}(\mathbb{R}^d) = \{f \in \mathcal{D}'(\mathbb{R}^d); \nabla^m f \in$

$L^2(\mathbb{R}^d)\}$, m integer, which contains polynomials of degree less than m , applying this to higher order Kolmogorov operators \mathcal{K}_m^\pm . For weak solutions, ellipticity (1.8) should be in the sense of a Gårding inequality on $\dot{W}_v^{m,2}$. Embedding (on \mathbb{R} or $[0, \infty)$) and uniqueness of weak solutions on \mathbb{R} are obtained modulo polynomials of degree less than m .

7. EXTENSIONS AND LINK WITH OTHER KINETIC EMBEDDINGS

This section is devoted to extensions of our results in two directions: in the case where $\gamma > 2\beta$ and in the case where $\beta > d/2$.

7.1. Beyond the limitation $\gamma \leq 2\beta$. As mentioned in the introduction there is a notion of kinetic spaces developed in [28] where embeddings similar to ours are proved without restriction on γ , while we are limited with $0 \leq \gamma \leq 2\beta$. It is worth explaining the links between the two works. For this, we need to introduce more spaces and make some extensions (mostly without proofs, which are adaptations of the previous arguments).

7.1.1. Homogeneous source spaces revisited. We first extend the range of parameter γ in our spaces \dot{X}_β^γ . We just mention at this stage that we did not do it in the article because it would not have helped to prove the existence of weak solutions in Theorem 1.12, where we only impose ellipticity in the v -variable.

In the proof of Lemma 2.9, we established the following isomorphism via the Fourier transform in \mathbb{R}^{2d}

$$\dot{X}_\beta^\gamma \rightarrow L_{\varphi, \xi}^2(w^{2\gamma} d\varphi d\xi),$$

where $w(\varphi, \xi) = \sup(|\xi|, |\varphi|^{\frac{1}{1+2\beta}})$, provided $\gamma < d/2$. When $\gamma > 0$, the only reason for the restriction $\gamma < d/2$ is our definition of \dot{X}_β^γ as the intersection of two spaces, separating the variables in some sense. We remark that a simple calculation shows for $\gamma \in \mathbb{R}$,

$$w^{-\gamma} \in L_{\text{loc}}^2(\mathbb{R}^{2d}) \iff \gamma < (\beta + 1)d.$$

This allows us to define \dot{X}_β^γ for $\gamma < (\beta + 1)d$ in the spirit of homogeneous Sobolev spaces by

$$\dot{X}_\beta^\gamma = \{f \in \mathcal{S}'(\mathbb{R}^{2d}); \exists g \in L^2(\mathbb{R}^{2d}), \hat{f} = w^{-\gamma} \hat{g}\}$$

with norm $\|f\|_{\dot{X}_\beta^\gamma} = \|g\|_{L_{\varphi, \xi}^2}$. In this range, it is a well-defined Hilbert space, contained in $\mathcal{S}'(\mathbb{R}^{2d})$ with elements having Fourier transforms in $L_{\text{loc}}^1(\mathbb{R}^{2d})$. When $\gamma < d/2$, this space is equal, as a set and with equivalent norm, to the one defined in (2.2) and (2.3).

With this definition, our proofs on the Fourier transform side (where there is no restriction on exponents) show that the statements of Lemma 2.9, Proposition 2.11 hold for $\gamma < (\beta + 1)d$. Also the uniqueness as in Lemma 3.1 applies with $f \in L_t^2 \dot{X}_\beta^\gamma$ and $\gamma < (\beta + 1)d$, thanks to Remark 3.2.

7.1.2. *More homogeneous kinetic spaces.* We remark that even with this new definition of \dot{X}_β^γ , the upper bound on γ in the definitions of our kinetic spaces $\dot{\mathcal{F}}_\beta^\gamma$, $\dot{\mathcal{G}}_\beta^\gamma$, $\dot{\mathcal{L}}_\beta^\gamma$ is $d/2$ because we impose them to be subspaces of $L_{t,x}^2 \dot{H}_v^\gamma$. Nevertheless, changing this condition to $f \in L_t^2 \dot{X}_\beta^\gamma$ allows to introduce when $\gamma < (\beta + 1)d$,

$$(7.1) \quad \dot{\mathcal{T}}_\beta^\gamma = \{f \in \mathcal{D}'(\Omega); f \in L_t^2 \dot{X}_\beta^\gamma \text{ \& } (\partial_t + v \cdot \nabla_x)f \in L_t^2 \dot{X}_\beta^{\gamma-2\beta}\}$$

with Hilbertian norm $\|f\|_{\dot{\mathcal{T}}_\beta^\gamma}$ defined by

$$\|f\|_{\dot{\mathcal{T}}_\beta^\gamma}^2 = \|f\|_{L_t^2 \dot{X}_\beta^\gamma}^2 + \|(\partial_t + v \cdot \nabla_x)f\|_{L_t^2 \dot{X}_\beta^{\gamma-2\beta}}^2,$$

and the larger space

$$(7.2) \quad \dot{\mathcal{U}}_\beta^\gamma = \{f \in \mathcal{D}'(\Omega); f \in L_t^2 \dot{X}_\beta^\gamma \text{ \& } (\partial_t + v \cdot \nabla_x)f \in L_t^2 \dot{X}_\beta^{\gamma-2\beta} + L_t^1 \dot{X}_\beta^{\gamma-\beta}\}$$

with norm $\|f\|_{\dot{\mathcal{U}}_\beta^\gamma}$ defined by

$$\|f\|_{\dot{\mathcal{U}}_\beta^\gamma} = \|f\|_{L_t^2 \dot{X}_\beta^\gamma} + \|(\partial_t + v \cdot \nabla_x)f\|_{L_t^2 \dot{X}_\beta^{\gamma-2\beta} + L_t^1 \dot{X}_\beta^{\gamma-\beta}}.$$

Note that the x -regularity is already included in the definition of these spaces, hence $\dot{\mathcal{T}}_\beta^\gamma \subset \dot{\mathcal{G}}_\beta^\gamma$ and $\dot{\mathcal{U}}_\beta^\gamma \subset \dot{\mathcal{L}}_\beta^\gamma$ when $0 \leq \gamma < d/2$. When $\gamma \leq 0$, the opposite inclusions hold.

There is an immediate observation that $(-\Delta_v)^\beta$ maps $L_t^2 \dot{X}_\beta^\gamma$ into $L_t^2 \dot{X}_\beta^{\gamma-2\beta}$ when $\gamma < (\beta + 1)d$. Indeed, writing $\hat{f} = w^{-\gamma} \hat{g}$ with $\hat{g} \in L_{t,\varphi,\xi}^2$, we have $|\xi|^{2\beta} \hat{f} = w^{-\gamma+2\beta} (|\xi|^{2\beta} w^{-2\beta} \hat{g})$. As $|\xi|^{2\beta} w^{-2\beta} \hat{g}$ is in $L_{t,\varphi,\xi}^2$ and $\gamma - 2\beta < \gamma$, we have that $|\xi|^{2\beta} \hat{f}$ is a tempered distribution with inverse (partial) Fourier transform in $L_t^2 \dot{X}_\beta^{\gamma-2\beta}$.

7.1.3. *Another homogeneous kinetic embedding.* In this scale, the kinetic embedding statement becomes the following one.

Theorem 7.1 (Homogeneous kinetic embedding: extension). *Assume $\gamma < (\beta + 1)d$.*

- (i) $\dot{\mathcal{T}}_\beta^\gamma \subset \dot{\mathcal{U}}_\beta^\gamma \leftrightarrow C_0(\mathbb{R}_t; \dot{X}_\beta^{\gamma-\beta})$ with $\sup_{t \in \mathbb{R}} \|f(t)\|_{\dot{X}_\beta^{\gamma-\beta}} \lesssim_{d,\beta,\gamma} \|f\|_{\dot{\mathcal{U}}_\beta^\gamma}$.
- (ii) The subspaces $\mathcal{K}_\beta^\pm(\mathcal{S}_K)$ of $\mathcal{S}(\Omega)$ in Theorem 2.2 are dense in $\dot{\mathcal{T}}_\beta^\gamma$ and $\dot{\mathcal{U}}_\beta^\gamma$.
- (iii) The family of spaces $(\dot{\mathcal{T}}_\beta^\gamma)_\gamma$ has the complex interpolation property.

Again this is only the definition of spaces that imposes the upper bound on γ . The inclusion (i) in this statement is weaker than (ii) in Theorem 2.2 but for a larger range.

We also have modifications of the statements of Theorem 1.1 and of Theorem 3.6 assuming $f \in \dot{\mathcal{U}}_\beta^{\beta+\varepsilon}$ and $\tilde{f} \in \dot{\mathcal{U}}_\beta^{\beta-\varepsilon}$, with only constraint $\beta \pm \varepsilon < (\beta + 1)d$. In the latter statement, as there is no transfer of regularity, (i) and (ii) only contain the sup norm estimates. Next, (iii) has to be modified slightly with assumptions on the sums $S_1 + S_2$, $\tilde{S}_1 + \tilde{S}_2$ rather than individual assumptions. The dualities in the formula for the derivative also have to be adapted. This is left to the reader.

The isomorphism statement Lemma 3.3 is modified as follows: when $\gamma < (\beta + 1)d$, \mathcal{K}_β^\pm are isomorphisms from $L_t^2 \dot{X}_\beta^{\gamma-2\beta}$ onto $\dot{\mathcal{T}}_\beta^\gamma$ and from $L_t^2 \dot{X}_\beta^{\gamma-2\beta} + L_t^1 \dot{X}_\beta^{\gamma-\beta}$ onto $\dot{\mathcal{U}}_\beta^\gamma$. This is what induces the interpolation property for the scale of spaces $\dot{\mathcal{T}}_\beta^\gamma$.

The above discussion also extends to homogeneous spaces on Ω_+ .

7.1.4. *Inhomogeneous versions.* Consider the inhomogeneous versions $\mathcal{T}_\beta^\gamma(I)$, $\mathcal{U}_\beta^\gamma(I)$ on Ω , Ω_+ or on strips $(0, T) \times \mathbb{R}^d \times \mathbb{R}^d$ where $I = \mathbb{R}$, $(0, \infty)$ or $(0, T)$, respectively. With these spaces, there are no conditions on $\gamma \in \mathbb{R}$. The natural generalizations to Theorem 7.1 and absolute continuity hold. In particular, one has

$$(7.3) \quad \mathcal{T}_\beta^\gamma(0, T) \hookrightarrow C([0, T]; X_\beta^{\gamma-\beta}) \text{ with } \sup_{t \in [0, T]} \|f(t)\|_{X_\beta^{\gamma-\beta}} \lesssim_{d, \beta, \gamma, T} \|f\|_{\mathcal{T}_\beta^\gamma(0, T)}.$$

If $T = \infty$, the homogeneous estimate is the one corresponding to (i) in Theorem 7.1 (with $\gamma < (\beta + 1)d$).

Proposition 7.2. *If $0 \leq \gamma \leq 2\beta$, then $\mathcal{G}_\beta^\gamma(0, T) = \mathcal{T}_\beta^\gamma(0, T)$. As a consequence, Theorem 5.5(i), and (7.3) are the same statements in this range of γ .*

Proof. When $\gamma \geq 0$, the definition yields the inclusion $\mathcal{T}_\beta^\gamma(0, T) \subset \mathcal{G}_\beta^\gamma(0, T)$. The inhomogeneous version of the transfer of regularity contained in Theorem 5.5(ii) shows that if $0 \leq \gamma \leq 2\beta$ then $\mathcal{G}_\beta^\gamma(0, T) \subset \mathcal{T}_\beta^\gamma(0, T)$. \square

When $\gamma \notin [0, 2\beta]$, this equality of spaces does not seem true, hence the kinetic embedding for $\mathcal{G}_\beta^\gamma(0, T)$ is not clear as well.

7.1.5. *Comparison with other embeddings.* We can now make a precise link with [28] where the estimate (7.3) was proved (see Theorem 5.13 in this work). More precisely, the spaces $\mathcal{T}_\beta^\gamma(0, T)$ are introduced (with different notation and in the range $\gamma \geq \beta$: a close examination shows this restriction was not necessary and their arguments work for all γ) as part of a more general framework (with temporal weights) for solving the kinetic Cauchy problem for the Kolmogorov equation with constant diffusion ($0 < \beta \leq 1$) and (7.3) is a key ingredient in the argument. While the techniques used there relied on Fourier transform as well, the method is different. It is proved by first establishing a weaker embedding in $C([0, T]; X)$ where X is a Sobolev space with a large enough negative exponent. This argument is possible when working with inhomogeneous spaces. Then by an interpolation argument, the exact trace space was characterized as $X_\beta^{\gamma-\beta}$. Some estimates were mentioned to hold when $T = \infty$ but not the homogeneous embedding that would correspond to (7.3).

7.2. Beyond the half-dimension condition. We describe here a way to remove the constraint $\beta < d/2$ while keeping homogeneous estimates.

7.2.1. *The two main results revisited.* The limitation $\beta < d/2$ in our two main results in the homogeneous situation is mostly a matter of definition in trying to avoid superfluous assumptions. We have seen that this limitation disappears under inhomogeneous assumptions (Theorem 5.7). This is because there is a control of $\|f(t)\|_{L_{x,v}^2}$ in L_t^2 sense. But then the price we paid in the inhomogeneous setting is that this control appears in the estimates.

If one wants to keep homogeneous estimates, then it suffices to add a mild qualitative inhomogeneous information that is not used quantitatively. More precisely, instead of working with general distributions, we assume at the beginning a control on $\|f(t)\|_{L_{x,v}^2}$ in $L_{loc,t}^2$ sense, which allows us to apply positive fractional powers of D_v and D_x to f .

We now indicate to the reader how things should be adapted. We follow the scheme of our discussion and start with tools allowing us to prove Theorem 1.1 when $\beta \geq d/2$.

To begin with, the mapping properties of the Kolmogorov operators \mathcal{K}_β^\pm from Proposition 2.11 read

$$(7.4) \quad \mathcal{K}_\beta^\pm : L_{t,x}^2 \dot{H}_v^{-\beta} + L_{t,v}^2 \dot{H}_x^{-\frac{\beta}{2\beta+1}} + L_t^1 L_{x,v}^2 \rightarrow L_{t,x}^2 \dot{H}_v^\beta \cap L_{t,v}^2 \dot{H}_x^{\frac{\beta}{2\beta+1}} \cap C_0(\mathbb{R}_t; L_{x,v}^2).$$

The norm estimates that go with make sense for all $\beta > 0$ if we take into account that the target space is contained in $L_{loc,t}^2 L_{x,v}^2$.

Next, the proof of uniqueness (Lemma 3.1) holds with $\gamma = \beta$ and assuming $f \in L_{loc,t}^2 L_{x,v}^2$, see Remark 3.2.

The isomorphism properties of Lemma 3.3 hold at $\gamma = \beta \geq d/2$ with modification of the kinetic spaces at $\gamma = \beta$: replace $f \in \mathcal{D}'(\Omega)$ by $f \in L_{loc,t}^2 L_{x,v}^2$, but keep the same norms.

The proof of Theorem 1.1 holds when $\beta \geq d/2$ assuming $f \in L_{loc,t}^2 L_{x,v}^2$ and taking into account the above modifications (it also uses the density Lemma 2.10 for exponent $-\beta$ which is unchanged).

Next, we come to weak solutions. We extend their definitions (Definition 1.9 on \mathbb{R} , Definition 5.1 on $(0, \infty)$ and its variant on $(0, T)$) when $\beta \geq d/2$ by requiring that a weak solution also belongs to $L_{loc,t}^2 L_{x,v}^2$.

Theorem 1.12 and Theorem 5.2 are valid with uniqueness class $L_{loc,t}^2 L_{x,v}^2 \cap L_{t,x}^2 \dot{H}_v^\beta$ when $\beta \geq d/2$. The proof of uniqueness uses Lemma 4.1 which is a mere corollary of Theorem 1.1. But now the proof of existence does not apply. Instead, we proceed the other way around and we adapt the method used to prove for existence in Theorem 6.7. That is, we start by solving the kinetic Cauchy problem on $(0, T)$ with inhomogeneous spaces for $(\partial_t + v \cdot \nabla_x) f_\varepsilon + \mathcal{A} f_\varepsilon + \varepsilon f_\varepsilon = e^{-\varepsilon t} S$ when $S \in L_{t,x}^2 \dot{H}_v^{-\beta}$ and $\psi \in L_{x,v}^2$. We observe that we have uniform bounds for $\|D_v^\beta(e^{\varepsilon t} f_\varepsilon)\|_{L^2(0,T; L_{x,v}^2)} + \sup_{t \in [0,T]} \|e^{\varepsilon t} f_\varepsilon(t)\|_{L_{x,v}^2}$ in $\varepsilon > 0$. Hence, we extract with $\varepsilon \rightarrow 0$ a weak*-limit $f^T \in L_t^\infty L_{x,v}^2 \cap L_{t,x}^2 \dot{H}_v^\beta$ that is a weak solution on $\Omega_{(0,T)}$ of $(\partial_t + v \cdot \nabla_x) f^T + \mathcal{A} f^T = S$ with same data. Then, the compatibility for different T implied by uniqueness allows us to take $T = \infty$, giving a proof for Theorem 5.2.

Now, we push backward the initial time to $-\infty$ with zero initial condition to obtain, again with a weak*-limit argument, a weak solution in Ω .

Eventually, we consider sources in $L_{t,v}^2 \dot{H}_x^{-\frac{\beta}{2\beta+1}} \cap L_t^1 L_{x,v}^2$ on Ω first by the duality scheme and next on Ω_+ using the causality principle. This finishes the proof for Theorem 1.12 when $\beta \geq d/2$.

The operator $\mathcal{K}_\mathcal{A}$ makes sense even when $\beta \geq d/2$ with the modification of the definition of weak solutions. Then Corollary 4.3 (boundedness and isomorphism) extends to $\beta \geq d/2$ with the modification of the homogeneous kinetic spaces as described above. Causality in Corollary 5.3 is still valid with this new definition.

Remark 7.3. When $\beta = 1$, the weak solutions built in the above paragraph are the same as the ones in Theorem 6.7 when $d \geq 1$.

7.2.2. *Concerning scales of homogeneous kinetic spaces.* A non-optimal way to address the case $\gamma \geq d/2$ in Theorem 2.2 is to impose in the definition of kinetic spaces

the condition $f \in L^2_{\text{loc},t} L^2_{x,v}$, keeping the norms unchanged. Then, homogeneous inequalities in (i) and (ii) remain valid. Density in (iii) also holds. But interpolation is probably lost (when $\gamma \geq d/2$) as is completeness. The proof and conclusion of Theorem 3.6 with this modification are also valid.

However, to modify Lemma 3.3 we need a definition ensuring completeness and this is not clear.

REFERENCES

- [1] Dallas Albritton, Scott Armstrong, Jean-Christophe Mourrat, and Matthew Novack. Variational methods for the kinetic Fokker-Planck equation, 2021. arXiv:1902.04037. [1.6.2](#), [1.6.3](#)
- [2] Francesca Anceschi and Annalaura Rebusci. On the obstacle problem associated to the Kolmogorov-Fokker-Planck operator with rough coefficients, 2023. arXiv:2302.11889. [1.6.2](#)
- [3] Pascal Auscher and Moritz Egert. A universal variational framework for parabolic equations and systems. *Calc. Var. Partial Differential Equations*, 62(9):Paper No. 249, 59, 2023. [1.5](#)
- [4] Pascal Auscher, Cyril Imbert, and Lukas Niebel. Fundamental solutions to Kolmogorov-Fokker-Planck equations with rough coefficients: existence, uniqueness, upper estimates. <https://arxiv.org/abs/2403.17468> [1.7](#)
- [5] Pascal Auscher, Sylvie Monniaux, and Pierre Portal. On existence and uniqueness for non-autonomous parabolic Cauchy problems with rough coefficients. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 19(2):387–471, 2019. [1.5](#)
- [6] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011. [2.5](#)
- [7] Jöran Bergh and Jörgen Löfström. *Interpolation spaces. An introduction*, volume No. 223 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin-New York, 1976. [2.3.9](#), [3.2.1](#)
- [8] François Bouchut. Hypocoelliptic regularity in kinetic equations. *J. Math. Pures Appl. (9)*, 81(11):1135–1159, 2002. [1.2](#), [1.6.3](#), [5.6](#), [6](#)
- [9] Luis Caffarelli, Chi Hin Chan, and Alexis Vasseur. Regularity theory for parabolic nonlinear integral operators. *J. Amer. Math. Soc.*, 24(3):849–869, 2011. [1.1](#)
- [10] Luis Caffarelli and Luis Silvestre. Regularity theory for fully nonlinear integro-differential equations. *Comm. Pure Appl. Math.*, 62(5):597–638, 2009. [1.1](#)
- [11] José A. Carrillo. Global weak solutions for the initial-boundary-value problems to the Vlasov-Poisson-Fokker-Planck system. *Math. Methods Appl. Sci.*, 21(10):907–938, 1998. [1.6.2](#)
- [12] François Golse, Cyril Imbert, Clément Mouhot, and Alexis F. Vasseur. Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and application to the Landau equation. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 19(1):253–295, 2019. [1.11](#), [1.6.4](#)
- [13] François Golse, Pierre-Louis Lions, Benoît Perthame, and Rémi Sentis. Regularity of the moments of the solution of a transport equation. *J. Funct. Anal.*, 76(1):110–125, 1988. [1.6.3](#)
- [14] François Golse, Benoît Perthame, and Rémi Sentis. Un résultat de compacité pour les équations de transport et application au calcul de la limite de la valeur propre principale d’un opérateur de transport. *C. R. Acad. Sci. Paris Sér. I Math.*, 301(7):341–344, 1985. [1.6.3](#)
- [15] Jessica Guerand and Cyril Imbert. Log-transform and the weak Harnack inequality for kinetic Fokker-Planck equations. *J. Inst. Math. Jussieu*, 22(6):2749–2774, 2023. [1.11](#), [1.6.4](#)
- [16] Jessica Guerand and Clément Mouhot. Quantitative De Giorgi methods in kinetic theory. *J. Éc. polytech. Math.*, 9:1159–1181, 2022. [1.11](#), [1.6.4](#)
- [17] Lars Hörmander. Hypocoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967. [1.6.1](#)
- [18] Cyril Imbert and Clément Mouhot. The Schauder estimate in kinetic theory with application to a toy nonlinear model. *Ann. H. Lebesgue*, 4:369–405, 2021. [1.6.4](#)
- [19] Cyril Imbert and Luis Silvestre. The weak Harnack inequality for the Boltzmann equation without cut-off. *J. Eur. Math. Soc. (JEMS)*, 22(2):507–592, 2020. [1.1](#), [1.1](#), [1.8](#), [1.11](#)

- [20] Cyril Imbert and Luis Silvestre. The Schauder estimate for kinetic integral equations. *Anal. PDE*, 14(1):171–204, 2021. [1.6.4](#)
- [21] Moritz Kassmann. A priori estimates for integro-differential operators with measurable kernels. *Calc. Var. Partial Differential Equations*, 34(1):1–21, 2009. [1.1](#)
- [22] A. Kolmogoroff. Zufällige Bewegungen (zur Theorie der Brownschen Bewegung). *Ann. of Math. (2)*, 35(1):116–117, 1934. [1.1](#), [1.6.1](#), [1.6.3](#)
- [23] Ermanno Lanconelli and Sergio Polidoro. On a class of hypoelliptic evolution operators. volume 52, pages 29–63. 1994. *Partial differential equations, II (Turin, 1993)*. [1.6.1](#), [1.7](#)
- [24] Jacques-Louis Lions. Sur les problèmes mixtes pour certains systèmes paraboliques dans des ouverts non cylindriques. *Ann. Inst. Fourier (Grenoble)*, 7:143–182, 1957. [1.2](#), [1.5](#), [4](#), [4.1](#)
- [25] Malte Litsgård and Kaj Nyström. The Dirichlet problem for Kolmogorov-Fokker-Planck type equations with rough coefficients. *J. Funct. Anal.*, 281(10):Paper No. 109226, 39, 2021. [1.6.2](#)
- [26] Amélie Loher. Quantitative De Giorgi methods in kinetic theory for non-local operators. *J. Funct. Anal.*, 286(6):110312, 2024. [1.8](#)
- [27] Lukas Niebel. Kinetic maximal $L^p_\mu(L^p)$ -regularity for the fractional Kolmogorov equation with variable density. *Nonlinear Anal.*, 214:Paper No. 112517, 21, 2022. [1.1](#), [1.6.4](#)
- [28] Lukas Niebel and Rico Zacher. Kinetic maximal L^2 -regularity for the (fractional) Kolmogorov equation. *J. Evol. Equ.*, 21(3):3585–3612, 2021. [1.1](#), [1.6.2](#), [2.3.1](#), [7.1](#), [7.1.5](#)
- [29] Lukas Niebel and Rico Zacher. Kinetic maximal L^p -regularity with temporal weights and application to quasilinear kinetic diffusion equations. *J. Differential Equations*, 307:29–82, 2022. [1.6.4](#), [1.7](#)
- [30] Lukas Niebel and Rico Zacher. On a kinetic Poincaré inequality and beyond, 2023. arXiv:2212.03199. [1.6.4](#)
- [31] Andrea Pascucci and Sergio Polidoro. The Moser’s iterative method for a class of ultraparabolic equations. *Commun. Contemp. Math.*, 6(3):395–417, 2004. [1.6.4](#)
- [32] Linda P. Rothschild and Elias M. Stein. Hypoelliptic differential operators and nilpotent groups. *Acta Math.*, 137(3-4):247–320, 1976. [1.6.4](#)
- [33] Wendong Wang and Liqun Zhang. The Hölder continuity of a class of 3-dimension ultraparabolic equations. *Acta Math. Sci. Ser. B (Engl. Ed.)*, 29(3):527–538, 2009. [1.6.4](#)

UNIVERSITÉ PARIS-SACLAY, CNRS, LABORATOIRE DE MATHÉMATIQUES D’ORSAY, 91405 ORSAY, FRANCE

Email address: `pascal.auscher@universite-paris-saclay.fr`

DÉPARTEMENT DE MATHÉMATIQUES ET APPLICATIONS, ÉCOLE NORMALE SUPÉRIEURE, UNIVERSITÉ PSL, CNRS, 75005 PARIS, FRANCE

Email address: `cyril.imbert@ens.psl.eu`

INSTITUT FÜR ANALYSIS UND NUMERIK, WESTFÄLISCHE WILHELMS-UNIVERSITÄT MÜNSTER, ORLÉANS-RING 10, 48149 MÜNSTER, GERMANY.

Email address: `lukas.niebel@uni-muenster.de`