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ON THE \mathbb{Z}_p -EXTENSIONS OF A TOTALLY p -ADIC IMAGINARY QUADRATIC FIELD

GEORGES GRAS

WITH AN APPENDIX BY JEAN-FRANÇOIS JAULENT

ABSTRACT. Let k be an imaginary quadratic field, and let p be an odd prime split in k . We analyze some properties of arbitrary \mathbb{Z}_p -extensions K/k . These properties are governed by the norm residue symbols of the fundamental p -unit x of k , in terms of the p -valuation, $\delta_p(k)$, of the Fermat quotient of x , which also determines the order of the logarithmic class group (Theorem 2.2, Appendix A) and leads to some generalizations of Gold’s criterion characterizing $\lambda_p(K/k) = 1$ (Theorems 5.1, 5.3 in general, then 5.5 for the cyclotomic case). This uses the higher rank Chevalley–Herbrand formulas, for the filtrations of the p -class groups in K , without any argument of Iwasawa’s theory. This study is in connection with articles of Gold, Sands, Dummit–Ford–Kisilevsky–Sands, Hubbard–Washington, Ozaki, Kataoka, Ray, Jaulent, Fujii and others, and provides a different and elementary perspective on these questions. Illustrations (with PARI/GP programs) are given in Appendix B.

1. INTRODUCTION

This work falls within the framework of Jaulent study of “totally p -adic” number fields k [Jau2024^a, Section 3], restricted, in particular, to the case of imaginary quadratic fields (in other words, we assume that p totally splits in k).

Our approach is not of Iwasawa’s theory type and uses classical arguments of class field theory by means of the Chevalley–Herbrand formula [Che1933], with some properties of the filtrations¹ attached to the p -class groups in a cyclic p -extension K/k that we gave in the 1994’s; this point of view allows to find some precise results and to understand algorithmic p -adic aspects that purely algebraic ones do not detect; it is now accepted that deep p -adic obstructions cannot be removed by means of a strict algebraic way, and, actually, a majority of articles supplement the usual techniques of $\mathbb{Z}_p[[T]]$ -modules with arguments of class field theory type, most often using, in a more or less hidden way, the genus theory form of Chevalley–Herbrand formula, improperly referenced through articles of Iwasawa or others

This gives rise to compelling heuristics for Greenberg’s conjectures [Gree1998], in the spirit of those for the real case [Gree1976], and to some general results of Gold’s criterion type [Gold1974, San1993] about Iwasawa’s invariants λ_p, μ_p .

More precisely we obtain, in a more simple way, the classical Gold’s criterion, and new generalizations which apply to any \mathbb{Z}_p -extensions K/k only ramified at all the p -places from some layers, with particular cases of non-trivial p -class group for k (recall that Gold’s techniques are applied to the cyclotomic \mathbb{Z}_p -extensions k^{cyc}/k , and are in fact essentially within the scope of class field theory). We suggest that these K/k ’s behave very differently from the cyclotomic one; so we conjecture that $\lambda_p(K/k) = 1$

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¹Let $G = \text{Gal}(K/k) = \langle \sigma \rangle$; the filtration is $\{\mathcal{H}_K^i := \{c \in \mathcal{H}_K, c^{(1-\sigma)^i} = 1\}\}_{i \geq 0}$.

and $\mu_p(K/k) = 0$ for them. Finally, we show that Jaulent's logarithmic class group, linked more generally to the Gross–Kuz'min conjecture, is the most appropriate object for these studies; see § 1.2 for more details.

1.1. Logarithmic writings. Let k be an imaginary quadratic field and let p be an odd prime split in k ; set $(p) = \mathfrak{p}\bar{\mathfrak{p}}$, where $z \rightarrow \bar{z}$ denotes complex conjugation for any object z . Let $\log_{\mathfrak{p}}$ and $\log_{\bar{\mathfrak{p}}}$ be the corresponding log-functions in the two p -completions of k , with the convention $\log_{\mathfrak{p}}(p) = \log_{\bar{\mathfrak{p}}}(p) = 0$. Recall that complex conjugation operates on $k_{\mathfrak{p}} \times k_{\bar{\mathfrak{p}}}$ by density of k in this product. Let h_k be the class number of k , and let $x_{\mathfrak{p}} =: x \in k^{\times}$ be the fundamental \mathfrak{p} -unit of k (obtained, up to sign, from $\mathfrak{p}^h =: (x)$, $h \mid h_k$ being the order of the class of \mathfrak{p}); so, $x_{\bar{\mathfrak{p}}} = \bar{x}_{\mathfrak{p}} = \bar{x}$. Since $x\bar{x} = p^h$, this defines:

$$(1.1) \quad \begin{cases} \log_{\mathfrak{p}}(\bar{x}) \text{ (usual definition since } \bar{x} \text{ is a unit at } \mathfrak{p}), \\ \log_{\bar{\mathfrak{p}}}(x) \text{ (conjugate of } \log_{\mathfrak{p}}(\bar{x})), \\ \log_{\mathfrak{p}}(x) = -\log_{\bar{\mathfrak{p}}}(\bar{x}), \log_{\bar{\mathfrak{p}}}(x) = -\log_{\mathfrak{p}}(\bar{x}) \text{ (with } \log(p) = 0). \end{cases}$$

Thus, this defines, without any ambiguity, in terms of p -adic valuations on \mathbb{Q}_p^{\times} :

$$\begin{cases} \delta_p(k) := v_p\left(\frac{1}{p} \log_{\mathfrak{p}}(\bar{x})\right) = v_p\left(\frac{1}{p} \log_{\bar{\mathfrak{p}}}(x)\right) \\ \quad = v_p\left(\frac{1}{p} \log_{\mathfrak{p}}(x)\right) = v_p\left(\frac{1}{p} \log_{\bar{\mathfrak{p}}}(\bar{x})\right) = v_p(\bar{x}^{p-1} - 1) - 1. \end{cases}$$

from arbitrary choices, the practical definition and computation of $\delta_p(k)$ being:

$$(1.2) \quad \delta_p(k) = v_p(\bar{x}^{p-1} - 1) - 1, \text{ with } \mathfrak{p}^h =: (x).$$

Definition 1.1. Let h_k be the class number of k and let $h \mid h_k$ be the order of the class of \mathfrak{p} . Put $\mathfrak{p}^h =: (x)$ and $\mathfrak{p}^{h_k} =: (X)$ (whence $X = x^{h_k \cdot h^{-1}}$), and define the integer $\tilde{\delta}_p(k) := v_p(\bar{X}^{p-1} - 1) - 1$, fulfilling, from (1.2), the obvious relation $\tilde{\delta}_p(k) = \delta_p(k) + [v_p(h_k) - v_p(h)]$. Note that $\tilde{\delta}_p(k) = \delta_p(k)$ for almost all primes p .

1.2. Main results. Let k be an imaginary quadratic field, let p be an odd prime number split in k , and set $(p) = \mathfrak{p}\bar{\mathfrak{p}}$. Denote by \mathcal{H}_k the p -class group of k . For any \mathbb{Z}_p -extension K/k , let $K_n \subset K$ be the layer of degree p^n over k , and let \mathcal{H}_{K_n} be its p -class group.

(a) (Theorem 2.2 and Appendix A). Let $\widetilde{\mathcal{H}}_k$ be the Jaulent logarithmic class group of k (i.e., $\text{Gal}(H_k^{\text{lc}}/k^{\text{cyc}})$, where k^{cyc} is the cyclotomic \mathbb{Z}_p -extension of k and H_k^{lc} the maximal abelian pro- p -extension of k , locally cyclotomic; see Section 2.1).

Then $\#\widetilde{\mathcal{H}}_k = p^{\tilde{\delta}_p(k)}$ (see Definition 1.1 for the explicit value of $\tilde{\delta}_p(k)$).

(b) (Theorems 3.3). Assume that \mathfrak{p} and $\bar{\mathfrak{p}}$ are ramified from some layers K_e (resp. $K_{\bar{e}}$) of K/k ($e \geq \bar{e} \geq 0$). The norm residue symbols $\left(\frac{x, K_n/k}{\mathfrak{p}}\right)$ and $\left(\frac{x, K_n/k}{\bar{\mathfrak{p}}}\right)$ are of orders $p^{n-e-\delta_p(k)}$ (resp. $p^{n-\bar{e}-\delta_p(k)}$), for $n \geq e + \delta_p(k)$ (cf. Diagram 1.3.3).

(c) (Theorem 5.1). Under the ramification of \mathfrak{p} , $\bar{\mathfrak{p}}$ from some layers $K_e, K_{\bar{e}}$, the order of the logarithmic class group \mathcal{H}_k is given by $(\mathcal{H}_{K_n}^2 : \mathcal{H}_{K_n}^1)$, for all $n \geq n_0$ (n_0 explicit), where $\{\mathcal{H}_{K_n}^i\}_{i \geq 0}$ is the filtration of \mathcal{H}_{K_n} (see (4.3)). Thus $\widetilde{\mathcal{H}}_k = 1$ leads to the Iwasawa's formula $\#\mathcal{H}_{K_n} = \#\mathcal{H}_{K_n}^1 = p^{n+v_p(h_k)-e-\bar{e}}$ for all $n \geq n_0$.

(d) (Theorem 5.3). Assume that the p -class group \mathcal{H}_k of k is generated by the p -class of $\mathfrak{p} \mid p$ (so, $\tilde{\delta}_p(k) = \delta_p(k)$), and that \mathfrak{p} and $\bar{\mathfrak{p}}$ totally ramify from k ($e = \bar{e} = 0$). Then, the two following claims are equivalent:

- $\#\mathcal{H}_{K_n} = p^{n+v_p(h_k)}$, for all $n \geq 0$,
- $\delta_p(k) = 0$.

(e) (*Theorem 5.5*). If $\mathcal{d}(\mathfrak{p})$ is of prime-to- p order, then $\lambda_p(k^{\text{cyc}}/k) \geq 2$ if and only if $p \mid h_k$ or $\delta_p(k) \geq 1$ (*Gold–Sands criterion*), or $\widetilde{\mathcal{H}}_k$ is non-trivial.

Many pioneering works dealing with analogues of Gold’s criterion consider k^{cyc} assuming $\mathcal{H}_k = 1$. More recent literature succeeds in obtaining similar results in more general cases. See § 5.4.3 for complementary comments.

These results are based, whatever the method, on assuming that k has a smooth arithmetic complexity, e.g., k is p -principal ($\mathcal{H}_k = 1$), or logarithmically principal ($\widetilde{\mathcal{H}}_k = 1$), or p -rational (i.e., the maximal p -ramified abelian pro- p -extension H_k^{pr} of k coincides with the compositum \widetilde{k} of the \mathbb{Z}_p -extension of k), or the filtration of the \mathcal{H}_{K_n} ’s is limited to the first element given by Chevalley–Herbrand formula.

The general case seems to be based on random p -adic numerical circumstances, but subject to accessible probability laws, as done in [KP2022]. We discuss this aspect in § 5.3 to try to understand the nature of the main obstructions, which clearly depends on the groups of units, in the \mathbb{Z}_p -extensions.

1.3. Notations and recalls on class field theory.

1.3.1. *The cyclotomic and anti-cyclotomic \mathbb{Z}_p -extensions.* Let $k = \mathbb{Q}(\sqrt{-m})$ ($m \in \mathbb{Z}_{>1}$, square-free) be an imaginary quadratic field and set $g = \text{Gal}(k/\mathbb{Q}) =: \langle \tau \rangle$. Let p be an odd prime number, split in k into \mathfrak{p} and $\bar{\mathfrak{p}} := \mathfrak{p}^\tau$. Let \widetilde{k} be the compositum of the \mathbb{Z}_p -extensions of k (the case $p = 2$ gives rise to more complex diagrams, but would yield similar results; for this one may use [Gra1983] where all possibilities for the structure of \widetilde{k}/k are enumerated).

We denote by \mathcal{H}_k (resp. H_k^{nr}) the p -class group (resp. the p -Hilbert class field) of k , and put $\widetilde{k}^{\text{nr}} := \widetilde{k} \cap H_k^{\text{nr}}$. Let $\mathcal{d}(\mathfrak{a})$ (resp. $\mathcal{d}'(\mathfrak{a})$), be the image of an ideal \mathfrak{a} in the whole class group (resp. in the p -class group) of k .

We will use the Jaulent logarithmic class group $\widetilde{\mathcal{H}}_k = \text{Gal}(H_k^{\text{lc}}/k^{\text{cyc}})$ where the field H_k^{lc} is defined as the maximal abelian pro- p -extension of k , locally cyclotomic (thus such that p totally splits in $H_k^{\text{lc}}/k^{\text{cyc}}$); these fields are contained in H_k^{pr} , for which $\text{Gal}(H_k^{\text{pr}}/\widetilde{k})$ is denoted \mathcal{T}_k .

The group g operates on $\Gamma := \text{Gal}(\widetilde{k}/k) \simeq \mathbb{Z}_p^2$ and we define Γ^+ (resp. Γ^-), the sub-module annihilated by $1 - \tau$ (resp. $1 + \tau$). Since $p \neq 2$, $\Gamma = \Gamma^+ \oplus \Gamma^-$ and $\Gamma^+ \cap \Gamma^- = 1$, leading to define the two following *fundamental* \mathbb{Z}_p -extensions yielding \widetilde{k} as direct compositum over k ; they are the only ones to be Galois over \mathbb{Q} :

- The cyclotomic \mathbb{Z}_p -extension k^{cyc} (fixed by Γ^-), equal to the compositum $k \mathbb{Q}^{\text{cyc}}$, where the cyclotomic \mathbb{Z}_p -extension \mathbb{Q}^{cyc} is linearly disjoint from k ; so k^{cyc}/k is totally ramified at \mathfrak{p} and $\bar{\mathfrak{p}}$.

- The anti-cyclotomic \mathbb{Z}_p -extension k^{acyc} (fixed by Γ^+) is a pro-diedral \mathbb{Z}_p -extension of \mathbb{Q} . Since $p \neq 2$, τ operates on a class, $\mathcal{d}(\mathfrak{a})$, of k , by $\mathcal{d}(\mathfrak{a})^\tau = \mathcal{d}(\mathfrak{a})^{-1}$, and operates similarly on $\text{Gal}(H_k^{\text{nr}}/k)$; thus $\widetilde{k}^{\text{nr}} = k^{\text{acyc}} \cap H_k^{\text{nr}}$, the extension $\widetilde{k}^{\text{nr}}/k$ is cyclic, and $k^{\text{acyc}}/\widetilde{k}^{\text{nr}}$ is totally ramified at p . In the same way, τ operates by inversion on $\widetilde{\mathcal{H}}_k$ and \mathcal{T}_k .

1.3.2. *Abelian p -ramification over k .* We introduce the inertia fields L of \mathfrak{p} and \bar{L} of $\bar{\mathfrak{p}}$, in \widetilde{k}/k , so that L/k is unramified at \mathfrak{p} and \bar{L}/k is unramified at $\bar{\mathfrak{p}}$; then, $\bar{L} \neq L$, $L \cap \bar{L} = \widetilde{k}^{\text{nr}}$ and $L\bar{L} = \widetilde{k}$.

The inertia groups, $I_{\mathfrak{p}}(\widetilde{k}/k)$ and $I_{\bar{\mathfrak{p}}}(\widetilde{k}/k)$, of \mathfrak{p} and $\bar{\mathfrak{p}}$ in \widetilde{k}/k , are images of the groups of principal local units $U_{\mathfrak{p}}$ and $U_{\bar{\mathfrak{p}}}$ of the completions $k_{\mathfrak{p}}$ and $k_{\bar{\mathfrak{p}}}$, respectively, by the local reciprocity maps, according to the following first diagram of the maximal p -sub-extension k^{ab} of the abelian closure of k , where H_k^{pr} is the maximal p -ramified

abelian pro- p -extension of k and H_k^{ta} the maximal tamely ramified abelian pro- p -extension of k , respectively:

$$\begin{array}{ccc}
 H_k^{\text{pr}} & \xrightarrow{\quad\quad\quad} & H_k^{\text{pr}} H_k^{\text{ta}} = k^{\text{ab}} \\
 I_{\mathfrak{p}} \oplus I_{\bar{\mathfrak{p}}} \Big\downarrow & & \Big\downarrow U_k = U_{\mathfrak{p}} \oplus U_{\bar{\mathfrak{p}}} \\
 k & \xrightarrow{\quad\quad\quad} & H_k^{\text{ta}}
 \end{array}$$

In [Gra2005, III.4.4.1, III.2.6.1 (Fig. 2)], we have proved that in this diagram, $\text{Gal}(k^{\text{ab}}/H_k^{\text{pr}} H_k^{\text{ta}})$ is isomorphic to $E_k \otimes \mathbb{Z}_p$, where E_k is the group of units of k . In the present case, it is reduced to the group μ_k of roots of unity of k ; for $p \neq 2$, $\mu_k \otimes \mathbb{Z}_p = 1$, except for $p = 3$ and $k = \mathbb{Q}(\mu_3)$, but the assumption p split in k does not hold and $H_k^{\text{pr}} H_k^{\text{ta}} = k^{\text{ab}}$ with $\text{Gal}(H_k^{\text{pr}}/H_k^{\text{nr}}) \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, since $\text{tor}_{\mathbb{Z}_p}(U_k) = 1$. We deduce that \tilde{k}/k^{cyc} is unramified at \mathfrak{p} and $\bar{\mathfrak{p}}$ (or apply [Jau2024^a, Lemme 4]).

The second diagram, corresponding to H_k^{pr}/k , admits also some simplifications with $\text{tor}_{\mathbb{Z}_p}(U_k) = 1$, which implies $\mathcal{W}_k := \text{tor}_{\mathbb{Z}_p}(U_k)/\mu_p(k) = 1$, and H_k^{pr} coincides with the Bertrandias–Payan field H_k^{bp} (maximal abelian pro- p -extension of k in which any cyclic extension is embeddable in cyclic p -extensions of k of arbitrary degree). The normalized regulator \mathcal{R}_k (defined in [Gra2018, Section 5]) is trivial since $E_k \otimes \mathbb{Z}_p = 1$, whence $H_k^{\text{pr}} = \tilde{k} H_k^{\text{nr}}$. Thus, the torsion group $\mathcal{T}_k := \text{Gal}(H_k^{\text{pr}}/\tilde{k})$ is isomorphic to a subgroup of \mathcal{H}_k and $H_k^{\text{pr}}/\tilde{k}$ is unramified and totally split at p ; indeed, since the Gross–Kuz’min conjecture holds for k , the decomposition groups $D_{\mathfrak{p}}$ and $D_{\bar{\mathfrak{p}}}$, in $H_k^{\text{pr}}/k^{\text{cyc}}$, are isomorphic to \mathbb{Z}_p and can not intersect \mathcal{T}_k :

$$\begin{array}{ccccc}
 & & \mathcal{T}_k & & \\
 & & \curvearrowright & & \\
 \tilde{k} & \xrightarrow{\text{totally split}} & \tilde{k} H_k^{\text{nr}} & \xrightarrow{\mathcal{R}_k=1} & H_k^{\text{bp}} & \xrightarrow{\mathcal{W}_k=1} & H_k^{\text{pr}} \\
 \mathbb{Z}_p \times \mathbb{Z}_p \Big\downarrow & & \Big\downarrow & & & & \\
 k & \xrightarrow{\quad\quad\quad} & \tilde{k}^{\text{nr}} & \xrightarrow{\quad\quad\quad} & H_k^{\text{nr}} & \xrightarrow{U_k} & H_k^{\text{pr}} \\
 & & \mathcal{H}_k & & & &
 \end{array}$$

Remark 1.2. Recall that the computation of this group \mathcal{T}_k in the imaginary quadratic case is easy from [Gra2005, Theorem III.2.6, Remark III.2.6.2], giving the formula $[\tilde{k}^{\text{nr}} : k] = (\mathbb{Z}_p \text{Log}(I_k) : \mathbb{Z}_p \text{Log}(P_k))$, where I_k is the group of prime-to- p ideals of k and P_k the sub-group of principal ideals; in the case of imaginary quadratic fields, the Log-function coincides with the usual p -adic logarithm:

(i) $k = \mathbb{Q}(\sqrt{-191})$, $p = 13$. We have $\mathcal{H}_k = \langle \mathfrak{d}(\mathfrak{l}) \rangle$ of order 13, where $\mathfrak{l} \mid 2$. We have $\mathfrak{l}^{13} = \frac{1}{2}(153 + 7\sqrt{-191})$, and since $(\frac{1}{2}(153 + 7\sqrt{-191}))^{12} \equiv 1 + 4 \times 13\sqrt{-191} \pmod{13^2}$, we obtain, in $\mathbb{Q}_{13} \times \mathbb{Q}_{13}$, $\text{Log}(\mathfrak{l}) = \frac{1}{13} \text{Log}(\frac{1}{2}(153 + 7\sqrt{-191})) = (u, \log(2) - u)$, where u is a 13-adic unit.

Since $\text{Log}(P_k) = 13\mathbb{Z}_{13} \oplus 13\mathbb{Z}_{13}$, we obtain, from $I_k = \langle \mathfrak{l} \rangle P_k$, $(\mathbb{Z}_{13} \text{Log}(I_k) : \mathbb{Z}_{13} \text{Log}(P_k)) = 13$, which yields $\mathcal{T}_k = 1$ and $H_k^{\text{nr}} \subset \tilde{k}$, whence $H_k^{\text{nr}} \subset k^{\text{acyc}}$.

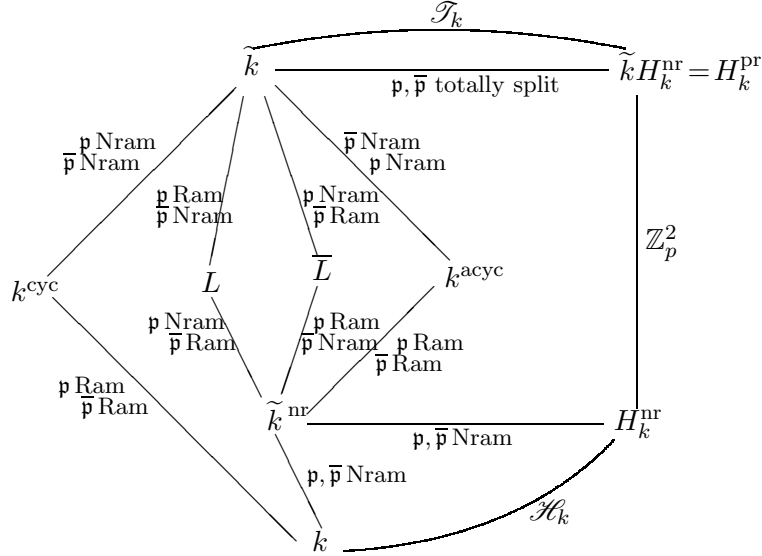
(ii) $k = \mathbb{Q}(\sqrt{-383})$, $p = 17$. We have $\mathcal{H}_k = \langle \mathfrak{d}(\mathfrak{l}) \rangle$ of order 17, where $\mathfrak{l} \mid 2$. We have $\mathfrak{l}^{17} = \frac{1}{2}(711 + 7\sqrt{-383})$, and we obtain $(\mathbb{Z}_{17} \text{Log}(I_k) : \mathbb{Z}_{17} \text{Log}(P_k)) = 1$, because $\text{Log}(\mathfrak{l}) \in 17(\mathbb{Z}_{17} \oplus \mathbb{Z}_{17})$ since $(\frac{1}{2}(711 + 7\sqrt{-383}))^{16} \equiv 1 \pmod{17^2}$.

So, $\#\mathcal{T}_k = 17$ and $\tilde{k}^{\text{nr}} = k$; the p -Hilbert class field is linearly disjoint from \tilde{k} .

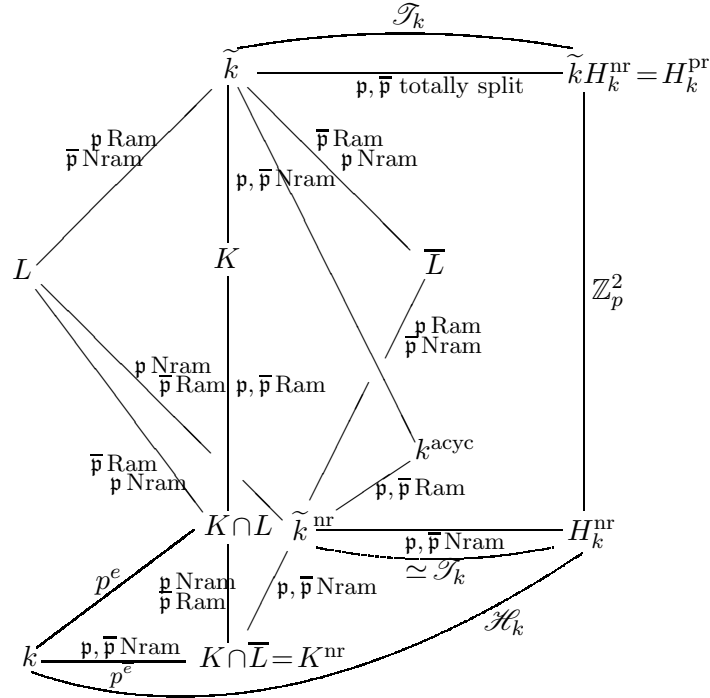
In the following diagram, ramification (resp. non-ramification) of \mathfrak{p} and $\bar{\mathfrak{p}}$ is made explicit with indications, as for instance $\mathfrak{p} \text{ Ram}$ (resp. $\mathfrak{p} \text{ Nram}$).

From [Jau2024^a, Lemme 4] about totally p -adic number fields, the extension \tilde{k}/k^{cyc} is unramified at p , and the decomposition group of p in this extension (isomorphic to \mathbb{Z}_p) is of finite index in $\text{Gal}(\tilde{k}/k^{\text{cyc}})$; similarly, p does not ramify in $\tilde{k}/k^{\text{acyc}}$, and the decomposition group of p is of finite index.

Then $\text{Gal}(L/k) \simeq \mathbb{Z}_p \oplus T$ and $\text{Gal}(\bar{L}/k) \simeq \mathbb{Z}_p \oplus \bar{T}$, where T and \bar{T} are cyclic groups; this corresponds to the case where $\text{Gal}(L/k)$ or/and $\text{Gal}(\bar{L}/k)$ are not free:



1.3.3. *General diagram including a \mathbb{Z}_p -extension K/k .* We consider in the following diagram an arbitrary \mathbb{Z}_p -extension $K := \bigcup_{n \geq 0} K_n$.



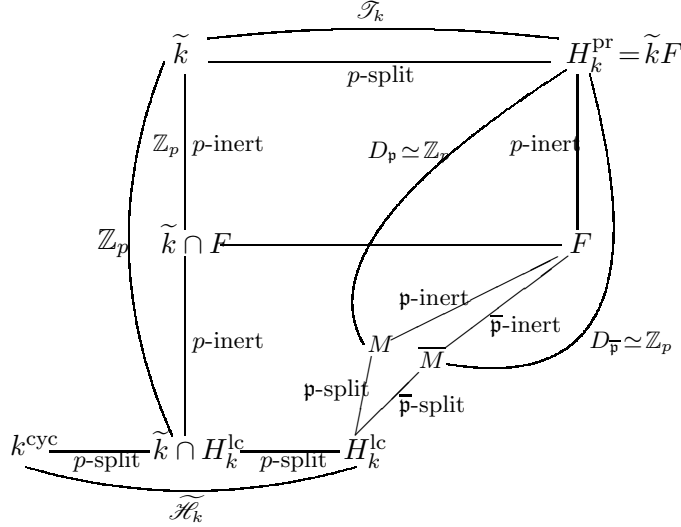
We assume that K is not contained in L , nor in \bar{L} . In the diagram, we have supposed, for instance, that $K \cap \bar{L} \subseteq K \cap L$, since \mathfrak{p} and $\bar{\mathfrak{p}}$ play symmetric roles. Since $L \cap \bar{L} = \tilde{k}^{\text{nr}}$, we have $K \cap L \cap \bar{L} = K \cap \bar{L} = K \cap \tilde{k}^{\text{nr}}$ that we denote by K^{nr} , which is a subfield of \tilde{k}^{nr} , thus of k^{acyc} . We put $[K \cap \bar{L} : k] = p^e$ & $[K \cap L : k] = p^e$,

$e \geq \bar{e}$, whence, $\#\mathcal{H}_k = p^{\bar{e}} \times [\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{T}_k$, since H_k^{pr} is the direct compositum of H_k^{nr} and \tilde{k} over \tilde{k}^{nr} .

2. A LOGARITHMIC INVARIANT IN ABELIAN p -RAMIFICATION THEORY

2.1. The Jaulent logarithmic class group $\widetilde{\mathcal{H}}_k$. For the definition and properties of $\widetilde{\mathcal{H}}_k$, see [Jau1986, Jau1994^b, Jau1998, BJ2016] and subsequent papers as [Jau2019^a], [Gra2021], about Greenberg's conjectures ([Gree1976] on Iwasawa's invariants in the totally real case, and [Gree1998] in more general setting).

An application in the case of totally p -adic imaginary quadratic fields is the following. Denote by \mathcal{H}_k the p -class group of \tilde{k} , defined as the projective limit, for the norms, of the p -class groups \mathcal{H}_F , for finite extensions F/k such that $\tilde{k} = \cup F$. The field k being totally p -adic, the extension \tilde{k}/k^{cyc} is unramified ([Jau2024^a, Lemme 4], [JS1995] for similar context). In [Jau2024^a, Proposition 12], a general result claims that $\mathcal{H}_k = 1$ if and only if $\widetilde{\mathcal{H}}_k = 1$. If so (whence $\delta_p(k) = 0$ and $\mathcal{H}_k = \langle \mathcal{C}'(\mathfrak{p}) \rangle$ from Theorem 2.2), this implies in particular that for $K = k^{\text{cyc}}$, $\tilde{k}H_{K_n}^{\text{nr}}$ is included in \tilde{k} , for all $n \geq 0$; since \tilde{k}/K is pro-cyclic, this proves that $\lambda_p(K/k) = 1$ and $\mu_p(K/k) = 0$. The following diagram introduces H_k^{lc} :



In the extension $H_k^{\text{pr}}/H_k^{\text{lc}}$ the decomposition groups $D_{\mathfrak{p}}$ and $D_{\bar{\mathfrak{p}}}$ are pro-cyclic groups fixing M and \bar{M} , respectively, such that $M \cap \bar{M} = H_k^{\text{lc}}$, $M\bar{M} = F \subset H_k^{\text{pr}}$. Necessarily, $[M : \tilde{k} \cap H_k^{\text{lc}}] = [\bar{M} : \tilde{k} \cap H_k^{\text{lc}}] = \#\mathcal{T}_k$, to get total splitting in $H_k^{\text{pr}}/\tilde{k}$.

The numerical examples given by the PARI/GP [Pari2013] program in Appendix B.1, may illustrate the above diagram; they show that \mathcal{T}_k (in $\mathbb{T}_{\mathfrak{p}}$) may be trivial, contrary to \mathcal{H}_k (in $\mathbb{H}_{\mathfrak{p}}$) and to the logarithmic class group $\widetilde{\mathcal{H}}_k$ (in Clog).

When $\mathcal{T}_k = 1$ (i.e., k is p -rational), $H_k^{\text{lc}} \subset \tilde{k}$, $\widetilde{\mathcal{H}}_k$ is cyclic of order $p^{\delta_p(k)}$ (Definition 1.1); moreover, $H_k^{\text{nr}} = \tilde{k}^{\text{nr}}$, the p -Hilbert class field is contained in k^{acyc} , and \mathcal{H}_k is cyclic. Some illustrations of p -rationality, with non-trivial $\widetilde{\mathcal{H}}_k$ and \mathcal{H}_k , are:

m=239	p=3	Clog=[[3], [], [3]]	T_p=[]	H_p=[3]
m=1238	p=3	Clog=[[9], [3], [3]]	T_p=[]	H_p=[3]
m=2239	p=5	Clog=[[25], [5], [5]]	T_p=[]	H_p=[5]

Remark 2.1. *The structures of the invariants $\widetilde{\mathcal{H}}_k$, \mathcal{T}_k and \mathcal{H}_k are most often obvious from the data; but some examples are interesting as the following one:*

m=78731	p=3	Clog=[[6561, 3], [6561], [3]]	T_p=[27]	H_p=[27]
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for which $\mathcal{d}(\mathfrak{p})$ is of order 9; surprisingly, $\mathfrak{p}^9 = \frac{1}{2}(1 + \sqrt{-78731})$, which is exceptional since, in general, coefficients may have thousands of digits. Since $\widetilde{\mathcal{H}}_k \simeq \mathbb{Z}/3^8\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, H_k^{lc} can not be contained in k ; so $[H_k^{\text{lc}} : k \cap H_k^{\text{lc}}] =: 3^r$, $r \geq 1$, and $\widetilde{\mathcal{H}}_k/\text{Gal}(H_k^{\text{lc}}/k \cap H_k^{\text{lc}})$ must be cyclic of order 3^{9-r} . Two cases may occur:

- $H_k^{\text{pr}} = \widetilde{k}H_k^{\text{lc}}$. In this case, $r = 3$, $M = \overline{M} = H_k^{\text{lc}} = F$, and $\text{Gal}(\widetilde{k} \cap H_k^{\text{lc}}/k^{\text{cyc}})$ must be a cyclic quotient of order 3^6 of $\mathbb{Z}/3^8\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} =: \langle a, b \rangle$ (a of order 3^8 , b of order 3); one checks that the unique solution is the quotient by $\langle a^{3^5} \cdot b \rangle$ (of order 3^3), giving a cyclic group of order 3^6 .

- $[H_k^{\text{pr}} : \widetilde{k}H_k^{\text{lc}}] \geq 3$. In this case, $r \leq 2$, and 3 can not be totally inert in $H_k^{\text{pr}}/H_k^{\text{lc}}$; so, this suggests that the two decomposition groups of \mathfrak{p} and $\overline{\mathfrak{p}}$ do not coincide in $\text{Gal}(H_k^{\text{pr}}/H_k^{\text{lc}})$, but are of index a suitable power of 3 (see the diagram where all possibilities are represented).

This situation does exist for all p ; for instance with $p = 5$ one gets the case:

$$m=30911 \quad p=5 \quad \text{Clog}=[[125,5],[125],[5]] \quad T_{-p}=[25] \quad H_{-p}=[25]$$

Even if most examples give either cyclic H_k^{lc} or else bi-cyclic ones, we have found some examples of rank 3, restricting the search to $p = 3$; of course, this implies that \mathcal{T}_k is at least of rank 2. Numerical results are given in Appendix B.2.

2.2. The order of $\widetilde{\mathcal{H}}_k$ – Fermat quotient $p^{\delta_p(k)}$ of the p -unit x . Our purpose was to give a class field theory computation (from a method yet used in [Gra2017^b, Gra2019^a, Gra2019^b]) of the order of the Hasse norm residue symbols $\left(\frac{x, K_n/k}{\mathfrak{p}}\right)$ and $\left(\frac{x, K_n/k}{\overline{\mathfrak{p}}}\right)$ for any \mathbb{Z}_p -extension $K = \bigcup_n K_n$ of k , where the p -unit x is the generator of \mathfrak{p}^h , $\mathfrak{p} \mid p$, where h is the order of $\mathcal{d}(\mathfrak{p})$. We show, Section 3, that the order of these symbols especially depends on the integer $\delta_p(k)$ (cf. (1.2)) which intervenes in the higher rank Chevalley–Herbrand formulas, in direction of a Gold criterion. But the PARI/GP computations of the main p -adic invariants of k have shown a remarkable numerical coincidence.

More precisely, consider, instead of x , the p -unit X defined by $\mathfrak{p}^{h_k} =: (X)$; then:

$$\widetilde{\delta}_p(k) := v_{\mathfrak{p}}(\overline{X}^{p-1} - 1) - 1 = v_{\mathfrak{p}}(\overline{x}^{p-1} - 1) - 1 + [v_p(h_k) - v_p(h)],$$

the valuation of the Fermat quotient of \overline{X} , giving $\widetilde{\delta}_p(k) = \delta_p(k) + [v_p(h_k) - v_p(h)]$ (Definition 1.1), is such that $p^{\widetilde{\delta}_p(k)}$ is quite simply the order of the logarithmic class group $\widetilde{\mathcal{H}}_k$, defined in Section 2.1, for totally p -adic imaginary quadratic fields. From his techniques of logarithmic class field theory [Jau1986, Jau1994^a, Jau1994^b, Jau1998, Jau2017], Jaulent proposes a more general framework where this property holds for some semi-simple totally p -adic imaginary abelian number fields (this is done Appendix A and developed in [Jau2024^b, Théorème 2] in direction of a Gold version about $\widetilde{\mathcal{H}}_k$ [Jau2024^b, Théorème 4]); for $k = \mathbb{Q}(\sqrt{-m})$, $\#\widetilde{\mathcal{H}}_k$ becomes:

Theorem 2.2. *Let k be an imaginary quadratic field of class number h_k , and let $p > 2$ be a prime number split in k . Let $X \in k^\times$ be the generator of \mathfrak{p}^{h_k} , and define $\widetilde{\delta}_p(k) := \delta_p(k) + [v_p(h_k) - v_p(h)]$, where $h \mid h_k$ is the order of $\mathcal{d}(\mathfrak{p})$.*

Then $\#\widetilde{\mathcal{H}}_k = p^{\widetilde{\delta}_p(k)} = p^{\delta_p(k)} \times \frac{\#\mathcal{H}_k}{\#\langle \mathcal{d}'(\mathfrak{p}) \rangle} = p^{\delta_p(k)} \times \frac{[k^{\text{nr}} : k]}{\#\langle \mathcal{d}'(\mathfrak{p}) \rangle} \times \#\mathcal{T}_k$; so, $\widetilde{\mathcal{H}}_k$ is trivial if and only if k is p -rational, p does not split in k^{acyc}/k and $\delta_p(k) = 0$.

We give in Appendix B.3 the checking of this property, for $p = 3$; then we give the successive maxima of possible $\widetilde{\delta}_p(k)$, $p \in \{3, 5\}$, in Appendix B.4.

Remark 2.3. *The order of $\widetilde{\mathcal{H}}_k$, for $k = \mathbb{Q}(\sqrt{-m})$ and p split, is given by means of the valuation of the Fermat quotient of an algebraic number, but depending of p ;*

indeed, $\frac{1}{p} \log_{\mathfrak{p}}(\bar{x}) \sim \frac{1}{p}(\bar{x}^{p-1} - 1)$ is not a “standard” Fermat’s quotient, like the case of $\frac{1}{p} \log_{\mathfrak{p}}(a) \sim \frac{1}{p}(a^{p-1} - 1)$, for the fixed prime-to- p algebraic number a .

Thus, at first glance, it seems difficult to apply the p -adic heuristics worked out in [Gra2016, Section 8] and to extend, to totally real number fields k , the conjecture [Jau2017, Conjecture 11] stated for the real quadratic case. In a numerical viewpoint, we see that for $k = \mathbb{Q}(\sqrt{-3})$, one finds the following large list of primes p , such that $\delta_p(k) \neq 0$, $p \in \{13, 181, 2521, 76543, 489061, 6811741, 1321442641, 18405321661\}$, but nothing else after 3 days of calculations. For $k = \mathbb{Q}(\sqrt{-5})$, one obtains the unique solution $p = 5881$, up to 10^9 . The case of $m = 47$, also giving many small solutions $p \in \{3, 17, 157, 1193, 1493, 1511, 19564087\}$, doesn’t give more up to 10^9 .

So this raises the question of the finiteness or not of the solutions p . If $x = u + v\sqrt{-m}$, then $\bar{x} = u - v\sqrt{-m} \equiv 2u \pmod{\mathfrak{p}^h}$, and $\delta_p(k)$ depends on the Fermat quotient q of $a = 2u$, when $h \geq 2$ (if $h = 1$ one computes the valuation of $(2u)^2q + 1$ (see Appendix B.5)). Such a non-zero valuation is in general related to probabilities $\frac{1}{p}$, the Borel–Cantelli heuristic giving a priori infiniteness; but the trace $2u$ is subject to the strong relationship $(u + v\sqrt{-m}) = \mathfrak{p}^h$, which does not define random coefficients contrary to the probability draw of a solution (u_0, v_0) of the congruence $U + V\sqrt{-m} \equiv 0 \pmod{\mathfrak{p}^h}$ (among \mathfrak{p}^h ones, characterized by only u_0 since $\sqrt{-m} \in \mathbb{Z}_{\mathfrak{p}}^{\times}$). So we may think that the corresponding probability for $\delta_p(k) \neq 0$ is much less than $\frac{1}{p}$.

Nevertheless, we suggest the following conjecture that we extend to any number field k , where $\widetilde{\mathcal{H}}_k(p)$ denotes the logarithmic class group for any prime p ; this is coherent with our similar conjectures about the $\mathcal{T}_k(p)$ ’s ([Gra2016, Conjecture 8.11], [Gra2019^a]):

Conjecture 2.4. *For any number field k , fulfilling universally the Leopoldt and Gross–Kuz’min conjectures, the set of primes p such that $\widetilde{\mathcal{H}}_k(p) \neq 1$ is finite.*

3. HASSE’S NORM RESIDUE SYMBOLS IN \widetilde{k}/k

Let K_n be the n th layer of a \mathbb{Z}_p -extension K/k . For $a \in k^{\times}$ and for any prime ideal $\mathfrak{q} \nmid p$, the Hasse norm residue symbol $\left(\frac{a, K_n/k}{\mathfrak{q}}\right)$ is given by the Artin symbol $\left(\frac{K_n/k}{\mathfrak{q}}\right)^{v_{\mathfrak{q}}(a)}$; but we need the less obvious symbol $\left(\frac{x, K_n/k}{\mathfrak{p}}\right)$ for $x \equiv 0 \pmod{\mathfrak{p}}$.

3.1. Computation of the symbol $\left(\frac{x, \widetilde{k}/k}{\mathfrak{p}}\right)$. Recall, from [Gra2005, II.4.4.3], a method to compute $\left(\frac{x, \widetilde{k}/k}{\mathfrak{p}}\right)$, for p split into $(p) = \mathfrak{p}\bar{\mathfrak{p}}$, when x is the fundamental \mathfrak{p} -unit of k . Put $\mathfrak{p}^h =: (x)$ for the order h of \mathfrak{p} . Then, $x\bar{x} = p^h$ since $E_k = 1$. Of course, $\left(\frac{\bullet, \widetilde{k}/k}{\bullet}\right)$ will be considered as the pro-limit for $N \rightarrow \infty$ of $\left(\frac{\bullet, F_N/k}{\bullet}\right)$, where F_N is the subfield of \widetilde{k} of norm conductor $(p)^N$; this implies that $\widetilde{k} = \bigcup_N F_N$, but any other system may be used; note that $\bigcup_N (K \cap F_N) = K$. Let $y_{N,\mathfrak{p}} \in k^{\times}$ (called a \mathfrak{p} -associate of x) fulfilling the following conditions (where the modulus may be taken larger than the norm conductors):

- (i) $y_{N,\mathfrak{p}} x^{-1} \equiv 1 \pmod{\mathfrak{p}^N}$,
- (ii) $y_{N,\mathfrak{p}} \equiv 1 \pmod{\bar{\mathfrak{p}}^N}$.

For N large enough, $v_{\mathfrak{p}}(y_{N,\mathfrak{p}}) = v_{\mathfrak{p}}(x) = h$, where $v_{\mathfrak{p}}$ is the \mathfrak{p} -adic valuation.

Product formula for $y_{N,\mathfrak{p}}$ yields $\left(\frac{y_{N,\mathfrak{p}}, F_N/k}{\mathfrak{p}}\right) = \prod_{\mathfrak{q}, \mathfrak{q} \neq \mathfrak{p}} \left(\frac{y_{N,\mathfrak{p}}, F_N/k}{\mathfrak{q}}\right)^{-1}$, and since $\left(\frac{x, F_N/k}{\mathfrak{p}}\right) = \left(\frac{y_{N,\mathfrak{p}}, F_N/k}{\mathfrak{p}}\right)$ from (i), by definition of the \mathfrak{p} -conductor of F_N/k , then $\left(\frac{x, F_N/k}{\mathfrak{p}}\right) = \prod_{\mathfrak{q}, \mathfrak{q} \neq \mathfrak{p}} \left(\frac{y_{N,\mathfrak{p}}, F_N/k}{\mathfrak{q}}\right)^{-1}$. Let's compute these symbols:

- If $\mathfrak{q} = \bar{\mathfrak{p}}$, since $y_{N,\mathfrak{p}} \equiv 1 \pmod{\bar{\mathfrak{p}}^N}$, one has $\left(\frac{y_{N,\mathfrak{p}}, F_N/k}{\bar{\mathfrak{p}}}\right) = 1$,

- If $\mathfrak{q} \nmid p$, in this case, $\left(\frac{y_{N,\mathfrak{p}}, F_N/k}{\mathfrak{q}}\right) = \left(\frac{F_N/k}{\mathfrak{q}}\right)^{v_{\mathfrak{q}}(y_{N,\mathfrak{p}})}$, where $\left(\frac{F_N/k}{\mathfrak{q}}\right)$ is the Frobenius automorphism of \mathfrak{q} .

Finally, $\left(\frac{x, F_N/k}{\mathfrak{p}}\right) = \prod_{\mathfrak{q} \nmid p} \left(\frac{F_N/k}{\mathfrak{q}}\right)^{-v_{\mathfrak{q}}(y_{N,\mathfrak{p}})} =: \left(\frac{F_N/k}{\mathfrak{a}_{N,\mathfrak{p}}}\right)^{-1}$, inverse of the Artin symbol of the principal ideal:

$$(3.1) \quad \mathfrak{a}_{N,\mathfrak{p}} := \prod_{\mathfrak{q} \nmid p} \mathfrak{q}^{v_{\mathfrak{q}}(y_{N,\mathfrak{p}})} = (y_{N,\mathfrak{p}}) \mathfrak{p}^{-h} = (y_{N,\mathfrak{p}} \cdot x^{-1}).$$

3.2. Computation of the orders of $\left(\frac{x, K_n/k}{\mathfrak{p}}\right)$ and $\left(\frac{x, K_n/k}{\bar{\mathfrak{p}}}\right)$. We will compute the order of $\left(\frac{x, K_n/k}{\mathfrak{p}}\right)$ and $\left(\frac{x, K_n/k}{\bar{\mathfrak{p}}}\right)$, for n large enough. We only assume that K is not included in L or in \bar{L} (otherwise, a single p -place does ramify in K/k and all is known in this case). The principle is to show that these orders can be deduced from that of $\left(\frac{x, k_n^{\text{cyc}}/k}{\mathfrak{p}}\right)$. The following lemma does simplify the relations between all possible choices for $x, \bar{x}, \mathfrak{p}, \bar{\mathfrak{p}}$ and the corresponding symbols:

Lemma 3.1. *One has $\left(\frac{p, k^{\text{cyc}}/k}{\mathfrak{p}}\right) = \left(\frac{p, k^{\text{cyc}}/k}{\bar{\mathfrak{p}}}\right) = 1$. Which implies, by conjugation, $\left(\frac{\bar{x}, k^{\text{cyc}}/k}{\mathfrak{p}}\right) = \left(\frac{x, k^{\text{cyc}}/k}{\mathfrak{p}}\right)^{-1} = \left(\frac{\bar{x}, k^{\text{cyc}}/k}{\bar{\mathfrak{p}}}\right) = \left(\frac{x, k^{\text{cyc}}/k}{\bar{\mathfrak{p}}}\right)^{-1}$.*

Proof. In the cyclotomic extension $\mathbb{Q}(\mu_{p^{n+1}})/\mathbb{Q}$, $p = \mathbf{N}_{\mathbb{Q}(\mu_{p^{n+1}})/\mathbb{Q}}(1 - \zeta_{p^{n+1}})$, for all $n \geq 0$, whence p local norm at \mathfrak{p} and $\bar{\mathfrak{p}}$ in k^{cyc}/k , and, since $x\bar{x} = p^h$, the relations follow, showing equality of the four symbols or inverses. \square

An interesting remark is that the norm property of p corresponds, in some sense, to the convention $\log(p) = 0$ in Definitions 1.1, and elucidates the case where the valuation $v_{\mathfrak{p}}(x^{p-1} - 1) - 1$ is not defined from an usual Fermat quotient.

Lemma 3.2. *For $n \geq e$, the order of $\left(\frac{x, K_n/k}{\mathfrak{p}}\right)$ is that of $\left(\frac{x, k_{n-e}^{\text{cyc}}/k}{\mathfrak{p}}\right)$, and for $n \geq \bar{e}$, the order of $\left(\frac{x, K_n/k}{\bar{\mathfrak{p}}}\right)$ is that of $\left(\frac{x, k_{n-\bar{e}}^{\text{cyc}}/k}{\bar{\mathfrak{p}}}\right)$.*

Proof. We consider the following diagrams, where \tilde{k} is the direct compositum over k (resp. over $K \cap L$) of L and k^{cyc} (resp. of L and K), and similarly, \tilde{k} is the direct compositum over k (resp. over $K \cap \bar{L}$) of \bar{L} and k^{cyc} (resp. of \bar{L} and K):

$$\begin{array}{c} \text{Symbol } \left(\frac{\bar{x}, \tilde{k}/k}{\mathfrak{p}}\right): \\ \begin{array}{ccccc} L & \xrightarrow[p \text{ Ram}]{p^m} & L_m & \xrightarrow[p \text{ Ram}]{\tilde{k}} & \tilde{k} \\ \mathfrak{p} \text{ Nram} \downarrow & & \downarrow & & \downarrow \mathbb{Z}_p \\ k & \xrightarrow{\quad} & k_m^{\text{cyc}} & \xrightarrow{\quad} & k^{\text{cyc}} \end{array} \\ \\ \text{Symbol } \left(\frac{\bar{x}, \tilde{k}/k}{\mathfrak{p}}\right): \\ \begin{array}{ccccc} L & \xrightarrow[p \text{ Ram}]{p^{n-e}} & L_{n-e} & \xrightarrow[p \text{ Ram}]{\tilde{k}} & \tilde{k} \\ \mathfrak{p} \text{ Nram} \downarrow & & \downarrow & & \downarrow \mathbb{Z}_p \\ k & \xrightarrow[p^e]{\mathfrak{p} \text{ Nram}} & K \cap L & \xrightarrow{\quad} & K \end{array} \end{array}$$

$$\begin{array}{c}
\text{Symbol } \left(\frac{x, \tilde{k}/k}{\bar{\mathfrak{p}}} \right): \\
\begin{array}{ccccc}
\bar{L} & \xrightarrow{\frac{p^m}{\bar{\mathfrak{p}} \text{ Ram}}} & \bar{L}_m & \xrightarrow{\tilde{k}} & \tilde{k} \\
\downarrow \bar{\mathfrak{p}} \text{ Nram} & & \downarrow \bar{\mathfrak{p}} \text{ Ram} & & \downarrow \mathbb{Z}_p \\
k & \xrightarrow{\quad} & k_m^{\text{cyc}} & \xrightarrow{\quad} & k^{\text{cyc}} \\
\downarrow \bar{\mathfrak{p}} \text{ Nram} & & \downarrow \bar{\mathfrak{p}} \text{ Ram} & & \downarrow \mathbb{Z}_p \\
k & \xrightarrow{\frac{p^{n-e}}{\bar{\mathfrak{p}} \text{ Nram}}} & K \cap \bar{L} & \xrightarrow{\quad} & K
\end{array} \\
\text{Symbol } \left(\frac{x, \tilde{k}/k}{\bar{\mathfrak{p}}} \right): \\
\begin{array}{ccccc}
\bar{L} & \xrightarrow{\frac{p^{n-\bar{e}}}{\bar{\mathfrak{p}} \text{ Ram}}} & \bar{L}_{n-\bar{e}} & \xrightarrow{\tilde{k}} & \tilde{k} \\
\downarrow \bar{\mathfrak{p}} \text{ Nram} & & \downarrow \bar{\mathfrak{p}} \text{ Ram} & & \downarrow \mathbb{Z}_p \\
k & \xrightarrow{\quad} & k_m^{\text{cyc}} & \xrightarrow{\quad} & k^{\text{cyc}} \\
\downarrow \bar{\mathfrak{p}} \text{ Nram} & & \downarrow \bar{\mathfrak{p}} \text{ Ram} & & \downarrow \mathbb{Z}_p \\
k & \xrightarrow{\frac{p^{n-\bar{e}}}{\bar{\mathfrak{p}} \text{ Nram}}} & K \cap \bar{L} & \xrightarrow{\quad} & K
\end{array}
\end{array}$$

In the first and third diagram, $k_m^{\text{cyc}} = k\mathbb{Q}_m^{\text{cyc}}$, where $\mathbb{Q}_m^{\text{cyc}}$ is of conductor p^{m+1} .

In the second and fourth diagram, since $[K_n : k] = p^n$ and $K \cap L = K_e$ (resp. $K \cap \bar{L} = K_{\bar{e}}$), it follows that $[LK_n : L] = p^{n-e}$ (resp. $[\bar{L}K_n : \bar{L}] = p^{n-\bar{e}}$). Finally, $L_m = L_{n-e}$ for $n = m + e$ (resp. $\bar{L}_m = \bar{L}_{n-\bar{e}}$ for $n = m + \bar{e}$).

Recall that \mathfrak{p} (resp. $\bar{\mathfrak{p}}$) is totally ramified in $K/K \cap L$ (resp. $K/K \cap \bar{L}$) and unramified in $L/K \cap L$ (resp. $\bar{L}/K \cap \bar{L}$). So:

$$\begin{aligned}
& \left(\frac{\bar{x}, \tilde{k}/k}{\mathfrak{p}} \right) \text{ fixes } L \text{ since } \bar{x} \text{ is a local unit in an extension unramified at } \mathfrak{p}; \\
& \left(\frac{\bar{x}, k^{\text{cyc}}/k}{\mathfrak{p}} \right) \in \text{Gal}(k^{\text{cyc}}/k), \text{ by restriction;} \\
& \left(\frac{\bar{x}, K/k}{\mathfrak{p}} \right) \in \text{Gal}(K/K \cap L), \text{ by restriction of } \left(\frac{\bar{x}, \tilde{k}/k}{\mathfrak{p}} \right) \in \text{Gal}(\tilde{k}/L); \\
& \left(\frac{x, \tilde{k}/k}{\bar{\mathfrak{p}}} \right) \text{ fixes } \bar{L} \text{ since } x \text{ is a local unit in an extension unramified at } \bar{\mathfrak{p}}; \\
& \left(\frac{x, k^{\text{cyc}}/k}{\bar{\mathfrak{p}}} \right) \in \text{Gal}(k^{\text{cyc}}/k), \text{ by restriction;} \\
& \left(\frac{x, K/k}{\bar{\mathfrak{p}}} \right) \in \text{Gal}(K/K \cap \bar{L}), \text{ by restriction of } \left(\frac{x, \tilde{k}/k}{\bar{\mathfrak{p}}} \right) \in \text{Gal}(\tilde{k}/\bar{L}).
\end{aligned}$$

Since these restrictions are isomorphisms, one obtains successively that the order of $\left(\frac{\bar{x}, k_m^{\text{cyc}}/k}{\mathfrak{p}} \right)$ is that of $\left(\frac{\bar{x}, L_m/k}{\mathfrak{p}} \right)$, and that the order of $\left(\frac{\bar{x}, K_n/k}{\mathfrak{p}} \right)$ is that of $\left(\frac{\bar{x}, L_{n-e}/k}{\mathfrak{p}} \right)$. For $n = m + e$, we get that the order of $\left(\frac{\bar{x}, K_n/k}{\mathfrak{p}} \right)$ is that of $\left(\frac{\bar{x}, k_{n-e}^{\text{cyc}}/k}{\mathfrak{p}} \right)$, whence that of $\left(\frac{x, k_{n-e}^{\text{cyc}}/k}{\mathfrak{p}} \right)$ from Lemma 3.1. Similarly, the order of $\left(\frac{x, K_n/k}{\bar{\mathfrak{p}}} \right)$ is that of $\left(\frac{x, k_{n-\bar{e}}^{\text{cyc}}/k}{\bar{\mathfrak{p}}} \right)$ and that of $\left(\frac{\bar{x}, k_{n-\bar{e}}^{\text{cyc}}/k}{\mathfrak{p}} \right)$. \square

But the orders of the symbols in k_m^{cyc}/k are easily computable:

Theorem 3.3. *Let k be an imaginary quadratic field and let $p > 2$ be a prime split in k . Let $x \in k^\times$ be the generator of \mathfrak{p}^h , where h is the order of $\mathfrak{cl}(\mathfrak{p})$, and let $\delta_p(k) := v_{\mathfrak{p}}(\bar{x}^{p-1} - 1) - 1$. Let K be a \mathbb{Z}_p -extension of k such that $\mathfrak{p}, \bar{\mathfrak{p}}$ are ramified from some layers $K_e, K_{\bar{e}}$, $e \geq \bar{e}$. The symbols $\left(\frac{x, K_n/k}{\mathfrak{p}} \right)$ and $\left(\frac{x, K_n/k}{\bar{\mathfrak{p}}} \right)$ are of orders $p^{n-e-\delta_p(k)}$ ($n \geq e + \delta_p(k)$) and $p^{n-\bar{e}-\delta_p(k)}$ ($n \geq \bar{e} + \delta_p(k)$), respectively.*

Proof. For this, it suffices to compute, for m large enough, the order of the Artin symbol $\left(\frac{k_m^{\text{cyc}}/k}{\mathfrak{a}_{N,\mathfrak{p}}} \right) \in \text{Gal}(k_m^{\text{cyc}}/k)$ obtained in expression (3.1), for $N \gg 0$. It is equivalent to compute the order of the canonical image of $\left(\frac{k_m^{\text{cyc}}/k}{\mathfrak{a}_{N,\mathfrak{p}}} \right)$ in $\text{Gal}(\mathbb{Q}_m^{\text{cyc}}/\mathbb{Q}) \simeq \{a \in (\mathbb{Z}/p^{m+1}\mathbb{Z})^\times, a \equiv 1 \pmod{p}\} \simeq \mathbb{Z}/p^m\mathbb{Z}$, which is the Artin symbol $\left(\frac{\mathbb{Q}_m^{\text{cyc}}/\mathbb{Q}}{\mathbf{N}_{k/\mathbb{Q}}(\mathfrak{a}_{N,\mathfrak{p}})} \right)$, whose order is that of the Artin symbol of the integer $\mathbf{N}_{k/\mathbb{Q}}(\mathfrak{a}_{N,\mathfrak{p}})^{p-1}$ of the form $1 + p \cdot p^\delta u$, with a unit u and $\delta \geq 0$ to be computed. Before, let's prove that

$\mathfrak{a}_{N,\mathfrak{p}} = (y_{N,\mathfrak{p}}) \cdot \mathfrak{p}^{-h}$ is such that, for $N \gg 0$, $\mathbf{N}_{k/\mathbb{Q}}(\mathfrak{a}_{N,\mathfrak{p}}) \equiv x \cdot p^{-h} \pmod{\mathfrak{p}^{N+1}}$. Indeed, one has $\mathbf{N}_{k/\mathbb{Q}}(\mathfrak{a}_{N,\mathfrak{p}}) = \mathbf{N}_{k/\mathbb{Q}}(y_{N,\mathfrak{p}}) \cdot p^{-h}$; by definition of an associate, $y_{N,\mathfrak{p}} x^{-1} \equiv 1 \pmod{\mathfrak{p}^{N+1}}$, $y_{N,\mathfrak{p}} \equiv 1 \pmod{\bar{\mathfrak{p}}^{N+1}}$, thus $\bar{y}_{N,\mathfrak{p}} \equiv 1 \pmod{\mathfrak{p}^{N+1}}$, which leads to $\mathbf{N}_{k/\mathbb{Q}}(y_{N,\mathfrak{p}}) \cdot x^{-1} \equiv 1 \pmod{\mathfrak{p}^{N+1}}$.

Finally, $\mathbf{N}_{k/\mathbb{Q}}(\mathfrak{a}_{N,\mathfrak{p}}) \equiv x \cdot p^{-h} \pmod{\mathfrak{p}^{N+1}}$, whence $v_p(\mathbf{N}_{k/\mathbb{Q}}(\mathfrak{a}_{N,\mathfrak{p}})^{p-1} - 1) = v_p((x \cdot p^{-h})^{p-1} - 1) = 1 + \delta_p(k)$; since $\mathbf{N}_{k/\mathbb{Q}}(\mathfrak{a}_{N,\mathfrak{p}})$ is rational, the number $\delta_p(k)$ gives the order of $\left(\frac{k_m^{\text{cyc}}/k}{\mathfrak{a}_{N,\mathfrak{p}}}\right)$ in $\text{Gal}(k_m^{\text{cyc}}/k)$, as soon as $m \geq \delta_p(k)$, and this order is $p^{m-\delta_p(k)}$. \square

4. CHEVALLEY–HERBRAND FORMULA IN K_n/k

4.1. Expression of $\#\mathcal{H}_{K_n}^{G_n}$. Let K be a \mathbb{Z}_p -extension not contained in L or \bar{L} (whence K/k ramified at all the p -places from some layers K_e and $K_{\bar{e}}$ of K). We have, from the general diagram 1.3.3, $K \cap \bar{L} = K_{\bar{e}}$ and $K \cap L = K_e$ with $e \geq \bar{e}$.

Moreover, we have, from the Diagram:

$$\#\mathcal{H}_k = p^{\bar{e}} \times [\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{T}_k.$$

Recall the Chevalley–Herbrand formula [Che1933, pp. 402-406] for the extensions K_n/k , of Galois groups G_n , and for the p -class group \mathcal{H}_{K_n} of K_n , $n \geq e$:

$$\#\mathcal{H}_{K_n}^{G_n} = \frac{\#\mathcal{H}_k \times \prod_v p^{n_v}}{[K_n : k] \times (E_k : E_k \cap \mathbf{N}_{K_n/k}(K_n^\times))},$$

where p^{n_v} is the ramification index of the place v (that of \mathfrak{p} , $\bar{\mathfrak{p}}$, of ramification indices p^{n-e} , $p^{n-\bar{e}}$, respectively), and E_k the group of units of k (equal to 1).

Using the above expression of $\#\mathcal{H}_k$ gives the final expression for all $n \geq e$:

$$(4.1) \quad \#\mathcal{H}_{K_n}^{G_n} = \frac{\#\mathcal{H}_k \times p^{n-\bar{e}} p^{n-e}}{p^n} = [\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{T}_k \times p^{n-e},$$

then immediately the well-known result that (for K not contained in L or \bar{L}), $\#\mathcal{H}_{K_n}$ is unbounded, whence $\lambda_p(K/k) \geq 1$ or $\mu_p(K/k) \geq 1$.

4.2. Genus theory and filtration of \mathcal{H}_{K_n} . We still assume K not contained in L or \bar{L} . Denote by $G_{n,\mathfrak{p}}$ and $G_{n,\bar{\mathfrak{p}}}$ the inertia groups in K_n/k , of \mathfrak{p} and $\bar{\mathfrak{p}}$, for all $n \geq 0$. We define, for all $n \geq 0$:

$$\Omega_{K_n/k} := \{(s, \bar{s}) \in G_{n,\mathfrak{p}} \times G_{n,\bar{\mathfrak{p}}}, s \cdot \bar{s} = 1\};$$

since $G_{n,\mathfrak{p}} \subseteq G_{n,\bar{\mathfrak{p}}}$, we have, for $n \geq e$:

$$(4.2) \quad \Omega_{K_n/k} \simeq \mathbb{Z}/p^{n-e}\mathbb{Z},$$

and we consider the map $\omega_{K_n/k} : k^\times \rightarrow \Omega_{K_n/k}$, defined on the set of elements $a \in k^\times$, assumed to be local norm in K_n/k for all place $v \nmid p$, by:

$$\omega_{K_n/k}(a) = \left(\left(\frac{a, K_n/k}{\mathfrak{p}} \right), \left(\frac{a, K_n/k}{\bar{\mathfrak{p}}} \right) \right) \in \Omega_{K_n/k},$$

since these symbols fulfill the product formula of class field theory.

The basic principle from Chevalley–Herbrand formula for p -class groups is to define a filtration of the form $\{\mathcal{H}_{K_n}^i =: \mathcal{O}_n(\mathcal{I}_{K_n}^i)\}_{i \geq 0}$, where $\mathcal{H}_{K_n}^0 := 1$, and:

$$(4.3) \quad \mathcal{H}_{K_n}^1 := \mathcal{H}_{K_n}^{G_n}, \quad \mathcal{H}_{K_n}^{i+1} / \mathcal{H}_{K_n}^i := (\mathcal{H}_{K_n} / \mathcal{H}_{K_n}^i)^{G_n},$$

up to a minimal bound $i = b_n$ for which $\mathcal{H}_{K_n}^{b_n} = \mathcal{H}_{K_n}$ (see the improved english translation [Gra2017^a] of our 1994's paper and its transcript into the idelic form in [LiYu2020]).

The general formula (giving the most powerful Chevalley–Herbrand formula that may be called genus theory form of Chevalley–Herbrand formula) writes, for any submodule $\mathcal{H} =: \mathcal{C}_n(\mathcal{I})$ of K_n , and for all $n \geq 0$ [Gra2017^a, Corollary 3.7]:

$$\#(\mathcal{H}_{K_n}/\mathcal{H})^{G_n} = \frac{\#\mathbf{N}_{K_n/k}(\mathcal{H}_{K_n})}{\#\mathbf{N}_{K_n/k}(\mathcal{H})} \times \frac{\#\Omega_{K_n/k}}{\#\omega_{K_n/k}(\mathbf{\Lambda})}, \quad \mathbf{\Lambda} := \{a \in k^\times, (a) \in \mathbf{N}_{K_n/k}(\mathcal{I})\}.$$

From (4.1) and the above formula, we get more precisely, where $\#\mathbf{N}_{K_n/k}(\mathcal{H}_{K_n}) = [H_k^{\text{nr}} : K_n \cap H_k^{\text{nr}}]$ takes into account the non-ramified layers for small values of n :

$$(4.4) \quad \left\{ \begin{array}{l} \# \mathcal{H}_{K_n}^1 = \# \mathcal{H}_{K_n}^{G_n} = \#\mathbf{N}_{K_n/k}(\mathcal{H}_{K_n}) \times \#\Omega_{K_n/k}, \\ \quad \left[= [\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{I}_k \times p^{n-e}, \text{ if } n \geq e \right], \\ \#(\mathcal{H}_{K_n}^{i+1}/\mathcal{H}_{K_n}^i) = \frac{\#\mathbf{N}_{K_n/k}(\mathcal{H}_{K_n}^i)}{\#\mathbf{N}_{K_n/k}(\mathcal{H}_{K_n}^{i+1})} \times \frac{\#\Omega_{K_n/k}}{\#\omega_{K_n/k}(\mathbf{\Lambda}_{K_n/k}^i)}, \quad i \geq 1, \\ \quad \left[= \frac{[\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{I}_k}{\#\mathbf{N}_{K_n/k}(\mathcal{H}_{K_n}^i)} \times \frac{p^{n-e}}{\#\omega_{K_n/k}(\mathbf{\Lambda}_{K_n/k}^i)}, \text{ if } n \geq e \right], \end{array} \right.$$

where $\mathbf{\Lambda}_{K_n/k}^i := \{a \in k^\times, (a) \in \mathbf{N}_{K_n/k}(\mathcal{I}_{K_n}^i)\}$ is a subgroup of finite type of k^\times , so that $\omega_{K_n/k}(\mathbf{\Lambda}_{K_n/k}^i) \subseteq \Omega_{K_n/k}$ from the product formula (recall that any $a \in k^\times$, norm of an ideal in K_n/k is local norm at the non-ramified places).

4.3. Properties of the filtration. It is the behavior of the filtrations, regarding $n \rightarrow \infty$, and the number of steps b_n , which determines the parameters $\lambda_p(K/k)$, $\mu_p(K/k)$; for this we will recall some general properties that have also been used in [Gra2017^b, Gra2019^b, Gra2021] in another context.

To ease notations, put $\mathcal{H}_n := \mathcal{H}_{K_n}$, $\mathbf{\Lambda}_n^i := \mathbf{\Lambda}_{K_n/k}^i$, $\mathbf{N}_n := \mathbf{N}_{K_n/k}$, and so on; let σ_n be a generator of G_n ; note that k may be seen as K_0 giving for instance $\mathcal{H}_k = \mathcal{H}_0$. The first obvious properties are, for n fixed:

- (i) $\mathcal{H}_n^i = \{c \in \mathcal{H}_n, c^{(1-\sigma_n)^i} = 1\}$, for all $i \geq 0$.
- (ii) The i -sequence $\#(\mathcal{H}_n^{i+1}/\mathcal{H}_n^i)$, $0 \leq i \leq b_n$, is decreasing from $\#\mathcal{H}_n^1$ to 1 since $1 - \sigma_n$ defines the injections $\mathcal{H}_n^{i+1}/\mathcal{H}_n^i \hookrightarrow \mathcal{H}_n^i/\mathcal{H}_n^{i-1}$, for all $i \geq 1$.

Remark 4.1. Set $\mathcal{H}_n^i = \mathcal{C}_n(\mathcal{I}_n^i)$ for an ideal group \mathcal{I}_n^i of K_n ; then $\mathbf{N}_n(\mathcal{I}_n^i)$ allows to find solutions $\alpha = \mathbf{N}_n(\beta) \in \mathbf{\Lambda}_n^i \cap \mathbf{N}_n(K_n^\times)$, by definition such that $(\alpha) = \mathbf{N}_n(\mathfrak{A})$, $\mathfrak{A} \in \mathcal{I}_n^i$, yielding $(\beta) = \mathfrak{B}^{1-\sigma_n} \cdot \mathfrak{A}$ to create $\mathfrak{B} \in \mathcal{I}_n^{i+1}$ and $\mathcal{C}_k(\mathbf{N}_n(\mathfrak{B}))$ in \mathcal{H}_k , whence building $\mathcal{H}_n^{i+1} = \mathcal{C}_n(\mathcal{I}_n^{i+1})$ and giving $\mathbf{N}_n(\mathcal{H}_n^{i+1}) \supseteq \mathbf{N}_n(\mathcal{H}_n^i)$ for the next step of the algorithm.

We have, for all n , the following diagram where norms \mathbf{N}_{K_{n+1}/K_n} , on \mathcal{H}_{n+1} and $(\mathcal{H}_{n+1})^{(1-\sigma_{n+1})^i}$, are onto; on \mathcal{H}_{n+1}^i , the norm may be not surjective nor injective:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{H}_{n+1}^i & \longrightarrow & \mathcal{H}_{n+1} & \xrightarrow{(1-\sigma_{n+1})^i} & (\mathcal{H}_{n+1})^{(1-\sigma_{n+1})^i} \longrightarrow 1 \\ & & \downarrow & & \mathbf{N}_{K_{n+1}/K_n} \downarrow & & \mathbf{N}_{K_{n+1}/K_n} \downarrow \\ 1 & \longrightarrow & \mathcal{H}_n^i & \longrightarrow & \mathcal{H}_n & \xrightarrow{(1-\sigma_n)^i} & (\mathcal{H}_n)^{(1-\sigma_n)^i} \longrightarrow 1. \end{array}$$

One has $\mathbf{N}_{K_{n+1}/K_n}(\mathcal{H}_{n+1}^i) \subseteq \mathcal{H}_n^i$, and one may assume, modifying \mathcal{I}_n^i modulo principal ideals (which does not modify norm indices), that $\mathbf{N}_{K_{n+1}/K_n}(\mathcal{I}_{n+1}^i) \subseteq \mathcal{I}_n^i$, implying $\mathbf{\Lambda}_{n+1}^i \subseteq \mathbf{\Lambda}_n^i$ from $\mathbf{N}_{K_{n+1}/k} = \mathbf{N}_{K_n/k} \circ \mathbf{N}_{K_{n+1}/K_n}$.

We assume now that p splits in k and that K/k is any \mathbb{Z}_p -extension not contained in L or \bar{L} ; this means that \mathfrak{p} and $\bar{\mathfrak{p}}$ ramify from some layers K_e and $K_{\bar{e}}$ in K/k .

Lemma 4.2. *For $i \geq 0$ fixed, and $n \geq e$, the $\#(\mathcal{H}_n^{i+1}/\mathcal{H}_n^i)$'s and the $\#\mathcal{H}_n^i$'s define stationary increasing n -sequences.*

Proof. Consider, for $i \geq 0$ fixed, the n -sequence defined, for $n \geq e$ (see (4.2)), by $\#(\mathcal{H}_n^{i+1}/\mathcal{H}_n^i) = \frac{[H_k^{\text{nr}} : K^{\text{nr}}]}{\#\mathbf{N}_n(\mathcal{H}_n^i)} \cdot \frac{p^{n-e}}{\omega_n(\Lambda_n^i)}$. The first factor is the ‘‘class factor’’ and the second factor is the ‘‘norm factor’’. We get $\mathbf{N}_{n+1}(\mathcal{H}_{n+1}^i) \subseteq \mathbf{N}_n(\mathcal{H}_n^i)$; thus, the n -sequence $\frac{[H_k^{\text{nr}} : K^{\text{nr}}]}{\#\mathbf{N}_n(\mathcal{H}_n^i)} =: p^{c_n^i}$ is increasing, stationary from a maximal value $p^{c_\infty^i}$ dividing $[H_k^{\text{nr}} : K^{\text{nr}}]$. Consider the norm factors $\frac{p^{n-e}}{\#\omega_n(\Lambda_n^i)} =: p^{\rho_n^i}$; then $p^{\rho_{n+1}^i - \rho_n^i} = p \cdot \frac{\#\omega_n(\Lambda_n^i)}{\#\omega_{n+1}(\Lambda_{n+1}^i)}$, and using Remark 4.1, one gets $\Lambda_{n+1}^i \subseteq \Lambda_n^i$, then $\#\omega_{n+1}(\Lambda_{n+1}^i) \leq \#\omega_{n+1}(\Lambda_n^i)$, and $p^{\rho_{n+1}^i - \rho_n^i} \geq p \cdot \frac{\#\omega_n(\Lambda_n^i)}{\#\omega_{n+1}(\Lambda_n^i)}$; under the restriction of Hasse’s norm residue symbols, one gets the map $\Omega_{n+1} \twoheadrightarrow \Omega_n$ (whose kernel is of order p), the image of $\omega_{n+1}(\Lambda_n^i)$ is $\omega_n(\Lambda_n^i)$, whence the result for the n -sequence $p^{\rho_n^i}$, stationary from $p^{\rho_\infty^i}$. The first point follows; we have $\lim_{n \rightarrow \infty} \#(\mathcal{H}_n^{i+1}/\mathcal{H}_n^i) = p^{c_\infty^i} \cdot p^{\rho_\infty^i}$, where the i -sequences $p^{c_\infty^i}$ and $p^{\rho_\infty^i}$ are decreasing stationary.

Assuming by induction that the n -sequence $\#\mathcal{H}_n^i$ is increasing stationary, the property follows for the n -sequence $\#\mathcal{H}_n^{i+1}$. \square

Lemma 4.3. *For $i \geq 0$ fixed and $n \geq e$, we have $\frac{p^{n-e}}{\#\omega_n(\Lambda_n^i)} \leq p^{\delta_p(k)}$ for $i \geq 1$ (resp. $\frac{p^{n-e}}{\#\omega_n(\Lambda_n^0)} = p^{n-e}$) and $\frac{[H_k^{\text{nr}} : K^{\text{nr}}]}{\#\mathbf{N}_n(\mathcal{H}_n^i)} \leq [H_k^{\text{nr}} : K^{\text{nr}}]$ for all i . Whence $p^{\rho_\infty^i} \leq p^{\delta_p(k)}$ (resp. $p^{\rho_\infty^0} = p^{n-e}$) and $p^{c_\infty^i} \leq [H_k^{\text{nr}} : K^{\text{nr}}]$. So, the n -sequence $\#(\mathcal{H}_n^{i+1}/\mathcal{H}_n^i)$ is increasing stationary bounded by $[H_k^{\text{nr}} : K^{\text{nr}}] \times p^{\delta_p(k)}$ for all $i \geq 1$.*

Proof. Since Λ_n^i contains x , for $i \geq 1$, with $\omega_n(x)$ of order $p^{n-e-\delta_p(k)}$, this yields $\frac{p^{n-e}}{\#\omega_n(\Lambda_n^i)} \leq \frac{p^{n-e}}{\#\omega_n(\langle x \rangle)} = p^{\delta_p(k)}$. Since $p^{c_\infty^i} \mid [H_k^{\text{nr}} : K^{\text{nr}}]$, the conclusion holds. The case $i = 0$ gives the bounds p^{n-e} and $[H_k^{\text{nr}} : K^{\text{nr}}]$, respectively. \square

Remark 4.4. *From Lemma 4.3 and $\#(\mathcal{H}_n^{i+1}/\mathcal{H}_n^i) = [H_k^{\text{nr}} : K^{\text{nr}}]p^{n-e}$, for $i = 0$, then $\#(\mathcal{H}_n^{i+1}/\mathcal{H}_n^i) \geq p$, for all $i \in [1, b_n - 1]$, we obtain, for $n \gg 0$ and the writing $\#\mathcal{H}_n = \prod_{i=0}^{b_n-1} \#(\mathcal{H}_n^{i+1}/\mathcal{H}_n^i)$:*

$$(4.5) \quad \begin{cases} v_p([H_k^{\text{nr}} : K^{\text{nr}}]) + (b_n - 1) + n - e \leq \lambda_p n + \mu_p p^n + \nu_p \\ \leq b_n \cdot v_p([H_k^{\text{nr}} : K^{\text{nr}}]) + (b_n - 1) \delta_p(k) + n - e, \end{cases}$$

which may justify the conjecture $\mu_p(K/k) = 0$ since, otherwise, the number b_n of steps would be $O(p^n)$, which is incredible in the algorithmic point of view described in Remark 4.1. Moreover, if b_n is less than $O(n)$, then $\lambda_p(K/k) = 1$ which leads to:

$$v_p([H_k^{\text{nr}} : K^{\text{nr}}]) + (b_n - 1) \leq \nu_p + e \leq b_n \cdot v_p([H_k^{\text{nr}} : K^{\text{nr}}]) + (b_n - 1) \delta_p(k).$$

So, b_n is in fact a constant for n large enough, Nevertheless, with very specific ‘‘non semi-simple’’ constructions of k using \mathbb{Z}_p -extensions K/k in which some tame prime numbers totally split, one may force to get $\mu_p(K/k) \neq 0$ (see details in § 5.4).

Apart from this case (no other one is known), from our point of view, for $i > 1$, the filtration does not depend on canonical elements as the p -units of k for $i = 1$, but of random groups Λ_n^i ; so using standard laws of probabilities, one may estimate the bounds b_n for which $\mathcal{H}_{K_n}^{b_n} = \mathcal{H}_{K_n}$, which ones depend on how fast, in an algorithmic sense, $\#\mathbf{N}_n(\mathcal{H}_n^i) \rightarrow [H_k^{\text{nr}} : K^{\text{nr}}]$ and $\#\omega_n(\Lambda_n^i) \rightarrow p^{n-e}$, as i increases, for $n \gg 0$ fixed (see Lemmas 4.2, 4.3); of course, Iwasawa’s theory corroborates the fact that such an heuristic makes sense, but it does not give an algorithm likely to ‘‘construct’’ the

λ, μ, ν invariants from the first layers. Many heuristics are suggested for $\lambda_p(k^{\text{cyc}}/k)$ as that of [EJV2011] using random matrices.

5. GENERALIZATIONS OF GOLD'S CRITERION

5.1. The first and second element of the filtration. We focus on the elements $\mathcal{H}_{K_n}^1$ and $\mathcal{H}_{K_n}^2$ of the filtration; they are somewhat canonical and bring a maximal information in a very general setting about ramification in K/k and the p -class group of k ; we will see that they give the order of the logarithmic class group, which shows, once again, that many results, obtained from classical Iwasawa's theory of $\mathbb{Z}_p[[T]]$ -modules, have an easy and computable interpretation via class field theory.

Consider formulas (4.4) for $i \leq 1$ and $n \geq e$, still using the previous simplified notations depending on n ; for the moment we are not making any hypotheses, except K not contained in L or \bar{L} (cf. Diagram 1.3.3):

$$(5.1) \quad \begin{cases} \#\mathcal{H}_n^1 = [\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{T}_k \times p^{n-e}, \\ \#(\mathcal{H}_n^2 / \mathcal{H}_n^1) = \frac{[\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{T}_k}{\#\mathbf{N}_n(\mathcal{H}_n^1)} \times \frac{p^{n-e}}{\#\omega_n(\mathbf{\Lambda}_n^1)}, \\ \mathbf{\Lambda}_n^1 = \{a \in k^\times, (a) = \mathbf{N}_n(\mathfrak{A}_n), \mathcal{C}_n(\mathfrak{A}_n) \in \mathcal{H}_n^1\}. \end{cases}$$

We have the following classical exact sequence, where \mathbf{J}_n is the transfer map and $\mathcal{H}_n^{\text{ram}}$ the subgroup generated by the classes of the ramified primes (more precisely by the product of the form $\mathfrak{p}_n^1 \cdots \mathfrak{p}_n^{p^g}$, and its conjugate, with $\mathfrak{p}_n^j | \mathfrak{p} | p$ in $K_n/k/\mathbb{Q}$):

$$(5.2) \quad 1 \rightarrow \mathbf{J}_n(\mathcal{H}_k) \cdot \mathcal{H}_{K_n}^{\text{ram}} \rightarrow \mathcal{H}_{K_n}^1 \rightarrow E_k \cap \mathbf{N}_n(K_n^\times) / \mathbf{N}_n(E_{K_n}) \rightarrow 1.$$

In the case of a totally p -adic imaginary quadratic field, one obtains the isomorphism $\mathcal{H}_n^1 \simeq \mathbf{J}_n(\mathcal{H}_k) \cdot \mathcal{H}_n^{\text{ram}} =: \mathcal{C}_n(\mathcal{I}_n^1)$ for a suitable ideal group \mathcal{I}_n^1 generated by extension of ideals of k generating $\mathbf{J}_n(\mathcal{H}_k)$ and by the products $\mathfrak{p}_n^1 \cdots \mathfrak{p}_n^{p^g}$. Taking the norm, we obtain $\mathbf{N}_n(\mathcal{H}_n^1) = \mathcal{H}_k^{p^n} \cdot \mathbf{N}_n(\mathcal{H}_n^{\text{ram}})$.

Let $\mathcal{I}_k = \langle \mathfrak{a}_1, \dots, \mathfrak{a}_r \rangle$, $r \geq 0$, be a subgroup of prime-to- p ideals of k , whose classes generate the $S_p(k)$ -class group $\mathcal{H}_k / \langle \mathcal{C}'(\mathfrak{p}) \rangle$ of k (recall that $\langle \mathcal{C}'(\bar{\mathfrak{p}}) \rangle = \langle \mathcal{C}'(\mathfrak{p}) \rangle$); in other words, the ideal class group \mathcal{I}_k^1 corresponding to $\mathbf{N}_n(\mathcal{H}_n^1)$ for the computation of $\mathbf{\Lambda}_n^1$, may be for instance $\mathcal{I}_k^1 := \mathcal{I}_k \oplus \mathbf{N}_n(\mathcal{I}_n^{\text{ram}})$ (i.e., $\mathcal{C}_n(\mathcal{I}_n^{\text{ram}}) = \mathcal{H}_n^{\text{ram}}$, $\mathbf{N}_n(\mathcal{I}_n^{\text{ram}}) \subseteq \langle \mathfrak{p}, \bar{\mathfrak{p}} \rangle$) (Remark 4.1).

Then $\mathbf{N}_n(\mathbf{J}_n(\mathcal{I}_k)) = \langle \mathfrak{a}_1^{p^n}, \dots, \mathfrak{a}_r^{p^n} \rangle$, where $\mathfrak{a}_j^{p^n}$ is of the form $(a_j^{p^{n-\tilde{e}_j}})$, where $a_j \in k^\times$ and $p^{\tilde{e}_j} \leq p^{\tilde{e}}$, the exponent of $\mathcal{H}_k / \langle \mathcal{C}'(\mathfrak{p}) \rangle$ (it is important to note that when \mathcal{H}_k is generated by $\mathcal{C}'(\mathfrak{p})$, $\mathcal{I}_k = 1$, $\mathcal{I}_k^1 \subseteq \langle \mathfrak{p}, \bar{\mathfrak{p}} \rangle$, and $\tilde{e} = 0$).

We then get $\langle (a_1^{p^{n-\tilde{e}_1}}), \dots, (a_r^{p^{n-\tilde{e}_r}}) \rangle \subset \mathbf{N}_n(\mathcal{I}_n^1)$, and these $a_j^{p^{n-\tilde{e}_j}}$ are elements of $\mathbf{\Lambda}_n^1$ that will be completed by some $S_p(k)$ -unit y coming from $\mathbf{N}_n(\mathcal{I}_n^{\text{ram}})$; but, for n large enough, $\omega_n(a_j^{p^{n-\tilde{e}_j}})$ would be of order less than $p^{\tilde{e}_j}$ in the cyclic group Ω_n (of order p^{n-e} from (4.2)); whence, in the computation of the norm factor:

$$\frac{p^{n-e}}{\#\omega_n(\mathbf{\Lambda}_n^1)} = \frac{p^{n-e}}{\#\omega_n(\langle \dots, a_j^{p^{n-\tilde{e}_j}}, \dots; y \rangle)},$$

only the p -unit y will intervene since $\omega_n(y)$ will have the large order $p^{n-e-\delta_p(y)}$ for a constant $\delta_p(y) \leq \tilde{\delta}_p(k)$ computed in (5.3) (one sees that if $\delta_p(y) = 0$, then the norm factor is trivial whatever the a_j 's).

Let p^f (resp. p^g) be the residue degree of \mathfrak{p} (resp. the index in G_n of the decomposition group of \mathfrak{p}), in K_n/k for n large enough; then $\mathbf{N}_n(\mathcal{H}_n^{\text{ram}})$ is generated by the classes of the ideals $\mathfrak{p}^{p^f+g} = \mathfrak{p}^{[K \cap L:k]} = \mathfrak{p}^{p^e}$ and $\bar{\mathfrak{p}}^{p^f+g} = \bar{\mathfrak{p}}^{[K \cap \bar{L}:k]} = \bar{\mathfrak{p}}^{p^e}$ since $\text{Gal}(K/K \cap L)$ and $\text{Gal}(K/K \cap \bar{L})$ are the inertia groups in K/k . Using the relation

$\mathfrak{p}\bar{\mathfrak{p}} = (p)$, $\mathbf{N}_n(\mathcal{H}_n^{\text{ram}})$ is generated by $\mathcal{C}'(\mathfrak{p}^{p^{\bar{e}}})$, since $K^{\text{nr}} = K \cap \bar{L} \subseteq K \cap L$ (from Diagram 1.3.3); whence $\mathbf{N}_n(\mathcal{H}_n^{\text{ram}}) = \langle \mathcal{C}'(\mathfrak{p}^{p^{\bar{e}}}) \rangle$.

So, $\omega_n(\mathbf{A}_n^1)$ will be generated by the image of a suitable p -unit y such that $(y) \in \mathbf{N}_n(\mathcal{S}_n^1)$; thus, if $v_p(h) \geq \bar{e}$, $y = x$, the generator of \mathfrak{p}^h , otherwise, $y = x^{p^{\bar{e} \cdot h^{-1}}}$ is the generator of $\mathfrak{p}^{p^{\bar{e}}}$. This will give, respectively:

$$(5.3) \quad \begin{cases} \delta_{\mathfrak{p}}(y) = \delta_p(k), & \text{if } v_p(h) \geq \bar{e}, \\ \delta_{\mathfrak{p}}(y) = \delta_p(k) + [\bar{e} - v_p(h)], & \text{if } v_p(h) < \bar{e}. \end{cases}$$

Then $\omega_n(y)$ is of order $p^{n-e-\delta_{\mathfrak{p}}(y)}$. Whence the formulas, where “ $m \geq \delta_{\mathfrak{p}}(x) = \delta_p(k)$ ” in the proof of Theorem 3.3 translates here to “ $n \geq e + \delta_{\mathfrak{p}}(y)$ ” (in fact we must have $n \geq \tilde{e} + e + \delta_{\mathfrak{p}}(y)$, where $p^{\tilde{e}}$ is the exponent of $\mathcal{H}_k / \langle \mathcal{C}'(\mathfrak{p}) \rangle$). Taking into account the two cases in (5.3) leads to:

$$(5.4) \quad \left\{ \begin{aligned} \#\mathcal{H}_n^1 &= [\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{T}_k \times p^{n-e} = [H_k^{\text{nr}} : K \cap H_k^{\text{nr}}] \times p^{n-e}; \\ \#(\mathcal{H}_n^2 / \mathcal{H}_n^1) &= \frac{[\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{T}_k}{\#\langle \mathcal{C}'(\mathfrak{p}^{p^{\bar{e}}}) \rangle} \times p^{\delta_{\mathfrak{p}}(y)} \\ &= \begin{cases} \frac{[\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{T}_k}{p^{v_p(h)} \cdot [K^{\text{nr}} : k]^{-1}} \times p^{\delta_p(k)} & \text{[if } v_p(h) \geq \bar{e}] \\ \frac{[\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{T}_k}{1} \times \frac{[K^{\text{nr}} : k]}{p^{v_p(h)}} \times p^{\delta_p(k)} & \text{[if } v_p(h) < \bar{e}] \end{cases} \\ &= \frac{[\tilde{k}^{\text{nr}} : k] \times \#\mathcal{T}_k}{p^{v_p(h)}} \times p^{\delta_p(k)} = \frac{\#\mathcal{H}_k}{p^{v_p(h)}} \times p^{\delta_p(k)} = \#\widetilde{\mathcal{H}}_k. \end{aligned} \right.$$

Theorem 5.1. *Let K/k be a \mathbb{Z}_p -extension with \mathfrak{p} , $\bar{\mathfrak{p}}$ ramified from some layers K_e , $K_{\bar{e}}$, with $e \geq \bar{e}$, and set $\Delta_p(k) := \delta_p(k) + \max(0, \bar{e} - v_p(h))$; let $p^{\tilde{e}}$ be the exponent of the $S_p(k)$ -class group $\mathcal{H}_k / \langle \mathcal{C}'(\mathfrak{p}) \rangle$. Assume $n \geq \tilde{e} + e + \Delta_p(k)$ in the following claims:*

- (i) *The order of the logarithmic class group $\widetilde{\mathcal{H}}_k$ is given by that of $\mathcal{H}_n^2 / \mathcal{H}_n^1$.*
- (ii) *The condition $\#\mathcal{H}_{K_n}^2 = \#\mathcal{H}_{K_n}^1$ (giving $\mathcal{H}_{K_n} = \mathcal{H}_{K_n}^1$), holds if and only if $\widetilde{\mathcal{H}}_k = 1$, which is equivalent to satisfy the two conditions: $\delta_p(k) = 0$ and $\mathcal{H}_k = \langle \mathcal{C}'(\mathfrak{p}) \rangle$ (see also the characterization given by the last claim of Theorem 2.2).*
- (iii) *We have the Iwasawa formula $\#\mathcal{H}_{K_n} = p^{n+v_p(h_k)-e-\bar{e}}$.*

These conditions, often required in the literature, are nothing else than the stability for $i = 1$. A consequence of the above formulas (5.4) is that $\#\mathcal{H}_n^1$ and $\#\mathcal{H}_n^2$ do not depend on the \mathbb{Z}_p -extension K/k , assumed to be ramified at the p -places from some explicit layers.

We shall give more details and consequences of the above result, applied to classical contexts.

5.2. Generalization of Gold's criterion.

5.2.1. *The λ -stability theorem in the totally ramified case.* We will use the theorem of λ -stability, applied to the modules $\mathcal{X}_n = \mathcal{H}_n$ and $X_k = E_k$, giving the number $\lambda := \max(0, \#S_p(k) - 1 - \rho(k))$ (not to be confused with the Iwasawa invariant $\lambda_p(K/k)$), with $\rho(k) = \text{rank}_{\mathbb{Z}}(X_k)$ (thus giving $\lambda = 1$ for totally p -adic imaginary quadratic fields).

We then have the following properties of λ -stability in any \mathbb{Z}_p -extension K/k in which the p -places totally ramify (i.e., $e = \bar{e} = 0$), without any other assumption on k and \mathcal{H}_k :

Proposition 5.2. [Gra2022, Theorem 3.1]. *We have $\#\mathcal{H}_{K_n} = \#\mathcal{H}_k \cdot p^{\lambda \cdot n}$ for all $n \geq 0$ if and only if $\#\mathcal{H}_{K_1} = \#\mathcal{H}_k \cdot p^\lambda$. If this property holds, we have, for all $n \geq 0$, $\mathcal{H}_{K_n} = \mathcal{H}_{K_n}^{G_n}$, $\mathbf{J}_{K_n/k}(\mathcal{H}_k) = \mathcal{H}_{K_n}^{p^n}$ and $\text{Ker}(\mathbf{J}_{K_n/k}) = \mathbf{N}_{K_n/k}(\mathcal{H}_{K_n}[p^n])$, where $\mathcal{H}_{K_n}[p^n] := \{c \in \mathcal{H}_{K_n}, c^{p^n} = 1\}$.*

5.2.2. *The \mathbb{Z}_p -extensions such that $\#\mathcal{H}_n = \#\mathcal{H}_k \cdot p^n$, $\forall n \geq 0$.* We obtain a variant of Gold's criterion which applies to all totally ramified \mathbb{Z}_p -extensions K/k , assuming only that $\mathcal{H}_k = \langle \mathcal{C}'(\mathfrak{p}) \rangle$ (i.e., the $S_p(k)$ -class group of k is trivial, equivalent to $\tilde{\delta}_p(k) = \delta_p(k)$):

Theorem 5.3. *Let k be an imaginary quadratic field and let $p > 2$ be a prime number split in k . We assume that \mathcal{H}_k is generated by $\mathcal{C}'(\mathfrak{p})$, where $\mathfrak{p} \mid p$ (i.e., $\tilde{e} = 0$). Let K/k be a \mathbb{Z}_p -extension in which \mathfrak{p} and $\bar{\mathfrak{p}}$ are totally ramified (i.e., $e = \bar{e} = 0$). Then, the two following claims are equivalent:*

- $\delta_p(k) = 0$ (k is logarithmically principal),
- $\#\mathcal{H}_{K_n} = p^{n+v_p(h_k)}$, for all $n \geq 0$.

Proof. We begin by a translation of the assumptions. If p is totally ramified in K/k with \mathcal{H}_k generated by $\mathcal{C}'(\mathfrak{p})$, then $h_k \sim h$, $\#\mathcal{H}_k = p^{v_p(h)}$, $\tilde{\delta}_p(k) = \delta_p(k)$, the condition about n in Theorem 5.1 is $n \geq \delta_p(k)$ since $\tilde{e} = e = 0$, and, from (5.4):

$$(5.5) \quad \begin{cases} \#\mathcal{H}_n^1 = p^{v_p(h)} \cdot p^n, \quad \forall n \geq \delta_p(k), \\ \#\mathcal{H}_n^2 = \#\mathcal{H}_n^1 \times \#\tilde{\mathcal{H}}_k = p^{n+v_p(h)} \cdot p^{\delta_p(k)} = p^{n+v_p(h)+\delta_p(k)}, \quad \forall n \geq \delta_p(k). \end{cases}$$

(i) If $\delta_p(k) = 0$, $\#\mathcal{H}_n^2 = \#\mathcal{H}_n^1$ from (5.5), the filtration stops, and then $\#\mathcal{H}_n = \#\mathcal{H}_n^1 = p^{v_p(h)} \cdot p^n$ for all $n \geq 0$, whence $\lambda_p(K/k) = 1$ and $\nu_p(K/k) = v_p(h)$.

(ii) Assume that $\#\mathcal{H}_n = p^{v_p(h)} \cdot p^n$, for all $n \geq 0$. Proposition 5.2 applies with $\lambda := \max(0, \#S_p(k) - 1 - \rho(k)) = 1$, thus, $\#\mathcal{H}_n = p^{v_p(h)} \cdot p^n$ for all $n \geq 0$, if and only if this holds for $n = 1$. So, we must verify that the relation is fulfilled for $n = 1$; we have by assumption $\#\mathcal{H}_1 = p^{v_p(h)} \cdot p$, which is also $\#\mathcal{H}_1^1$ from (5.5) (stabilization at $i = 1$), which implies, in particular, that $\#\mathcal{H}_1^2 = \#\mathcal{H}_1^1 = p^{v_p(h)} \cdot p$, but $\#\mathcal{H}_1^2 = p^{v_p(h)} \cdot p^{1+\delta_p(k)}$ from (5.5), whence $\delta_p(k) = 0$, yielding the reciprocal and the theorem. \square

5.2.3. *The cyclotomic \mathbb{Z}_p -extension k^{cyc} with $\mathcal{C}'(\mathfrak{p}) = 1$.* The case $\mathcal{H}_k = 1$, often considered in the literature, may be replaced by the interesting Sands remark [San1993, Proposition 2.1, Theorem 3.3, Corollary 3.4], only using the assumption $h \not\equiv 0 \pmod{p}$ (equivalent to $\tilde{\delta}_p(k) = \delta_p(k) + v_p(h_k)$, whence to $\#\tilde{\mathcal{H}}_k = \#\mathcal{H}_k \times p^{\delta_p(k)}$). We recall first some historical results.

(a) Case $\mathcal{H}_k = 1$. Formulas (5.4) become:

$$(5.6) \quad \#\mathcal{H}_n^1 = p^n \quad \& \quad \#\mathcal{H}_n^2 = p^{n+\delta_p(k)}, \quad \text{for all } n \geq \delta_p(k),$$

and this is the framework of Gold criterion (as particular case of Theorem 5.3) dealing with the cyclotomic \mathbb{Z}_p -extension k^{cyc} and Iwasawa's invariants $\lambda_p(k^{\text{cyc}}/k)$, $\nu_p(k^{\text{cyc}}/k)$, since $\mu_p(k^{\text{cyc}}/k) = 0$:

Corollary 5.4 (Gold's criterion [Gold1974]). *Let k^{cyc} be the cyclotomic \mathbb{Z}_p -extension of k and assume that $\mathcal{H}_k = 1$. Then $\lambda_p(k^{\text{cyc}}/k) = 1$ & $\nu_p(k^{\text{cyc}}/k) = 0$, if and only if $\delta_p(k) = 0$. If so, $\mathcal{H}_{K_n} = \langle \mathcal{C}'(\mathfrak{p}_n) \rangle$, for all $n \geq 0$.*

(b) Case $v_p(h) = 0$. Properties of the filtration in the case $p \nmid h$ and $K = k^{\text{cyc}}$ give the Gold–Sands criterion; as we will see, this property is very specific of k^{cyc} (recall that k^{cyc}/k is totally ramified at the p -places):

Theorem 5.5. (*Gold–Sands criterion*). *Let k^{cyc} be the cyclotomic \mathbb{Z}_p -extension of k and let \mathcal{H}_k be the p -class group of k , of order h_k . When $\mathcal{C}(\mathfrak{p})$ is of prime-to- p order (i.e., $v_p(h) = 0$), $\lambda_p(k^{\text{cyc}}/k) \geq 2$ if and only if $v_p(h_k) \geq 1$ or $\delta_p(k) \geq 1$, whence if and only if $\widetilde{\mathcal{H}}_k \neq 1$.*

Proof. We still put $K = k^{\text{cyc}}$. Prove first that $\widetilde{\mathcal{H}}_k \neq 1$ implies $\lambda_p(k^{\text{cyc}}/k) \geq 2$; so, assume that $\#\mathcal{H}_k p^{\delta_p(k)} \neq 1$ and $\lambda_p(K/k) = 1$ to get a contradiction. Let $m \gg n \gg 0$, whence $\#\mathcal{H}_{K_m} = p^{m+\nu}$, $\#\mathcal{H}_{K_n} = p^{n+\nu}$, with $\nu = \nu_p(K/k)$.

One considers the relative formulas for the filtration of \mathcal{H}_{K_m} with respect to $G_n^m := \text{Gal}(K_m/K_n)$, $\omega_n^m := \omega_{K_m/K_n}$, and so on for the norms \mathbf{N}_n^m and transfers \mathbf{J}_n^m ; let $\mathcal{H}_{K_m}^{(1)}$ and $\mathcal{H}_{K_m}^{(2)}$ be the first and second steps of the filtration regarding G_n^m . We use the fact that the group of units E_{K_n} of K_n is that of \mathbb{Q}_n for which $\omega_n^m(E_{\mathbb{Q}_n}) = 1$ and $E_{\mathbb{Q}_n} \cap \mathbf{N}_{\mathbb{Q}_m/\mathbb{Q}_n}(\mathbb{Q}_m^\times) = \mathbf{N}_{\mathbb{Q}_m/\mathbb{Q}_n}(E_{\mathbb{Q}_m})$:

$$\begin{cases} \#\mathcal{H}_{K_m}^{(1)} = \#\mathcal{H}_{K_n} \cdot \frac{p^{m-n}}{\omega_n^m(E_{K_n})} = p^{(n+\nu)+(m-n)} = p^{m+\nu}, \\ \mathcal{H}_{K_m}^{(1)} \simeq \mathbf{J}_n^m(\mathcal{H}_{K_n}) \cdot \mathcal{H}_{K_m}^{\text{ram}} = \mathbf{J}_n^m(\mathcal{H}_{K_n}) \cdot \langle \mathcal{C}_m(\mathfrak{p}_m) \rangle, \end{cases}$$

giving $\mathbf{N}_n^m(\mathcal{H}_{K_m}^{(1)}) = \mathcal{H}_{K_n}^{p^{m-n}} \cdot \langle \mathcal{C}_n(\mathfrak{p}_n) \rangle = \langle \mathcal{C}_n(\mathfrak{p}_n) \rangle$ for $m \gg n$, and the following inequality about the case $i = 1$ of the relative filtration:

$$\#(\mathcal{H}_{K_m}^{(2)}/\mathcal{H}_{K_m}^{(1)}) = \frac{\#\mathcal{H}_{K_n}}{\#\langle \mathcal{C}_n(\mathfrak{p}_n) \rangle} \times \frac{p^{m-n}}{\#\omega_n^m(\mathbf{\Lambda}^1)} \geq \frac{p^{n+\nu}}{\#\langle \mathcal{C}_n(\mathfrak{p}_n) \rangle} \times 1.$$

But $\mathfrak{p}_n^{p^n} = \mathbf{J}_{K_n/k}(\mathfrak{p})$ is p -principal and $\mathcal{C}_n(\mathfrak{p}_n)$ is of order a divisor of p^n ; then we get $\#(\mathcal{H}_{K_m}^{(2)}/\mathcal{H}_{K_m}^{(1)}) \geq p^\nu$; then $\#\mathcal{H}_{K_m}^{(2)} \geq \#\mathcal{H}_{K_m}^{(1)} \cdot p^\nu = p^{m+2\nu}$, whence the inequality $p^{m+\nu} = \#\mathcal{H}_{K_m} \geq \#\mathcal{H}_{K_m}^{(2)} \geq p^{m+2\nu}$, which implies $\nu = 0$ and $\#\mathcal{H}_{K_n} = p^n$ for all $n \gg 0$; but using the filtration with respect to $G_n = \text{Gal}(K_n/k)$, this gives $\#\mathcal{H}_{K_n} = p^n \geq \#\mathcal{H}_{K_n}^{(2)} = (\#\mathcal{H}_k)^2 \times p^{n+\delta_p(k)}$ from formulas (5.4) with $v_p(h) = 0$, whence $(\#\mathcal{H}_k)^2 \times p^{\delta_p(k)} = 1$ (contradiction).

For the reciprocal, let's assume $\lambda_p(K/k) \geq 2$ and $v_p(h_k) = 0$; then usual Gold's criterion (Corollary 5.4) implies $\delta_p(k) \geq 1$. \square

Note that the proof of the Sands improvement of Gold's criterion comes from a deep general analytic result of Federer–Gross–Sinnott [FGS1981, Corollary 5.8 to Proposition 5.6].

The filtration may be somewhat important, giving large $\lambda_p(k^{\text{cyc}}/k)$'s (see examples in Dummit–Ford–Kisilevsky–Sands [DFKS1991] or Ozaki [Oza2001]). For the three examples of $\lambda_p(k^{\text{cyc}}/k) \geq 2$ given in [Oza2001, Section 1, (4)], we compute $\#\widetilde{\mathcal{H}}_k = p^{\tilde{\delta}_p(k)} = p^{\delta_p(k)}$, in the case $v_p(h) = 0$ and see that $\delta_p(k) \neq 0$ ($\delta_p(k) = 1$ for $m = 52391$, $p = 5$, $\delta_p(k) = 2$ for $m = 1371$, $p = 7$, $\delta_p(k) = 3$ for $m = 23834$, $p = 3$). In the case $m = 239$ and $p = 3$ for which $\#\mathcal{H}_k = 3$, we find $\tilde{\delta}_p(k) = 1$, even if \mathfrak{p} is 3-principal and $\delta_p(k) = 0$ (thus $\tilde{\delta}_3(k) = \delta_3(k) + v_3(h_k) - v_3(h) = 1$, as expected). Statistical results on $\lambda_p(k^{\text{cyc}}/k)$, regarding $\text{rank}_p(\mathcal{H}_k)$, are given in [Ray2023], varying imaginary quadratic fields k , for p fixed.

5.3. Specificity of k^{cyc} regarding the non-Galois K 's. We will explain why the unit groups, even for the non-Galois totally imaginary \mathbb{Z}_p -extensions, play a fundamental role for Iwasawa's invariants, contrary to the unique case of $K = k^{\text{cyc}}$. Indeed, one may ask why Theorem 5.5 does not imply that $\delta_p(k) \geq 1$ is equivalent to $\lambda_p(K/k) \geq 2$.

We propose the following explanations, in relation with various phenomena of capitulation in a \mathbb{Z}_p -extension. A first remark being that minus parts of p -class groups and logarithmic class groups do not capitulate in k^{cyc}/k for k imaginary, and it is probably the unique case, which suggests that $\widetilde{\mathcal{H}}_k$ and \mathcal{H}_k behave analogously. Nevertheless, recall that the modules \mathcal{T}_k , of p -ramification theory, never capitulate in K/k , whatever the p -extension K/k (under Leopoldt's conjecture).

We describe the following four cases of units groups, where k is a quadratic field (real or imaginary), but this applies in any real or imaginary abelian field:

(a) $K = k^{\text{cyc}}$, for k **real**. For this, recall that Greenberg's conjecture [Gree1976] ($\lambda_p(K/k) = \mu_p(K/k) = 0$ for totally real base field k , whatever the splitting of p), is equivalent to the capitulation of $\widetilde{\mathcal{H}}_k$ in K [Jau2019^a]. The phenomenon is due to the existence of unit groups E_{K_n} of maximal \mathbb{Z} -rank $[K_n : \mathbb{Q}] - 1 = 2p^n - 1$.

We note for the sequel that for $K = k^{\text{cyc}}$, $E_{K_n} = E_{K_n}^{\frac{1+\tau}{2}} \oplus E_{K_n}^{\frac{1-\tau}{2}}$, where $E_{K_n}^{\frac{1+\tau}{2}} = E_{\mathbb{Q}_{n_0}^{\text{cyc}}}$ never intervenes since in any relative layers in $\mathbb{Q}^{\text{cyc}}/\mathbb{Q}$, units are norms of units; only the non-trivial isotypic component, of \mathbb{Z} -rank p^n , does give non-trivial formulas. Real Greenberg's conjecture has been largely checked in Kraft–Schoof–Pagani [KS1995, Pag2022], for $p = 3$ and $p = 2$, respectively.

(b) $K = k^{\text{cyc}}$, for k **imaginary**. In this case, $E_{K_n}^{\frac{1-\tau}{2}} = 1$, and we can say, by abuse of language, that, in the norm point of view (norm factor), K has no proper units, thus giving largest orders of the elements $\mathcal{H}_{K_n}^i$ (e.g., $\#\mathcal{H}_{K_n}^{G_n^m} = \#\mathcal{H}_{K_n} \cdot p^{(m-n)(\#S_p(k)-1)}$), and, as we have seen, $\lambda_p(K/k)$ may be large.

(c) K is a **non-Galois \mathbb{Z}_p -extension of k imaginary**. Since K is totally imaginary, we have $\text{rank}_{\mathbb{Z}}(E_{K_n}) = p^n - 1$, and for n large enough, $E_{K_n} \cap E_{\mathbb{Q}^{\text{cyc}}} = E_{\mathbb{Q}_{n_0}^{\text{cyc}}}$, where n_0 depends on $K \cap k^{\text{cyc}}$; this group of real units is not necessarily norm in relative K_m/K_n 's and does not matter. In other words, one may consider that in the case of imaginary quadratic fields, any K/k distinct from k^{cyc}/k , behaves like the totally real case for k^{cyc}/k because, conjecturally, $\omega_n^m(E_{K_n}) = \Omega_n^m$ for $m \gg n \gg 0$, which yields the conjecture 5.6 that we give hereafter.

(d) $K = k^{\text{acyc}}$, for k **imaginary**. In this case, $\text{rank}_{\mathbb{Z}}(E_{K_n}) = p^n - 1$, and, in fact, the context is analogous to the case (iii) of a general non-Galois \mathbb{Z}_p -extension. The fact that in the previous imaginary cases there is no proper units is related to the non-capitulation of $\widetilde{\mathcal{H}}_k$ and \mathcal{H}_k in K ; on the contrary, capitulation phenomena need units and lead to Greenberg's conjecture (see [Jau2019^a] for the totally real case, and [Jau2019^b] for another aspect implying capitulation).

In conclusion of this study and taking into account all available results, including numerical ones, we propose the following conjecture in the semi-simple case (an essential context since examples of $\mu_p(K/k) \geq 1$ may be obtained; see § 5.4):

Conjecture 5.6. *Let k be an imaginary abelian field, and let $p \nmid [k : \mathbb{Q}]$ be a prime totally split in k (semi-simple totally p -adic case). Let K/k be a \mathbb{Z}_p -extension and let $S_p(K/k)$ be the set of p -places ramified in K . If $K \neq k^{\text{cyc}}$:*

- (i) $\mu_p(K/k) = 0$ & $\lambda_p(K/k) = \#S_p(K/k) - 1 \in \{0, [k : \mathbb{Q}] - 1\}$.
- (ii) Point (i) is equivalent to the capitulation of $\widetilde{\mathcal{H}}_k$ in K .

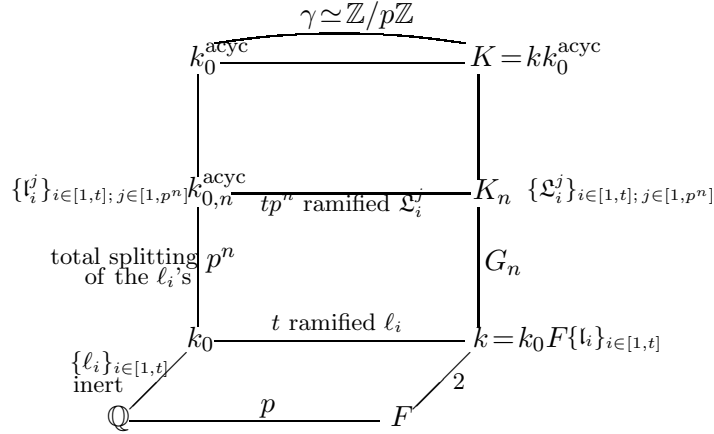
In relation with the deep results of Sands [San1991, San1993], Ozaki [Oza2001, Oza2004], Fujii [Fu2013], Itoh–Takakura [IT2014] and others, there is no contradiction in the quadratic case since these authors deduce, from specific assumptions (involving properties of k^{cyc} in general), that $\lambda_p(K/k)$ is less than some integer λ_0 .

In Ozaki [Oza2001], the main theorem claims, under the assumption $\mathcal{H}_k = 1$, that “ $\lambda_p(K/k) = 1$ and $\mu_p(K/k) = 0$ for all but finitely many \mathbb{Z}_p -extension K/k ”

(including of course k^{cyc}/k). Because it is difficult to conceive this situation, since the cardinality of the set of K/k 's is that of \mathbb{Z}_p , this result enforces the conjecture. In Kataoka [Kat2017, Theorems 1.3, 5.3, 5.9] many particular cases are proved by means of defining a topology and a measure on the set of \mathbb{Z}_p^i -extensions of k .

5.4. Non semi-simple imaginary abelian field k .

5.4.1. *Classical construction of an example with $\mu_p(K/k)$ arbitrary large.* Let k_0 be an imaginary quadratic field. Let $\{\ell_1, \dots, \ell_t\}$, $t \geq 2$, be a set of distinct primes $\ell_i \equiv 1 \pmod{p}$, totally inert in $k_0^{\text{cyc}}/\mathbb{Q}$. Let F be a degree- p cyclic field of conductor $\ell_1 \cdots \ell_t$ and set $k := k_0 F$. Since the ℓ_i 's are inert in k_0/\mathbb{Q} , there is t ramified primes in k/k_0 . Being unique, the decomposition group $D_i \simeq \mathbb{Z}_p$ of ℓ_i in \tilde{k}_0/k_0 is invariant by τ ; so $D_i = \text{Gal}(\tilde{k}_0/k_0^{\text{acyc}})$, and ℓ_i totally splits in k_0^{acyc}/k_0 . Consider the \mathbb{Z}_p -extension $K := k k_0^{\text{acyc}}$ of k :



The \mathbb{Z} -rank of the unit group $E_{k_0^{\text{acyc}}}$ is $p^n - 1$ and the module $\Omega_{K_n/k_0^{\text{acyc}}}$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{t \cdot p^n - 1}$. The Chevalley–Herbrand formula, in K_n/k_0^{acyc} , yields:

$$\#\mathcal{H}_n \geq \#\mathcal{H}_n^\gamma = \#\mathcal{H}_{k_0^{\text{acyc}}} \times \frac{p^{t \cdot p^n - 1}}{\omega(E_{k_0^{\text{acyc}}})} \geq \frac{p^{t \cdot p^n - 1}}{p^{p^n - 1}} \geq p^{(t-1) \cdot p^n},$$

giving $\mu_p(K/k) \geq t - 1$, only by considering the first step of the filtration.

Many variants of this process are in the literature from Iwasawa’s examples [Iw1973], e.g., Hubbard–Washington [HW2018] where numerical computations are done, then Ozaki [Oza2004] where λ and μ can be prescribed; in any case, the genus theory form of Chevalley–Herbrand formula is needed in suitable finite p -sub-extensions of K_n (here K_n/k_0^{acyc}) in which the number of tame ramified primes is for instance of order $O(p^n)$, due to splitting. See important generalizations by Perbet [Per2011], Hajir–Maire [HM2019] where it is shown that large μ_p 's are related to the existence of tame primes that split completely in the tower.

5.4.2. *Complexity of a \mathbb{Z}_p -extension.* We note that for the previous example giving $\mu_p(K/k) \geq t - 1$, inequalities (4.5) show that the number b_n of steps in the algorithm at the layer K_n is of the order of p^n , which probably defines the biggest “complexity” that may exist, the problem being to define this notion. An heuristic, analyzed and numerically tested in [Gra2024, Theorems 1.1, 1.2] for totally ramified cyclic p -extensions L/k , suggests that if the complexity of \mathcal{H}_L is at a low level and $[L : k] \gg 0$, then, in a not very intuitive way, \mathcal{H}_k capitulates in L . We give in Appendix B.6 some numerical results.

We propose to state a general definition, from [Gra2024, Definition 2.12]:

Definition 5.7. Let L/k be any totally ramified cyclic p -extension of degree p^N , $N \geq 1$, of Galois group $G = \langle \sigma \rangle$. Let $b(L) \geq 0$ be the minimal integer such that $(\sigma - 1)^{b(L)}$ annihilates \mathcal{H}_L and let $p^{e(L)}$, $e(L) \geq 0$, be the exponent of \mathcal{H}_L . We say that L/k is of smooth complexity if $e(L) \leq N - s(L)$ if $b(L) \in [p^{s(L)}, p^{s(L)+1} - 1]$ for $s(L) \in [0, N - 1]$. If K/k is a totally ramified \mathbb{Z}_p -extension, we say that K/k is of smooth complexity if for all N large enough, K_N is of smooth complexity ($e_N \leq N - s_N$ if $b_N \in [p^{s_N}, p^{s_N+1} - 1]$ for $s_N \in [0, N - 1]$, where p^{e_N} is the exponent of \mathcal{H}_{K_N} , and b_N is the length of the filtration of \mathcal{H}_{K_N}).

For instance, this is the case of the stability $\#\mathcal{H}_{K_N} = \#\mathcal{H}_k$ for all $N \geq 0$ (filtrations of minimal length), in which case \mathcal{H}_k capitulates.

We have proven in [Gra2024, Corollary 2.11] a sufficient condition of capitulation from the property that $\nu_{L/k}(\mathcal{H}_L) = \mathbf{J}_{L/k} \circ \mathbf{N}_{L/k}(\mathcal{H}_L) = \mathbf{J}_{L/k}(\mathcal{H}_k)$ in the totally ramified case giving $\mathbf{N}_{L/k}(\mathcal{H}_L) = \mathcal{H}_k$ by class field theory:

Theorem 5.8. Let L/k be any totally ramified cyclic p -extension of degree p^N , $N \geq 1$, of Galois group $G = \langle \sigma \rangle$. Let $b(L) \geq 0$ be the minimal integer such that $(\sigma - 1)^{b(L)}$ annihilates \mathcal{H}_L and let $p^{e(L)}$, $e(L) \geq 0$, be the exponent of \mathcal{H}_L .

Then a sufficient condition of capitulation of \mathcal{H}_k in L is that $e(L) \in [1, N - s(L)]$ if $b(L) \in [p^{s(L)}, p^{s(L)+1} - 1]$ for $s(L) \in [0, N - 1]$.

It would remain to see if the invariant $\widetilde{\mathcal{H}}_k$ is more suitable for capitulation aspects (see Conjecture 5.6) and if the smoothness of K/k implies that $\lambda_p(K/k)$ and $\mu_p(K/k)$ are subject to certain constraints.

These constructions, in a non semi-simple split context, are atypical since k depends on p , and one may think that, in our framework, such a case does not hold; moreover, to our knowledge no similar results are known for non-Galois \mathbb{Z}_p -extensions K/k .

5.4.3. *Complementary remarks.* In [Ho1987] was proved that there exist infinitely many imaginary quadratic fields k in which p does not split, and such that $\lambda(k^{\text{cyc}}/k) = 0$; but the reason relies on the non-trivial existence of infinitely many fields k such that $p \nmid h_k$; whence the result from Chevalley–Herbrand formula. In Kundu–Washington [KuWa2022], heuristics and conjectures are given for k^{acyc}/k , also in the non-split case, which is of another nature compared to our study of the totally p -adic case, but suggests that the filtrations are bounded so that, in general, $\lambda_p(k^{\text{acyc}}/k)$ and $\mu_p(k^{\text{acyc}})$ vanish for these imaginary quadratic fields, contrary to the study done in [HW2018], where examples of $\mu_p(K/k) \geq 1$ need a non semi-simple context as we have seen.

The triviality of $\mu_p(K/k)$ is known for some K/k 's (see, e.g., [Gil1985, Sch1987, OV2016]). A good overview of these properties may be found in Fujii's paper [Fu2013] giving results for any \mathbb{Z}_p -extension of k .

Some studies are of cohomological nature, generalizing Gold's criterion by means of Massey products, depending also of $\delta_p(k)$; see [Pei2024, § 5.1.2], where an algorithm is given and looks like the determination of a filtration, by means of norm computations; but the comparison should be explored in greater depth. In [Sto2024] a weaker interpretation of Gold's criterion is given from the theory of elliptic curves. Other studies are based on the General Greenberg Conjecture [Gree1998, § 4, Conjecture 3.5], as for instance [Oza2001, Tak2020, Mur2023], among many others; in [LiQi2023, Theorem 1.4], links, between the General Greenberg Conjecture and p -rationality (i.e., $\mathcal{T}_k = 1$), are given and generalized in [Jau2024^a, Théorèmes 6, 10]. But it seems that any approach has analogous limits separating accessible algebraic

informations, from random ones and many proofs using Iwasawa's theory are of high technicality.

The strange "transfer" of the properties of k^{cyc} onto K/k is illustrated, in our method, by the computation of the orders of the norm symbols (Theorem 3.3) and the universal existence of $\delta_p(k)$ governing the order of the logarithmic class group.

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A. LOGARITHMIC CLASSES OF IMAGINARY ABELIAN NUMBER FIELDS BY JEAN-FRANÇOIS JAULENT

Let k be an abelian number field. It is well known that such a field satisfies the so-called Gross–Kuz'min conjecture for every prime number p , i.e. that its logarithmic class group $\widetilde{\mathcal{H}}_k$ has finite order, see [Jau1994^b]. The computation of this last group for arbitrary number fields via the PARI/GP system is given in [BJ2016].

We are interested, in this appendix, to give an explicit formula for the orders of the imaginary components of $\widetilde{\mathcal{H}}_k$ in the semi-simple case.

We assume from now on that k is an imaginary abelian number field and that the prime number p does not divide $d = [k : \mathbb{Q}]$. Thus the p -adic algebra $\mathbb{Z}_p[\Delta]$ of the Galois group $\Delta = \text{Gal}(k/\mathbb{Q})$ is a direct product indexed by the primitive idempotents $e_\varphi = \frac{1}{d} \sum_{\tau \in \Delta} \varphi(\tau) \tau^{-1}$ associated to the \mathbb{Q}_p -irreducible characters φ :

$$\mathbb{Z}_p[\Delta] = \prod_{\varphi} \mathbb{Z}_p[\Delta] e_\varphi = \prod_{\varphi} \mathbb{Z}_\varphi,$$

where $\mathbb{Z}_\varphi = \mathbb{Z}_p[\Delta] e_\varphi$ is an unramified extension of \mathbb{Z}_p of degree $d_\varphi = \deg \varphi$.

As a consequence, the p -class group \mathcal{H}_k , as well as the logarithmic p -class group $\widetilde{\mathcal{H}}_k$, decompose as the product of their φ -components; likewise do their respective wild subgroups \mathcal{H}_k^w and $\widetilde{\mathcal{H}}_k^w$ constructed on the places \mathfrak{p} over p . Now, we have:

Lemma A.1. *For any irreducible character φ the φ -components of the tame quotients $\mathcal{H}'_k := \mathcal{H}_k / \mathcal{H}_k^w$ and $\widetilde{\mathcal{H}}'_k := \widetilde{\mathcal{H}}_k / \widetilde{\mathcal{H}}_k^w$ are \mathbb{Z}_φ -isomorphic. From where the equality between orders: $\#(\widetilde{\mathcal{H}}'_k)^{e_\varphi} / \#(\widetilde{\mathcal{H}}_k^w)^{e_\varphi} = \#(\mathcal{H}'_k)^{e_\varphi} / \#(\mathcal{H}_k^w)^{e_\varphi}$.*

Proof. Let us denote H_k^{lc} the maximal abelian pro- p -extension of k which completely splits, at every place, over the cyclotomic \mathbb{Z}_p -extension k^{cyc} ; and let H'_k be its maximal sub-extension which completely splits over k at every place dividing p . By p -adic class field theory (see [Jau1998]), we have the isomorphisms:

$$\widetilde{\mathcal{H}}_k \simeq \text{Gal}(H_k^{\text{lc}}/k^{\text{cyc}}) \quad \& \quad \mathcal{H}'_k \simeq \text{Gal}(H'_k/k).$$

Now, for the canonical map $\text{Gal}(H_k^{\text{lc}}/k^{\text{cyc}}) \rightarrow \text{Gal}(H'_k/k)$ given by the restriction of Galois automorphisms, the kernel is isomorphic to the wild subgroup $\widetilde{\mathcal{H}}_k^w$ of $\widetilde{\mathcal{H}}_k$ and the cokernel $\text{Gal}((k^{\text{cyc}} \cap H'_k)/k)$ is here trivial, since we have $p \nmid d$.

Let us assume now that φ is imaginary and denote $[\mathfrak{p}_\varphi] = [\mathfrak{p}]^{e_\varphi}$ the φ -component of the class in \mathcal{H}_k of any prime \mathfrak{p} above p . Let $w_\varphi \in \mathbb{N}$ be the order of $[\mathfrak{p}_\varphi]$. As a consequence, taking the p -adifications $\mathcal{D}_k := D_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $\mathcal{E}'_k := E'_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$ of the groups of divisors and p -units of k , we can write:

$$\mathfrak{p}_\varphi^{w_\varphi} = \mathfrak{p}^{w_\varphi e_\varphi} = (\alpha_\varphi), \text{ with } \alpha_\varphi \text{ isotypic in } \mathcal{E}'_k.$$

Let us observe that α_φ is uniquely defined (except in the special case where k contains the p th roots of unity and φ is the Teichmüller character ω , where it is defined up to a root of unity) and that α_φ generates the φ -component of \mathcal{E}'_k . Thus:

$$\#(\mathcal{H}_k^w)^{e_\varphi} = (\mathbb{Z}_\varphi : w_\varphi \mathbb{Z}_\varphi) = w_\varphi^{\deg \varphi}.$$

Likewise the logarithmic valuation of α_φ is $\tilde{\nu}_\mathfrak{p}(\alpha_\varphi) := \frac{1}{\deg \mathfrak{p}} \log_p(N_{k_\mathfrak{p}/\mathbb{Q}_p}(\alpha_\varphi))$, where \log_p is the Iwasawa logarithm in \mathbb{Q}_p (see [Jau1994^b]). So, we get (with the convention $x \sim y$ whenever x/y is a p -adic unit):

$$\#(\widetilde{\mathcal{H}}_k^w)^{e_\varphi} = (\mathbb{Z}_\varphi : \tilde{w}_\varphi \mathbb{Z}_\varphi) = \tilde{w}_\varphi^{\deg \varphi},$$

where $\tilde{w}_\varphi \sim \frac{1}{\deg \mathfrak{p}} \log_p(N_{k_\mathfrak{p}/\mathbb{Q}_p}(\alpha_\varphi))$ is the order in \mathbb{N} of the class $[\tilde{\mathfrak{p}}_\varphi] = e_\varphi[\tilde{\mathfrak{p}}]$ of the logarithmic divisor constructed on \mathfrak{p} .

In summary, we have:

Theorem A.2. *Let k be an abelian imaginary number field, $p \nmid d = [k : \mathbb{Q}]$ be a prime number, Δ_p be the decomposition subgroup of p in $\Delta = \text{Gal}(k/\mathbb{Q})$ and $\chi_p := \text{Ind}_{\Delta_p}^{\Delta} 1_{\Delta_p}$ the character of Δ induced by the unit character of Δ_p . Then:*

- (i) *For each irreducible character φ of Δ that does not appear in χ_p , the φ components of the logarithmic class group $\widetilde{\mathcal{H}}_k^{e_\varphi}$ and of the classical class group $\mathcal{H}_k^{e_\varphi}$ are \mathbb{Z}_φ -isomorphic; so, they have same orders: $\tilde{h}_\varphi^{\deg \varphi} = h_\varphi^{\deg \varphi}$.*
- (ii) *For each imaginary φ that does appear in χ_p , the integer \tilde{h}_φ is given by the formula: $\tilde{h}_\varphi \sim \frac{1}{\deg \mathfrak{p}} \log_p(N_{k_\mathfrak{p}/\mathbb{Q}_p}(\beta_\varphi))$, where β_φ is an isotypic generator of the principal divisor $\mathfrak{p}^{h_\varphi e_\varphi}$.*

Proof. Let us observe first that every finite \mathbb{Z}_φ -module is the direct sum of cyclic submodules. So, we can write its order as the $\deg \varphi$ -th power of an integer.

(i) If the irreducible character φ does not appear in χ_p , the φ -components of the wild subgroups $\widetilde{\mathcal{H}}_k^w$ and \mathcal{H}_k^w are trivial. So, the Lemma A.1 gives directly:

$$\widetilde{\mathcal{H}}_k \simeq \mathcal{H}_k.$$

(ii) Otherwise, we get however, accordingly to Lemma A.1:

$$\tilde{h}_\varphi = \frac{h_\varphi \tilde{w}_\varphi}{w_\varphi} \sim \frac{h_\varphi}{w_\varphi} \left(\frac{1}{\deg \mathfrak{p}} \log_p(N_{k_\mathfrak{p}/\mathbb{Q}_p}(\alpha_\varphi)) \right) = \frac{1}{\deg \mathfrak{p}} \log_p(N_{k_\mathfrak{p}/\mathbb{Q}_p}(\beta_\varphi)),$$

where $\beta_\varphi = \alpha^{h_\varphi/w_\varphi}$ is an isotypic generator of the principal divisor $\mathfrak{p}^{h_\varphi} = \mathfrak{p}^{h_\varphi e_\varphi}$.

Corollary A.3. *In case the prime p completely splits in k , we get anyway for every imaginary irreducible character φ :*

$$\tilde{h}_\varphi \sim \left(\frac{1}{p} \log_p(\beta_\varphi) \right)^{\deg \varphi}.$$

Corollary A.4. *In the particular case k is an imaginary quadratic field and the prime p decomposes in k as $(p) = \mathfrak{p}\bar{\mathfrak{p}}$, the order $\tilde{h} = \#\widetilde{\mathcal{H}}_k$ satisfies the identity:*

$$\tilde{h} \sim \left(\frac{1}{p} \log_p(\beta) \right), \text{ where } \beta \text{ is a generator of } (\mathfrak{p}/\bar{\mathfrak{p}})^h \text{ with } h = \#\mathcal{H}_k.$$

Proof. In this later case there are exactly two irreducible characters 1 and φ . The 1-components, i.e. the real components of ordinary or logarithmic p -class groups coincide with the respective groups of \mathbb{Q} , which are trivial. So, the whole groups coincide with their φ -components, i.e. the imaginary ones, and we have the equality between the whole orders: $h := h_1 h_\varphi = h_\varphi$; and $\tilde{h} := \tilde{h}_1 \tilde{h}_\varphi = \tilde{h}_\varphi$.

B. COMPUTATIONS OF $\widetilde{\mathcal{H}}_k$, \mathcal{T}_k , \mathcal{H}_k – PARI/GP PROGRAMS

B.1. General program. The table below only gives some excerpts in the cases where $\mathcal{H}_k \neq 1$:

```
{n=8;for(m=2,10^4,if(core(m)!=m,next);k=bnfinit(x^2+m);Hk=k.clgp[2];
forprime(p=3,10^4,if(kronecker(-m,p)!=1,next);Clog=bnflog(k,p);
if(Clog[1]!=[],Kn=bnrinit(k,p^n);CKn=Kn.cyc;Tk=List;d=matsize(CKn)[2];
for(j=1,d-2,c=CKn[d-j+1];w=valuation(c,p);if(w>0,listinsert(Tk,p^w,1)));
```

```

Hkp=List;dh=matsize(Hk)[2];for(j=1,dh,c=Hk[j];w=valuation(c,p);
if(w>0,listinsert(Hkp,p^w,1)));print("m=",m," p=",p," Clog=",Clog,
" T_p=",Tk," H_p=",Hkp))}
m=3    p=13    Clog=[[13],[13],[ ]]    T_p=[]    H_p=[]
m=3    p=181   Clog=[[181],[181],[ ]]    T_p=[]    H_p=[]
m=3    p=2521  Clog=[[2521],[2521],[ ]]  T_p=[]    H_p=[]
m=5    p=5881  Clog=[[5881],[5881],[ ]]  T_p=[]    H_p=[]
(...)
m=107  p=3     Clog=[[9],[9],[ ]]       T_p=[3]   H_p=[3]
m=166  p=5     Clog=[[5],[5],[ ]]       T_p=[5]   H_p=[5]
m=239  p=3     Clog=[[3],[ ],[3]]       T_p=[]    H_p=[3]
m=302  p=3     Clog=[[3],[3],[ ]]       T_p=[3]   H_p=[3]
m=362  p=3     Clog=[[3],[3],[ ]]       T_p=[3]   H_p=[9]
m=374  p=7     Clog=[[49],[49],[ ]]     T_p=[7]   H_p=[7]
m=383  p=17    Clog=[[17],[17],[ ]]     T_p=[17]  H_p=[17]
m=413  p=3     Clog=[[9],[9],[ ]]       T_p=[]    H_p=[]
m=974  p=3     Clog=[[3],[ ],[3]]       T_p=[3]   H_p=[3,3]
m=998  p=13    Clog=[[13],[13],[ ]]     T_p=[13]  H_p=[13]
m=1238 p=3     Clog=[[9],[3],[3]]       T_p=[]    H_p=[3]
m=1319 p=3     Clog=[[9],[9],[ ]]       T_p=[9]   H_p=[9]
m=1409 p=3     Clog=[[3,3],[3],[3]]     T_p=[3]   H_p=[9]
m=1967 p=3     Clog=[[9],[3],[3]]       T_p=[3]   H_p=[9]
m=2239 p=5     Clog=[[25],[5],[5]]      T_p=[]    H_p=[5]
m=2334 p=11    Clog=[[11],[11],[ ]]     T_p=[11]  H_p=[11]
m=2351 p=3     Clog=[[729],[729],[ ]]   T_p=[9]   H_p=[9]
m=2759 p=3     Clog=[[3],[3],[ ]]       T_p=[3]   H_p=[27]
m=2915 p=3     Clog=[[243],[243],[ ]]   T_p=[3]   H_p=[3]
m=2963 p=13    Clog=[[169],[169],[ ]]   T_p=[13]  H_p=[13]
m=1987 p=163   Clog=[[26569],[26569],[ ]] T_p=[]    H_p=[]

```

B.2. Examples of $\widetilde{\mathcal{H}}_k$'s of p -rank 3. This table uses the previous program for $p = 3$. We do not repeat identical structures for the three invariants:

```

p=3
m=207143  Clog=[[9,3,3],[9],[3,3]]  T_p=[3,3]  H_p=[3,3]
m=1654781 Clog=[[3,3,3],[3],[3,3]]  T_p=[3,3]  H_p=[9,3]
m=2688977 Clog=[[9,9,3],[9],[9,3]]  T_p=[9,3]  H_p=[9,3]
m=2750507 Clog=[[3,3,3],[3],[3,3]]  T_p=[3,3]  H_p=[3,3]
m=3334937 Clog=[[9,3,3],[3],[9,3]]  T_p=[3,3]  H_p=[9,3]
m=3527078 Clog=[[3,3,3],[3],[3,3]]  T_p=[3,3]  H_p=[3,3]
m=4201313 Clog=[[3,3,3],[3],[3,3]]  T_p=[3,3]  H_p=[9,3]
m=4455293 Clog=[[9,3,3],[9],[3,3]]  T_p=[3,3]  H_p=[3,3]
m=4996655 Clog=[[3,3,3],[3],[3,3]]  T_p=[3,3]  H_p=[243,3]
m=5176481 Clog=[[9,3,3],[3],[9,3]]  T_p=[9,3]  H_p=[9,3]
m=5546015 Clog=[[9,3,3],[9],[3,3]]  T_p=[9,3]  H_p=[27,3]
m=5894459 Clog=[[9,3,3],[3],[9,3]]  T_p=[3,3]  H_p=[27,3]
m=6493730 Clog=[[3,3,3],[3],[3,3]]  T_p=[3,3]  H_p=[27,3]
m=6740726 Clog=[[27,9,3],[9],[27,3]] T_p=[27,3] H_p=[27,3]
m=7054241 Clog=[[27,3,3],[27],[3,3]] T_p=[3,3]  H_p=[3,3]
m=7347419 Clog=[[3,3,3],[3],[3,3]]  T_p=[3,3]  H_p=[81,3]
m=8596733 Clog=[[81,3,3],[81],[3,3]] T_p=[3,3]  H_p=[3,3]
m=9098093 Clog=[[81,9,3],[81],[9,3]] T_p=[9,3]  H_p=[9,3]
m=9483965 Clog=[[9,3,3],[3],[3,3,3]] T_p=[3,3]  H_p=[3,3,3]
m=10446266 Clog=[[9,3,3],[3],[9,3]]  T_p=[9,3]  H_p=[9,3]
m=10566149 Clog=[[9,3,3],[9],[3,3]]  T_p=[9,3]  H_p=[9,3]
m=11458877 Clog=[[9,3,3],[3],[9,3]]  T_p=[3,3]  H_p=[9,3]
m=11584382 Clog=[[9,3,3],[9],[3,3]]  T_p=[9,3]  H_p=[81,3]
m=11702378 Clog=[[3,3,3],[3],[3,3]]  T_p=[3,3]  H_p=[81,3]
m=11896847 Clog=[[81,3,3],[27],[9,3]] T_p=[3,3]  H_p=[9,3]
m=12160511 Clog=[[27,3,3],[27],[3,3]] T_p=[3,3]  H_p=[3,3]

```

B.3. Checking of the relation $v_p(\#\widetilde{\mathcal{H}}_k) = \delta_p(k) + [v_p(h_k) - v_p(h)]$. The following program deals with p fixed and for a list of totally p -adic fields $k = \mathbb{Q}(\sqrt{-m})$ giving various examples. One may vary at will the prime p and the list M . The computation of $\widetilde{\mathcal{H}}_k$ is obtained from the definition and $\#\widetilde{\mathcal{H}}_k$ is obtained with formula of Theorem 2.2 from the computation of $\delta_p(k)$ (in $d(x)$) and $\widetilde{\delta}_p(k) = \delta_p(k) + [v_p(h_k) - v_p(h)]$ (in $d(X)$):

```
{p=3;n=12;M=[107,302,362,413,1238,1319,1967,2351,
2915,78731,4996655,8596733,9098093,14729261,15163271];
for(i=1,15,m=M[i];if(core(m)!=m,next);if(kronecker(-m,p)!=1,next);
P=x^2+m;k=bnfinit(P);Clog=bnflog(k,p);H=k.clgp[2];hk=k.no;
D=divisors(hk);N=numdiv(hk);F=component(idealfactor(k,p),1)[1];
for(j=1,N,hh=D[j];Y=idealpow(k,F,hh);Q=bnfisprincipal(k,Y);
dx=matsize(Q[1])[1];S=0;for(j=1,dx,S=S+Q[1][j]);if(S!=0,next);
h=hh;break);vno=valuation(hk,p);vh=valuation(h,p);
X=Mod(Q[2][1]*k.zk[1]+Q[2][2]*k.zk[2],P);qx=X^(p-1)-1;z=norm(qx);
delta=valuation(z,p)-1;Delta=delta+vno-vh;Kn=bnrinit(k,p^n);CKn=Kn.cyc;
Tk=List;d=matsize(CKn)[2];for(j=1,d-2,c=CKn[d-j+1];w=valuation(c,p);
if(w>0,listinsert(Tk,p^w,1));Hk=List;dh=matsize(H)[2];
for(j=1,dh,c=H[dh+1-j];w=valuation(c,p);if(w>0,listinsert(Hk,p^w,1)));
print("m=",m," hk=",hk," h=",h," Hp=",Hp," Clog=",Clog);
print(" d(x)=",delta," d(X)=",Delta," Tp=",Tk)}}
m=107      hk=3      h=3      Hp=[3]      Clog=[[9],[9],[ ]]      d(x)=2 d(X)=2 Tp=[3]
m=302      hk=12     h=12     Hp=[3]      Clog=[[3],[3],[ ]]      d(x)=1 d(X)=1 Tp=[3]
m=362      hk=18     h=9      Hp=[9]      Clog=[[3],[3],[ ]]      d(x)=1 d(X)=1 Tp=[3]
m=413      hk=20     h=10     Hp=[ ]      Clog=[[9],[9],[ ]]      d(x)=2 d(X)=2 Tp=[ ]
m=1238     hk=42     h=14     Hp=[3]      Clog=[[9],[3],[3]]      d(x)=1 d(X)=2 Tp=[ ]
m=1319     hk=45     h=45     Hp=[9]      Clog=[[9],[9],[ ]]      d(x)=2 d(X)=2 Tp=[9]
m=1967     hk=36     h=12     Hp=[9]      Clog=[[9],[3],[3]]      d(x)=1 d(X)=2 Tp=[3]
m=2351     hk=63     h=63     Hp=[9]      Clog=[[729],[729],[ ]]   d(x)=6 d(X)=6 Tp=[9]
m=2915     hk=24     h=6      Hp=[3]      Clog=[[243],[243],[ ]]   d(x)=5 d(X)=5 Tp=[3]
m=78731    hk=108    h=9      Hp=[27]     Clog=[[6561,3],[6561],[3]]d(x)=8 d(X)=9 Tp=[27]
m=4996655  hk=2916  h=324    Hp=[243,3]Clog=[[3,3,3],[3],[3,3]] d(x)=1 d(X)=3 Tp=[3,3]
m=8596733  hk=2664  h=74     Hp=[3,3]    Clog=[[81,3,3],[81],[3,3]]d(x)=4 d(X)=6 Tp=[3,3]
m=9098093  hk=2160  h=20     Hp=[9,3]    Clog=[[81,9,3],[81],[9,3]]d(x)=4 d(X)=7 Tp=[9,3]
m=14729261hk=4158 h=462    Hp=[3,3,3]Clog=[[3,3,3],[3],[3,3]] d(x)=1 d(X)=3 Tp=[3,3,3]
m=15163271hk=4050 h=225    Hp=[27,3]   Clog=[[27,3,3],[27],[3,3]]d(x)=3 d(X)=5 Tp=[27,3]
```

We give a simplified and faster program computing $\widetilde{\delta}_p(k)$ and $\widetilde{\mathcal{H}}_k$ ($p \geq 3$ is to be specified, as well as an interval for m):

```
{p=3;for(m=2,10^5,if(core(m)!=m,next);if(kronecker(-m,p)!=1,next);P=x^2+m;
k=bnfinit(P);Clog=bnflog(k,p);hk=k.no;F=component(idealfactor(k,p),1)[1];
Y=idealpow(k,F,hk);Q=bnfisprincipal(k,Y);X=Q[2][1]*k.zk[1]+Q[2][2]*k.zk[2];
Delta=valuation(norm(Mod(X,P)^(p-1)-1),p)-1;
print("m=",m," hk=",hk," Clog=",Clog," delta(X)=",Delta)}
```

B.4. Successive maxima of $\widetilde{\delta}_p(k)$, for $p = 3$ and for $p = 5$. The program shows that $\delta_p(k)$ is, as expected for a Fermat quotient, very small, giving possibly $\lambda_p(K/k) = 1$ for $p \gg 0$ (see Conjecture 2.4):

```
{p=3;n=16;Dmax=0;for(m=1,10^6,if(core(m)!=m,next);if(kronecker(-m,p)!=1,next);
P=x^2+m;k=bnfinit(P);H=k.clgp[2];hk=k.no;F=component(idealfactor(k,p),1)[1];
Y=idealpow(k,F,hk);Q=bnfisprincipal(k,Y);vno=valuation(hk,p);vh=valuation(h,p);
X=Mod(Q[2][1]*k.zk[1]+Q[2][2]*k.zk[2],P);qx=X^(p-1)-1;z=norm(qx);
Delta=valuation(z,p)-1+vno-vh;if(Delta>Dmax,Dmax=Delta;Kn=bnrinit(k,p^n);
CKn=Kn.cyc;Tk=List;d=matsize(CKn)[2];for(j=1,d-2,c=CKn[d-j+1];w=valuation(c,p);
if(w>0,listinsert(Tk,p^w,1));Hk=List;dh=matsize(H)[2];
for(j=1,dh,c=H[dh+1-j];w=valuation(c,p);if(w>0,listinsert(Hk,p^w,1)));
print("m=",m," delta(X)=",Delta," T_p=",Tk," Hkp=",Hk)}}
p=3
m=14      delta(X)=1 T_p=[ ] Hkp=[ ]      m=2351      delta(X)=6 T_p=[9] Hkp=[9]
```

m=41	delta(X)=3	T_p=[]	Hkp=[]	m=8693	delta(X)=7	T_p=[]	Hkp=[3]
m=365	delta(X)=4	T_p=[]	Hkp=[]	m=26243	delta(X)=8	T_p=[]	Hkp=[3]
m=971	delta(X)=5	T_p=[]	Hkp=[3]	m=30611	delta(X)=13	T_p=[]	Hkp=[]
p=5							
m=11	delta(X)=1	T_p=[]	Hkp=[]	m=16249	delta(X)=5	T_p=[5]	Hkp=[5]
m=51	delta(X)=3	T_p=[]	Hkp=[]	m=8865	delta(X)=6	T_p=[]	Hkp=[]
m=4821	delta(X)=4	T_p=[]	Hkp=[]	m=173611	delta(X)=9	T_p=[]	Hkp=[5]
p=37							
m=114	delta(X)=1	Tor_p=[]	Hkp=[]	m=154207	delta(X)=3	Tor_p=[37]	Hkp=[37]
m=238	delta(X)=2	Tor_p=[]	Hkp=[]				

B.5. Research of primes p such that $\delta_p(k) \geq 1$. The program below gives, for $k = \mathbb{Q}(\sqrt{-m})$, split primes p in a given interval, solutions to $\delta_p(k) \geq 1$, without computation of the precise value of $\lambda_p(k^{\text{cyc}}/k) \geq 2$, which needs to know approximations of the p -adic L -functions as in [DFKS1991].

Put $x = u + v\sqrt{-m}$. The principle is to test the principality of \mathfrak{p}^h , for increasing divisors h of h_k . If $h \neq 1$, $v\sqrt{-m} \equiv -u \pmod{\mathfrak{p}^2}$, then $\bar{x} \equiv 2u \pmod{\mathfrak{p}^2}$, and it suffices to compute the p -valuation of the Fermat quotient $q = \frac{1}{p}((2u)^{p-1} - 1)$. If $h = 1$, we use the relations $u^2 - mv^2 + 2uv\sqrt{-m} \equiv 0 \pmod{\mathfrak{p}^2}$ and $u^2 + mv^2 = p$. Then one gets $v\sqrt{-m} \equiv \frac{p}{2u} - u \pmod{\mathfrak{p}^2}$, and replacing in $\bar{x}^{p-1} - 1 = (u - v\sqrt{-m})^{p-1} - 1$ yields $\frac{1}{p}(\bar{x}^{p-1} - 1) \equiv \frac{(2u)^2q + 1}{(2u)^2} \pmod{\mathfrak{p}}$, whose valuation is that of $(2u)^2q + 1$, and gives $\delta_p(k) = 0$ or $\delta_p(k) \geq 1$.

We only give hkp , the order of \mathcal{H}_k , instead of h_k . We give an excerpt of the results; they coincide with that obtained in the table of [DFKS1991] by means of analytic methods with L_p -functions for computing $\lambda_p(k^{\text{cyc}}/k)$, and we can check all the cases, but our program may be used for any interval and for instance the case $m = 923$, $\#\mathcal{H}_k = 1$, $h = 10$, $p = 85985437$, does not appear in their table limited to $p < 10^7$. We then have $\delta_p(k) \geq 1$ (i.e., $\lambda_p(k^{\text{cyc}}/k) \geq 2$) if and only if the valuation w in the program is larger or equal to 1; the program being very fast we only give few excerpts:

```
{for(m=3,10^4,if(core(m)!=m,next);k=bnfinit(x^2+m);hk=k.no;
D=divisors(hk);N=numdiv(hk);forprime(p=3,10^5,if(kronecker(-m,p)!=1,next);
P=component(idealfactor(k,p),1)[1];for(n=1,N,h=D[n];Y=idealpow(k,P,h);
Q=bnfisprincipal(k,Y);d=matsize(Q[1])[1];S=0;for(j=1,d,S=S+Q[1][j]);if(S==0,
X=Q[2][1]*k.zk[1]+Q[2][2]*k.zk[2];a=Mod(2*polcoeff(X,0),p^2);
q=lift(a^(p-1)-1)/p;if(h!=1,z=q);if(h==1,z=a^2*q+1);w=valuation(z,p);if(w>=1,
print("m=",m," h_kp=",p^(valuation(hk,p))," h=",h," p=",p));break))}}
m=3 h_k=1 h=1 p=13 m=47 h_k=5 h=5 p=157
m=3 h_k=1 h=1 p=181 m=47 h_k=5 h=1 p=1193
m=3 h_k=1 h=1 p=2521 m=47 h_k=5 h=5 p=1493
m=3 h_k=1 h=1 p=76543 m=47 h_k=5 h=5 p=1511
m=5 h_k=2 h=1 p=5881 m=51 h_k=2 h=2 p=5
m=6 h_k=2 h=2 p=131 m=53 h_k=6 h=6 p=81439
m=7 h_k=1 h=1 p=19531 m=55 h_k=4 h=4 p=8447
m=11 h_k=1 h=1 p=5 m=57 h_k=4 h=2 p=79
m=13 h_k=2 h=1 p=113 m=57 h_k=4 h=1 p=13729
m=14 h_k=4 h=4 p=3 m=58 h_k=2 h=2 p=3037
m=15 h_k=2 h=1 p=1741 m=61 h_k=6 h=6 p=11
m=17 h_k=4 h=1 p=8521 m=65 h_k=8 h=4 p=3
m=19 h_k=1 h=1 p=11 m=65 h_k=8 h=4 p=19
m=21 h_k=4 h=2 p=107 m=65 h_k=8 h=4 p=103
m=21 h_k=4 h=2 p=173 m=65 h_k=8 h=4 p=1663
m=22 h_k=2 h=1 p=23 m=67 h_k=1 h=1 p=24421
m=22 h_k=2 h=2 p=29 m=71 h_k=7 h=7 p=29
m=26 h_k=6 h=6 p=5 m=73 h_k=4 h=4 p=7
m=29 h_k=6 h=1 p=5741 m=74 h_k=10 h=5 p=3
```

m=31	h_k=3	h=1	p=227	m=74	h_k=10	h=5	p=641
m=31	h_k=3	h=3	p=727	m=74	h_k=10	h=10	p=5711
m=34	h_k=4	h=4	p=5	m=78	h_k=4	h=2	p=3307
m=34	h_k=4	h=4	p=7	m=82	h_k=4	h=2	p=43
m=34	h_k=4	h=4	p=709	m=83	h_k=3	h=3	p=17
m=35	h_k=2	h=2	p=3	m=83	h_k=3	h=1	p=41
m=35	h_k=2	h=2	p=13	m=83	h_k=3	h=1	p=89431
m=37	h_k=2	h=2	p=23	m=86	h_k=10	h=10	p=3
m=37	h_k=2	h=2	p=1051	m=87	h_k=6	h=6	p=1187
m=38	h_k=6	h=6	p=211	m=89	h_k=12	h=4	p=7
m=38	h_k=6	h=2	p=6947	m=91	h_k=2	h=2	p=761
m=41	h_k=8	h=8	p=3	m=91	h_k=2	h=2	p=787
m=41	h_k=8	h=4	p=5	m=93	h_k=4	h=2	p=677
m=42	h_k=4	h=2	p=251	m=94	h_k=8	h=8	p=11
m=43	h_k=1	h=1	p=1741	m=94	h_k=8	h=8	p=13
(...)							
m=901	h_kp=1	h=12	p=1693	m=943	h_kp=1	h= 8	p=173
m=902	h_kp=1	h=7	p=23	m=946	h_kp=1	h= 8	p=13
m=902	h_kp=1	h= 14	p=8663	m=946	h_kp=1	h= 8	p=6869
m=903	h_kp=1	h= 8	p=17	m=946	h_kp=1	h= 4	p=55217
m=903	h_kp=1	h= 8	p=311	m=947	h_kp=1	h= 5	p=41
m=905	h_kp=1	h= 6	p=18269	m=949	h_kp=1	h= 3	p=2281189
m=905	h_kp=1	h= 12	p=91869119	m=951	h_kp=1	h= 2	p=509
m=906	h_kp=1	h= 14	p=971	m=951	h_kp=1	h= 26	p=797
m=907	h_kp=1	h= 3	p=19	m=951	h_kp=1	h= 13	p=1549
m=907	h_kp=1	h= 3	p=1229	m=955	h_kp=1	h= 4	p=167
m=911	h_kp=1	h= 31	p=5	m=958	h_kp=1	h= 16	p=347
m=914	h_kp=1	h= 36	p=401629	m=962	h_kp=1	h= 2	p=13077373
m=915	h_kp=1	h= 2	p=11777	m=966	h_kp=1	h= 6	p=5
m=917	h_kp=1	h= 10	p=467	m=966	h_kp=1	h= 2	p=111533
m=917	h_kp=1	h= 10	p=1247527	m=966	h_kp=1	h= 2	p=694091
m=919	h_kp=1	h= 19	p=23	m=967	h_kp=1	h= 11	p=139
m=923	h_kp=1	h= 10	p=85985437	m=967	h_kp=1	h= 1	p=1291
m=926	h_kp=1	h= 40	p=3	m=970	h_kp=1	h= 2	p=613
m=926	h_kp=1	h= 40	p=37	m=970	h_kp=1	h= 1	p=1019
m=929	h_kp=1	h= 18	p=5	m=971	h_kp=3	h= 5	p=3
m=930	h_kp=1	h= 6	p=43	m=971	h_kp=1	h= 5	p=3361
m=930	h_kp=1	h= 6	p=397	m=973	h_kp=1	h= 6	p=26399
m=930	h_kp=1	h= 6	p=74507	m=974	h_kp=1	h= 6	p=1151
m=933	h_kp=1	h= 4	p=2689	m=978	h_kp=1	h= 4	p=13604797
m=934	h_kp=1	h= 13	p=15287	m=979	h_kp=1	h= 8	p=7
m=935	h_kp=1	h= 14	p=3	m=982	h_kp=1	h= 10	p=19
m=939	h_kp=1	h= 1	p=367	m=982	h_kp=1	h= 10	p=4311317
m=939	h_kp=1	h= 1	p=192013	m=993	h_kp=1	h= 6	p=7
m=941	h_kp=1	h= 23	p=5	m=995	h_kp=1	h= 8	p=3
m=941	h_kp=1	h= 46	p=7	m=998	h_kp=1	h= 26	p=3
m=941	h_kp=1	h= 46	p=11	m=998	h_kp=13h=	26	p=13
m=942	h_kp=1	h= 6	p=467	m=998	h_kp=1	h= 13	p=1279
m=942	h_kp=1	h= 6	p=3527	m=998	h_kp=1	h= 26	p=847277

In [San1993, Theorem 3.3, Corollary 3.4], Sands proves that there are infinitely many totally p -adic imaginary quadratic fields such that $\lambda_p(k^{\text{cyc}}/k) \geq 2$; this is done under a variant of Gold's criterion when $p \nmid h$ (Theorem 5.5).

B.6. Table of \mathcal{H}_k and \mathcal{H}_{K_1} for the \mathbb{Z}_3 -extensions of Section 5.4. We use some polynomials defining K_1 given by Kim–Oh [KO2004, Table I], for $p = 3$, and the following program, where the two fields F of conductor $f = \ell_1 \ell_2$ are considered; we privilege the case p split in the base field k , even if the program may be used in any case, testing the decomposition of p in k_0 and F (by means of the Kronecker symbol $(d, 3)$ then factoring the defining polynomial of F in the completion \mathbb{Q}_3 ; in this context, p is totally ramified in K/k :

```

{L=[[-2,x^3-3*x-10],[ -11,x^3-3*x-46]];p=3;for(i=1,2,d=L[i][1];Pac=L[i][2];
print();print("d=",d," Pac=",Pac);forprime(ell1=1,100,forprime(ell2=ell1+1,100,
if(valuation(ell1-1,p)!=1 || valuation(ell2-1,p)!=1,next);
if(kronecker(d,ell1)!=-1 || kronecker(d,ell2)!=-1,next);f=ell1*ell2;
for(i=1,4,PF=polsubcyclo(f,p)[i];D=nfdisc(PF);if(D!=f^2,next);
dF=matsize(factor(PF+O(p^6)))[1];if(dF==1,next);print();
Pk=polcompositum(PF,x^2-d)[1];print("ell1=",ell1," ell2=",ell2," Pk=",Pk);
k=bnfinit(Pk,1);Ck=k.clgp;rk=matsize(Ck[2])[2];CkL=List;for(i=1,rk,c=Ck[2][i];
w=valuation(c,p);if(w!=0,listput(CkL,p^w,i));PK1=polcompositum(Pk,Pac)[1];
K1=bnfinit(PK1,1);CK1=K1.clgp;rk1=matsize(CK1[2])[2];CKL=List;
for(i=1,rk1,c=CK1[2][i];w=valuation(c,p);if(w!=0,listput(CKL,p^w,i));
print("Ck=",CkL," CK",1,"=",CKL);\\ Test de capitulation
))}}
      d=-2      (d,p)=1      Pac=x^3-3*x-10
ell1=7 ell2=31 Pk=x^6+2*x^5-137*x^4-586*x^3+4750*x^2+34820*x+62481
p split in F      Ck=[3]      CK1=[9,3,3,3,3]
ell1=13 ell2=31 Pk=x^6+2*x^5-261*x^4-1454*x^3+16778*x^2+166632*x+395793
p split in F      Ck=[3]      CK1=[9,3,3,3,3]
      d=-11      (d,p)=1      Pac=x^3-3*x-46
ell1=7 ell2=43 Pk=x^6+2*x^5-166*x^4-914*x^3+9627*x^2+98856*x+287631
p split in F      Ck=[3]      CK1=[9,3,3,3,3]
ell1=7 ell2=79 Pk=x^6+2*x^5-334*x^4-242*x^3+34323*x^2-21600*x+419175
p split in F      Ck=[3]) CK1=[9,3,3,3,3,3,3]
ell1=13 ell2=43 Pk=x^6+2*x^5-338*x^4-742*x^3+34567*x^2+86816*x+474423
p split in F      Ck=[3]      CK1=[9,9,3,3,3,3]

```

After the last instruction of the program, the reader may add the following, testing the algebraic norm $\nu_{K_1/k}(\mathcal{H}_{K_1})$; so it is possible to see if there is capitulation of \mathcal{H}_k in K_1 (the list G gives the set of elements of order 3 of $\text{Gal}(K_1/\mathbb{Q})$; then S fixes k , and NuA computes the algebraic norm of ideals A generating \mathcal{H}_{K_1}):

```

GG=nfgaloisconj(K1);rG=matsize(GG)[1];G=List;for(j=1,rG,S=GG[j];
S=nfgaloisapply(K1,S,S);if(S!=x,listput(G,S,j));X=nfsubfields(PK1,6);
dX=matsize(X)[2];for(j=1,dX,PX=X[j][1];Q=polcompositum(PX,Pk)[1];
if(poldegree(Q)==6,Y=Mod(X[j][2],PK1);for(kk=1,rG,S=G[kk];
YY=nfgaloisapply(K1,S,Y);if(YY==Y,break(2)))));
for(j=1,rK1,A0=CK1[3][j];A1=A0;As=nfgaloisapply(K1,S,A0);
Ass=nfgaloisapply(K1,S,As);NuA=idealmul(K1,A1,As);
NuA=idealmul(K1,NuA,Ass);B=bnfisprincipal(K1,NuA)[1];print(B))

```

Recall that 3 is totally split in k/\mathbb{Q} , which gives, a priori, a maximal complexity. One verifies that $\#\mathcal{H}_1 \geq 3^4$; most of the examples are of exponent 9; but we have found two cases of exponent 3 and large rank:

```

      d=-2      (d,p)=1      Pac=x^3-3*x-10
ell1=7 ell2=13 Pk=x^6+2*x^5-53*x^4+2*x^3+970*x^2-2056*x+2673
p split in F      Ck=[3]      CK1=[9,3,3,3,3]
Nu(CK1)=[[3,0,0,0,0],[0,0,0,0,0],[0,0,0,0,0],[0,0,0,0,0],[402,0,0,0,0]]

```

the data means $\mathbf{J}_1(\mathcal{H}_k) = \langle h_1^3, 1, 1, 1, 1 \rangle \simeq \mathbb{Z}/3\mathbb{Z}$; so, \mathcal{H}_k does not capitulate in K_1 .

```

ell1=67 ell2=103 Pk=x^6+2*x^5-4596*x^4-84852*x^3+5209749*x^2+184824970*x+1615631242
p split in F      Ck=[3]      CK1=[3,3,3,3,3,3]
Nu(CK1)=[[3,0,0,0,0,0],[0,13,1,1,1,0],[0,0,0,0,0,0],
[0,10,1,1,1,0],[0,10,1,1,1,0],[0,20,2,2,2,0]]

```

similarly, \mathcal{H}_k does not capitulate in K_1 .

Same conclusion with the following more complex cases:

```

      d=-2      (d,p)=1      Pac=x^3-3*x-10
ell1=7 ell2=157 Pk=x^6+2*x^5-725*x^4+2858*x^3+137554*x^2-1333960*x+3471369
p split in F      Ck=[3]      CK1=[9,9,9,3,3,3,3]
Nu(CK1)=[[3,0,0,0,0,0,0],[0,36,0,0,0,0,0],[462,0,0,0,0,0,0],[0,0,0,0,0,0,0],

```

```

[231,0,0,0,0,0], [462,0,0,0,0,0], [231,0,0,0,0,0]]
e111=31 e112=79 Pk=x^6+2*x^5-1625*x^4-13234*x^3+654262*x^2+9540164*x+35059497
p split in F Ck=[3] CK1=[9,9,3,3,3,3,3]
Nu(CK1)=[[549,0,0,0,0,0,0], [546,81,0,0,0,0,0], [273,0,0,0,0,0,0],
[546,0,0,0,0,0,0], [273,0,0,0,0,0,0], [273,0,0,0,0,0,0],
[546,0,0,0,0,0,0], [546,0,0,0,0,0,0]]
d=-11 Pac=x^3-3*x-46
e111=7 e112=13 Pk=x^6+2*x^5-26*x^4+38*x^3+1339*x^2-3820*x+18747
p split in F Ck=[3] CK1=[9,3,3,3,3]
Nu(CK1)=[[3,0,0,0,0], [0,3,0,0,0], [0,0,0,0,0], [78,0,0,0,0], [78,0,0,0,0]]
e111=13 e112=241 Pk=x^6+2*x^5-2054*x^4-8542*x^3+1083823*x^2+6975620*x+22870875
p split in F Ck=[3] CK1=[9,9,9,3,3,3,3]
Nu(CK1)=[[3,0,0,0,0,0,0], [19878,0,0,0,0,0,0], [19878,0,0,0,0,0,0],
[0,0,0,0,0,0,0], [0,0,0,0,0,0,0], [0,0,0,0,0,0,0],
[0,0,0,0,0,0,0], [0,0,0,0,0,0,0]]
e111=61 e112=151 Pk=x^6+2*x^5-6106*x^4-84560*x^3+9346821*x^2+243406494*x+1644431220
p split in F Ck=[3] CK1=[3,3,3,3,3,3,3]
Nu(CK1)=[[30137,0,2,2,2,1,0], [60268,3,4,4,1,2,0], [60268,0,1,4,1,2,0],
[30134,0,2,5,2,1,0], [0,0,0,0,0,0,0], [30134,0,2,2,2,1,0], [60268,0,4,4,1,2,0]]

```

In the selected intervals no cases yielded capitulation in K_1 , which enforces the heuristic regarding the huge complexity of \mathcal{H}_{K_1} when p totally split; but we can not say if a capitulation does exist in larger layers, and the sufficient condition of Theorem 5.8 does not hold. Note that capitulation in K_1 is indeed possible when k is not totally 3-adic; let's give two examples giving $\mathbf{J}_1(\mathcal{H}_k) = 1$:

```

d=-1 (d,p)=-1 Pac=x^3-3*x-4
e111=7 e112=31 Pk=x^6+2*x^5-140*x^4-590*x^3+4739*x^2+33608*x+56405
p split in F Ck=[3] CK1=[27,9,9,9,3]
Nu(CK1)=[[108,0,0,0,0], [0,18,0,0,0], [0,0,0,0,0], [0,0,0,0,0], [0,0,0,0,0]]
d=-2 (d,p)=1 Pac=x^3-3*x-10
e111=7 e112=13 Pk=x^6+2*x^5-53*x^4-180*x^3+788*x^2+4496*x+6404
p inert in F Ck=[3] CK1=[9,3,3,3]
Nu(CK1)=[[27,0,0,0], [0,3,0,0], [0,0,0,0], [0,0,0,0]]

```

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