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ON THE \mathbb{Z}_p -EXTENSIONS OF A TOTALLY p -ADIC IMAGINARY QUADRATIC FIELD

GEORGES GRAS

WITH AN APPENDIX BY JEAN-FRANÇOIS JAULENT

ABSTRACT. Let k be an imaginary quadratic field, and let p be an odd prime number split in k . We analyze some properties of arbitrary \mathbb{Z}_p -extensions K/k . These properties are governed by the Hase norm residue symbol of the fundamental p -unit of k , in terms of the valuation $\delta_p(k)$ of a Fermat quotient, which determines the order of the logarithmic class group $\widetilde{\mathcal{H}}_k$ (Theorem 2.2) and leads, under some conditions, to generalizations of Gold’s criterion characterizing $\lambda_p(K/k) = 1$ (Theorems 3.3, 5.1, 5.3). This uses the higher rank Chevalley–Herbrand formulas, for the filtrations of the p -class groups in K , that we gave in the 1994’s, and the theorem of λ -stability (2022). This study is in connection with articles of Gold, Sands, Dummit–Ford–Kisilevsky–Sands, Ozaki, Jaulent, Fujii. In Appendix A, is given a general proof, by Jaulent, of the link between $\widetilde{\mathcal{H}}_k$ and $\delta_p(k)$ in a broader context. Numerical illustrations (with PARI/GP programs) are given.

1. INTRODUCTION

This work falls within the framework of Jaulent study of “totally p -adic” number fields [Jau2022, Section 3] (in other words, we assume that p totally splits in k), restricted to the particularly case of imaginary quadratic fields.

Our approach is not of Iwasawa’s theory type and only uses classical arguments of class field theory with the Herbrand–Chevalley formula (published in 1933) and with some properties associated to the filtrations of the p -class groups in a \mathbb{Z}_p -extensions K/k ; this point of view allows to understand algorithmic p -adic aspects that purely algebraic ones do not detect.

This yields to compelling heuristics about generalized Greenberg’s conjectures [Gree1998], in the spirit of those for the real case [Gree1976], and to some general results of Gold’s criterion type [Gold1974], about Iwasawa’s invariants $\lambda_p(K/k)$, $\mu_p(K/k)$. More precisely, we prove again the classical Gold’s criterion with our method (Gold’s techniques being also essentially class field theory applied to the cyclotomic \mathbb{Z}_p -extension), and obtain a variant of Gold’s criterion, but which applies to any \mathbb{Z}_p -extensions K/k , non Galois over \mathbb{Q} , totally ramified at p from some layer, with particular cases of non-trivial p -class group for k . We obtain that theses K/k ’s behave very differently from the cyclotomic one; so we conjecture that $\lambda_p(K/k) = 1$ and $\mu_p(K/k) = 0$ for them (Conjecture 5.6).

1.1. Logarithmic writings. Let k be an imaginary quadratic field and let p be an odd prime number split in k , and set $(p) = \mathfrak{p}\bar{\mathfrak{p}}$. We denote by $\log_{\mathfrak{p}}$ and $\log_{\bar{\mathfrak{p}}}$ the corresponding log-functions in the two p -completions of k , with the convention $\log_{\mathfrak{p}}(p) = \log_{\bar{\mathfrak{p}}}(p) = 0$. Let h_k be the class number of k , and let $x_{\mathfrak{p}} =: x \in k^{\times}$ be

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the fundamental \mathfrak{p} -unit of k (obtained from $\mathfrak{p}^h =: (x), h \mid h_k, h$ minimal); of course, $x_{\bar{\mathfrak{p}}} = \bar{x}_{\mathfrak{p}} = \bar{x}$. Since $x\bar{x} = p^h$, this defines (using conjugation and $\log(p) = 0$):

$$\begin{cases} \log_{\bar{\mathfrak{p}}}(x) \text{ (usual definition since } x \text{ is a unit at } \bar{\mathfrak{p}}), \\ \log_{\mathfrak{p}}(\bar{x}) \text{ (conjugate of } \log_{\bar{\mathfrak{p}}}(x)), \\ \log_{\mathfrak{p}}(x) = -\log_{\mathfrak{p}}(\bar{x}), \quad \log_{\bar{\mathfrak{p}}}(\bar{x}) = -\log_{\bar{\mathfrak{p}}}(x). \end{cases}$$

Thus, this yields, without any ambiguity, in terms of p -adic valuations:

$$(1.1) \quad \begin{cases} \delta_p(k) := v_p\left(\frac{1}{p} \log_{\bar{\mathfrak{p}}}(x)\right) = v_p\left(\frac{1}{p} \log_{\mathfrak{p}}(\bar{x})\right) = v_p\left(\frac{1}{p} \log_{\mathfrak{p}}(x)\right) = v_p\left(\frac{1}{p} \log_{\bar{\mathfrak{p}}}(\bar{x})\right) \\ = v_p((x p^{-h})^{p-1} - 1) - 1 = v_p(\bar{x}^{p-1} - 1) - 1. \end{cases}$$

from arbitrary choices, the practical computation being:

$$\delta_p(k) = v_p(\bar{x}^{p-1} - 1) - 1.$$

Definition 1.1. We consider X defined by $\mathfrak{p}^{h_k} =: (X)$ (whence $X := x^{h_k \cdot h^{-1}}$), and $\tilde{\delta}_p(k) := v_p((X p^{-h_k})^{p-1} - 1) - 1 = v_p(\bar{X}^{p-1} - 1) - 1$, fulfilling, from (1.1), the obvious relation $\tilde{\delta}_p(k) = \delta_p(k) + [v_p(h_k) - v_p(h)]$.

1.2. Main results. Let k be an imaginary quadratic field, let p be an odd prime number split in k , and set $(p) = \mathfrak{p}\bar{\mathfrak{p}}$. Let h_k be the class number of k , and let $h \mid h_k$ be the order of the class of \mathfrak{p} . Let $h_{k,p}$ and h_p be the p -parts of h_k and h , respectively. Let $K_n \subset K$ be the layer of degree p^n over k and let \mathcal{H}_{K_n} be its p -class group.

(a) (Theorem 2.2 and Appendix A). Let $\widetilde{\mathcal{H}}_k$ be the Jaulent logarithmic class group of k (i.e., $\text{Gal}(H_k^{\text{lc}}/k^{\text{cyc}})$, where k^{cyc} is the cyclotomic \mathbb{Z}_p -extension and H_k^{lc} the maximal abelian pro- p -extension of k , locally cyclotomic; see Section 2.2).

Then $\#\widetilde{\mathcal{H}}_k = p^{\tilde{\delta}_p(k)}$; thus $v_p(\#\widetilde{\mathcal{H}}_k) = \delta_p(k) + [v_p(h_k) - v_p(h)]$.

(b) (Theorems 3.3). Assume that \mathfrak{p} and $\bar{\mathfrak{p}}$ are totally ramified from some layer of K . From a class field theory computation in Section 3, of the Hasse norm residue symbols of the form $\left(\frac{x, K_n/k}{\mathfrak{p}}\right)$ and $\left(\frac{x, K_n/k}{\bar{\mathfrak{p}}}\right)$, we prove that they depend essentially of the invariant $\delta_p(k)$, and are of orders $p^{n-e-\bar{e}-\delta_p(k)}$ and $p^{n-\bar{e}-\delta_p(k)}$, respectively (see Diagram 2.1 for the definition of the constants e, \bar{e}). This will be crucial in view of Chevalley–Herbrand formula and the computation of the second step of the filtration of the \mathcal{H}_{K_n} 's.

(c) (Theorem 5.1). Assume \mathfrak{p} and $\bar{\mathfrak{p}}$ totally ramified from some layer of K . Then $\tilde{\delta}_p(k) = 0$ implies $\lambda_p(K/k) = 1$.

(d) (Theorem 5.3). We assume that the p -class group \mathcal{H}_k of k is generated by the p -class of the prime $\mathfrak{p} \mid p$ (whence $h_{k,p} = h_p$ and $\tilde{\delta}_p(k) = \delta_p(k)$). Let K/k be a \mathbb{Z}_p -extension in which the two primes dividing p are totally ramified from k .

Then, $\#\mathcal{H}_{K_n} = h_p \cdot p^n$ for all $n \geq 0$, if and only if $\delta_p(k) = 0$ (i.e., if and only if, under the above assumptions, k is logarithmically principal).

Many pioneering works dealing with analogues of Gold's criterion consider k^{cyc} assuming $\mathcal{H}_k = 1$. More recent literature succeeds in obtaining similar results in more general cases. A good overview of this new literature may be found in Fujii's paper [Fu2013] giving results for any \mathbb{Z}_p -extension of k .

In our opinion, these results, whatever the method, were based on the assumption that k has a smooth arithmetic complexity, e.g., k is p -principal ($\mathcal{H}_k = 1$), or logarithmically principal ($\widetilde{\mathcal{H}}_k = 1$), or p -rational (i.e., when the maximal p -ramified abelian pro- p -extension H_k^{Pr} of k coincides with the compositum, \widetilde{k} , of the \mathbb{Z}_p -extension of k).

The completely general case seems to be based on random p -adic numerical circumstances, but subject to accessible probability laws. We give an overview of this aspect in §5.1 to try to understand the nature of the main obstructions.

1.3. Notations and recalls about class field theory.

1.3.1. *The cyclotomic and anti-cyclotomic \mathbb{Z}_p -extensions.* Let $k = \mathbb{Q}(\sqrt{-m})$ ($m \in \mathbb{Z}_{>1}$, square-free) be an imaginary quadratic field and set $g = \text{Gal}(k/\mathbb{Q}) =: \langle \tau \rangle$. Let p be an odd prime number, split in k into \mathfrak{p} and $\bar{\mathfrak{p}} := \mathfrak{p}^\tau$. Let \tilde{k} be the compositum of the \mathbb{Z}_p -extensions of k (the case $p = 2$ gives rise to more complex diagrams, but would yield similar results; for this one may use [Gra1983] where all the possibilities for the structure of \tilde{k}/k are enumerated).

We denote by \mathcal{H}_k (resp. H_k^{nr}) the p -class group (resp. the p -Hilbert class field) of k , and put $\tilde{k}^{\text{nr}} := \tilde{k} \cap H_k^{\text{nr}}$.

We will use the logarithmic class group $\tilde{\mathcal{H}}_k = \text{Gal}(H_k^{\text{lc}}/k^{\text{cyc}})$ where the field H_k^{lc} is defined as the maximal abelian pro- p -extension of k , locally cyclotomic (thus such that p totally splits in $H_k^{\text{lc}}/k^{\text{cyc}}$); these fields are contained in H_k^{pr} , for which $\text{Gal}(H_k^{\text{pr}}/\tilde{k})$ is denoted \mathcal{T}_k .

The group g operates on $\Gamma := \text{Gal}(\tilde{k}/k) \simeq \mathbb{Z}_p^2$ and we define Γ^+ (resp. Γ^-), the sub-module annihilated by $1 - \tau$ (resp. $1 + \tau$). Since $p \neq 2$, $\Gamma = \Gamma^+ \oplus \Gamma^-$ and $\Gamma^+ \cap \Gamma^- = 1$, leading to define the two following *fundamental* \mathbb{Z}_p -extensions yielding \tilde{k} as direct compositum over k ; they are the only ones to be Galois over \mathbb{Q} :

- The cyclotomic \mathbb{Z}_p -extension k^{cyc} (fixed by Γ^-), equal to the compositum $k^{\mathbb{Q}^{\text{cyc}}}$, where the cyclotomic \mathbb{Z}_p -extension \mathbb{Q}^{cyc} is linearly disjoint from k ; so k^{cyc}/k is totally ramified at \mathfrak{p} and $\bar{\mathfrak{p}}$.
- The anti-cyclotomic \mathbb{Z}_p -extension k^{acyc} (fixed by Γ^+) is a pro-diedral \mathbb{Z}_p -extension of \mathbb{Q} for which the complex conjugation τ acts by $\tau\sigma\tau = \sigma^{-1}$ for all $\sigma \in \text{Gal}(k^{\text{acyc}}/k)$. Since $p \neq 2$, τ operates on the class, $\mathcal{C}(\mathfrak{a})$, of an ideal \mathfrak{a} of k , by $\mathcal{C}(\mathfrak{a})^\tau = \mathcal{C}(\mathfrak{a})^{-1}$, and operates similarly on $\text{Gal}(H_k^{\text{nr}}/k)$; thus $\tilde{k}^{\text{nr}} = k^{\text{acyc}} \cap H_k^{\text{nr}}$, and this cyclic extension of k is Galois over \mathbb{Q} , and may be non-trivial; then $k^{\text{acyc}}/\tilde{k}^{\text{nr}}$ is totally ramified at the prime ideals dividing p . In the same way, τ operates by inversion on $\tilde{\mathcal{H}}_k$.

1.3.2. *Abelian p -ramification over k .* We introduce the inertia fields L of \mathfrak{p} and \bar{L} of $\bar{\mathfrak{p}}$, in \tilde{k}/k , so that L/k is unramified at \mathfrak{p} and \bar{L}/k is unramified at $\bar{\mathfrak{p}}$; then, $\bar{L} \neq L$, $L \cap \bar{L} = \tilde{k}^{\text{nr}}$ and $L\bar{L} = \tilde{k}$ because $\text{Gal}(\tilde{k}/L\bar{L})$ is free of finite degree since $\bar{L} \neq L$. In the particular case $\mathcal{H}_k = 1$, \tilde{k} is the direct compositum of L and \bar{L} over k , and the ideal \mathfrak{p} (resp. $\bar{\mathfrak{p}}$) is totally ramified in \bar{L}/k (resp. in L/k). A consequence is that, when $\mathcal{H}_k = 1$, any pair in the set $\{k^{\text{cyc}}, k^{\text{acyc}}, L, \bar{L}\}$ makes \tilde{k} as a direct compositum over k .

The inertia groups, $I_{\mathfrak{p}}$ and $I_{\bar{\mathfrak{p}}}$, of \mathfrak{p} and $\bar{\mathfrak{p}}$ in \tilde{k}/k , are images of the groups of principal local units $U_{\mathfrak{p}}$ and $U_{\bar{\mathfrak{p}}}$ of the completions $k_{\mathfrak{p}}$ and $k_{\bar{\mathfrak{p}}}$, respectively, by the local reciprocity maps, according to the following first diagram of the maximal p -sub-extension k^{ab} of the abelian closure of k , where H_k^{pr} is the maximal p -ramified abelian pro- p -extension of k and H_k^{ta} the maximal tamely ramified abelian pro- p -extension of k , respectively.

In [Gra2005, III.4.4.1, III.2.6.1 (Fig. 2)], we have proved that in this diagram, $\text{Gal}(k^{\text{ab}}/H_k^{\text{pr}}H_k^{\text{ta}})$ is isomorphic to $E_k \otimes \mathbb{Z}_p$, where E_k is the group of units of k . In the present case, it is reduced to the group μ_k of roots of unity of k ; for $p \neq 2$, $\mu_k \otimes \mathbb{Z}_p = 1$, except for $p = 3$ and $k = \mathbb{Q}(\mu_3)$, but the assumption p split in k

does not hold and the diagram reduces to the following one with $H_k^{\text{pr}} H_k^{\text{ta}} = k^{\text{ab}}$ and $\text{Gal}(H_k^{\text{pr}}/H_k^{\text{nr}}) \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, since $\text{tor}_{\mathbb{Z}_p}(U_k) = 1$:

$$\begin{array}{ccc} H_k^{\text{pr}} & \xrightarrow{\quad\quad\quad} & H_k^{\text{pr}} H_k^{\text{ta}} = k^{\text{ab}} \\ I_{\mathfrak{p}} \oplus I_{\bar{\mathfrak{p}}} \downarrow & & \downarrow U_k = U_{\mathfrak{p}} \oplus U_{\bar{\mathfrak{p}}} \\ k & \xrightarrow{\quad\quad\quad} & H_k^{\text{nr}} \xrightarrow{\quad\quad\quad} H_k^{\text{ta}} \end{array}$$

The second diagram, corresponding to H_k^{pr}/k , admits also some simplifications with $\text{tor}_{\mathbb{Z}_p}(U_k) = 1$, which implies $\mathscr{W}_k := \text{tor}_{\mathbb{Z}_p}(U_k)/\mu_p(k) = 1$, and H_k^{pr} coincides with the Bertrandias–Payan field H_k^{bp} (maximal abelian pro- p -extension of k in which any cyclic extension is embeddable in cyclic p -extensions of k of arbitrary degree).

The normalized regulator \mathscr{R}_k (defined in [Gra2018, Section 5]) is trivial since $E_k \otimes \mathbb{Z}_p = 1$, whence $H_k^{\text{pr}} = \tilde{k} H_k^{\text{nr}}$.

Thus, the torsion group $\mathscr{T}_k := \text{Gal}(H_k^{\text{pr}}/\tilde{k})$ is isomorphic to a subgroup of \mathscr{H}_k and the extension $H_k^{\text{pr}}/\tilde{k}$ is unramified; moreover, it is totally split at p (indeed, by any p -completion of H_k^{pr}/k one gets an abelian p -ramified pro- p -extension of \mathbb{Q}_p , whose Galois group is free of \mathbb{Z}_p -rank 2):

$$\begin{array}{ccccccc} & & & \mathscr{T}_k & & & \\ & & & \curvearrowright & & & \\ & & \tilde{k} & \xrightarrow{\text{unramified}} & \tilde{k} H_k^{\text{nr}} & \xrightarrow{\mathscr{R}_k = 1} & H_k^{\text{bp}} & \xrightarrow{\mathscr{W}_k = 1} & H_k^{\text{pr}} \\ \mathbb{Z}_p \times \mathbb{Z}_p \downarrow & & \downarrow & & \downarrow & & & & \\ k & \xrightarrow{\quad\quad\quad} & \tilde{k}^{\text{nr}} & \xrightarrow{\quad\quad\quad} & H_k^{\text{nr}} & \xrightarrow{\quad\quad\quad} & U_k & & \\ & & \mathscr{H}_k & & & & & & \end{array}$$

Remark 1.2. Recall that the computation of this group \mathscr{T}_k in the imaginary quadratic case is easy from [Gra2005, Theorem III.2.6, Remark III.2.6.2], giving the formula $[\tilde{k}^{\text{nr}} : k] = (\mathbb{Z}_p \text{Log}(I_k) : \mathbb{Z}_p \text{Log}(P_k))$, where I_k is the group of prime-to- p ideals of k and P_k the sub-group of principal ideals; in the case of imaginary quadratic fields, the Log-function coincides with the usual p -adic logarithm.

The following numerical examples give the process:

(i) $k = \mathbb{Q}(\sqrt{-191})$, $p = 13$. We have $\mathscr{H}_k = \langle \mathfrak{d}(\mathfrak{l}) \rangle$ of order 13, where $\mathfrak{l} \mid 2$. We have $\mathfrak{l}^{13} = \frac{1}{2}(153 + 7\sqrt{-191})$, and since $(\frac{1}{2}(153 + 7\sqrt{-191}))^{12} \equiv 1 + 4 \times 13\sqrt{-191} \pmod{13^2}$, we obtain, in $\mathbb{Q}_{13} \times \mathbb{Q}_{13}$, $\text{Log}(\mathfrak{l}) = \frac{1}{13} \text{Log}(\frac{1}{2}(153 + 7\sqrt{-191})) = (u, \log(2) - u)$, where u is a 13-adic unit.

Since $\text{Log}(P_k) = 13\mathbb{Z}_{13} \oplus 13\mathbb{Z}_{13}$, we obtain, from $I_k = \langle \mathfrak{l} \rangle P_k$, $(\mathbb{Z}_{13} \text{Log}(I_k) : \mathbb{Z}_{13} \text{Log}(P_k)) = 13$, which yields $\mathscr{T}_k = 1$ and $H_k^{\text{nr}} \subset \tilde{k}$; the p -Hilbert class field of k is contained in k^{acyc} .

(ii) $k = \mathbb{Q}(\sqrt{-383})$, $p = 17$. We have $\mathscr{H}_k = \langle \mathfrak{d}(\mathfrak{l}) \rangle$ of order 17, where $\mathfrak{l} \mid 2$. We have $\mathfrak{l}^{17} = \frac{1}{2}(711 + 7\sqrt{-383})$, and we obtain $(\mathbb{Z}_{17} \text{Log}(I_k) : \mathbb{Z}_{17} \text{Log}(P_k)) = 1$, because $\text{Log}(\mathfrak{l}) \in 17(\mathbb{Z}_{17} \oplus \mathbb{Z}_{17})$ since $(\frac{1}{2}(711 + 7\sqrt{-383}))^{16} \equiv 1 \pmod{17^2}$.

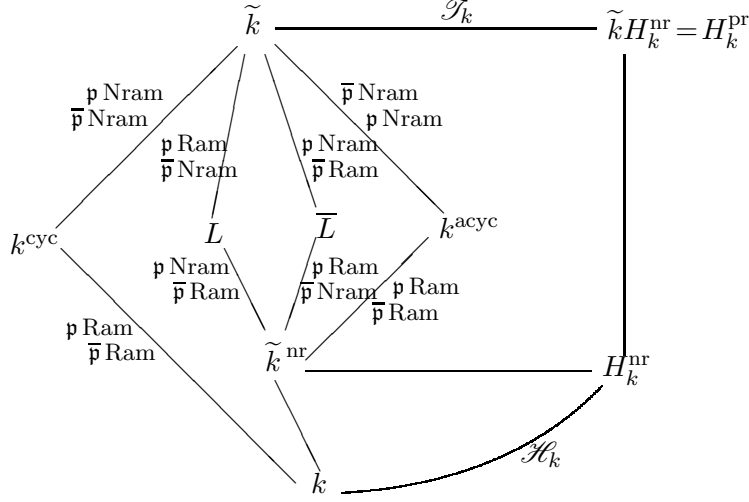
So, $\#\mathscr{T}_k = 17$ and $\tilde{k}^{\text{nr}} = k$; the p -Hilbert class field is linearly disjoint from \tilde{k} .

In the following diagram, ramification (resp. non-ramification) of \mathfrak{p} and $\bar{\mathfrak{p}}$ is made explicit with indications, as for instance \mathfrak{p} Ram (resp. \mathfrak{p} Nram).

From [Jau2022, Lemme 4] about totally p -adic number fields, the extension \tilde{k}/k^{cyc} is unramified at p , and the decomposition group of p in this extension (isomorphic to \mathbb{Z}_p) is of finite index in $\text{Gal}(\tilde{k}/k^{\text{cyc}})$; similarly, p does not ramify in $\tilde{k}/k^{\text{acyc}}$, and the decomposition group of p is of finite index. Then:

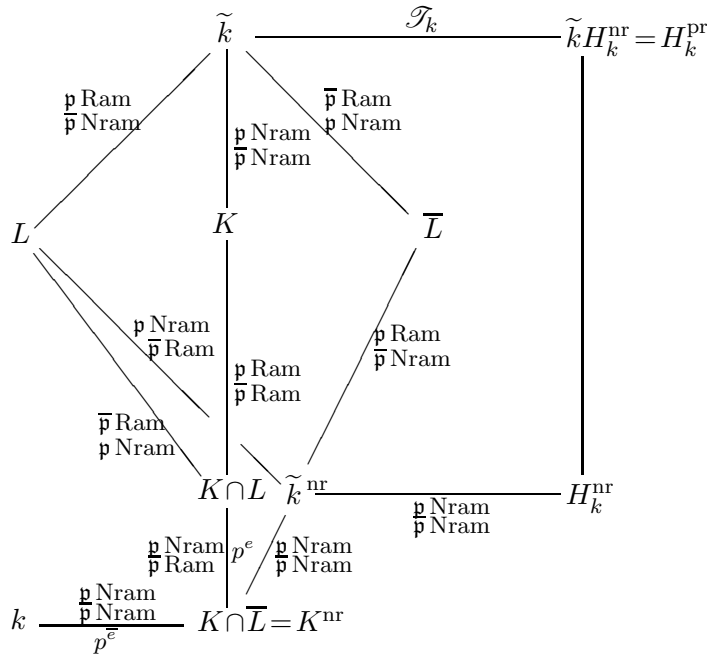
$$\text{Gal}(L/k) \simeq \mathbb{Z}_p \oplus T \quad \& \quad \text{Gal}(\bar{L}/k) \simeq \mathbb{Z}_p \oplus \bar{T},$$

where T and \bar{T} are finite cyclic groups; this corresponds to the case where $\text{Gal}(L/k)$ or/and $\text{Gal}(\bar{L}/k)$, of \mathbb{Z}_p -rank 1, are not free:



2. ANOTHER INVARIANT OF THE ABELIAN p -RAMIFICATION THEORY

2.1. General diagram including a \mathbb{Z}_p -extension K/k . We consider an arbitrary \mathbb{Z}_p -extension $K := \bigcup_{n \geq 0} K_n$, using the notation K_n for the layer of degree p^n over k ; this is necessary since we consider simultaneously other \mathbb{Z}_p -extensions of k , like $k^{\text{cyc}} = \bigcup_{n \geq 0} k_n^{\text{cyc}}$ and $k^{\text{acyc}} = \bigcup_{n \geq 0} k_n^{\text{acyc}}$. So, the notation k_∞ , usually equivalent to k^{cyc} , will not be used. We assume that K is not contained in L , nor in \bar{L} . Let $G_n := \text{Gal}(K_n/k) \simeq \mathbb{Z}/p^n\mathbb{Z}$:

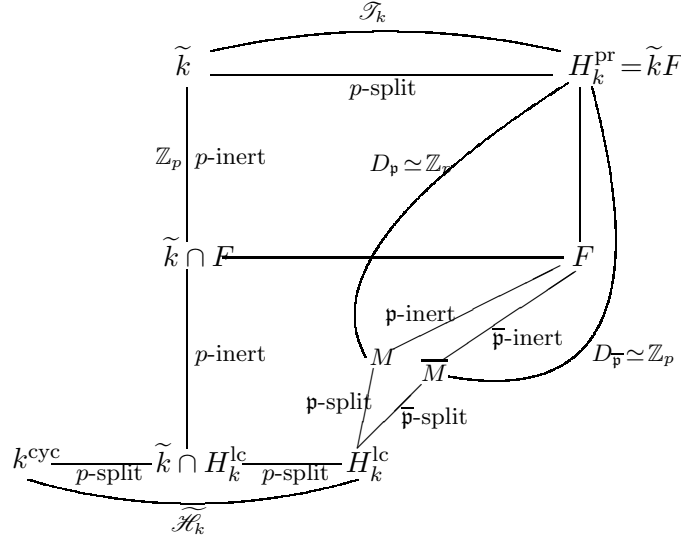


In the diagram, we have supposed, for instance, that $K \cap \bar{L} \subseteq K \cap L$, since \mathfrak{p} and $\bar{\mathfrak{p}}$ play symmetric roles by means of complex conjugation. Since $L \cap \bar{L} = \tilde{k}^{\text{nr}}$, we have $K \cap L \cap \bar{L} = K \cap \bar{L} = K \cap \tilde{k}^{\text{nr}}$ that we denote by K^{nr} , which is a subfield of \tilde{k}^{nr} , thus of k^{acyc} . We put $[K \cap \bar{L} : k] = p^{\bar{e}}$ & $[K \cap L : K \cap \bar{L}] = p^e$, whence:

$$(2.1) \quad \#\mathcal{H}_k = p^{\bar{e}} \times [\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{T}_k.$$

2.2. The Jaulent logarithmic class group $\widetilde{\mathcal{H}}_k$. For the definition and properties of $\widetilde{\mathcal{H}}_k$, see [Jau1986, Jau1994^b, Jau1998, BJ2016] and subsequent papers as [Jau2019^a], [Gra2021], about Greenberg's conjectures ([Gree1976] on Iwasawa's invariants in the totally real case, and [Gree1998] in more general setting).

An application in the case of totally p -adic imaginary quadratic fields is the following. Denote by $\mathcal{H}_{\tilde{k}}$ the p -class group of \tilde{k} , defined as the projective limit, for the norms, of the p -class groups \mathcal{H}_{F_n} , for finite extensions F_n/k such that $\tilde{k} = \bigcup_n F_n$. The field k being totally p -adic, the extension \tilde{k}/k^{cyc} is unramified [Jau2022, Lemme 4] (see also [JS1995] for similar context). Moreover, in [Jau2022, Proposition 12], a general result claims that $\mathcal{H}_{\tilde{k}} = 1$ if and only if the logarithmic class group $\widetilde{\mathcal{H}}_k$ is trivial. If so, this implies that the compositum $\tilde{k}H_{k_n^{\text{cyc}}}^{\text{nr}}$ is included in \tilde{k} , for all $n \geq 0$ such that $[k_n^{\text{cyc}} : k] = p^n$; since \tilde{k}/k^{cyc} is pro-cyclic, this proves that $\lambda_p(k^{\text{cyc}}/k) = 1$ and $\mu_p(k^{\text{cyc}}/k) = 0$ under the triviality of $\widetilde{\mathcal{H}}_k$. The reciprocal is true, under some conditions (Theorem 5.3). The following diagram introduces H_k^{lc} :



In the extension $H_k^{\text{pr}}/H_k^{\text{lc}}$ the decomposition groups $D_{\mathfrak{p}}$ and $D_{\bar{\mathfrak{p}}}$ are pro-cyclic groups fixing M and \bar{M} , respectively, such that $M \cap \bar{M} = H_k^{\text{lc}}$, $M\bar{M} = F \subset H_k^{\text{pr}}$.

The numerical examples given by the following PARI/GP program [Pari2013], may illustrate the above diagram; they show that \mathcal{T}_k (in $\text{Tor}_{\mathfrak{p}}$) may be trivial, contrary to \mathcal{H}_k (in $\text{H}_{\mathfrak{p}}$) and to the logarithmic class group $\widetilde{\mathcal{H}}_k$ (in Clog):

```
{n=8;for(m=3,10^4,if(core(m)!=m,next);k=bnfinit(x^2+m);Hk=k.clgp[2];
forprime(p=3,10^4,if(kronecker(-m,p)!=1,next);Clog=bnflog(k,p);
if(Clog[1]!=[],Kn=bnrinit(k,p^n);CKn=Kn.cyc;Tk=List;d=matsize(CKn)[2];
for(j=1,d-2,c=CKn[d-j+1];w=valuation(c,p);if(w>0,listinsert(Tk,p^w,1)));
Hkp=List;dh=matsize(Hk)[2];for(j=1,dh,c=Hk[j];w=valuation(c,p);
if(w>0,listinsert(Hkp,p^w,1));print("m=",m," p=",p," Clog=",Clog,
" Tor_p=",Tk," H_p=",Hkp))}}
```

m=3 p=13 Clog=[[13],[13],[]] Tor_p=[] H_p=[]

m=3	p=181	Clog=[[181], [181], []]	Tor_p=[]	H_p=[]
m=3	p=2521	Clog=[[2521], [2521], []]	Tor_p=[]	H_p=[]
m=5	p=5881	Clog=[[5881], [5881], []]	Tor_p=[]	H_p=[]
(...)				
m=107	p=3	Clog=[[9], [9], []]	Tor_p=[3]	H_p=[3]
m=166	p=5	Clog=[[5], [5], []]	Tor_p=[5]	H_p=[5]
m=239	p=3	Clog=[[3], [], [3]]	Tor_p=[]	H_p=[3]
m=302	p=3	Clog=[[3], [3], []]	Tor_p=[3]	H_p=[3]
m=362	p=3	Clog=[[3], [3], []]	Tor_p=[3]	H_p=[9]
m=374	p=7	Clog=[[49], [49], []]	Tor_p=[7]	H_p=[7]
m=383	p=17	Clog=[[17], [17], []]	Tor_p=[17]	H_p=[17]
m=413	p=3	Clog=[[9], [9], []]	Tor_p=[]	H_p=[]
m=974	p=3	Clog=[[3], [], [3]]	Tor_p=[3]	H_p=[3, 3]
m=998	p=13	Clog=[[13], [13], []]	Tor_p=[13]	H_p=[13]
m=1238	p=3	Clog=[[9], [3], [3]]	Tor_p=[]	H_p=[3]
m=1319	p=3	Clog=[[9], [9], []]	Tor_p=[9]	H_p=[9]
m=1409	p=3	Clog=[[3, 3], [3], [3]]	Tor_p=[3]	H_p=[9]
m=1967	p=3	Clog=[[9], [3], [3]]	Tor_p=[3]	H_p=[9]
m=2239	p=5	Clog=[[25], [5], [5]]	Tor_p=[]	H_p=[5]
m=2334	p=11	Clog=[[11], [11], []]	Tor_p=[11]	H_p=[11]
m=2351	p=3	Clog=[[729], [729], []]	Tor_p=[9]	H_p=[9]
m=2759	p=3	Clog=[[3], [3], []]	Tor_p=[3]	H_p=[27]
m=2915	p=3	Clog=[[243], [243], []]	Tor_p=[3]	H_p=[3]
m=2963	p=13	Clog=[[169], [169], []]	Tor_p=[13]	H_p=[13]
m=1987	p=163	Clog=[[26569], [26569], []]	Tor_p=[]	H_p=[]

Remark 2.1. *The structures of the invariants $\widetilde{\mathcal{H}}_k$, \mathcal{T}_k and \mathcal{H}_k are most often obvious from the data; but some examples are interesting as the following one:*

$$m = -78731 \quad p = 3 \quad \text{Clog} = [[6561, 3], [6561], [3]] \quad \text{Tor}_p = [27] \quad \text{H}_p = [27],$$

for which the class of \mathfrak{p} is of order 9; surprisingly, we note that $\mathfrak{p}^9 = \frac{1}{2}(1 + \sqrt{-78731})$, which is exceptional since, in general, coefficients may have thousands of digits. Since $\widetilde{\mathcal{H}}_k \simeq \mathbb{Z}/3^8\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, H_k^{lc} can not be contained in \widetilde{k} ; so $[H_k^{\text{lc}} : \widetilde{k} \cap H_k^{\text{lc}}] = 3^\rho$, $\rho \geq 1$, and the quotient $\widetilde{\mathcal{H}}_k/\text{Gal}(H_k^{\text{lc}}/\widetilde{k} \cap H_k^{\text{lc}})$ must be cyclic of order $3^{9-\rho}$.

Two cases may occur:

- $H_k^{\text{pr}} = \widetilde{k}H_k^{\text{lc}}$. In this case, $\rho = 3$, and $\text{Gal}(\widetilde{k} \cap H_k^{\text{lc}}/k^{\text{cyc}})$ must be a cyclic quotient of order 3^6 of $\mathbb{Z}/3^8\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} =: \langle a, b \rangle$ (a of order 3^8 , b of order 3); one checks that the unique solution is the quotient by $\langle a^{3^5} \cdot b \rangle$ (of order 3^3), giving a cyclic group of order 3^6 .

- $[H_k^{\text{pr}} : \widetilde{k}H_k^{\text{lc}}] \geq 3$. In this case, 3 can not be totally inert in $H_k^{\text{pr}}/H_k^{\text{lc}}$, since decomposition groups are pro-cyclic; so, this suggests that the two decomposition groups of \mathfrak{p} and $\bar{\mathfrak{p}}$ do not coincide in $\text{Gal}(H_k^{\text{pr}}/H_k^{\text{lc}})$, but are of index a suitable power of 3 (see the diagram where all possibilities are represented).

This situation does occur for all p ; for instance with $p = 5$ one gets the case:

$$m = -30911 \quad p = 5 \quad \text{Clog} = [[125, 5], [125], [5]] \quad \text{Tor}_p = [25] \quad \text{H}_p = [25].$$

Even if most examples give either cyclic H_k^{lc} or else bi-cyclic ones, we have found some examples of rank 3, restricting the search to $p = 3$; of course, this implies that \mathcal{T}_k is at least of rank 2. We do not repeat identical structures for the three invariants:

m=207143	Clog=[[9, 3, 3], [9], [3, 3]]	Tor_p=[3, 3]	H_p=[3, 3]
m=1654781	Clog=[[3, 3, 3], [3], [3, 3]]	Tor_p=[3, 3]	H_p=[9, 3]
m=2688977	Clog=[[9, 9, 3], [9], [9, 3]]	Tor_p=[9, 3]	H_p=[9, 3]
m=2750507	Clog=[[3, 3, 3], [3], [3, 3]]	Tor_p=[3, 3]	H_p=[3, 3]
m=3334937	Clog=[[9, 3, 3], [3], [9, 3]]	Tor_p=[3, 3]	H_p=[9, 3]
m=3527078	Clog=[[3, 3, 3], [3], [3, 3]]	Tor_p=[3, 3]	H_p=[3, 3]
m=4201313	Clog=[[3, 3, 3], [3], [3, 3]]	Tor_p=[3, 3]	H_p=[9, 3]

m=4455293	Clog=[[9,3,3],[9],[3,3]]	Tor_p=[3,3]	H_p=[3,3]
m=4996655	Clog=[[3,3,3],[3],[3,3]]	Tor_p=[3,3]	H_p=[243,3]
m=5176481	Clog=[[9,3,3],[3],[9,3]]	Tor_p=[9,3]	H_p=[9,3]
m=5546015	Clog=[[9,3,3],[9],[3,3]]	Tor_p=[9,3]	H_p=[27,3]
m=5894459	Clog=[[9,3,3],[3],[9,3]]	Tor_p=[3,3]	H_p=[27,3]
m=6493730	Clog=[[3,3,3],[3],[3,3]]	Tor_p=[3,3]	H_p=[27,3]
m=6740726	Clog=[[27,9,3],[9],[27,3]]	Tor_p=[27,3]	H_p=[27,3]
m=7054241	Clog=[[27,3,3],[27],[3,3]]	Tor_p=[3,3]	H_p=[3,3]
m=7347419	Clog=[[3,3,3],[3],[3,3]]	Tor_p=[3,3]	H_p=[81,3]
m=8596733	Clog=[[81,3,3],[81],[3,3]]	Tor_p=[3,3]	H_p=[3,3]
m=9098093	Clog=[[81,9,3],[81],[9,3]]	Tor_p=[9,3]	H_p=[9,3]
m=9483965	Clog=[[9,3,3],[3],[3,3,3]]	Tor_p=[3,3]	H_p=[3,3,3]
m=10446266	Clog=[[9,3,3],[3],[9,3]]	Tor_p=[9,3]	H_p=[9,3]
m=10566149	Clog=[[9,3,3],[9],[3,3]]	Tor_p=[9,3]	H_p=[9,3]
m=11458877	Clog=[[9,3,3],[3],[9,3]]	Tor_p=[3,3]	H_p=[9,3]
m=11584382	Clog=[[9,3,3],[9],[3,3]]	Tor_p=[9,3]	H_p=[81,3]
m=11702378	Clog=[[3,3,3],[3],[3,3]]	Tor_p=[3,3]	H_p=[81,3]
m=11896847	Clog=[[81,3,3],[27],[9,3]]	Tor_p=[3,3]	H_p=[9,3]
m=12160511	Clog=[[27,3,3],[27],[3,3]]	Tor_p=[3,3]	H_p=[3,3]

2.3. Logarithmic class group and Fermat quotients. Our purpose was to give a class field theory computation (from a method yet used in [Gra2017^b, Gra2019^a, Gra2019^b]) of the order of the Hasse norm residue symbols $\left(\frac{x, K_n/k}{\mathfrak{p}}\right)$ and $\left(\frac{x, K_n/k}{\bar{\mathfrak{p}}}\right)$ for any \mathbb{Z}_p -extension $K = \bigcup_n K_n$ of k , where the p -unit x is the generator of the minimal principal power \mathfrak{p}^h of $\mathfrak{p} \mid p$. We show, Section 3, that the order of these symbols especially depends on the integer $\delta_p(k)$ (see defining relations (1.1)) which intervenes in generalizations of the Herbrand–Chevalley formula, in direction of a Gold criterion. But the PARI/GP computations of the main p -adic invariants of k have shown a remarkable numerical coincidence. More precisely, consider, instead of the p -unit x generating \mathfrak{p}^h and related to $\delta_p(k)$, the p -unit X defined by $\mathfrak{p}^{h_k} =: (X)$, where h_k is the class number; then the valuation:

$$\tilde{\delta}_p(k) := v_{\mathfrak{p}}((X p^{-h_k})^{p-1} - 1) - 1 = v_{\mathfrak{p}}((x p^{-h})^{p-1} - 1) - 1 + [v_p(h_k) - v_p(h)],$$

of the Fermat quotient of X , giving $\tilde{\delta}_p(k) = \delta_p(k) + [v_p(h_k) - v_p(h)]$, is such that $p^{\tilde{\delta}_p(k)}$ is quite simply the order of the logarithmic class group $\widetilde{\mathcal{H}}_k$ (defined in Section 2.2) for totally p -adic imaginary quadratic fields.

Having all the techniques of the logarithmic class field theory [Jau1986, Jau1994^a, Jau1994^b, Jau1998, Jau2017] to prove this, Jaulent proposes a more general framework where this property holds for some totally p -adic imaginary abelian number fields (this is done in the Appendix A and will be developed in [Jau2024]); in our particular case, this result writes:

Theorem 2.2. *Let k be an imaginary quadratic field and let $p > 2$ be a prime number split in k . Let $X \in k^\times$ be the generator of the principal ideal \mathfrak{p}^{h_k} , where h_k is the class number of k , and let $\tilde{\delta}_p(k) \geq 0$, defined by $\tilde{\delta}_p(k) := \delta_p(k) + [v_p(h_k) - v_p(h)]$, where $h \mid h_k$ is the order of the class of \mathfrak{p} . Then $\#\widetilde{\mathcal{H}}_k = p^{\tilde{\delta}_p(k)}$.*

We give the following checking of this property, for $p = 3$ and a list of integers m that may be modified at will:

```
{p=3;n=12;M=List([14,17,107,239,302,362,413,974,1238,1319,1967,2351,
2759,2915,78731,4996655,8596733,9098093,14729261,15163271,16433942]);
for(i=1,21,m=M[i];if(core(m)!=m,next);if(kronecker(-m,p)!=1,next);
P=x^2+m;k=bnfinit(P);Clog=bnflog(k,p);H=k.clgp[2];hk=k.no;
D=divisors(hk);N=numdiv(hk);F=component(idealfactor(k,p),1)[1];
for(j=1,N,hh=D[j];Y=idealpow(k,F,hh);Q=bnfisprincipal(k,Y);
dx=matsize(Q[1])[1];S=0;for(j=1,dx,S=S+Q[1][j]);if(S!=0,next);
```

```

h=hh;break);vno=valuation(hk,p);vh=valuation(h,p);
X=Mod(Q[2][1]*k.zk[1]+Q[2][2]*k.zk[2],P);qx=X^(p-1)-1;z=norm(qx);
delta=valuation(z,p)-1;Delta=delta+vno-vh;Kn=bnrinit(k,p^n);CKn=Kn.cyc;
Tk=List;d=matsize(CKn)[2];for(j=1,d-2,c=CKn[d-j+1];w=valuation(c,p);
if(w>0,listinsert(Tk,p^w,1));Hk=List;dh=matsize(H)[2];
for(j=1,dh,c=H[dh+1-j];w=valuation(c,p);if(w>0,listinsert(Hk,p^w,1)));
print("m=",m," hk=",hk," h=",h," H_p=",Hk);
print(" delta(x)=", delta," delta(X)=", Delta," Tor_p=",Tk)}

```

m=14 hk=4	h=4	H_p=[]	delta(x)=1	delta(X)=1	Tor_p=[]
Clog=[[3],[3],[[]]					
m=17 hk=4	h=4	H_p=[]	delta(x)=0	delta(X)=0	Tor_p=[]
Clog=[[[]],[[]],[[]]					
m=107 hk=3	h=3	H_p=[3]	delta(x)=2	delta(X)=2	Tor_p=[3]
Clog=[[9],[9],[[]]					
m=239 hk=15	h=5	H_p=[3]	delta(x)=0	delta(X)=1	Tor_p=[]
Clog=[[3],[[]],[3]]					
m=302 hk=12	h=12	H_p=[3]	delta(x)=1	delta(X)=1	Tor_p=[3]
Clog=[[3],[3],[[]]					
m=362 hk=18	h=9	H_p=[9]	delta(x)=1	delta(X)=1	Tor_p=[3]
Clog=[[3],[3],[[]]					
m=413 hk=20	h=10	H_p=[]	delta(x)=2	delta(X)=2	Tor_p=[]
Clog=[[9],[9],[[]]					
m=974 hk=36	h=12	H_p=[3,3]	delta(x)=0	delta(X)=1	Tor_p=[3]
Clog=[[3],[[]],[3]]					
m=1238 hk=42	h=14	H_p=[3]	delta(x)=1	delta(X)=2	Tor_p=[]
Clog=[[9],[3],[3]]					
m=1319 hk=45	h=45	H_p=[9]	delta(x)=2	delta(X)=2	Tor_p=[9]
Clog=[[9],[9],[[]]					
m=1967 hk=36	h=12	H_p=[9]	delta(x)=1	delta(X)=2	Tor_p=[3]
Clog=[[9],[3],[3]]					
m=2351 hk=63	h=63	H_p=[9]	delta(x)=6	delta(X)=6	Tor_p=[9]
Clog=[[729],[729],[[]]					
m=2759 hk=54	h=54	H_p=[27]	delta(x)=1	delta(X)=1	Tor_p=[3]
Clog=[[3],[3],[[]]					
m=2915 hk=24	h=6	H_p=[3]	delta(x)=5	delta(X)=5	Tor_p=[3]
Clog=[[243],[243],[[]]					
m=78731 hk=108	h=9	H_p=[27]	delta(x)=8	delta(X)=9	Tor_p=[27]
Clog=[[6561,3],[6561],[3]]					
m=4996655 hk=2916	h=324	H_p=[243,3]	delta(x)=1	delta(X)=3	Tor_p=[3,3]
Clog=[[3,3,3],[3],[3,3]]					
m=8596733 hk=2664	h=74	H_p=[3,3]	delta(x)=4	delta(X)=6	Tor_p=[3,3]
Clog=[[81,3,3],[81],[3,3]]					
m=9098093 hk=2160	h=20	H_p=[9,3]	delta(x)=4	delta(X)=7	Tor_p=[9,3]
Clog=[[81,9,3],[81],[9,3]]					
m=14729261 hk=4158	h=462	H_p=[3,3,3]	delta(x)=1	delta(X)=3	Tor_p=[3,3,3]
Clog=[[3,3,3],[3],[3,3]]					
m=15163271 hk=4050	h=225	H_p=[27,3]	delta(x)=3	delta(X)=5	Tor_p=[27,3]
Clog=[[27,3,3],[27],[3,3]]					
m=16433942 hk=3168	h=176	H_p=[3,3]	delta(x)=4	delta(X)=6	Tor_p=[3,3]
Clog=[[81,3,3],[81],[3,3]]					

The successive maxima of $\tilde{\delta}_p(k)$, up to $m = 10^6$, are, for $p = 3$, then for $p = 5$:

```

{p=3;n=16;Dmax=0;for(m=1,10^9,if(core(m)!=m,next);if(kronecker(-m,p)!=1,next);
P=x^2+m;k=bnfinit(P);H=k.clgp[2];hk=k.no;F=component(idealfactor(k,p),1)[1];
Y=idealpow(k,F,hk);Q=bnfisprincipal(k,Y);vno=valuation(hk,p);vh=valuation(h,p);
X=Mod(Q[2][1]*k.zk[1]+Q[2][2]*k.zk[2],P);qx=X^(p-1)-1;z=norm(qx);
Delta=valuation(z,p)-1+vno-vh;if(Delta>Dmax,Dmax=Delta;Kn=bnrinit(k,p^n);
CKn=Kn.cyc;Tk=List;d=matsize(CKn)[2];for(j=1,d-2,c=CKn[d-j+1];w=valuation(c,p);
if(w>0,listinsert(Tk,p^w,1));Hk=List;dh=matsize(H)[2];
for(j=1,dh,c=H[dh+1-j];w=valuation(c,p);if(w>0,listinsert(Hk,p^w,1));

```

```

print("m=",m," delta(X)=", Delta," Tor_p=",Tk," Hkp=",Hk))}
p=3
m=14  delta(X)=1 Tor_p=[] Hkp=[] m=2351  delta(X)=6 Tor_p=[9] Hkp=[9]
m=41  delta(X)=3 Tor_p=[] Hkp=[] m=8693  delta(X)=7 Tor_p=[] Hkp=[3]
m=365 delta(X)=4 Tor_p=[] Hkp=[] m=26243 delta(X)=8 Tor_p=[] Hkp=[3]
m=971 delta(X)=5 Tor_p=[] Hkp=[3] m=30611 delta(X)=13 Tor_p=[] Hkp=[]
p=5
m=11  delta(X)=1 Tor_p=[] Hkp=[] m=16249 delta(X)=5 Tor_p=[5] Hkp=[5]
m=51  delta(X)=3 Tor_p=[] Hkp=[] m=8865  delta(X)=6 Tor_p=[] Hkp=[]
m=4821 delta(X)=4 Tor_p=[] Hkp=[] m=173611 delta(X)=9 Tor_p=[] Hkp=[5]

```

In order for the reader to verify these statements, we give, for any $p > 2$, a simplified program computing $\widetilde{\delta}_p(k)$ and the structure of $\widetilde{\mathcal{H}}_k$:

```

{p=3;for(m=2,10^5,if(core(m)!=m,next);if(kronecker(-m,p)!=1,next);P=x^2+m;
k=bnfinit(P);Clog=bnflog(k,p);hk=k.no;F=component(idealfactor(k,p),1)[1];
Y=idealpow(k,F,hk);Q=bnfisprincipal(k,Y);X=Q[2][1]*k.zk[1]+Q[2][2]*k.zk[2];
Delta=valuation(norm(Mod(X,P)^(p-1)-1),p)-1;
print("m=",m," hk=",hk," Clog=",Clog," delta(X)=", Delta)}

```

Remark 2.3. *The order of $\widetilde{\mathcal{H}}_k$, for $k = \mathbb{Q}(\sqrt{-m})$ and p split, is given by means of the valuation of the Fermat quotient of an algebraic number, but depending of p ; indeed, $\frac{1}{p} \log_p(\bar{x}) \sim \frac{1}{p}(\bar{x}^{p-1} - 1)$ is not a “standard” Fermat’s quotient, like the case of $\frac{1}{p} \log_p(a) \sim \frac{1}{p}(a^{p-1} - 1)$, for the fixed prime-to- p algebraic number a .*

Thus, at first glance, it seems difficult to apply the p -adic heuristics worked out in [Gra2016, Section 8] and to extend, to totally real number fields k , the conjecture [Jau2017, Conjecture 11] stated for the real quadratic case. In a numerical viewpoint, we see that for $k = \mathbb{Q}(\sqrt{-3})$, one finds the following large list of primes p , such that $\delta_p(k) \neq 0$, $p \in \{13, 181, 2521, 76543, 489061, 6811741, 1321442641, 18405321661\}$, but nothing else after 3 days of calculations. For $k = \mathbb{Q}(\sqrt{-5})$, one obtains the unique solution $p = 5881$, up to 10^9 . The case of $m = 47$, also giving many small solutions $p \in \{3, 17, 157, 1193, 1493, 1511, 19564087\}$, doesn’t give more up to 10^9 .

So this raises the question of the finiteness or not of the solutions p . If $x = u + v\sqrt{-m}$, then $\bar{x} = u - v\sqrt{-m} \equiv 2u \pmod{\mathfrak{p}^h}$, and $\delta_p(k)$ depends on the Fermat quotient q of $a = 2u$, when $h \geq 2$ (if $h = 1$ one computes the valuation of $(2u)^2q + 1$ (see Section 6)).

Such a non-zero valuation is in general related to probabilities $\frac{1}{p}$, the Borel–Cantelli heuristic giving a priori infiniteness; but the trace $2u$ is subject to the strong relationship $(u + v\sqrt{-m}) = \mathfrak{p}^h$, which does not define random coefficients contrary to the probability draw of a solution (u_0, v_0) of the congruence $U + V\sqrt{-m} \equiv 0 \pmod{\mathfrak{p}^h}$ (among \mathfrak{p}^h ones, characterized by only u_0 since $\sqrt{-m} \in \mathbb{Z}_p^\times$). So we may think that the corresponding probability for $\delta_p(k) \neq 0$ is much less than $\frac{1}{p}$.

Nevertheless, we suggest the following conjecture that we extend to any number field k , where $\widetilde{\mathcal{H}}_k(p)$ denotes the logarithmic class group for any prime p :

Conjecture 2.4. *For any number field k fulfilling, everywhere, the Leopoldt and Gross–Kuz’min conjectures, the set of primes p such that $\widetilde{\mathcal{H}}_k(p) \neq 1$ is finite.*

3. HASSE’S NORM RESIDUE SYMBOLS IN \widetilde{k}/k

Let K_n be the n th layer of a \mathbb{Z}_p -extension K/k . For $a \in k^\times$ and for any prime ideal $\mathfrak{q} \nmid p$, the Hasse norm residue symbol $\left(\frac{a, K_n/k}{\mathfrak{q}}\right)$ is given by the Artin symbol $\left(\frac{K_n/k}{\mathfrak{q}}\right)^{v_{\mathfrak{q}}(a)}$; but we need the less obvious symbol $\left(\frac{x, K_n/k}{\mathfrak{p}}\right)$ for $x \equiv 0 \pmod{\mathfrak{p}}$.

3.1. Computation of the symbol $(\frac{x, \tilde{k}/k}{\mathfrak{p}}$. Recall, from [Gra2005, II.4.4.3], a method to compute $(\frac{x, \tilde{k}/k}{\mathfrak{p}})$, for p split into $(p) = \mathfrak{p}\bar{\mathfrak{p}}$, when x is the fundamental \mathfrak{p} -unit of k . Put $\mathfrak{p}^h =: (x)$ for the order h of $\mathcal{O}(\mathfrak{p})$. Then, $x\bar{x} = p^h$ since $E_k = 1$. Of course, $(\frac{\bullet, \tilde{k}/k}{\bullet})$ will be considered as the pro-limit for $N \rightarrow \infty$ of $(\frac{\bullet, F_N/k}{\bullet})$, where F_N is the subfield of \tilde{k} of norm conductor $(p)^N$; this implies that $\tilde{k} = \bigcup_N F_N$, but any other system may be used; note that $\bigcup_N (K \cap F_N) = K$.

Let $y_{N,\mathfrak{p}} \in k^\times$ (called a \mathfrak{p} -associate of x) fulfilling the following conditions (where the modulus may be taken larger than the norm conductors):

- (i) $y_{N,\mathfrak{p}} x^{-1} \equiv 1 \pmod{\mathfrak{p}^N}$,
- (ii) $y_{N,\mathfrak{p}} \equiv 1 \pmod{\bar{\mathfrak{p}}^N}$.

For N large enough, $v_{\mathfrak{p}}(y_{N,\mathfrak{p}}) = v_{\mathfrak{p}}(x) = h$, where $v_{\mathfrak{p}}$ is the \mathfrak{p} -adic valuation.

Product formula for $y_{N,\mathfrak{p}}$ yields $(\frac{y_{N,\mathfrak{p}}, F_N/k}{\mathfrak{p}}) = \prod_{\mathfrak{q}, \mathfrak{q} \neq \mathfrak{p}} (\frac{y_{N,\mathfrak{p}}, F_N/k}{\mathfrak{q}})^{-1}$, and since $(\frac{x, F_N/k}{\mathfrak{p}}) = (\frac{y_{N,\mathfrak{p}}, F_N/k}{\mathfrak{p}})$ from (i), by definition of the \mathfrak{p} -conductor of F_N/k , then $(\frac{x, F_N/k}{\mathfrak{p}}) = \prod_{\mathfrak{q}, \mathfrak{q} \neq \mathfrak{p}} (\frac{y_{N,\mathfrak{p}}, F_N/k}{\mathfrak{q}})^{-1}$. Let's compute the symbols of this product:

- If $\mathfrak{q} = \bar{\mathfrak{p}}$, since $y_{N,\mathfrak{p}} \equiv 1 \pmod{\bar{\mathfrak{p}}^N}$, one has $(\frac{y_{N,\mathfrak{p}}, F_N/k}{\bar{\mathfrak{p}}}) = 1$,
- If $\mathfrak{q} \nmid p$, in this case, $(\frac{y_{N,\mathfrak{p}}, F_N/k}{\mathfrak{q}}) = (\frac{F_N/k}{\mathfrak{q}})^{v_{\mathfrak{q}}(y_{N,\mathfrak{p}})}$, where $(\frac{F_N/k}{\mathfrak{q}})$ is the Frobenius automorphism of \mathfrak{q} .

Finally, $(\frac{x, F_N/k}{\mathfrak{p}}) = \prod_{\mathfrak{q} \nmid p} (\frac{F_N/k}{\mathfrak{q}})^{-v_{\mathfrak{q}}(y_{N,\mathfrak{p}})} =: (\frac{F_N/k}{\mathfrak{a}_{N,\mathfrak{p}}})^{-1}$, inverse of the Artin symbol of the principal ideal:

$$(3.1) \quad \mathfrak{a}_{N,\mathfrak{p}} := \prod_{\mathfrak{q} \nmid p} \mathfrak{q}^{v_{\mathfrak{q}}(y_{N,\mathfrak{p}})} = (y_{N,\mathfrak{p}}) \mathfrak{p}^{-h} = (y_{N,\mathfrak{p}} \cdot x^{-1}).$$

3.2. Computation of the order of $(\frac{x, K_n/k}{\mathfrak{p}})$ and $(\frac{x, K_n/k}{\bar{\mathfrak{p}}})$. We will compute the order of $(\frac{x, K_n/k}{\mathfrak{p}})$ and $(\frac{x, K_n/k}{\bar{\mathfrak{p}}})$, regarding n large enough. We only assume that K is not included in L or in \bar{L} (otherwise, a single p -place does ramify in K/k and all is known in this case). The principle is to show that these orders can be deduced from that of $(\frac{x, k_n^{\text{cyc}}/k}{\mathfrak{p}})$. The following lemma does simplify the relations between all possible choices for $x, \bar{x}, \mathfrak{p}, \bar{\mathfrak{p}}$ and the corresponding symbols:

Lemma 3.1. *One has $(\frac{p, k^{\text{cyc}}/k}{\mathfrak{p}}) = (\frac{p, k^{\text{cyc}}/k}{\bar{\mathfrak{p}}}) = 1$. Which implies, using the conjugation of symbols, $(\frac{\bar{x}, k^{\text{cyc}}/k}{\mathfrak{p}}) = (\frac{x, k^{\text{cyc}}/k}{\mathfrak{p}})^{-1} = (\frac{\bar{x}, k^{\text{cyc}}/k}{\bar{\mathfrak{p}}}) = (\frac{x, k^{\text{cyc}}/k}{\bar{\mathfrak{p}}})^{-1}$.*

Proof. In the cyclotomic extension $\mathbb{Q}(\mu_{p^{n+1}})/\mathbb{Q}$, $p = \mathbf{N}_{\mathbb{Q}(\mu_{p^{n+1}})/\mathbb{Q}}(1 - \zeta_{p^{n+1}})$, for all $n \geq 0$, whence p local norm at \mathfrak{p} and $\bar{\mathfrak{p}}$ in k^{cyc}/k , and, since $x\bar{x} = p^h$, the relations follow, showing equality of the four symbols or inverses. \square

An interesting remark is that the norm property of p corresponds, in some sense, to the convention $\log(p) = 0$ in Definitions 1.1, and elucidates the case where the valuation $v_{\mathfrak{p}}(x^{p-1} - 1) - 1$ is not defined from an usual Fermat quotient.

Lemma 3.2. *For $n \geq e + \bar{e}$, the order of $(\frac{x, K_n/k}{\mathfrak{p}})$ is that of $(\frac{x, k_{n-e-\bar{e}}^{\text{cyc}}/k}{\mathfrak{p}})$, and for $n \geq \bar{e}$, the order of $(\frac{x, K_n/k}{\bar{\mathfrak{p}}})$ is that of $(\frac{x, k_{n-\bar{e}}^{\text{cyc}}/k}{\bar{\mathfrak{p}}})$.*

Proof. We consider the following diagrams, where \tilde{k} is the direct compositum over k (resp. over $K \cap L$) of L and k^{cyc} (resp. of L and K), and similarly, \bar{k} is the direct compositum over k (resp. over $K \cap \bar{L}$) of \bar{L} and k^{cyc} (resp. of \bar{L} and K):

$$\text{Symbol } \left(\frac{\bar{x}, \tilde{k}/k}{\mathfrak{p}} \right): \quad \begin{array}{ccccc} L & \xrightarrow[p \text{ Ram}]{p^m} & L_m & \xrightarrow[p \text{ Ram}}{\quad} & \tilde{k} \\ \downarrow \mathfrak{p} \text{ Nram} & & \downarrow & & \downarrow \mathbb{Z}_p \\ k & \xrightarrow{\quad} & k_m^{\text{cyc}} & \xrightarrow{\quad} & k^{\text{cyc}} \end{array}$$

$$\text{Symbol } \left(\frac{\bar{x}, \tilde{k}/k}{\mathfrak{p}} \right): \quad \begin{array}{ccccc} L & \xrightarrow[p \text{ Ram}]{p^{n-e-\bar{e}}} & L_{n-e-\bar{e}} & \xrightarrow[p \text{ Ram}}{\quad} & \tilde{k} \\ \downarrow \mathfrak{p} \text{ Nram} & & \downarrow & & \downarrow \mathbb{Z}_p \\ k & \xrightarrow[p^{e+\bar{e}}]{\mathfrak{p} \text{ Nram}} & K \cap L & \xrightarrow{\quad} & K_n & \xrightarrow{\quad} & K \end{array}$$

$$\text{Symbol } \left(\frac{x, \bar{k}/k}{\bar{\mathfrak{p}}} \right): \quad \begin{array}{ccccc} \bar{L} & \xrightarrow[\bar{\mathfrak{p}} \text{ Ram}]{p^m} & \bar{L}_m & \xrightarrow[\bar{\mathfrak{p}} \text{ Ram}}{\quad} & \bar{k} \\ \downarrow \bar{\mathfrak{p}} \text{ Nram} & & \downarrow & & \downarrow \mathbb{Z}_p \\ k & \xrightarrow{\quad} & k_m^{\text{cyc}} & \xrightarrow{\quad} & k^{\text{cyc}} \end{array}$$

$$\text{Symbol } \left(\frac{x, \bar{k}/k}{\bar{\mathfrak{p}}} \right): \quad \begin{array}{ccccc} \bar{L} & \xrightarrow[\bar{\mathfrak{p}} \text{ Ram}]{p^{n-\bar{e}}} & \bar{L}_{n-\bar{e}} & \xrightarrow[\bar{\mathfrak{p}} \text{ Ram}}{\quad} & \bar{k} \\ \downarrow \bar{\mathfrak{p}} \text{ Nram} & & \downarrow & & \downarrow \mathbb{Z}_p \\ k & \xrightarrow[p^e]{\bar{\mathfrak{p}} \text{ Nram}} & K \cap \bar{L} & \xrightarrow{\quad} & K_n & \xrightarrow{\quad} & K \end{array}$$

In the first and third diagram, $k_m^{\text{cyc}} = k\mathbb{Q}_m^{\text{cyc}}$, where $\mathbb{Q}_m^{\text{cyc}}$ is of conductor p^{m+1} .

In the second and fourth diagram, since $[K_n : k] = p^n$ and $[K \cap L : k] = p^{e+\bar{e}}$ (resp. $[K \cap \bar{L} : k] = p^{\bar{e}}$), it follows that $[LK_n : L] = p^{n-e-\bar{e}}$ (resp. $[\bar{L}K_n : \bar{L}] = p^{n-\bar{e}}$). Finally, $L_m = L_{n-e-\bar{e}}$ for $n = m + e + \bar{e}$ (resp. $\bar{L}_m = \bar{L}_{n-\bar{e}}$ for $n = m + \bar{e}$).

Recall that \mathfrak{p} (resp. $\bar{\mathfrak{p}}$) is totally ramified in $K/K \cap L$ (resp. $K/K \cap \bar{L}$) and unramified in $L/K \cap L$ (resp. $\bar{L}/K \cap \bar{L}$). So:

$\left(\frac{\bar{x}, \tilde{k}/k}{\mathfrak{p}} \right)$ fixes L since \bar{x} is a local unit in an extension unramified at \mathfrak{p} ;

$\left(\frac{\bar{x}, k^{\text{cyc}}/k}{\mathfrak{p}} \right) \in \text{Gal}(k^{\text{cyc}}/k)$, by restriction;

$\left(\frac{\bar{x}, K/k}{\mathfrak{p}} \right) \in \text{Gal}(K/K \cap L)$, by restriction of $\left(\frac{\bar{x}, \tilde{k}/k}{\mathfrak{p}} \right) \in \text{Gal}(\tilde{k}/L)$;

$\left(\frac{x, \bar{k}/k}{\bar{\mathfrak{p}}} \right)$ fixes \bar{L} since x is a local unit in an extension unramified at $\bar{\mathfrak{p}}$;

$\left(\frac{x, k^{\text{cyc}}/k}{\bar{\mathfrak{p}}} \right) \in \text{Gal}(k^{\text{cyc}}/k)$, by restriction;

$\left(\frac{x, K/k}{\bar{\mathfrak{p}}} \right) \in \text{Gal}(K/K \cap \bar{L})$, by restriction of $\left(\frac{x, \bar{k}/k}{\bar{\mathfrak{p}}} \right) \in \text{Gal}(\bar{k}/\bar{L})$.

Since these restrictions are isomorphisms, one obtains successively that the order of $\left(\frac{\bar{x}, k_m^{\text{cyc}}/k}{\mathfrak{p}} \right)$ is that of $\left(\frac{\bar{x}, L_m/k}{\mathfrak{p}} \right)$, and that the order of $\left(\frac{\bar{x}, K_n/k}{\mathfrak{p}} \right)$ is that of $\left(\frac{\bar{x}, L_{n-e-\bar{e}}/k}{\mathfrak{p}} \right)$. For $n = m + e + \bar{e}$, we get that the order of $\left(\frac{\bar{x}, K_n/k}{\mathfrak{p}} \right)$ is that of

$\left(\frac{\bar{x}, k_{n-e-\bar{e}}^{\text{cyc}}/k}{\mathfrak{p}}\right)$, whence that of $\left(\frac{x, k_{n-e-\bar{e}}^{\text{cyc}}/k}{\mathfrak{p}}\right)$ from Lemma 3.1. Similarly, the order of $\left(\frac{x, K_n/k}{\bar{\mathfrak{p}}}\right)$ is that of $\left(\frac{x, k_{n-\bar{e}}^{\text{cyc}}/k}{\bar{\mathfrak{p}}}\right)$ and that of $\left(\frac{\bar{x}, k_{n-\bar{e}}^{\text{cyc}}/k}{\bar{\mathfrak{p}}}\right)$. \square

But the orders of the symbols in k_m^{cyc}/k are easily computable:

Theorem 3.3. *Let k be an imaginary quadratic field and let $p > 2$ be a prime split in k . Let $x \in k^\times$ be the generator of \mathfrak{p}^h , where h is the order of $\mathfrak{cl}(\mathfrak{p})$, and let $\delta_p(k) := v_{\mathfrak{p}}((xp^{-h})^{p-1} - 1) - 1$. Let K be a \mathbb{Z}_p -extension of k such that \mathfrak{p} and $\bar{\mathfrak{p}}$ are totally ramified from some layer. The symbols $\left(\frac{x, K_n/k}{\mathfrak{p}}\right)$ and $\left(\frac{x, K_n/k}{\bar{\mathfrak{p}}}\right)$ are of orders $p^{n-e-\bar{e}-\delta_p(k)}$ and $p^{n-\bar{e}-\delta_p(k)}$, respectively.*

Proof. For this, it suffices to compute, for any integer m , the order of the Artin symbol $\left(\frac{k_m^{\text{cyc}}/k}{\mathfrak{a}_{N,\mathfrak{p}}}\right) \in \text{Gal}(k_m^{\text{cyc}}/k)$ obtained in expression (3.1), for N large enough, regarding the m th layer k_m^{cyc} . It is equivalent to compute the order of the canonical image of $\left(\frac{k_m^{\text{cyc}}/k}{\mathfrak{a}_{N,\mathfrak{p}}}\right)$ in $\text{Gal}(\mathbb{Q}_m^{\text{cyc}}/\mathbb{Q}) \simeq \{a \in (\mathbb{Z}/p^{m+1}\mathbb{Z})^\times, a \equiv 1 \pmod{p}\} \simeq \mathbb{Z}/p^m\mathbb{Z}$, which is the Artin symbol $\left(\frac{\mathbb{Q}_m^{\text{cyc}}/\mathbb{Q}}{\mathbf{N}_{k/\mathbb{Q}}(\mathfrak{a}_{N,\mathfrak{p}})}\right)$, whose order is that of the Artin symbol of $\mathbf{N}_{k/\mathbb{Q}}(\mathfrak{a}_{N,\mathfrak{p}})^{p-1}$. Before, let's prove that $\mathfrak{a}_{N,\mathfrak{p}} = (y_{N,\mathfrak{p}}) \cdot \mathfrak{p}^{-h}$ is such that:

$$\mathbf{N}_{k/\mathbb{Q}}(\mathfrak{a}_{N,\mathfrak{p}}) \equiv x \cdot p^{-h} \pmod{\mathfrak{p}^{N+1}}.$$

Indeed, one has $\mathbf{N}_{k/\mathbb{Q}}(\mathfrak{a}_{N,\mathfrak{p}}) = \mathbf{N}_{k/\mathbb{Q}}(y_{N,\mathfrak{p}}) \cdot p^{-h}$; by definition of an associate, $y_{N,\mathfrak{p}}x^{-1} \equiv 1 \pmod{\mathfrak{p}^{N+1}}$, and $y_{N,\mathfrak{p}} \equiv 1 \pmod{\bar{\mathfrak{p}}^{N+1}}$, thus $\bar{y}_{N,\mathfrak{p}} \equiv 1 \pmod{\mathfrak{p}^{N+1}}$, which leads to $\mathbf{N}_{k/\mathbb{Q}}(y_{N,\mathfrak{p}}) \cdot x^{-1} \equiv 1 \pmod{\mathfrak{p}^{N+1}}$. Finally, $\mathbf{N}_{k/\mathbb{Q}}(\mathfrak{a}_{N,\mathfrak{p}}) \equiv x \cdot p^{-h} \pmod{\mathfrak{p}^{N+1}}$, whence $v_{\mathfrak{p}}(\mathbf{N}_{k/\mathbb{Q}}(\mathfrak{a}_{N,\mathfrak{p}})^{p-1} - 1) = v_{\mathfrak{p}}((x \cdot p^{-h})^{p-1} - 1) = \delta_p(k) + 1$, for all N large enough; since $\mathbf{N}_{k/\mathbb{Q}}(\mathfrak{a}_{N,\mathfrak{p}})$ is rational, this gives the order of $\left(\frac{k_m^{\text{cyc}}/k}{\mathfrak{a}_{N,\mathfrak{p}}}\right)$ in $\text{Gal}(k_m^{\text{cyc}}/k)$, as soon as m is strictly larger than $\delta_p(k)$, and this order is $p^{m-\delta_p(k)}$. \square

4. CHEVALLEY–HERBRAND FORMULA IN K_n/k

4.1. Expression of $\#\mathcal{H}_{K_n}^{G_n}$. Let K be a \mathbb{Z}_p -extension not contained in L or \bar{L} (whence K/k totally ramified at p from some layer of K). We have, from the general diagram 2.1, $[K \cap \bar{L} : k] := p^{\bar{e}}$, $[K \cap L : k] := p^{e+\bar{e}}$; in other words, $K \cap \bar{L} = K_{\bar{e}}$ and $K \cap L = K_{e+\bar{e}}$. Moreover, we have, from expression (2.1):

$$\#\mathcal{H}_k = p^{\bar{e}} \times [\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{T}_k.$$

Recall the Chevalley–Herbrand formula [Che1933, pp. 402-406] for the extensions K_n/k , of Galois groups G_n , and for the p -class group \mathcal{H}_{K_n} of K_n , $n \geq e + \bar{e}$:

$$\#\mathcal{H}_{K_n}^{G_n} = \frac{\#\mathcal{H}_k \times \prod_v p^{n\nu}}{[K_n : k] \times (E_k : E_k \cap \mathbf{N}_{K_n/k}(K_n^\times))},$$

where $p^{n\nu}$ is the ramification index of the place v (whence that of \mathfrak{p} , $\bar{\mathfrak{p}}$, of ramification indices $p^{n-e-\bar{e}}$, $p^{n-\bar{e}}$, respectively), and E_k the group of units of k (equal to 1).

Using formula (2.1) gives the final expression for all $n \geq e + \bar{e}$:

$$(4.1) \quad \#\mathcal{H}_{K_n}^{G_n} = \frac{\#\mathcal{H}_k \times p^{n-\bar{e}} p^{n-e-\bar{e}}}{p^n} = [\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{T}_k \times p^{n-e-\bar{e}},$$

then immediately the well-known result that (for K not contained in L or \bar{L}), $\#\mathcal{H}_{K_n}$ is unbounded, whence $\lambda_p(K/k) \geq 1$ or $\mu_p(K/k) \neq 0$.

4.2. Genus theory and filtration of \mathcal{H}_{K_n} . We still assume K not contained in L or \bar{L} . Denote by $G_{n,\mathfrak{p}}$ and $G_{n,\bar{\mathfrak{p}}}$ the inertia groups in K_n/k , of \mathfrak{p} and $\bar{\mathfrak{p}}$, for all $n \geq 0$; of course, $G_{n,\mathfrak{p}} = \text{Gal}(K_n/K_{e+\bar{e}})$ and $G_{n,\bar{\mathfrak{p}}} = \text{Gal}(K_n/K_{\bar{e}})$ only for $n \geq e + \bar{e}$.

We define, for all $n \geq 0$:

$$\Omega_{K_n/k} := \{(s, \bar{s}) \in G_{n,\mathfrak{p}} \times G_{n,\bar{\mathfrak{p}}}, s \cdot \bar{s} = 1\};$$

since $G_{n,\mathfrak{p}} \subseteq G_{n,\bar{\mathfrak{p}}}$, $\Omega_{K_n/k}$ is cyclic (of order $p^{n-e-\bar{e}}$ if $n \geq e + \bar{e}$), and we consider the map $\omega_{K_n/k} : k^\times \rightarrow \Omega_{K_n/k}$, defined on the set of elements $a \in k^\times$, assumed to be local norm in K_n/k for all place $v \nmid p$, by:

$$\omega_{K_n/k}(a) = \left(\left(\frac{a, K_n/k}{\mathfrak{p}} \right), \left(\frac{a, K_n/k}{\bar{\mathfrak{p}}} \right) \right) \in \Omega_{K_n/k},$$

since these symbols fulfill the product formula of class field theory.

The basic principle from Chevalley–Herbrand formula for p -class groups is to define a filtration of the form $\{\mathcal{H}_{K_n}^i\}_{i \geq 0}$, where $\mathcal{H}_{K_n}^0 := 1$, and:

$$\mathcal{H}_{K_n}^1 := \mathcal{H}_{K_n}^{G_n}, \quad \mathcal{H}_{K_n}^{i+1}/\mathcal{H}_{K_n}^i := (\mathcal{H}_{K_n}/\mathcal{H}_{K_n}^i)^{G_n},$$

up to a minimal bound $i = b_n$ for which $\mathcal{H}_{K_n}^{b_n} = \mathcal{H}_{K_n}$ (see the improved english translation [Gra2017^a] of our 1994's paper, and its transcript into the idelic form in [LiYu2020]); from (4.1) and the formula giving the orders of the $\mathcal{H}_{K_n}^{i+1}/\mathcal{H}_{K_n}^i$'s, we get more precisely, where $\#\mathbf{N}_{K_n/k}(\mathcal{H}_{K_n}) = [H_k^{\text{nr}} : K_n \cap H_k^{\text{nr}}]$ takes into account the non-ramified layers for small values of n :

$$(4.2) \quad \left\{ \begin{array}{l} \#\mathcal{H}_{K_n}^1 = \#\mathcal{H}_{K_n}^{G_n} = \#\mathbf{N}_{K_n/k}(\mathcal{H}_{K_n}) \times \#\Omega_{K_n/k}, \\ \quad \left[= [\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{T}_k \times p^{n-e-\bar{e}}, \text{ if } n \geq e + \bar{e} \right], \\ \#(\mathcal{H}_{K_n}^{i+1}/\mathcal{H}_{K_n}^i) = \frac{\#\mathbf{N}_{K_n/k}(\mathcal{H}_{K_n})}{\#\mathbf{N}_{K_n/k}(\mathcal{H}_{K_n}^i)} \times \frac{\#\Omega_{K_n/k}}{\#\omega_{K_n/k}(\Lambda_{K_n/k}^i)}, \quad i \geq 1, \\ \quad \left[= [\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{T}_k \times \frac{p^{n-e-\bar{e}}}{\#\omega_{K_n/k}(\Lambda_{K_n/k}^i)}, \text{ if } n \geq e + \bar{e} \right], \end{array} \right.$$

where $\Lambda_{K_n/k}^i := \{a \in k^\times, (a) = \mathbf{N}_{K_n/k}(\mathfrak{A}_n), \mathfrak{d}(\mathfrak{A}_n) \in \mathcal{H}_{K_n}^i\}$ is a subgroup of finite type of k^\times , so that $\omega_{K_n/k}(\Lambda_{K_n/k}^i) \subseteq \Omega_{K_n/k}$ from the product formula.

4.3. Properties of the filtration. It is the behavior of the filtrations, regarding $n \rightarrow \infty$, and the number of steps b_n , which determines the parameters $\lambda_p(K/k)$, $\mu_p(K/k)$; for this we will recall some general properties that have also been used in [Gra2017^b, Gra2019^b, Gra2021] in another context.

To ease notations, put $\mathcal{H}_n := \mathcal{H}_{K_n}$, $\Lambda_n^i := \Lambda_{K_n/k}^i$, $\mathbf{N}_n := \mathbf{N}_{K_n/k}$, and so on; let σ_n be a generator of G_n ; note that k may be seen as K_0 giving for instance $\mathcal{H}_k = \mathcal{H}_0$. The first obvious properties are, for n fixed:

- (i) $\mathcal{H}_n^i = \{c \in \mathcal{H}_n, c^{(1-\sigma_n)^i} = 1\}$, for all $i \geq 0$.
- (ii) $\#(\mathcal{H}_n^{i+1}/\mathcal{H}_n^i) \leq \#\mathcal{H}_n^1$ for all $i \geq 0$.

(iii) The i -sequence $\#(\mathcal{H}_n^{i+1}/\mathcal{H}_n^i)$, $0 \leq i \leq b_n$, is decreasing to 1 since $1 - \sigma_n$ defines the injections $\mathcal{H}_n^{i+1}/\mathcal{H}_n^i \hookrightarrow \mathcal{H}_n^i/\mathcal{H}_n^{i-1}$, for all $i \geq 1$.

Remark 4.1. Set $\mathcal{H}_n^i = \mathfrak{d}_n(\mathcal{I}_n^i)$ for an ideal group \mathcal{I}_n^i of K_n ; then $\mathbf{N}_n(\mathcal{I}_n^i)$ allows to find solutions $\alpha = \mathbf{N}_n(\beta) \in \Lambda_n^i \cap \mathbf{N}_n(K_n^\times)$, by definition such that $(\alpha) = \mathbf{N}_n(\mathfrak{A})$, $\mathfrak{A} \in \mathcal{I}_n^i$, yielding $(\beta) = \mathfrak{B}^{1-\sigma_n} \cdot \mathfrak{A}$ to create $\mathfrak{B} \in \mathcal{I}_n^{i+1}$ and $\mathfrak{d}(\mathbf{N}_n(\mathfrak{B}))$ in \mathcal{H}_k (whence building $\mathcal{H}_n^{i+1} = \mathfrak{d}(\mathcal{I}_n^{i+1})$ and $\mathbf{N}_n(\mathcal{H}_n^{i+1})$ for the next step of the algorithm).

We have, for all n , the following diagram where norms \mathbf{N}_{K_{n+1}/K_n} , on \mathcal{H}_{n+1} and $(\mathcal{H}_{n+1})^{(1-\sigma_{n+1})^i}$, are onto; on \mathcal{H}_{n+1}^i , the norm may be not surjective nor injective:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{H}_{n+1}^i & \longrightarrow & \mathcal{H}_{n+1} & \xrightarrow{(1-\sigma_{n+1})^i} & (\mathcal{H}_{n+1})^{(1-\sigma_{n+1})^i} \longrightarrow 1 \\ & & \downarrow & & \mathbf{N}_{K_{n+1}/K_n} \downarrow & & \mathbf{N}_{K_{n+1}/K_n} \downarrow \\ 1 & \longrightarrow & \mathcal{H}_n^i & \longrightarrow & \mathcal{H}_n & \xrightarrow{(1-\sigma_n)^i} & (\mathcal{H}_n)^{(1-\sigma_n)^i} \longrightarrow 1. \end{array}$$

One has $\mathbf{N}_{K_{n+1}/K_n}(\mathcal{H}_{n+1}^i) \subseteq \mathcal{H}_n^i$, and one may assume, modifying \mathcal{I}_n^i modulo principal ideals (which does not modify norm indices), that $\mathbf{N}_{K_{n+1}/K_n}(\mathcal{I}_{n+1}^i) \subseteq \mathcal{I}_n^i$, implying $\mathbf{\Lambda}_{n+1}^i \subseteq \mathbf{\Lambda}_n^i$ from $\mathbf{N}_{K_{n+1}/k} = \mathbf{N}_{K_n/k} \circ \mathbf{N}_{K_{n+1}/K_n}$.

We assume now that p splits in k and that K/k is any \mathbb{Z}_p -extension not contained in L or \bar{L} ; this means that \mathfrak{p} and $\bar{\mathfrak{p}}$ totally ramify from some layer in K/k .

Lemma 4.2. *For $i \geq 0$ fixed, and $n \geq e + \bar{e}$, the $\#(\mathcal{H}_n^{i+1}/\mathcal{H}_n^i)$'s and the $\#\mathcal{H}_n^i$'s define stationary increasing n -sequences.*

Proof. Consider, for $i \geq 0$ fixed, the n -sequence defined, for $n \geq e + \bar{e}$, by:

$$(4.3) \quad \#(\mathcal{H}_n^{i+1}/\mathcal{H}_n^i) = \frac{[H_k^{\text{nr}} : K^{\text{nr}}]}{\#\mathbf{N}_n(\mathcal{H}_n^i)} \cdot \frac{p^{n-e-\bar{e}}}{\omega_n(\mathbf{\Lambda}_n^i)}.$$

The first factor is the ‘‘class factor’’ and the second factor is the ‘‘norm factor’’. We get $\mathbf{N}_{n+1}(\mathcal{H}_{n+1}^i) \subseteq \mathbf{N}_n(\mathcal{H}_n^i)$; thus, the n -sequence $\frac{[H_k^{\text{nr}} : K^{\text{nr}}]}{\#\mathbf{N}_n(\mathcal{H}_n^i)} =: p^{c_n^i}$ is increasing, stationary from a maximal value $p^{c_\infty^i}$ dividing $[H_k^{\text{nr}} : K^{\text{nr}}]$. The norm factors $\frac{p^{n-e-\bar{e}}}{\#\omega_n(\mathbf{\Lambda}_n^i)} =: p^{\rho_n^i}$ define an increasing n -sequence. Indeed, $p^{\rho_{n+1}^i - \rho_n^i} = p \cdot \frac{\#\omega_n(\mathbf{\Lambda}_n^i)}{\#\omega_{n+1}(\mathbf{\Lambda}_{n+1}^i)}$, and using Remark 4.1, one gets $\mathbf{\Lambda}_{n+1}^i \subseteq \mathbf{\Lambda}_n^i$, then $\#\omega_{n+1}(\mathbf{\Lambda}_{n+1}^i) \leq \#\omega_{n+1}(\mathbf{\Lambda}_n^i)$, and $p^{\rho_{n+1}^i - \rho_n^i} \geq p \cdot \frac{\#\omega_n(\mathbf{\Lambda}_n^i)}{\#\omega_{n+1}(\mathbf{\Lambda}_n^i)}$; under the restriction of Hasse’s norm residue symbols, one gets the map $\Omega_{n+1} \longrightarrow \Omega_n$ (whose kernel is of order p), the image of $\omega_{n+1}(\mathbf{\Lambda}_n^i)$ is $\omega_n(\mathbf{\Lambda}_n^i)$, whence the result for the n -sequence $p^{\rho_n^i}$, stationary from $p^{\rho_\infty^i}$. The first point of the Lemma follows; we have $\lim_{n \rightarrow \infty} \#(\mathcal{H}_n^{i+1}/\mathcal{H}_n^i) = p^{c_\infty^i} \cdot p^{\rho_\infty^i}$.

Assuming by induction that the n -sequence $\#\mathcal{H}_n^i$ is increasing stationary, the property follows for the n -sequence $\#\mathcal{H}_n^{i+1}$. \square

Lemma 4.3. *For $i \geq 0$ fixed and $n \geq e + \bar{e}$, we have $\frac{p^{n-e-\bar{e}}}{\#\omega_n(\mathbf{\Lambda}_n^i)} \leq p^{\delta_p(k)}$ for $i \geq 1$ (resp. $= p^{n-e-\bar{e}}$ for $i = 0$) and $\frac{[H_k^{\text{nr}} : K^{\text{nr}}]}{\#\mathbf{N}_n(\mathcal{H}_n^i)} \leq [H_k^{\text{nr}} : K^{\text{nr}}]$ for all i . Whence $p^{\rho_\infty^i} \leq p^{\delta_p(k)}$ (resp. $= p^{n-e-\bar{e}}$) and $p^{c_\infty^i} \leq [H_k^{\text{nr}} : K^{\text{nr}}]$. So, the n -sequence $\#(\mathcal{H}_n^{i+1}/\mathcal{H}_n^i)$ is increasing stationary bounded by $[H_k^{\text{nr}} : K^{\text{nr}}] \times p^{\delta_p(k)}$.*

Proof. Since $\mathbf{\Lambda}_n^i$ contains x , for $i \geq 1$, with $\omega_n(x)$ of order $p^{n-e-\bar{e}-\delta_p(k)}$, this yields $\frac{p^{n-e-\bar{e}}}{\#\omega_n(\mathbf{\Lambda}_n^i)} \leq \frac{p^{n-e-\bar{e}}}{\#\omega_n(\langle x \rangle)} = p^{\delta_p(k)}$. Since $p^{c_\infty^i} \mid [H_k^{\text{nr}} : K^{\text{nr}}]$, the conclusion holds. The case $i = 0$ gives the bounds $p^{n-e-\bar{e}}$ and $[H_k^{\text{nr}} : K^{\text{nr}}]$, respectively. \square

Lemma 4.4. *The i -sequences $p^{c_\infty^i}$, $p^{\rho_\infty^i}$, and $\lim_{n \rightarrow \infty} \#(\mathcal{H}_n^{i+1}/\mathcal{H}_n^i) = p^{c_\infty^i} \cdot p^{\rho_\infty^i}$ are decreasing stationary. There exists $i_{\min} \geq 0$ and constants $c \in \mathbb{Z}_{\geq 0}$ and $\rho \in \mathbb{Z}_{\geq 0}$, such that $c_\infty^i = c$ and $\rho_\infty^i = \rho$ for all $i \geq i_{\min}$.*

Proof. Come easily from expressions of the two factors of (4.3) for $n \geq e + \bar{e}$. \square

Remark 4.5. From Lemma 4.3 and $\#(\mathcal{H}_n^{i+1}/\mathcal{H}_n^i) \geq p$, for all $i \in [1, b_n - 1]$ (resp. $= [H_k^{\text{nr}} : K^{\text{nr}}] p^{n-e-\bar{e}}$ for $i = 0$), we obtain, for $n \gg 0$ and $\#\mathcal{H}_n = \prod_{i=0}^{b_n-1} \#(\mathcal{H}_n^{i+1}/\mathcal{H}_n^i)$:

$$(4.4) \quad \begin{cases} \nu_p([H_k^{\text{nr}} : K^{\text{nr}}]) + (b_n - 1) + n - e - \bar{e} \leq \lambda_p n + \mu_p p^n + \nu_p \\ \leq b_n \cdot \nu_p([H_k^{\text{nr}} : K^{\text{nr}}]) + (b_n - 1) \delta_p(k) + n - e - \bar{e}, \end{cases}$$

which justifies the conjecture $\mu_p(K/k) = 0$ since, otherwise, the number b_n of steps would be $O(p^n)$, which is incredible in the algorithmic point of view described in Remark 4.1. Moreover, if b_n is small, then $\lambda_p(K/k) = 1$ leads to:

$$\nu_p([H_k^{\text{nr}} : K^{\text{nr}}]) + (b_n - 1) \leq \nu_p + e + \bar{e} \leq b_n \cdot \nu_p([H_k^{\text{nr}} : K^{\text{nr}}]) + (b_n - 1) \delta_p(k).$$

This is enforced by the fact that $\delta_p(k)$ is very small (see Conjecture 2.4), and only intervenes for $p \nmid h_k$, giving possibly $\lambda_p(K/k) = 1$ for all $p \gg 0$.

In our opinion, for $i > 1$, the filtration does not depend on canonical elements as the p -units of k for $i = 1$, but of random groups $\mathbf{\Lambda}_n^i$; so using standard laws of probabilities, one may estimate the bounds b_n for which $\mathcal{H}_{K_n}^{b_n} = \mathcal{H}_{K_n}$, which ones depend on how fast, in an algorithmic sense, $\#\mathbf{N}_n(\mathcal{H}_n^i) \rightarrow [H_k^{\text{nr}} : K^{\text{nr}}]$ and $\#\omega_n(\mathbf{\Lambda}_n^i) \rightarrow p^{n-e-\bar{e}}$, as i increases, for $n \gg 0$ fixed (see Lemmas 4.2, 4.3, 4.4); of course, Iwasawa's theory corroborates the fact that such an heuristic makes sense, but it does not give an algorithm likely to "construct" the λ, μ, ν invariants.

5. GOLD'S CRITERION AND GENERALIZATIONS

5.1. The first and second element of the filtration. We focus on the elements $\mathcal{H}_{K_n}^1$ and $\mathcal{H}_{K_n}^2$ of the filtration; they are somewhat canonical and bring a maximal information, depending on some assumptions on the ramification in K/k and on the p -class group of k . Consider formulas (4.2) for $i \leq 1$ and $n \geq e + \bar{e}$, still using the previous simplified notations depending on n :

$$(5.1) \quad \begin{cases} \#\mathcal{H}_n^1 = [\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{T}_k \times p^{n-e-\bar{e}}, \\ \#(\mathcal{H}_n^2/\mathcal{H}_n^1) = \frac{[\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{T}_k}{\#\mathbf{N}_n(\mathcal{H}_n^1)} \times \frac{p^{n-e-\bar{e}}}{\#\omega_n(\mathbf{\Lambda}_n^1)}, \\ \mathbf{\Lambda}_n^1 = \{a \in k^\times, (a) = \mathbf{N}_n(\mathfrak{A}_n), \mathfrak{d}(\mathfrak{A}_n) \in \mathcal{H}_n^1\}. \end{cases}$$

In the case of a totally p -adic imaginary quadratic field, we have, from the classical exact sequence:

$$1 \rightarrow \mathbf{J}_n(\mathcal{H}_k) \cdot \mathcal{H}_n^{\text{ram}} \rightarrow \mathcal{H}_n^1 \rightarrow E_k \cap \mathbf{N}_n(K_n^\times)/\mathbf{N}_n(E_n) \rightarrow 1,$$

the isomorphism $\mathcal{H}_n^1 \simeq \mathbf{J}_n(\mathcal{H}_k) \cdot \mathcal{H}_n^{\text{ram}}$, where \mathbf{J}_n is the transfer map and $\mathcal{H}_n^{\text{ram}}$ is the subgroup generated by the classes of the ramified primes (more precisely by the products of the form $\mathfrak{P}_n^1 \cdots \mathfrak{P}_n^{p^g}$, with $\mathfrak{P}_n^j \mid \mathfrak{p} \mid p$ in $K_n/k/\mathbb{Q}$).

Taking the norm, we obtain, for n large enough (such that p^n be larger than the exponent of \mathcal{H}_k), $\mathbf{N}_n(\mathcal{H}_n^1) = \mathcal{H}_k^{p^n} \cdot \mathbf{N}_n(\mathcal{H}_n^{\text{ram}}) = \mathbf{N}_n(\mathcal{H}_n^{\text{ram}})$. Let $\mathcal{I}_k =: \langle \mathfrak{a}_1, \dots, \mathfrak{a}_r \rangle$ be a subgroup of prime-to- p ideals of k , whose classes generate \mathcal{H}_k ; thus $\mathbf{N}_n(\mathbf{J}_n(\mathcal{I}_k)) = \langle \mathfrak{a}_1^{p^n}, \dots, \mathfrak{a}_r^{p^n} \rangle$, where each $\mathfrak{a}_j^{p^n}$ is of the form $(a_j^{p^{n-\rho_j}})$, where $a_j \in k^\times$ and p^{ρ_j} is less than the exponent of \mathcal{H}_k .

We then have $\langle (a_1^{p^{n-\rho_1}}), \dots, (a_r^{p^{n-\rho_r}}) \rangle \subset \mathbf{N}_n(\mathcal{I}_n^1)$, the ideal group $\mathbf{N}_n(\mathcal{I}_n^1)$ being completed by some S_p -units y (cf. Remark 4.1); but, for n large enough, $\omega_n(a_j^{p^{n-\rho_j}})$ would be of order less than p^{ρ_j} in the cyclic group Ω_n of order $p^{n-e-\bar{e}}$; whence, in the computation of the norm factor $\frac{p^{n-e-\bar{e}}}{\#\omega_n(\mathbf{\Lambda}_n^1)}$, only the p -units y will intervene since $\omega_n(y)$ will have order $p^{n-e-\bar{e}-\delta_p(y)}$ for a small constant $\delta_p(y)$ computed in (5.2).

More precisely, let p^f (resp. p^g) be the residue degree of \mathfrak{p} (resp. the index in G_n of the decomposition group of \mathfrak{p}), in K_n/k for n large enough; then $\mathbf{N}_n(\mathcal{H}_n^{\text{ram}})$ is generated by the classes of the ideals $\mathfrak{p}^{p^{f+g}} = \mathfrak{p}^{[K \cap L:k]} = \mathfrak{p}^{p^{e+\bar{e}}}$ and $\bar{\mathfrak{p}}^{p^{f+\bar{g}}} = \bar{\mathfrak{p}}^{[K \cap \bar{L}:k]} = \bar{\mathfrak{p}}^{p^{\bar{e}}}$ since $\text{Gal}(K/K \cap L)$ and $\text{Gal}(K/K \cap \bar{L})$ are the inertia groups in K/k . Using the relation $\mathfrak{p}\bar{\mathfrak{p}} = (p)$, $\mathbf{N}_n(\mathcal{H}_n^{\text{ram}})$ is generated by the class of $\mathfrak{p}^{[K^{\text{nr}}:k]}$, since $K^{\text{nr}} = K \cap \bar{L} \subseteq K \cap L$ (see Diagram 2.1); whence $\mathbf{N}_n(\mathfrak{P}_n^1 \cdots \mathfrak{P}_n^{p^g}) = \langle \mathfrak{p}^{[K^{\text{nr}}:k]} \rangle$.

So, $\omega_n(\Lambda_n^1)$ will be generated by the image of a suitable p -unit y such that $(y) \in \mathbf{N}_n(\mathcal{S}_n^1)$; thus, if $h_p \geq [K^{\text{nr}} : k]$, $y = x$, the generator of \mathfrak{p}^h , otherwise, $y = x^{[K^{\text{nr}}:k] \cdot h^{-1}}$ is the generator of $\mathfrak{p}^{[K^{\text{nr}}:k] \cdot h^{-1}}$. This will give, respectively:

$$(5.2) \quad \begin{cases} \delta_{\mathfrak{p}}(y) = \delta_p(k), & \text{if } h_p \geq [K^{\text{nr}} : k], \\ \delta_{\mathfrak{p}}(y) = v_p([K^{\text{nr}} : k]) - v_p(h) + \delta_p(k), & \text{if } h_p < [K^{\text{nr}} : k]. \end{cases}$$

Then $\omega_n(y)$ is of order $p^{n-e-\bar{e}-\delta_{\mathfrak{p}}(y)}$. Whence the formulas, from Theorem 3.3, for n large enough, taking into account the two cases in (5.2):

$$(5.3) \quad \left\{ \begin{array}{l} \#\mathcal{H}_n^1 = [\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{T}_k \times p^{n-e-\bar{e}}; \\ \#(\mathcal{H}_n^2 / \mathcal{H}_n^1) = \frac{[\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{T}_k}{\#\langle \mathcal{cl}(\mathfrak{p}^{[K^{\text{nr}}:k]}) \rangle} \times p^{\delta_{\mathfrak{p}}(y)} \\ = \begin{cases} \frac{[\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{T}_k}{h_p \cdot [K^{\text{nr}} : k]^{-1}} \times p^{\delta_p(k)}, \\ \frac{[\tilde{k}^{\text{nr}} : K^{\text{nr}}] \times \#\mathcal{T}_k}{1} \times \frac{[K^{\text{nr}} : k]}{h_p} \times p^{\delta_p(k)}, \text{ respectively} \end{cases} \\ = \frac{[\tilde{k}^{\text{nr}} : k] \times \#\mathcal{T}_k}{h_p} \times p^{\delta_p(k)} = \frac{\#\mathcal{H}_k}{h_p} \times p^{\delta_p(k)} = \#\widetilde{\mathcal{H}}_k. \end{array} \right.$$

Theorem 5.1. *The condition $\#\mathcal{H}_n^2 = \#\mathcal{H}_n^1$, for n large enough (giving $\mathcal{H}_n = \mathcal{H}_n^1$), holds if and only if $\widetilde{\mathcal{H}}_k = 1$, which is equivalent to satisfy the two conditions:*

- (i) $\delta_p(k) = 0$;
- (ii) $\widetilde{\mathcal{H}}_k = \langle \mathcal{cl}(\mathfrak{p}) \rangle$ (i.e., for $S_p(k) := \{\mathfrak{p}, \bar{\mathfrak{p}}\}$, the $S_p(k)$ -class group is trivial).

When this stability holds, the \mathbb{Z}_p -extension K/k is such that $\lambda_p(K/k) = 1$.

Which explains that these conditions are often required in the literature are nothing else than the stability for $i = 1$; but we note that we do not assume p totally ramified. An interesting consequence of the above formulas is that the orders of the two elements \mathcal{H}_n^1 and \mathcal{H}_n^2 of the filtration of \mathcal{H}_n do not depend on the \mathbb{Z}_p -extension K/k , assumed to be totally ramified from some explicit layers.

5.2. Generalization of Gold's criterion.

5.2.1. *The λ -stability theorem in the totally ramified case.* We will use the theorem of λ -stability [Gra2022, Theorem 3.1], applied to the modules $\mathcal{X}_n = \mathcal{H}_n$ and $X_k = E_k = 1$, giving the number $\lambda := \max(0, \#S_p(k) - 1 - \rho(k))$ (not to be confused with the Iwasawa invariant $\lambda_p(K/k)$), with $S_p(k) = \{\mathfrak{p}, \bar{\mathfrak{p}}\}$ and $\rho(k) = \text{rank}_{\mathbb{Z}}(X_k) = 0$, whence $\lambda = 1$.

We then have the following properties of λ -stability in any \mathbb{Z}_p -extension K/k in which the p -places totally ramify from k :

Theorem 5.2. *If $\#\mathcal{H}_{K_1} = \#\mathcal{H}_k \cdot p^\lambda$, the following properties hold:*

- (i) $\#\mathcal{H}_{K_n} = \#\mathcal{H}_k \cdot p^{\lambda \cdot n}$ and $\mathcal{H}_{K_n} = \mathcal{H}_{K_n}^{G_n}$, for all $n \geq 0$.

(ii) $\mathbf{J}_{K_n/k}(\mathcal{H}_k) = \mathcal{H}_{K_n}^{p^n}$ and $\text{Ker}(\mathbf{J}_{K_n/k}) = \mathbf{N}_{K_n/k}(\mathcal{H}_{K_n}[p^n])$, for all $n \geq 0$, where $\mathcal{H}_{K_n}[p^n] := \{c \in \mathcal{H}_{K_n}, c^{p^n} = 1\}$.

5.2.2. *The \mathbb{Z}_p -extensions such that $\#\mathcal{H}_n = \#\mathcal{H}_k \cdot p^n$, $\forall n \geq 0$.* We obtain a variant of Gold's criterion, but which applies for all totally ramified \mathbb{Z}_p -extensions K/k , as follows, assuming only that $\mathcal{H}_k = \langle \mathcal{d}(\mathfrak{p}) \rangle$ (i.e., the $S_p(k)$ -class group of k is trivial, equivalent to $\widetilde{\delta}_p(k) = \delta_p(k)$).

Theorem 5.3. *Let k be an imaginary quadratic field and let $p > 2$ be a prime number split in k . We assume that \mathcal{H}_k is generated by $\mathcal{d}(\mathfrak{p})$, where $\mathfrak{p} \mid p$.*

Let K/k be a \mathbb{Z}_p -extension in which \mathfrak{p} and $\bar{\mathfrak{p}}$ are totally ramified. Then, we have $\#\mathcal{H}_{K_n} = \#\mathcal{H}_k \cdot p^n$ for all $n \geq 0$, if and only if $\delta_p(k) = 0$; in other words, when $\mathcal{H}_k = \langle \mathcal{d}(\mathfrak{p}) \rangle$, k logarithmically principal is equivalent to the Iwasawa formula $\#\mathcal{H}_{K_n} = \#\mathcal{H}_k \cdot p^n$ fulfilled for all $n \geq 0$.

Proof. We begin by a translation of the assumptions. If p is totally ramified in K/k with \mathcal{H}_k generated by $\mathcal{d}(\mathfrak{p})$, then $\#\mathcal{H}_k = h_p$, and:

$$(5.4) \quad \begin{cases} \#\mathcal{H}_n^1 = h_p \cdot p^n, \quad \forall n \geq 0, \\ \#\mathcal{H}_n^2 = \frac{(\#\mathcal{H}_k)^2}{\#(\mathcal{H}_k^{p^n} \cdot \langle \mathcal{d}(\mathfrak{p}) \rangle)} p^{n+\delta_p(k)} = h_p \cdot p^{n+\delta_p(k)}, \quad \forall n \geq 0. \end{cases}$$

Let's prove the equivalence:

(i) If $\delta_p(k) = 0$, $\#\mathcal{H}_n^2 = \#\mathcal{H}_n^1$ from (5.4), the filtration stops, and then $\#\mathcal{H}_n = \#\mathcal{H}_n^1 = h_p \cdot p^n$ for all $n \geq 0$, whence $\lambda_p(K/k) = 1$ and $\nu_p(K/k) = \nu_p(h_p)$.

(ii) For the reciprocal, assume that $\#\mathcal{H}_n = h_p \cdot p^n$, for all $n \geq 0$.

We apply Theorem 5.2 with $\lambda := \max(0, \#S_p(k) - 1 - \rho(k)) = 1$. Thus we can say that $\#\mathcal{H}_n = h_p \cdot p^{\lambda n}$ for all $n \geq 0$, if and only if this holds for $n = 1$; since we have by assumption $\#\mathcal{H}_1 = h_p \cdot p$, one obtains from (5.4) that $\#\mathcal{H}_1^1 = \#\mathcal{H}_1$ (stabilization at $i = 1$), which implies, in particular, that $\#\mathcal{H}_1^2 = \#\mathcal{H}_1^1 = h_p \cdot p$, but $\#\mathcal{H}_1^2 = h_p \cdot p^{1+\delta_p(k)}$ from (5.4), whence $\delta_p(k) = 0$, yielding the reciprocal and the theorem. \square

5.2.3. *Case of the cyclotomic \mathbb{Z}_p -extension k^{cyc} with trivial \mathcal{H}_k .* In this case, often considered in the literature, the formulas become, for all $n \geq 0$:

$$(5.5) \quad \#\mathcal{H}_n^1 = p^n \quad \& \quad \#\mathcal{H}_n^2 = p^{n+\delta_p(k)},$$

and this is the framework of Gold criterion dealing with the cyclotomic \mathbb{Z}_p -extension k^{cyc} and Iwasawa's invariants $\lambda_p(k^{\text{cyc}}/k)$, $\nu_p(k^{\text{cyc}}/k)$, since $\mu_p(k^{\text{cyc}}/k) = 0$:

Theorem 5.4 (Gold's criterion [Gold1974]). *Let k^{cyc} be the cyclotomic \mathbb{Z}_p -extension of k and assume that $\mathcal{H}_k = 1$. Then $\lambda_p(k^{\text{cyc}}/k) = 1$ & $\nu_p(k^{\text{cyc}}/k) = 0$, if and only if $\delta_p(k) = 0$.*

Proof. If $\delta_p(k) = 0$, then, from relations (5.5), the filtration stops at $i = 1$ and for all n , $\mathcal{H}_n = \mathcal{H}_n^1$ of order p^n generated by $\mathcal{H}_n^{\text{ram}}$, thus by the class of $\mathfrak{P}_n \mid \mathfrak{p}$ in K_n ; whence $\lambda_p(k^{\text{cyc}}/k) = 1$ and $\nu_p(k^{\text{cyc}}/k) = 0$. The reciprocal is point (ii) of Theorem 5.3 in the more general case where \mathcal{H}_k is generated by $\mathcal{d}(\mathfrak{p})$. \square

Properties of the filtration in the simplest case $\mathcal{H}_k = 1$ and $K = k^{\text{cyc}}$ give the second part of Gold's criterion; as we will see, this property is very specific of k^{cyc} :

Corollary 5.5. *When $\mathcal{H}_k = 1$, the non-triviality of $\widetilde{\mathcal{H}}_k$ (i.e., $\delta_p(k) \geq 1$) implies $\lambda_p(k^{\text{cyc}}/k) \geq 2$. It results that, $\delta_p(k) \geq 1$ is equivalent to $\lambda_p(k^{\text{cyc}}/k) \geq 2$.*

Proof. We still put $K = k^{\text{cyc}}$ to simplify notations. Let $m \gg n \gg 0$ and assume that $\delta_p(k) \geq 1$ & $\lambda_p(K/k) = 1$, whence $\#\mathcal{H}_{K_m} = p^{m+\nu}$ and $\#\mathcal{H}_{K_n} = p^{n+\nu}$, with $\nu = \nu_p(K/k)$.

One has the relative formulas for the filtration of \mathcal{H}_{K_m} with respect to $G_n^m := \text{Gal}(K_m/K_n)$, $\omega_n^m := \omega_{K_m/K_n}$, and so on for the norms \mathbf{N}_n^m and transfers \mathbf{J}_n^m ; we use the fact that the group of units E_{K_n} of K_n is that of \mathbb{Q}_n for which $\omega_n^m(E_{\mathbb{Q}_n}) = 1$ and $E_{\mathbb{Q}_n} \cap \mathbf{N}_{\mathbb{Q}_m/\mathbb{Q}_n}(\mathbb{Q}_m^\times) = \mathbf{N}_{\mathbb{Q}_m/\mathbb{Q}_n}(E_{\mathbb{Q}_m})$:

$$(5.6) \quad \left\{ \begin{array}{l} \#\mathcal{H}_{K_m}^{G_n^m} = \#\mathcal{H}_{K_n} \cdot \frac{p^{m-n}}{\omega_n^m(E_{K_n})} = p^{(n+\nu)+(m-n)} = p^{m+\nu}, \\ \mathcal{H}_{K_m}^{G_n^m} \simeq \mathbf{J}_n^m(\mathcal{H}_{K_n}) \cdot \mathcal{H}_{K_m}^{\text{ram}} = \mathbf{J}_n^m(\mathcal{H}_{K_n}) \cdot \langle \mathcal{C}_m(\mathfrak{p}_m) \rangle, \\ \mathbf{N}_n^m(\mathcal{H}_{K_m}^{G_n^m}) = \mathcal{H}_{K_n}^{p^{m-n}} \cdot \langle \mathcal{C}_n(\mathfrak{p}_n) \rangle = \langle \mathcal{C}_n(\mathfrak{p}_n) \rangle, \end{array} \right.$$

and the inequality about the case $i = 1$ of the relative filtration:

$$\#\left[\left(\mathcal{H}_{K_m} / \mathcal{H}_{K_m}^{G_n^m} \right)^{G_n^m} \right] = \frac{\#\mathcal{H}_{K_n}}{\#\langle \mathcal{C}_n(\mathfrak{p}_n) \rangle} \times \frac{p^{m-n}}{\#\omega_n^m(\Lambda^1)} \geq \frac{p^{n+\nu}}{\#\langle \mathcal{C}_n(\mathfrak{p}_n) \rangle}.$$

But $\mathfrak{p}_n^{p^n} = \mathbf{J}_{K_n/k}(\mathfrak{p})$ is principal and $\mathcal{C}_n(\mathfrak{p}_n)$ is of order a divisor of p^n ; then we get $\#(\mathcal{H}_{K_m}^{(2)} / \mathcal{H}_{K_m}^{(1)}) \geq p^\nu$, where $\mathcal{H}_{K_m}^{(1)}$ and $\mathcal{H}_{K_m}^{(2)}$ are the first and second steps of the filtration regarding G_n^m ; then $\#\mathcal{H}_{K_m}^{(2)} \geq \#\mathcal{H}_{K_m}^{(1)} \cdot p^\nu = p^{m+2\nu}$, whence the inequality $p^{m+\nu} = \#\mathcal{H}_{K_m} \geq \#\mathcal{H}_{K_m}^{(2)} \geq p^{m+2\nu}$, which implies $\nu = 0$ and $\#\mathcal{H}_{K_n} = p^n$ for all $n \gg 0$; but using the filtration with respect to $G_n = \text{Gal}(K_n/k)$, this gives $\#\mathcal{H}_{K_n} = p^n \geq \#\mathcal{H}_{K_n}^2 = p^{n+\delta_p(k)} \geq p^{n+1}$, from formulas (5.5) (absurd). \square

The filtration may be somewhat important, giving large $\lambda_p(k^{\text{cyc}}/k)$'s (see examples in Dummit–Ford–Kisilevsky–Sands [DFKS1991] or Ozaki [Oza2001]). For the three examples of $\lambda_p(k^{\text{cyc}}/k) \geq 2$ given in [Oza2001, Section 1, (4)], we compute $\#\widetilde{\mathcal{H}}_k = p^{\tilde{\delta}_p(k)} = p^{\delta_p(k)}$, in the case $h_p = 1$ and see that $\delta_p(k) \neq 0$ ($\delta_p(k) = 1$ for $m = 52391$, $p = 5$, $\delta_p(k) = 2$ for $m = 1371$, $p = 7$, $\delta_p(k) = 3$ for $m = 23834$, $p = 3$). In the case $m = 239$ and $p = 3$ for which $\#\mathcal{H}_k = 3$, we find $\tilde{\delta}_p(k) = 1$, even if \mathfrak{p} is 3-principal and $\delta_p(k) = 0$ (thus $\tilde{\delta}_3(k) = \delta_3(k) + v_3(h_k) - v_3(h) = 1$, as expected).

5.2.4. *Remarks on the specificity of k^{cyc} regarding non-Galois K 's.* We will explain why the unit groups, even in the totally imaginary \mathbb{Z}_p -extension, play a fundamental role for Iwasawa's invariants.

(i) One may ask why Theorem 5.3 does not imply that $\delta_p(k) \neq 0$ is equivalent to $\lambda_p(K/k) \geq 2$, whence ask if Corollary 5.5 applies for a non-Galois totally ramified \mathbb{Z}_p -extension K/k , with $\mathcal{H}_k = \langle \mathcal{C}(\mathfrak{p}) \rangle$.

Many properties of k^{cyc}/k are not fulfilled for K/k , as the fact that units of K_n are not the real units of $\mathbb{Q}_n^{\text{cyc}}$ (first relation in (5.6)); then the second and third relations do not hold since $\mathcal{H}_{K_m}^{G_n^m}$ does not depend only on $\mathcal{H}_{K_m}^{\text{ram}}$, but also on units.

(ii) We propose the following explanation, in relation with various phenomena of capitulation in a \mathbb{Z}_p -extension; a first remark being that, in k^{cyc}/k , p -class groups and logarithmic class groups do not capitulate.

For this, recall that Greenberg's conjecture ($\lambda_p(k^{\text{cyc}}/k) = \mu_p(k^{\text{cyc}}/k) = 0$ for totally real base field k , whatever the splitting of p [Gree1976]), is equivalent to the capitulation of $\widetilde{\mathcal{H}}_k$ in k^{cyc} [Jau2019^a]; the phenomenon is due to the existence of unit

groups $E_{k_n^{\text{cyc}}}$ of maximal \mathbb{Z} -rank $[k_n^{\text{cyc}} : \mathbb{Q}] - 1 = 2p^n - 1$, contrary to k^{cyc} for imaginary k 's. Real Greenberg's conjecture has been largely checked in Kraft–Schoof–Pagani [KS1995, Pag2022], for $p = 3$ and $p = 2$, respectively, using cyclotomic units.

In other words, one may consider that in the case of imaginary quadratic fields (or minus parts in imaginary abelian base fields k), any K/k distinct from k^{cyc}/k , behaves like the totally real case, which yields the conjecture 5.6 that we give hereafter.

(iii) In [KuWa2022] Kundu–Washington, heuristics and conjectures are given for k^{acyc} , but in the case p non-split, which is of another nature, but suggests that the filtrations are bounded so that in general, $\lambda_p(k^{\text{acyc}}/k)$ and $\mu_p(k^{\text{acyc}})$ vanish for imaginary quadratic fields.

(iv) Some studies are of cohomological nature, generalizing Gold's criterion by means of Massey products, depending also of $\delta_p(k)$; see [Pei2024, § 5.1.2], where an algorithm is given and looks like the determination of a filtration, by means of norm computations; but the comparison should be explored in greater depth. In [Sto2023] an interpretation of Gold's results are given from the theory of elliptic curves.

Other studies are based on the General Greenberg Conjecture [Gree1998, 4, Conjecture 3.5], as for instance [Oza2001, Tak2020, Mur2023], among many other that may be found in each bibliographies; but it seems that any approach must have analogous limits separating accessible algebraic informations, from random ones.

In conclusion of this study and taking into account all available results, including numerical ones, we propose the following conjecture in the simplest case:

Conjecture 5.6. *Let k be an imaginary quadratic field, and let $p > 2$ be a prime. Let K/k be a \mathbb{Z}_p -extension and let $S_p(K/k)$ be the set of p -places totally ramified in K from some layer. If K is not the cyclotomic \mathbb{Z}_p -extension of k , then:*

$$\mu_p(K/k) = 0 \quad \& \quad \lambda_p(K/k) = \#S_p(K/k) - 1 \in \{0, 1\},$$

which is equivalent to the capitulation of $\widetilde{\mathcal{H}}_k$ in K .

In relation with the deep results of Sands [San1991], Ozaki [Oza2001], Fujii [Fu2013] and others, there is no contradiction since these authors deduce, from specific assumptions (involving properties of k^{cyc} in general), that $\lambda_p(K/k)$ is less than some constant $C > 1$. Naturally, the conjecture may be extended at least for imaginary abelian fields in a suitable manner.

The strange “transfer” of the properties of k^{cyc} onto K/k is illustrated, in our method, by the computation of the orders of the norm symbols and the universal existence of $\delta_p(k)$ governing the order of the logarithmic class group (Theorem 3.3).

In Ozaki [Oza2001], the main theorem claims, under the assumption $\mathcal{H}_k = 1$, that “ $\lambda_p(K/k) = 1$ and $\mu_p(K/k) = 0$ for all but finitely many \mathbb{Z}_p -extension K/k ” (including of course k^{cyc}/k). Because it is difficult to conceive this situation, since the cardinality of the set of K/k 's is that of \mathbb{Z}_p , this result enforces the conjecture.

The conjectural capitulation of $\widetilde{\mathcal{H}}_k$, in non-Galois K 's is in accordance with the results of more general capitulation given in [Jau2019^b].

6. COMPUTATION OF $\delta_p(k)$

The following program gives, for the fields $k = \mathbb{Q}(\sqrt{-m})$, split primes p in a given interval, solutions to $\delta_p(k) \geq 1$, without computation of the precise value of $\lambda_p(k^{\text{cyc}}/k) \geq 2$, which needs to know approximations of the p -adic L -functions.

Put $x = u + v\sqrt{-m}$. The principle of the program is to test the principality of \mathfrak{p}^h , for increasing divisors h of $h_k = \#\mathcal{H}_k$.

When $h \neq 1$, then $v\sqrt{-m} \equiv -u \pmod{\mathfrak{p}^2}$, then $\bar{x} \equiv 2u \pmod{\mathfrak{p}^2}$, and in this case it suffices to compute the p -valuation of the Fermat quotient $q = \frac{1}{p}((2u)^{p-1} - 1)$.

In the case $h = 1$, we use instead the relations $u^2 - mv^2 + 2uv\sqrt{-m} \equiv 0 \pmod{\mathfrak{p}^2}$ and $u^2 + mv^2 = p$. Then one gets $v\sqrt{-m} \equiv \frac{p}{2u} - u \pmod{\mathfrak{p}^2}$, and replacing in the computation of $\bar{x}^{p-1} - 1 = (u - v\sqrt{-m})^{p-1} - 1$ yields $\frac{1}{p}(\bar{x}^{p-1} - 1) \equiv \frac{(2u)^2q + 1}{(2u)^2} \pmod{\mathfrak{p}}$, whose valuation is that of $(2u)^2q + 1$.

We then have $\delta_p(k) \geq 1$ if and only if the valuation w in the program is larger or equal to 1.

The order of $\mathcal{O}(\mathfrak{p})$ is denoted h , then h_k is the order of \mathcal{H}_k , and for large m 's, we only give h_{kp} , the p -part of the order. We give an excerpt of the results; they coincide with that obtained in the table of [DFKS1991] by means of analytic methods with L_p -functions for computing $\lambda_p(k^{cyc}/k)$, and we can check all the cases, but our program may be used for any interval and for instance the case $m = 923$, $\#\mathcal{H}_k = 1$, $h = 10$, $p = 85985437$, does not appear in their table limited to $p < 10^7$:

```
\p 200
{for(m=3,10^3,if(core(m)!=m,next);k=bnfinit(x^2+m);hk=k.no;
D=divisors(hk);N=numdiv(hk);forprime(p=3,10^5,if(kronecker(-m,p)!=1,next);
P=component(idealfactor(k,p),1)[1];for(n=1,N,h=D[n];Y=idealpow(k,P,h);
Q=bnfisprincipal(k,Y);d=matsize(Q[1])[1];S=0;for(j=1,d,S=S+Q[1][j]);if(S==0,
X=Q[2][1]*k.zk[1]+Q[2][2]*k.zk[2];a=Mod(2*polcoeff(X,0),p^2);
q=lift(a^(p-1)-1)/p;if(h!=1,z=q);if(h==1,z=a^2*q+1);w=valuation(z,p);if(w>=1,
print("m=",m," h_kp=",p^(valuation(hk,p))," h = ",h," p=",p));break))})
```

m=3	h_k=1	h=1	p=13	m=47	h_k=5	h=5	p=157
m=3	h_k=1	h=1	p=181	m=47	h_k=5	h=1	p=1193
m=3	h_k=1	h=1	p=2521	m=47	h_k=5	h=5	p=1493
m=3	h_k=1	h=1	p=76543	m=47	h_k=5	h=5	p=1511
m=5	h_k=2	h=1	p=5881	m=51	h_k=2	h=2	p=5
m=6	h_k=2	h=2	p=131	m=53	h_k=6	h=6	p=81439
m=7	h_k=1	h=1	p=19531	m=55	h_k=4	h=4	p=8447
m=11	h_k=1	h=1	p=5	m=57	h_k=4	h=2	p=79
m=13	h_k=2	h=1	p=113	m=57	h_k=4	h=1	p=13729
m=14	h_k=4	h=4	p=3	m=58	h_k=2	h=2	p=3037
m=15	h_k=2	h=1	p=1741	m=61	h_k=6	h=6	p=11
m=17	h_k=4	h=1	p=8521	m=65	h_k=8	h=4	p=3
m=19	h_k=1	h=1	p=11	m=65	h_k=8	h=4	p=19
m=21	h_k=4	h=2	p=107	m=65	h_k=8	h=4	p=103
m=21	h_k=4	h=2	p=173	m=65	h_k=8	h=4	p=1663
m=22	h_k=2	h=1	p=23	m=67	h_k=1	h=1	p=24421
m=22	h_k=2	h=2	p=29	m=71	h_k=7	h=7	p=29
m=26	h_k=6	h=6	p=5	m=73	h_k=4	h=4	p=7
m=29	h_k=6	h=1	p=5741	m=74	h_k=10	h=5	p=3
m=31	h_k=3	h=1	p=227	m=74	h_k=10	h=5	p=641
m=31	h_k=3	h=3	p=727	m=74	h_k=10	h=10	p=5711
m=34	h_k=4	h=4	p=5	m=78	h_k=4	h=2	p=3307
m=34	h_k=4	h=4	p=7	m=82	h_k=4	h=2	p=43
m=34	h_k=4	h=4	p=709	m=83	h_k=3	h=3	p=17
m=35	h_k=2	h=2	p=3	m=83	h_k=3	h=1	p=41
m=35	h_k=2	h=2	p=13	m=83	h_k=3	h=1	p=89431
m=37	h_k=2	h=2	p=23	m=86	h_k=10	h=10	p=3
m=37	h_k=2	h=2	p=1051	m=87	h_k=6	h=6	p=1187
m=38	h_k=6	h=6	p=211	m=89	h_k=12	h=4	p=7
m=38	h_k=6	h=2	p=6947	m=91	h_k=2	h=2	p=761
m=41	h_k=8	h=8	p=3	m=91	h_k=2	h=2	p=787

m=41	h_k=8	h=4	p=5	m=93	h_k=4	h=2	p=677
m=42	h_k=4	h=2	p=251	m=94	h_k=8	h=8	p=11
m=43	h_k=1	h=1	p=1741	m=94	h_k=8	h=8	p=13
(...)							
m=901	h_kp=1	h=12	p=1693	m=943	h_kp=1	h= 8	p=173
m=902	h_kp=1	h=7	p=23	m=946	h_kp=1	h= 8	p=13
m=902	h_kp=1	h= 14	p=8663	m=946	h_kp=1	h= 8	p=6869
m=903	h_kp=1	h= 8	p=17	m=946	h_kp=1	h= 4	p=55217
m=903	h_kp=1	h= 8	p=311	m=947	h_kp=1	h= 5	p=41
m=905	h_kp=1	h= 6	p=18269	m=949	h_kp=1	h= 3	p=2281189
m=905	h_kp=1	h= 12	p=91869119	m=951	h_kp=1	h= 2	p=509
m=906	h_kp=1	h= 14	p=971	m=951	h_kp=1	h= 26	p=797
m=907	h_kp=1	h= 3	p=19	m=951	h_kp=1	h= 13	p=1549
m=907	h_kp=1	h= 3	p=1229	m=955	h_kp=1	h= 4	p=167
m=911	h_kp=1	h= 31	p=5	m=958	h_kp=1	h= 16	p=347
m=914	h_kp=1	h= 36	p=401629	m=962	h_kp=1	h= 2	p=13077373
m=915	h_kp=1	h= 2	p=11777	m=966	h_kp=1	h= 6	p=5
m=917	h_kp=1	h= 10	p=467	m=966	h_kp=1	h= 2	p=111533
m=917	h_kp=1	h= 10	p=1247527	m=966	h_kp=1	h= 2	p=694091
m=919	h_kp=1	h= 19	p=23	m=967	h_kp=1	h= 11	p=139
m=923	h_kp=1	h= 10	p=85985437	m=967	h_kp=1	h= 1	p=1291
m=926	h_kp=1	h= 40	p=3	m=970	h_kp=1	h= 2	p=613
m=926	h_kp=1	h= 40	p=37	m=970	h_kp=1	h= 1	p=1019
m=929	h_kp=1	h= 18	p=5	m=971	h_kp=3	h= 5	p=3
m=930	h_kp=1	h= 6	p=43	m=971	h_kp=1	h= 5	p=3361
m=930	h_kp=1	h= 6	p=397	m=973	h_kp=1	h= 6	p=26399
m=930	h_kp=1	h= 6	p=74507	m=974	h_kp=1	h= 6	p=1151
m=933	h_kp=1	h= 4	p=2689	m=978	h_kp=1	h= 4	p=13604797
m=934	h_kp=1	h= 13	p=15287	m=979	h_kp=1	h= 8	p=7
m=935	h_kp=1	h= 14	p=3	m=982	h_kp=1	h= 10	p=19
m=939	h_kp=1	h= 1	p=367	m=982	h_kp=1	h= 10	p=4311317
m=939	h_kp=1	h= 1	p=192013	m=993	h_kp=1	h= 6	p=7
m=941	h_kp=1	h= 23	p=5	m=995	h_kp=1	h= 8	p=3
m=941	h_kp=1	h= 46	p=7	m=998	h_kp=1	h= 26	p=3
m=941	h_kp=1	h= 46	p=11	m=998	h_kp=13h=	26	p=13
m=942	h_kp=1	h= 6	p=467	m=998	h_kp=1	h= 13	p=1279
m=942	h_kp=1	h= 6	p=3527	m=998	h_kp=1	h= 26	p=847277

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A. LOGARITHMIC CLASSES OF IMAGINARY ABELIAN NUMBER FIELDS
BY JEAN-FRANÇOIS JAULENT

Let k be an abelian number field. It is well known that such a field satisfies the so-called Gross–Kuz'min conjecture for every prime number p , i.e. that its logarithmic class group $\widetilde{\mathcal{H}}_k$ has finite order, see [Jau1994^b]. The computation of this last group for arbitrary number fields via the PARI/GP system is given in [BJ2016].

We are interested, in this appendix, to give an explicit formula for the orders of the imaginary components of $\widetilde{\mathcal{H}}_k$ in the semi-simple case.

We assume from now on that k is an imaginary abelian number field and that the prime number p does not divide $d = [k : \mathbb{Q}]$. Thus the p -adic algebra $\mathbb{Z}_p[\Delta]$ of the Galois group $\Delta = \text{Gal}(k/\mathbb{Q})$ is a direct product indexed by the primitive idempotents $e_\varphi = \frac{1}{d} \sum_{\tau \in \Delta} \varphi(\tau) \tau^{-1}$ associated to the \mathbb{Q}_p -irreducible characters φ :

$$\mathbb{Z}_p[\Delta] = \prod_{\varphi} \mathbb{Z}_p[\Delta] e_\varphi = \prod_{\varphi} \mathbb{Z}_\varphi,$$

where $\mathbb{Z}_\varphi = \mathbb{Z}_p[\Delta] e_\varphi$ is an unramified extension of \mathbb{Z}_p of degree $d_\varphi = \deg \varphi$.

As a consequence, the p -class group \mathcal{H}_k , as well as the logarithmic p -class group $\widetilde{\mathcal{H}}_k$, decompose as the product of their φ -components; likewise do their respective wild subgroups \mathcal{H}_k^w and $\widetilde{\mathcal{H}}_k^w$ constructed on the places \mathfrak{p} over p . Now, we have:

Lemma A.1. *For any irreducible character φ the φ -components of the tame quotients $\mathcal{H}'_k := \mathcal{H}_k / \mathcal{H}_k^w$ and $\widetilde{\mathcal{H}}'_k := \widetilde{\mathcal{H}}_k / \widetilde{\mathcal{H}}_k^w$ are \mathbb{Z}_φ -isomorphic. From where the equality between orders: $\#(\widetilde{\mathcal{H}}_k)^{e_\varphi} / \#(\widetilde{\mathcal{H}}_k^w)^{e_\varphi} = \#(\mathcal{H}_k)^{e_\varphi} / \#(\mathcal{H}_k^w)^{e_\varphi}$.*

Proof. Let us denote H_k^{lc} the maximal abelian pro- p -extension of k which completely splits, at every place, over the cyclotomic \mathbb{Z}_p -extension k^{cyc} ; and let H'_k be its maximal sub-extension which completely splits over k at every place dividing p . By p -adic class field theory (see [Jau1998]), we have the isomorphisms:

$$\widetilde{\mathcal{H}}_k \simeq \text{Gal}(H_k^{\text{lc}}/k^{\text{cyc}}) \quad \& \quad \mathcal{H}'_k \simeq \text{Gal}(H'_k/k).$$

Now, for the canonical map $\text{Gal}(H_k^{\text{lc}}/k^{\text{cyc}}) \rightarrow \text{Gal}(H'_k/k)$ given by the restriction of Galois automorphisms, the kernel is isomorphic to the wild subgroup $\widetilde{\mathcal{H}}_k^w$ of $\widetilde{\mathcal{H}}_k$ and the cokernel $\text{Gal}((k^{\text{cyc}} \cap H'_k)/k)$ is here trivial, since we have $p \nmid d$.

Let us assume now that φ is imaginary and denote $[\mathfrak{p}_\varphi] = [\mathfrak{p}]^{e_\varphi}$ the φ -component of the class in \mathcal{H}_k of any prime \mathfrak{p} above p . Let $w_\varphi \in \mathbb{N}$ be the order of $[\mathfrak{p}_\varphi]$. As a consequence, taking the p -adifications $\mathcal{D}_k := D_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $\mathcal{E}'_k := E'_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$ of the groups of divisors and p -units of k , we can write:

$$\mathfrak{p}_\varphi^{w_\varphi} = \mathfrak{p}^{w_\varphi e_\varphi} = (\alpha_\varphi), \text{ with } \alpha_\varphi \text{ isotypic in } \mathcal{E}'_k.$$

Let us observe that α_φ is uniquely defined (except in the special case where k contains the p th roots of unity and φ is the Teichmüller character ω , where it is defined up to a root of unity) and that α_φ generates the φ -component of \mathcal{E}'_k . Thus:

$$\#(\mathcal{H}_k^w)^{e_\varphi} = (\mathbb{Z}_\varphi : w_\varphi \mathbb{Z}_\varphi) = w_\varphi^{\deg \varphi}.$$

Likewise the logarithmic valuation of α_φ is $\tilde{\nu}_\mathfrak{p}(\alpha_\varphi) := \frac{1}{\deg \mathfrak{p}} \log_p(N_{k_\mathfrak{p}/\mathbb{Q}_p}(\alpha_\varphi))$, where \log_p is the Iwasawa logarithm in \mathbb{Q}_p (see [Jau1994^b]). So, we get (with the convention $x \sim y$ whenever x/y is a p -adic unit):

$$\#(\widetilde{\mathcal{H}}_k^w)^{e_\varphi} = (\mathbb{Z}_\varphi : \tilde{w}_\varphi \mathbb{Z}_\varphi) = \tilde{w}_\varphi^{\deg \varphi},$$

where $\tilde{w}_\varphi \sim \frac{1}{\deg \mathfrak{p}} \log_p(N_{k_\mathfrak{p}/\mathbb{Q}_p}(\alpha_\varphi))$ is the order in \mathbb{N} of the class $[\tilde{\mathfrak{p}}_\varphi] = e_\varphi[\tilde{\mathfrak{p}}]$ of the logarithmic divisor constructed on \mathfrak{p} .

In summary, we have:

Theorem A.2. *Let k be an abelian imaginary number field, $p \nmid d = [k : \mathbb{Q}]$ be a prime number, Δ_p be the decomposition subgroup of p in $\Delta = \text{Gal}(k/\mathbb{Q})$ and $\chi_p := \text{Ind}_{\Delta_p}^{\Delta} 1_{\Delta_p}$ the character of Δ induced by the unit character of Δ_p . Then:*

- (i) *For each irreducible character φ of Δ that does not appear in χ_p , the φ components of the logarithmic class group $\widetilde{\mathcal{H}}_k^{e_\varphi}$ and of the classical class group $\mathcal{H}_k^{e_\varphi}$ are \mathbb{Z}_φ -isomorphic; so, they have same orders: $\widetilde{h}_\varphi^{\text{deg } \varphi} = h_\varphi^{\text{deg } \varphi}$.*
- (ii) *For each imaginary φ that does appear in χ_p , the integer \widetilde{h}_φ is given by the formula: $\widetilde{h}_\varphi \sim \frac{1}{\text{deg } \mathfrak{p}} \log_p(N_{k_{\mathfrak{p}}/\mathbb{Q}_p}(\beta_\varphi))$, where β_φ is an isotypic generator of the principal divisor $\mathfrak{p}^{h_\varphi e_\varphi}$.*

Proof. Let us observe first that every finite \mathbb{Z}_φ -module is the direct sum of cyclic submodules. So, we can write its order as the $\text{deg } \varphi$ -th power of an integer.

(i) If the irreducible character φ does not appear in χ_p , the φ -components of the wild subgroups $\widetilde{\mathcal{H}}_k^w$ and \mathcal{H}_k^w are trivial. So, the Lemma A.1 gives directly:

$$\widetilde{\mathcal{H}}_k \simeq \mathcal{H}_k.$$

(ii) Otherwise, we get however, accordingly to Lemma A.1:

$$\widetilde{h}_\varphi = \frac{h_\varphi \widetilde{w}_\varphi}{w_\varphi} \sim \frac{h_\varphi}{w_\varphi} \left(\frac{1}{\text{deg } \mathfrak{p}} \log_p(N_{k_{\mathfrak{p}}/\mathbb{Q}_p}(\alpha_\varphi)) \right) = \frac{1}{\text{deg } \mathfrak{p}} \log_p(N_{k_{\mathfrak{p}}/\mathbb{Q}_p}(\beta_\varphi)),$$

where $\beta_\varphi = \alpha^{h_\varphi/w_\varphi}$ is an isotypic generator of the principal divisor $\mathfrak{p}_\varphi^{h_\varphi} = \mathfrak{p}^{h_\varphi e_\varphi}$.

Corollary A.3. *In case the prime p completely splits in k , we get anyway for every imaginary irreducible character φ :*

$$\widetilde{h}_\varphi \sim \left(\frac{1}{p} \log_p(\beta_\varphi) \right)^{\text{deg } \varphi}.$$

Corollary A.4. *In the particular case k is an imaginary quadratic field and the prime p decomposes in k as $(p) = \mathfrak{p}\bar{\mathfrak{p}}$, the order $\widetilde{h} = \#\widetilde{\mathcal{H}}_k$ satisfies the identity:*

$$\widetilde{h} \sim \left(\frac{1}{p} \log_p(\beta) \right), \text{ where } \beta \text{ is a generator of } (\mathfrak{p}/\bar{\mathfrak{p}})^h \text{ with } h = \#\mathcal{H}_k.$$

Proof. In this later case there are exactly two irreducible characters 1 and φ . The 1-components, i.e. the real components of ordinary or logarithmic p -class groups coincide with the respective groups of \mathbb{Q} , which are trivial. So, the whole groups coincide with their φ -components, i.e. the imaginary ones, and we have the equality between the whole orders: $h := h_1 h_\varphi = h_\varphi$; and $\widetilde{h} := \widetilde{h}_1 \widetilde{h}_\varphi = \widetilde{h}_\varphi$.

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