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# Scattering of electromagnetic waves by small obstacles

Sébastien Tordeux

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## Scattering of electromagnetic waves by small obstacles

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EPI Makutu, INRIA, Université de Pau, CNRS

September 4, AIP 2023



# Outline

## 1. Introduction

## 2. Foldy-Lax-based models

## 3. Spectral models

## 4. Numerical solution for large number of scatterers

## 5. Conclusion and perspectives

# Electromagnetic scattering problem by small obstacles

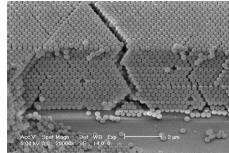
### Atmospheric particles



### Cosmic dust



### Photonic crystal



(by Allan Dyer)

(by Koen Clays)



- $\lambda$  from 400 to 800nm
- $\delta$  from 10 to 400nm

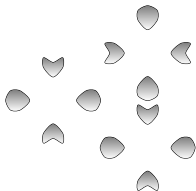
- $\lambda$  from 400 to 700nm
- $\delta$  from 1 to 100nm

- $\lambda$  from 400 to 800nm
- $\delta$  from 10 to 200nm

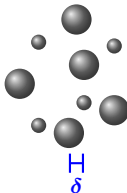
$\delta$  size of particle,  $\lambda$  wavelength

# Electromagnetic scattering problem by small obstacles

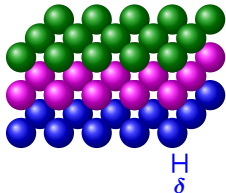
Arbitrary structure



Spherical scatterers



Periodic structure



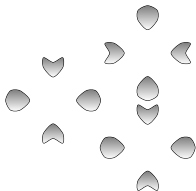
Electromagnetic source

- Time-harmonic  $\propto \exp(-i\omega t)$
- Wavelength  $\lambda = \frac{2\pi c}{\omega}$
- Homogeneous and isotropic

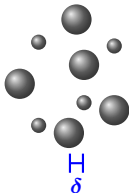


# Electromagnetic scattering problem by small obstacles

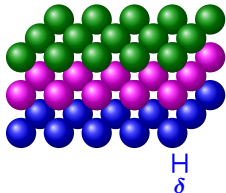
Arbitrary structure



Spherical scatterers

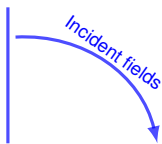
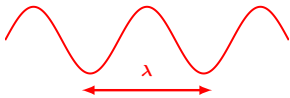


Periodic structure



Electromagnetic source

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- Wavelength  $\lambda = \frac{2\pi c}{\omega}$
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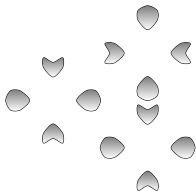
Reduced Maxwell equations

$$\begin{aligned} \text{curl } \mathbf{E}^i - i\kappa \mathbf{H}^i &= 0 & \text{in } \mathbb{R}^3 \\ \text{curl } \mathbf{H}^i + i\kappa \mathbf{E}^i &= \mathbf{J} & \text{in } \mathbb{R}^3 \end{aligned}$$

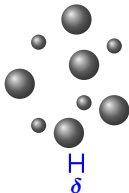
$$\kappa = \frac{\omega}{c} \text{ constant}$$

# Electromagnetic scattering problem by small obstacles

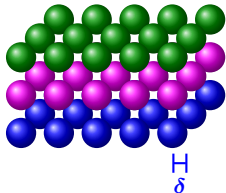
Arbitrary structure



Spherical scatterers

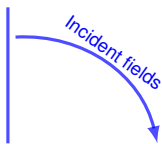
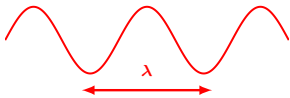


Periodic structure



Electromagnetic source

- Time-harmonic  $\propto \exp(-i\omega t)$
- Wavelength  $\lambda = \frac{2\pi c}{\omega}$
- **Homogeneous** and **isotropic**



Asymptotic assumption  
 $\delta \ll \lambda$

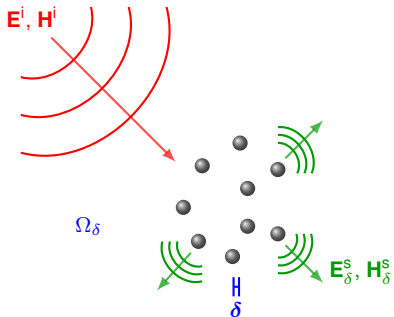
Reduced Maxwell equations

$$\kappa = \frac{\omega}{c} \text{ constant}$$

$$\text{curl } \mathbf{E}^i - i\kappa \mathbf{H}^i = 0 \quad \text{in } \mathbb{R}^3$$

$$\text{curl } \mathbf{H}^i + i\kappa \mathbf{E}^i = \mathbf{J} \quad \text{in } \mathbb{R}^3$$

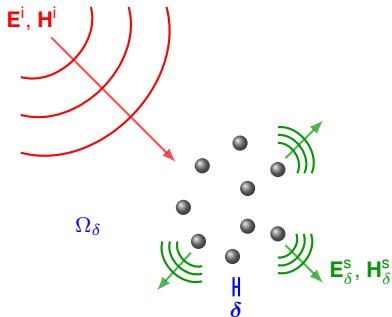
## Model problem



- Obstacles  $\mathcal{B}_\delta^k = \mathcal{B}(c_k, \delta)$
- Exterior domain  $\Omega_\delta = \mathbb{R}^3 \setminus \bigcup \overline{\mathcal{B}_\delta^k}$
- Boundary  $\Gamma_\delta = \bigcup \Gamma_\delta^k$



## Model problem



- Obstacles  $\mathcal{B}_\delta^k = \mathcal{B}(c_k, \delta)$
- Exterior domain  $\Omega_\delta = \mathbb{R}^3 \setminus \overline{\cup \mathcal{B}_\delta^k}$
- Boundary  $\Gamma_\delta = \cup \Gamma_\delta^k$

Total electromagnetic fields

$$\mathbf{E}_\delta = \mathbf{E}^i + \mathbf{E}_\delta^s \quad \mathbf{H}_\delta = \mathbf{H}^i + \mathbf{H}_\delta^s$$

Time-harmonic Maxwell equations

$$\begin{cases} \mathbf{curl} \mathbf{E}_\delta^s - i\kappa \mathbf{H}_\delta^s = 0 & \text{in } \Omega_\delta \\ \mathbf{curl} \mathbf{H}_\delta^s + i\kappa \mathbf{E}_\delta^s = 0 & \text{in } \Omega_\delta \end{cases}$$

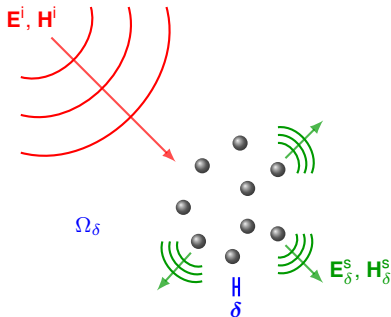
Silver-Müller condition

$$r (\mathbf{H}_\delta \times \frac{\mathbf{x}}{r} - \mathbf{E}_\delta) \xrightarrow[r \rightarrow \infty]{} 0 \quad \text{unif. in } \frac{\mathbf{x}}{r}$$

Perfect conductor condition

$$\mathbf{n} \times \mathbf{E}_\delta^s = -\mathbf{n} \times \mathbf{E}^i \quad \text{on } \Gamma_\delta$$

## Model problem



- Obstacles  $\mathcal{B}_\delta^k = \mathcal{B}(c_k, \delta)$
- Exterior domain  $\Omega_\delta = \mathbb{R}^3 \setminus \bigcup \overline{\mathcal{B}_\delta^k}$
- Boundary  $\Gamma_\delta = \bigcup \Gamma_\delta^k$

Total electromagnetic fields

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Time-harmonic Maxwell equations

$$\begin{cases} \mathbf{curl} \mathbf{E}_\delta^S - i\kappa \mathbf{H}_\delta^S = 0 & \text{in } \Omega_\delta \\ \mathbf{curl} \mathbf{H}_\delta^S + i\kappa \mathbf{E}_\delta^S = 0 & \text{in } \Omega_\delta \end{cases}$$

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$$r (\mathbf{H}_\delta \times \frac{\mathbf{x}}{r} - \mathbf{E}_\delta) \xrightarrow[r \rightarrow \infty]{} 0 \quad \text{unif. in } \frac{\mathbf{x}}{r}$$

Perfect conductor condition

$$\mathbf{n} \times \mathbf{E}_\delta^S = -\mathbf{n} \times \mathbf{E}^i \quad \text{on } \Gamma_\delta$$

For  $\mathbf{E}^i, \mathbf{H}^i \in \mathbf{H}_{\text{loc}}(\mathbf{curl}, \Omega_\delta)$  there is a unique solution  
 $\mathbf{E}_\delta^S, \mathbf{H}_\delta^S \in \mathbf{H}_{\text{loc}}(\mathbf{curl}, \Omega_\delta)$

Well-posedness

# Numerical strategies

## Mesh-dependent methods

- Finite differences
- Finite element method
- Discontinuous Galerkin
- Boundary element method

} Too expensive

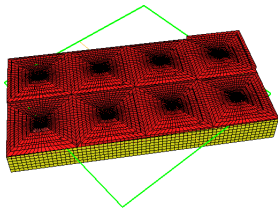
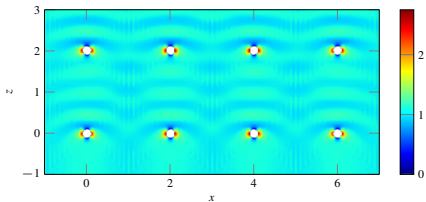
# Numerical strategies

## Mesh-dependent methods

- Finite differences
- **Finite element method**
- Discontinuous Galerkin
- Boundary element method

## Montjoie on Plafrim

- Order 2
- 2 436 832 dof
- Memory  $\approx$  40 GB
- Time  $\approx$  2h on 4 cores



# Numerical strategies

## Mesh-dependent methods

- Finite differences
- Finite element method
- Discontinuous Galerkin
- Boundary element method

## Mesh-less methods

- Asymptotic methods
- Spectral-based algorithms

# Numerical strategies

## Mesh-dependent methods

- Finite differences
- Finite element method
- Discontinuous Galerkin
- Boundary element method

## Mesh-less methods

- **Asymptotic methods**
- Spectral-based algorithms

## Asymptotic expansions

- Single scattering
- Restricted to small obstacles
- Low computational cost

## Foldy-Lax model

- Multiple scattering
- Restricted to small obstacles
- Low computational cost

## Numerical strategies

### Mesh-dependent methods

- Finite differences
- Finite element method
- Discontinuous Galerkin
- Boundary element method

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- Asymptotic methods
- Spectral-based algorithms

### Asymptotic expansions

- Single scattering
- Restricted to small obstacles
- Low computational cost

### Foldy-Lax model

- Multiple scattering
- Restricted to small obstacles
- Low computational cost

### Spectral method

- Single and multiple scattering
- All sizes of obstacles
- Analytical for spheres

## (Non-exhaustive) List of references

- Historic references
  - Rayleigh (1884), Foldy (1945), Lax (1951)
- **Small defect** theory
  - Il'in (1992), Maz'ya *et al.* (2000)
- Acoustic obstacle
  - Ammari and Kang (2003), Ramm (2005), Claeys (2008)
- Time-dependent domain
  - Mattes (2014), Korikov (2015), Marmorat (2015)
- **Electromagnetic** obstacle
  - Vogelius and Volkov (2000), Ammari *et al.* (2001), Korikov and Plamenevskii (2017)
- **Foldy** theory
  - Martin (2004), Cassier and Hazard (2013), Bendali *et al.* (2014), Challa *et al.* (2014), Bouzekri and Sini (2019)
- High-order **spectral** algorithms
  - Xu (1995), Ganesh and Hawkins (2009), Thierry (2011), Ammari *et al.* (2013), Barucq *et al.* (2017), Egel *et al.* (2017)
- Inverse problem
  - Volkov (2001), Ammari and Kang (2004), Challa and Sini (2012)



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# Outline

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**2. Foldy-Lax-based models**

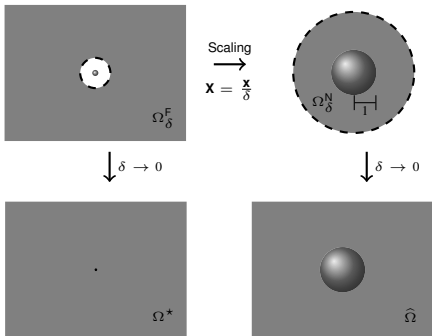
3. Spectral models

4. Numerical solution for large number of scatterers

5. Conclusion and perspectives

# Approximate solution to single scattering by a sphere

- Asymptotic method: the **matched asymptotic expansions**



- Domain decomposition
- Local approximations
- Matching procedure

## Approximate solution to single scattering by a sphere

- **Far from the obstacle**, the solution is approximated by

$$\mathbf{E}_\delta^s \approx \delta^3 \tilde{\mathbf{E}}_3 + \delta^5 \tilde{\mathbf{E}}_5 + \dots$$

$$\mathbf{H}_\delta^s \approx \delta^3 \tilde{\mathbf{H}}_3 + \delta^5 \tilde{\mathbf{H}}_5 + \dots$$

## Approximate solution to single scattering by a sphere

- Far from the obstacle, the solution is approximated by

$$\mathbf{E}_g^S \approx \delta^3 \tilde{\mathbf{E}}_3 + \delta^5 \tilde{\mathbf{E}}_5 + \dots \qquad \mathbf{H}_g^S \approx \delta^3 \tilde{\mathbf{H}}_3 + \delta^5 \tilde{\mathbf{H}}_5 + \dots$$

$$\tilde{\mathbf{E}}_3(x) = -\kappa^3 \left( \tilde{h}_1^{(1)}(\kappa r) \gamma_n[\mathbf{d}_e] + 2 \frac{h_1^{(1)}(\kappa r)}{i\kappa r} \gamma_n[\mathbf{d}_e] \frac{x}{r} - h_1^{(1)}(\kappa r) \gamma_\times[\mathbf{d}_h] \right)$$

$$\tilde{\mathbf{H}}_3(x) = -\kappa^3 \left( \tilde{h}_1^{(1)}(\kappa r) \gamma_t[\mathbf{d}_h] + 2 \frac{h_1^{(1)}(\kappa r)}{i\kappa r} \gamma_n[\mathbf{d}_h] \frac{x}{r} + h_1^{(1)}(\kappa r) \gamma_\times[\mathbf{d}_e] \right)$$

$$\tilde{\mathbf{E}}_5(x) = \dots + \frac{\kappa^4}{4} \left( \tilde{h}_2^{(1)}(\kappa r) \gamma_t[\mathbf{Q}_e x] + 3 \frac{h_2^{(1)}(\kappa r)}{i\kappa r} \gamma_n[\mathbf{Q}_e x] \frac{x}{r} + h_2^{(1)}(\kappa r) \gamma_\times[\mathbf{Q}_h x] \right)$$

$$\tilde{\mathbf{H}}_5(x) = \dots + \frac{\kappa^4}{4} \left( \tilde{h}_2^{(1)}(\kappa r) \gamma_t[\mathbf{Q}_h x] + 3 \frac{h_2^{(1)}(\kappa r)}{i\kappa r} \gamma_n[\mathbf{Q}_h x] \frac{x}{r} - h_2^{(1)}(\kappa r) \gamma_\times[\mathbf{Q}_e x] \right)$$

where

$$\mathbf{d}_e = \mathbf{E}^i(0) \qquad \mathbf{d}_h = \mathbf{H}^i(0) \qquad \mathbf{Q}_e = -\frac{2}{3} \mathbf{J}_{\mathbf{E}^i}^{\text{sym}}(0) \qquad \mathbf{Q}_h = \frac{4}{3} \mathbf{J}_{\mathbf{H}^i}^{\text{sym}}(0)$$

## Approximate solution to single scattering by a sphere

- Far from the obstacle, the solution is approximated by

$$\mathbf{E}_g^s \approx \delta^3 \tilde{\mathbf{E}}_3 + \delta^5 \tilde{\mathbf{E}}_5 + \dots$$

$$\mathbf{H}_g^s \approx \delta^3 \tilde{\mathbf{H}}_3 + \delta^5 \tilde{\mathbf{H}}_5 + \dots$$

$$\tilde{\mathbf{E}}_3(x) = -\kappa^3 \left( \overbrace{\tilde{h}_1^{(1)}(\kappa r) \gamma_t[\mathbf{d}_e]}^{\text{electric dipole}} + 2 \frac{h_1^{(1)}(\kappa r)}{i\kappa r} \gamma_n[\mathbf{d}_e] \frac{x}{r} \overbrace{-h_1^{(1)}(\kappa r) \gamma_\times[\mathbf{d}_h]}^{\text{magnetic dipole}} \right)$$

$$\tilde{\mathbf{H}}_3(x) = -\kappa^3 \left( \tilde{h}_1^{(1)}(\kappa r) \gamma_t[\mathbf{d}_h] + 2 \frac{h_1^{(1)}(\kappa r)}{i\kappa r} \gamma_n[\mathbf{d}_h] \frac{x}{r} + h_1^{(1)}(\kappa r) \gamma_\times[\mathbf{d}_e] \right)$$

$$\tilde{\mathbf{E}}_5(x) = \text{dipole} + \frac{\kappa^4}{4} \left( \overbrace{\tilde{h}_2^{(1)}(\kappa r) \gamma_t[\mathbf{Q}_e x]}^{\text{electric quadrupole}} + 3 \frac{h_2^{(1)}(\kappa r)}{i\kappa r} \gamma_n[\mathbf{Q}_e x] \frac{x}{r} \overbrace{+ h_2^{(1)}(\kappa r) \gamma_\times[\mathbf{Q}_h x]}^{\text{magnetic quadrupole}} \right)$$

$$\tilde{\mathbf{H}}_5(x) = \text{dipole} + \frac{\kappa^4}{4} \left( \tilde{h}_2^{(1)}(\kappa r) \gamma_t[\mathbf{Q}_h x] + 3 \frac{h_2^{(1)}(\kappa r)}{i\kappa r} \gamma_n[\mathbf{Q}_h x] \frac{x}{r} - h_2^{(1)}(\kappa r) \gamma_\times[\mathbf{Q}_e x] \right)$$

## Approximate solution to single scattering by a sphere

- **Far from the obstacle**, the solution is approximated by

$$\mathbf{E}_\delta^s \approx \delta^3 \tilde{\mathbf{E}}_3 + \delta^5 \tilde{\mathbf{E}}_5 + \dots$$

$$\mathbf{H}_\delta^s \approx \delta^3 \tilde{\mathbf{H}}_3 + \delta^5 \tilde{\mathbf{H}}_5 + \dots$$

$$\tilde{\mathbf{E}}_3(x) = \mathcal{E}_{\text{elec}}^{\text{dip}}[\mathbf{d}_e](x) + \mathcal{E}_{\text{mag}}^{\text{dip}}[\mathbf{d}_h](x)$$

$$\tilde{\mathbf{H}}_3(x) = \mathcal{H}_{\text{elec}}^{\text{dip}}[\mathbf{d}_e](x) + \mathcal{H}_{\text{mag}}^{\text{dip}}[\mathbf{d}_h](x)$$

$$\tilde{\mathbf{E}}_5(x) = \frac{3\kappa^2}{10} \mathcal{E}_{\text{elec}}^{\text{dip}}[\mathbf{d}_e](x) - \frac{3\kappa^2}{5} \mathcal{E}_{\text{mag}}^{\text{dip}}[\mathbf{d}_h](x) + \mathcal{E}_{\text{elec}}^{\text{quad}}[\mathbf{Q}_e](x) + \mathcal{E}_{\text{mag}}^{\text{quad}}[\mathbf{Q}_h](x)$$

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## Approximate solution to single scattering by a sphere

- Far from the obstacle, the solution is approximated by

$$\mathbf{E}_\delta^s \approx \delta^3 \tilde{\mathbf{E}}_3 + \delta^5 \tilde{\mathbf{E}}_5 + \dots$$

$$\mathbf{H}_\delta^s \approx \delta^3 \tilde{\mathbf{H}}_3 + \delta^5 \tilde{\mathbf{H}}_5 + \dots$$

- By using multipole expansion

$$\mathbf{E}_\delta^s \approx \alpha_e(\delta) \mathcal{E}_{\text{elec}}^{\text{dip}}[\mathbf{d}_e] + \alpha_h(\delta) \mathcal{E}_{\text{mag}}^{\text{dip}}[\mathbf{d}_h] + \beta_e(\delta) \mathcal{E}_{\text{elec}}^{\text{quad}}[\mathbf{Q}_e] + \beta_h(\delta) \mathcal{E}_{\text{mag}}^{\text{quad}}[\mathbf{Q}_h] + \dots$$

$$\mathbf{H}_\delta^s \approx \alpha_e(\delta) \mathcal{H}_{\text{elec}}^{\text{dip}}[\mathbf{d}_e] + \alpha_h(\delta) \mathcal{H}_{\text{mag}}^{\text{dip}}[\mathbf{d}_h] + \beta_e(\delta) \mathcal{H}_{\text{elec}}^{\text{quad}}[\mathbf{Q}_e] + \beta_h(\delta) \mathcal{H}_{\text{mag}}^{\text{quad}}[\mathbf{Q}_h] + \dots$$

### Multipole moments

- $\mathbf{d}_e, \mathbf{d}_h$  : dipole moment
- $\mathbf{Q}_e, \mathbf{Q}_h$  : quadrupole moment tensor

depend on



- the incident fields
- the **center** of the obstacle
- the **shape** of the obstacle

$$\mathbf{d}_e = \mathbf{E}^i(\mathbf{0}) \quad \mathbf{d}_h = -\frac{1}{2} \mathbf{H}^i(\mathbf{0})$$

$$\mathbf{Q}_e = -\frac{2}{3} \mathbb{J}_{\mathbf{E}^i}^{\text{sym}}(\mathbf{0}) \quad \mathbf{Q}_h = \frac{4}{3} \mathbb{J}_{\mathbf{H}^i}^{\text{sym}}(\mathbf{0})$$



## Approximate solution to single scattering by a sphere

- Far from the obstacle, the solution is approximated by

$$\mathbf{E}_\delta^s \approx \delta^3 \tilde{\mathbf{E}}_3 + \delta^5 \tilde{\mathbf{E}}_5 + \dots$$

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- By using multipole expansion

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$$\mathbf{H}_\delta^s \approx \alpha_e(\delta) \mathcal{H}_{\text{elec}}^{\text{dip}}[\mathbf{d}_e] + \alpha_h(\delta) \mathcal{H}_{\text{mag}}^{\text{dip}}[\mathbf{d}_h] + \beta_e(\delta) \mathcal{H}_{\text{elec}}^{\text{quad}}[\mathbf{Q}_e] + \beta_h(\delta) \mathcal{H}_{\text{mag}}^{\text{quad}}[\mathbf{Q}_h] + \dots$$

### Multipole moments

- $\mathbf{d}_e, \mathbf{d}_h$  : dipole moment
- $\mathbf{Q}_e, \mathbf{Q}_h$  : quadrupole moment tensor

### Order of approximation

- Order 3:  $\beta_e = \beta_h = 0$  and  $\alpha_e = \alpha_h = \delta^3$
- Order 5:  $\beta_e = \beta_h = \delta^5$ ,  $\alpha_e = \delta^3 \left(1 + \frac{3(\kappa\delta)^2}{10}\right)$   
 $\alpha_h = \delta^3 \left(1 - \frac{3(\kappa\delta)^2}{5}\right)$

$$\mathbf{d}_e = \mathbf{E}^i(\mathbf{0}) \quad \mathbf{d}_h = -\frac{1}{2} \mathbf{H}^i(\mathbf{0})$$

$$\mathbf{Q}_e = -\frac{2}{3} \mathbb{J}_{\mathbf{E}^i}^{\text{sym}}(\mathbf{0}) \quad \mathbf{Q}_h = \frac{4}{3} \mathbb{J}_{\mathbf{H}^i}^{\text{sym}}(\mathbf{0})$$

## Approximate solution to single scattering by a sphere

- Far from the obstacle, the solution is approximated by

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$$\mathbf{H}_\delta^s \approx \delta^3 \tilde{\mathbf{H}}_3 + \delta^5 \tilde{\mathbf{H}}_5 + \dots$$

- By using multipole expansion

$$\mathbf{E}_\delta^s \approx \alpha_e(\delta) \mathcal{E}_{\text{elec}}^{\text{dip}}[\mathbf{d}_e] + \alpha_h(\delta) \mathcal{E}_{\text{mag}}^{\text{dip}}[\mathbf{d}_h] + \beta_e(\delta) \mathcal{E}_{\text{elec}}^{\text{quad}}[\mathbf{Q}_e] + \beta_h(\delta) \mathcal{E}_{\text{mag}}^{\text{quad}}[\mathbf{Q}_h] + \dots$$

$$\mathbf{H}_\delta^s \approx \alpha_e(\delta) \mathcal{H}_{\text{elec}}^{\text{dip}}[\mathbf{d}_e] + \alpha_h(\delta) \mathcal{H}_{\text{mag}}^{\text{dip}}[\mathbf{d}_h] + \beta_e(\delta) \mathcal{H}_{\text{elec}}^{\text{quad}}[\mathbf{Q}_e] + \beta_h(\delta) \mathcal{H}_{\text{mag}}^{\text{quad}}[\mathbf{Q}_h] + \dots$$

### Multipole moments

- $\mathbf{d}_e, \mathbf{d}_h$  : dipole moment
- $\mathbf{Q}_e, \mathbf{Q}_h$  : quadrupole moment tensor

$$\mathbf{d}_e = \mathbf{E}^i(\mathbf{0}) \quad \mathbf{d}_h = -\frac{1}{2} \mathbf{H}^i(\mathbf{0})$$

$$\mathbf{Q}_e = -\frac{2}{3} \mathbb{J}_{\mathbf{E}^i}^{\text{sym}}(\mathbf{0}) \quad \mathbf{Q}_h = \frac{4}{3} \mathbb{J}_{\mathbf{H}^i}^{\text{sym}}(\mathbf{0})$$

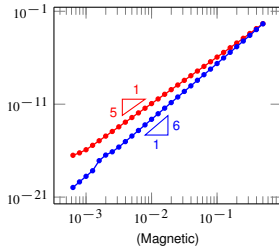
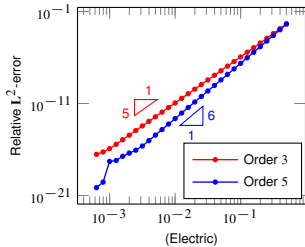
### Order of approximation

- Order 3:  $\beta_e = \beta_h = 0$  and  $\alpha_e = \alpha_h = \delta^3$
- Order 5:  $\beta_e = \beta_h = \delta^5$ ,  $\alpha_e = \delta^3 \left(1 + \frac{3(\kappa\delta)^2}{10}\right)$   
 $\alpha_h = \delta^3 \left(1 - \frac{3(\kappa\delta)^2}{5}\right)$

intermediate approximations

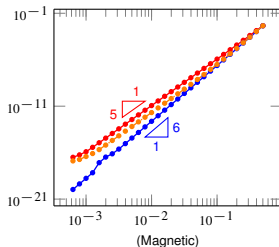
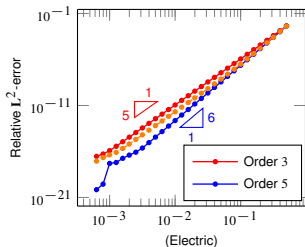
- Collected dipole
- Modified approximation

# Complements on single scattering



- Data
- $\lambda = 5.0$
  - Plane wave
  - $\delta$  varies

## Complements on single scattering



### Data

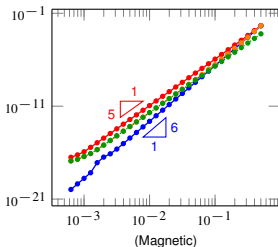
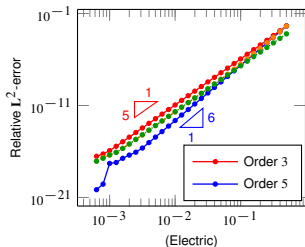
- $\lambda = 5.0$
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$$\mathbf{E}_\delta^s \approx \alpha_e(\delta) \mathcal{E}_{\text{dip}}^{\text{elec}}[\mathbf{d}_e] + \alpha_h(\delta) \mathcal{E}_{\text{dip}}^{\text{mag}}[\mathbf{d}_h]$$

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## Complements on single scattering



## Data

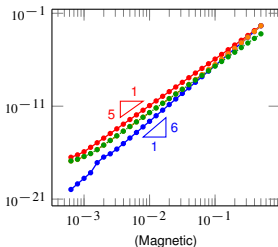
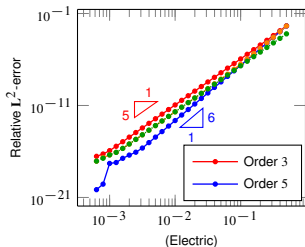
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- **Modified approximation:**  $\beta_e = \beta_h = 0$ ,  $\alpha_e = \frac{3i}{2(\kappa\delta)^3} \frac{j_1(\kappa\delta)}{h_1(\kappa\delta)}$  and  $\alpha_h = -\frac{3i}{(\kappa\delta)^3} \frac{\tilde{j}_1(\kappa\delta)}{h_1(\kappa\delta)}$

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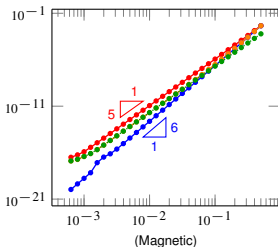
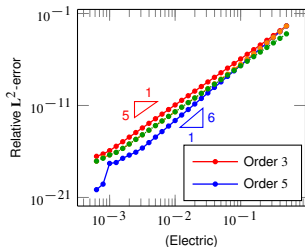
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Obstacle of arbitrary shape

- Approximation of order 3:  $\mathbf{d}_e = \mathbf{M}_e \mathbf{E}^i(\mathbf{c})$  and  $\mathbf{d}_h = -\frac{1}{2} \mathbf{M}_h \mathbf{H}^i(\mathbf{c})$

## Complements on single scattering



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- Approximation of order 3:  $\mathbf{d}_e = \mathbb{M}_e \mathbf{E}^i(\mathbf{c})$  and  $\mathbf{d}_h = -\frac{1}{2} \mathbb{M}_h \mathbf{H}^i(\mathbf{c})$
- High-order approximation:

$$\tilde{\mathbf{E}}_4 = \text{dipole} + \text{quadrupole}$$

$$\tilde{\mathbf{E}}_5 = \text{dipole} + \text{quadrupole} + \text{octupole}$$

## Approximate solution to multiple scattering by small spheres

- The electromagnetic fields are decomposed by superposition principle

$$\mathbf{E}_{\delta}^{\text{S}}(x) = \sum_{k=1}^{N_{\text{obs}}} \mathbf{E}_{\delta,k}^{\text{S}}(x - \mathbf{c}_k) \quad \mathbf{H}_{\delta}^{\text{S}}(x) = \sum_{k=1}^{N_{\text{obs}}} \mathbf{H}_{\delta,k}^{\text{S}}(x - \mathbf{c}_k)$$

- Each obstacle is modeled by a dipolar source around  $\mathbf{c}_k$

$$\mathbf{E}_{\delta,k}^{\text{S}} \approx \mathcal{E}_{\text{elec}}^{\text{dip}}[\mathbf{d}_{\delta,e}^k] + \mathcal{E}_{\text{mag}}^{\text{dip}}[\mathbf{d}_{\delta,h}^k] \quad \mathbf{H}_{\delta,k}^{\text{S}} \approx \mathcal{H}_{\text{elec}}^{\text{dip}}[\mathbf{d}_{\delta,e}^k] + \mathcal{H}_{\text{mag}}^{\text{dip}}[\mathbf{d}_{\delta,h}^k]$$



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### Born approximation

$$\mathbf{d}_{\delta,e}^k = \alpha_e(\delta) \mathbf{E}^i(\mathbf{c}_k)$$

$$\mathbf{d}_{\delta,h}^k = -\frac{1}{2} \alpha_h(\delta) \mathbf{H}^i(\mathbf{c}_k)$$



- ✓ Explicit formulation
- ✗ Interactions are not taken into account

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- ✓ Explicit formulation
- ✗ Interactions are not taken into account

### Foldy-Lax approximation

$$\mathbf{d}_{\delta,e}^k = \alpha_e(\delta) \left( \mathbf{E}^{\text{i}}(\mathbf{c}_k) + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N_{\text{obs}}} \mathbf{E}_{\delta,\ell}^{\text{S}}(\mathbf{c}_k) \right)$$

$$\mathbf{d}_{\delta,h}^k = -\frac{1}{2} \alpha_h(\delta) \left( \mathbf{H}^{\text{i}}(\mathbf{c}_k) + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N_{\text{obs}}} \mathbf{H}_{\delta,\ell}^{\text{S}}(\mathbf{c}_k) \right)$$



- ✗ Implicit formulation
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$$\mathbf{H}_{\delta,k}^{\text{S}} \approx \mathcal{H}_{\text{elec}}^{\text{dip}}[\mathbf{d}_{\delta,e}^k] + \mathcal{H}_{\text{mag}}^{\text{dip}}[\mathbf{d}_{\delta,h}^k]$$

### Vectorial formulation

$$\mathbf{d}_{\delta} = (\mathbf{d}_{\delta,e}^1, \dots, \mathbf{d}_{\delta,e}^{N_{\text{obs}}}, \mathbf{d}_{\delta,h}^1, \dots, \mathbf{d}_{\delta,h}^{N_{\text{obs}}}) \in \mathbb{C}^{6N_{\text{obs}}}$$

$$(\mathbb{I} - \alpha(\delta)\mathbb{A}) \mathbf{d}_{\delta} = \alpha(\delta)\mathbf{f}$$

where  $\mathbb{A}$  is the “interaction” matrix

- Order 5:  $\alpha(\delta) = \delta^3$
- Collected and Modified:

$$\alpha(\delta) = \begin{pmatrix} \alpha_e(\delta)\mathbb{I} & \mathbf{0} \\ \mathbf{0} & \alpha_h(\delta)\mathbb{I} \end{pmatrix}$$

### Foldy-Lax approximation

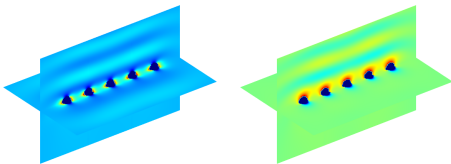
$$\mathbf{d}_{\delta,e}^k = \alpha_e(\delta) \left( \mathbf{E}^{\text{i}}(\mathbf{c}_k) + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N_{\text{obs}}} \mathbf{E}_{\delta,\ell}^{\text{S}}(\mathbf{c}_k) \right)$$

$$\mathbf{d}_{\delta,h}^k = -\frac{1}{2} \alpha_h(\delta) \left( \mathbf{H}^{\text{i}}(\mathbf{c}_k) + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N_{\text{obs}}} \mathbf{H}_{\delta,\ell}^{\text{S}}(\mathbf{c}_k) \right)$$



- ✗ Implicit formulation
- ✓ Interactions are taken into account

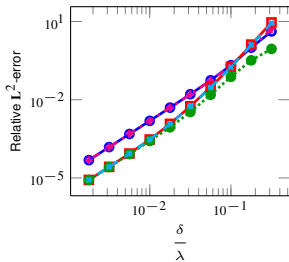
# Validation of asymptotic models



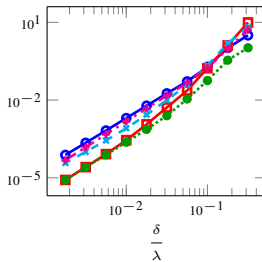
## Data

- $\lambda = 1.0$
- Plane wave
- $\delta$  varies

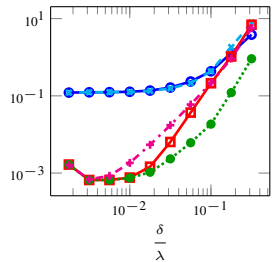
- Born
- Foldy
- +--+-- Collected Born
- Collected Foldy
- Modified Foldy



(a) Distance  $\propto \lambda$



(b) Distance  $\propto \sqrt{\lambda\delta}$



(c) Distance  $\propto \delta$

# Outline

1. Introduction

2. Foldy-Lax-based models

**3. Spectral models**

4. Numerical solution for large number of scatterers

5. Conclusion and perspectives

## Principle of the spectral method

- The electromagnetic fields are represented by the **Stratton-Chu formula** for  $x \in \Omega_\delta$

$$\mathbf{E}_\delta^s(x) = \sum_{k=1}^{N_{\text{obs}}} \mathbf{curl} \int_{\Gamma_\delta^k} \Phi(x - \mathbf{c}_k, y - \mathbf{c}_k) \mathbf{p}_k(y) ds_y$$

where  $\Phi(x, y) = \frac{\exp(i\kappa|x-y|)}{4\pi|x-y|}$

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- Each  $\mathbf{p}_\ell \in \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_\delta^\ell}, \Gamma_\delta^\ell)$  satisfies the **boundary integral equation**

$$\sum_{\ell=1}^{N_{\text{obs}}} \langle \mathcal{M}_\Gamma^{k\ell} \mathbf{p}_\ell, \mathbf{v}_k \rangle_{\Gamma_\delta^k} = - \langle \mathbf{n} \times \mathbf{E}^{\text{inc}}, \mathbf{v}_k \rangle_{\Gamma_\delta^k} \quad \forall \mathbf{v}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\text{curl}_{\Gamma_\delta^k}, \Gamma_\delta^k)$$

where  $\mathcal{M}_\Gamma^{k\ell} : \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_\delta^\ell}, \Gamma_\delta^\ell) \longrightarrow \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_\delta^k}, \Gamma_\delta^k)$  is the extension of

$$\mathcal{M}_\Gamma^{k\ell} \boldsymbol{\lambda}(x_\Gamma) = \mathbf{n}(x_\Gamma) \times \lim_{x \rightarrow x_\Gamma} \mathbf{curl} \int_{\Gamma_\delta^\ell} \Phi(x - \mathbf{c}_\ell, y - \mathbf{c}_\ell) \boldsymbol{\lambda}(y) ds_y \quad \boldsymbol{\lambda} \in \mathcal{C}^\infty(\Gamma_\delta^\ell) \quad x_\Gamma \in \Gamma_\delta^k$$

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- Galerkin discretization** of the BIE on **local spectral basis** with  $N_{\text{mod}}$  modes

$$\mathbf{p}_\ell(x) = \sum_{n=1}^{N_{\text{mod}}} \sum_{m=-n}^n p_{n,m}^{\ell,\perp} a_n(\delta) \nabla_{S^2} Y_{n,m}(\hat{x}_\ell) + p_{n,m}^{\ell,\times} b_n(\delta) \mathbf{curl}_{S^2} Y_{n,m}(\hat{x}_\ell) \quad x \in \Gamma_\delta^\ell$$

with  $\hat{x}_\ell = \frac{x - \mathbf{c}_\ell}{|x - \mathbf{c}_\ell|}$  and  $\nabla_{S^2} Y_{n,m}, \mathbf{curl}_{S^2} Y_{n,m}$ : complex-valued vector spherical harmonics



## Vectorial formulation

The variational formulation: Find  $(\mathbf{p}_k) \in \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_\delta^k}, \Gamma_\delta^k)$  such that

$$\langle \mathcal{M}_\Gamma^{kk} \mathbf{p}_k, \mathbf{v}_k \rangle_{\Gamma_\delta^k} + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N_{\text{obs}}} \langle \mathcal{M}_\Gamma^{k\ell} \mathbf{p}_\ell, \mathbf{v}_k \rangle_{\Gamma_\delta^k} = -\langle \mathbf{n} \times \mathbf{E}^{\text{inc}}, \mathbf{v}_k \rangle_{\Gamma_\delta^k} \quad \forall \mathbf{v}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\text{curl}_{\Gamma_\delta^k}, \Gamma_\delta^k)$$

Can be put under vectorial form  $\mathbf{p} = ((p_{n,m}^{1,\perp}), \dots, (p_{n,m}^{N_{\text{obs}},\perp}), (p_{n,m}^{1,\times}), \dots, (p_{n,m}^{N_{\text{obs}},\times}))^\top \in \mathbb{C}^N$

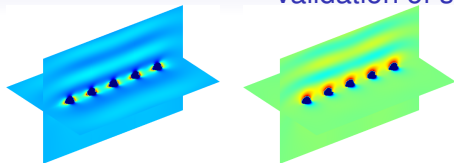
$$\mathbb{M} \mathbf{p} = \mathbf{f}$$

with  $N = 2N_{\text{mod}}(N_{\text{mod}} + 2)N_{\text{obs}}$

$$\mathbb{M} = \begin{pmatrix} \mathbb{M}_{\perp\perp} & \mathbb{M}_{\perp\times} \\ \mathbb{M}_{\times\perp} & \mathbb{M}_{\times\times} \end{pmatrix} \quad \text{with} \quad \mathbb{M}_{\alpha\beta} = \begin{pmatrix} \mathbb{M}_{\alpha\beta}^{11} & \mathbb{M}_{\alpha\beta}^{12} & \dots & \mathbb{M}_{\alpha\beta}^{1N_{\text{obs}}} \\ \mathbb{M}_{\alpha\beta}^{21} & \mathbb{M}_{\alpha\beta}^{22} & \dots & \mathbb{M}_{\alpha\beta}^{2N_{\text{obs}}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{M}_{\alpha\beta}^{N_{\text{obs}}1} & \mathbb{M}_{\alpha\beta}^{N_{\text{obs}}2} & \dots & \mathbb{M}_{\alpha\beta}^{N_{\text{obs}}N_{\text{obs}}} \end{pmatrix}$$

The analytical blocks  $\mathbb{M}_{\alpha\beta}^{k\ell}$  depend on  $\delta$  and  $(c_k - c_\ell)$

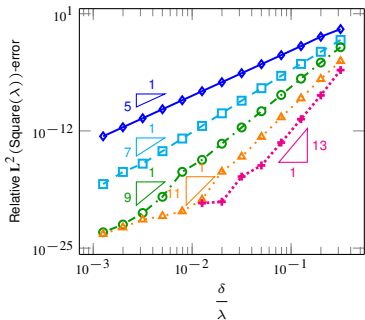
## Validation of spectral models



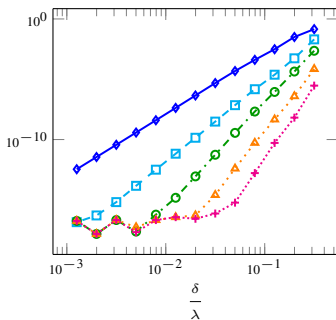
### Data

- $\lambda = 1.0$
- Plane wave
- $\delta$  varies

- ◆— Spectral 1
- -□- - Spectral 2
- -○- - Spectral 3
- ...▲... Spectral 4
- ...+... Spectral 5

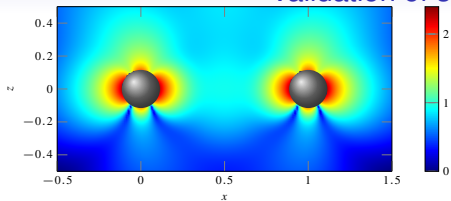


(d) Distance  $\propto \lambda$



(e) Distance  $\propto \sqrt{\lambda\delta}$

## Validation of spectral models



### Data

- $\lambda = 1.0$
- Plane wave
- $h$  varies

- ◆ Finite Element 1
- Finite Element 2
- Finite Element 3

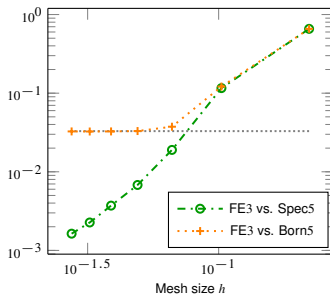
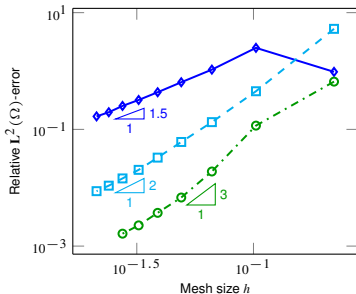
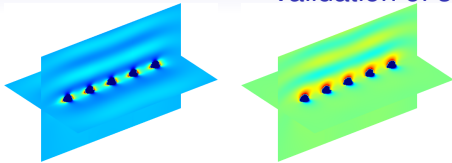


Figure: Spectral solutions vs. Finite element solutions (Montjoie)

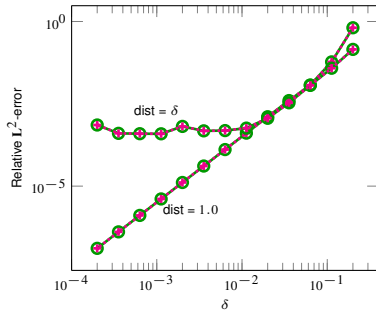
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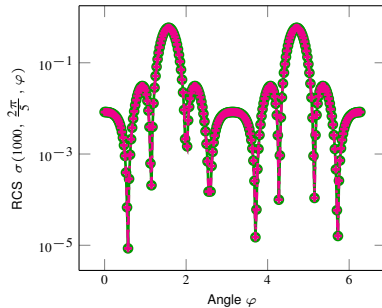
## Data

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- Plane wave
- $\delta$  varies

- Modified Foldy
- +--- Spectral 1



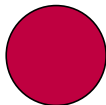
(a) Foldy vs Spectral 1



(b) Radar cross section

## However . . .

The spectral method has some disadvantages . . .



**Dense** matrix inherited from BIEs

Increasing in **number of unknowns** as the number of obstacles grows

Quickly limited with **memory resources**



Ill-conditioned system requiring **preconditioning**

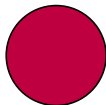
**Smart storage and assembling** in specific configurations

Implementation of an **iterative** resolution

. . . and that can be applied on the Foldy systems too

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# Outline

1. Introduction

2. Foldy-Lax-based models

3. Spectral models

**4. Numerical solution for large number of scatterers**

5. Conclusion and perspectives

## Iterative solver

- Simple calculations improve condition number associated with the matrix

$$\mathbf{M} \mathbf{p} = \mathbf{f}$$

- The matrix is decomposed as

$$\mathbf{M} = \mathbf{I} + \mathbf{A} + \mathbf{B}$$

where  $(\mathbf{I} + \mathbf{A})$  invertible contains the *main* interactions

$$A_{ij} = M_{ij} \quad \text{if } |M_{ij}| \geq \text{tolerance}$$

We solve iteratively

$$\begin{cases} \mathbf{p}^{(0)} = (\mathbf{I} + \mathbf{A})^{-1} \mathbf{f} \\ (\mathbf{I} + \mathbf{A}) \mathbf{p}^{(n+1)} = \mathbf{f} - \mathbf{B} \mathbf{p}^{(n)} \end{cases}$$

- $\mathbf{I}$  is the **identity** matrix
- $\mathbf{A}$  is a **sparse** matrix
- $\mathbf{B}$  is **dense** and **never** assembled

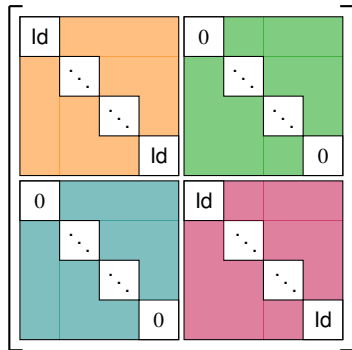


## Smart storage and assembling

- $M_{\alpha\beta}^{k\ell}$  depends only on  $\delta$  and  $(c_k - c_\ell)$
- Example: 4 aligned obstacles

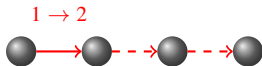


$M =$



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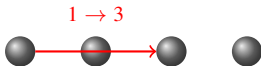


$$M =$$

Id	1 - 2			0			
	⋮				⋮		
		⋮			⋮		
			Id				0
0				Id			
	⋮			⋮			
		⋮			⋮		
			0				Id

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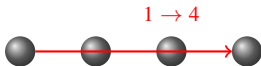


$M =$

Id	1 - 2	1 - 3		0			
	⋮				⋮		
		⋮				⋮	
			Id				0
0				Id			
	⋮				⋮		
		⋮				⋮	
			0				Id

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Id	1 - 2	1 - 3	1 - 4	0			
	⋮				⋮		
		⋮				⋮	
			Id				0
0				Id			
	⋮				⋮		
		⋮				⋮	
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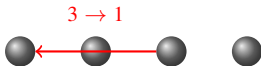


$M =$

Id	1 - 2	1 - 3	1 - 4	0			
2 - 1	• • •				• • •		
		• • •				• • •	
			Id				0
0				Id			
	• • •				• • •		
		• • •				• • •	
			0				Id

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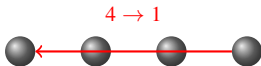


$M =$

Id	1 - 2	1 - 3	1 - 4	0			
2 - 1	•••						
3 - 1		•••					
			Id				0
0				Id			
	•••						
		•••					
			0				Id

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- $M_{\alpha\beta}^{k\ell}$  depends only on  $\delta$  and  $(c_k - c_\ell)$
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$$M =$$

Id	1 - 2	1 - 3	1 - 4	0			
2 - 1	•••						
3 - 1		•••					
4 - 1			Id				0
0				Id			
	•••						
		•••					
			0				Id





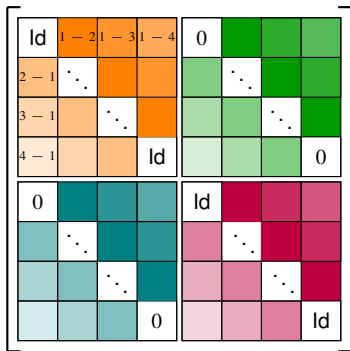
## Smart storage and assembling

- $M_{\alpha\beta}^{kl}$  depends only on  $\delta$  and  $(c_k - c_\ell)$

Instead of storing wholly  $M$ , we only keep the (two-by-two) different blocks

$$M_{\perp\perp}^{kl}, M_{\perp\times}^{kl}, M_{\times\perp}^{kl} \text{ and } M_{\times\times}^{kl}$$

$$M =$$



- The decomposition becomes  $M_{\text{Block}} = A_{\text{Block}} + B_{\text{Block}}$
- Assembling of  $A_{\text{Block}}$  in sparse matrix
- Define the action of  $B_{\text{Block}}$

$$\mathcal{A}(B_{\text{Block}}, \mathbf{p}) = B \mathbf{p}$$



## Numerical results

- Test n°1: 1 000 aligned obstacles

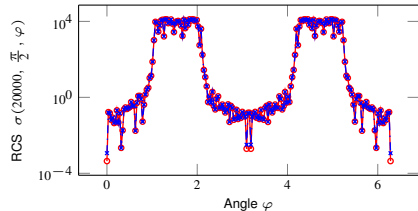
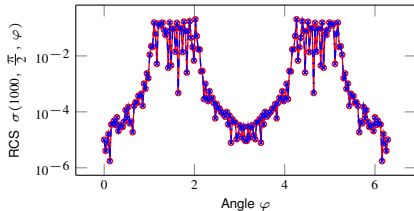
$$\delta = 0.1, d = 1.0, \lambda = 5.0$$

	Modified Foldy	Spectral 1
Solver Density	Direct 100%	
Linear system	21.25s	44.59s
Post-processing	16.12s	21.19s
Total time	37.37s	65.78s

- Test n°2: 10 000 aligned obstacles

$$\delta = 0.5, d = 2.0, \lambda = 5.0$$

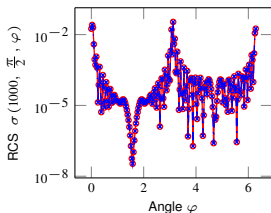
	Modified Foldy	Spectral 1
Solver Density	Iterative 4.95%	
Linear system	29.74s	163.25s
Post-processing	76.64s	262.97s
Total time	106.46s	426.34s



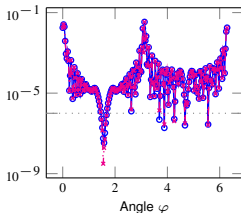
## Numerical results

- Test n°3:  $50 \times 50 = 2\,500$  obstacles uniformly distributed on a plane  
 $\delta = 0.1, d = 1.0, \lambda = 5.0$

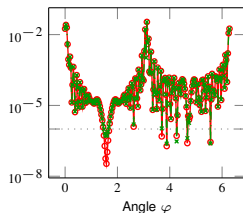
Solver	Modified Foldy		Spectral 1	
	Direct	Iterative	Direct	Iterative
Density	100%	29.66%	100%	29.66%
Linear system	595.10s	38.32s	613.10s	234.84s
Post-processing	22.96s	19.00s	62.69s	58.47s
Total time	618.48s	57.32s	676.09s	293.31s



(f) Foldy vs Spectral: direct solver



(g) Foldy Iterative vs Direct



(h) Spectral Iterative vs Direct

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- ✓ High-order asymptotic expansions to single scattering (Labat, Péron and Tordeux, In revision 2019)
- ✓ Low-order Born and Foldy-Lax models to multiple scattering
- ✓ High-order spectral models equivalent to the *Generalized Multiparticle Mie-solution* theory (Xu, 1995)
- ✓ Fast resolution using few memory to the multiple scattering problem by a large number of spheres

### On-going work

- ~ Comparison of preconditionners and iterative solvers
- ~ Smart assembling for obstacles uniformly distributed into a volume

### Perspectives

- ✗ Definition of high-order asymptotic models to multiple scattering
- ✗ Extension to obstacles of arbitrary shape
- ✗ Extension to time-dependent domain

Thank you for your attention

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