Covariance Stability Conditions for Linear Systems under Unbounded Additive and Parametric Uncertainties in Stochastic MPC applications

Kaouther Moussa, Mirko Fiacchini

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Kaouther Moussa[1,2], Mirko Fiacchini[3]

Abstract— **This paper proposes new expressions of the error covariance dynamics for discrete-time linear systems affected by unbounded additive and parametric uncertainties. In tube-based stochastic MPC approaches, this dynamics is important for the computation of stochastic invariant sets that are used for time-varying constraints tightening. Furthermore, this paper addresses stability conditions of the error covariance dynamics that can be used either in checking a posteriori the stability or to design the prestabilizing feedback gain depending on the level of uncertainties. Numerical examples are proposed to compare the evolution of the theoretical and the empirical error covariance.**

Index Terms— **Uncertain systems, invariance, stochastic MPC**

I. Introduction

Model Predictive Control (MPC) being particularly interesting for dynamical systems subject to different types of constraints (state and input for example), many theoretical results have been already established [11], specifically for the case of systems without uncertainties and those with bounded ones. Robust MPC approaches have been significantly developed in the last decades, where the objective is to design optimal control strategies handling different types of uncertainties. In this case, the uncertainties are considered to be lying in a known bounded set. Therefore, the control strategies require to ensure the satisfaction of the constraints for all the uncertainties realizations, which might lead to pessimistic and conservative results since the worst case scenario of uncertainties realizations is considered.

One of the main approaches to handle uncertainties in MPC applications is the tube-based one [10]. It consists in separating the state into a deterministic and an uncertain component and to design a prestabilizing feedback allowing to handle the uncertainties and their effects on chance constraints. This approach has been firstly developed for uncertainties of bounded nature, and was later extended to unbounded stochastic ones, which allowed a less pessimistic control design regarding systems for which probabilistic uncertainties can be adequately characterized.

Tube-based stochastic MPC (SMPC) approaches focused so far on considering disturbances that are additive to the dynamics, for instance in [8], [4] and [9], which means that a detailed knowledge of the systems parameters is required. However, many systems arising in practice involve parameters that are uncertain by nature.

The literature on uncertain parameters and multiplicative disturbances, in general, is less developed than for the additives ones [7]. In [5], a stochastic MPC framework considering both additive and multiplicative uncertainties has been proposed, though defined by probability distributions, the uncertainties were considered lying in a bounded support.

One of the recent works on the topic of multiplicative uncertainties is [1], where a stochastic MPC framework is presented, although the uncertainties on the parameters are assumed to be bounded within a polytopic set. Furthermore, in [7], a numerical solution of the covariance steering problem, for linear systems affected by unbounded additive and parametric uncertainties, has been proposed. In this case the parameters are considered mutually independent and the dependence between the state and the parameters is taken into account, since the parameters are considered to be constant.

In this paper, we examine the problem of characterizing the dynamical evolution of the covariance of a linear system subject to unbounded additive and parametric uncertainties, this dynamical expression being fundamental in the constraint tightening in the context of SMPC applications. The main originality of this paper is to propose an exact representation of the error covariance dynamics. This expression turns out to be a generalization of the commonly considered error covariance, when only additive disturbances are dealt with, for example in [1] and [6]. Furthermore, it brings out a particular representation of the parameters covariance effect on the evolution of the error covariance. When the prestabilizing feedback gain is designed, this expression can be used to check the stability of the error covariance in presence of uncertainties. Moreover, it can be used to design the prestabilizing feedback gain knowing the level of parametric uncertainties.

Based on the knowledge of the parameters covariance matrix, the state covariance evolution is exactly characterized in form of a linear difference equation and stability conditions for the related error covariance dynamics is provided. Besides the presence of parametric uncertainty, also additive stochastic disturbances are considered for which stability conditions are given too. To the best of our knowledge, this is the first time that such conditions are derived. Therefore, this work opens many perspectives on the design of controllers in the presence of uncertainties,

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specifically in the stochastic MPC context.

This paper is structured as follows: Section II presents the formulation of the problem, Section III describes the main results concerning the error covariance expressions. Section IV presents a stability analysis on the error covariance dynamics. Section V presents some numerical examples to compare the empirical error covariance with the theoretical one. Finally Section VI concludes the paper with some perspectives for future works.

Notation: The sets of real and integer numbers are denoted, respectively, $\mathbb{R}$ and $\mathbb{N}$, $\mathbb{E}[x]$ stands for the expectation of a random variable x. Given a random vector y, $\mathrm{cov}(y) = \mathbb{E}[(y-\mathbb{E}[y])(y-\mathbb{E}[y])^T]$ stands for the covariance of y, if the later has a zero mean ($\mathbb{E}[y] = 0$), then the covariance of y is simply $\mathrm{cov}(y) = \mathbb{E}[yy^T]$. The normal distribution of mean μ and covariance matrix Σ is denoted $\mathcal{N}(\mu, \Sigma)$. The Kronecker product is denoted by $\otimes$, $\mathrm{vec}(\cdot)$ stands for the vectorization operator and $\mathrm{vec}(\cdot)^{-1}$ stands for the inverse of the vectorization operator. $\rho(M)$ stands for the spectral radius of a square matrix $M \in \mathbb{R}^{n\times n}$ with $n \in \mathbb{N}$. The zero matrix with n rows and m columns, with $n, m \in \mathbb{N}$, is denoted $0_{n,m}$. The identity matrix of n rows and columns is denoted I_n. Given a symmetric matrix M, $M \succeq 0$ means that M is positive semidefinite. Given a time-varying vector x, in this paper the time dependence is omitted except for the initial value that is denoted x_0, and the index x_i, with $i \in \mathbb{N}_{\neq 0}$, stands for the the i-th element of the vector, when no ambiguity may raise.

II. Problem formulation

Let's consider the following discrete-time linear system, affected by additive and parametric uncertainties:

$$x^+ = A(p)x + Bu + w, \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ represent, respectively the state and the control input. The additive disturbance $w \in \mathbb{R}^n$ is an i.i.d. sequence of random variables with $\mathbb{E}[w] = 0$ and covariance $\mathrm{cov}(w) = \mathbb{E}[(w-\mathbb{E}[w])(w-\mathbb{E}[w])^T] = \mathbb{E}[ww^T] = W$, with $W \succeq 0$. Let's denote by $p \in \mathbb{R}^\theta$ an i.i.d. sequence of random variables representing the uncertain parameters, affecting the terms of the state matrix $A(p)$ in an affine way, and having as covariance $\Sigma \succeq 0$. Therefore, the state matrix $A(p)$ can be written as:

$$A(p) = A_0 + \sum_{i=1}^{\theta} A_i p_i = A_0 + \bar{A}(p),$$

where A_0 represents the known (or nominal) and deterministic component of $A(p)$, whereas $\bar{A}(p)$ represents the stochastic time-varying component. We can consider without loss of generality that $\mathbb{E}[p_j] = 0$ for all $j = 1, \cdots, \theta$, and then $\mathbb{E}[\bar{A}(p)] = 0$, since the parameters means can always be accounted for by appropriately adding an offset to A_0. The pair (A_0, B) is assumed stabilizable.

Notice that since we do not assume that Σ and W are identity matrices, then the parameters and the additive disturbances might be mutually dependent, meaning that $\mathbb{E}[p_i p_j]$ and $\mathbb{E}[w_i w_j]$ can be different than zero for all $i \neq j$. Furthermore, we assume that the elements of $\bar{A}(p)$ are mutually independent of the elements of w. Moreover, since the sequences of p and w are i.i.d, then the elements of $\bar{A}(p)$ and those of w are also independent of the state for the same time step k, implying $\mathbb{E}[x\bar{A}(p)] = \mathbb{E}[x]\mathbb{E}[\bar{A}(p)]$ and $\mathbb{E}[xw] = \mathbb{E}[x]\mathbb{E}[w]$.

Given system (1) and the assumptions formulated above, the problem that will be addressed in this paper is finding an exact expression of the error covariance dynamics related to system (1), often useful in the context of tube-based stochastic MPC applications. Generally, in this context, the state is expressed as a sum of a deterministic and a random component. This paper addresses the problem of finding the error covariance dynamics from the same point of view.

Moreover, in this paper, we are interested in studying the stability of the error covariance in order to derive conditions allowing both to test and to design constant feedback gains while guaranteeing the stability of the error covariance in the presence of unbounded parametric and additive uncertainties.

III. Main results

Firstly, let's consider the following discrete-time linear dynamical system with uncertainties on the state parameters p only, without additive uncertainties

$$x^+ = A(p)x + Bu, \tag{2}$$

with $A(p) = A_0 + \bar{A}(p)$ and $\mathbb{E}[\bar{A}(p)] = 0$ such that the pair (A_0, B) is stabilizable.

The state x can be expressed as the sum of a deterministic component z and a random component e that is

$$x = z + e, \tag{3}$$

such that

$$z^+ = A_0 z + Bv, \tag{4}$$

with $z_0 = x_0$ and then $e_0 = 0$. From $e = x - z$ and by considering $u = Ke + v$ we have:

$$\begin{aligned} e^+ = x^+ - z^+ &= (A_0 + BK)e + \bar{A}(p)x \\ &= (A(p) + BK)e + \bar{A}(p)z. \end{aligned} \tag{5}$$

The following standard assumptions are functional to the subsequent results.

Assumption 1: The constant feedback gain K is designed such that $\rho(A_0 + BK) < 1$.

Assumption 2: The system (4) is exponentially stabilized by the control v.

These assumptions are not restrictive being (A_0, B) stabilizable and are commonly used in standard MPC methods.

Proposition 1: From $e_0 = 0$ it follows that $\mathbb{E}[e] = 0$ for all time instants.

Proof: The expectation of the error is

$$\begin{aligned}\mathbb{E}[e^+] &= \mathbb{E}[(A(p)+BK)e+\bar{A}(p)z]\\ &= \mathbb{E}[(A(p)+BK)e]+\mathbb{E}[\bar{A}(p)z].\end{aligned}$$

Since that $A(p)$ is independent of both e and z, being independent of x, and the expectation of the product of two independent random variables is the product of their respective expectations [3] then it follows:

$$\mathbb{E}[e^+] = \mathbb{E}[(A(p)+BK)]\mathbb{E}[e]+\mathbb{E}[\bar{A}(p)]z. \tag{6}$$

Moreover, since $\mathbb{E}[\bar{A}(p)]=0$, then

$$\mathbb{E}[e^+] = \mathbb{E}[(A(p)+BK)]\mathbb{E}[e]$$

and therefore, from $e_0=0$, we have that $\mathbb{E}[e]=0$ for all time instants. ■

A direct implication of Proposition 1 is that $\mathrm{cov}(e) = \mathbb{E}[ee^T]$. The following property, see [2], is used hereafter.

Property 1: Given three matrices A, B and V, the mixed Kronecker matrix-vector product can be written as:

$$\mathrm{vec}(BVA^T) = (A\otimes B)\mathrm{vec}(V). \tag{7}$$

The exact evolution of the error covariance matrix is given below.

Theorem 1: The dynamics of the error covariance related to system (2) is given by the following expression:

$$\begin{aligned}\mathrm{cov}(e^+) =&(A_0+BK)\mathrm{cov}(e)(A_0+BK)^T\\ &+\mathrm{vec}^{-1}[\mathbb{E}[\bar{A}(p)\otimes\bar{A}(p)]\mathrm{vec}\left(\mathrm{cov}(e)+zz^T\right)].\end{aligned} \tag{8}$$

Proof: From (5) and Proposition 1, it follows

$$\begin{aligned}\mathrm{cov}(e^+) &= \mathbb{E}[e^+{e^+}^T] = \mathbb{E}\Big[\left(\left(A_0+BK\right)e+\bar{A}(p)x\right)\\ &\qquad\cdot\left(\left(A_0+BK\right)e+\bar{A}(p)x\right)^T\Big]\\ &= \left(A_0+BK\right)\mathbb{E}[ee^T]\left(A_0+BK\right)^T+\mathbb{E}[\bar{A}(p)xx^T\bar{A}(p)^T]\\ &\quad+\left(A_0+BK\right)\mathbb{E}[ex^T\bar{A}(p)^T]+\mathbb{E}[\bar{A}(p)xe^T]\left(A_0+BK\right)^T.\end{aligned}$$

Since $\mathbb{E}[ex^T\bar{A}(p)^T] = \mathbb{E}[ex^T]\mathbb{E}[\bar{A}(p)^T] = 0$ and $\mathbb{E}[\bar{A}(p)xe^T]=\mathbb{E}[\bar{A}(p)]\mathbb{E}[xe^T]=0$, it turns out that:

$$\begin{aligned}\mathrm{cov}(e^+) &= \left(A_0+BK\right)\mathbb{E}[ee^T]\left(A_0+BK\right)^T\\ &\quad+\mathbb{E}[\bar{A}(p)xx^T\bar{A}(p)^T].\end{aligned}$$

The first term is dependent on the error covariance $\mathrm{cov}(e) = \mathbb{E}[ee^T]$, while the second one, resulting from the presence of uncertain parameters, is given by

$$\begin{aligned}\mathbb{E}[\bar{A}(p)xx^T\bar{A}(p)^T] &= \mathbb{E}[\bar{A}(p)(e+z)(e+z)^T\bar{A}(p)^T]\\ &= \mathbb{E}[\bar{A}(p)ee^T\bar{A}(p)^T]+\mathbb{E}[\bar{A}(p)zz^T\bar{A}(p)^T]\\ &\quad+\mathbb{E}[\bar{A}(p)ez^T\bar{A}(p)^T]+\mathbb{E}[\bar{A}(p)ze^T\bar{A}(p)^T].\end{aligned} \tag{9}$$

By using Property 1 on the different terms of (9), from the linearity of the vectorization operator, and the fact that the expectation of a matrix is the matrix of expectations, implying that the vectorization operator and the expectation can commute, we obtain the following result:

$$\begin{aligned}&\mathrm{vec}\left(\mathbb{E}[\bar{A}(p)xx^T\bar{A}(p)^T]\right) = \mathrm{vec}\Big(\mathbb{E}[\bar{A}(p)ee^T\bar{A}(p)^T]\\ &+\mathbb{E}[\bar{A}(p)zz^T\bar{A}(p)^T]+\mathbb{E}[\bar{A}(p)ez^T\bar{A}(p)^T]+\mathbb{E}[\bar{A}(p)ze^T\bar{A}(p)^T]\Big)\\ &= \mathrm{vec}\left(\mathbb{E}[\bar{A}(p)ee^T\bar{A}(p)^T]\right)+\mathrm{vec}\left(\mathbb{E}[\bar{A}(p)zz^T\bar{A}(p)^T]\right)\\ &+\mathrm{vec}\left(\mathbb{E}[\bar{A}(p)ez^T\bar{A}(p)^T]\right)+\mathrm{vec}\left(\mathbb{E}[\bar{A}(p)ze^T\bar{A}(p)^T]\right).\end{aligned}$$

From Proposition 1 and Property 1 it follows

$$\begin{aligned}&\mathrm{vec}\left(\mathbb{E}[\bar{A}(p)ez^T\bar{A}(p)^T]\right) = \mathbb{E}\left[\mathrm{vec}\left(\bar{A}(p)ez^T\bar{A}(p)^T\right)\right]\\ &=\mathbb{E}[\left(\bar{A}(p)\otimes\bar{A}(p)\right)\mathrm{vec}\left(ez^T\right)]=\mathbb{E}[\left(\bar{A}(p)\otimes\bar{A}(p)\right)]\mathrm{vec}\left(\mathbb{E}[ez^T]\right)\\ &=\mathbb{E}[\left(\bar{A}(p)\otimes\bar{A}(p)\right)]\mathrm{vec}\left(\mathbb{E}[e]z^T\right)=0\end{aligned} \tag{10}$$

Analogous results hold for the term $\mathbb{E}[\bar{A}(p)ze^T\bar{A}(p)^T]$, then

$$\begin{aligned}\mathrm{vec}\left(\mathbb{E}[\bar{A}(p)xx^T\bar{A}(p)^T]\right) =&\mathrm{vec}\left(\mathbb{E}[\bar{A}(p)ee^T\bar{A}(p)^T]\right)\\ &+\mathrm{vec}\left(\mathbb{E}[\bar{A}(p)zz^T\bar{A}(p)^T]\right)\end{aligned}$$

and hence, following the same steps as in (10), one has:

$$\begin{aligned}\mathrm{vec}\left(\mathbb{E}[\bar{A}(p)xx^T\bar{A}(p)^T]\right) &= \mathbb{E}[\bar{A}(p)\otimes\bar{A}(p)]\mathrm{vec}\left(\mathbb{E}[ee^T]\right)\\ &+\mathbb{E}[\bar{A}(p)\otimes\bar{A}(p)]\mathrm{vec}\left(zz^T\right)\end{aligned} \tag{11}$$

z being deterministic. Finally, from (11) it follows

$$\begin{aligned}&\mathrm{cov}(e^+) = (A_0+BK)\mathrm{cov}(e)(A_0+BK)^T\\ &+\mathrm{vec}^{-1}[\mathbb{E}[\bar{A}(p)\otimes\bar{A}(p)]\mathrm{vec}\left(\mathrm{cov}(e)+zz^T\right)].\end{aligned}$$

■

Theorem 1 provides to exact characterisation of the dynamics of the covariance of mismatch between the nominal state z and the real one x.

A. Presence of additive uncertainties

Let's consider now the case of additive disturbances in addition to the parametric ones as follows:

$$x^+ = A(p)x+Bu+w, \tag{12}$$

where w is defined such that:

$$\mathbb{E}[w]=0 \qquad \text{and} \qquad \mathrm{cov}(w)=\mathbb{E}[ww^T]=W.$$

In this case e^+ has the following expressions:

$$\begin{aligned}e^+ = x^+-z^+ &= (A_0+BK)e+\bar{A}(p)x+w\\ &= (A(p)+BK)e+\bar{A}(p)z+w\end{aligned} \tag{13}$$

and its covariance matrix dynamics can be characterized.

Theorem 2: The dynamics of the error covariance related to system (12) is given by the following expression:

$$\begin{aligned}\mathrm{cov}(e^+) &= (A_0+BK)\mathrm{cov}(e)(A_0+BK)^T+W\\ &+\mathrm{vec}^{-1}\left(\mathbb{E}[\bar{A}(p)\otimes\bar{A}(p)]\left(\mathrm{cov}(e)+zz^T\right)\right).\end{aligned} \tag{14}$$

Proof: From (13) and Proposition 1, it follows that

$$\mathrm{cov}(e^+) = \mathbb{E}[e^+ e^{+T}] = \mathbb{E}\Big[\left((A_0 + BK)\, e + \bar{A}(p)x + w\right) \cdot \left((A_0 + BK)\, e + \bar{A}(p)x + w\right)^T \Big]$$

and therefore, by using Theorem 1, we can prove that:

$$\begin{aligned}\mathrm{cov}(e^+) =&(A_0 + BK)\mathbb{E}[ee^T])(A_0 + BK)^T \\ &+ \mathrm{vec}^{-1}[\mathbb{E}[\bar{A}(p) \otimes \bar{A}(p)]\mathrm{vec}\left(\mathbb{E}[ee^T] + zz^T\right)] \\ &+ \mathbb{E}[w\left((A_0 + BK)\, e + \bar{A}(p)x\right)^T] \\ &+ \mathbb{E}[\left((A_0 + BK)\, e + \bar{A}(p)x\right) w^T] + \mathbb{E}[ww^T].\end{aligned}$$

It can be proved that $\mathbb{E}[\left((A_0 + BK)\, e + \bar{A}(p)x\right) w^T] = \mathbb{E}[w\left((A_0 + BK)\, e + \bar{A}(p)x\right)^T] = 0$, since w is independent of e, x and $\bar{A}(p)$ and $\mathbb{E}[w] = 0$, which implies (14). ■

Theorem 2 is therefore an extension of results on the covariance dynamics already presented in the literature, for example in [1] and [6], which considered only additive disturbances. It shows thereby the effect that uncertain parameters have on the error covariance dynamics through the specific matrix $C_p = \mathbb{E}[\bar{A}(p) \otimes \bar{A}(p)]$.

Remark 1: The matrix $C_p = \mathbb{E}[\bar{A}(p) \otimes \bar{A}(p)]$ is constant, since it contains the parameters variances $\mathbb{E}[p_i^2], \quad i = 1, \cdots, \theta$ as well as their mutual covariances $\mathbb{E}[p_i p_j], \quad i,j = 1, \cdots, \theta$ with $i \neq j$, or an affine combination of the laters in the case where the matrix $\bar{A}(p)$ elements contain affine combinations of the uncertain parameters $p_i, \quad i = 1, \cdots, \theta$. Therefore, this matrix is a representation of the covariance matrix with a different structure.

Example 1: Consider the following two dimensional example:

$$A(p) = A_0 + \bar{A}(p) \qquad \text{with} \qquad \bar{A}(p) = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix},$$

in which case we have:

$$C_p = \begin{pmatrix} \mathbb{E}[p_1^2] & \mathbb{E}[p_1 p_2] & \mathbb{E}[p_1 p_2] & \mathbb{E}[p_2^2] \\ \mathbb{E}[p_1 p_3] & \mathbb{E}[p_1 p_4] & \mathbb{E}[p_2 p_3] & \mathbb{E}[p_2 p_4] \\ \mathbb{E}[p_1 p_3] & \mathbb{E}[p_2 p_3] & \mathbb{E}[p_1 p_4] & \mathbb{E}[p_2 p_4] \\ \mathbb{E}[p_3^2] & \mathbb{E}[p_3 p_4] & \mathbb{E}[p_3 p_4] & \mathbb{E}[p_4^2] \end{pmatrix}. \quad (15)$$

Note that C_p contains all the parameters variances as well as their covariances, but it has not the same structure as the covariance matrix, which is symmetric positive definite.

IV. Covariance stability analysis

Consider first the case of system (2), without additive uncertainties. Sufficient conditions for the convergence of the error covariance can be given. From Theorem 1, the covariance of e^+ can be exactly computed from the covariance matrix of e, the value of z and the knowledge of A_0, B, K and C_p.

Consider the evolution of $\mathrm{vec}\left(\mathbb{E}[ee^T]\right)$, and notice that, provided that the dynamics of z is exponentially stable (Assumption 2), then the effect of the value of z on $\mathrm{cov}(e)$ is vanishing. Defining

$$\epsilon = \mathrm{vec}(\mathrm{cov}(e)) \in \mathbb{R}^{n^2}, \qquad \zeta = \mathrm{vec}((zz^T)) \in \mathbb{R}^{n^2}.$$

From Property 1 and since the vectorization is a linear transformation, then the evolution of the vectorization form of the covariance matrix provided in (8) is given by the following difference equation:

$$\begin{aligned}\epsilon^+ &= \mathrm{vec}\left((A_0 + BK)\mathrm{cov}(e)(A_0 + BK)^T\right) \\ &\qquad + \mathbb{E}[\bar{A}(p) \otimes \bar{A}(p)](\epsilon + \zeta) \\ &= \Big((A_0 + BK) \otimes (A_0 + BK)\Big)\mathrm{vec}(\mathrm{cov}(e)) + C_p\epsilon + C_p\zeta \\ &= \Big((A_0 + BK) \otimes (A_0 + BK)\Big)\epsilon + C_p\epsilon + C_p\zeta \\ &= \Big((A_0 + BK) \otimes (A_0 + BK) + C_p\Big)\epsilon + C_p\zeta \qquad (16)\end{aligned}$$

that is a linear time invariant system. Defining

$$M = \Big((A_0 + BK) \otimes (A_0 + BK) + C_p\Big),$$

the following corollary holds.

Corollary 1: Let Assumption 2 hold. Then the covariance matrix of e for system (2) converges to the null matrix if

$$\rho\,(M) < 1. \qquad (17)$$

Remark 2: The necessity of condition (17) is not direct since the matrix M might involve unstable modes provided that are not excited by the input $C_p\zeta$, since $\epsilon_0 = 0$. A less conservative condition could consist in imposing that the controllable subspace of the pair (M, C_p) is contained in the subspace related to the asymptotically stable modes of M. Neither this condition might be necessary, though, since the vector ζ could evolve in a subspace smaller than the controllable one, from its structure. A careful analysis is required to formulate necessary and sufficient conditions for the covariance convergence, in our opinion.

Note that condition (17) can be used to check if with the designed feedback gain K, the covariance of the error remains stable knowing the covariance matrix Σ of the parametric uncertainties p. Furthermore, for a given feedback gain K, condition (17) can also be used to obtain bounds on the covariance matrix of the parametric uncertainties that the system can afford while remaining stable. Indeed, considering $A_K = A_0 + BK$, since the set of eigenvalues of $A_K \otimes A_K$ is given by $\lambda_i(A_K)\lambda_j(A_K)$ for all $i, j = 1, \ldots, n$, see [2], which means that in discrete time context, the stability of $A_K \otimes A_K$ is equivalent to the stability of A_K. This can help to derive other sufficient stability conditions allowing to facilitate the design of a feedback gain K that satisfies (17). Deriving constructive stability conditions from (17) might be more complex, though, due to the presence of joint products of the elements of K.

A. Presence of additive uncertainties

Consider now the more general case of system (1), involving both additive and parametric uncertainties. Denote by ω the vectorization of the additive disturbance covariance W as follows:

$$\omega = \mathrm{vec}\left(\mathrm{cov}(w)\right) = \mathrm{vec}\left(W\right) \in \mathbb{R}^{n^2}.$$

The difference equation (16) can be rewritten in the case of system (1) as follows:

$$\epsilon^{+} = M\epsilon + C_p\zeta + \omega,$$

which is a linear time invariant system with an exponentially vanishing input and a constant one, then the following corollary holds.

Corollary 2: Suppose Assumptions 1 and 2 hold. The covariance matrix of e for system (1) converges to the matrix $\mathrm{vec}^{-1}\left(\left(I_{n^2} - M\right)^{-1}\omega\right)$ if (17) holds.

The comments on the necessity of condition (17) and its practical use that have been formulated for system (2) hold also for system (1).

V. Numerical examples

We present in this section two examples illustrating both the case of systems affected by parametric uncertainties only and those affected also by additive uncertainties.

A. Parametric uncertain systems

Consider the following dynamical system:

$$x^{+} = \begin{pmatrix} 2+p_1 & 1+p_2 \\ p_3 & 0.1+p_4 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u, \tag{18}$$

leading to matrices A_0 and $\bar{A}(p)$, with $p = (p_1, p_2, p_3, p_4)^T$ as follows:

$$A_0 = \begin{pmatrix} 2 & 1 \\ 0 & 0.1 \end{pmatrix}, \qquad \bar{A}(p) = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix}.$$

The parameter vector p follows a multivariate normal distribution with zero mean and a covariance matrix Σ, i.e. $p \sim \mathcal{N}(0, \Sigma)$, where

$$\Sigma = \begin{pmatrix} 1.58 & 1.48 & 1.49 & 1.63 \\ 1.48 & 3.14 & 2.78 & 2.85 \\ 1.49 & 2.78 & 2.58 & 2.54 \\ 1.63 & 2.85 & 2.54 & 2.72 \end{pmatrix} \cdot 0.01$$

has been randomly generated. From (15), the matrix C_p then results in

$$C_p = \begin{pmatrix} 1.58 & 1.48 & 1.48 & 3.14 \\ 1.49 & 1.63 & 2.78 & 2.85 \\ 1.49 & 2.78 & 1.63 & 2.85 \\ 2.58 & 2.54 & 2.54 & 2.72 \end{pmatrix} \cdot 0.01. \tag{19}$$

We design the feedback gain K, using pole placement with desired poles vector $(0.5, 0.9)^T$ for the matrix $(A_0 + BK)$, giving $K = (-0.569, -0.131)$. Of course the gain K can also be designed using other methods. Furthermore, the same designed gain K is used also for stabilizing the dynamics of z, i.e. $v = Kz$, although v should be given by a deterministic MPC control to implement a stochastic MPC for system (1). The vectorized error covariance is

$$\epsilon = (\epsilon_{11}, \quad \epsilon_{12}, \quad \epsilon_{12}, \quad \epsilon_{22})^T.$$

We compute the evolution of the vectorization of the error covariance using the difference equation in (16), as well as the empirical covariance based on $N = 10000$ trials.

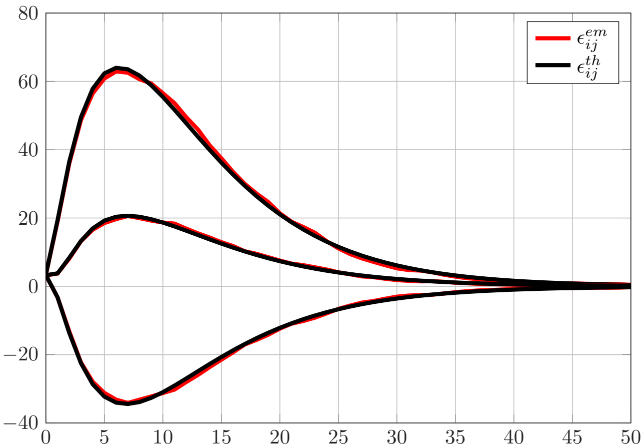

Fig. 1. Theoretical and empirical error covariance evolution related to system (18) with C_p as in (19).

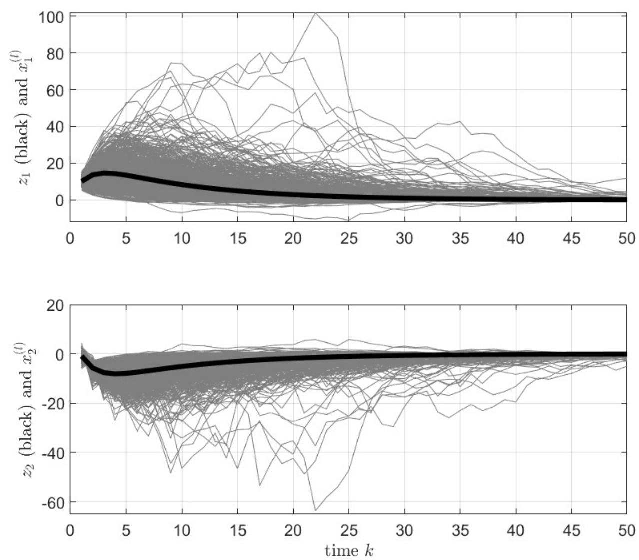

Fig. 2. The nominal and stochastic states z and x corresponding to system (18) using C_p as in (19), (up) The first components z_1 (in black) and $x_1^{(l)}$ (in grey) for $N = 1000$ trials with $l = 1, \cdots, N$, (bottom) The second components z_2 (in black) and $x_2^{(l)}$ (in grey).

We denote by ϵ_{ij}^{th} and ϵ_{ij}^{em}, respectively, the theoretical and the empirical elements of the error covariance matrix, for $i, j \in \{1, 2\}$. Fig. 1 shows that the empirical error covariance matches the theoretical one. Furthermore, they both converge to the null matrix since there is no additive disturbance in this case ($w = 0$).

Moreover, one can check the stability condition in (17) by defining $M = (A_0 + BK) \otimes (A_0 + BK) + C_p$ and computing its spectral radius $\rho(M) = 0.8702$, which is a sufficient condition for the stability of the error covariance.

Fig. 2 shows the nominal states z_1 and z_2 as well as the stochastic ones x_1 and x_2 for N trials and $x_0 = (1, 10)^T$, with which the empirical covariance has been computed. We can see that, since the stability conditions are satisfied, both

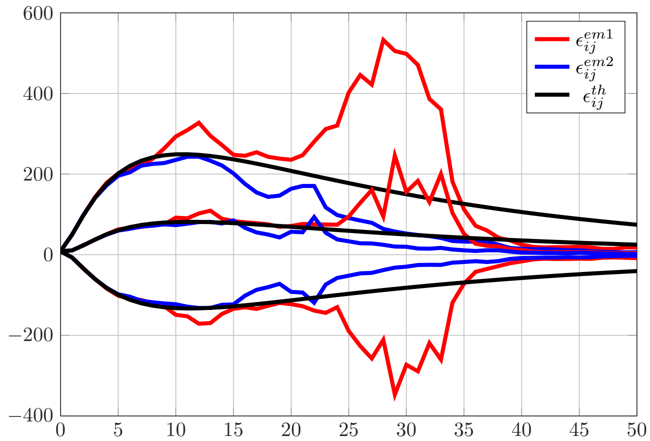

Fig. 3. Theoretical and empirical error covariance evolution related to system (21) with C_p as in (20) and the samplings $\mathcal{S}_1$ (in red) and $\mathcal{S}_2$ (in blue).

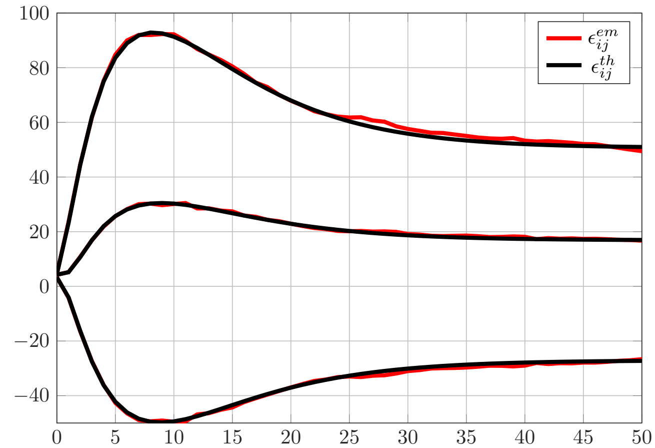

Fig. 4. Theoretical and empirical error covariance evolution related to system (21) with C_p as in (19).

the nominal and the real state vectors converge to 0.

Consider now higher uncertainties levels with:

$$C_p = \begin{pmatrix} 3.94 & 3.70 & 3.70 & 7.85 \\ 3.72 & 4.08 & 6.95 & 7.12 \\ 3.72 & 6.95 & 4.08 & 7.12 \\ 6.46 & 6.34 & 6.34 & 6.80 \end{pmatrix} \cdot 0.01. \tag{20}$$

In this case the stability condition on $\rho(M)$ still holds since $\rho(M) = 0.9652$. Consider two batches of N random trajectories generated with the uncertain parameters using C_p as in (20), and denote them $\mathcal{S}_1$ and $\mathcal{S}_2$. Fig. 3 shows the empirical and the theoretical covariances for $\mathcal{S}_1$ and $\mathcal{S}_2$. We can notice that the theoretical and empirical covariances are more affected by the uncertainties in this case, and that depending on the sampling of uncertainties, the empirical covariance can be either worse or better than the theoretical one, although the convergence remains guaranteed.

B. Presence of additive uncertainties

Consider the same system as in (18) with also an additive disturbance w:

$$x^+ = \begin{pmatrix} 2+p_1 & 1+p_2 \\ p_3 & 0.1+p_4 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u + w, \tag{21}$$

where the covariance of w is defined as $\mathbb{E}[ww^T] = W = I_n$. The same setup as for system (18) is considered, with C_p as in (19). Fig. 4 shows that the error covariance is asymptotically stable, since the same stability condition $\rho(M) = 0.8702$ holds in this case too. Furthermore we can compute the matrix to which the error covariance converges theoretically:

$$\mathrm{vec}^{-1}\left(\left(I_4 - M\right)^{-1}\omega\right) = \begin{pmatrix} 50.6293 & -27.0591 \\ -27.0591 & 16.8446 \end{pmatrix} \tag{22}$$

Fig. 4 shows that both the empirical and theoretical covariances converge to the matrix in (22).

VI. Conclusion and Perspectives

In this paper, we provide the exact characterization of the dynamics of the error covariance for discrete-time linear systems under unbounded additive and parametric uncertainties and presented sufficient conditions for asymptotic stability, useful in the context of stochastic tube-based MPC approaches. The numerical examples proposed show that the theoretical and the empirical error covariance converge to the same value when the stability conditions are satisfied. Future works would focus on the problem of designing the prestabilizing feedback gain K that satisfies the stability conditions in presence of uncertainties.

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