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# Scattering of a Scalar Time-Harmonic Wave by a Penetrable Obstacle with a Thin Layer

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This work looks at the asymptotic behaviour of the solution to the Helmholtz equation in a penetrable domain of  $\mathbb{R}^3$  with a thin layer of thickness  $\delta$  which tends to 0. We use the method of multiscale expansion to derive and justify an asymptotic expansion of the solution with respect to the thickness  $\delta$  up to any order. We then provide approximate transmission conditions of order two defined on an interface located inside the thin layer, with accuracy up to  $O(\delta^2)$ , which allow one to take into account the influence of the thin layer.

**Key Words:** Asymptotic analysis, Asymptotic expansions, Wave scattering, Helmholtz equation, Thin structures.

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## 1 Introduction

In this work we study the asymptotic behaviour of the solution to the Helmholtz equation

$$\begin{cases} \operatorname{div}(\sigma_\delta \nabla u_\delta) + k_\delta^2 u_\delta = 0 & \text{in } \mathbb{R}^3, \\ \lim_{|x| \rightarrow +\infty} |x| (\partial_{|x|} - ik_{\text{ext}})(u_\delta - u_{\text{inc}}) = 0, \end{cases} \quad (1.1)$$

where  $\sigma_\delta$  and  $k_\delta^2$  are piecewise constant functions defined by

$$\sigma_\delta(x) = \begin{cases} \sigma_{\text{ext}} & \text{if } x \in \Omega_{\text{ext},\delta}, \\ \tilde{\sigma}_\delta & \text{if } x \in \Omega_\delta, \\ \sigma_{\text{int}} & \text{if } x \in \Omega_{\text{int},\delta}, \end{cases} \quad ; \quad k_\delta^2(x) = \begin{cases} k_{\text{ext}}^2 & \text{if } x \in \Omega_{\text{ext},\delta}, \\ \tilde{k}_\delta^2 & \text{if } x \in \Omega_\delta, \\ k_{\text{int}}^2 & \text{if } x \in \Omega_{\text{int},\delta}, \end{cases}$$

where  $\tilde{\sigma}_\delta$  and  $\sigma_{\text{int}}$  are two strictly positive constants describing the contrast properties of  $\Omega_\delta$  and  $\Omega_{\text{int},\delta}$  relative to the exterior propagation domain  $\Omega_{\text{ext},\delta}$ . The refractive properties of the media are defined by  $k_{\text{int}}^2$  and  $k_\delta^2$  which are two complex numbers with strictly positive real parts and positive imaginary parts. We also assume that  $\sigma_{\text{ext}}$  and  $k_{\text{ext}}$  are strictly positive constants and that  $\sigma_{\text{ext}}, \sigma_{\text{int}}, k_{\text{ext}}, k_{\text{int}}$  are independent of  $\delta$ . The domain  $\Omega_{\text{int},\delta}$  is a three dimensional bounded domain with regular boundary  $\Gamma_{\delta,1}$ , surrounded by a thin layer  $\Omega_\delta$  of thickness  $\delta$  (which tends to 0) and  $\Omega_{\text{ext},\delta}$  is the exterior domain defined by  $\Omega_{\text{ext},\delta} = \mathbb{R}^3 \setminus (\overline{\Omega_{\text{int},\delta} \cup \Gamma_{\delta,1} \cup \Omega_\delta \cup \Gamma_{\delta,2}})$  (see Fig. 1). This work looks at the scattering of an incident wave  $u_{\text{inc}}(x) = e^{ik_{\text{ext}}(x \cdot d)/\sigma_{\text{ext}}}$  by the penetrable domain  $(\overline{\Omega_{\text{int},\delta} \cup \Gamma_{\delta,1} \cup \Omega_\delta})$  where  $d$  is a unit vector of  $\mathbb{R}^3$  giving the direction of the plane wave  $u_{\text{inc}}$ .

Numerical simulations of scattering problems as the one considered here need to mesh the thin layer. Since this can be a very costly task [39], it is of great interest to take into account the effect of such thin layer thanks to suitable approximate boundary conditions. The latter can be derived by studying the asymptotic behaviour, as  $\delta \rightarrow 0$ , of the total

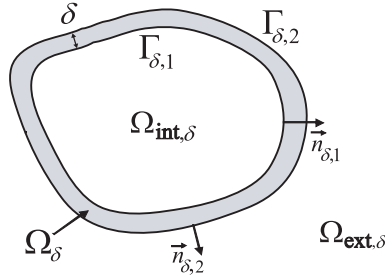


FIGURE 1. Geometric data

field  $u_\delta$ . The asymptotic behaviour of solutions to problems with thin layers has been addressed by many authors in the last decade (cf., e.g., [5, 7, 8, 12, 33, 15, 17, 20]...). Many different techniques have been used in these papers and a variety of results have been obtained. More precisely, approximate transmission conditions have been derived for the electro-quasistatic equations in [29] and time-harmonic Maxwell equations in [28] for thin layer and in [11, 12] for the Laplace equation in the case of thin periodic coating. Higher order approximation were derived in smooth geometries of conductive thin sheets for the Helmholtz equation in [37] and for the eddy current problem in [38]. The case of a thin ring with regularly spaced inhomogeneities has been treated in [15, 17] for the 2D Helmholtz equation.

Here, we derive transmission conditions to approximate the solution  $u_\delta$  to Problem (1.1) by a solution  $u_\delta^{ap}$  to a problem  $(\mathcal{P}_\delta^{ap})$  with the Helmholtz equation defined in a domain without a thin layer with Ventcel-type transmission conditions, involving tangential differential operators of order two, with accuracy up to  $O(\delta^2)$ . We propose a technique (see [8]) that consists of dividing  $\Omega_\delta$  into two thin layers separated by a surface  $\Gamma$  parallel to  $\Gamma_{\delta,1}$  and  $\Gamma_{\delta,2}$  (see Fig. 2 and Fig. 3) and choosing it in such a way that transmission conditions ensure existence and uniqueness of the solution  $u_\delta^{ap}$  to the approximate

problem. The main difficulty here, compared to the studies performed in [7, 8], is that the Helmholtz equation is not elliptic implying, for example, that we do not readily have a stability result which is uniform with respect to  $\delta$ . Another difficulty comes from the unbounded setting of the study.

In order to accomplish our goal, we derive an asymptotic expansion of the solution  $u_\delta$  to Problem (1.1). Two different approaches are often used: the matched asymptotic expansions method (cf., e.g., [2, 3, 15, 21, 24]) and the method of multiscale expansion (cf., e.g., [4, 7, 37]). However the problem is not defined on the exterior of the thin layer and thus the multiscale expansion method is more suitable since we are dealing with the transmission problem [7, 8].

The paper is organized as follows. In Section 2, we give the statement of the problem considered, the existence and uniqueness theorem for the solution to Problem (1.1) together with a uniform stability estimate for  $u_\delta$ . Section 3 recalls some basic definitions and notation from the differential geometry of surfaces.

In Section 4, we construct a formal asymptotic expansion for the solution to Problem (1.1), while Section 5 focuses on the justification of the asymptotics and the convergence of this ansatz. In Section 6, we model the effect of the thin layer by a problem with Ventcel-type transmission conditions. The well-posedness of Problem  $(\mathcal{P}_\delta^{ap})$  will also be proved.

Finally, in Section 7, we extend our results to the case of materials having high magnetic permittivity in the domain  $\Omega_\delta$ .

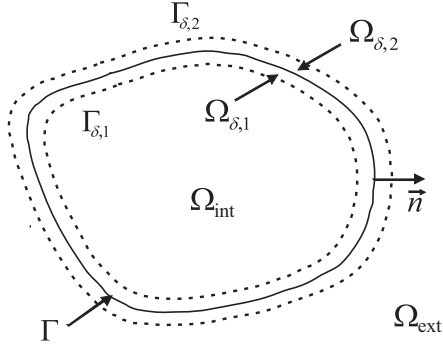


FIGURE 2. Geometry of the studied problem

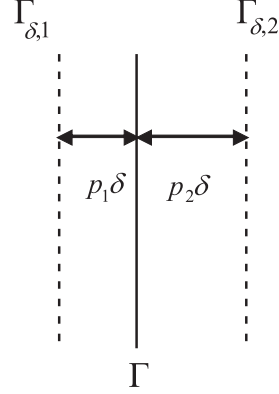


FIGURE 3. The thin layer  $\Omega_\delta$

## 2 Problem setting

We consider a parallel surface  $\Gamma$  to  $\Gamma_{\delta,1}$  and  $\Gamma_{\delta,2}$  dividing  $\Omega_\delta$  into two thin layers  $\Omega_{\delta,1}$  and  $\Omega_{\delta,2}$  of respective thickness  $p_1\delta$  and  $p_2\delta$ , where  $p_1$  and  $p_2$  are strictly positive real numbers satisfying  $p_1 + p_2 = 1$  and such that  $p_1$  and  $p_2$  belong to a small neighbourhood of  $1/2$  (see Fig. 2 and Fig. 3). The term *small neighbourhood* means that the constants  $p_1$  and  $p_2$  are not too close to 1 or 0, in order to avoid having a layer too thin compared to the other because the following analysis does not lend itself to this case. Let us denote by  $u_{\text{ext},\delta}$ ,  $u_{d_2,\delta}$ ,  $u_{d_1,\delta}$  and  $u_{\text{int},\delta}$  the restrictions of  $u_\delta$  respectively to the domains  $\Omega_{\text{ext},\delta}$ ,  $\Omega_{\delta,2}$ ,  $\Omega_{\delta,1}$  and  $\Omega_{\text{int},\delta}$ . Under the aforementioned assumptions, we investigate in  $H_{loc}^1(\mathbb{R}^3)$  the solution  $u_\delta$  to the following equivalent problem

$$\begin{cases} \operatorname{div}(\sigma_{\text{ext}} \nabla u_{\text{ext},\delta}) + k_{\text{ext}}^2 u_{\text{ext},\delta} = 0 & \text{in } \Omega_{\text{ext},\delta}, & (2.1 a) \\ \operatorname{div}(\tilde{\sigma}_\delta \nabla u_{d_2,\delta}) + \tilde{k}_\delta^2 u_{d_2,\delta} = 0 & \text{in } \Omega_{\delta,2}, & (2.1 b) \\ \operatorname{div}(\tilde{\sigma}_\delta \nabla u_{d_1,\delta}) + \tilde{k}_\delta^2 u_{d_1,\delta} = 0 & \text{in } \Omega_{\delta,1}, & (2.1 c) \\ \operatorname{div}(\sigma_{\text{int}} \nabla u_{\text{int},\delta}) + k_{\text{int}}^2 u_{\text{int},\delta} = 0 & \text{in } \Omega_{\text{int},\delta}, & (2.1 d) \end{cases}$$

with transmission conditions

$$\left\{ \begin{array}{ll} u_{d_2, \delta}|_{\Gamma_{\delta, 2}} = u_{\text{ext}, \delta}|_{\Gamma_{\delta, 2}} & \text{on } \Gamma_{\delta, 2}, \quad (2.1 e) \\ \tilde{\sigma}_\delta \partial_{\mathbf{n}_{\delta, 2}} u_{d_2, \delta}|_{\Gamma_{\delta, 2}} = \sigma_{\text{ext}} \partial_{\mathbf{n}_{\delta, 2}} u_{\text{ext}, \delta}|_{\Gamma_{\delta, 2}} & \text{on } \Gamma_{\delta, 2}, \quad (2.1 f) \\ u_{d_1, \delta}|_\Gamma = u_{d_2, \delta}|_\Gamma & \text{on } \Gamma, \quad (2.1 g) \\ \partial_{\mathbf{n}} u_{d_1, \delta}|_\Gamma = \partial_{\mathbf{n}} u_{d_2, \delta}|_\Gamma & \text{on } \Gamma, \quad (2.1 h) \\ u_{\text{int}, \delta}|_{\Gamma_{\delta, 1}} = u_{d_1, \delta}|_{\Gamma_{\delta, 1}} & \text{on } \Gamma_{\delta, 1}, \quad (2.1 i) \\ \sigma_{\text{int}} \partial_{\mathbf{n}_{\delta, 1}} u_{\text{int}, \delta}|_{\Gamma_{\delta, 1}} = \tilde{\sigma}_\delta \partial_{\mathbf{n}_{\delta, 1}} u_{d_1, \delta}|_{\Gamma_{\delta, 1}} & \text{on } \Gamma_{\delta, 1}, \quad (2.1 j) \end{array} \right.$$

and radiation condition

$$\lim_{|x| \rightarrow +\infty} |x| (\partial_{|x|} - ik_{\text{ext}}) (u_{\text{ext}, \delta} - u_{\text{inc}}) = 0, \quad (2.1 k)$$

where  $\partial_{\mathbf{n}_{\delta, 1}}$ ,  $\partial_{\mathbf{n}}$ ,  $\partial_{\mathbf{n}_{\delta, 2}}$  and  $\partial_{\mathbf{n}_e}$  denote the derivatives in the direction of the unit normal vectors  $\mathbf{n}$ ,  $\mathbf{n}_{\delta, 1}$ ,  $\mathbf{n}_{\delta, 2}$  and  $\mathbf{n}_e$  to  $\Gamma_{\delta, 1}$ ,  $\Gamma$ ,  $\Gamma_{\delta, 2}$  and  $\partial\Omega$  respectively (see Fig. 1). The following theorem gives the well-posedness of (2.1).

**Theorem 2.1** *Problem (2.1) has one and only one solution  $u_\delta$  in  $H_{loc}^1(\mathbb{R}^3)$ .*

**Proof** Uniqueness follows by Rellich's lemma (cf. [35, 13]). Existence of a solution is obtained by standard arguments involving the limiting absorption principle (cf., e.g., [30]).  $\square$

We now rewrite the problem in a truncated domain (see [4, 5, 3] for a similar reduction) in order to get a uniform stability result with respect to  $\delta$ . The latter is actually going



to be useful for proving error estimates between  $u_\delta$  and the asymptotic expansion that is going to be built in the next sections.

Let  $\Omega$  be a bounded domain of class  $\mathcal{C}^\infty$  which contains the thin layer  $\Omega_\delta$  as depicted in Fig. 4.

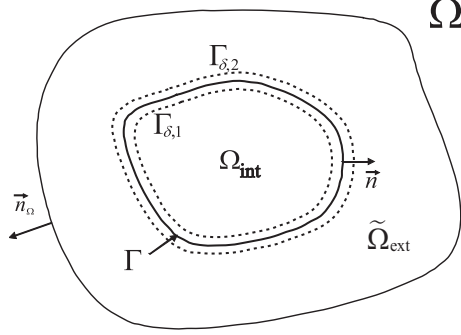


FIGURE 4. The truncated domain

We denote by  $T$  the DtN operator (Dirichlet-to-Neumann) defined on  $H^{1/2}(\partial\Omega)$  by  $T\varphi := -\partial_{\mathbf{n}_\Omega}\omega$ , where  $\mathbf{n}_\Omega$  is the unit normal to  $\partial\Omega$  directed out of  $\Omega$  and  $\omega$  is the unique solution to the following problem

$$\left\{ \begin{array}{ll} \text{Find } \omega \in H_{loc}^1(\mathbb{R}^3 \setminus \bar{\Omega}) & \\ \operatorname{div}(\sigma_{\text{ext}} \nabla \omega) + k_{\text{ext}}^2 \omega = 0 & \text{in } \mathbb{R}^3 \setminus \bar{\Omega}, \\ \omega|_{\partial\Omega} = \varphi & \text{on } \partial\Omega, \\ \lim_{|x| \rightarrow +\infty} |x| (\partial_{|x|} - ik_{\text{ext}}) \omega = 0. & \end{array} \right. \quad (2.2)$$

The DtN operator  $T$  is a pseudodifferential operator of order one [4] and is linearly continuous from  $H^{1/2}(\partial\Omega)$  to  $H^{-1/2}(\partial\Omega)$ . The next lemma, whose proof uses standard elliptic regularity (cf., e.g., [4]), gives a useful decomposition of the DtN operator.

**Lemma 2.1** *Let  $\phi \in H^{1/2}(\partial\Omega)$  and  $\varphi_0 \in H^1(\mathbb{R}^3 \setminus \bar{\Omega})$  be the unique solution to the*

following coercive scattering problem

$$\begin{cases} \Delta\varphi_0 - \varphi_0 = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega} \\ \varphi_0 = \phi & \text{on } \partial\Omega. \end{cases}$$

Now let us consider  $T_0\phi = -\partial_{\mathbf{n}\Omega}\varphi_0$ . Then  $T_0$  is bounded and coercive from  $H^{1/2}(\partial\Omega)$  into  $H^{-1/2}(\partial\Omega)$ . In addition, there exists a compact operator  $K$  acting from  $H^{1/2}(\partial\Omega)$  into  $H^{3/2}(\partial\Omega)$  such that

$$T = T_0 + K. \quad (2.3)$$

Using the DtN operator  $T$ , Problem (1.1) can be written as

$$\begin{cases} \text{Find } u_\delta \in H^1(\Omega) \text{ such that} \\ \operatorname{div}(\sigma_\delta \nabla u_\delta) + k_\delta^2 u_\delta = 0 & \text{in } \Omega, \\ (\partial_{\mathbf{n}\Omega} + T) u_\delta = (\partial_{\mathbf{n}\Omega} + T) u_{inc} & \text{on } \partial\Omega. \end{cases} \quad (2.4)$$

A variational formulation of (2.4) is then given by

$$\begin{cases} \text{Find } u_\delta \in H^1(\Omega), \forall v \in H^1(\Omega) \\ a_\delta(u_\delta, v) := \int_{\Omega} \sigma_\delta \nabla u_\delta \cdot \nabla \bar{v} - k_\delta^2 u_\delta \bar{v} \, d\Omega + \sigma_{\text{ext}} \langle T u_\delta, \bar{v} \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} \\ = l_\delta(v), \end{cases} \quad (2.5)$$

where  $\langle \cdot, \cdot \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)}$  denotes the duality pairing between  $H^{-1/2}(\partial\Omega)$  and

$H^{1/2}(\partial\Omega)$  and  $l_\delta$  is an arbitrary linear form on  $H^1(\Omega)$ . For Problem (2.4),  $l_\delta$  is defined

by

$$l_\delta(v) := \sigma_{\text{ext}} \int_{\partial\Omega} (\partial_{\mathbf{n}\Omega} + T) u_{inc} \bar{v} \, d\sigma.$$

For our purpose, one needs to know about the dependance of  $u_\delta$  as  $\delta$  goes to zero.

**Theorem 2.2** [Uniform stability.] Suppose that

$$\exists \varepsilon > 0, \tilde{\sigma}_\delta = O(\delta^{-\frac{1}{2}+\varepsilon}), \tilde{k}_\delta^2 = O(\delta^{-\frac{1}{2}+\varepsilon}); \quad (2.6)$$

$$\exists \alpha > 0, \forall \delta > 0, \Re(\sigma_\delta) > \alpha; \quad (2.7)$$

then, for all  $l_\delta$  in  $(H^1(\Omega))'$ , Problem (2.5) admits a unique solution in  $H^1(\Omega)$ . Furthermore, there exists a positive constant  $c$  independent of  $\delta$  such that

$$\|u_\delta\|_{H^1(\Omega)} \leq c \|l_\delta\|_{(H^1(\Omega))'}.$$

**Proof** We need to prove that

$$\|u_\delta\|_{H^1(\Omega)} \leq C \sup_{v \in H^1(\Omega)} \frac{|a_\delta(u_\delta, v)|}{\|v\|_{H^1(\Omega)}}.$$

To do so, we use a standard proof (cf. e.g., [23, 16]), and proceed by contradiction by assuming that there exists sequences  $(\delta_n)_{n \geq 0}$  and  $(u_{\delta_n})_{n \geq 0}$  (denoted by  $(u_n)_{n \geq 0}$ ) such that

$$\lim_{n \rightarrow +\infty} \delta_n \rightarrow 0, \|u_n\|_{H^1(\Omega)} = 1, \forall n \in \mathbb{N}, \lim_{n \rightarrow +\infty} \sup_{\|\varphi\|_{H^1(\Omega)}=1} |a_{\delta_n}(u_n, \varphi)| = 0.$$

From Rellich's embedding theorem, we can extract a subsequence, still denoted by  $(u_n)_{n \geq 0}$ , such that

$$\begin{cases} u_n \rightarrow u_0 \text{ in } L^2(\Omega), \\ u_n \rightharpoonup u_0 \text{ in } H^1(\Omega). \end{cases}$$

Since  $\sigma_\delta$  and  $k_\delta^2$  satisfy (2.6) one gets

$$\begin{cases} \sigma_\delta \rightarrow \sigma_0 := \sigma_{\text{int}} \chi_{\Omega_{\text{int}}}(x) + \sigma_{\text{ext}} \chi_{\tilde{\Omega}_{\text{ext}}}(x) \text{ in } L^2(\Omega), \\ k_\delta^2 \rightarrow k_0^2 := k_{\text{int}}^2 \chi_{\Omega_{\text{int}}}(x) + k_{\text{ext}}^2 \chi_{\tilde{\Omega}_{\text{ext}}}(x) \text{ in } L^2(\Omega), \end{cases} \quad (2.8)$$

where  $\Omega_{\text{int}} = \lim_{\delta \rightarrow 0} \Omega_{\text{int},\delta}$  and  $\tilde{\Omega}_{\text{ext}} = \Omega \setminus \overline{\Omega_{\text{int}}}$  (see Fig. 2) and  $\chi_{\mathcal{O}}$  denotes the indicator function of the open set  $\mathcal{O}$ . Indeed,

$$\|\sigma_{\delta} - \sigma_0\|_{L^2(\Omega)}^2 = \int_{\Omega_{\delta}} |\sigma_{\delta} - \sigma_0|^2 d\Omega_{\delta} \leq \int_{\Omega_{\delta,1}} |\sigma_{\text{int}}|^2 d\Omega_{\delta,1} + \int_{\Omega_{\delta}} |\tilde{\sigma}_{\delta}|^2 d\Omega_{\delta} + \int_{\Omega_{\delta,2}} |\sigma_{\text{ext}}|^2 d\Omega_{\delta,2}.$$

Using Lebesgue dominated convergence theorem, we obtain

$$\lim_{\delta \rightarrow 0} \int_{\Omega_{\delta,1}} |\sigma_{\text{int}}|^2 d\Omega_{\delta,1} = \lim_{\delta \rightarrow 0} \int_{\Omega_{\delta,2}} |\sigma_{\text{ext}}|^2 d\Omega_{\delta,2} = 0.$$

Moreover, we have

$$\int_{\Omega_{\delta}} |\tilde{\sigma}_{\delta}|^2 d\Omega_{\delta} \leq C \left( \delta^{-\frac{1}{2} + \varepsilon} \right)^2 \text{meas}(\Omega_{\delta}) = C \delta^{2\varepsilon} \text{meas}(\Gamma) \rightarrow_{\delta \rightarrow 0} 0.$$

Now, upon using (2.8), we get

$$\lim_{n \rightarrow +\infty} a_{\delta_n}(u_n, \varphi) = \int_{\Omega} \sigma_0 \nabla u_0 \cdot \overline{\nabla \varphi} - k_0^2 u_0 \overline{\varphi} d\Omega \text{ for all } \varphi \text{ in } H^1(\Omega),$$

so  $u_0 \in H^1(\Omega)$  satisfies

$$\begin{cases} \operatorname{div}(\sigma_0 \nabla u_0) + k_0^2 u_0 = 0 & \text{in } \Omega, \\ (\partial_{\mathbf{n}_{\Omega}} + T) u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

As a result, well-known properties of uniqueness of the solution to this type of problem based on Rellich's lemma and the operator  $T$  imply that  $u_0 = 0$ . It only remains to show that  $\lim_{n \rightarrow +\infty} \|u_n\|_{H^1(\Omega)} = 0$ . Note that, since  $u_0$  is uniquely determined, the whole sequence  $(u_n)_{n \geq 0}$  that converges to  $u_0$ . To obtain the contradiction, we now show that

$\lim_{n \rightarrow +\infty} \|\nabla u_n\|_{L^2(\Omega)} = 0$ . From (2.7), we have

$$\begin{aligned} \|\nabla u_n\|_{L^2(\Omega)}^2 &\leq C \int_{\Omega} \sigma_{\delta} |\nabla u_n|^2 d\Omega \\ &= C \Re \left( a_n(u_n, u_n) + \int_{\Omega} k_{\delta_n}^2 |u_n|^2 d\Omega - \sigma_{\text{ext}} \langle T u_n, \overline{u_n} \rangle_{H^{-1/2} \times H^{1/2}} \right). \end{aligned}$$

Using Lemma 2.1, we infer

$$\|\nabla u_n\|_{L^2(\Omega)}^2 \leq C \left\{ \Re \left[ a_n(u_n, u_n) + \int_{\Omega} k_{\delta_n}^2 |u_n|^2 d\Omega - \sigma_{\text{ext}} \langle K u_n, \bar{u}_n \rangle_{H^{-1/2} \times H^{1/2}} \right] \right\}.$$

As  $u_n \xrightarrow{n \rightarrow +\infty} u_0 = 0$  in  $L^2(\Omega)$  and  $k_{\delta_n}^2 \xrightarrow{n \rightarrow +\infty} k_0^2$  in  $L^2(\Omega)$ , it follows that

$$\int_{\Omega} k_{\delta_n}^2 |u_n|^2 d\Omega \xrightarrow{n \rightarrow +\infty} 0.$$

Since  $K$  is compact and  $u_n \rightarrow 0$  in  $H^1(\Omega)$ ,  $\langle K u_n, \bar{u}_n \rangle_{H^{-1/2} \times H^{1/2}} \xrightarrow{n \rightarrow +\infty} 0$ . Finally,

the hypothesis  $\lim_{n \rightarrow +\infty} \Re[a_n(u_n, u_n)] = 0$  yields  $\lim_{n \rightarrow +\infty} \|\nabla u_n\|_{L^2(\Omega)} = 0$  contradicting

$$\|u_n\|_{H^1(\Omega)} = 1. \quad \square$$

**Remark 2.1** *In the proof of Theorem 2.2, we require the convergence in  $L^2$  of  $\sigma_{\delta}$  and  $k_{\delta}$ . This justifies assumption (2.6) used to prove that (2.8) holds.*

### 3 Tools of differential geometry

The goal of this section is to define and to collect the main features of differential geometry [19] (see also [27]) in order to formulate our problem in a fixed domain (independent of  $\delta$ ). This technique is a key tool to determine the asymptotic expansion of the solution  $u_{\delta}$ .

#### 3.1 Parametrization of the surface $\Gamma$

Let  $(\mathcal{U}, \varphi)$  be a local coordinate patch for the surface  $\Gamma$ , with  $\mathcal{U}$  being an open domain of  $\mathbb{R}^2$  and

$$\begin{aligned} \varphi: \quad \mathcal{U} &\rightarrow \Gamma \\ (\xi^1, \xi^2) &\rightarrow m = \varphi(\xi^1, \xi^2). \end{aligned}$$

A basis of the tangent plane  $T_m(\Gamma)$  to  $\Gamma$  at the point  $m \in \Gamma$  is given by

$$\tau_\alpha(m) := \frac{\partial \varphi(\xi^1, \xi^2)}{\partial \xi^\alpha}; \quad \alpha = 1, 2.$$

We assume that the coordinate patch  $\{\tau_\alpha\}_{\alpha=1,2}$  is compatible with the orientation, namely, the unit normal  $\mathbf{n}(m)$  to  $\Gamma$  at point  $m$  is given by

$$\mathbf{n}(m) := \frac{\tau_1 \times \tau_2}{|\tau_1 \times \tau_2|},$$

where  $\times$  and  $|\cdot|$  are respectively the usual cross product and norm in  $\mathbb{R}^3$ .

We denote by  $\mathcal{R}$  the symmetric linear operator of the tangent plane  $T_m(\Gamma)$  that characterizes the curvature of  $\Gamma$  at point  $m$ , and defined by

$$\frac{\partial \mathbf{n}(m)}{\partial \xi^\alpha} := \mathcal{R}(m)\tau_\alpha; \quad \alpha = 1, 2.$$

Let  $\Pi_m$  be the orthogonal projector from  $\mathbb{R}^3$  into  $T_m(\Gamma)$  and  $\mathbf{w}$  a vector of  $\mathbb{R}^3$ , we have

$$\mathbf{w} = \mathbf{w}_T + w_n \mathbf{n} = \Pi_m \mathbf{w} + w_n \mathbf{n},$$

where  $\mathbf{w}_T = \Pi_m \mathbf{w}$  is the tangential component and  $w_n$  is the normal component of  $\mathbf{w}$ .

### 3.2 Differential operators on $\Gamma$

Let  $v$  be a smooth function defined on  $\Gamma$ . The surfacic gradient  $\nabla_\Gamma v(m)$  of  $v$  at  $m \in \Gamma$  is defined by

$$\nabla_\Gamma v(m) := \sum_{\lambda=1}^2 \left[ \sum_{\alpha=1}^2 G^{\lambda\alpha}(m) \frac{\partial}{\partial \xi^\alpha} (v \circ \varphi)(\xi^1, \xi^2) \right] \tau_\lambda(m),$$

where  $(G^{\lambda\alpha}(m))$  is the inverse of the metric tensor  $(\tau_\lambda(m) \cdot \tau_\alpha(m))_{\lambda, \alpha=1,2}$ .

If  $\hat{v}$  is a function defined in a neighbourhood of  $\Gamma$ , we have

$$\nabla_\Gamma(\hat{v})(m) := \Pi_m(\nabla \hat{v}(m)); \quad m \in \Gamma.$$

Let  $\mathbf{w}_T$  be a smooth tangent vector field defined on  $\Gamma$ . The surfacic divergence of  $\mathbf{w}_T$  is the scalar function defined on  $\Gamma$  through Stokes formula

$$\int_{\Gamma} \phi \operatorname{div}_{\Gamma} \mathbf{w}_T \, d\Gamma := - \int_{\Gamma} \nabla_{\Gamma} \phi \cdot \mathbf{w}_T \, d\Gamma,$$

where  $\phi$  is any regular function on  $\Gamma$  and  $d\Gamma = |\tau_1 \times \tau_2| d\xi^1 d\xi^2$  denotes the surfacic measure on  $\Gamma$ . The scalar Laplace-Beltrami operator on  $\Gamma$  is finally given by

$$\Delta_{\Gamma} := \operatorname{div}_{\Gamma} (\nabla_{\Gamma}).$$

### 3.3 Parametrization of $\Omega_{\delta,\beta}$

In what follows, the Greek indice  $\beta$  takes the values 1 and 2. Let  $I_{\delta,1} = (-\delta, 0)$  and  $I_{\delta,2} = (0, \delta)$ . We parameterize the thin shell  $\Omega_{\delta,\beta}$  by the manifold  $\Gamma \times I_{\delta,\beta}$  through the mapping  $\psi_{\beta}$  defined by

$$\begin{cases} \Gamma \times I_{\delta,\beta} & \xrightarrow{\psi_{\beta}} & \Omega_{\delta,\beta} \\ (m, \eta_{\beta}) & \rightarrow & x := m + p_{\beta} \eta_{\beta} \mathbf{n}(m). \end{cases}$$

As is well known [19], if the thickness of  $\Omega_{\delta,\beta}$  is small enough,  $\psi_{\beta}$  is a  $C^{\infty}$ -diffeomorphism of manifolds and it is also known [32, Remark 2.1] that the normal vector  $\mathbf{n}_{\delta,\beta}$  to  $\Gamma_{\delta,\beta}$  can be identified with  $\mathbf{n}$ .

With each function  $v_{\beta}$  defined on  $\Omega_{\delta,\beta}$ , we associate the function  $\tilde{v}_{\beta}$  defined on  $\Gamma \times I_{\delta,\beta}$  by

$$\begin{cases} \tilde{v}_{\beta}(m, \eta_{\beta}) & := v_{\beta}(x), \\ x & = \psi_{\beta}(m, \eta_{\beta}). \end{cases}$$

One then has

$$\frac{\partial \tilde{v}_{\beta}}{\partial \xi^{\alpha}} = \sum_{i=1}^3 \frac{\partial v_{\beta}}{\partial x^i} \frac{\partial x^i}{\partial \xi^{\alpha}} = \nabla v_{\beta} \cdot (I + p_{\beta} \eta_{\beta} \mathcal{R}) \tau_{\alpha}; \quad \alpha = 1, 2$$

and

$$\frac{\partial \tilde{v}_\beta}{\partial \eta_\beta} = \sum_{i=1}^3 \frac{\partial v_\beta}{\partial x^i} \frac{\partial x^i}{\partial \eta_\beta} = p_\beta \nabla v_\beta \cdot \mathbf{n},$$

where  $I$  is the identity operator on the tangent plane  $T_m(\Gamma)$ .

Since the vector  $(I + p_\beta \eta_\beta \mathcal{R})\tau_\alpha$  is in  $T_m(\Gamma)$  and  $(I + p_\beta \eta_\beta \mathcal{R})$  is a symmetric operator,

we can write

$$\frac{\partial \tilde{v}_\beta}{\partial \xi^\alpha} = (I + p_\beta \eta_\beta \mathcal{R}) \Pi_m \nabla v_\beta \cdot \tau_\alpha,$$

or equivalently [18]

$$\Pi_m \nabla v_\beta = (I + p_\beta \eta_\beta \mathcal{R})^{-1} \nabla_\Gamma \tilde{v}_\beta.$$

One gets

$$\nabla v_\beta = (I + p_\beta \eta_\beta \mathcal{R})^{-1} \nabla_\Gamma \tilde{v}_\beta + p_\beta^{-1} \frac{\partial \tilde{v}_\beta}{\partial \eta_\beta} \mathbf{n}.$$

The volume element on the thin shell  $\Omega_{\delta,\beta}$  is given by

$$d\Omega_{\delta,\beta} = \frac{\partial x}{\partial \xi^1} \times \frac{\partial x}{\partial \xi^2} \cdot \frac{\partial x}{\partial \eta_\beta} d\xi^1 d\xi^2 d\eta_\beta.$$

As

$$\frac{\partial x}{\partial \xi^1} \times \frac{\partial x}{\partial \xi^2} = (I + p_\beta \eta_\beta \mathcal{R}) \tau_1 \times (I + p_\beta \eta_\beta \mathcal{R}) \tau_2 = \det(I + p_\beta \eta_\beta \mathcal{R}) (\tau_1 \times \tau_2),$$

and

$$|\tau_1 \times \tau_2| d\xi^1 d\xi^2 = d\Gamma,$$

we obtain

$$d\Omega_{\delta,\beta} = p_\beta \det(I + p_\beta \eta_\beta \mathcal{R}) d\Gamma d\eta_\beta.$$



Now, we introduce the scaling  $s_\beta = \eta_\beta/\delta$ , and the intervals  $I_1 = (-1, 0)$  and  $I_2 = (0, 1)$

such that the  $C^\infty$ -diffeomorphism  $\Phi_\beta$ , defined by

$$\begin{cases} \Omega^\beta := \Gamma \times I_\beta & \xrightarrow{\Phi_\beta} & \Omega_{\delta,\beta} \\ (m, s_\beta) & \rightarrow & x := m + \delta p_\beta s_\beta \mathbf{n}(m), \end{cases}$$

parameterizes the thin shell  $\Omega_{\delta,\beta}$ .

To any function  $v_\beta$  defined on  $\Omega_{\delta,\beta}$ , the function  $v^{[\beta]}$  defined on  $\Omega^\beta$  is associated through

$$\begin{cases} v^{[\beta]}(m, s_\beta) & := v_\beta(x), \\ x & = \Phi_\beta(m, s_\beta). \end{cases}$$

Then, in local coordinates  $(\xi^1, \xi^2, s_\beta)$ , the gradient takes the form

$$\nabla v_\beta = (I + \delta p_\beta s_\beta \mathcal{R})^{-1} \nabla_\Gamma v^{[\beta]} + p_\beta^{-1} \delta^{-1} \frac{\partial v^{[\beta]}}{\partial s_\beta} \mathbf{n}. \quad (3.1)$$

The volume element on the thin shell  $\Omega_{\delta,\beta}$  becomes

$$d\Omega_{\delta,\beta} = p_\beta \delta \det J_{\delta,\beta} d\Gamma ds_\beta, \quad (3.2)$$

where

$$J_{\delta,\beta} := I + p_\beta \delta s_\beta \mathcal{R},$$

and the surfacic measure on  $\Gamma_{\delta,\beta}$  is

$$d\Gamma_{\delta,\beta} = \det (I + (-1)^\beta p_\beta \delta \mathcal{R}) d\Gamma.$$

Let  $u_\beta$  and  $v_\beta$  be two regular functions defined on  $\Omega_{\delta,\beta}$ . From (3.1) and (3.2), we get the change of variables formula

$$\begin{aligned} \int_{\Omega_{\delta,\beta}} \nabla u_\beta \cdot \nabla v_\beta d\Omega_{\delta,\beta} &= p_\beta \delta \int_{\Omega^\beta} J_{\delta,\beta}^{-2} \nabla_\Gamma u^{[\beta]} \cdot \nabla_\Gamma v^{[\beta]} \det J_{\delta,\beta} d\Gamma ds_\beta \\ &+ p_\beta^{-1} \delta^{-1} \int_{\Omega^\beta} \partial_{s_\beta} u^{[\beta]} \partial_{s_\beta} v^{[\beta]} \det J_{\delta,\beta} d\Gamma ds_\beta, \end{aligned} \quad (3.3)$$

$$\int_{\Omega_{\delta,\beta}} u_\beta v_\beta \, d\Omega_{\delta,\beta} = p_\beta \delta \int_{\Omega^\beta} u^{[\beta]} v^{[\beta]} \det J_{\delta,\beta} \, d\Gamma ds_\beta. \quad (3.4)$$

**Remark 3.1** For any function  $u$  defined in a neighbourhood of  $\Gamma$ , we denote, for convenience, by  $u|_\Gamma$  the trace of  $u$  on  $\Gamma$  indifferently in local coordinates or in Cartesian coordinates.

#### 4 The asymptotic analysis

This section is devoted to the asymptotic analysis of the solution to Problem (2.1). From now on, we assume that  $\tilde{\sigma}_\delta$  and  $\tilde{k}_\delta$  are independent of  $\delta$  (denoted by  $\tilde{\sigma}$  and  $\tilde{k}$  respectively) to simplify the overall presentation. We give a hierarchy of variational equations defined in a domain that does not depend on  $\delta$  suited to the construction of a formal asymptotic expansion up to any order. We then calculate the first two terms and we conclude with a convergence theorem ensuring the validity of the ansatz.

Let  $v_d$  be in  $H^1(\Omega_\delta)$ . We denote by  $v_{d,\beta}$  its restriction to  $\Omega_{\delta,\beta}$ . Multiplying Equation

$$\operatorname{div}(\tilde{\sigma} \nabla u_{d,\delta}) + \tilde{k}^2 u_{d,\delta} = 0 \text{ in } \Omega_\delta,$$

by test functions  $v_d$ , using (2.1 f)-(2.1 h), (2.1 j) and Green's formula, we get

$$\begin{aligned} & \langle \sigma_{\text{int}} \partial_{\mathbf{n}_{\delta,1}} u_{\text{int},\delta}|_{\Gamma_{\delta,1}}, v_{d1}|_{\Gamma_{\delta,1}} \rangle_{H^{-1/2}(\Gamma_{\delta,1}) \times H^{1/2}(\Gamma_{\delta,1})} \\ & - \langle \sigma_{\text{ext}} \partial_{\mathbf{n}_{\delta,2}} u_{\text{ext},\delta}|_{\Gamma_{\delta,2}}, v_{d2}|_{\Gamma_{\delta,2}} \rangle_{H^{-1/2}(\Gamma_{\delta,2}) \times H^{1/2}(\Gamma_{\delta,2})} \\ & + \tilde{\sigma} \int_{\Omega_\delta} \nabla u_{d,\delta} \cdot \nabla v_d \, d\Omega_\delta - \tilde{k}^2 \int_{\Omega_\delta} u_{d,\delta} v_d \, d\Omega_\delta = 0. \end{aligned}$$

Using the dilation of the thin layers, (3.3) and (3.4), one obtains

$$\langle \sigma_{\text{int}} \partial_{\mathbf{n}_{\delta,1}} u_{\text{int},\delta}|_{\Gamma_{\delta,1}} \circ \Phi_1(m, -1), v_d^{[1]}(m, -1) \rangle_{H^{-1/2}(\Gamma \times \{-1\}) \times H^{1/2}(\Gamma \times \{-1\})}$$

$$\begin{aligned}
& - \langle \sigma_{\text{ext}} \partial_{\mathbf{n}_{\delta,2}} u_{\text{ext},\delta}|_{\Gamma_{\delta,2}} \circ \Phi_2(m, 1), v_d^{[2]}(m, 1) \rangle_{H^{-1/2}(\Gamma \times \{1\}) \times H^{1/2}(\Gamma \times \{1\})} \\
& + \sum_{\beta=1}^2 \left[ \delta a_\delta^{[\beta]} \left( u_{d,\delta}^{[\beta]}, v_d^{[\beta]} \right) + \delta b_\delta^{[\beta]} \left( u_{d,\delta}^{[\beta]}, v_d^{[\beta]} \right) \right] = 0, \tag{4.1}
\end{aligned}$$

where the bilinear forms  $a_\delta^{[\beta]}(\cdot, \cdot)$  and  $b_\delta^{[\beta]}(\cdot, \cdot)$  ( $\beta = 1, 2$ ) are defined by

$$\begin{aligned}
a_\delta^{[\beta]} \left( u^{[\beta]}, v^{[\beta]} \right) & := \tilde{\sigma} p_\beta \int_{\Omega^\beta} J_{\delta,\beta}^{-2} \nabla_\Gamma u^{[\beta]} \cdot \nabla_\Gamma v^{[\beta]} \det J_{\delta,\beta} \, d\Gamma ds_\beta \\
& + \tilde{\sigma} p_\beta^{-1} \delta^{-2} \int_{\Omega^\beta} \partial_{s_\beta} u^{[\beta]} \partial_{s_\beta} v^{[\beta]} \det J_{\delta,\beta} \, d\Gamma ds_\beta, \tag{4.2}
\end{aligned}$$

and

$$b_\delta^{[\beta]} \left( u^{[\beta]}, v^{[\beta]} \right) := -\tilde{k}^2 p_\beta \int_{\Omega^\beta} u^{[\beta]} v^{[\beta]} \det J_{\delta,\beta} \, d\Gamma ds_\beta, \tag{4.3}$$

for every  $u^{[\beta]}$  and  $v^{[\beta]}$  in  $H^1(\Omega^\beta)$ . Standard regularity results for elliptic problems (see e.g. [1]) ensure that the trace of  $u_\delta$  on  $\Gamma_{\delta,1}$  or  $\Gamma_{\delta,2}$  is  $\mathcal{C}^\infty$ . This fact allows us to write

Problem (4.1) as

$$\begin{aligned}
& \int_\Gamma \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},\delta}|_{\Gamma_{\delta,1}} \circ \Phi_1(m, -1) v_d^{[1]}(m, -1) \det(I - p_1 \delta \mathcal{R}) \, d\Gamma \\
& - \int_\Gamma \sigma_{\text{ext}} \partial_{\mathbf{n}_{\delta,2}} u_{\text{ext},\delta}|_{\Gamma_{\delta,2}} \circ \Phi_2(m, 1) v_d^{[2]}(m, 1) \det(I + p_2 \delta \mathcal{R}) \, d\Gamma \\
& + \sum_{\beta=1}^2 \left[ \delta a_\delta^{[\beta]} \left( u_{d,\delta}^{[\beta]}, v_d^{[\beta]} \right) + \delta b_\delta^{[\beta]} \left( u_{d,\delta}^{[\beta]}, v_d^{[\beta]} \right) \right] = 0, \tag{4.4}
\end{aligned}$$

which is the starting point of the asymptotic analysis.

#### 4.1 Hierarchy of the variational equations

To carry out an asymptotic expansion of the solution  $u_\delta$  of (2.1) in powers of  $\delta$ , we consider two asymptotic expansions. Exterior expansions corresponding to the expansion of  $u_\delta$  restricted to  $\Omega_{\text{ext},\delta}$  and to  $\Omega_{\text{int},\delta}$  are characterized by the ansatz

$$u_{\text{ext},\delta} = u_{\text{ext},0} + \delta u_{\text{ext},1} + \cdots, \tag{4.5}$$

$$u_{\text{int},\delta} = u_{\text{int},0} + \delta u_{\text{int},1} + \dots, \quad (4.6)$$

where the terms  $u_{\text{ext},n}$  and  $u_{\text{int},n}$  ( $n \in \mathbb{N}$ ) are independent of  $\delta$  and respectively defined on  $\Omega_{\text{ext}} := \Omega_{\text{ext},\delta} \cup \Gamma_{\delta,2} \cup \Omega_{\delta,2}$ , and on  $\Omega_{\text{int}} := \Omega_{\text{int},\delta} \cup \Gamma_{\delta,1} \cup \Omega_{\delta,1}$ . The latter are the limits of  $\Omega_{\text{ext},\delta}$  and  $\Omega_{\text{int},\delta}$  when  $\delta \rightarrow 0$ . They fulfill

$$\begin{cases} \operatorname{div}(\sigma_{\text{int}} \nabla u_{\text{int},n}) + k_{\text{int}}^2 u_{\text{int},n} = 0 & \text{in } \Omega_{\text{int}}, \\ \operatorname{div}(\sigma_{\text{ext}} \nabla u_{\text{ext},n}) + k_{\text{ext}}^2 u_{\text{ext},n} = 0 & \text{in } \Omega_{\text{ext}}, \\ \lim_{|x| \rightarrow +\infty} |x| (\partial_{|x|} - ik_{\text{ext}}) (u_{\text{ext},n} - \delta_{0,n} u_{\text{inc}}) = 0, \end{cases} \quad (4.7)$$

where  $\delta_{0,n}$  indicates the Kronecker symbol. An interior expansion corresponding to the asymptotic expansion of  $u_{d,\delta}$  written in a fixed domain is now defined by the ansatz

$$u_{d,\delta}^{[\beta]} = u_0^{[\beta]} + \delta u_1^{[\beta]} + \dots, \quad \text{in } \Omega^\beta, \quad (4.8)$$

where the terms  $u_n^{[\beta]}$ ,  $n \in \mathbb{N}$ , are independent of  $\delta$ . Using a Taylor expansion in the normal variable, we formally infer

$$u_{\text{int},\delta|_{\Gamma_{\delta,1}}} = u_{\text{int},0|_{\Gamma}} + \delta(u_{\text{int},1|_{\Gamma}} - p_1 \partial_{\mathbf{n}} u_{\text{int},0|_{\Gamma}}) + \dots,$$

$$\partial_{\mathbf{n}_{\delta,1}} u_{\text{int},\delta|_{\Gamma_{\delta,1}}} = \partial_{\mathbf{n}} u_{\text{int},0|_{\Gamma}} + \delta(\partial_{\mathbf{n}} u_{\text{int},1|_{\Gamma}} - p_1 \partial_{\mathbf{n}}^2 u_{\text{int},0|_{\Gamma}}) + \dots,$$

and

$$u_{\text{ext},\delta|_{\Gamma_{\delta,2}}} = u_{\text{ext},0|_{\Gamma}} + \delta(u_{\text{ext},1|_{\Gamma}} + p_2 \partial_{\mathbf{n}} u_{\text{ext},0|_{\Gamma}}) + \dots,$$

$$\partial_{\mathbf{n}_{\delta,2}} u_{\text{ext},\delta|_{\Gamma_{\delta,2}}} = \partial_{\mathbf{n}} u_{\text{ext},0|_{\Gamma}} + \delta(\partial_{\mathbf{n}} u_{\text{ext},1|_{\Gamma}} + p_2 \partial_{\mathbf{n}}^2 u_{\text{ext},0|_{\Gamma}}) + \dots.$$

Transmission conditions (2.1 e), (2.1 g) and (2.1 i) become

$$u_{\text{ext},0|_{\Gamma}} + \delta(u_{\text{ext},1|_{\Gamma}} + p_2 \partial_{\mathbf{n}} u_{\text{ext},0|_{\Gamma}}) + \dots = u_{0|_{s_2=1}}^{[2]} + \delta u_{1|_{s_2=1}}^{[2]} + \dots, \quad (4.9)$$

$$u_{0|_{s_1=0}}^{[1]} + \delta u_{1|_{s_1=0}}^{[1]} + \dots = u_{0|_{s_2=0}}^{[2]} + \delta u_{1|_{s_2=0}}^{[2]} + \dots, \quad (4.10)$$

$$u_{\text{int},0|\Gamma} + \delta(u_{\text{int},1|\Gamma} - p_1 \partial_{\mathbf{n}} u_{\text{int},0|\Gamma}) + \cdots = u_{0|s_1=-1}^{[1]} + \delta u_{1|s_1=-1}^{[1]} + \cdots \quad (4.11)$$

We now use the identity

$$\det J_{\delta,\beta} = 1 + 2p_\beta s_\beta \delta \mathcal{H} + (p_\beta s_\beta \delta)^2 \mathcal{K},$$

where  $2\mathcal{H} := \text{tr}\mathcal{R}$  and  $\mathcal{K} := \det \mathcal{R}$  are respectively the mean and the Gaussian curvatures of the surface  $\Gamma$ . We obtain

$$\begin{aligned} & \int_{\Gamma} \sigma_{\text{int}} \partial_{\mathbf{n}_{\delta,1}} u_{\text{int},\delta|\Gamma_{\delta,1}} \circ \Phi_1(m, -1) v_d^{[1]}(m, -1) \det(I - p_1 \delta \mathcal{R}) \, d\Gamma \\ &= \int_{\Gamma} \sigma_{\text{int}} [\partial_{\mathbf{n}} u_{\text{int},0|\Gamma} + \delta (\partial_{\mathbf{n}} u_{\text{int},1|\Gamma} - p_1 \partial_{\mathbf{n}}^2 u_{\text{int},0|\Gamma} - 2p_1 \mathcal{H} \partial_{\mathbf{n}} u_{\text{int},0|\Gamma}) \\ &+ \delta^2 \left( \partial_{\mathbf{n}} u_{\text{int},2|\Gamma} - p_1 \partial_{\mathbf{n}}^2 u_{\text{int},1|\Gamma} + \frac{1}{2} p_1^2 \partial_{\mathbf{n}}^3 u_{\text{int},0|\Gamma} - 2p_1 \mathcal{H} \partial_{\mathbf{n}} u_{\text{int},1|\Gamma} \right. \\ &+ \left. 2p_1^2 \mathcal{H} \partial_{\mathbf{n}}^2 u_{\text{int},0|\Gamma} + p_1^2 \mathcal{K} \partial_{\mathbf{n}} u_{\text{int},0|\Gamma} \right) + \cdots] v_d^{[1]}(m, -1) \, d\Gamma, \end{aligned} \quad (4.12)$$

and

$$\begin{aligned} & \int_{\Gamma} \sigma_{\text{ext}} \partial_{\mathbf{n}_{\delta,2}} u_{\text{ext},\delta|\Gamma_{\delta,2}} \circ \Phi_2(m, 1) v_d^{[2]}(m, 1) \det(I + p_2 \delta \mathcal{R}) \, d\Gamma \\ &= \int_{\Gamma} \sigma_{\text{ext}} [\partial_{\mathbf{n}} u_{\text{ext},0|\Gamma} + \delta (\partial_{\mathbf{n}} u_{\text{ext},1|\Gamma} + p_2 \partial_{\mathbf{n}}^2 u_{\text{ext},0|\Gamma} + 2p_2 \mathcal{H} \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma}) \\ &+ \delta^2 \left( \partial_{\mathbf{n}} u_{\text{ext},2|\Gamma} + p_2 \partial_{\mathbf{n}}^2 u_{\text{ext},1|\Gamma} + \frac{1}{2} p_2^2 \partial_{\mathbf{n}}^3 u_{\text{ext},0|\Gamma} + 2p_2 \mathcal{H} \partial_{\mathbf{n}} u_{\text{ext},1|\Gamma} \right. \\ &+ \left. 2p_2^2 \mathcal{H} \partial_{\mathbf{n}}^2 u_{\text{ext},0|\Gamma} + p_2^2 \mathcal{K} \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma} \right) + \cdots] v_d^{[2]}(m, 1) \, d\Gamma. \end{aligned} \quad (4.13)$$

It remains to give the expansion of  $a_\delta^{[\beta]}(\cdot, \cdot)$  defined in (4.2) in powers of  $\delta$ . We use the identity (see [4, p. 1680])

$$J_{\delta,\beta}^{-2} := I - 2s_\beta p_\beta \delta \mathcal{R} + 3(p_\beta s_\beta \delta \mathcal{R})^2 + \cdots + n(-p_\beta s_\beta \delta \mathcal{R})^{n-1} + (-s_\beta p_\beta \delta \mathcal{R})^n [nJ_{\delta,\beta}^{-1} + J_{\delta,\beta}^{-2}].$$

The bilinear form  $a_\delta^{[\beta]}(\cdot, \cdot)$  then admits the expansion

$$a_\delta^{[\beta]}(\cdot, \cdot) = \delta^{-2} a_{0,2}^{[\beta]} + \delta^{-1} a_{1,2}^{[\beta]} + \left( a_{2,2}^{[\beta]} + a_{0,1}^{[\beta]} \right) + \delta a_{1,1}^{[\beta]} + \cdots + \delta^{n-1} a_{n-1,1}^{[\beta]} + \delta^n r_n^{[\beta]}(\cdot, \cdot), \quad (4.14)$$

where the forms  $a_{k,l}^{[\beta]}$  are independent of  $\delta$  and are given by

$$\begin{aligned}
 a_{0,2}^{[\beta]}(u^{[\beta]}, v^{[\beta]}) &:= \int_{\Omega^\beta} p_\beta^{-1} \tilde{\sigma} \partial_{s_\beta} u^{[\beta]} \partial_{s_\beta} v^{[\beta]} d\Gamma ds_\beta, \\
 a_{1,2}^{[\beta]}(u^{[\beta]}, v^{[\beta]}) &:= \int_{\Omega^\beta} 2\tilde{\sigma} \mathcal{H} s_\beta \partial_{s_\beta} u^{[\beta]} \partial_{s_\beta} v^{[\beta]} d\Gamma ds_\beta, \\
 a_{2,2}^{[\beta]}(u^{[\beta]}, v^{[\beta]}) &:= \int_{\Omega^\beta} p_\beta \tilde{\sigma} \mathcal{K} s_\beta^2 \partial_{s_\beta} u^{[\beta]} \partial_{s_\beta} v^{[\beta]} d\Gamma ds_\beta, \\
 a_{0,1}^{[\beta]}(u^{[\beta]}, v^{[\beta]}) &:= \int_{\Omega^\beta} p_\beta \tilde{\sigma} \nabla_\Gamma u^{[\beta]} \cdot \nabla_\Gamma v^{[\beta]} d\Gamma ds_\beta, \\
 a_{1,1}^{[\beta]}(u^{[\beta]}, v^{[\beta]}) &:= \int_{\Omega^\beta} 2p_\beta^2 \tilde{\sigma} s_\beta (\mathcal{H}I - \mathcal{R}) \nabla_\Gamma u^{[\beta]} \cdot \nabla_\Gamma v^{[\beta]} d\Gamma ds_\beta, \\
 a_{2,1}^{[\beta]}(u^{[\beta]}, v^{[\beta]}) &:= \int_{\Omega^\beta} p_\beta^3 \tilde{\sigma} (\mathcal{K}I - 4\mathcal{H}\mathcal{R} + 3\mathcal{R}^2) s_\beta^2 \nabla_\Gamma u^{[\beta]} \cdot \nabla_\Gamma v^{[\beta]} d\Gamma ds_\beta, \\
 a_{n-1,1}^{[\beta]}(u^{[\beta]}, v^{[\beta]}) &:= \int_{\Omega^\beta} p_\beta^n \tilde{\sigma} [(n-2)\mathcal{K}\mathcal{R}^{n-3} - (n-1)2\mathcal{H}\mathcal{R}^{n-2} \\
 &\quad + n\mathcal{R}^{n-1}] (-s_\beta)^{n-1} \nabla_\Gamma u^{[\beta]} \cdot \nabla_\Gamma v^{[\beta]} d\Gamma ds_\beta, \quad n > 3,
 \end{aligned}$$

in which the index 1 in the bilinear forms  $a_{k,1}^{[\beta]}(u^{[\beta]}, v^{[\beta]})$  corresponds to the derivatives of  $u^{[\beta]}$  and  $v^{[\beta]}$  with respect to the tangential variables and the index 2 refers to the derivatives of  $u^{[\beta]}$  and  $v^{[\beta]}$  with respect to the normal variable  $s_\beta$ . The remainder of Expansion (4.14) is expressed as follows

$$r_n^{[\beta]}(u^{[\beta]}, v^{[\beta]}) := \int_{\Omega^\beta} \tilde{\sigma} (B_{n,\delta} + 2\mathcal{H}B_{n-1,\delta} + \mathcal{K}B_{n-2,\delta}) s_\beta^n \nabla_\Gamma u^{[\beta]} \cdot \nabla_\Gamma v^{[\beta]} d\Gamma ds_\beta,$$

with

$$B_{n,\delta} := \begin{cases} (-\mathcal{R})^n (nJ_{\delta,\beta}^{-1} + J_{\delta,\beta}^{-2}) & \text{if } n \geq 0, \\ J_{\delta,\beta}^{-2} & \text{otherwise.} \end{cases}$$

The form  $b_\delta^{[\beta]}(\cdot, \cdot)$  has the finite expansion

$$b_\delta^{[\beta]}(\cdot, \cdot) = b_0^{[\beta]}(\cdot, \cdot) + \delta b_1^{[\beta]}(\cdot, \cdot) + \delta^2 b_2^{[\beta]}(\cdot, \cdot), \quad (4.15)$$

where

$$\begin{aligned} b_0^{[\beta]}(u^{[\beta]}, v^{[\beta]}) &:= \int_{\Omega^\beta} -p_\beta \tilde{k}^2 u^{[\beta]} v^{[\beta]} d\Gamma ds_\beta, \\ b_1^{[\beta]}(u^{[\beta]}, v^{[\beta]}) &:= \int_{\Omega^\beta} -p_\beta^2 \tilde{k}^2 2s_\beta \mathcal{H} u^{[\beta]} v^{[\beta]} d\Gamma ds_\beta, \\ b_2^{[\beta]}(u^{[\beta]}, v^{[\beta]}) &:= \int_{\Omega^\beta} -p_\beta^3 \tilde{k}^2 s_\beta^2 \mathcal{K} u^{[\beta]} v^{[\beta]} d\Gamma ds_\beta. \end{aligned}$$

Now inserting expansions (4.8) and (4.12)-(4.14) in (4.4) and matching the same powers of  $\delta$ , we obtain the following variational equations, which hold for all  $v^{[\beta]}$  in  $H^1(\Gamma \times I_\beta)$  such that  $v^{[1]}(., 0) = v^{[2]}(., 0)$ ,

$$a_{0,2}^{[1]}(u_0^{[1]}, v^{[1]}) + a_{0,2}^{[2]}(u_0^{[2]}, v^{[2]}) = 0, \quad (4.16)$$

$$\begin{aligned} & \int_{\Gamma} \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},0|\Gamma} v^{[1]}(m, -1) d\Gamma + \sum_{\beta=1}^2 \left[ a_{1,2}^{[\beta]}(u_0^{[\beta]}, v^{[\beta]}) + a_{0,2}^{[\beta]}(u_1^{[\beta]}, v^{[\beta]}) \right] \\ & - \int_{\Gamma} \sigma_{\text{ext}} \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma} v^{[2]}(m, 1) d\Gamma = 0, \end{aligned} \quad (4.17)$$

$$\begin{aligned} & \int_{\Gamma} \sigma_{\text{int}} \left[ -2p_1 \mathcal{H} \partial_{\mathbf{n}} u_{\text{int},0|\Gamma} + \partial_{\mathbf{n}} u_{\text{int},1|\Gamma} - p_1 \partial_{\mathbf{n}}^2 u_{\text{int},0|\Gamma} \right] v^{[1]}(m, -1) d\Gamma \\ & + \sum_{\beta=1}^2 \left[ a_{1,2}^{[\beta]}(u_1^{[\beta]}, v^{[\beta]}) + \left( a_{2,2}^{[\beta]} + a_{0,1}^{[\beta]} + b_0^{[\beta]} \right) (u_0^{[\beta]}, v^{[\beta]}) + a_{0,2}^{[\beta]}(u_2^{[\beta]}, v^{[\beta]}) \right] \\ & - \int_{\Gamma} \sigma_{\text{ext}} \left[ 2p_2 \mathcal{H} \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma} + \partial_{\mathbf{n}} u_{\text{ext},1|\Gamma} + p_2 \partial_{\mathbf{n}}^2 u_{\text{ext},0|\Gamma} \right] v^{[2]}(m, 1) d\Gamma = 0, \end{aligned} \quad (4.18)$$

$$\begin{aligned} & \int_{\Gamma} \sigma_{\text{int}} \left[ p_1^2 \mathcal{K} \partial_{\mathbf{n}} u_{\text{int},0|\Gamma} - 2p_1 \mathcal{H} (\partial_{\mathbf{n}} u_{\text{int},1|\Gamma} - p_1 \partial_{\mathbf{n}}^2 u_{\text{int},0|\Gamma}) + \partial_{\mathbf{n}} u_{\text{int},2|\Gamma} \right. \\ & \left. - p_1 \partial_{\mathbf{n}}^2 u_{\text{int},1|\Gamma} + \frac{p_1^2}{2} \partial_{\mathbf{n}}^3 u_{\text{int},0|\Gamma} \right] v^{[1]}(m, -1) d\Gamma + \sum_{\beta=1}^2 \left[ a_{0,2}^{[\beta]}(u_3^{[\beta]}, v^{[\beta]}) \right. \\ & \left. + a_{1,2}^{[\beta]}(u_2^{[\beta]}, v^{[\beta]}) \left( a_{2,2}^{[\beta]} + a_{0,1}^{[\beta]} + b_0^{[\beta]} \right) (u_1^{[\beta]}, v^{[\beta]}) + \left( a_{1,1}^{[\beta]} + b_1^{[\beta]} \right) (u_0^{[\beta]}, v^{[\beta]}) \right] \\ & - \int_{\Gamma} \sigma_{\text{ext}} \left[ p_2^2 \mathcal{K} \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma} + 2p_2 \mathcal{H} (\partial_{\mathbf{n}} u_{\text{ext},1|\Gamma} + p_2 \partial_{\mathbf{n}}^2 u_{\text{ext},0|\Gamma}) + \partial_{\mathbf{n}} u_{\text{ext},2|\Gamma} \right. \\ & \left. + p_2 \partial_{\mathbf{n}}^2 u_{\text{ext},1|\Gamma} + \frac{p_2^2}{2} \partial_{\mathbf{n}}^3 u_{\text{ext},0|\Gamma} \right] v^{[2]}(m, 1) d\Gamma = 0, \end{aligned} \quad (4.19)$$

$$\begin{aligned}
& \int_{\Gamma} \sigma_{\text{int}} [\partial_{\mathbf{n}} u_{\text{int},n|\Gamma} + \dots] v^{[1]}(m, -1) d\Gamma + \sum_{\beta=1}^2 \left[ a_{0,2}^{[\beta]} (u_{n+1}^{[\beta]}, v^{[\beta]}) + a_{1,2}^{[\beta]} (u_n^{[\beta]}, v^{[\beta]}) \right. \\
& + \left( a_{0,1}^{[\beta]} + b_0^{[\beta]} \right) (u_{n-1}^{[\beta]}, v^{[\beta]}) + \left( a_{1,1}^{[\beta]} + b_1^{[\beta]} \right) (u_{n-2}^{[\beta]}, v^{[\beta]}) + \left( a_{2,1}^{[\beta]} + b_2^{[\beta]} \right) (u_{n-3}^{[\beta]}, v^{[\beta]}) \\
& \left. + \sum_{l=4}^n a_{l-1,1}^{[\beta]} (u_{n-l}^{[\beta]}, v^{[\beta]}) \right] - \int_{\Gamma} \sigma_{\text{ext}} [\partial_{\mathbf{n}} u_{\text{ext},n|\Gamma} + \dots] v^{[2]}(m, 1) d\Gamma = 0, \quad n \geq 4. \quad (4.20)
\end{aligned}$$

## 4.2 Computation of the first two terms

In this paragraph, we compute explicitly the first two terms in order to present a recursive method to define successively the asymptotic expansions. We need the following theorem whose proof can be found in [36].

**Theorem 4.1** *Let  $h \in H^{1/2}(\Gamma)$  and  $\zeta \in H^{-1/2}(\Gamma)$ . Then the following problem*

$$\left\{ \begin{array}{ll} \operatorname{div}(\sigma_{\text{int}} \nabla U_{\text{int}}) + k_{\text{int}}^2 U_{\text{int}} = 0 & \text{in } \Omega_{\text{int}}, \\ \operatorname{div}(\sigma_{\text{ext}} \nabla U_{\text{ext}}) + k_{\text{ext}}^2 U_{\text{ext}} = 0 & \text{in } \Omega_{\text{ext}}, \\ U_{\text{int}}|_{\Gamma} - U_{\text{ext}}|_{\Gamma} = h & \text{on } \Gamma, \\ \sigma_{\text{int}} \partial_{\mathbf{n}} U_{\text{int}}|_{\Gamma} - \sigma_{\text{ext}} \partial_{\mathbf{n}} U_{\text{ext}}|_{\Gamma} = \zeta & \text{on } \Gamma, \\ \lim_{|x| \rightarrow +\infty} |x| (\partial_{|x|} - ik_{\text{ext}}) U_{\text{ext}} = 0, & \end{array} \right. \quad (4.21)$$

admits a unique solution  $(U_{\text{int}}, U_{\text{ext}})$  in  $H^1(\Omega_{\text{int}}) \times H_{\text{loc}}^1(\overline{\Omega_{\text{ext}}})$ . Moreover, for  $k_0 \in \mathbb{N}$ ,  $h \in H^{k_0-1/2}(\Gamma)$ ,  $\zeta \in H^{k_0-3/2}(\Gamma)$  and  $\Gamma \cup \partial\Omega$   $C^{k_0}$ -continuous, let  $(U_{\text{int}}, U_{\text{ext}}) \in H^1(\Omega_{\text{int}}) \times H_{\text{loc}}^1(\overline{\Omega_{\text{ext}}})$  be the solution of (4.21). For any positive integer  $k \leq k_0$ , there exists a constant  $c_k$  such that

$$\|U_{\text{int}}\|_{H^k(\Omega_{\text{int}})} + \|U_{\text{ext}}\|_{H^k(\tilde{\Omega}_{\text{ext}})} \leq c_k \left( \|h\|_{H^{k-1/2}(\Gamma)} + \|\zeta\|_{H^{k-3/2}(\Gamma)} \right).$$



We also need the following technical result to determine terms of asymptotic expansions whose proof is obtained in a straightforward way.

**Lemma 4.1** *For  $\beta = 1, 2$ , let  $q^{[\beta]}$  be a given function in  $L^2(\Gamma)$  and let  $k^{[\beta]}$  be a vectorial function in  $L^2(\Omega^\beta, \mathbb{C}^3)$  such that the partial application  $s_\beta \rightarrow k^{[\beta]}(\cdot, s_\beta)$  is valued in the space of vectorial fields tangent to  $\Gamma$  and also  $\operatorname{div}_\Gamma k^{[\beta]} \in L^2(\Omega^\beta)$ . Then the solution  $h^{[\beta]}$  of the variational equation*

$$\begin{aligned} \mathcal{L}^{[\beta]} v^{[\beta]} &:= \int_{\Omega^\beta} h^{[\beta]}(m, s_\beta) \partial_{s_\beta} v^{[\beta]}(m, s_\beta) \, d\Gamma ds_\beta \\ &+ \int_{\Omega^\beta} k^{[\beta]}(m, s_\beta) \cdot \nabla_\Gamma v^{[\beta]}(m, s_\beta) + \theta^{[\beta]}(m, s_\beta) v^{[\beta]}(m, s_\beta) \, d\Gamma ds_\beta \\ &+ \int_\Gamma q^{[\beta]}(m) v^{[\beta]}(m, (-1)^\beta) \, d\Gamma = 0 \end{aligned}$$

for all  $v^{[\beta]} \in H^1(\Omega^\beta)$ ;  $v^{[\beta]}(\cdot, 0) = 0$  is explicitly given by

$$h^{[\beta]}(m, s_\beta) = (-1)^{\beta+1} q^{[\beta]}(m) + \int_{s_\beta}^{(-1)^\beta} \left( \operatorname{div}_\Gamma k^{[\beta]} - \theta^{[\beta]} \right)(m, \lambda) \, d\lambda.$$

Moreover, for all  $v^{[\beta]} \in H^1(\Omega^\beta)$ , we have

$$\begin{aligned} \mathcal{L}^{[\beta]} v^{[\beta]} &= (-1)^{\beta+1} \int_\Gamma h^{[\beta]}(m, 0) v^{[\beta]}(m, 0) \, d\Gamma \\ &= \int_\Gamma \left[ q^{[\beta]}(m) - (-1)^\beta \int_0^{(-1)^\beta} \left( \operatorname{div}_\Gamma k^{[\beta]} - \theta^{[\beta]} \right)(m, s_\beta) \, ds_\beta \right] v^{[\beta]}(m, 0) \, d\Gamma. \end{aligned}$$

#### 4.2.1 Term of order 0

Equation (4.16) implies that  $\partial_{s_\beta} u_0^{[\beta]} = 0$ . Using (4.9), (4.10) and (4.11), we obtain

$$u_{\text{int},0|\Gamma} = u_0^{[1]}(m, s_1) = u_0^{[2]}(m, s_2) = u_{\text{ext},0|\Gamma}, \quad m \in \Gamma. \quad (4.22)$$

The choice of  $v^{[1]} = 0$  in (4.17) gives

$$a_{1,2}^{[2]}(u_0^{[2]}, v^{[2]}) + a_{0,2}^{[2]}(u_1^{[2]}, v^{[2]}) - \sigma_{\text{ext}} \int_{\Gamma} \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma} v^{[2]}(m, 1) d\Gamma = 0.$$

We apply Lemma 4.1 with  $h^{[2]} = p_2^{-1} \tilde{\sigma} \partial_{s_2} u_1^{[2]}$ ,  $q^{[2]}(m) = -\sigma_{\text{ext}} \partial_{\mathbf{n}} u_{e,0|\Gamma}$ ,  $k^{[2]} = 0$  and  $\theta^{[2]} = 0$ , to obtain

$$p_2^{-1} \tilde{\sigma} \partial_{s_2} u_1^{[2]} = \sigma_{\text{ext}} \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma}. \quad (4.23)$$

Similarly, choosing  $v^{[2]} = 0$  in (4.17) gives

$$a_{1,2}^{[1]}(u_0^{[1]}, v^{[1]}) + a_{0,2}^{[1]}(u_1^{[1]}, v^{[1]}) + \sigma_{\text{int}} \int_{\Gamma} \partial_{\mathbf{n}} u_{\text{int},0|\Gamma} v^{[1]}(m, -1) d\Gamma = 0.$$

We apply Lemma 4.1 with  $h^{[1]} = p_1^{-1} \tilde{\sigma} \partial_{s_1} u_1^{[1]}$ ,  $q^{[1]}(m) = \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},0|\Gamma}$ ,  $k^{[1]} = 0$  and  $\theta^{[1]} = 0$ , to find

$$p_1^{-1} \tilde{\sigma} \partial_{s_1} u_1^{[1]} = \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},0|\Gamma}. \quad (4.24)$$

From the second part of Lemma 4.1, one gets

$$\int_{\Gamma} \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},0|\Gamma} v^{[1]}(m, 0) d\Gamma = \int_{\Gamma} \sigma_{\text{ext}} \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma} v^{[1]}(m, 0) d\Gamma,$$

then

$$\sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},0|\Gamma} = \sigma_{\text{ext}} \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma}. \quad (4.25)$$

Let us define  $\sigma_0$ ,  $u_n$  and  $k_0$  by

$$\sigma_0(x) = \begin{cases} \sigma_{\text{ext}} & \text{if } x \in \Omega_{\text{ext}}, \\ \sigma_{\text{int}} & \text{if } x \in \Omega_{\text{int}}, \end{cases} \quad ; \quad k_0^2(x) = \begin{cases} k_{\text{ext}}^2 & \text{if } x \in \Omega_{\text{ext}}, \\ k_{\text{int}}^2 & \text{if } x \in \Omega_{\text{int}}, \end{cases}$$

and

$$u_n = \begin{cases} u_{\text{ext},n} & \text{in } \Omega_{\text{ext}}, \\ u_{\text{int},n} & \text{in } \Omega_{\text{int}}. \end{cases}$$

Therefore, with (4.7), (4.22), (4.25) and Theorem 4.1,  $u_0$  satisfies the following problem

$$\begin{cases} \operatorname{div}(\sigma_0 \nabla u_0) + k_0^2 u_0 = 0 & \text{in } \mathbb{R}^3, \\ \lim_{|x| \rightarrow +\infty} |x| (\partial_{|x|} - ik_{\text{ext}}) (u_0 - u_{\text{inc}}) = 0. \end{cases}$$

The zeroth-order term is then determined. Note that  $u_0$  is nothing but the solution to the scattering problem where there is no thin layer.

#### 4.2.2 Term of order 1

Integrating Relations (4.23) and (4.24) in  $s_\beta$  and identifying terms of order 1 in (4.9) and (4.11), yields

$$u_1^{[1]}(m, s_1) = u_{\text{int},1|\Gamma} + p_1 [(s_1 + 1)\sigma_{\text{int}}\tilde{\sigma}^{-1} - 1] \partial_{\mathbf{n}} u_{\text{int},0|\Gamma}, \quad \forall (m, s_1) \in \Omega^1,$$

and

$$u_1^{[2]}(m, s_2) = u_{\text{ext},1|\Gamma} + p_2 [(s_2 - 1)\sigma_{\text{ext}}\tilde{\sigma}^{-1} + 1] \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma}, \quad \forall (m, s_2) \in \Omega^2.$$

The identification of first-order terms of (4.10) gives a first transmission condition on  $\Gamma$

$$u_{\text{int},1|\Gamma} - u_{\text{ext},1|\Gamma} = p_1 (1 - \sigma_{\text{int}}\tilde{\sigma}^{-1}) \partial_{\mathbf{n}} u_{\text{int},0|\Gamma} + p_2 (1 - \sigma_{\text{ext}}\tilde{\sigma}^{-1}) \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma}. \quad (4.26)$$

The second one follows the same lines as for order 0. Indeed, we apply Lemma 4.1 to Equation (4.18) once for  $\beta = 2$  and another for  $\beta = 1$ , and using the identity [27, p. 75]

$$\Delta u = \Delta_\Gamma u + 2\mathcal{H}\partial_{\mathbf{n}} u + \partial_{\mathbf{n}}^2 u,$$

we obtain

$$\begin{aligned} \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},1|\Gamma} - \sigma_{\text{ext}} \partial_{\mathbf{n}} u_{\text{ext},1|\Gamma} &= p_1 (\tilde{\sigma} - \sigma_{\text{int}}) \Delta_\Gamma u_{\text{int},0|\Gamma} + p_2 (\tilde{\sigma} - \sigma_{\text{ext}}) \Delta_\Gamma u_{\text{ext},0|\Gamma} \\ &+ p_1 \left( \tilde{k}^2 - k_{\text{int}}^2 \right) u_{\text{int},0|\Gamma} + p_2 \left( \tilde{k}^2 - k_{\text{ext}}^2 \right) u_{\text{ext},0|\Gamma} \end{aligned} \quad (4.27)$$

It follows from (4.7), (4.26), (4.27) and Theorem 4.1 that  $u_1$  is the unique solution to the following problem

$$\begin{cases} \operatorname{div}(\sigma_{\text{int}} \nabla u_{\text{int},1}) + k_{\text{int}}^2 u_{\text{int},1} = 0 & \text{in } \Omega_{\text{int}}, \\ \operatorname{div}(\sigma_{\text{ext}} \nabla u_{\text{ext},1}) + k_{\text{ext}}^2 u_{\text{ext},1} = 0 & \text{in } \Omega_{\text{ext}}, \\ \lim_{|x| \rightarrow +\infty} |x| (\partial_{|x|} - ik_{\text{ext}}) u_{\text{ext},1} = 0, \end{cases}$$

with the following transmission conditions on  $\Gamma$

$$u_{\text{int},1|\Gamma} - u_{\text{ext},1|\Gamma} = p_1(1 - \sigma_{\text{int}} \tilde{\sigma}^{-1}) \partial_{\mathbf{n}} u_{\text{int},0|\Gamma} + p_2(1 - \sigma_{\text{ext}} \tilde{\sigma}^{-1}) \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma},$$

$$\begin{aligned} \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},1|\Gamma} - \sigma_{\text{ext}} \partial_{\mathbf{n}} u_{\text{ext},1|\Gamma} &= p_1(\tilde{\sigma} - \sigma_{\text{int}}) \Delta_{\Gamma} u_{\text{int},0|\Gamma} + p_2(\tilde{\sigma} - \sigma_{\text{ext}}) \Delta_{\Gamma} u_{\text{ext},0|\Gamma} \\ &\quad + p_1 \left( \tilde{k}^2 - k_{\text{int}}^2 \right) u_{\text{int},0|\Gamma} + p_2 \left( \tilde{k}^2 - k_{\text{ext}}^2 \right) u_{\text{ext},0|\Gamma}, \end{aligned}$$

or

$$\begin{aligned} u_{\text{int},1|\Gamma} - u_{\text{ext},1|\Gamma} &= \frac{p_1 \sigma_{\text{ext}} \tilde{\sigma} + p_2 \sigma_{\text{int}} \tilde{\sigma} - \sigma_{\text{int}} \sigma_{\text{ext}}}{2 \sigma_{\text{int}} \sigma_{\text{ext}} \tilde{\sigma}} \left( \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},0|\Gamma} + \sigma_{\text{ext}} \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma} \right), \\ \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},1|\Gamma} - \sigma_{\text{ext}} \partial_{\mathbf{n}} u_{\text{ext},1|\Gamma} &= \frac{1}{2} (\tilde{\sigma} - p_1 \sigma_{\text{int}} - p_2 \sigma_{\text{ext}}) \left( \Delta_{\Gamma} u_{\text{int},0|\Gamma} + \Delta_{\Gamma} u_{\text{ext},0|\Gamma} \right) \\ &\quad + \frac{1}{2} \left( \tilde{k}^2 - p_1 k_{\text{int}}^2 - p_2 k_{\text{ext}}^2 \right) \left( u_{\text{int},0|\Gamma} + u_{\text{ext},0|\Gamma} \right), \end{aligned}$$

## 5 Optimal error estimates

The process described in the previous section can be continued up to any order provided that the data are smooth enough. Here the source term is given by a plane wave which is  $\mathcal{C}^\infty$ . We can also estimate the error made by truncating the series after a finite number

of terms. Let  $n$  be in  $\mathbb{N}$ , we set  $\tilde{\Omega}_{\text{ext},\delta} = \tilde{\Omega}_{\text{ext}} \setminus \bar{\Omega}_{\delta,2}$ ,

$$u_{\text{int},\delta}^{(n)} := \sum_{j=0}^n \delta^j u_{\text{int},j}, \quad u_{\text{ext},\delta}^{(n)} := \sum_{j=0}^n \delta^j u_{\text{ext},j} \quad \text{and} \quad u_{d,\delta}^{(n)} := \begin{cases} u_{d_1,\delta}^{(n)} := \sum_{j=0}^n \delta^j u_{d_1,j} & \text{in } \Omega_{\delta,1}, \\ u_{d_2,\delta}^{(n)} := \sum_{j=0}^n \delta^j u_{d_2,j} & \text{in } \Omega_{\delta,2}, \end{cases}$$

where  $u_{d_\beta,j}(x) := \tilde{u}_{d_\beta,j}(m, \delta s_\beta) := u_j^{[\beta]}(m, s_\beta)$ ;  $\forall x = \Phi_\beta(m, s_\beta) \in \Omega_{\delta,\beta}$ .

**Theorem 5.1** (Convergence of the asymptotic expansion) *For all integers  $n$ , there exists a constant  $c$  independent of  $\delta$  such that*

$$\left\| u_{\text{int},\delta} - u_{\text{int},\delta}^{(n)} \right\|_{H^1(\Omega_{\text{int},\delta})} + \delta^{1/2} \left\| u_{d,\delta} - u_{d,\delta}^{(n)} \right\|_{H^1(\Omega_\delta)} + \left\| u_{\text{ext},\delta} - u_{\text{ext},\delta}^{(n)} \right\|_{H^1(\tilde{\Omega}_{\text{ext},\delta})} \leq c\delta^{n+1}.$$

**Proof** Let us define the remainders  $R_{D_1,n}, R_{D_2,n}, R_{N_1,n}$  and  $R_{N_2,n}$  of Taylor expansions in the normal variable with respect to  $\delta$  up to order  $n$  of  $u_{\text{int},\delta|\Gamma_{\delta,1}}^{(n)}, u_{\text{ext},\delta|\Gamma_{\delta,2}}^{(n)}, \partial_{\mathbf{n}_{\delta,1}} u_{\text{int},\delta|\Gamma_{\delta,1}}^{(n)}$  and  $\partial_{\mathbf{n}_{\delta,2}} u_{\text{ext},\delta|\Gamma_{\delta,2}}^{(n)}$  respectively by

$$\begin{aligned} R_{D_1,n} &:= u_{\text{int},\delta|\Gamma_{\delta,1}}^{(n)} - \sum_{j=0}^n \sum_{l=0}^{n-j} \frac{(-1)^l \delta^{j+l}}{l!} p_1^l \partial_{\mathbf{n}}^l u_{\text{int},j|\Gamma} \\ &= \sum_{j=0}^n \delta^j \tilde{u}_{\text{int},j}(m, -\delta) - \sum_{j=0}^n \sum_{l=0}^{n-j} \frac{(-1)^l \delta^{j+l}}{l!} \partial_{s_1}^l \tilde{u}_{\text{int},j}(m, 0), \end{aligned} \quad (5.1)$$

$$\begin{aligned} R_{D_2,n} &:= u_{\text{ext},\delta|\Gamma_{\delta,2}}^{(n)} - \sum_{j=0}^n \sum_{l=0}^{n-j} \frac{\delta^{j+l}}{l!} p_2^l \partial_{\mathbf{n}}^l u_{\text{ext},j|\Gamma} \\ &= \sum_{j=0}^n \delta^j \tilde{u}_{\text{ext},j}(m, \delta) - \sum_{j=0}^n \sum_{l=0}^{n-j} \frac{\delta^{j+l}}{l!} \partial_{s_2}^l \tilde{u}_{\text{ext},j}(m, 0), \end{aligned} \quad (5.2)$$

$$\begin{aligned} R_{N_1,n} &:= \partial_{\mathbf{n}_{\delta,1}} u_{\text{int},\delta|\Gamma_{\delta,1}}^{(n)} - \sum_{j=0}^n \sum_{l=0}^{n-j} \frac{(-1)^l \delta^{j+l}}{l!} p_1^l \partial_{\mathbf{n}}^{l+1} u_{\text{int},j|\Gamma} \\ &= \sum_{j=0}^n \delta^j p_1^{-1} \partial_{s_1} \tilde{u}_{\text{int},j}(m, -\delta) - \sum_{j=0}^n \sum_{l=0}^{n-j} p_1^{-1} \frac{(-1)^l \delta^{j+l}}{l!} \partial_{s_1}^{l+1} \tilde{u}_{\text{int},j}(m, 0), \end{aligned} \quad (5.3)$$

and

$$\begin{aligned}
R_{N_2,n} &:= \partial_{\mathbf{n}_{\delta,2}} u_{\text{ext},\delta}^{(n)}|_{\Gamma_{\delta,2}} - \sum_{j=0}^n \sum_{l=0}^{n-j} \frac{\delta^{j+l}}{l!} p_2^l \partial_{\mathbf{n}}^{l+1} u_{\text{ext},j}|_{\Gamma} \\
&= \sum_{j=0}^n \delta^j p_2^{-1} \partial_{s_2} \tilde{u}_{\text{ext},j}(m, \delta) - \sum_{j=0}^n \sum_{l=0}^{n-j} p_2^{-1} \frac{\delta^{j+l}}{l!} \partial_{s_2}^{l+1} \tilde{u}_{\text{ext},j}(m, 0). \quad (5.4)
\end{aligned}$$

We shall rely on the following proposition to show the estimates of the remainders  $R_{D_\beta,n}$  and  $R_{N_\beta,n}$ . The steps of its proof are very similar to those given in [37, Section 5].  $\square$

**Proposition 5.1** *There exists a constant  $c > 0$ , independent of  $\delta$ , such as*

$$\begin{aligned}
\|R_{N_\beta,n}\|_{L^2(\Gamma_{\delta,\beta})} &\leq c\delta^{n+1/2}, \\
\|\nabla_{\Gamma}^{(j)} R_{D_\beta,n}\|_{L^2(\Gamma)} &\leq c\delta^{n+1/2}, \quad \text{for } j = 0, 1.
\end{aligned}$$

Moreover, there exists an extension  $\mathcal{P}R$  of  $R_{D_\beta,n}$  into  $\Omega_\delta$  with

$$\partial_{\eta_\beta} \widetilde{\mathcal{P}R}(m, \eta_\beta)|_{\eta_\beta = (-1)^\beta p_\beta \delta} = 0 \quad \text{and} \quad \|\mathcal{P}R\|_{H^1(\Omega_\delta)} \leq c\delta^n.$$

**Continuation of the proof of Theorem 5.1** Let  $r_{\text{int},\delta}^n$ ,  $r_{d,\delta}^n$  and  $r_{\text{ext},\delta}^n$  be the remainders got by truncating Series (4.5), (4.6) and (4.8)

$$r_{\text{int},\delta}^n := u_{\text{int},\delta} - u_{\text{int},\delta}^{(n)}, \quad r_{\text{ext},\delta}^n := u_{\text{ext},\delta} - u_{\text{ext},\delta}^{(n)}, \quad r_{d,\delta}^n := u_{d,\delta} - u_{d,\delta}^{(n)},$$

and  $\mathcal{L}_\delta$  be the linear form defined on  $H^1(\Omega)$

$$\begin{aligned}
\mathcal{L}_\delta v &:= \int_{\Omega_{\text{int},\delta}} \sigma_{\text{int}} \nabla r_{\text{int},\delta}^n \cdot \nabla \bar{v}_{\text{int}} - k_{\text{int}}^2 r_{\text{int},\delta}^n \bar{v}_{\text{int}} \, d\Omega_{\text{int},\delta} \\
&+ \int_{\Omega_\delta} \tilde{\sigma} \nabla (r_{d,\delta}^n - \mathcal{P}R) \cdot \nabla \bar{v}_d - \tilde{k}^2 (r_{d,\delta}^n - \mathcal{P}R) \bar{v}_d \, d\Omega_\delta \\
&+ \int_{\tilde{\Omega}_{\text{ext},\delta}} \sigma_{\text{ext}} \nabla r_{\text{ext},\delta}^n \cdot \nabla \bar{v}_{\text{ext}} - k_{\text{ext}}^2 r_{\text{ext},\delta}^n \bar{v}_{\text{ext}} \, d\tilde{\Omega}_{\text{ext},\delta} \\
&+ \sigma_{\text{ext}} \left\langle T r_{\text{ext},\delta}^n|_{\partial\Omega}, \bar{v}|_{\partial\Omega} \right\rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)}, \quad (5.5)
\end{aligned}$$

in which  $\mathcal{P}R$  is the extension function of  $R_{D_{\beta,n}}$  into  $\Omega_\delta$  and  $v_{\text{int}}$ ,  $v_d$  and  $v_{\text{ext}}$ , are the restrictions of  $v$  respectively to the domains  $\Omega_{\text{int},\delta}$ ,  $\Omega_\delta$  and  $\tilde{\Omega}_{\text{ext},\delta}$ . Using Green's formula in  $\Omega_{\text{int},\delta}$  and in  $\tilde{\Omega}_{\text{ext},\delta}$  with the help of (4.7), we obtain

$$\begin{aligned} \mathcal{L}_\delta v &= - \int_{\Gamma_{\delta,1}} \sigma_{\text{int}} \left( \partial_{\mathbf{n}_{\delta,1}} u_{\text{int},0|\Gamma_{\delta,1}} + \cdots + \delta^n \partial_{\mathbf{n}_{\delta,1}} u_{\text{int},n|\Gamma_{\delta,1}} \right) \bar{v}_{\text{int}|\Gamma_{\delta,1}} d\Gamma_{\delta,1} \\ &\quad - \sum_{\beta=1}^2 \left[ \delta a^{[\beta]} (u_0^{[\beta]} + \cdots + \delta^n u_n^{[\beta]}, \bar{v}^{[\beta]}) + \delta b^{[\beta]} \left( u_0^{[\beta]} + \cdots + \delta^n u_n^{[\beta]}, \bar{v}^{[\beta]} \right) \right] \\ &\quad + \int_{\Gamma_{\delta,2}} \sigma_{\text{ext}} \left( \partial_{\mathbf{n}_{\delta,2}} u_{\text{ext},0|\Gamma_{\delta,2}} + \cdots + \delta^n \partial_{\mathbf{n}_{\delta,2}} u_{\text{ext},n|\Gamma_{\delta,2}} \right) \bar{v}_{\text{ext}|\Gamma_{\delta,2}} d\Gamma_{\delta,2} \\ &\quad - \int_{\Omega_\delta} \tilde{\sigma} \nabla \mathcal{P}R \cdot \nabla \bar{v}_d - \tilde{k}^2 \mathcal{P}R \bar{v}_d d\Omega_\delta. \end{aligned}$$

It follows, from (5.1)-(5.4), that

$$\begin{aligned} \mathcal{L}_\delta v &= - \int_{\Gamma_{\delta,1}} \sigma_{\text{int}} R_{N_{1,n}} \bar{v}_{\text{int}|\Gamma_{\delta,1}} d\Gamma_{\delta,1} + \int_{\Gamma_{\delta,2}} \sigma_{\text{ext}} R_{N_{2,n}} \bar{v}_{\text{ext}|\Gamma_{\delta,2}} d\Gamma_{\delta,2} \\ &\quad - \sum_{\beta=1}^2 \left[ \delta a_\delta^{[\beta]} (u_0^{[\beta]} + \cdots + \delta^n u_n^{[\beta]}, \bar{v}^{[\beta]}) + \delta b_\delta^{[\beta]} (u_0^{[\beta]} + \cdots + \delta^n u_n^{[\beta]}, \bar{v}^{[\beta]}) \right] \\ &\quad - \int_{\Gamma} \sigma_{\text{int}} \left( \sum_{k=0}^n \sum_{l=0}^{n-k} \frac{(-1)^l \delta^{k+l}}{l!} p_1^l \partial_{\mathbf{n}}^{l+1} u_{\text{int},k|\Gamma} \right) \bar{v}^{[1]}(m, -1) \det(1 - 2\delta p_1 \mathcal{H} + \delta^2 p_1^2 \mathcal{K}) d\Gamma \\ &\quad + \int_{\Gamma} \sigma_{\text{ext}} \left( \sum_{k=0}^n \sum_{l=0}^{n-k} \frac{\delta^{k+l}}{l!} p_1^l \partial_{\mathbf{n}}^{l+1} u_{\text{ext},k|\Gamma} \right) \bar{v}^{[2]}(m, 1) \det(1 + 2\delta p_2 \mathcal{H} + \delta^2 p_2^2 \mathcal{K}) d\Gamma \\ &\quad - \int_{\Omega_\delta} \tilde{\sigma} \nabla \mathcal{P}R \cdot \nabla \bar{v}_d - \tilde{k}^2 \mathcal{P}R \bar{v}_d d\Omega_\delta, \end{aligned}$$

where  $R_{N_{1,n}}$  and  $R_{N_{2,n}}$  are respectively the remainders of Taylor expansions in the normal variable with respect to  $\delta$  up to order  $n$  of  $\partial_{\mathbf{n}_{\delta,1}} u_{\text{int},\delta|\Gamma_{\delta,1}}^{(n)}$  and  $\partial_{\mathbf{n}_{\delta,2}} u_{\text{ext},\delta|\Gamma_{\delta,2}}^{(n)}$ ;  $2\mathcal{H}$  and  $\mathcal{K}$  are respectively the mean and the Gaussian curvatures of the surface  $\Gamma$ . Now, we use that  $u_0^{[\beta]}, \dots, u_{n+1}^{[\beta]}$ , ( $\beta = 1, 2$ ) are solutions of Equations (4.16)-(4.20), and obtain

$$\begin{aligned} \mathcal{L}_\delta v &= \delta^{n+1} \sum_{\beta=1}^2 \left\{ \delta^{-1} a_{0,2}^{[\beta]} \left( u_{n+1}^{[\beta]}, \bar{v}^{[\beta]} \right) - \left( a_{2,2}^{[\beta]} + a_{0,1}^{[\beta]} + b_0^{[\beta]} \right) \left( u_n^{[\beta]}, \bar{v}^{[\beta]} \right) \right. \\ &\quad \left. - \left( a_{1,1}^{[\beta]} + b_1^{[\beta]} \right) \left( u_{n-1}^{[\beta]} + \delta u_n^{[\beta]}, \bar{v}^{[\beta]} \right) - \left( a_{2,1}^{[\beta]} + b_2^{[\beta]} \right) \left( u_{n-2}^{[\beta]} + \delta u_{n-1}^{[\beta]} + \delta^2 u_n^{[\beta]}, \bar{v}^{[\beta]} \right) \right\} \end{aligned}$$

$$\begin{aligned}
 & - \dots - a_{n-1,1}^{[\beta]} \left( u_1^{[\beta]} + \dots + \delta^{n-1} u_{n-1}^{[\beta]}, \overline{v^{[\beta]}} \right) - r_n^{[\beta]} \left( u_1^{[\beta]} + \dots + \delta^n u_n^{[\beta]}, \overline{v^{[\beta]}} \right) \} \\
 & + \int_{\Gamma_{\delta,1}} \sigma_{\text{int}} R_{N_1,n} \overline{v_{\text{int}}|_{\Gamma_{\delta,1}}} d\Gamma_{\delta,1} - \int_{\Gamma_{\delta,2}} \sigma_{\text{ext}} R_{N_2,n} \overline{v_{\text{ext}}|_{\Gamma_{\delta,2}}} d\Gamma_{\delta,2} \\
 & - \int_{\Omega_\delta} \tilde{\sigma} \nabla \mathcal{P} R \cdot \nabla \overline{v}_d - \tilde{k}^2 \mathcal{P} R \overline{v}_d d\Omega_\delta.
 \end{aligned}$$

By the estimates based on the explicit expressions of the bilinear forms  $a_{k,l}^{[\beta]}(\cdot, \cdot)$  and those from Propositions 5.1, we have

$$\begin{aligned}
 |\mathcal{L}_\delta v| & \leq c\delta^{n+1} \sum_{\beta=1}^2 \left( \left\| \nabla_\Gamma v^{[\beta]} \right\|_{L^2(\Omega^\beta)} + \delta^{-1} \left\| \partial_{s_\beta} v^{[\beta]} \right\|_{L^2(\Omega^\beta)} + \left\| v^{[\beta]} \right\|_{L^2(\Omega^\beta)} \right) \\
 & + c\delta^n \left( \left\| v_{\text{int}} \right\|_{H^1(\Omega_{\text{int},\delta})} + \sum_{\beta=1}^2 \left\| v_\beta \right\|_{H^1(\Omega_{\delta,\beta})} + \left\| v_{\text{ext}} \right\|_{H^1(\tilde{\Omega}_{\text{ext},\delta})} \right).
 \end{aligned}$$

This implies,

$$\begin{aligned}
 |\mathcal{L}_\delta v| & \leq c\delta^{n+\frac{1}{2}} \sum_{\beta=1}^2 \left( \delta^{\frac{1}{2}} \left\| \nabla_\Gamma v^{[\beta]} \right\|_{L^2(\Omega^\beta)} + \delta^{-\frac{1}{2}} \left\| \partial_{s_\beta} v^{[\beta]} \right\|_{L^2(\Omega^\beta)} + \delta^{\frac{1}{2}} \left\| v^{[\beta]} \right\|_{L^2(\Omega^\beta)} \right) \\
 & + c\delta^n \left( \left\| v_{\text{int}} \right\|_{H^1(\Omega_{\text{int},\delta})} + \sum_{\beta=1}^2 \left\| v_\beta \right\|_{H^1(\Omega_{\delta,\beta})} + \left\| v_{\text{ext}} \right\|_{H^1(\tilde{\Omega}_{\text{ext},\delta})} \right).
 \end{aligned}$$

Therefore

$$|\mathcal{L}_\delta v| \leq c\delta^n \|v\|_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega). \quad (5.6)$$

We set in (5.5)  $v_{\text{int}} = r_{\text{int},\delta}^n$ ,  $v_d = r_{d,\delta}^n - \mathcal{P}R$  and  $v_{\text{ext}} = r_{\text{ext},\delta}^n$ . Then,  $v$  is continuous over the interfaces  $\Gamma_{\delta,1}$  and  $\Gamma_{\delta,2}$ . Hence,  $v \in H^1(\Omega)$ . Using (5.6) and the stability theorem 2.2, we obtain

$$\left\| r_{\text{int},\delta}^n \right\|_{H^1(\Omega_{\text{int},\delta})} + \left\| r_{d,\delta}^n - \mathcal{P}R \right\|_{H^1(\Omega_\delta)} + \left\| r_{\text{ext},\delta}^n \right\|_{H^1(\tilde{\Omega}_{\text{ext},\delta})} \leq c\delta^n.$$

Thanks to Proposition 5.1, we find

$$\left\| r_{\text{int},\delta}^n \right\|_{H^1(\Omega_{\text{int},\delta})} + \left\| r_{d,\delta}^n \right\|_{H^1(\Omega_\delta)} + \left\| r_{\text{ext},\delta}^n \right\|_{H^1(\tilde{\Omega}_{\text{ext},\delta})} \leq c\delta^n. \quad (5.7)$$



Finally, since  $\|u_{\text{ext},j}\|_{H^1(\tilde{\Omega}_{\text{ext},\delta})} = O(1)$ ,  $\|u_{\text{int},j}\|_{H^1(\Omega_{\text{int},\delta})} = O(1)$  and  $\|u_{d,\beta,j}\|_{H^1(\Omega_{\delta,\beta})} = O(\delta^{-1/2})$ , one gets

$$\begin{aligned} \|r_{\text{int},\delta}^n\|_{H^1(\Omega_{\text{int},\delta})} &= \left\| \delta^{n+1} u_{\text{int},n+1} + r_{\text{int},\delta}^{n+1} \right\|_{H^1(\Omega_{\text{int},\delta})} \\ &\stackrel{(5.7)}{\leq} c\delta^{n+1} + c\delta^{n+1} \leq c\delta^{n+1}, \\ \|r_{\text{ext},\delta}^n\|_{H^1(\tilde{\Omega}_{\text{ext},\delta})} &= \left\| \delta^{n+1} u_{\text{ext},n+1} + r_{\text{ext},\delta}^{n+1} \right\|_{H^1(\tilde{\Omega}_{\text{ext},\delta})} \\ &\stackrel{(5.7)}{\leq} c\delta^{n+1} + c\delta^{n+1} \leq c\delta^{n+1}, \\ \|r_{d,\delta}^n\|_{H^1(\Omega_\delta)} &= \left\| r_{d,\delta}^{n+1} + \delta^{n+1} u_{d,n+1} \right\|_{H^1(\Omega_\delta)} \\ &\stackrel{(5.7)}{\leq} c\delta^{n+1} + c\delta^{n+1/2} \leq c\delta^{n+1/2}, \end{aligned}$$

which completes the proof.  $\square$

## 6 The first-order approximate transmission conditions

In this section, we model the effect of the thin layer by a problem with appropriate transmission conditions and prove that the modelling error is of order two in  $\delta$ . We begin to truncate the series defining the asymptotic expansions, keeping only the first two terms. This yields

$$\begin{aligned} u_{\text{int},\delta} &\simeq u_{\text{int},\delta}^{(1)} := u_{\text{int},0} + \delta u_{\text{int},1} \quad \text{in } \Omega_{\text{ext}}, \\ u_{\text{ext},\delta} &\simeq u_{\text{ext},\delta}^{(1)} := u_{\text{ext},0} + \delta u_{\text{ext},1} \quad \text{in } \Omega_{\text{ext}}, \\ u_{d_1,\delta}(x) &\simeq u_{d_1,\delta}^{(1)}(m, s_1) := u_0^{[1]}(m, s_1) + \delta u_1^{[1]}(m, s_1), \quad \forall x = \Phi_1(m, s_1) \in \Omega_{\delta,1}, \\ u_{d_2,\delta}(x) &\simeq u_{d_2,\delta}^{(1)}(m, s_2) := u_0^{[2]}(m, s_2) + \delta u_1^{[2]}(m, s_2), \quad \forall x = \Phi_2(m, s_2) \in \Omega_{\delta,2}, \end{aligned}$$

where

$$U_\delta^{(1)} := \begin{cases} u_{\text{ext},\delta}^{(1)} & \text{in } \Omega_{\text{ext}}, \\ u_{\text{int},\delta}^{(1)} & \text{in } \Omega_{\text{int}}, \end{cases}$$

is the solution to

$$\begin{cases} \operatorname{div} \left( \sigma_{\text{int}} \nabla u_{\text{int},\delta}^{(1)} \right) + k_{\text{int}}^2 u_{\text{int},\delta}^{(1)} = 0 & \text{in } \Omega_{\text{int}}, \\ \operatorname{div} \left( \sigma_{\text{ext}} \nabla u_{\text{ext},\delta}^{(1)} \right) + k_{\text{ext}}^2 u_{\text{ext},\delta}^{(1)} = 0 & \text{in } \Omega_{\text{ext}}, \\ u_{\text{int},\delta|_\Gamma}^{(1)} - u_{\text{ext},\delta|_\Gamma}^{(1)} = \delta \mathcal{A} \left( u_{\text{int},\delta}^{(1)}, u_{\text{ext},\delta}^{(1)} \right) - \delta^2 \xi_\delta & \text{on } \Gamma, \\ \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},\delta|_\Gamma}^{(1)} - \sigma_{\text{ext}} \partial_{\mathbf{n}} u_{\text{ext},\delta|_\Gamma}^{(1)} = \delta \mathcal{B} \left( u_{\text{int},\delta}^{(1)}, u_{\text{ext},\delta}^{(1)} \right) - \delta^2 \rho_\delta & \text{on } \Gamma, \\ \lim_{|x| \rightarrow +\infty} |x| \left( \partial_{|x|} - ik_{\text{ext}} \right) \left( u_{\text{ext},\delta}^{(1)} - u_{\text{inc}} \right) = 0, \end{cases} \quad (6.1)$$

with

$$\begin{aligned} \mathcal{A}(u, v) &:= \frac{p_1 \sigma_{\text{ext}} \tilde{\sigma} + p_2 \sigma_{\text{int}} \tilde{\sigma} - \sigma_{\text{int}} \sigma_{\text{ext}}}{2 \sigma_{\text{int}} \sigma_{\text{ext}} \tilde{\sigma}} \left( \sigma_{\text{int}} \partial_{\mathbf{n}} u|_\Gamma + \sigma_{\text{ext}} \partial_{\mathbf{n}} v|_\Gamma \right), \\ \mathcal{B}(u, v) &:= \frac{1}{2} \left( \tilde{\sigma} - p_1 \sigma_{\text{int}} - p_2 \sigma_{\text{ext}} \right) \left( \Delta_\Gamma u|_\Gamma + \Delta_\Gamma v|_\Gamma \right) \\ &\quad + \frac{1}{2} \left( \tilde{k}^2 - p_1 k_{\text{int}}^2 - p_2 k_{\text{ext}}^2 \right) \left( u|_\Gamma + v|_\Gamma \right), \\ \xi_\delta &:= \frac{p_1 \sigma_{\text{ext}} \tilde{\sigma} + p_2 \sigma_{\text{int}} \tilde{\sigma} - \sigma_{\text{int}} \sigma_{\text{ext}}}{2 \sigma_{\text{int}} \sigma_{\text{ext}} \tilde{\sigma}} \left( \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},1|_\Gamma} + \sigma_{\text{ext}} \partial_{\mathbf{n}} u_{\text{ext},1|_\Gamma} \right), \\ \rho_\delta &:= \frac{1}{2} \left( \tilde{\sigma} - p_1 \sigma_{\text{int}} - p_2 \sigma_{\text{ext}} \right) \left( \Delta_\Gamma u_{\text{int},1|_\Gamma} + \Delta_\Gamma u_{\text{ext},1|_\Gamma} \right) \\ &\quad + \frac{1}{2} \left( \tilde{k}^2 - p_1 k_{\text{int}}^2 - p_2 k_{\text{ext}}^2 \right) \left( u_{\text{int},1|_\Gamma} + u_{\text{ext},1|_\Gamma} \right). \end{aligned}$$

The first-order approximation is defined by

$$U_\delta^{ap} := \begin{cases} u_{\text{ext},\delta}^{ap} & \text{in } \Omega_{\text{ext}}, \\ u_{\text{int},\delta}^{ap} & \text{in } \Omega_{\text{int}}, \end{cases}$$

where  $U_\delta^{ap}$  is the solution of (6.1) with  $\rho_\delta = 0$  and  $\xi_\delta = 0$ . The approximate problem

$(\mathcal{P}_\delta^{ap})$  is then defined by equation (6.1) with the following transmission conditions

$$\begin{cases} u_{\text{int},\delta|\Gamma}^{ap} - u_{\text{ext},\delta|\Gamma}^{ap} = \delta \frac{p_1 \sigma_{\text{ext}} \tilde{\sigma} + p_2 \sigma_{\text{int}} \tilde{\sigma} - \sigma_{\text{int}} \sigma_{\text{ext}}}{2 \sigma_{\text{int}} \sigma_{\text{ext}} \tilde{\sigma}} \left( \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},\delta|\Gamma}^{ap} + \sigma_{\text{ext}} \partial_{\mathbf{n}} u_{\text{ext},\delta|\Gamma}^{ap} \right), \\ \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},\delta|\Gamma}^{ap} - \sigma_{\text{ext}} \partial_{\mathbf{n}} u_{\text{ext},\delta|\Gamma}^{ap} = \delta \frac{1}{2} (\tilde{\sigma} - p_1 \sigma_{\text{int}} - p_2 \sigma_{\text{ext}}) \left( \Delta_\Gamma u_{\text{int},\delta|\Gamma}^{ap} + \Delta_\Gamma u_{\text{ext},\delta|\Gamma}^{ap} \right) \\ + \delta \frac{1}{2} \left( \tilde{k}^2 - p_1 k_{\text{int}}^2 - p_2 k_{\text{ext}}^2 \right) \left( u_{\text{int},\delta|\Gamma}^{ap} + u_{\text{ext},\delta|\Gamma}^{ap} \right). \end{cases} \quad (6.2)$$

Before proving that  $U_\delta^{ap}$  is indeed an approximation of the field  $u_\delta$  far from the thin layer with error  $O(\delta^2)$ , we study the well-posedness of  $(\mathcal{P}_\delta^{ap})$ . However, the bilinear form associated to  $(\mathcal{P}_\delta^{ap})$  is neither positive nor negative. To show the existence and uniqueness of the solution  $U_\delta^{ap}$ , we reformulate Problem  $(\mathcal{P}_\delta^{ap})$  into a nonlocal equation on the interface  $\Gamma$  (cf. e.g., [6, 9] for different problems). We introduce the DtN operators (Dirichlet-to-Neumann)  $S_{\text{int}}$  and  $S_{\text{ext}}$  defined from  $H^{1/2}(\Gamma)$  onto  $H^{-1/2}(\Gamma)$  by  $S_{\text{int}}\varphi := \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int}}|_\Gamma$ , where  $u_{\text{int}}$  is the solution to the boundary value problem

$$\begin{cases} \operatorname{div}(\sigma_{\text{int}} \nabla u_{\text{int}}) + k_{\text{int}}^2 u_{\text{int}} = 0 & \text{in } \Omega_{\text{int}}, \\ u_{\text{int}}|_\Gamma = \varphi & \text{on } \Gamma, \end{cases}$$

and by  $S_{\text{ext}}\psi := \sigma_{\text{ext}} \partial_{-\mathbf{n}} u_{\text{ext}}|_\Gamma$ , where  $u_{\text{ext}}$  is the solution to the boundary value problem

$$\begin{cases} \operatorname{div}(\sigma_{\text{ext}} \nabla u_{\text{ext}}) + k_{\text{ext}}^2 u_{\text{ext}} = 0 & \text{in } \Omega_{\text{ext}}, \\ u_{\text{ext}}|_\Gamma = \psi & \text{on } \Gamma, \\ \lim_{|x| \rightarrow +\infty} |x| (\partial_{|x|} - ik_{\text{ext}}) u_{\text{ext}} = 0. \end{cases}$$

**Remark 6.1** *The function  $u_{\text{int}}$  is defined only in the case where the constant  $k_{\text{int}}^2/\sigma_{\text{int}}$  does not belong to the spectrum of the closed operator  $(-\Delta, H_0^1(\Omega_{\text{int}}))$ . Fortunately, its*

spectrum is discrete since this operator has a compact resolvent and is composed only of real numbers so we can always assume that  $u_{\text{int}}$  is well-defined.

The following theorem gives the uniqueness of the solution  $U_\delta^{ap}$  to Problem  $(\mathcal{P}_\delta^{ap})$ .

**Theorem 6.1** *Assume that the following hypotheses hold*

$$\Im \left( \tilde{k}^2 - p_1 k_{\text{int}}^2 \right) \geq 0, \quad (6.3)$$

$$\lambda_{\delta,1} := \frac{1}{\delta} \frac{2\sigma_{\text{ext}}\tilde{\sigma}\sigma_{\text{int}}}{p_1\sigma_{\text{ext}}\tilde{\sigma} + p_2\sigma_{\text{int}}\tilde{\sigma} - \sigma_{\text{int}}\sigma_{\text{ext}}} \notin \sigma(S_{\text{int}}), \quad (6.4)$$

problem  $(\mathcal{P}_\delta^{ap})$  admits at most one solution.

**Remark 6.2** *Note that we can always choose  $p_1$  and  $p_2$  in such a manner that the condition on  $\lambda_{\delta,1}$  is fulfilled.*

**Proof** Let us consider the homogeneous problem associated to  $(\mathcal{P}_\delta^{ap})$ :

$$\begin{cases} \operatorname{div} \left( \sigma_{\text{int}} \nabla u_{\text{int},\delta}^{ap} \right) + k_{\text{int}}^2 u_{\text{int},\delta}^{ap} = 0 & \text{in } \Omega_{\text{int}}, \\ \operatorname{div} \left( \sigma_{\text{ext}} \nabla u_{\text{ext},\delta}^{ap} \right) + k_{\text{ext}}^2 u_{\text{ext},\delta}^{ap} = 0 & \text{in } \Omega_{\text{ext}}, \\ \lim_{|x| \rightarrow +\infty} |x| \left( \partial_{|x|} - ik_{\text{ext}} \right) u_{\text{ext},\delta}^{ap} = 0, \end{cases} \quad (6.5)$$

$$\begin{cases} \operatorname{div} \left( \sigma_{\text{int}} \nabla u_{\text{int},\delta}^{ap} \right) + k_{\text{int}}^2 u_{\text{int},\delta}^{ap} = 0 & \text{in } \Omega_{\text{int}}, \\ \operatorname{div} \left( \sigma_{\text{ext}} \nabla u_{\text{ext},\delta}^{ap} \right) + k_{\text{ext}}^2 u_{\text{ext},\delta}^{ap} = 0 & \text{in } \Omega_{\text{ext}}, \\ \lim_{|x| \rightarrow +\infty} |x| \left( \partial_{|x|} - ik_{\text{ext}} \right) u_{\text{ext},\delta}^{ap} = 0, \end{cases} \quad (6.6)$$

with transmission conditions on the interface  $\Gamma$

$$\begin{cases} u_{\text{int},\delta|\Gamma}^{ap} - u_{\text{ext},\delta|\Gamma}^{ap} = \lambda_{\delta,1}^{-1} \left( \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},\delta|\Gamma}^{ap} + \sigma_{\text{ext}} \partial_{\mathbf{n}} u_{\text{ext},\delta|\Gamma}^{ap} \right), \\ \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},\delta|\Gamma}^{ap} - \sigma_{\text{ext}} \partial_{\mathbf{n}} u_{\text{ext},\delta|\Gamma}^{ap} = \delta \frac{1}{2} (\tilde{\sigma} - p_1 \sigma_{\text{int}} - p_2 \sigma_{\text{ext}}) \left( \Delta_\Gamma u_{\text{int},\delta|\Gamma}^{ap} + \Delta_\Gamma u_{\text{ext},\delta|\Gamma}^{ap} \right) \\ \quad + \delta \frac{1}{2} \left( \tilde{k}^2 - p_1 k_{\text{int}}^2 - p_2 k_{\text{ext}}^2 \right) \left( u_{\text{int},\delta|\Gamma}^{ap} + u_{\text{ext},\delta|\Gamma}^{ap} \right). \end{cases} \quad (6.7)$$

Standard regularity results for elliptic problems (see e.g. [1]) show that  $(u_{\text{int},\delta}^{ap}, u_{\text{ext},\delta}^{ap}) \in \mathcal{C}^\infty(\overline{\Omega_{\text{int}}}) \times \mathcal{C}^\infty(\overline{\Omega_{\text{ext}}})$ . Let  $B_R$  denote the ball with centre  $O$  and radius  $R$  large enough to contain  $\Omega_{\text{int}}$  and  $\Omega_R$  be the domain of  $\mathbb{R}^3$  defined by  $\Omega_R := B_R \cap \Omega_{\text{ext}}$ . Multiplying equations (6.5) and (6.6) respectively by  $\overline{u_{\text{int},\delta}^{ap}}$  and  $\overline{u_{\text{ext},\delta}^{ap}}$ , integrating in  $B_R$  and using Green formula, we obtain

$$\begin{aligned}
& \sigma_{\text{int}} \int_{\Omega_{\text{int}}} \left| \nabla u_{\text{int},\delta}^{ap} \right|^2 d\Omega_{\text{int}} - k_{\text{int}}^2 \int_{\Omega_{\text{int}}} \left| u_{\text{int},\delta}^{ap} \right|^2 d\Omega_{\text{int}} + \sigma_{\text{ext}} \int_{\Omega_R} \left| \nabla u_{\text{ext},\delta}^{ap} \right|^2 d\Omega_R \\
& - k_{\text{ext}}^2 \int_{\Omega_R} \left| u_{\text{ext},\delta}^{ap} \right|^2 d\Omega_R + \gamma_1 \int_{\Gamma} \left( \left| u_{\text{int},\delta|_{\Gamma}}^{ap} \right|^2 + \left| u_{\text{ext},\delta|_{\Gamma}}^{ap} \right|^2 \right) d\Gamma \\
& + \delta \frac{1}{4} (\tilde{\sigma} - p_1 \sigma_{\text{int}} - p_2 \sigma_{\text{ext}}) \int_{\Gamma} \left| \nabla_{\Gamma} u_{\text{int},\delta|_{\Gamma}}^{ap} + \nabla_{\Gamma} u_{\text{ext},\delta|_{\Gamma}}^{ap} \right|^2 d\Gamma \\
& + 2\gamma_2 \int_{\Gamma} \Re \left( u_{\text{int},\delta|_{\Gamma}}^{ap} \overline{u_{\text{ext},\delta|_{\Gamma}}^{ap}} \right) d\Gamma = \sigma_{\text{ext}} \int_{S_R} \partial_{\mathbf{R}} u_{\text{ext},\delta|_{S_R}}^{ap} \overline{u_{\text{ext},\delta|_{S_R}}^{ap}} dS_R, \tag{6.8}
\end{aligned}$$

where

$$\begin{aligned}
\gamma_1 &:= -\frac{\sigma_{\text{int}} \sigma_{\text{ext}} \tilde{\sigma}}{\delta (p_1 \sigma_{\text{ext}} \tilde{\sigma} + p_2 \sigma_{\text{int}} \tilde{\sigma} - \sigma_{\text{int}} \sigma_{\text{ext}})} - \delta \frac{\tilde{k}^2 - p_1 k_{\text{int}}^2 - p_2 k_{\text{ext}}^2}{4}, \\
\gamma_2 &:= \frac{\sigma_{\text{int}} \sigma_{\text{ext}} \tilde{\sigma}}{\delta (p_1 \sigma_{\text{ext}} \tilde{\sigma} + p_2 \sigma_{\text{int}} \tilde{\sigma} - \sigma_{\text{int}} \sigma_{\text{ext}})} - \delta \frac{\tilde{k}^2 - p_1 k_{\text{int}}^2 - p_2 k_{\text{ext}}^2}{4},
\end{aligned}$$

and  $S_R$  denotes the sphere with centre  $O$  and radius  $R$ . Hence, taking the imaginary part of (6.8) and using (6.3), we have

$$\Im \left( \int_{S_R} \partial_{\mathbf{R}} u_{\text{ext},\delta|_{S_R}}^{ap} \overline{u_{\text{ext},\delta|_{S_R}}^{ap}} dS_R \right) \leq 0. \tag{6.9}$$

It follows from Rellich's lemma and radiation condition (2.1 k) that

$$u_{\text{ext},\delta}^{ap} = 0 \quad \text{on } \Omega_{\text{ext}}. \tag{6.10}$$

Problem (6.6)-(6.7) is reduced to

$$\left\{ \begin{array}{ll} \operatorname{div} \left( \sigma_{\text{int}} \nabla u_{\text{int},\delta}^{\text{ap}} \right) + k_{\text{int}}^2 u_{\text{int},\delta}^{\text{ap}} = 0 & \text{in } \Omega_{\text{int}}, \\ u_{\text{int},\delta|_{\Gamma}}^{\text{ap}} = \lambda_{\delta,1}^{-1} \partial_{\mathbf{n}} u_{\text{int},\delta|_{\Gamma}}^{\text{ap}} & \text{on } \Gamma, \\ \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},\delta|_{\Gamma}}^{\text{ap}} = \delta \frac{1}{2} (\tilde{\sigma} - p_1 \sigma_{\text{int}} - p_2 \sigma_{\text{ext}}) \Delta_{\Gamma} u_{\text{int},\delta|_{\Gamma}}^{\text{ap}} \\ + \delta \frac{1}{2} \left( \tilde{k}^2 - p_1 k_{\text{int}}^2 - p_2 k_{\text{ext}}^2 \right) u_{\text{int},\delta|_{\Gamma}}^{\text{ap}} & \text{on } \Gamma. \end{array} \right. \quad (6.11)$$

The equation

$$u_{\text{int},\delta|_{\Gamma}}^{\text{ap}} = \lambda_{\delta,1}^{-1} \partial_{\mathbf{n}} u_{\text{int},\delta|_{\Gamma}}^{\text{ap}}, \quad (6.12)$$

implies

$$(S_{\text{int}} - \lambda_{\delta,1} I) \varphi_{\text{int}} = 0, \quad (6.13)$$

where  $\varphi_{\text{int}}$  is the trace of  $u_{\text{int},\delta}^{\text{ap}}$  on the surface  $\Gamma$ . By virtue of (6.4), we get  $u_{\text{int},\delta}^{\text{ap}} = 0$  on  $\Gamma$  and therefore  $u_{\text{int},\delta}^{\text{ap}} = 0$  in  $\Omega_{\text{int}}$ .  $\square$

The existence of  $U_{\delta}^{\text{ap}}$  is based on properties of Laplace-Beltrami and Dtn operators.

The latter are given in the next lemma whose proof can be found, for example, in [40].

**Lemma 6.1** (1) *The Laplace-Beltrami operator  $-\Delta_{\Gamma}$  on  $\Gamma$  is a pseudodifferential operator of real symbol of order 2. It is Fredholm of index 0.*

(2) *The Dirichlet-to-Neumann operators  $S_{\text{int}}$  and  $S_{\text{ext}}$  are elliptic pseudodifferential operators of real symbol of order 1.*

Using the definition of  $S_{\text{int}}$  and  $S_{\text{ext}}$ , Problem  $(\mathcal{P}_\delta^{ap})$  is equivalent to the boundary equations

$$\begin{cases} (S_{\text{int}} - \lambda_{\delta,1}I)\omega - (S_{\text{ext}} - \lambda_{\delta,1}I)\varkappa = g & (6.14) \\ S_{\text{int}}\omega + S_{\text{ext}}\varkappa - \delta\frac{1}{2}(\tilde{\sigma} - p_1\sigma_{\text{int}} - p_2\sigma_{\text{ext}})(\Delta_\Gamma\omega + \Delta_\Gamma\varkappa) \\ -\delta\frac{1}{2}\left(\tilde{k}^2 - p_1k_{\text{int}}^2 - p_2k_{\text{ext}}^2\right)(\omega + \varkappa) = -g, & (6.15) \end{cases}$$

where

$$g := -\sigma_{\text{ext}}\partial_{\mathbf{n}}u_{\text{inc}}|_\Gamma - S_{\text{ext}}u_{\text{inc}}|_\Gamma \in \mathcal{C}^\infty(\Gamma), \quad (6.16)$$

$\omega$  and  $\varkappa$  are the traces of  $u_{\text{int},\delta}^{ap}$  and  $u_{\text{ext},\delta}^{ap}$  on the surface  $\Gamma$  respectively. From (6.4),  $-\lambda_{\delta,1} \notin \sigma(S_{\text{int}})$  thus the next pseudodifferential operator of order  $-1$  is well-defined

$$K_\delta := (S_{\text{int}} - \lambda_{\delta,1}I)^{-1}. \quad (6.17)$$

Equation (6.14) then reduces to

$$\omega = K_\delta(S_{\text{ext}} - \lambda_{\delta,1}I)\varkappa + K_\delta g, \quad (6.18)$$

and Problem (6.14)-(6.15) is equivalent to the boundary equation

$$A_\delta\varkappa := B_\delta\varkappa - \lambda_{\delta,2}\Delta_\Gamma K_\delta(S_{\text{ext}} + S_{\text{int}} - 2\lambda_{\delta,1}I)\varkappa = \theta, \quad (6.19)$$

where

$$\lambda_{\delta,2} := \delta\frac{1}{2}(\tilde{\sigma} - p_1\sigma_{\text{int}} - p_2\sigma_{\text{ext}}), \quad (6.20)$$

$$\theta := -g - S_{\text{int}}K_\delta g + \lambda_{\delta,2}\Delta_\Gamma K_\delta g + \lambda_{\delta,3}K_\delta g,$$

$$B_\delta := S_{\text{int}}K_\delta S_{\text{ext}} - \lambda_{\delta,1}S_{\text{int}}K_\delta + S_{\text{ext}} - \lambda_{\delta,3}K_\delta S_{\text{ext}} + \lambda_{\delta,3}\lambda_{\delta,1}K_\delta - \lambda_{\delta,3}I,$$

$$\lambda_{\delta,3} := \delta\frac{1}{2}\left(\tilde{k}^2 - p_1k_{\text{int}}^2 - p_2k_{\text{ext}}^2\right).$$

Some properties of the operator  $A_\delta$  are given in the next proposition.

**Proposition 6.1** *For all integers  $k$  in  $\mathbb{N}$ , the operator  $A_\delta$  defined from  $H^{k+1/2}(\Gamma)$  to  $H^{k-3/2}(\Gamma)$  is Fredholm with index zero.*

**Proof** Let  $k$  be an integer in  $\mathbb{N}$ . Since  $S_{\text{int}}$  and  $S_{\text{ext}}$  are pseudodifferential operators of order 1, they map  $H^k(\Gamma)$  to  $H^{k-1}(\Gamma)$ .  $K_\delta$  being a pseudodifferential operator of order  $-1$ , it maps  $H^k(\Gamma)$  to  $H^{k+1}(\Gamma)$ . As a consequence,  $B_\delta$  maps  $H^k(\Gamma)$  to  $H^{k-1}(\Gamma)$ . The injection  $H^{k-1}(\Gamma) \hookrightarrow H^{k-2}(\Gamma)$  being compact, the operator  $A_\delta$  defined from  $H^{k+1/2}(\Gamma)$  to  $H^{k-3/2}(\Gamma)$  is a compact perturbation of  $\lambda_{\delta,2}\Delta_\Gamma K_\delta (S_{\text{ext}} + S_{\text{int}} - 2\lambda_{\delta,1}I)$ . Since  $\Delta_\Gamma$  is Fredholm with index zero, to show that  $A_\delta$  is Fredholm with index zero, it suffices to show that  $S_{\text{ext}} + S_{\text{int}} - 2\lambda_{\delta,1}I$  is invertible.

Let us consider the equation

$$(S_{\text{ext}} + S_{\text{int}} - 2\lambda_{\delta,1}I)\varphi = \psi, \quad \psi \in H^{k-1/2}(\Gamma), \quad k \in \mathbb{N}. \quad (6.21)$$

Using the definition of  $S_{\text{ext}}$  and  $S_{\text{int}}$ , Equation (6.21) is equivalent to the following problem

$$\left\{ \begin{array}{ll} \operatorname{div}(\sigma_{\text{int}}\nabla V_{\text{int}}) + k_{\text{int}}^2 V_{\text{int}} = 0 & \text{in } \Omega_{\text{int}}, \\ \operatorname{div}(\sigma_{\text{ext}}\nabla V_{\text{ext}}) + k_{\text{ext}}^2 V_{\text{ext}} = 0 & \text{in } \Omega_{\text{ext}}, \\ V_{\text{int}|_\Gamma} - V_{\text{ext}|_\Gamma} = 0 & \text{in } \Gamma \\ \sigma_{\text{int}}\partial_{\mathbf{n}}V_{\text{int}|_\Gamma} - \sigma_{\text{ext}}\partial_{\mathbf{n}}V_{\text{ext}|_\Gamma} = 2\lambda_{\delta,1}V_{\text{int}|_\Gamma} + \psi & \text{in } \Gamma \\ \lim_{|x| \rightarrow +\infty} |x|(\partial_{|x|} - ik_{\text{ext}})V_{\text{ext}} = 0, & \end{array} \right. \quad (6.22)$$

where  $\varphi = V_{\text{int}|_\Gamma} = V_{\text{ext}|_\Gamma}$ . Standard arguments involving Rellich's lemma and the Fredholm alternative show that, for all  $k$  in  $\mathbb{N}$ , if  $\psi \in H^{k-1/2}(\Gamma)$ , then Problem (6.22) admits a unique solution  $(V_{\text{int}}, V_{\text{ext}})$  in  $H^{k+1}(\Omega_{\text{int}}) \times H_{\text{loc}}^{k+1}(\overline{\Omega_{\text{ext}}})$ , and hence a unique



trace  $\varphi \in H^{k+1/2}(\Gamma)$ . As a consequence, the operator  $S_{\text{ext}} + S_{\text{int}} - 2\lambda_{\delta,1}I$ , defined from  $H^{k+1/2}(\Gamma)$  to  $H^{k-1/2}(\Gamma)$ , is invertible.  $\square$

We are now in position to state the existence theorem.

**Theorem 6.2** *Under the assumptions of Theorem 6.1, Problem  $(\mathcal{P}_\delta^{ap})$  admits a unique solution  $(u_{\text{int},\delta}^{ap}, u_{\text{ext},\delta}^{ap})$  in  $H^{k+1}(\Omega_{\text{int}}) \times H_{loc}^{k+1}(\overline{\Omega_{\text{ext}}})$ ,  $\forall k \in \mathbb{N}$ .*

**Proof** It follows from Proposition 6.1 that the uniqueness of  $U_\delta^{ap}$  implies the existence.

From Theorem 6.1, we then infer that, for all  $k$  in  $\mathbb{N}$ , there exists a unique solution  $(u_{\text{int},\delta}^{ap}, u_{\text{ext},\delta}^{ap})$  in  $H^{k+1}(\Omega_{\text{int}}) \times H_{loc}^{k+1}(\overline{\Omega_{\text{ext}}})$ .  $\square$

Let us denote by  $u_\delta^{ap}$  the approximate solution defined on  $\Omega$  by

$$u_\delta^{ap} := \begin{cases} u_{\text{int},\delta}^{ap} & \text{in } \Omega_{\text{int},\delta}, \\ u_{d_\beta,\delta}^{ap} & \text{in } \Omega_{\delta,\beta}, \ (\beta = 1 \text{ or } 2), \\ u_{\text{ext},\delta}^{ap} & \text{in } \Omega_{\text{ext},\delta}, \end{cases}$$

such that  $u_{d_\beta,\delta}^{ap}$  are defined on  $\Omega_{\delta,\beta}$  by

$$u_{d_1,\delta}^{ap}(x) := u_{d_1,\delta}^{[1],ap}(m, s_1) := u_{\text{int},\delta|\Gamma}^{ap} + \delta p_1 [(s_1 + 1)\sigma_{\text{int}}\tilde{\sigma}^{-1} - 1] \partial_{\mathbf{n}} u_{\text{int},\delta|\Gamma}^{ap},$$

$$\forall x = \Phi_1(m, s_1) \in \Omega_{\delta,1},$$

and

$$u_{d_2,\delta}^{ap}(x) := u_{d_2,\delta}^{[2],ap}(m, s_2) := u_{\text{ext},\delta|\Gamma}^{ap} + \delta p_2 [(s_2 - 1)\sigma_{\text{ext}}\tilde{\sigma}^{-1} + 1] \partial_{\mathbf{n}} u_{\text{ext},\delta|\Gamma}^{ap},$$

$$\forall x = \Phi_2(m, s_2) \in \Omega_{\delta, 2}.$$

Finally, we want to derive an error estimate between  $u_\delta$  and the approximate solution  $u_\delta^{ap}$ . To do so, we need once again a uniform stability result for the approximate problem.

Let  $\mathbb{H}^1(\Omega)$  be the Hilbert space defined by

$$\mathbb{H}^1(\Omega) := \left\{ v = (v_{\text{int}}, v_{\text{ext}}) \in H^1(\Omega_{\text{int}}) \times H^1(\tilde{\Omega}_{\text{ext}}) \right\},$$

equipped with its natural norm and  $b_\delta$  be a bilinear form defined on  $\mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega)$  by

$$\begin{aligned} b_\delta(u, v) &:= \sigma_{\text{int}} \int_{\Omega_{\text{int}}} \nabla u_{\text{int}} \cdot \nabla v_{\text{int}} \, d\Omega_{\text{int}} - k_{\text{int}}^2 \int_{\Omega_{\text{int}}} u_{\text{int}} v_{\text{int}} \, d\Omega_{\text{int}} \\ &+ \sigma_{\text{ext}} \int_{\tilde{\Omega}_{\text{ext}}} \nabla u_{\text{ext}} \cdot \nabla v_{\text{ext}} \, d\tilde{\Omega}_{\text{ext}} - k_{\text{ext}}^2 \int_{\tilde{\Omega}_{\text{ext}}} u_{\text{ext}} v_{\text{ext}} \, d\tilde{\Omega}_{\text{ext}} \\ &- \frac{1}{2} \lambda_{\delta, 1} \int_{\Gamma} (u_{\text{int}|_{\Gamma}} - u_{\text{ext}|_{\Gamma}}) (v_{\text{int}|_{\Gamma}} - v_{\text{ext}|_{\Gamma}}) \, d\Gamma \\ &+ \sigma_{\text{ext}} \langle T u_{\text{ext}|_{\partial\Omega}}, v|_{\partial\Omega} \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)}. \end{aligned}$$

We have the following lemma

**Lemma 6.2** 1) For all  $h_\delta$  in  $(\mathbb{H}^1(\Omega))'$ , there exists a positive constant  $c$  independent of  $\delta$  such that the solution to the variational problem

$$\left\{ \begin{array}{l} \text{Find } u_\delta \in \mathbb{H}^1(\Omega), \forall v \in \mathbb{H}^1(\Omega), \\ b_\delta(u_\delta, v) = h_\delta(v), \end{array} \right.$$

satisfies

$$\|u_\delta\|_{\mathbb{H}^1(\Omega)} \leq c \delta^{-1/2} \|h_\delta\|_{(\mathbb{H}^1(\Omega))'}. \quad (6.23)$$

2) Furthermore, if  $p_1 \sigma_{\text{ext}} \tilde{\sigma}_\delta + p_2 \sigma_{\text{int}} \tilde{\sigma}_\delta - \sigma_{\text{int}} \sigma_{\text{ext}} \leq 0$ , one has

$$\|u_\delta\|_{\mathbb{H}^1(\Omega)} \leq c \|h_\delta\|_{(\mathbb{H}^1(\Omega))'}. \quad (6.24)$$

**Proof 1)** We need to prove that

$$\|u_\delta\|_{\mathbb{H}^1(\Omega)} \leq c\delta^{-1/2} \sup_{v \in \mathbb{H}^1(\Omega)} \frac{|b_\delta(u_\delta, v)|}{\|v\|_{\mathbb{H}^1(\Omega)}}.$$

We proceed by contradiction, assuming there exist sequences  $(\delta_n)_{n \geq 0}$  and  $(u_{\delta_n})_{n \geq 0}$ , denoted by  $(u_n)_{n \geq 0}$ , such that

$$\lim_{n \rightarrow +\infty} \delta_n = 0, \quad \left\| \sqrt{\delta} u_n \right\|_{\mathbb{H}^1(\Omega)} = 1, \quad \forall n \in \mathbb{N} \quad \text{and} \quad \lim_{n \rightarrow +\infty} \sup_{\|\varphi\|_{\mathbb{H}^1(\Omega)}=1} |b_{\delta_n}(u_n, \varphi)| = 0. \quad (6.25)$$

We can extract a subsequence of  $(\sqrt{\delta} u_n)_{n \geq 0}$ , still denoted by  $(\sqrt{\delta} u_n)_{n \geq 0}$ , such that

$$\begin{cases} \sqrt{\delta} u_n \rightarrow u_0 \text{ in } \mathbb{L}^2(\Omega), \\ \sqrt{\delta} u_n \rightharpoonup u_0 \text{ in } \mathbb{H}^1(\Omega). \end{cases} \quad (6.26)$$

Furthermore, for all  $v$  in  $\mathcal{C}^\infty(\overline{\Omega_{\text{int}}}) \times \mathcal{C}^\infty(\overline{\tilde{\Omega}_{\text{ext}}})$ , we have

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \frac{1}{2} \sqrt{\delta} \lambda_{\delta,1} \int_{\Gamma} (u_{\text{int},n|\Gamma} - u_{\text{ext},n|\Gamma}) (v_{\text{int}|\Gamma} - v_{\text{ext}|\Gamma}) \, d\Gamma \\ &= \sigma_{\text{int}} \int_{\Omega_{\text{int}}} \sqrt{\delta} \nabla u_{\text{int},n} \cdot \nabla v_{\text{int}} \, d\Omega_{\text{int}} - k_{\text{int}}^2 \int_{\Omega_{\text{int}}} \sqrt{\delta} u_{\text{int},n} v_{\text{int}} \, d\Omega_{\text{int}} \\ &+ \sigma_{\text{ext}} \int_{\tilde{\Omega}_{\text{ext}}} \sqrt{\delta} \nabla u_{\text{ext},n} \cdot \nabla v_{\text{ext}} \, d\tilde{\Omega}_{\text{ext}} - k_{\text{ext}}^2 \int_{\tilde{\Omega}_{\text{ext}}} \sqrt{\delta} u_{\text{ext},n} v_{\text{ext}} \, d\tilde{\Omega}_{\text{ext}} \\ &- \sqrt{\delta} b_\delta(u_n, v) + \sigma_{\text{ext}} \left\langle \sqrt{\delta} T u_{\text{ext},n} |_{\partial\Omega}, v |_{\partial\Omega} \right\rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} \\ &= \sigma_{\text{int}} \int_{\Omega_{\text{int}}} \nabla u_0 \cdot \nabla v_{\text{int}} \, d\Omega_{\text{int}} - k_{\text{int}}^2 \int_{\Omega_{\text{int}}} u_0 v_{\text{int}} \, d\Omega_{\text{int}} \\ &+ \sigma_{\text{ext}} \int_{\tilde{\Omega}_{\text{ext}}} \nabla u_0 \cdot \nabla v_{\text{ext}} \, d\tilde{\Omega}_{\text{ext}} - k_{\text{ext}}^2 \int_{\tilde{\Omega}_{\text{ext}}} u_0 v_{\text{ext}} \, d\tilde{\Omega}_{\text{ext}} \\ &+ \sigma_{\text{ext}} \left\langle T u_{\text{ext},0} |_{\partial\Omega}, v |_{\partial\Omega} \right\rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)}. \end{aligned}$$

As the right-hand side is independent of  $\delta$ , we have

$$\frac{1}{2} \sqrt{\delta} \lambda_{\delta,1} \int_{\Gamma} (u_{\text{int},n|\Gamma} - u_{\text{ext},n|\Gamma}) (v_{\text{int}|\Gamma} - v_{\text{ext}|\Gamma}) \, d\Gamma = O(1), \quad \forall v \in \mathcal{C}^\infty(\overline{\Omega_{\text{int}}}) \times \mathcal{C}^\infty(\overline{\tilde{\Omega}_{\text{ext}}}), \quad (6.27)$$

and by density of  $\mathcal{C}^\infty(\overline{\Omega_{\text{int}}}) \times \mathcal{C}^\infty(\overline{\tilde{\Omega}_{\text{ext}}})$  in  $\mathbb{H}^1(\Omega)$ , we conclude that the equality is true

for all  $v$  in  $\mathbb{H}^1(\Omega)$ . Setting  $v = \overline{u_n}$ , we obtain

$$\|u_{\text{int},n}|_\Gamma - u_{\text{ext},n}|_\Gamma\|_{L^2(\Gamma)} \leq c\delta^{1/4}. \quad (6.28)$$

It follows that  $u_{\text{int},0} = u_{\text{ext},0}$  on  $\Gamma$  and, for all  $v$  in  $H^1(\Omega)$ , one gets

$$\begin{aligned} \lim_{n \rightarrow +\infty} \sqrt{\delta} b_\delta(u, v) &:= \sigma_{\text{int}} \int_{\Omega_{\text{int}}} \nabla u_{\text{int},0} \cdot \nabla v_{\text{int}} \, d\Omega_{\text{int}} - k_{\text{int}}^2 \int_{\Omega_{\text{int}}} u_{\text{int},0} v_{\text{int}} \, d\Omega_{\text{int}} \\ &+ \sigma_{\text{ext}} \int_{\tilde{\Omega}_{\text{ext}}} \nabla u_{\text{ext},0} \cdot \nabla v_{\text{ext}} \, d\tilde{\Omega}_{\text{ext}} - k_{\text{ext}}^2 \int_{\tilde{\Omega}_{\text{ext}}} u_{\text{ext},0} v_{\text{ext}} \, d\tilde{\Omega}_{\text{ext}} \\ &+ \sigma_{\text{ext}} \langle T u_{\text{ext},0}|_{\partial\Omega}, v|_{\partial\Omega} \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} = 0. \end{aligned} \quad (6.29)$$

Theorem 4.1 ensures that the problem: *Find  $u_0$  in  $H^1(\Omega)$  satisfying (6.29),  $\forall v \in H^1(\Omega)$ ,*

is well-posed. We then infer that  $u_0 = 0$  and it only remains to show that  $\lim_{n \rightarrow +\infty} \|\sqrt{\delta} u_n\|_{\mathbb{H}^1(\Omega)} =$

0. Note that, since  $u_0$  is uniquely determined, the whole sequence converges to  $u_0 = 0$  in

$\mathbb{L}^2(\Omega)$ . To obtain a contradiction, we have to show that  $\lim_{n \rightarrow +\infty} \|\sqrt{\delta} \nabla u_n\|_{\mathbb{L}^2(\Omega)} = 0$ .

One has

$$\begin{aligned} \|\sqrt{\delta} \nabla u_n\|_{\mathbb{L}^2(\Omega)}^2 &\leq c\delta \left( \sigma_{\text{int}} \int_{\Omega_{\text{int}}} |\nabla u_{\text{int},n}|^2 \, d\Omega_{\text{int}} + \sigma_{\text{ext}} \int_{\Omega_{\text{ext}}} |\nabla u_{\text{ext},n}|^2 \, d\Omega_{\text{ext}} \right) \\ &= c\Re \left( \delta b_\delta(u_n, \overline{u_n}) + k_{\text{int}}^2 \int_{\Omega_{\text{int}}} \delta |u_{\text{int},n}|^2 \, d\Omega_{\text{int}} + k_{\text{ext}}^2 \int_{\Omega_{\text{ext}}} \delta |u_{\text{ext},n}|^2 \, d\Omega_{\text{ext}} \right. \\ &+ \frac{1}{2} \delta \lambda_{\delta,1} \int_\Gamma |u_{\text{int},n}|_\Gamma - u_{\text{ext},n}|_\Gamma|^2 \, d\Gamma \\ &\left. - \delta \sigma_{\text{ext}} \langle T u_{\text{ext},n}|_{\partial\Omega}, \overline{u_{\text{ext},n}|_{\partial\Omega}} \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} \right). \end{aligned}$$

Using Lemma 2.1, we infer

$$\begin{aligned} \|\sqrt{\delta} \nabla u_n\|_{L^2(\Omega)}^2 &\leq c\Re \left[ \delta b_n(u_n, \overline{u_n}) + k_{\text{int}}^2 \int_{\Omega_{\text{int}}} \delta |u_{\text{int},n}|^2 \, d\Omega_{\text{int}} + k_{\text{ext}}^2 \int_{\Omega_{\text{ext}}} \delta |u_{\text{ext},n}|^2 \, d\Omega_{\text{ext}} \right. \\ &+ \frac{1}{2} \delta \lambda_{\delta,1} \int_\Gamma |u_{\text{int},n}|_\Gamma - u_{\text{ext},n}|_\Gamma|^2 \, d\Gamma \\ &\left. - \sigma_{\text{ext}} \langle \sqrt{\delta} K u_{\text{ext},n}|_{\partial\Omega}, \sqrt{\delta} \overline{u_{\text{ext},n}|_{\partial\Omega}} \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} \right]. \end{aligned}$$

Since  $K$  is compact and  $\sqrt{\delta}u_n \rightharpoonup 0$  in  $\mathbb{H}^1(\Omega)$ ,

$$\left\langle \sqrt{\delta}K u_{\text{ext},n|\partial\Omega}, \sqrt{\delta}\overline{u_{\text{ext},n|\partial\Omega}} \right\rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} \rightarrow 0.$$

Finally, the assumption  $\lim_{n \rightarrow +\infty} \Re[b_n(u_n, \overline{u_n})] = 0$  and (6.28) yield  $\lim_{n \rightarrow +\infty} \left\| \sqrt{\delta}\nabla u_n \right\|_{\mathbb{L}^2(\Omega)} = 0$  contradicting  $\left\| \sqrt{\delta}u_n \right\|_{\mathbb{H}^1(\Omega)} = 1$ .

2) Similar arguments to those used to prove (6.23) guarantee Inequality (6.24).  $\square$

We can now prove optimal error estimates.

**Theorem 6.3** *There exists a constant  $c$  independent of  $\delta$  such that*

$$\begin{aligned} \left\| u_{\text{int},\delta} - u_{\text{int},\delta}^{\text{ap}} \right\|_{H^1(\Omega_{\text{int},\delta})} + \delta^{1/2} \sum_{\beta=1}^2 \left\| u_{d\beta,\delta} - u_{d\beta,\delta}^{\text{ap}} \right\|_{H^1(\Omega_{\delta,\beta})} \\ + \left\| u_{\text{ext},\delta} - u_{\text{ext},\delta}^{\text{ap}} \right\|_{H^1(\tilde{\Omega}_{\text{ext},\delta})} \leq c\delta^2. \end{aligned}$$

**Proof** According to the Convergence Theorem, it is enough to estimate the error  $U_\delta^{\text{ap}} - U_\delta^{(1)}$ . Therefore, as in [41], we perform an asymptotic expansion for  $U_\delta^{\text{ap}}$  which amounts to postulating the ansatz

$$U_\delta^{\text{ap}} = \sum_{j \geq 0} \delta^j w_j, \tag{6.30}$$



which gives, for all  $v = (v_{\text{int}}, v_{\text{ext}})$  in  $\mathbb{H}^1(\Omega)$ ,

$$\begin{aligned}
& \sigma_{\text{int}} \int_{\Omega_{\text{int}}} \nabla \mathcal{R}_{\text{int},w} \cdot \nabla v_{\text{int}} \, d\Omega_{\text{int}} - k_{\text{int}}^2 \int_{\Omega_{\text{int}}} \mathcal{R}_{\text{int},w} v_{\text{int}} \, d\Omega_{\text{int}} \\
& + \sigma_{\text{ext}} \int_{\tilde{\Omega}_{\text{ext}}} \nabla \mathcal{R}_{\text{ext},w} \cdot \nabla v_{\text{ext}} \, d\tilde{\Omega}_{\text{ext}} - k_{\text{ext}}^2 \int_{\tilde{\Omega}_{\text{ext}}} \mathcal{R}_{\text{ext},w} v_{\text{ext}} \, d\tilde{\Omega}_{\text{ext}} \\
& - \frac{1}{2} \lambda_{\delta,1} \int_{\Gamma} (\mathcal{R}_{\text{int},w|_{\Gamma}} - \mathcal{R}_{\text{ext},w|_{\Gamma}}) (v_{\text{int}|_{\Gamma}} - v_{\text{ext}|_{\Gamma}}) \, d\Gamma \\
& - \delta \frac{1}{4} \left( \tilde{k}_{\delta}^2 - p_1 k_{\text{int}}^2 - p_2 k_{\text{ext}}^2 \right) \int_{\Gamma} (\mathcal{R}_{\text{int},w|_{\Gamma}} + \mathcal{R}_{\text{ext},w|_{\Gamma}}) (v_{\text{int}|_{\Gamma}} + v_{\text{ext}|_{\Gamma}}) \, d\Gamma \\
& - \delta \frac{1}{4} (\tilde{\sigma}_{\delta} - p_1 \sigma_{\text{int}} - p_2 \sigma_{\text{ext}}) \int_{\Gamma} (\Delta_{\Gamma} \mathcal{R}_{\text{int},w|_{\Gamma}} + \Delta_{\Gamma} \mathcal{R}_{\text{ext},w|_{\Gamma}}) (v_{\text{int}|_{\Gamma}} + v_{\text{ext}|_{\Gamma}}) \, d\Gamma \\
& + \sigma_{\text{ext}} \langle T \mathcal{R}_{\text{ext},w|_{\partial\Omega}}, v|_{\partial\Omega} \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} \\
& = \frac{1}{2} \delta^4 \int_{\Gamma} \mathcal{B}(w_{\text{int},3}, w_{\text{ext},3}) (v_{\text{int}|_{\Gamma}} + v_{\text{ext}|_{\Gamma}}) \, d\Gamma \\
& + \frac{1}{2} \delta^4 \int_{\Gamma} \lambda_{\delta,1} \mathcal{A}(w_{\text{int},3}, w_{\text{ext},3}) (v_{\text{int}|_{\Gamma}} - v_{\text{ext}|_{\Gamma}}) \, d\Gamma.
\end{aligned}$$

Putting all terms of order 1 in  $\delta$  on the right hand side, we get

$$b_{\delta}(\mathcal{R}_w, v) = h_{\delta}(v),$$

where

$$\begin{aligned}
h_{\delta}(v) & := \delta \frac{1}{4} \left( \tilde{k}_{\delta}^2 - p_1 k_{\text{int}}^2 - p_2 k_{\text{ext}}^2 \right) \int_{\Gamma} (\mathcal{R}_{\text{int},w|_{\Gamma}} + \mathcal{R}_{\text{ext},w|_{\Gamma}}) (v_{\text{int}|_{\Gamma}} + v_{\text{ext}|_{\Gamma}}) \, d\Gamma \\
& + \delta \frac{1}{4} (\tilde{\sigma}_{\delta} - p_1 \sigma_{\text{int}} - p_2 \sigma_{\text{ext}}) \int_{\Gamma} (\Delta_{\Gamma} \mathcal{R}_{\text{int},w|_{\Gamma}} + \Delta_{\Gamma} \mathcal{R}_{\text{ext},w|_{\Gamma}}) (v_{\text{int}|_{\Gamma}} + v_{\text{ext}|_{\Gamma}}) \, d\Gamma \\
& + \frac{1}{2} \delta^4 \int_{\Gamma} \mathcal{B}(w_{\text{int},3}, w_{\text{ext},3}) (v_{\text{int}|_{\Gamma}} + v_{\text{ext}|_{\Gamma}}) \, d\Gamma \\
& + \delta^4 \int_{\Gamma} \lambda_{\delta,1} \mathcal{A}(w_{\text{int},3}, w_{\text{ext},3}) (v_{\text{int}|_{\Gamma}} - v_{\text{ext}|_{\Gamma}}) \, d\Gamma.
\end{aligned}$$

From Lemma 6.2, there exists a constant  $c$  independent of  $\delta$  such that

$$\|\mathcal{R}_w\|_{\mathbb{H}^1(\Omega)} \leq c \delta^{-1/2} \|h_{\delta}\|_{(\mathbb{H}^1(\Omega))'}.$$

Hence, we obtain

$$\|\mathcal{R}_w\|_{\mathbb{H}^1(\Omega)} \leq c \left( \delta^{1/2} \|\mathcal{R}_w\|_{\mathbb{H}^1(\Omega)} + \delta^{5/2} \|w_3\|_{\mathbb{H}^1(\Omega)} \right),$$

so

$$\|\mathcal{R}_w\|_{\mathbb{H}^1(\Omega)} \leq \frac{\delta^{5/2} c}{(1 - c\delta^{1/2})} \|w_3\|_{\mathbb{H}^1(\Omega)}.$$

Since  $\delta$  is small enough, we have

$$\|\mathcal{R}_w\|_{\mathbb{H}^1(\Omega)} \leq c\delta^2 \|w_3\|_{\mathbb{H}^1(\Omega)},$$

which gives the desired result.  $\square$

**Remark 6.3** *There is a particularly interesting case when  $\sigma_{\text{ext}}$ ,  $\tilde{\sigma}$  and  $\sigma_{\text{int}}$  are strictly positive constants satisfying  $\sigma_{\text{int}} < \tilde{\sigma} < \sigma_{\text{ext}}$  or  $\sigma_{\text{ext}} < \tilde{\sigma} < \sigma_{\text{int}}$ , it corresponds to the case where the solution  $U_\delta^{ap}$  is continuous when crossing  $\Gamma$ . Indeed, if we set*

$$\sigma_{\text{int}}\sigma_{\text{ext}} - p_1\sigma_{\text{ext}}\tilde{\sigma} - p_2\sigma_{\text{int}}\tilde{\sigma} = 0,$$

we obtain

$$p_1 = \frac{\sigma_{\text{int}}(\sigma_{\text{ext}} - \tilde{\sigma})}{\tilde{\sigma}(\sigma_{\text{ext}} - \sigma_{\text{int}})} \text{ and } p_2 = \frac{\sigma_{\text{ext}}(\tilde{\sigma} - \sigma_{\text{int}})}{\tilde{\sigma}(\sigma_{\text{ext}} - \sigma_{\text{int}})}.$$

Then, the transmission conditions (6.2) become

$$\begin{cases} u_{\text{int},\delta}^{ap} - u_{\text{ext},\delta}^{ap} = 0, \\ \sigma_{\text{int}}\partial_{\mathbf{n}}u_{\text{int},\delta}^{ap} - \sigma_{\text{ext}}\partial_{\mathbf{n}}u_{\text{ext},\delta}^{ap} = \delta \frac{(\sigma_{\text{ext}} - \tilde{\sigma})(\sigma_{\text{int}} - \tilde{\sigma})}{\tilde{\sigma}} \Delta_\Gamma u_{\text{int},\delta}^{ap} \\ + \delta \frac{\tilde{\sigma}k_{\text{ext}}^2(\sigma_{\text{ext}} - \sigma_{\text{int}}) - \sigma_{\text{int}}k_{\text{int}}^2(\sigma_{\text{ext}} - \tilde{\sigma}) - \sigma_{\text{ext}}k_{\text{ext}}^2(\tilde{\sigma} - \sigma_{\text{int}})}{\tilde{\sigma}(\sigma_{\text{ext}} - \sigma_{\text{int}})} u_{\text{ext},\delta}^{ap}. \end{cases}$$

Problem  $(\mathcal{P}_\delta^{ap})$  is equivalent to the boundary equation

$$B_\delta\omega := -\Delta_\Gamma\omega + \frac{\tilde{\sigma}\sigma_{\text{int}}}{\delta(\sigma_{\text{ext}} - \tilde{\sigma})(\sigma_{\text{int}} - \tilde{\sigma})} S_{\text{int}}\omega + \frac{\tilde{\sigma}\sigma_{\text{ext}}}{\delta(\sigma_{\text{ext}} - \tilde{\sigma})(\sigma_{\text{int}} - \tilde{\sigma})} S_{\text{ext}}\omega$$



$$\begin{aligned}
& + \frac{\sigma_{\text{int}} k_{\text{int}}^2 (\sigma_{\text{ext}} - \tilde{\sigma}) + \sigma_{\text{ext}} k_{\text{ext}}^2 (\tilde{\sigma} - \sigma_{\text{int}}) - \tilde{\sigma} \tilde{k}^2 (\sigma_{\text{ext}} - \sigma_{\text{int}})}{(\sigma_{\text{ext}} - \tilde{\sigma})(\sigma_{\text{int}} - \tilde{\sigma})(\sigma_{\text{ext}} - \sigma_{\text{int}})} \omega \\
& = \frac{\sigma_{\text{ext}} \tilde{\sigma}}{\delta (\sigma_{\text{ext}} - \tilde{\sigma})(\sigma_{\text{int}} - \tilde{\sigma})} \partial_{\mathbf{n}} u_{\text{inc}}|_{\Gamma} + \frac{\sigma_{\text{ext}} \tilde{\sigma}}{\delta (\sigma_{\text{ext}} - \tilde{\sigma})(\sigma_{\text{int}} - \tilde{\sigma})} S_{\text{ext}} u_{\text{inc}}|_{\Gamma} \text{ on } \Gamma,
\end{aligned}$$

where  $\omega$  is the trace of  $u_{\text{int},\delta}^{\text{ap}}$  on the surface  $\Gamma$ . As above, the existence and uniqueness are obtained with a Fredholm alternative, and similar error estimates can be shown.

## 7 Extension to thin layer with high magnetic permittivity

In this section, we consider the case of a high value of magnetic permittivity of the domain  $\Omega_{\delta}$  (cf. e.g. [25, 34] for similar problems). More precisely, we consider the case where  $\tilde{\sigma}_{\delta} := \tilde{\sigma}/\delta$  and  $\tilde{k}_{\delta}^2 := \tilde{k}^2/\delta$  where  $\tilde{\sigma}$  is a strictly positive constant and  $\tilde{k}^2$  is a complex number with strictly positive real part and positive imaginary part.

The asymptotic analysis can be done in the same way and we are thus going to only give the approximate transmission conditions without doing all computations. Although the derivation of these new conditions can be done without additional difficulties, the uniform stability estimate, which is the basis for optimal error estimates, can not be proved as Theorem 2.2. Actually, the singularity of both the contrast and the refractive index of the thin layer yield a limiting equation that involves Ventcel-like transmission condition. All the well-posedness and regularity results used below to get such a uniform stability estimate are postponed to the appendix. The non-standard nature of transmission conditions of the Ventcel problem lead us to introduce the Sobolev spaces  $H^{1,1}(\Omega_{\text{int}})$ ,  $H_{\text{loc}}^{1,1}(\overline{\Omega_{\text{ext}}})$  and  $H_{\Gamma}^1(\Omega)$  defined by

$$H^{1,1}(\Omega_{\text{int}}) := \{v \in H^1(\Omega_{\text{int}}), v|_{\Gamma} \in H^1(\Gamma)\},$$

$$H_{\Gamma}^1(\Omega) := \{v \in H^1(\Omega), v|_{\Gamma} \in H^1(\Gamma)\},$$

$$H_{loc}^{1,1}(\overline{\Omega_{\text{ext}}}) := \{v \in H_{loc}^1(\overline{\Omega_{\text{ext}}}), v|_{\Gamma} \in H^1(\Gamma)\},$$

where  $\Omega$  is a bounded domain of  $\mathbb{R}^3$  containing  $\Omega_{\text{int}}$  (cf. Figure 4), equipped with their natural norms and semi-norm.

This section is then organized as follows, we first prove the uniform stability estimate and give next the first-order transmission conditions to take into account the effect of the thin layer.

### 7.1 Uniform stability estimate

We prove here the uniform stability result for the high-permittivity case.

**Theorem 7.1** (Uniform stability) *If  $l_{\delta} \in (H^1(\Omega))'$ , then Problem (2.5) admits a unique solution in  $H^1(\Omega)$ . Furthermore, there exists a positive constant  $c$  independent of  $\delta$  such that*

$$\|u_{\delta}\|_{H^1(\Omega)} \leq c \|l_{\delta}\|_{(H^1(\Omega))'}. \quad (7.1)$$

**Proof** We recall below the definition of the bilinear form

$$a_{\delta}(u_{\delta}, v) := \int_{\Omega} \sigma_{\delta} \nabla u_{\delta} \cdot \nabla \bar{v} - k_{\delta}^2 u_{\delta} \bar{v} \, d\Omega + \sigma_{\text{ext}} \langle T u_{\delta}|_{\partial\Omega}, \bar{v}|_{\partial\Omega} \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)}, \quad (7.2)$$

and we need to prove that

$$\|u_{\delta}\|_{H^1(\Omega)} \leq c \sup_{v \in H^1(\Omega)} \frac{|a_{\delta}(u_{\delta}, v)|}{\|v\|_{H^1(\Omega)}}. \quad (7.3)$$

To do so, we proceed by contradiction, assuming there exist sequences  $(\delta_n)_{n \geq 0}$  and  $(u_{\delta_n})_{n \geq 0}$  such that

$$\lim_{n \rightarrow +\infty} \delta_n = 0, \quad \|u_{\delta_n}\|_{H^1(\Omega)} = 1, \quad \forall n \in \mathbb{N} \quad \text{and} \quad \lim_{n \rightarrow +\infty} \sup_{\|\varphi\|_{H^1(\Omega)}=1} |a_{\delta_n}(u_{\delta_n}, \varphi)| = 0. \quad (7.4)$$

In the following, we assume that there exist two positive constants  $\varepsilon$  et  $\delta_1$  such that

$$\delta_n < \varepsilon < \delta_1 \leq \delta_0. \quad (7.5)$$

Note that this can be done at least by extracting a subsequence of  $(\delta_n)_{n \geq 0}$ . We show in three steps that there exists a subsequence of  $(u_{\delta_n})$  such that  $\lim_{n \rightarrow +\infty} \|u_{\delta_n}\|_{H^1(\Omega)} = 0$ , which will lead to a contradiction.

**Step 1:** *There exists a subsequence  $(u_{\delta_n})_{n \geq 0}$  such that  $\|u_{d_\beta, \delta_n}\|_{H^1(\Omega_{\delta_n, \beta})} \leq c\delta_n^{1/2}$ .*

From (7.4) and Rellich's theorem, we can extract a subsequence of  $(u_{\delta_n})_{n \geq 0}$ , still denoted by  $(u_{\delta_n})_{n \geq 0}$ , such that

$$\begin{cases} u_{\delta_n} \rightarrow u_0 \text{ in } L^2(\Omega), \\ u_{\delta_n} \rightharpoonup u_0 \text{ in } H^1(\Omega). \end{cases} \quad (7.6)$$

Furthermore, for all  $v$  in  $C^\infty(\overline{\Omega})$ ,  $a_\delta(\cdot, \cdot)$  can be written as

$$\begin{aligned} a_\delta(u_\delta, v) &= \int_{\Omega} \sigma_0 \nabla u_\delta \cdot \nabla \overline{v} - k_0^2 u_\delta \overline{v} \, d\Omega + \sigma_{\text{ext}} \langle Tu_{\delta|\partial\Omega}, \overline{v}|_{\partial\Omega} \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} \\ &\quad - \int_{\Omega_{\delta,1}} \sigma_{\text{int}} \nabla u_{d_1, \delta} \cdot \nabla \overline{v_{d_1}} \, d\Omega_{\delta,1} + \int_{\Omega_{\delta,1}} k_{\text{int}}^2 u_{d_1, \delta} \overline{v_{d_1}} \, d\Omega_{\delta,1} - \int_{\Omega_{\delta,2}} \sigma_{\text{ext}} \nabla u_{d_2, \delta} \cdot \nabla \overline{v_{d_2}} \, d\Omega_{\delta,2} \\ &\quad + \int_{\Omega_{\delta,2}} k_{\text{ext}}^2 u_{d_2, \delta} \overline{v_{d_2}} \, d\Omega_{\delta,2} + \int_{\Omega_\delta} \frac{\tilde{\sigma}}{\delta} \nabla u_{d, \delta} \cdot \nabla \overline{v_d} \, d\Omega_\delta - \int_{\Omega_\delta} \frac{\tilde{k}^2}{\delta} u_{d, \delta} \overline{v_d} \, d\Omega_\delta, \end{aligned} \quad (7.7)$$

with

$$\begin{cases} \sigma_0 := \sigma_{\text{int}} \chi_{\Omega_{\text{int}}}(x) + \sigma_{\text{ext}} \chi_{\Omega_{\text{ext}}}(x), \\ k_0^2 := k_{\text{int}}^2 \chi_{\Omega_{\text{int}}}(x) + k_{\text{ext}}^2 \chi_{\Omega_{\text{ext}}}(x), \end{cases} \quad (7.8)$$

and  $u_{d_\beta, \delta}$  and  $v_{d_\beta}$  are respectively the restriction of  $u_\delta$  and  $v$  to the domain  $\Omega_{\delta, \beta}$ . Applying the Cauchy-Schwarz and triangular inequalities, we get

$$\begin{aligned} \left| \int_{\Omega_{\delta, 1}} \sigma_{\text{int}} \nabla u_{d_1, \delta} \cdot \nabla \bar{v}_{d_1} - k_{\text{int}}^2 u_{d_1, \delta} \bar{v}_{d_1} \, d\Omega_{\delta, 1} \right| &\leq c \|u_{d_1, \delta}\|_{H^1(\Omega_{\delta, 1})} \|v_{d_1}\|_{W^{1, \infty}(\Omega_{\delta, 1})} \sqrt{|\Omega_{\delta, 1}|} \\ &\leq c \|u_\delta\|_{H^1(\Omega)} \|v\|_{W^{1, \infty}(\Omega)} \sqrt{|\Omega_{\delta, 1}|}. \end{aligned}$$

Using  $\|u_{\delta_n}\|_{H^1(\Omega)} = 1$  and that  $|\Omega_{\delta_n, 1}| = c\delta_n$ , we infer

$$\lim_{n \rightarrow +\infty} \left| \int_{\Omega_{\delta_n, 1}} \sigma_{\text{int}} \nabla u_{d_1, \delta} \cdot \nabla \bar{v}_{d_1} - k_{\text{int}}^2 u_{d_1, \delta} \bar{v}_{d_1} \, d\Omega_{\delta_n, 1} \right| = 0, \quad \forall v \in \mathcal{C}^\infty(\bar{\Omega}). \quad (7.9)$$

Similarly, we show that

$$\lim_{n \rightarrow +\infty} \left| \int_{\Omega_{\delta_n, 2}} \sigma_{\text{ext}} \nabla u_{d_2, \delta} \cdot \nabla \bar{v}_{d_2} - k_{\text{ext}}^2 u_{d_2, \delta} \bar{v}_{d_2} \, d\Omega_{\delta_n, 2} \right| = 0, \quad \forall v \in \mathcal{C}^\infty(\bar{\Omega}). \quad (7.10)$$

As a consequence, we infer from (7.4)-(7.10)

$$\begin{aligned} &\lim_{n \rightarrow +\infty} \int_{\Omega_{\delta_n}} \frac{\tilde{\sigma}}{\delta_n} \nabla u_{d, \delta_n} \cdot \nabla \bar{v}_d - \frac{\tilde{k}^2}{\delta_n} u_{d, \delta_n} \bar{v}_d \, d\Omega_{\delta_n} \\ &= \lim_{n \rightarrow +\infty} \left[ a_{\delta_n}(u_{\delta_n}, v) - \int_{\Omega} \sigma_0 \nabla u_{\delta_n} \cdot \nabla \bar{v} - k_0^2 u_{\delta_n} \bar{v} \, dx \right. \\ &\quad - \sigma_{\text{ext}} \langle T u_{\delta_n}|_{\partial\Omega}, \bar{v}|_{\partial\Omega} \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} + \int_{\Omega_{\delta, 1}} \sigma_{\text{int}} \nabla u_{d_1, \delta} \cdot \nabla \bar{v}_{d_1} - k_{\text{int}}^2 u_{d_1, \delta} \bar{v}_{d_1} \, d\Omega_{\delta, 1} \\ &\quad \left. + \int_{\Omega_{\delta_n, 2}} \sigma_{\text{ext}} \nabla u_{d_2, \delta} \cdot \nabla \bar{v}_{d_2} - k_{\text{ext}}^2 u_{d_2, \delta} \bar{v}_{d_2} \, d\Omega_{\delta_n, 2} \right] \\ &= - \int_{\Omega} \sigma_0 \nabla u_0 \cdot \nabla \bar{v} - k_0^2 u_0 \bar{v} \, dx - \sigma_{\text{ext}} \langle T u_0|_{\partial\Omega}, \bar{v}|_{\partial\Omega} \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)}, \quad \forall v \in \mathcal{C}^\infty(\bar{\Omega}). \end{aligned}$$

As the right-hand side is independent of  $\delta$ , one has

$$\int_{\Omega_{\delta_n}} \frac{\tilde{\sigma}}{\delta_n} \nabla u_{d, \delta_n} \cdot \nabla \bar{v}_d - \frac{\tilde{k}^2}{\delta_n} u_{d, \delta_n} \bar{v}_d \, d\Omega_{\delta_n} = O(1), \quad \forall v \in \mathcal{C}^\infty(\bar{\Omega}), \quad (7.11)$$

and by density, we conclude that the equality is true for all  $v$  in  $H^1(\Omega)$ .

Taking now  $v \in \mathcal{C}^\infty(\bar{\Omega})$  one has for all  $x = \psi_\beta(m, \eta_\beta)$  in  $\Omega_{\delta, \beta}$ ,

$$v_{d_\beta}(x) = \tilde{v}_{d_\beta}(m, \eta_\beta) = \tilde{v}_{d_\beta}(m, 0) + \int_0^{\eta_\beta} \partial_{\eta_\beta} \tilde{v}_{d_\beta}(m, \lambda) \, d\lambda, \quad (7.12)$$

so

$$|\tilde{v}_{d_\beta}(m, \eta_\beta)|^2 \leq 2 |\tilde{v}_{d_\beta}(m, 0)|^2 + 2 \left| \int_0^{\eta_\beta} \partial_{\eta_\beta} \tilde{v}_{d_\beta}(m, \lambda) d\lambda \right|^2. \quad (7.13)$$

By integrating on  $\Gamma$ , we find

$$\int_\Gamma |\tilde{v}_{d_\beta}(m, \eta_\beta)|^2 d\Gamma \leq 2 \int_\Gamma |\tilde{v}_{d_\beta}(m, 0)|^2 d\Gamma + 2 \int_\Gamma \int_0^{p_\beta \delta} |\partial_{\eta_\beta} \tilde{v}_{d_\beta}(m, \lambda)|^2 d\lambda d\Gamma, \quad (7.14)$$

so

$$\int_\Gamma |\tilde{v}_{d_\beta}(m, \eta_\beta)|^2 d\Gamma \leq 2 \|v_{d_\beta}|_\Gamma\|_{L^2(\Gamma)}^2 + 2 \|\partial_{\eta_\beta} \tilde{v}_{d_\beta}\|_{L^2(\Omega_{\delta, \beta})}^2. \quad (7.15)$$

Integrating a second time with respect with  $\eta_\beta$ , we obtain

$$\int_0^{p_\beta \delta} \int_\Gamma |\tilde{v}_{d_\beta}(m, \eta_\beta)|^2 d\Gamma d\eta_\beta \leq c\delta \left( \|v_{d_\beta}|_\Gamma\|_{L^2(\Gamma)}^2 + \|\partial_{\eta_\beta} \tilde{v}_{d_\beta}\|_{L^2(\Omega_{\delta, \beta})}^2 \right), \quad (7.16)$$

so

$$\|v_{d_\beta}\|_{L^2(\Omega_{\delta, \beta})} \leq c\delta^{1/2} \|v\|_{H^1(\Omega)}. \quad (7.17)$$

Since  $v \in \mathcal{C}^\infty(\bar{\Omega})$  is arbitrary, we conclude, by density, that the last estimate is true for all  $v$  in  $H^1(\Omega)$ . Hence, for  $v = u_{\delta_n}$ , we have

$$\|u_{d_\beta, \delta_n}\|_{L^2(\Omega_{\delta_n, \beta})} \leq c\delta_n^{1/2}. \quad (7.18)$$

Using both (7.11) and (7.18), one gets

$$\|u_{d_\beta, \delta_n}\|_{H^1(\Omega_{\delta_n, \beta})} \leq c\delta_n^{1/2}, \quad (7.19)$$

which proves the first claim.

**Step 2:** We show that  $u_0 = 0$  in  $\Omega$ .

In view of (7.19) and (3.4), one has

$$\begin{aligned} \|u_\delta\|_{H^1(\Omega_{\delta, \beta})}^2 &= p_\beta \delta \int_{\Omega^\beta} J_{\delta, \beta}^{-2} \left| \nabla_\Gamma u_\delta^{[\beta]} \right|^2 \det J_{\delta, \beta} d\Gamma ds_\beta \\ &\quad + p_\beta^{-1} \delta^{-1} \int_{\Omega^\beta} \left| \partial_{s_\beta} u_\delta^{[\beta]} \right|^2 \det J_{\delta, \beta} d\Gamma ds_\beta + \delta p_\beta \int_{\Omega^\beta} \left| u_\delta^{[\beta]} \right|^2 \det J_{\delta, \beta} d\Gamma ds_\beta, \end{aligned}$$

we then infer the estimates

$$\left\| \nabla_{\Gamma} u_{\delta_n}^{[\beta]} \right\|_{L^2(\Omega^{\beta})} \leq c, \quad (7.20)$$

$$\left\| \partial_{s_{\beta}} u_{\delta_n}^{[\beta]} \right\|_{L^2(\Omega^{\beta})} \leq c\delta_n, \quad (7.21)$$

$$\left\| u_{\delta_n}^{[\beta]} \right\|_{L^2(\Omega^{\beta})} \leq c. \quad (7.22)$$

To compute the limiting equation, we introduce  $X$  as the Hilbert space defined by

$$X := \left\{ V := \left( v, v^{[1]}, v^{[2]} \right) \in H^1(\Omega) \times H^1(I_1, H^1(\Gamma)) \times H^1(I_2, H^1(\Gamma)) ; \right. \\ \left. v^{[\beta]}(m, 0) = v|_{\Gamma}, \beta = 1, 2 \right\},$$

It follows from (7.4) and (7.20)-(7.22) that the sequence  $(U_{\delta_n})_n$  defined by  $U_{\delta_n} := (u_{\delta_n}, u_{\delta_n}^{[1]}, u_{\delta_n}^{[2]})$  is bounded in  $X$ . Therefore, there exists a subsequence of  $(U_{\delta_n})_{n \geq 0}$ , still denoted by  $(U_{\delta_n})_{n \geq 0}$ , such that  $U_{\delta_n} \rightharpoonup U_0 := (u_0, \omega_0^{[1]}, \omega_0^{[2]})$  in  $X$ . Inequality (7.21) implies  $\partial_{s_{\beta}} \omega_0^{[\beta]} = 0$  resulting in

$$\omega_0^{[\beta]}(m, s_{\beta}) = u_0|_{\Gamma}, \quad \forall (m, s_{\beta}) \in \Omega^{\beta}. \quad (7.23)$$

Let now  $v$  be a smooth function  $v$  in  $H^1(\Omega)$ ,  $a_{\delta}(\cdot, \cdot)$  becomes

$$a_{\delta}(u_{\delta}, v) = \int_{\Omega} \sigma_0 \nabla u_{\delta} \cdot \nabla \bar{v} - k_0^2 u_{\delta} \bar{v} \, d\Omega + \sigma_{\text{ext}} \langle T u_{\delta}|_{\partial\Omega}, \bar{v}|_{\partial\Omega} \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} \\ - \int_{\Omega_{\delta,1}} \sigma_{\text{int}} \nabla u_{d_1, \delta} \cdot \nabla \bar{v}_{d_1} \, d\Omega_{\delta,1} + \int_{\Omega_{\delta,1}} k_{\text{int}}^2 u_{d_1, \delta} \bar{v}_{d_1} \, d\Omega_{\delta,1} - \int_{\Omega_{\delta,2}} \sigma_{\text{ext}} \nabla u_{d_2, \delta} \cdot \nabla \bar{v}_{d_2} \, d\Omega_{\delta,2} \\ + \int_{\Omega_{\delta,2}} k_{\text{ext}}^2 u_{d_2, \delta} \bar{v}_{d_2} \, d\Omega_{\delta,2} + \sum_{\beta=1}^2 \left[ p_{\beta} \tilde{\sigma} \int_{\Omega^{\beta}} J_{\delta, \beta}^{-2} \nabla_{\Gamma} u_{\delta}^{[\beta]} \cdot \nabla_{\Gamma} \bar{v}^{[\beta]} \det J_{\delta, \beta} \, d\Gamma ds_{\beta} \right. \\ \left. + p_{\beta}^{-1} \delta^{-2} \tilde{\sigma} \int_{\Omega^{\beta}} \partial_{s_{\beta}} u_{\delta}^{[\beta]} \partial_{s_{\beta}} \bar{v}^{[\beta]} \det J_{\delta, \beta} \, d\Gamma ds_{\beta} - p_{\beta} \tilde{k}^2 \int_{\Omega^{\beta}} u_{\delta}^{[\beta]} \bar{v}^{[\beta]} \det J_{\delta, \beta} \, d\Gamma ds_{\beta} \right] \quad (7.24)$$

Now choosing a smooth  $v$  in  $H(\Omega)$  (see Lemma 9.1) leads to

$$p_{\beta} \tilde{\sigma} \int_{\Omega^{\beta}} J_{\delta_n, \beta}^{-2} \nabla_{\Gamma} u_{\delta_n}^{[\beta]} \cdot \nabla_{\Gamma} \bar{v}^{[\beta]} \det J_{\delta_n, \beta} \, d\Gamma ds_{\beta} + p_{\beta}^{-1} \delta_n^{-2} \tilde{\sigma} \int_{\Omega^{\beta}} \partial_{s_{\beta}} u_{\delta_n}^{[\beta]} \partial_{s_{\beta}} \bar{v}^{[\beta]} \det J_{\delta_n, \beta} \, d\Gamma ds_{\beta}$$

$$\begin{aligned}
& - p_\beta \tilde{k}^2 \int_{\Omega^\beta} u_{\delta_n}^{[\beta]} \overline{v^{[\beta]}} \det J_{\delta_n, \beta} d\Gamma ds_\beta \\
& \xrightarrow{n \rightarrow +\infty} p_\beta \tilde{\sigma} \int_{\Omega^\beta} \nabla_\Gamma \omega_0^{[\beta]} \cdot \nabla_\Gamma \overline{v^{[\beta]}} d\Gamma ds_\beta - p_\beta \tilde{k}^2 \int_\Gamma \omega_0^{[\beta]} \overline{v^{[\beta]}} d\Gamma ds_\beta, \tag{7.25}
\end{aligned}$$

otherwise, as  $\omega_0^{[\beta]}$  is independent of  $s_\beta$ , we would have

$$p_\beta \tilde{\sigma} \int_{\Omega^\beta} \nabla_\Gamma \omega_0^{[\beta]} \cdot \nabla_\Gamma \overline{v^{[\beta]}} d\Gamma ds_\beta = p_\beta \tilde{\sigma} \int_\Gamma \nabla_\Gamma u_{0|\Gamma} \cdot \nabla_\Gamma \overline{v|_\Gamma} d\Gamma \tag{7.26}$$

$$-p_\beta \tilde{k}^2 \int_{\Omega^\beta} \omega_0^{[\beta]} \overline{v^{[\beta]}} d\Gamma ds_\beta = -p_\beta \tilde{k}^2 \int_\Gamma u_{0|\Gamma} \overline{v|_\Gamma} d\Gamma. \tag{7.27}$$

Since  $\omega_0^{[\beta]}$  is independent of  $s_\beta$  and  $U_0 \in X$ ,  $\omega_0^{[\beta]}(m, s_\beta) = \omega_0^{[\beta]}(m, 0) = u_{0|\Gamma} \in H^1(\Gamma)$ ,

which gives meaning to the last two equalities. As a consequence, in view of (7.4) and

(7.25)-(7.27), we obtain

$$\begin{aligned}
0 &= \lim_{n \rightarrow +\infty} a_{\delta_n}(u_{\delta_n}, v) \\
&= \int_\Omega \sigma_0 \nabla u_0 \cdot \nabla \bar{v} - k_0^2 u_0 \bar{v} d\Omega + \sigma_{\text{ext}} \langle T u_{0|\partial\Omega}, \bar{v}|_{\partial\Omega} \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} \\
&+ \tilde{\sigma} \int_\Gamma \nabla_\Gamma u_{0|\Gamma} \cdot \nabla_\Gamma \overline{v|_\Gamma} d\Gamma - \tilde{k}^2 \int_\Gamma u_{0|\Gamma} \overline{v|_\Gamma} d\Gamma. \tag{7.28}
\end{aligned}$$

By density (Lemma 9.1), we deduce that (7.28) is true for all  $v$  in  $H_\Gamma^1(\Omega)$ . It follows

from Theorem 9.1 that the problem: Find  $u_0$  in  $H_\Gamma^1(\Omega)$  satisfying (7.28),  $\forall v \in H_\Gamma^1(\Omega)$ ,

is well-posed, moreover  $u_0 = 0$ .

**Step 3:** *Getting the contradiction.*

To obtain the contradiction, we show that  $\lim_{n \rightarrow +\infty} \|u_{\delta_n}\|_{H^1(\Omega)} = 0$ . Since  $u_0$  is uniquely

determined, the whole sequence  $(U_\delta)_\delta$  converges to  $U_0 = 0$ , then the whole sequence  $(u_\delta)_\delta$

converges to  $u_0 = 0$  in  $L^2(\Omega)$ . It only remains to show that  $\lim_{n \rightarrow +\infty} \|\nabla u_{\delta_n}\|_{L^2(\Omega)} = 0$ . We

have

$$\|\nabla u_{\delta_n}\|_{L^2(\Omega)}^2 \leq c \int_\Omega \sigma_{\delta_n} |\nabla u_{\delta_n}|^2 d\Omega$$

$$\begin{aligned}
&= c\Re\{ a_{\delta_n}(u_{\delta_n}, u_{\delta_n}) + \int_{\Omega} k_{\delta_n}^2 |u_{\delta_n}|^2 d\Omega \\
&\quad - \sigma_{\text{ext}} \left\langle Tu_{\delta_n}|_{\partial\Omega}, \overline{u_{\delta_n}}|_{\partial\Omega} \right\rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} \}.
\end{aligned}$$

Using Lemma 2.1, we infer

$$\begin{aligned}
\|\nabla u_{\delta_n}\|_{\mathbb{L}^2(\Omega)}^2 &\leq c\Re\{ a_{\delta_n}(u_{\delta_n}, u_{\delta_n}) + \int_{\Omega} k_{\delta_n}^2 |u_{\delta_n}|^2 d\Omega \\
&\quad - \sigma_{\text{ext}} \left\langle Ku_{\delta_n}|_{\partial\Omega}, \overline{u_{\delta_n}}|_{\partial\Omega} \right\rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} \}.
\end{aligned}$$

As

$$\begin{aligned}
\int_{\Omega} k_{\delta_n}^2 |u_{\delta_n}|^2 d\Omega &= \int_{\Omega_{\text{int},\delta_n}} k_{\text{int}}^2 |u_{\text{int},\delta_n}|^2 d\Omega_{\text{int},\delta_n} + \int_{\Omega_{\text{ext},\delta_n}} k_{\text{ext}}^2 |u_{\text{ext},\delta_n}|^2 d\Omega_{\text{ext},\delta_n} \\
&\quad + \delta_n^{-1} \sum_{\beta=1}^2 \int_{\Omega_{\delta_n,\beta}} \tilde{k}^2 |u_{d_{\beta},\delta_n}|^2 d\Omega_{\delta_n,\beta} \\
&= \int_{\Omega_{\text{int},\delta_n}} k_{\text{int}}^2 |u_{\text{int},\delta_n}|^2 d\Omega_{\text{int},\delta_n} + \int_{\Omega_{\text{ext},\delta_n}} k_{\text{ext}}^2 |u_{\text{ext},\delta_n}|^2 d\Omega_{\text{ext},\delta_n} \\
&\quad + \sum_{\beta=1}^2 \int_{\Omega^{\beta}} \tilde{k}^2 |u_{\delta_n}^{[\beta]}|^2 \det J_{\delta_n,\beta} d\Gamma ds_{\beta} \\
&= \int_{\Omega} k_0^2 |u_{\delta_n}|^2 d\Omega + \sum_{\beta=1}^2 \int_{\Omega^{\beta}} \tilde{k}^2 |u_{\delta_n}^{[\beta]}|^2 \det J_{\delta_n,\beta} d\Gamma ds_{\beta} \\
&\quad - \delta_n \int_{\Omega^1} k_{\text{int}}^2 |u_{\delta_n}^{[1]}|^2 \det J_{\delta,1} d\Gamma ds_1 - \int_{\Omega^2} k_{\text{ext}}^2 |u_{\delta_n}^{[2]}|^2 \det J_{\delta,2} d\Gamma ds_2,
\end{aligned}$$

$u_{\delta_n} \xrightarrow{n \rightarrow +\infty} u_0 = 0$  in  $L^2(\Omega)$  and  $u_{\delta_n}^{[\beta]} \xrightarrow{n \rightarrow +\infty} u_{\text{int},0}^{[\beta]} = u_0|_{\Gamma} = 0$  in  $L^2(I_{\beta}, H^1(\Gamma))$ , it follows

$\int_{\Omega} k_{\delta_n}^2 |u_{\delta_n}|^2 d\Omega \xrightarrow{n \rightarrow +\infty} 0$ . Since  $K$  is compact and  $u_{\delta_n} \rightharpoonup 0$  in  $H^1(\Omega)$ ,

$\left\langle Ku_{\delta_n}|_{\partial\Omega}, \overline{u_{\delta_n}}|_{\partial\Omega} \right\rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} \xrightarrow{n \rightarrow +\infty} 0$ . Finally, the hypothesis  $\lim_{n \rightarrow +\infty} \Re[a_{\delta_n}(u_{\delta_n}, u_{\delta_n})] =$

0 leads to  $\lim_{n \rightarrow +\infty} \|\nabla u_{\delta_n}\|_{\mathbb{L}^2(\Omega)} = 0$  contradicting  $\|u_{\delta_n}\|_{H^1(\Omega)} = 1$ .  $\square$



## 7.2 Approximate transmission conditions of order 2

Using the same techniques as in the previous section, we derive an asymptotic expansion of the total field  $u_\delta$  and establish a convergence theorem. We now give the first two terms of the asymptotic expansion, denoted by,  $(u_{\text{int},n}, u_{\text{ext},n})_{0 \leq n \leq 1}$  satisfy the following problem

$$\begin{cases} \operatorname{div}(\sigma_{\text{ext}} \nabla u_{\text{ext},n}) + k_{\text{ext}}^2 u_{\text{ext},n} = 0 & \text{in } \Omega_{\text{ext}}, \\ \operatorname{div}(\sigma_{\text{int}} \nabla u_{\text{int},n}) + k_{\text{int}}^2 u_{\text{int},n} = 0 & \text{in } \Omega_{\text{int}}, \\ \lim_{|x| \rightarrow +\infty} |x| (\partial_{|x|} - ik_{\text{ext}}) (u_{\text{ext},n} - \delta_{0,n} u_{\text{inc}}) = 0, \end{cases}$$

where  $\delta_{0,n}$  indicates the Kronecker symbol. The transmission conditions on  $\Gamma$  are described below:

*At order 0.*

$$u_{\text{int},0|\Gamma} = u_{\text{ext},0|\Gamma}, \quad (7.29)$$

$$\sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},0|\Gamma} - \sigma_{\text{ext}} \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma} = \tilde{\sigma} \Delta_{\Gamma} u_{\text{int},0|\Gamma} + \tilde{k}^2 u_{\text{int},0|\Gamma}. \quad (7.30)$$

*At order 1.*

$$u_{\text{int},1|\Gamma} - u_{\text{ext},1|\Gamma} = p_1 \partial_{\mathbf{n}} u_{\text{int},0|\Gamma} + p_2 \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma}, \quad (7.31)$$

$$\begin{aligned} \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},1|\Gamma} - \sigma_{\text{ext}} \partial_{\mathbf{n}} u_{\text{ext},1|\Gamma} &= p_1 \tilde{\sigma} \Delta_{\Gamma} u_{\text{int},1|\Gamma} + p_2 \tilde{\sigma} \Delta_{\Gamma} u_{\text{ext},1|\Gamma} + p_1 \tilde{k}^2 u_{\text{int},1|\Gamma} \\ &+ p_2 \tilde{k}^2 u_{\text{ext},1|\Gamma} - p_2 \sigma_{\text{ext}} \Delta_{\Gamma} u_{\text{ext},0|\Gamma} - p_1 \sigma_{\text{int}} \Delta_{\Gamma} u_{\text{int},0|\Gamma} \\ &- p_1^2 \tilde{k}^2 \mathcal{H} u_{\text{int},0|\Gamma} - p_1 k_{\text{int}}^2 u_{\text{int},0|\Gamma} + p_2^2 \tilde{k}^2 \mathcal{H} u_{\text{ext},0|\Gamma} \\ &- p_2 k_{\text{ext}}^2 u_{\text{ext},0|\Gamma} + p_2^2 \tilde{\sigma} \operatorname{div}_{\Gamma} ((\mathcal{H}I - \mathcal{R}) \nabla_{\Gamma} u_{\text{ext},0|\Gamma}) \\ &- p_1^2 \tilde{\sigma} \operatorname{div}_{\Gamma} ((\mathcal{H}I - \mathcal{R}) \nabla_{\Gamma}) u_{\text{int},0|\Gamma} + p_2^2 \tilde{\sigma} \Delta_{\Gamma} (\partial_{\mathbf{n}} u_{\text{ext},0|\Gamma}) \end{aligned}$$

$$+ p_2^2 \tilde{k}^2 \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma} - p_1^2 \tilde{\sigma} \Delta_{\Gamma} (\partial_{\mathbf{n}} u_{\text{int},0|\Gamma}) - p_1^2 \tilde{k}^2 \partial_{\mathbf{n}} u_{\text{int},0|\Gamma}. \quad (7.32)$$

Moreover  $(u_n^{[\beta]})_{0 \leq n \leq 1}$  is determined by the following expressions

$$\begin{aligned} u_0^{[1]}(m, s_1) &= u_0^{[2]}(m, s_2) = u_{\text{int},0|\Gamma} = u_{\text{ext},0|\Gamma}, \\ u_1^{[1]}(m, s_1) &= u_1^{[2]}(m, s_2) = u_{\text{ext},1|\Gamma} + p_2 \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma} \\ &= u_{\text{int},1|\Gamma} - p_1 \partial_{\mathbf{n}} u_{\text{int},0|\Gamma}, \quad \forall (m, s_{\beta}) \in \Omega^{\beta}. \end{aligned}$$

We follow the approach used in Section 6 to derive an approximate problem of order 1. The proof of the uniqueness of the solution is then going to encounter two difficulties. The first comes from terms  $\Delta_{\Gamma} (\partial_{\mathbf{n}} u_{\text{ext},0|\Gamma})$  and  $\Delta_{\Gamma} (\partial_{\mathbf{n}} u_{\text{int},0|\Gamma})$  in Condition (7.32). To bypass this difficulty, we determine constants  $p_1$  and  $p_2$  making these terms vanish. From (7.29) and (7.30), Condition (7.32) becomes

$$\begin{aligned} \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},1|\Gamma} - \sigma_{\text{ext}} \partial_{\mathbf{n}} u_{\text{ext},1|\Gamma} &= p_1 \tilde{\sigma} \Delta_{\Gamma} u_{\text{int},1|\Gamma} + p_2 \tilde{\sigma} \Delta_{\Gamma} u_{\text{ext},1|\Gamma} + p_1 \tilde{k}^2 u_{\text{int},1|\Gamma} \\ &+ p_2 \tilde{k}^2 u_{\text{ext},1|\Gamma} - p_2 \sigma_{\text{ext}} \Delta_{\Gamma} u_{\text{ext},0|\Gamma} - p_1 \sigma_{\text{int}} \Delta_{\Gamma} u_{\text{int},0|\Gamma} \\ &+ p_2^2 \tilde{k}^2 \mathcal{H} u_{\text{ext},0|\Gamma} - p_2 k_{\text{ext}}^2 u_{\text{ext},0|\Gamma} - p_1^2 \tilde{k}^2 \mathcal{H} u_{\text{int},0|\Gamma} - p_1 k_{\text{int}}^2 u_{\text{int},0|\Gamma} \\ &+ p_2^2 \tilde{\sigma} \text{div}_{\Gamma} ((\mathcal{H}I - \mathcal{R}) \nabla_{\Gamma} u_{\text{ext},0|\Gamma}) - p_1^2 \tilde{\sigma} \text{div}_{\Gamma} ((\mathcal{H}I - \mathcal{R}) \nabla_{\Gamma}) u_{\text{int},0|\Gamma} \\ &+ (\sigma_{\text{int}} p_2^2 - p_1^2 \sigma_{\text{ext}}) \frac{\tilde{\sigma}}{\sigma_{\text{int}}} \Delta_{\Gamma} (\partial_{\mathbf{n}} u_{\text{ext},0|\Gamma}) + (\sigma_{\text{int}} p_2^2 - p_1^2 \sigma_{\text{ext}}) \frac{\tilde{k}^2}{\sigma_{\text{int}}} \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma} \\ &- 2p_1^2 \tilde{\sigma} \frac{\tilde{k}^2}{\sigma_{\text{int}}} \Delta_{\Gamma} u_{\text{int},0|\Gamma} - p_1^2 \frac{\tilde{\sigma}}{\sigma_{\text{int}}} (\Delta_{\Gamma}^2 u_{\text{int},0|\Gamma}) - p_1^2 \frac{\tilde{k}^4}{\sigma_{\text{int}}} u_{\text{int},0|\Gamma}. \end{aligned} \quad (7.33)$$

Then, by setting  $\sigma_{\text{int}} p_2^2 - p_1^2 \sigma_{\text{ext}} = 0$ , one obtains

$$p_1 = \frac{\sqrt{\sigma_{\text{int}}} (\sqrt{\sigma_{\text{int}}} - \sqrt{\sigma_{\text{ext}}})}{\sigma_{\text{int}} - \sigma_{\text{ext}}} \quad \text{and} \quad p_2 = \frac{\sqrt{\sigma_{\text{ext}}} (\sqrt{\sigma_{\text{int}}} - \sqrt{\sigma_{\text{ext}}})}{\sigma_{\text{int}} - \sigma_{\text{ext}}},$$

As a consequence, (7.33) becomes

$$\sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},1|\Gamma} - \sigma_{\text{ext}} \partial_{\mathbf{n}} u_{\text{ext},1|\Gamma} = p_1 \tilde{\sigma} \Delta_{\Gamma} u_{\text{int},1|\Gamma} + p_2 \tilde{\sigma} \Delta_{\Gamma} u_{\text{ext},1|\Gamma}$$

$$\begin{aligned}
& + p_1 \tilde{k}^2 u_{\text{int},1|\Gamma} + p_2 \tilde{k}^2 u_{\text{ext},1|\Gamma} - p_2 \sigma_{\text{ext}} \Delta_\Gamma u_{\text{ext},0|\Gamma} - p_1 \sigma_{\text{int}} \Delta_\Gamma u_{\text{int},0|\Gamma} \\
& + p_2^2 \tilde{k}^2 \mathcal{H} u_{\text{ext},0|\Gamma} - p_2 k_{\text{ext}}^2 u_{\text{ext},0|\Gamma} - p_1^2 \tilde{k}^2 \mathcal{H} u_{\text{int},0|\Gamma} - p_1 k_{\text{int}}^2 u_{\text{int},0|\Gamma} \\
& + p_2^2 \tilde{\sigma} \operatorname{div}_\Gamma [(\mathcal{H}I - \mathcal{R}) \nabla_\Gamma u_{\text{ext},0|\Gamma}] - p_1^2 \tilde{\sigma} \operatorname{div}_\Gamma [(\mathcal{H}I - \mathcal{R}) \nabla_\Gamma] u_{\text{int},0|\Gamma} \\
& - 2p_1^2 \tilde{\sigma} \frac{\tilde{k}^2}{\sigma_{\text{int}}} \Delta_\Gamma u_{\text{int},0|\Gamma} - p_1^2 \frac{\tilde{\sigma}}{\sigma_{\text{int}}} (\Delta_\Gamma^2 u_{\text{int},0|\Gamma}) - p_1^2 \frac{\tilde{k}^4}{\sigma_{\text{int}}} u_{\text{int},0|\Gamma}. \quad (7.34)
\end{aligned}$$

As a result, we assume in what follows that the following constraints hold

$$p_1 = \frac{\sqrt{\sigma_{\text{int}}} (\sqrt{\sigma_{\text{int}}} - \sqrt{\sigma_{\text{ext}}})}{\sigma_{\text{int}} - \sigma_{\text{ext}}} \quad \text{and} \quad p_2 = \frac{\sqrt{\sigma_{\text{ext}}} (\sqrt{\sigma_{\text{int}}} - \sqrt{\sigma_{\text{ext}}})}{\sigma_{\text{int}} - \sigma_{\text{ext}}}.$$

The second difficulty comes from complexity of Condition (7.34) which can be overcome by rewriting the whole transmission condition. Thus, from (7.29) and (7.34), we deduce a form, albeit longer but better adapted to treat, still, the uniqueness of the solution of the approximate problem.

$$\begin{aligned}
& \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},1|\Gamma} - \sigma_{\text{ext}} \partial_{\mathbf{n}} u_{\text{ext},1|\Gamma} = p_1 \tilde{k}^2 u_{\text{int},1|\Gamma} + p_2 \tilde{k}^2 u_{\text{ext},1|\Gamma} \\
& + p_1 \tilde{\sigma} \Delta_\Gamma u_{\text{int},1|\Gamma} + p_2 \tilde{\sigma} \Delta_\Gamma u_{\text{ext},1|\Gamma} \\
& + \frac{\sigma_{\text{ext}} p_1}{\sigma_{\text{int}} p_2 + \sigma_{\text{ext}} p_1} \left( p_2^2 \tilde{k}^2 \mathcal{H} - p_1^2 \tilde{k}^2 \mathcal{H} - p_2 k_{\text{ext}}^2 - p_1 k_{\text{int}}^2 - p_1^2 \frac{\tilde{k}^4}{\sigma_{\text{int}}} \right) u_{\text{ext},0|\Gamma} \\
& + \frac{\sigma_{\text{int}} p_2}{\sigma_{\text{int}} p_2 + \sigma_{\text{ext}} p_1} \left( p_2^2 \tilde{k}^2 \mathcal{H} - p_1^2 \tilde{k}^2 \mathcal{H} - p_2 k_{\text{ext}}^2 - p_1 k_{\text{int}}^2 - p_1^2 \frac{\tilde{k}^4}{\sigma_{\text{int}}} \right) u_{\text{int},0|\Gamma} \\
& + \frac{\sigma_{\text{ext}} p_1}{\sigma_{\text{int}} p_2 + \sigma_{\text{ext}} p_1} \left( -p_2 \sigma_{\text{ext}} - p_1 \sigma_{\text{int}} - 2p_1^2 \tilde{\sigma} \frac{\tilde{k}^2}{\sigma_{\text{int}}} \right) \Delta_\Gamma u_{\text{ext},0|\Gamma} \\
& + \frac{\sigma_{\text{int}} p_2}{\sigma_{\text{int}} p_2 + \sigma_{\text{ext}} p_1} \left( -p_2 \sigma_{\text{ext}} - p_1 \sigma_{\text{int}} - 2p_1^2 \tilde{\sigma} \frac{\tilde{k}^2}{\sigma_{\text{int}}} \right) \Delta_\Gamma u_{\text{int},0|\Gamma} \\
& + \frac{\sigma_{\text{ext}} p_1}{\sigma_{\text{int}} p_2 + \sigma_{\text{ext}} p_1} (p_2 - p_1) \tilde{\sigma} \operatorname{div}_\Gamma [(\mathcal{H}I - \mathcal{R}) \nabla_\Gamma] u_{\text{ext},0|\Gamma} \\
& + \frac{\sigma_{\text{int}} p_2}{\sigma_{\text{int}} p_2 + \sigma_{\text{ext}} p_1} (p_2 - p_1) \tilde{\sigma} \operatorname{div}_\Gamma [(\mathcal{H}I - \mathcal{R}) \nabla_\Gamma] u_{\text{int},0|\Gamma} \\
& + \frac{\sigma_{\text{ext}} p_1}{\sigma_{\text{int}} p_2 + \sigma_{\text{ext}} p_1} \left( -p_1^2 \frac{\tilde{\sigma}}{\sigma_{\text{int}}} \right) (\Delta_\Gamma^2 u_{\text{ext},0|\Gamma})
\end{aligned}$$

$$+ \frac{\sigma_{\text{int}} p_2}{\sigma_{\text{int}} p_2 + \sigma_{\text{ext}} p_1} \left( -p_1^2 \frac{\tilde{\sigma}}{\sigma_{\text{int}}} \right) (\Delta_{\Gamma}^2 u_{\text{int},0|\Gamma}).$$

We are now in position to give the first-order model. Once again (see Section 6) we define the approximate solution  $u_{\delta}^{ap}$  on  $\Omega$  by

$$u_{\delta}^{ap} := \begin{cases} u_{\text{int},\delta}^{ap} & \text{in } \Omega_{\text{int},\delta}, \\ u_{d_{\beta},\delta}^{ap} & \text{in } \Omega_{\delta,\beta}, (\beta = 1 \text{ or } 2), \\ u_{\text{ext},\delta}^{ap} & \text{in } \Omega_{\text{ext},\delta}, \end{cases}$$

where  $u_{d_{\beta},\delta}^{ap}$  are defined on  $\Omega_{\delta,\beta}$  by

$$\begin{aligned} u_{d_{\beta},\delta}^{ap}(x) &:= u_{d_{\beta},\delta}^{[\beta],ap}(m, s_{\beta}) := u_{\text{int},\delta|\Gamma}^{ap} - \delta p_1 \partial_{\mathbf{n}} u_{\text{int},\delta|\Gamma}^{ap} \\ &:= u_{\text{ext},\delta|\Gamma}^{ap} + \delta p_2 \partial_{\mathbf{n}} u_{\text{ext},\delta|\Gamma}^{ap}, \quad \forall x = \Phi_{\beta}(m, s_{\beta}) \in \Omega_{\delta,\beta}, \end{aligned}$$

and  $(u_{\text{int},\delta}^{ap}, u_{\text{ext},\delta}^{ap})$  is the solution to the following problem

$$\begin{cases} \operatorname{div}(\sigma_{\text{ext}} \nabla u_{\text{ext},\delta}^{ap}) + k_{\text{ext}}^2 u_{\text{ext},\delta}^{ap} = 0 & \text{in } \Omega_{\text{ext}}, \\ \operatorname{div}(\sigma_{\text{int}} \nabla u_{\text{int},\delta}^{ap}) + k_{\text{int}}^2 u_{\text{int},\delta}^{ap} = 0 & \text{in } \Omega_{\text{int}}, \\ \lim_{|x| \rightarrow +\infty} |x| (\partial_{|x|} - ik_{\text{ext}}) (u_{\text{ext},\delta}^{ap} - u_{\text{inc}}) = 0, \end{cases} \quad (7.35)$$

with Ventcel-type transmission conditions  $\Gamma$

$$\begin{cases} u_{\text{int},\delta|\Gamma}^{ap} - u_{\text{ext},\delta|\Gamma}^{ap} = \delta p_1 \partial_{\mathbf{n}} u_{\text{int},\delta|\Gamma}^{ap} + \delta p_2 \partial_{\mathbf{n}} u_{\text{ext},\delta|\Gamma}^{ap}, \\ \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},\delta|\Gamma}^{ap} - \sigma_{\text{ext}} \partial_{\mathbf{n}} u_{\text{ext},\delta|\Gamma}^{ap} = \mathcal{K} (u_{\text{int},\delta|\Gamma}^{ap}, u_{\text{ext},\delta|\Gamma}^{ap}), \end{cases} \quad (7.36)$$

where

$$\begin{aligned} \mathcal{K} (u_{|\Gamma}, v_{|\Gamma}) &:= \alpha_{\delta,1} u_{|\Gamma} + \alpha_{\delta,2} v_{|\Gamma} + \alpha_{\delta,3} \Delta_{\Gamma} u_{|\Gamma} \\ &+ \alpha_{\delta,4} \Delta_{\Gamma} v_{|\Gamma} - \alpha_{\delta,7} \Delta_{\Gamma}^2 u_{|\Gamma} - \alpha_{\delta,8} \Delta_{\Gamma}^2 v_{|\Gamma} \\ &+ \alpha_{\delta,5} \operatorname{div}_{\Gamma} [(\mathcal{H}I - \mathcal{R}) \nabla_{\Gamma}] u_{|\Gamma} \\ &+ \alpha_{\delta,6} \operatorname{div}_{\Gamma} [(\mathcal{H}I - \mathcal{R}) \nabla_{\Gamma}] v_{|\Gamma} \end{aligned}$$

and

$$\begin{aligned}
\alpha_{\delta,1} &:= p_1 \tilde{k}^2 + \delta \frac{\sigma_{\text{int}} p_2}{\sigma_{\text{int}} p_2 + \sigma_{\text{ext}} p_1} \left( p_2 \tilde{k}^2 \mathcal{H} - p_1 \tilde{k}^2 \mathcal{H} - p_2 k_{\text{ext}}^2 - p_1 k_{\text{int}}^2 - p_1^2 \frac{\tilde{k}^4}{\sigma_{\text{int}}} \right), \\
\alpha_{\delta,2} &:= p_2 \tilde{k}^2 + \delta \frac{\sigma_{\text{ext}} p_1}{\sigma_{\text{int}} p_2 + \sigma_{\text{ext}} p_1} \left( p_2 \tilde{k}^2 \mathcal{H} - p_1 \tilde{k}^2 \mathcal{H} - p_2 k_{\text{ext}}^2 - p_1 k_{\text{int}}^2 - p_1^2 \frac{\tilde{k}^4}{\sigma_{\text{int}}} \right), \\
\alpha_{\delta,3} &:= p_1 \tilde{\sigma} - \delta \frac{\sigma_{\text{int}} p_2}{\sigma_{\text{int}} p_2 + \sigma_{\text{ext}} p_1} \left( p_2 \sigma_{\text{ext}} + p_1 \sigma_{\text{int}} + 2p_1^2 \tilde{\sigma} \frac{\tilde{k}^2}{\sigma_{\text{int}}} \right), \\
\alpha_{\delta,4} &:= p_2 \tilde{\sigma} - \delta \frac{\sigma_{\text{ext}} p_1}{\sigma_{\text{int}} p_2 + \sigma_{\text{ext}} p_1} \left( p_2 \sigma_{\text{ext}} + p_1 \sigma_{\text{int}} + 2p_1^2 \tilde{\sigma} \frac{\tilde{k}^2}{\sigma_{\text{int}}} \right), \\
\alpha_{\delta,5} &:= \delta \frac{\sigma_{\text{int}} p_2}{\sigma_{\text{int}} p_2 + \sigma_{\text{ext}} p_1} (p_2 - p_1) \tilde{\sigma}, \quad \alpha_{\delta,6} := \delta \frac{\sigma_{\text{ext}} p_1}{\sigma_{\text{int}} p_2 + \sigma_{\text{ext}} p_1} (p_2 - p_1) \tilde{\sigma}, \\
\alpha_{\delta,7} &:= \delta \frac{\sigma_{\text{int}} p_2}{\sigma_{\text{int}} p_2 + \sigma_{\text{ext}} p_1} \left( p_1^2 \frac{\tilde{\sigma}^2}{\sigma_{\text{int}}} \right), \quad \alpha_{\delta,8} := \delta \frac{\sigma_{\text{ext}} p_1}{\sigma_{\text{int}} p_2 + \sigma_{\text{ext}} p_1} \left( p_1^2 \frac{\tilde{\sigma}^2}{\sigma_{\text{int}}} \right),
\end{aligned}$$

Similar ideas to those used in Section 6 guarantee the existence and the uniqueness of the solution to Problem (7.35)-(7.36). Optimal error estimates can also be obtained. Nevertheless, note that we suppose here that  $\sigma_\delta$  and  $k_\delta^2$  are strictly positive constants.

## 8 Conclusion

In this work, we determined and justified an asymptotic expansion of the exact solution to Problem (1.1) for different values of contrast and wavenumber. For each case we derive approximate transmission conditions, validated thanks to optimal error estimates (see Theorem 6.3 and the end of Section 7) to take into account the effect of the thin layer. Ventcel-type transmission conditions, involving tangential differential operators of order two, have also been obtained in the case of high values of magnetic permittivity and wavenumber.

An interesting continuation of this work could be to consider the full Maxwell's system describing the scattering of electromagnetic waves by an obstacle.

## 9 Appendix

This appendix contains some technical results needed in the proof of the uniform stability estimate in the high-permittivity case (see Section 7). We first give the well-posedness and regularity results for a Helmholtz equation with Ventcel-type transmission conditions and provide next a density result.

### Well-posedness and regularity results for a solution to a Ventcel transmission problem

Recalling that  $\tilde{\Omega}_{\text{ext}} = \Omega \setminus \overline{\Omega_{\text{int}}}$ , one has the following result.

**Theorem 9.1** *Let  $h \in H^1(\Gamma)$  and  $\zeta \in H^{-1}(\Gamma)$ . Then the following problem*

$$\left\{ \begin{array}{ll} \operatorname{div}(\sigma_{\text{int}} \nabla U_{\text{int}}) + k_{\text{int}}^2 U_{\text{int}} = 0 & \text{in } \Omega_{\text{int}}, \\ \operatorname{div}(\sigma_{\text{ext}} \nabla U_{\text{ext}}) + k_{\text{ext}}^2 U_{\text{ext}} = 0 & \text{in } \Omega_{\text{ext}}, \\ U_{\text{int}|_{\Gamma}} - U_{\text{ext}|_{\Gamma}} = h & \text{on } \Gamma, \\ \sigma_{\text{int}} \partial_{\mathbf{n}} U_{\text{int}|_{\Gamma}} - \sigma_{\text{ext}} \partial_{\mathbf{n}} U_{\text{ext}|_{\Gamma}} = p_1 \tilde{\sigma} \Delta_{\Gamma} U_{\text{int}|_{\Gamma}} + p_1 \tilde{k}^2 U_{\text{int}|_{\Gamma}} \\ + p_2 \tilde{\sigma} \Delta_{\Gamma} U_{\text{ext}|_{\Gamma}} + p_2 \tilde{k}^2 U_{\text{ext}|_{\Gamma}} + \zeta & \text{on } \Gamma, \\ \lim_{|x| \rightarrow +\infty} |x| (\partial_{|x|} - ik_{\text{ext}}) U_{\text{ext}} = 0, & \end{array} \right. \quad (9.1)$$

admits a unique solution  $(U_{\text{int}}, U_{\text{ext}})$  in  $H^{1,1}(\Omega_{\text{int}}) \times H_{\text{loc}}^{1,1}(\overline{\Omega_{\text{ext}}})$  satisfying the inequality

$$\|U_{\text{int}}\|_{H^{1,1}(\Omega_{\text{int}})} + \|U_{\text{ext}}\|_{H^{1,1}(\tilde{\Omega}_{\text{ext}})} \leq c_k \left( \|h\|_{H^1(\Gamma)} + \|\zeta\|_{H^{-1}(\Gamma)} \right). \quad (9.2)$$

Moreover, for all  $k \in \mathbb{N}^*$ , if  $h \in H^k(\Gamma)$ ,  $\zeta \in H^{k-2}(\Gamma)$  and  $\Gamma$   $\mathcal{C}^{k+1}$ -continuous, then

$$(U_{\text{int}}, U_{\text{ext}}) \in H^{k+1/2}(\Omega_{\text{int}}) \times H_{\text{loc}}^{k+1/2}(\overline{\Omega_{\text{ext}}}).$$

**Proof** Uniqueness follows by Rellich's lemma. Existence of a solution can be obtained by the Fredholm alternative. To show the regularity result, we proceed by induction in  $k$ . For  $k = 1$ , we showed above that if  $h \in H^1(\Gamma)$  and  $\zeta \in H^{-1}(\Gamma)$ , then Problem (9.1) admits a unique solution  $(U_{\text{int}}, U_{\text{ext}})$  in  $H^{1,1}(\Omega_{\text{int}}) \times H_{loc}^{1,1}(\overline{\Omega_{\text{ext}}})$ ; hence  $U_{\text{ext}|_\Gamma}, U_{\text{int}|_\Gamma} \in H^1(\Gamma)$ . Since

$$\operatorname{div}(\sigma_{\text{ext}} \nabla U_{\text{ext}}) + k_{\text{ext}}^2 U_{\text{ext}} = 0 \text{ in } \Omega_{\text{ext}},$$

$$\operatorname{div}(\sigma_{\text{int}} \nabla U_{\text{int}}) + k_{\text{int}}^2 U_{\text{int}} = 0 \text{ in } \Omega_{\text{int}}$$

and  $\Gamma$  is  $\mathcal{C}^2$ , we get (cf. [10])  $(U_{\text{int}}, U_{\text{ext}}) \in H^{3/2}(\Omega_{\text{int}}) \times H_{loc}^{3/2}(\overline{\Omega_{\text{ext}}})$ . Now assume that the assertion holds up to  $k - 1$ . Let  $h \in H^k(\Gamma)$  and  $\zeta \in H^{k-2}(\Gamma)$ . Since  $H^k(\Gamma) \subset H^{k-1}(\Gamma)$  and  $H^{k-2}(\Gamma) \subset H^{k-3}(\Gamma)$ ,  $h \in H^{k-1}(\Gamma)$  and  $\zeta \in H^{k-3}(\Gamma)$ . Then there exists a unique solution  $(U_{\text{int}}, U_{\text{ext}})$  of (9.1) in  $H^{k-1/2}(\Omega_{\text{int}}) \times H_{loc}^{k-1/2}(\overline{\Omega_{\text{ext}}})$ . Applying trace theorem of functions in  $H^{k-1/2}$  (cf. [10]), we obtain  $\partial_{\mathbf{n}} U_{|\Gamma}^\pm \in H^{k-2}(\Gamma)$ .

Now from

$$\sigma_{\text{int}} \partial_{\mathbf{n}} U_{\text{int}|_\Gamma} - \sigma_{\text{ext}} \partial_{\mathbf{n}} U_{\text{ext}|_\Gamma} = p_1 \tilde{\sigma} \Delta_\Gamma U_{\text{int}|_\Gamma} + p_1 \tilde{k}^2 U_{\text{int}|_\Gamma} + p_2 \tilde{\sigma} \Delta_\Gamma U_{\text{ext}|_\Gamma} + p_2 \tilde{k}^2 U_{\text{ext}|_\Gamma} + \zeta, \quad (9.3)$$

we can write

$$\tilde{\sigma} \Delta_\Gamma (p_1 U_{\text{int}|_\Gamma} + p_2 U_{\text{ext}|_\Gamma}) = \sigma_{\text{int}} \partial_{\mathbf{n}} U_{\text{int}|_\Gamma} - \sigma_{\text{ext}} \partial_{\mathbf{n}} U_{\text{ext}|_\Gamma} - p_1 \tilde{k}^2 U_{\text{int}|_\Gamma} - p_2 \tilde{k}^2 U_{\text{ext}|_\Gamma} - \zeta. \quad (9.4)$$

Thus  $\Delta_\Gamma (p_1 U_{\text{int}|_\Gamma} + p_2 U_{\text{ext}|_\Gamma}) \in H^{k-2}(\Gamma)$ . As  $p_1 U_{\text{int}|_\Gamma} + p_2 U_{\text{ext}|_\Gamma} \in H^{k-1}(\Gamma)$ ,  $\Delta_\Gamma (p_1 U_{\text{int}|_\Gamma} + p_2 U_{\text{ext}|_\Gamma}) \in H^{k-2}(\Gamma)$  and the operator  $\Delta_\Gamma$  is elliptic of order 2 on a compact manifold without boundary  $\Gamma$  of class  $\mathcal{C}^{k+2}$ ,  $p_1 U_{\text{int}|_\Gamma} + p_2 U_{\text{ext}|_\Gamma} \in H^k(\Gamma)$  but  $U_{\text{int}|_\Gamma} - U_{\text{ext}|_\Gamma} \in H^k(\Gamma)$ , then

$U_{\text{ext}}|_{\Gamma}, U_{\text{int}}|_{\Gamma} \in H^k(\Gamma)$ . Summarising, we have

$$\operatorname{div}(\sigma_{\text{ext}} \nabla U_{\text{ext}}) + k_{\text{ext}}^2 U_{\text{ext}} = 0 \text{ in } \Omega_{\text{ext}},$$

$$\operatorname{div}(\sigma_{\text{int}} \nabla U_{\text{int}}) + k_{\text{int}}^2 U_{\text{int}} = 0 \text{ in } \Omega_{\text{int}}$$

and  $U_{\text{ext}}|_{\Gamma}, U_{\text{int}}|_{\Gamma} \in H^k(\Gamma)$ , as a consequence  $(U_{\text{int}}, U_{\text{ext}}) \in H^{k+1/2}(\Omega_{\text{int}}) \times H_{loc}^{k+1/2}(\overline{\Omega_{\text{ext}}})$ .

□

### The density lemma

Recall that, in view of the thin shells assumption (cf. [18]), there exists  $\delta_0 > 0$  such that

$$\Omega_{\delta_0} = \{x \in \mathbb{R}^3 ; x := m + \eta \mathbf{n}(m) \text{ where } -\delta_0 \leq \eta \leq \delta_0 \text{ and } m \in \Gamma\} \quad (9.5)$$

defines a bijection between  $\overline{\Omega_{\delta_0}}$  and  $\Gamma \times [-\delta_0, \delta_0]$ . Let now  $\varepsilon > 0$ , satisfying  $\varepsilon < \delta_0$ . We

denote by  $H(\Omega)$  the space of functions defined by

$$H(\Omega) := \{v \in H_{\Gamma}^1(\Omega) / \exists \varepsilon > 0 \text{ such as } \partial_{\eta} v(m, \eta) = 0, \forall |\eta| < \varepsilon\}. \quad (9.6)$$

We have the following density lemma.

**Lemma 9.1**  $H(\Omega)$  is dense in  $H_{\Gamma}^1(\Omega)$ .

**Proof** Let  $v \in H_{\Gamma}^1(\Omega)$ . We construct a sequence  $(v_{\varepsilon})_{\varepsilon} \subset H(\Omega)$ ,  $\varepsilon > 0$ , such that

$\lim_{\varepsilon \rightarrow 0} v_{\varepsilon} \rightarrow v$  in  $H_{\Gamma}^1(\Omega)$ . Since  $C^{\infty}(\overline{\Omega})$  is dense in  $H_{\Gamma}^1(\Omega)$ , it is sufficient to construct such

a sequence for  $v \in C^{\infty}(\overline{\Omega})$ . Let  $\varepsilon > 0$ , we introduce the function  $\varphi_{\varepsilon}$  defined on  $[-\delta_0, \delta_0]$



by

$$\varphi_\varepsilon(\eta) := \begin{cases} 0 & \text{if } |\eta| < \varepsilon, \\ \delta_1 \frac{\eta - \varepsilon}{\delta_1 - \varepsilon} & \text{if } \varepsilon \leq \eta \leq \delta_1, \\ \delta_1 \frac{\eta + \varepsilon}{\delta_1 - \varepsilon} & \text{if } -\delta_1 \leq \eta \leq -\varepsilon, \\ \eta & \text{if } |\eta| > \delta_1, \end{cases} \quad (9.7)$$

where  $\delta_1$  satisfies  $\varepsilon < \delta_1 \leq \delta_0$ . Then we set

$$v_\varepsilon(x) := \begin{cases} v(m, \varphi_\varepsilon(\eta)) & \text{if } x = (m, \eta) \in \overline{\Omega_{\delta_0}}, \\ v(x) & \text{if } x \in \Omega_{\text{int}, \delta_0} \cup \tilde{\Omega}_{\text{ext}, \delta_0}. \end{cases} \quad (9.8)$$

It is easy to show that  $v_\varepsilon \in H(\Omega)$  and  $v_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} v$  in  $H_\Gamma^1(\Omega)$ . Hence the lemma.  $\square$

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