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## Signals and Systems

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## **Licence 3**

### **Working Sessions: Signals**

#### **Travaux Dirigés : UE Signaux**



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## Working Session n°1: Signals

### Properties of signals (linearity, periodicity, principle of superposition, dilation, time reversal, time shift, time invariance, memory, causality, stability), energy content and power, Fourier series and analysis

⇒ A mathematical appendix is located at the end of this work session. Thank you for reading it.

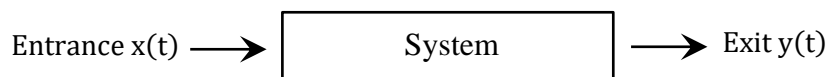
▪ **Exercise 1)** We consider the following signals  $f(t)$  et  $g(t)$ :

$$f(t) = A \cdot \cos(\omega t) \quad \text{and} \quad g(t) = \begin{cases} t + 7 & \text{for } -7 \leq t \leq -3 \\ 4 & \text{for } -3 \leq t \leq 3 \\ -t + 7 & \text{for } 3 \leq t \leq 7 \\ 0 & \text{Elsewhere} \end{cases}$$

1.1) Calculate the signal energy over one period for  $f(t)$  and the energy content of  $g(t)$ .

1.2) Calculate the average power of both signals.

▪ **Exercise 2)** Demonstrate whether or not the following systems are linear:



2.1)  $y(t) = A \cdot x(t)$  with,  $A$  constant.

2.2)  $y(t) = x^3(t)$ .

▪ **Exercise 3)** If  $x_1(t)$  and  $x_2(t)$  are two solutions of the following differential equation which represents a certain evolutionary process:

$$x''(t) + x'(t) + x(t) = \cos t,$$

Concerning  $x_1(t) + x_2(t)$ , is it solution of this equation?

What great principle manages this aspect in physics?

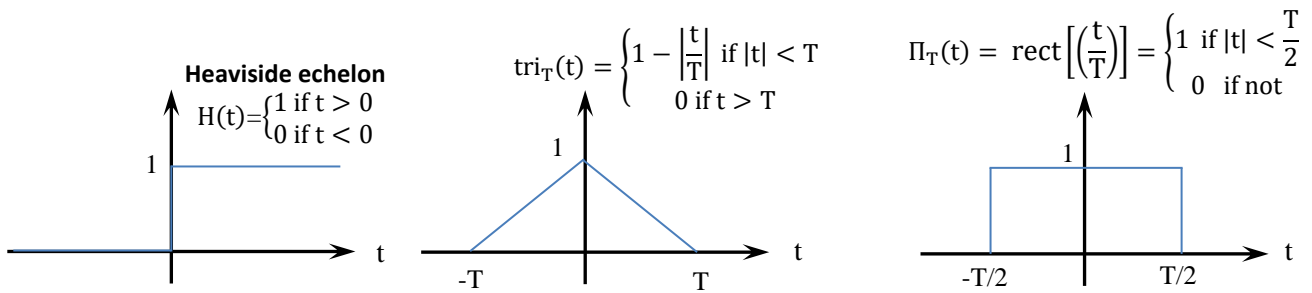
▪ **Exercise 4)** Graph the following signals:

4.1)  $x(t) = H(t + 2) - H(t - 1) + \text{Tri}_1(t - 1)$ , with  $H(t)$  the Heaviside echelon and  $\text{Tri}(t)$  the triangle distribution.

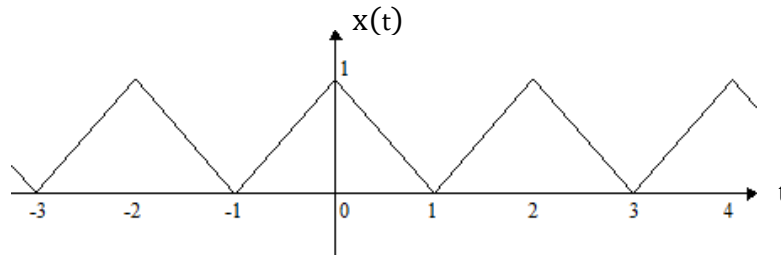
4.2)  $y(t) = x(-t)$ ,

4.3)  $z(t) = \frac{d}{dt}y(t)$ ,

4.4)  $\omega(t) = x\left(\frac{t}{2} + 1\right)$ .



▪ **Exercise 5)** Consider the following function  $x(t)$ :



- 5.1) Calculate the energy  $E$  of the signal  $x(t)$  over a period  $T$  (we will calculate between  $[0-2]$  as a  $T$  period). Then, determine its average power  $P$ .
- 5.2) Determine the Fourier series development of  $x(t)$ .
- 5.3) Represent the power spectrum of  $x(t)$  for the first five harmonics.

▪ **Exercise 6)** The purpose of this exercise is to determine the Fourier series of rectified sine signals (half-wave rectification and full-wave rectification).

- 6.1) Determine the Fourier series development of  $x(t)$  which represents the sinusoidal signal rectified in half-wave (such a signal is obtained by the use of a diode). The period is defined by  $T_0=1/\nu_0$  and we recall that by definition the average value of the signal gives the  $a_0$  coefficient:

$$\begin{cases} x(t) = \sin(2\pi\nu_0 t) & \text{for } 0 \leq t \leq \frac{T_0}{2} \\ x(t) = 0 & \text{for } \frac{T_0}{2} \leq t \leq T_0 \end{cases}$$

Represent its spectrum.

- 6.2) Determine the Fourier series development of  $x(t)$  which represents the sinusoidal signal rectified in full-wave (such a signal can be obtained by the use of a diode bridge). The period is defined by  $T_0=1/\nu_0$ .

$$x(t) = |\sin(2\pi\nu_0 t)| \quad \text{for } 0 \leq t \leq T_0$$

Represent its spectrum and conclude taking into account 6.1).

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⇒ **Mathematical form (decomposition in Fourier series, calculation of Fourier coefficients).**

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cdot \cos\left(\frac{2\pi n t}{T}\right) + b_n \cdot \sin\left(\frac{2\pi n t}{T}\right) \right]$$

with,

$$a_0 = \frac{1}{T} \int_0^T x(t) \cdot dt, \quad a_n = \frac{2}{T} \int_0^T x(t) \cdot \cos\left(\frac{2\pi n t}{T}\right) dt \quad \text{and} \quad b_n = \frac{2}{T} \int_0^T x(t) \cdot \sin\left(\frac{2\pi n t}{T}\right) dt \quad (n > 0)$$

**Remark on spectrum:** The power of the harmonic of n-rank will be:  $\frac{1}{2}(a_n^2 + b_n^2)$  and,  $a_0^2$  for the harmonic of 0-rank.

**Remark on the parity of the real signal  $x(t)$ :** If  $x(t)$  even, then  $b_n=0$  ; if  $x(t)$  odd  $a_n=0$ .

## Working Session n°2: Signals

### Fourier transforms, Distributions, Convolution, Correlation Diffractive optics application

⇒ A mathematical appendix is located at the end of this work session. Thank you for reading it.

▪ **Exercise 1** We recall that the Fourier Transform (FT) and its inverse definitions are written respectively  $\tilde{f}(v) = \text{FT}[f(t)] = \int_{-\infty}^{+\infty} f(t) \cdot e^{-i2\pi vt} dt$ , and  $f(t) = \text{FT}^{-1}[\tilde{f}(v)] = \int_{-\infty}^{+\infty} \tilde{f}(v) \cdot e^{i2\pi vt} dv$ .

Calculate the Fourier Transform (FT) of the following signals (for each case we will graphically represent the signal and its FT or module):

1.1) 'Door distribution' signal  $\Pi_T(t)$  or  $\text{rect}\left(\frac{t}{T}\right)$  defined by:  $\Pi_T(t) = \begin{cases} 1 & \text{if } |t| < \frac{T}{2} \\ 0 & \text{if not} \end{cases}$ . Demonstrate that  $\text{FT}[\Pi_T(t)] = T \cdot \text{sinc}(\pi v T)$ .

1.2) The signal  $f(t) = \begin{cases} 1 & \text{if } 0 < t < 1 \\ 0 & \text{if not} \end{cases}$  by two various methods (first the integral calculation, and then using a shift applied directly to the previous  $\Pi_{T=1}(t)$  signal). For the second method, we recall the specific property  $f(t) * \delta(t - t_0) = f(t - t_0)$ .

1.3) 'Exponential discharge' signal defined by (with  $a > 0$ ):  $f(t) = \begin{cases} e^{-at} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$ .

1.4) 'Symmetric exponential' signal defined by (with  $a > 0$ ):  $f(t) = e^{-a|t|}$

1.5) 'Damped oscillation' signal defined by:  $f(t) = \begin{cases} e^{-at} \cdot \sin(2\pi v_0 t) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$ .

1.6) 'Damped oscillation' signal defined by:  $f(t) = \begin{cases} e^{-at} \cdot \cos(2\pi v_0 t) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$ .

1.7) Gaussian signal defined by:  $f(t) = e^{-\left(\frac{\pi t^2}{a^2}\right)}$ . Demonstrate that the FT of a Gaussian is a Gaussian.

▪ **Exercise 2** We recall that the convolution and correlation definitions between two  $f(t)$  and  $g(t)$  signals are written respectively:

$$\text{Convolution: } f(t) * g(t) = \int_{-\infty}^{+\infty} f(\tau) \cdot g(t - \tau) d\tau = \int_{-\infty}^{+\infty} f(\tau) \cdot g(t - \tau) d\tau, \text{ and}$$

$$\text{Correlation (or cross-correlation): } \mathcal{C}_{fg}(\tau) = f(t) \circledast g(t) = f(t) * g^*(-t) = \int_{-\infty}^{+\infty} f(t) \cdot g^*(t + \tau) dt.$$

2.1) Calculate the 'auto-correlation' function  $\mathcal{C}_{\Pi\Pi}(\tau)$  of the door distribution (see 1.1) by drawing the common area. We will naturally call this signal and such a result:  $T \cdot \text{tri}_{2T}(t)$ .

2.2) Calculate by a simple way its FT, that is,  $\text{FT}[\mathcal{C}_{\Pi\Pi}(\tau)]$ , by using that:  $T \cdot \text{tri}_T(t) = \Pi_T(t) * \Pi_T(t)$ . What remarks can be said about this autocorrelation function which will be defined as the spectral energy density, later.

▪ **Exercise 3)** Let  $f(t)$  be the signal composed of  $\Pi_T(t)$  and an echo of  $\Pi_T(t)$  :  $f(t) = \Pi_T(t) + a.\Pi_T(t - t_0)$

3.1) Calculate  $\mathcal{C}_{ff}(\tau)$ .

3.2) Calculate the spectral density energy  $\text{FT}[\mathcal{C}_{ff}(\tau)] = |\tilde{f}(v)|^2$ , trace its general appearance.

▪ **Exercise 4)** We consider the signal  $f(t)$  shaped by the 'periodization' of  $\Pi_\tau(t)$  with a temporal step  $T$   $T > \tau$ , that is:  $f(t) = \Pi_\tau(t) * \text{III}_T(t)$  with the Dirac comb  $\text{III}_T(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$ .

4.1) Calculate its Fourier transform  $\text{FT}[f(t)]$ .

4.2) Calculate and trace its autocorrelation function and show that it is periodic.

▪ **Exercise 5) - In addition, will be not corrected in session -**

We consider both signals:  $f_1(t) = a_1.\Pi_T(t - T)$  and  $f_2(t) = a_2.\Pi_{T'}(t - T')$ , with  $a_{i=1,2}$  constants and  $T' < T$ .

5.1) Calculate and trace  $\mathcal{C}_{f_1f_2}(\tau)$  the cross-correlation function between the  $f_1$  and  $f_2$  signals.

5.2) Calculate  $\mathcal{S}_{f_1f_2}(v)$  the cross-energy spectrum function defined as  $\text{FT}[\mathcal{C}_{f_1f_2}(\tau)](v)$ .

5.3) We consider now that  $f_1$  undergoes a time shift:  $f_1^{\text{new}}(t) = f_1(t) * \delta(t - t_0)$ . What is the effect on the new calculation of the cross-correlation and the cross-spectrum functions respectively.

▪ **Exercise 6)** We consider the signal  $f(t) = e^{-at} H(t)$ , with  $H(t)$  the Heaviside echelon.

6.1) Calculate the auto-correlation function  $\mathcal{C}_{ff}(\tau)$ , with  $\tau > 0$ .

6.2) Calculate the spectral energy density  $\mathcal{S}_{ff}(v) = |\tilde{f}(v)|^2 = |\text{FT}[f(t)]|^2$ . We give  $\text{FT}[e^{-at} H(t)] = \frac{1}{a + i2\pi v}$  and that the primitive of  $\frac{1}{1+x^2}$  is  $\arctan$ .

6.3) Calculate the integral  $\int_{-\infty}^{+\infty} |f(t)|^2 dt$  and conclude with Parseval-Plancherel theorem.

▪ **Exercise 7) Diffractive optics application, towards optical filtering**

▪ **Notions required**

**Introduction to the optical signal and definition of two-dimensional Fourier Transform (FT):**

It is possible to define an object as 'source' of the optical signal and consider it in terms of amplitude  $a(x,y)$  and phase  $\varphi(x,y)$ , both varying spatially. We will then define a *transparency spatial function*  $P(x,y)$  associated with this *object* as  $P(x,y) = a(x,y).e^{i\varphi(x,y)}$ . In optics, such object could represent a blade, a lens, a diffracting pupil, a diaphragm...

By analogy to time signals (time  $s \Leftrightarrow$  frequency Hz), this two-dimensional spatial function  $P(x,y)$  (spatial dimensions : m, m) is associated with a *Fourier spectrum which will be a two-dimensional FT* ( $m^{-1}, m^{-1}$ ).

The two-dimensional FT is obtained by generalization of the one-dimensional integral, that is:

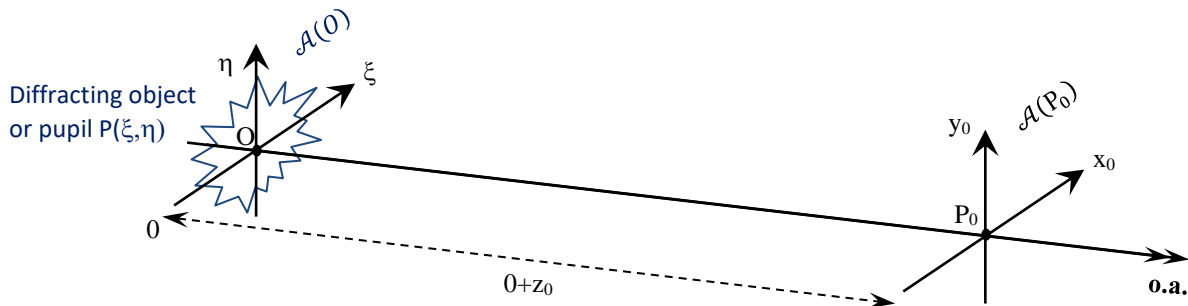
$$\tilde{P}(u, v) = \text{FT}[P(x, y)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P(x, y) \cdot e^{-i2\pi(u \cdot x + v \cdot y)} dx \cdot dy$$

This spectrum will be a spectrum of *spatial frequencies* of the object and will thus allow possible filtering, descrambling, contrast inversion operations into such a so-called Fourier plane.

**Theoretical reminder of diffraction (see Born and Wolf, 'Principles of Optics):**

The calculation of the complex diffracted amplitude  $\mathcal{A}(P_0(x_0, y_0, z_0))$  (or so-called Fresnel Kirchhoff integral) is written at a  $z_0$ -distance from an origin O:

$$\mathcal{A}(P_0(x_0, y_0, 0 + z_0)) = \frac{e^{ik \cdot z_0}}{i \cdot \lambda \cdot z_0} \iint_{\Sigma_0} \mathcal{A}(O(\xi, \eta, 0)) \cdot e^{ik \left[ \frac{(\xi - x_0)^2}{2 \cdot z_0} + \frac{(\eta - y_0)^2}{2 \cdot z_0} \right]} d\xi \cdot d\eta$$



Such an expression at a  $z_0$ -long distance (or Fraunhofer regime) is simplified by:

$$\mathcal{A}(P_0) = B \cdot \tilde{P} \left( u = \frac{x_0}{\lambda \cdot z_0}, v = \frac{y_0}{\lambda \cdot z_0} \right),$$

with,  $B_{\text{Fresnel}} = \text{constant} = \frac{e^{ik \cdot z_0}}{i \cdot \lambda \cdot z_0} \cdot e^{-i \frac{\pi(x_0^2 + y_0^2)}{\lambda \cdot z_0}}$  at a  $z_0$ -distance, and then  $B_{\text{Fraunhofer}} = \frac{e^{ik \cdot z_0}}{i \cdot \lambda \cdot z_0}$  (without dependence on  $x_0$  and  $y_0$ ) at a  $z_0$ -long distance, and  $k = 2\pi/\lambda$  the absolute wave vector.

This means that in Fraunhofer's regime (or at a long distance) the complex amplitude of the diffracted optical field is directly proportional to the FT of the pupil object! Is it necessary to recall that the illuminance (intensity) will be therefore  $E(P_0) = \frac{1}{\lambda^2 \cdot z_0^2} \left| \tilde{P} \left( \frac{x_0}{\lambda \cdot z_0}, \frac{y_0}{\lambda \cdot z_0} \right) \right|^2$ ?

### • Calculation of diffraction patterns

**7.1)** The object consists of a horizontal slit of  $a$ -width, inside which an amplitude network R has a sinusoidal intensity modulation  $R(\xi, \eta) = [1 + \cos(2\pi N_0 \xi)] \times 1_\eta$ .

Write the full pupil expression  $P(\xi, \eta)$ . Then calculate and represent the diffraction pattern at a long distance.

**7.2)** When the object is positioned attached to a lens L with focal length  $f$ , then the FT of such an object will be shaped and positioned at the focal point of the image.

We consider a diaphragm type object composed of two rectangular openings such as  $(\Delta\xi, \Delta\eta)$  the widths and heights):

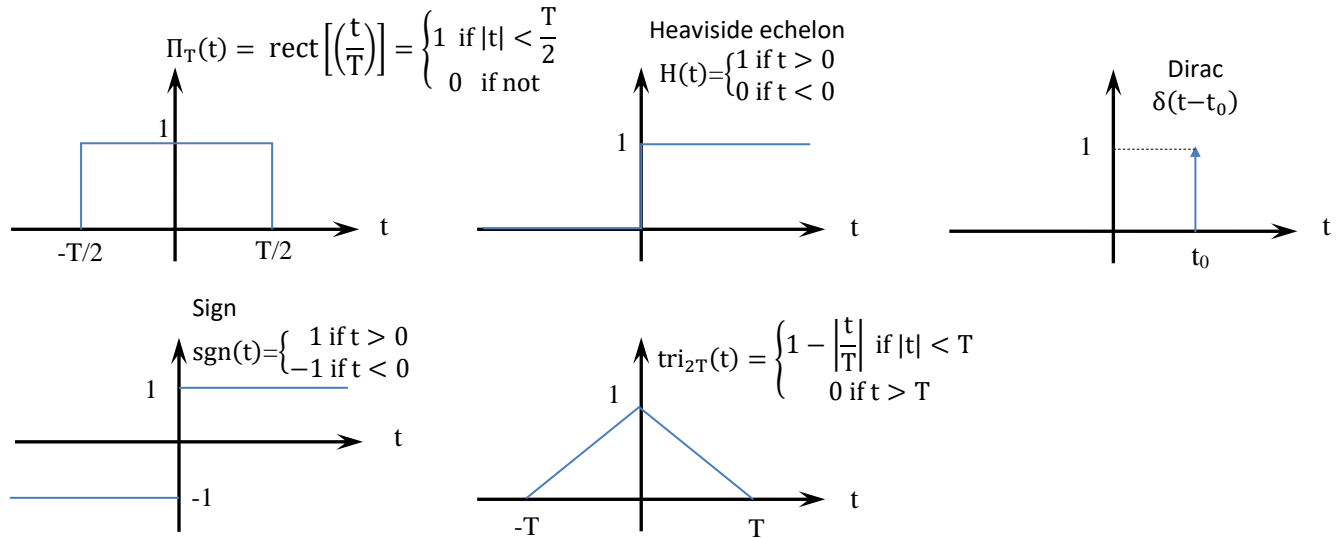
$$P(\xi, \eta) = \Pi_{\Delta\xi, \Delta\eta}(\xi, \eta) * \delta(\xi - \xi_0, \eta - \eta_0) \times_{\text{et}} \Pi_{\Delta\xi, \Delta\eta}(\xi, \eta) * \delta(\xi + \xi_0, \eta + \eta_0)$$

Draw and represent such a pupil or diaphragm. Then, calculate the diffraction pattern plus the intensity directly observed laying into the image focal plane (that is at a  $z_0=f$  distance).



⇒ **Mathematical form (FT and Distributions definitions).**

▪ **Distributions, some definitions:**



▪ **Some FT and properties:**

	<b>FT</b> →
$\delta(t)$	1
1	$\delta(v)$
$\Pi_T(t) = \text{rect}\left(\frac{t}{T}\right)$	$\frac{\sin(\pi v T)}{\pi v} = T \cdot \text{sinc}(\pi v T)$
$\text{sinc}(\pi b T)$	$\frac{1}{b} \Pi_{1/b}(v)$
$e^{i2\pi v_0 t}$	$\delta(v - v_0)$
$\delta(v - v_0)$	$e^{-i2\pi v_0 t}$
$a \cdot \cos(2\pi v_0 t)$	$\frac{a}{2} [\delta(v - v_0) + \delta(v + v_0)]$
$a \cdot \sin(2\pi v_0 t)$	$\frac{a}{2i} [\delta(v - v_0) - \delta(v + v_0)]$
$\text{III}_T(t)$	$\frac{1}{T} \text{III}_{1/T}(v)$
$a \cdot e^{-\pi \cdot b \cdot t^2}$	$\frac{a}{\sqrt{b}} e^{-\frac{\pi}{b} v^2}$
$a \cdot \Pi_T(t) \cdot e^{-\frac{ t }{b}}$ , with $b > 0$	$\frac{a \cdot b}{1 + i2\pi b v}$
$a \cdot e^{-\frac{ t }{b}}$ , with $b > 0$	$\frac{2ba}{1 + (2\pi b v)^2}$
$\text{tri}_{2T}(t)$	$T^2 \cdot \text{sinc}^2(\pi v T)$

$FT \left[ \frac{d^n}{dt^n} f(t) \right] = (i2\pi v)^n \cdot \tilde{f}(v)$  ;  $FT[f(t - t_0)] = e^{-i2\pi v t_0} \cdot \tilde{f}(v)$  ;  $FT[f(a \cdot t)] = \frac{1}{|a|} \cdot \tilde{f}\left(\frac{v}{a}\right)$ .

$FT[f(t) * g(t)] = \tilde{f}(v) \cdot \tilde{g}(v)$  ;  $FT[FT[f(t)]] = f(-t)$  ;  $FT[f^*(t)] = \tilde{f}^*(-v)$ .

$|\tilde{f}(v)|^2 = FT[\mathcal{C}_{ff}(\tau)]$ .

▪ **Parseval-Plancherel theorem:**

$\int_{-\infty}^{+\infty} f(t) \cdot g^*(t) dt = \int_{-\infty}^{+\infty} \tilde{f}(v) \cdot \tilde{g}^*(v) dv$  ; then  $\int_{-\infty}^{+\infty} |f(t)|^2 dt = \int_{-\infty}^{+\infty} |\tilde{f}(v)|^2 dv$ .

## Working Session n°3: Signals

### Systems and properties (linearity, time invariance, memory, causality, stability), Impulse/percussion response, Convolution, Transfer function, Discrete Fourier transforms (DFT) and series

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⇒ **Some definitions:**

**Memoryless:** a static system *without memory* (or memoryless) is such that the output signal  $s(t_0)$  at time  $t_0$  only depends on the input signal  $e(t_0)$  at the same time  $t_0$ ;  $s(t)$  function of  $e(t)$ .

Then, a dynamic system *with memory* can be defined by an output signal  $s(t_0)$  function of inputs  $e(t)$  at times with  $t \leq t_0$  and/or output  $s(t)$  with  $t < t_0$ .

**Causality:** Such causal system have an output signal  $s(t_0)$  which depend on the previous values (and also the present moment) of the input signal  $e(t)$ , with  $t \leq t_0$ .

**Stability:** a system will be stable if any bounded input  $e(t)$  gives a bounded output  $s(t)$ .

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▪ **Exercise 1)** Discuss the properties of memory, causality, norm (bounded input/output) and stability, temporal invariance of the following systems:

1.1)  $y(t) = x(t-6)$ ,                      1.2)  $y(t) = x^2(t)$ ,

1.3)  $y(t) = x(t) \cdot \sin(\omega_0 t)$ ,            1.4)  $y(t) = \frac{d}{dt}x(t)$ .

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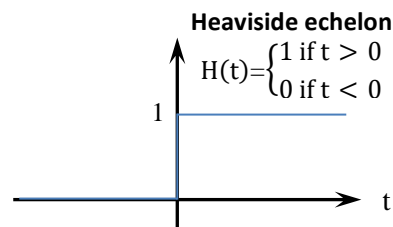
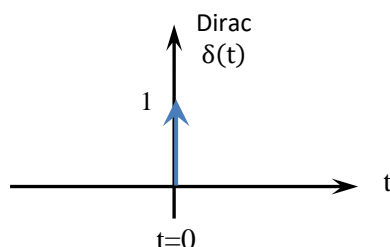
**Theorem:** For a system described by a convolution operator to be causal, it is necessary and sufficient that its impulse response  $h(t)$  is zero for the negative values of the variable  $t$ . The system is then causal; this means that the exit cannot precede the entry. Thus, for the entrance  $\delta(t)$  (Dirac impulse), corresponding to the non-zero entry at  $t=0$ , the impulse response  $h(t)$  is necessarily zero for  $t$  negative and  $s(t) = h(t) * \delta(t) = h(t)$ .

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▪ **Exercise 2)** Discuss the properties of memory, causality, norm and stability of the following systems and impulse responses:

2.1)  $h(t) = 4 \cdot \delta(t)$ ,

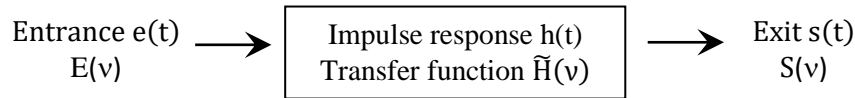
2.2)  $h(t) = \cos(\pi t) \cdot H(t + 1)$  with  $H(t)$  the Heaviside echelon.



▪ **Exercise 3)** For a given system we have respectively the input signal  $x(t) = H(t-1)$  and the impulse (or percussion) response of the system equal to  $h(t) = x(t) = e^{-t} \cdot H(t)$ ; deduce the output signal  $y(t)$ .

▪ **Exercise 4)** For a given system, we have  $x(t) = e^{-t} \cdot H(t + 2)$  and  $h(t) = e^{-t} \cdot H(t)$ , with  $H$  Heaviside echelon. Find the response function  $y(t)$ .

▪ **Exercise 5)** We consider the system whose input  $e(t)$  and output  $s(t)$  are connected by the relation:  $e(t) = h(t) * s(t)$ , with  $h(t) = \Pi_T \left( t - \frac{T}{2} \right) = \text{rect} \left[ \left( \frac{t}{T} \right) - \frac{1}{2} \right]$ .



5.1) Represent  $h(t)$ . We consider  $e(t) = A \cos(2\pi\nu_0 t)$ ; calculate  $s(t)$  and show that:

$$s(t) = A \cdot \frac{\sin(2\pi\nu_0 t) - \sin[2\pi\nu_0(-T + t)]}{2\pi\nu_0}$$

5.2) We can write  $s(t) = A(\nu_0) \cdot \cos[(2\pi\nu_0 t) - \phi(\nu_0)]$ . Calculate  $A(\nu_0)$  and  $\phi(\nu_0)$ . Represent them as a function of  $\nu_0$ .

5.3) We consider that the input signal  $e(t)$  is arbitrary. Express  $\tilde{S}(\nu)$  as a function of  $\tilde{H}(\nu)$ . Represent  $|\tilde{H}(\nu)|$  and  $\arg[\tilde{H}(\nu)]$ .

▪ **Exercise 6) - In addition, will be not corrected in session –**

We consider a system whose output and input are linked by  $(\forall t)$ :

$$\sum_{n=0}^N a_n \frac{d^n s(t)}{dt^n} = \sum_{m=0}^M b_m \frac{d^m e(t)}{dt^m}$$

6.1) Show that this system has the linear characteristic or property. Check that  $e^{i2\pi\nu_0 t}$  is an eigen-function of such system.

6.2) Assuming that the Fourier transforms (FT) of  $e(t)$  and  $s(t)$  plus their derivatives up to the order  $M$  and  $N$  have a meaning (or exist), give the relation linking the FT of  $s(t)$  to that of  $e(t)$ .

6.3) The system is supposed to be stable, in practice how can we calculate its impulse (or percussion) response  $h(t)$ ?

6.4) The system being assumed to be causal, what can we say about its impulse response? Give the expression of its transfer function  $\tilde{H}(\nu)$ .

⇒ **Reminder and analogy/ definitions:**

The Fourier series decomposition was  $f(t) = \sum_{k \rightarrow -\infty}^{+\infty} c_k \cdot e^{ik2\pi\nu t}$ , with  $c_k = \frac{1}{T} \int_0^T f(t) \cdot e^{-ik2\pi\nu t} dt$ .

By analogy, the DFT (N-points sampled at a period  $\left( \frac{v_{\text{sampling}}}{N} \right)$ ) will be then defined by:

$$\tilde{f}_N(n) = \sum_{k=0}^{N-1} f_N(k) \cdot e^{-i2\pi k \frac{n}{N}}$$

▪ **Exercise 7)** Find and write the Discrete Fourier transform (DFT) of the signal  $x[n] = \{2, 0, -1, 3\}$ .

▪ **Exercise 8)** Considering the signal sequence  $x[n] = \sum_{k=-\infty}^{+\infty} \delta(n - 4k)$ . Represent and sketch several period of  $x[n]$ . Calculate the Fourier coefficient  $c_k$  of  $x[n]$ .

## Working Session n°4: Signals

### Sampling operation, Discrete Fourier transforms (DFT), Filtering, Modulation/demodulation

- **Exercise 1)** We consider a sinusoidal signal function  $x(t) = \cos(2\pi\nu_0 t)$  with  $\nu_0$  frequency. Show that if there is synchronization between the signal and the sampling operation (it means  $\nu_0 = k \cdot \frac{\nu_s}{N}$ , with  $\nu_s$  the  $\nu_{\text{sampling}}$  and  $k$  integer), then the DFT reconstructs exactly the spectrum of the previous  $x(t)$  function.

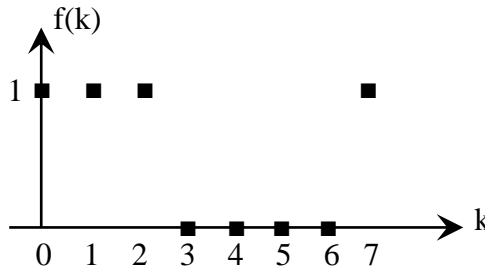
[Note: You can already write the sampled signal  $x_s(t)$ ; Then calculate  $\tilde{X}_s(\nu)$  and plot such a spectrum... Finally discuss the spectral aliasing to conclude]

- **Exercise 2)** The signal  $x(t) = \cos(200\pi t) + 2 \cos(320\pi t)$  is ideally sampled at the frequency of  $\nu_{\text{sampling}} = 300$  Hz. We process this sampled signal obtained by a low-pass filter whose cut-off frequency is  $\nu_{\text{cut}} = 250$  Hz. What are the spectral components collected at the filter output?

[Note: Write the sampled signal  $x_s(t)$ ; Then calculate  $\tilde{X}_s(\nu)$  and plot such a spectrum... Finally discuss on the positive side of the spectrum the positioning and effect of  $\nu_{\text{cut}}$ ]

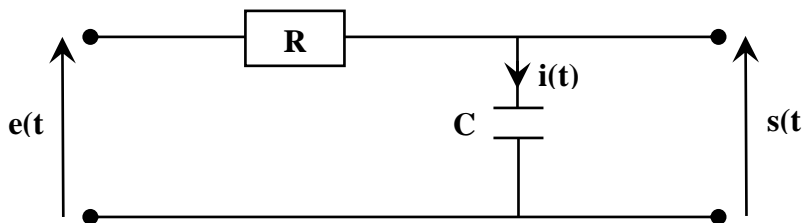
- **Exercise 3) - In addition, will be not corrected in session –**

By using the definition of the DFT, calculate the DFT of the following discrete sequence:



[Note:  $N = 8$  terms, then  $\tilde{f}_N(n) = \sum_{k=0}^{k=N-1=7} f_N(k) \cdot e^{-i2\pi k \frac{n}{N}}$ ]

- **Exercise 4)** The goal will be to study the filtering aspect of a known circuit by directly taking the FT from the differential equation. We consider the following RC circuit:



- 4.1)** Establish the differential equation that links the output voltage  $s(t)$  to the input voltage  $e(t)$ . Calculate the transfer function  $\tilde{H}(\nu)$  and the impulse response  $h(t)$ .

[Note: you have to apply FT directly to the whole differential equation to calculate  $\tilde{H}(\nu)$ . Then to

determinate  $h(t)$  we recall that:  $e^{-at} \cdot H(t) \xrightarrow[\text{FT}^{-1}]{\text{FT}} \frac{1}{a+i2\pi\nu}$ ]

4.2) Represent the amplitude spectrum  $|\tilde{H}(v)|$  and the phase spectrum  $\arg[\tilde{H}(v)]$ . Show that this system is a low pass filter.

4.3) We consider the input signal  $e(t) = A \cdot \Pi_T \left( t - \frac{T}{2} \right)$ . Draw the graph of the output signal  $s(t)$  in the following cases: a)  $0 < T < +\infty$  ; b)  $T \rightarrow +\infty$  ; c)  $T \rightarrow 0$ .

4.4) We consider  $e(t) = A \cdot e^{-at} H(t)$ , with  $H(t)$  Heaviside unity echelon and  $a > 0$ . Calculate the energy spectral density of the output signal  $s(t)$ .

[Note: This question can be answered directly after 4.2]

### ▪ Exercise 5) Chopper modulation/demodulation

We wish to send a signal  $x(t)$  on a communication channel using amplitude modulation (AM) at a carrier frequency of  $f_c$ . For this purpose, we propose to use a chopper modulation technique, which consists of passing the signal  $x(t)$  through a 'on/off' gate driven at the frequency called  $f_c$  ; this operation basically results in the multiplication of the signal  $x(t)$  with a periodic square signal with 50% duty cycle.

5.1) Show that this modulation technique can be used, along with appropriate filtering, to produce an amplitude modulated signal.

5.2) Corollary for coherent chopper demodulation: show that the same technique can be used to demodulate a AM-modulated signal  $x_{AM}(t) = [A_c + x(t)] \cdot \cos(2\pi f_c t + \varphi)$ , provided the 'on/off' square signal is well in phase with the carrier.

## Working Session n°5: Signals

### Laplace integral and transforms, Region of Convergence (RoC), Distributions, Systems, Causality and Stability, Impulse/percussion response, Transfer function, Z-transform....

⇒ A mathematical appendix is located at the end of this work session. Thank you for reading it.

▪ **Exercise 1)** We recall that the *unilateral* Laplace Transform ( $LT \equiv \mathcal{L}$ -operator) of a function  $f(t)$  locally summable and null for  $t < 0$ , is written :  $\hat{F}(p) = \mathcal{L}[f(t)] = \int_0^{+\infty} f(t) \cdot e^{-pt} dt$ , with  $p$  complex variable. Thus, this mathematical tool will be strongly and naturally used for the calculation of transfer functions from causal impulse responses  $h(t) = 0$  for  $t < 0$ .

Calculate the Laplace Transform ( $LT \equiv \mathcal{L}$ ),  $\hat{X}(p)$  of the following signals and plot onto the complex plane, respectively the pole and the zero plus the notion of global Region of Convergence (RoC) for such signal.

1.1)  $x(t) = t \cdot e^{-at} \cdot H(t)$  [Note: it will be necessary to integrate by part]

1.2)  $x(t) = e^{-2t} \cdot H(t) + e^{-3t} \cdot H(t)$  [Note: directly by adapting result from the appendix LT table]

1.3)  $x(t) = e^{-3t} \cdot H(t) + e^{2t} \cdot H(-t)$

1.4)  $x(t) = e^{2t} \cdot H(t) + e^{-3t} \cdot H(-t)$

▪ **Exercise 2)** Determine the inverse LT of the following spectral-function  $\hat{X}(p)$  corresponding to a causal system ( $\mathcal{R}e(p) > -1$ ).

2.1) At first, decompose the rational fraction  $\hat{X}(p) = \frac{2p+4}{p^2+4p+3}$  into simple elements.

[Note: if necessary, review the techniques and bases of decomposing fractions into simple elements].

[https://fr.wikiversity.org/wiki/Fractions\\_rationnelles/D%C3%A9composition\\_en\\_%C3%A9l%C3%A9ments\\_simples\\_dans\\_R](https://fr.wikiversity.org/wiki/Fractions_rationnelles/D%C3%A9composition_en_%C3%A9l%C3%A9ments_simples_dans_R)

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2.2) Calculate  $x(t)$ , which will be called a 'right-sided signal' according to  $\mathcal{R}e(p) > -1$ .

▪ **Exercise 3)** We consider the continuous Linear Time Invariant (LTI) system for which the input  $x(t)$  and the output  $y(t)$  are related by the differential temporal equation:

$$y''(t) + y'(t) - 2y(t) = x(t)$$

3.1) Taking the LT of the above equation, find the transfer system function  $\hat{H}(p)$ . Decompose such transfer function  $\hat{H}(p)$  into simple elements.

3.2) Starting from such  $\hat{H}(p)$ , determine the impulse response  $h(t)$  in the case of the system is causal ( $\mathcal{R}oC \mathcal{R}e(p) > 1$ ).

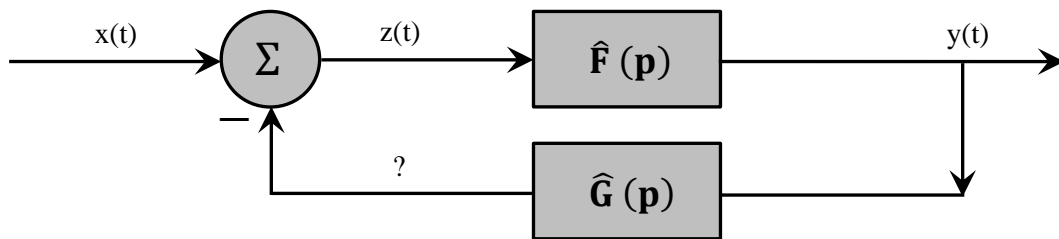
▪ **Exercise 4)** In mathematics, the two transformations FT and LT facilitate the resolution of the differential equations. They provide both an automatic means to obtain a particular solution. Let us illustrate this point knowing that LT appears more 'general' than FT. Consider the following differential equation:

$$-f''(t) + f(t) = \delta(t)$$

4.1) Taking LT of such equation, calculate respectively  $\hat{F}(p)$  and show that  $f(t) = -H(t) \cdot \text{sh}(t)$  with table in appendix. Then, taking FT of the differential equation, determine  $\tilde{F}(v)$  and show that  $f(t) = \frac{1}{2}e^{-|t|}$ .

4.2) By equalizing the more 'global' solution obtained by LT with, the 'particular' solution obtained by FT plus one a general solution of the homogeneous differential equation (i.e. second member = 0), then determine the two constants for such equality.

▪ **Exercise 5)** The feedback interconnection of two causal sub-systems with functions  $\hat{F}(p) = \frac{1}{p-2}$  and  $\hat{G}(p) = 3$  is describe below (the element  $\Sigma$  gives an output signal equal to the sum of both input signals, taking into account of the minus sign here).



5.1) Determine the transfer function  $\hat{H}(p)$  of the feedback system represented by the above diagram.

5.2) Determine the poles and the zeros (if any) of  $\hat{H}(p)$ . Discuss the both properties together 'causality plus stability' of the system.

5.3) We consider the input signal  $x(t) = e^{-3t} \cdot H(t)$ , calculate the output signal  $y(t)$ . [Note: we will already determine  $\hat{Y}(p)$ ]

▪ **Exercise 6) - To go further** - In numerical, the passage from the DFT to the Z-transform (ZT) is obtained by a change of complex variable  $z \equiv e^{i2\pi vt}$  and thus comes back to the calculations of series in mathematics.

Calculate the Z-transforms (ZT), which is written  $\check{X}(z)$ , of the following  $x(n)$  causal sequences; indicate their area of convergence.

6.1)  $x(n) = \begin{cases} 1 & (n = 0) \\ 0 & (n > 0) \end{cases}$

6.2)  $x(n) = a^n$ ,

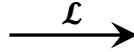
6.3)  $x(n) = n \cdot a^n$ .

[Note: for the third case we will use, k.  $x(k) \xrightarrow{ZT} -z \cdot \frac{d\check{X}(z)}{dz}$ ]

⇒ **Mathematical form (LT and definitions).**

The notations  $\mathcal{L}\{f(t)\} = \mathcal{L}(f(t)) = \hat{F}(p)$  are equivalent.

▪ **Some Laplace transforms LT ( $\equiv \mathcal{L}$ -operator) and properties:**



$\delta(t)$	1
<b><math>D^{(n)}(t) \equiv \frac{d^n}{dt^n} D(t)</math>, (<math>\forall D</math> Distribution)</b>  <i>Example:</i> $\delta^{(n)}(t) \equiv \frac{d^n}{dt^n} \delta(t)$	<b><math>p^n \cdot \mathcal{L}(D)</math> or <math>p^n \cdot \hat{D}(p)</math></b>  $p^n$
<b><math>H(t)</math></b>	<b><math>\frac{1}{p}</math></b>  (direct : $\hat{H}(p) = \int_0^{+\infty} 1 \cdot e^{-pt} \cdot dt$ , for $\mathcal{R}e(p) > 0$ )
<b><math>D(t - t_0)</math></b> , ( $\forall D$ Distribution)  <i>Example :</i> $H(t - t_0)$	<b><math>e^{-t_0 p} \cdot \mathcal{L}(D)</math> or <math>e^{-t_0 p} \cdot \hat{D}(p)</math></b>  $\frac{e^{-t_0 p}}{p}$
$t \cdot H(t)$	$\frac{1}{p^2}$
$t^n \cdot H(t)$	$\frac{n!}{p^{n+1}}$
<b><math>e^{p_0 t} \cdot t^n \cdot H(t)</math> (<math>\forall p_0</math> complex)</b>  <i>Examples:</i> $H(t) \cdot e^{\pm i\omega t}$  $H(t) \cdot \cos(\omega t) = \frac{H(t)}{2} \cdot (e^{i\omega t} + e^{-i\omega t})$  $H(t) \cdot \sin(\omega t) = \frac{H(t)}{2i} \cdot (e^{i\omega t} - e^{-i\omega t})$	<b><math>\frac{n!}{(p - p_0)^{n+1}}</math></b>  $\frac{1}{p \mp i\omega}$ $\frac{1}{2} \left( \frac{1}{p - i\omega} + \frac{1}{p + i\omega} \right) = \frac{p}{p^2 + \omega^2}$  $\frac{1}{2i} \left( \frac{1}{p - i\omega} - \frac{1}{p + i\omega} \right) = \frac{\omega}{p^2 + \omega^2}$
$H(t) \cdot e^{\pm t}$  <i>Examples:</i> $H(t) \cdot \text{ch}(t) = \frac{H(t)}{2} \cdot (e^t + e^{-t})$  $H(t) \cdot \text{sh}(t) = \frac{H(t)}{2} \cdot (e^t - e^{-t})$	$\frac{1}{p \mp 1}$  $\frac{1}{2} \left( \frac{1}{p - 1} + \frac{1}{p + 1} \right) = \frac{p}{p^2 - 1}$  $\frac{1}{2} \left( \frac{1}{p - 1} - \frac{1}{p + 1} \right) = \frac{1}{p^2 - 1}$

▪ **Plus all the properties of linearity, convolution and so on obviously... (as FT).**