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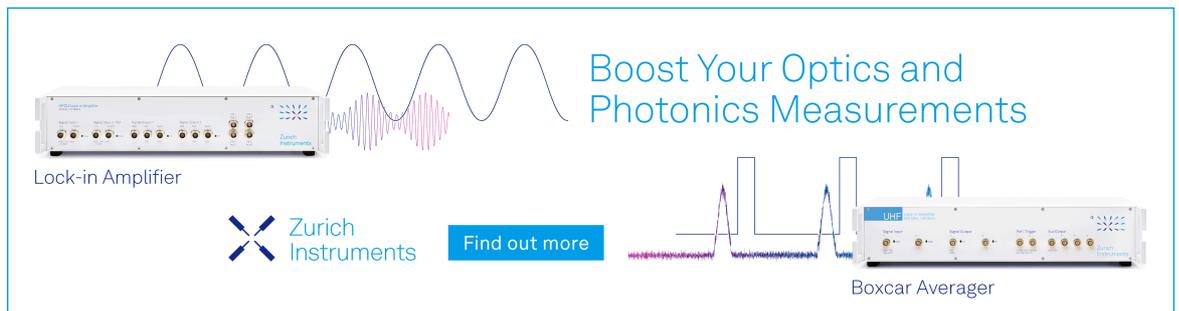


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ABSTRACT

To evaluate the long-distance magnetic field generated by neurons, we propose a reformulation of the brain's magneto-quasistatic approximation based on the Jefimenko's time-dependent generalization of the Biot-Savart law. This differs from the traditional approach relying on Maxwell's equations and not on their general solution. Instead of a typical length of the medium in the conventional approach, we use the signal traveling distance $|\mathbf{r} - \vec{r}'|$, from the farthest source point \vec{r}' to the field point \mathbf{r} , as the proper length to define the quasistatic dynamics. We consider relatively low frequencies below a typical value $f_{\max} = 100$ Hz. The quasistatic approximation is justified since $(2\pi f_{\max})^2 |\mathbf{r} - \vec{r}'|^2 \epsilon \mu_0 \approx 4 \times 10^{-7} \ll 1$. We take $|\mathbf{r} - \vec{r}'| \approx 0.1$ m, with the gray matter permittivity, previously underestimated, to be $\epsilon \approx 10^7 \epsilon_0$. A formalism for the long-distance magnetic field generated by neuronal populations is then developed. Each population is described as a region of small dimensions compared to the average distance from the field point. We split the impressed current density into synaptic and action potential contributions and study their magnetic field. Assuming a small contribution of the impressed currents at the region boundary surface, we obtain two equivalent expressions of the synaptic current dipole moment in terms of the scalar potential and/or its spatial derivatives. Using Maxwell-Wagner's time, the synaptic current dipole moment of a region is also shown to be related to the electric dipole moment of each part with uniform conductivity and permittivity. Finally, the long-distance magnetic field of action potential currents is expressed in terms of the magnetic dipole moment for these currents.

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I. INTRODUCTION

Magnetic field measurements constitute a very attractive method to measure neuronal activity, in particular, through magneto-encephalography (MEG), which measures the magnetic field at some distance from the brain surface, as reviewed previously.^{1,2} Contrary to the genesis of electric fields and potentials, membrane currents do not contribute much to extracellular magnetic fields, because they cancel due to the cylindrical shape of neuronal cables, but the main contribution to the extracellular magnetic field is the axial current in neuronal dendrites.¹ It was shown previously that the "microscopic" magnetic fields measured for muscle fibers using miniaturized sensors could be quantitatively

matched by computational models taking into account the axial currents.³ There is also a potentially strong contribution from the conduction currents,⁴ which are currents flowing in the extracellular medium. These currents cause a "secondary magnetic field" which also influences the magnetic field measured outside of the biological tissue.⁵

A main difference between electric and magnetic fields is that the electric potential around neurons crucially depends on the electric properties of the extracellular medium, while magnetic fields are believed to have negligible interactions with the extracellular medium.² Note that some studies have found that the electric properties of extracellular media also affect magnetic fields, because

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they influence the neuronal current sources that produce the field.⁶ We will assume here that there is no such dependence and will consider neurons embedded in resistive (Ohmic) media, with no capacitive or diffusive effects.

There exist well established theories to calculate the microscopic currents generating extracellular electric potentials, such as the neuronal cable theory.⁷ Such theories can be formulated macroscopically, using a mean-field formulation of Maxwell equations,⁸ yielding expressions for the extracellular potential at scales larger than the variations of resistivity and permittivity. Such theories can easily integrate the influence of multiple neuronal current sources and thus can be used to calculate the electric potential resulting from populations of neurons.

However, how to account for “macroscopic” neuronal magnetic fields is less clear. One of the reasons is that the vectorial nature of the magnetic field will lead to partial cancelation of the fields generated by single neurons, in a way that critically depends on their orientation. We derive a mean-field formulation to account for macroscopic neuronal magnetic fields. We start from the concepts of current dipole and current dipole moment. Given the nonstatic conditions that characterize brain activity, we consider the time-dependent generalization of the Biot–Savart law and propose a reformulation of the magneto-quasistatic approach to the brain. We next evaluate the contribution to different types of current sources, such as the synaptic currents and action potential currents, to generate the magnetic field located far away (centimeters) from neurons, such as typically in MEG measurements.

II. CURRENT DIPOLE AND CURRENT DIPOLE MOMENT CONCEPTS

In this section, we discuss the concepts of current dipole and current dipole moment and their relevance to neurons.

A current line segment can be seen as a current dipole. This is a combination of a current sink and a current source of equal magnitude.⁹ The current enters at one end—a current sink—and leaves the segment at the other end—a current source. Synaptic and action potential currents can be described as a set of tiny current dipoles.

In the case of synaptic currents, we have a single current dipole at each synapse. Here, the current either flows into or out of the cell. Transmitter molecules in the synapse change the membrane permeability for specific ions. In the excitatory (depolarization) case, Na⁺ channels open and current flows into the cell. For inhibitory (hyperpolarization) synaptic currents, K⁺ or Cl[−] channels open and current flows out of the cell.

Action potential currents, in turn, can be seen as two oppositely oriented current dipoles forming a current quadrupole.¹ Here two currents flow in opposite directions. Na⁺ ions go into the cell causing the membrane to depolarize and K⁺ ions move out of the cell causing repolarization. These currents produce a change in the membrane potential that lasts about 1 ms, a time scale 10 times smaller than that of a postsynaptic potential lasting tens of ms.¹

The opposite currents creating an action potential produce magnetic fields that tend to cancel each other. Hence, a relatively small magnetic field, compared to that of synaptic currents, is generated in that case. The brain magnetic field¹⁰ is typically up to $5 \times 10^{-13} \text{ T}$ ¹ which is still very small compared to the earth

magnetic field of about $5 \times 10^{-5} \text{ T}$.¹¹ Superconducting quantum interference device (SQUID) detectors are employed to record the MEG signals. These sensors work at liquid helium temperature (about 4 K = −269 °C) and are placed some centimeters away from the scalp.

Our goal is to obtain the long-distance magnetic field produced by ionic currents in small regions containing populations of neurons. The synaptic currents in each region define an overall (time-dependent) current dipole. The dimensions of each region must be small compared to the average distance from the region to the measurement point. Considering this distance in the order of centimeters, the region size can be thought in the order of millimeters. One of those relatively small regions can then have a volume of about 1 mm³. We can take the values of the cortical surface, the average cortical thickness, and the total number of neurons in the human cerebral cortex, given by about $1.8 \times 10^5 \text{ mm}^2$, 2.6 mm, and 16 billion neurons, respectively.¹² We then find about 4.7×10^5 regions of 1 mm³ in the human cerebral cortex, each one with an average estimate of about 3.4×10^4 neurons.

A. Long-distance magnetic field of a current line segment

The Biot–Savart law for a time-independent line current,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu}{4\pi} \mathcal{I} \int d\mathbf{l}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}, \quad (1)$$

can be employed to estimate the magnetic induction $\mathbf{B}_{\text{seg}}(\mathbf{r})$ of a current line segment at a relatively long distance. This leads to the concept of current dipole moment. The measurement point is located at \mathbf{r} . We consider a linear and isotropic medium with the same magnetic permeability μ throughout space. \mathcal{I} is the current and $d\mathbf{l}'$ is a vector element of length in the current flow direction. \mathbf{r}' refers to a given point in the segment.

Given a long distance to the point of interest when compared to the size of the current line segment, any point \mathbf{r}_0 in it can be used to characterize its average position. We have that $|\mathbf{r}' - \mathbf{r}_0| \ll |\mathbf{r} - \mathbf{r}_0|$ for every point in the segment. The long-distance magnetic field can be estimated by considering in Eq. (1) all $\mathbf{r}' \approx \mathbf{r}_0$. We then have for the given segment

$$\mathbf{B}_{\text{seg}}(\mathbf{r}) \approx \frac{\mu}{4\pi} \frac{\mathcal{Q}_{\text{seg}} \times \hat{\mathbf{R}}(\mathbf{r}, \mathbf{r}_0)}{|\mathbf{R}(\mathbf{r}, \mathbf{r}_0)|^2}, \quad (2)$$

where

$$\hat{\mathbf{R}}(\mathbf{r}, \mathbf{r}_0) \approx \frac{\mathbf{R}(\mathbf{r}, \mathbf{r}_0)}{|\mathbf{R}(\mathbf{r}, \mathbf{r}_0)|}, \quad \mathbf{R}(\mathbf{r}, \mathbf{r}_0) = \mathbf{r} - \mathbf{r}_0, \quad (3)$$

and

$$\mathcal{Q}_{\text{seg}} = \mathcal{I} \int_{\mathcal{L}_{\text{seg}}} d\mathbf{l}'. \quad (4)$$

Here, \mathcal{Q}_{seg} is the current dipole moment of the line segment \mathcal{L}_{seg} . Its magnitude is given by the product of the current times the

distance between the ends of the segment. This is similar to an electric dipole moment which is given by the product of the positive charge times the distance between the charges forming the dipole.

Equation (2) shows a decrease with the square of the distance for the magnetic field of a current dipole. This behavior can be expected for synaptic currents due to their dipolar nature.¹ Below, we reproduce that behavior and relate the synaptic current dipole moment of a region to the currents driven by the electric field. It is also shown that the synaptic current dipole moment of a region is related to the *electric* dipole moments of its homogeneous parts with uniform conductivity and permittivity.

Later, we study the long-distance magnetic field of action potential currents. It decreases faster—with the cube of the distance—than in the synaptic case. This is characteristic for the quadrupolar case.¹ We derive such a decrease by approximating to zero the current dipole moment for action potential currents. The long-distance magnetic field in this case is expressed in terms of the corresponding *magnetic* dipole moment.

B. Current dipole moment for volume currents

The current dipole moment \mathcal{Q}_{reg} of a given volume region \mathcal{V}_{reg} can be expressed in terms of the current density $\mathcal{J}(\mathbf{r}')$ at the different points \mathbf{r}' in the region. Considering (4), we write

$$\mathcal{Q}_{reg} = \int_{\mathcal{V}_{reg}} \delta \mathcal{Q}, \quad \delta \mathcal{Q} = \mathcal{I}' d\mathbf{l}' = \mathcal{J}(\mathbf{r}') ds' d\mathbf{l}', \quad (5)$$

where ds' is a surface element perpendicular to the current flow direction at point \mathbf{r}' . Then,

$$\delta \mathcal{Q}_{reg} = \mathcal{J}(\mathbf{r}') ds' d\mathbf{l}' = \mathcal{J}(\mathbf{r}') d^3 r', \quad (6)$$

where $d^3 r'$ is a volume element. The current density $\mathcal{J}(\mathbf{r}')$ is, therefore, the density per unit volume, at point \mathbf{r}' , of the current dipole moment. We thus have

$$\mathcal{Q}_{reg} = \int_{\mathcal{V}_{reg}} \mathcal{J}(\mathbf{r}') d^3 r'. \quad (7)$$

III. BIOT-SAVART LAW AND THE NONSTATIC CASE

Given the nonstatic conditions that characterize brain activity, we discuss here the time-dependent generalization of the Biot-Savart law. Based on the general solution obtained by Jefimenko, we next propose a reformulation of the brain's magneto-quasistatic approximation.

A. The Biot-Savart law in the static case and its time-dependent generalization

The Biot-Savart law for volume currents is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu}{4\pi} \int \mathcal{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r', \quad (8)$$

where the integration is over all space and \mathbf{J} is the free current density which is assumed to be confined to a finite region. As mentioned earlier, we consider a linear and isotropic medium with uniform permeability μ . Equation (8) can be proved to be valid in the static case where both \mathbf{J} and the free charge density ρ are time-independent. It can be shown¹¹ that $\mathbf{B}(\mathbf{r})$ given by (8) satisfies Ampère's law in a differential form

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu \mathbf{J}(\mathbf{r}), \quad (9)$$

provided

$$\nabla \cdot \mathbf{J}(\mathbf{r}) = 0. \quad (10)$$

This characterizes the magnetostatic case where ρ is independent of time t , i.e., $\partial \rho / \partial t = 0$. The continuity equation $\nabla \cdot \mathbf{J} + \partial \rho / \partial t = 0$, expressing charge conservation, becomes Eq. (10).

On the other hand, the brain tissue represents a nonstatic situation and the use of the Biot-Savart law must be justified. Here, the charge density, given by the ion concentrations, depends on time at the different points in space. As noted below, Eq. (8) still holds in the special nonstatic case where the free charge density changes linearly with time.¹³ The continuity equation here implies that the free current density does not depend on time and the same happens with \mathbf{B} as shown by Eq. (8). However, MEG measurements show time-varying magnetic signals necessarily corresponding to time-varying currents in the brain. In this connection, a time-dependent version of Eq. (8), given by

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu}{4\pi} \int \mathcal{J}(\mathbf{r}', t) \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r', \quad (11)$$

could be proposed. Nevertheless, as noted also below, Eq. (11) is warranted in the particular case where $\mathcal{J}(\mathbf{r}', t)$ is linear in time.¹³ Here \mathbf{B} , according to (11), is also linear in time which neither corresponds to the situation in MEG, where oscillating magnetic signals are recorded.

Therefore, the general nonstatic case, where ρ and \mathbf{J} depend arbitrarily on time, must be considered. Then, instead of (9), the Ampère-Maxwell equation,

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \mu \left[\mathcal{J}(\mathbf{r}, t) + \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} \right], \quad (12)$$

must be satisfied. Here, \mathbf{D} is the vector of electric displacement, such that

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = \rho(\mathbf{r}, t). \quad (13)$$

The continuity equation

$$\nabla \cdot \mathcal{J}(\mathbf{r}, t) + \frac{\partial \rho(\mathbf{r}, t)}{\partial t} = 0, \quad (14)$$

being a general statement of charge conservation, remains valid in

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this case. It implies, using Eq. (13), that

$$\nabla \cdot \left[\mathbf{J}(\mathbf{r}, t) + \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} \right] = 0, \tag{15}$$

instead of Eq. (10). This can also be obtained directly from Eq. (12). The terms inside the brackets form the generalized current density used previously to include charge accumulation in cable equations.^{6,8}

Aiming to obtain a time-dependent generalization of the Biot-Savart law, Maxwell's equation $\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0$ can be considered. This allows us to introduce the vector potential $\mathbf{A}(\mathbf{r}, t)$, so that

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t). \tag{16}$$

Substituting this into Eq. (12) results in a Biot-Savart-like expression,^{14,15} which reads

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu}{4\pi} \int \left[\mathbf{J}(\mathbf{r}', t) + \frac{\partial \mathbf{D}(\mathbf{r}', t)}{\partial t} \right] \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r'. \tag{17}$$

This satisfies Eq. (12) by taking into account Eq. (15) and reduces to the Biot-Savart law (8) in the static case.

From a practical point of view, however, Eq. (17) is not helpful as it turns out to be self-referential.^{13,15} Equation (17) requires the electric displacement \mathbf{D} to be known at all points of space. Considering a linear and isotropic medium with permittivity ϵ , we have $\mathbf{D} = \epsilon \mathbf{E}$, where \mathbf{E} is the vector of electric field. The latter, meanwhile, is determined by \mathbf{B} through Faraday's law of induction, $\nabla \times \mathbf{E}(\mathbf{r}, t) = -\partial \mathbf{B}(\mathbf{r}, t) / \partial t$. Therefore, in order to evaluate \mathbf{B} by employing Eq. (17), we must know \mathbf{B} itself at all points of space.

Thus, Eq. (17) cannot be actually used to obtain \mathbf{B} for time-dependent systems. It requires to know the displacement current density $\partial \mathbf{D} / \partial t$ which has \mathbf{B} as its own source.¹⁵ We employ below the proper general solution given to this problem.

B. Jefimenko's equation: The general nonstatic case

A suitable time-dependent generalization of the Biot-Savart law was derived by Jefimenko,^{11,15,16} given by

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu}{4\pi} \int \left[\mathbf{J}(\mathbf{r}', t_{ret}) + \frac{|\mathbf{r} - \mathbf{r}'|}{v} \frac{\partial \mathbf{J}(\mathbf{r}', t_{ret})}{\partial t} \right] \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r'. \tag{18}$$

An infinite, linear, and isotropic medium with uniform permittivity ϵ and permeability μ is assumed. Here,

$$t_{ret} = t - |\mathbf{r} - \mathbf{r}'| / v \tag{19}$$

is the retarded time, and

$$v = 1 / \sqrt{\epsilon \mu}, \tag{20}$$

is the velocity of light in the medium. Equation (18) can be

obtained from (16) with

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_{ret})}{|\mathbf{r} - \mathbf{r}'|} d^3 r'. \tag{21}$$

The current sources in the vector potential must thus be evaluated at the retarded time (19). The status of the sources at this time determines \mathbf{B} at the later time t . It takes a time $|\mathbf{r} - \mathbf{r}'| / v$ for an electromagnetic signal to travel from a source at point \mathbf{r}' to the point \mathbf{r} where \mathbf{B} is measured. In this sense, Eq. (18) is a causality-based general solution of Maxwell's equations.

Equation (18) simplifies to the static form (8) of the Biot-Savart law in the case of a time-independent current density. This does not require the free charge density to be time-independent as well but, according to the continuity Eq. (14), it may change linearly with time.¹³ It can also be verified that Eq. (18) takes the non-retarded form (11) when the current density is linear in time,¹³ that is, when $\mathbf{J}(\mathbf{r}', t) = \mathbf{J}(\mathbf{r}', 0) + t \mathbf{C}(\mathbf{r}')$, where $\mathbf{C}(\mathbf{r}')$ is a time-independent function.

C. Magneto-quasistatic approximation

It is worth considering Jefimenko's equation (18) in the quasistatic case where the current density $\mathbf{J}(\mathbf{r}', t)$, as well as the corresponding charge density, changes relatively slowly with time.¹³ The characteristic time for this change to occur must be large compared to the signal traveling time $t_{trav}(\mathbf{r}, \mathbf{r}') = |\mathbf{r} - \mathbf{r}'| / v$. Considering (20), we have

$$t_{trav}(\mathbf{r}, \mathbf{r}') = |\mathbf{r} - \mathbf{r}'| \sqrt{\epsilon \mu}. \tag{22}$$

Similar to the static case, the current density \mathbf{J} in Eq. (18) is assumed to be confined to a finite region of space. The source point \mathbf{r}' which is farthest from the field point \mathbf{r} determines the largest traveling time $t_{trav}(\mathbf{r}, \mathbf{r}')$.

In the quasistatic case, it is useful to expand the current density $\mathbf{J}(\mathbf{r}', t_{ret}) = \mathbf{J}[\mathbf{r}', t - t_{trav}(\mathbf{r}, \mathbf{r}')] in Eq. (18) as a Taylor's series, so that$

$$\begin{aligned} \mathbf{J}(\mathbf{r}', t_{ret}) &= \mathbf{J}(\mathbf{r}', t) - t_{trav}(\mathbf{r}, \mathbf{r}') \frac{\partial \mathbf{J}(\mathbf{r}', t)}{\partial t} \\ &+ \frac{1}{2} t_{trav}^2(\mathbf{r}, \mathbf{r}') \frac{\partial^2 \mathbf{J}(\mathbf{r}', t)}{\partial t^2} + \dots \end{aligned} \tag{23}$$

Consequently, the second term inside the brackets in Eq. (18), equal to $t_{trav}(\mathbf{r}, \mathbf{r}') \partial \mathbf{J}(\mathbf{r}', t_{ret}) / \partial t$, expands as

$$t_{trav}(\mathbf{r}, \mathbf{r}') \frac{\partial \mathbf{J}(\mathbf{r}', t)}{\partial t} - t_{trav}^2(\mathbf{r}, \mathbf{r}') \frac{\partial^2 \mathbf{J}(\mathbf{r}', t)}{\partial t^2} + \dots \tag{24}$$

The first order terms here and in the original expansion cancel

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each other. This leads to

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu}{4\pi} \int \left[\mathbf{J}(\mathbf{r}', t) - \frac{1}{2} t_{trav}^2(\mathbf{r}, \mathbf{r}') \frac{\partial^2 \mathbf{J}(\mathbf{r}', t)}{\partial t^2} + \dots \right] \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r'. \quad (25)$$

The time dependency of the current density, and correspondingly that of \mathbf{B} , can be described through the different frequency components. A component of angular frequency w corresponds to a time oscillation with period $T(w) = 2\pi/w$. Here, it is helpful to notice that the main contributions in neuromagnetism are from frequencies below a typical value $f_{max} = 100$ Hz.¹ We thus consider a maximum angular frequency

$$w_{max} = 2\pi f_{max}, \quad (26)$$

and write

$$\mathbf{J}(\mathbf{r}', t) \approx \int_0^{w_{max}} e^{iwt} \tilde{\mathbf{J}}(\mathbf{r}', w) dw. \quad (27)$$

Substituting this into Eq. (25) yields

$$\mathbf{B}(\mathbf{r}, t) \approx \int_0^{w_{max}} e^{iwt} \tilde{\mathbf{B}}(\mathbf{r}, w) dw, \quad (28)$$

where

$$\tilde{\mathbf{B}}(\mathbf{r}, w) = \frac{\mu}{4\pi} \int \left[\tilde{\mathbf{J}}(\mathbf{r}', w) + \frac{1}{2} w^2 t_{trav}^2(\mathbf{r}, \mathbf{r}') \times \tilde{\mathbf{J}}(\mathbf{r}', w) + \dots \right] \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r'. \quad (29)$$

In the quasistatic case, we must have at every source point \mathbf{r}'

$$w^2 t_{trav}^2(\mathbf{r}, \mathbf{r}') \ll 1, \quad \text{for all } 0 < w < w_{max}. \quad (30)$$

Since $w t_{trav}(\mathbf{r}, \mathbf{r}') \leq w_{max} t_{trav}(\mathbf{r}, \vec{r}')$, it is enough to require that

$$w_{max}^2 t_{trav}^2(\mathbf{r}, \vec{r}') \ll 1. \quad (31)$$

This describes the quasistatic case through the current frequency component with the shortest oscillation period $2\pi/w_{max} = 1/f_{max}$. This time must be large enough compared to the largest traveling time $t_{trav}(\mathbf{r}, \vec{r}')$. In that case, Eq. (29) can be replaced by

$$\tilde{\mathbf{B}}(\mathbf{r}, w) \approx \frac{\mu}{4\pi} \int \tilde{\mathbf{J}}(\mathbf{r}', w) \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r', \quad (32)$$

for all $0 < w < w_{max}$. Using (28) and (27), we then have

$$\mathbf{B}(\mathbf{r}, t) \approx \frac{\mu}{4\pi} \int \mathbf{J}(\mathbf{r}', t) \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r'. \quad (33)$$

This is similar to the static form (8) of the Biot–Savart law. The expression (33) is the quasistatic approximation of the general solution (18) of Maxwell’s equations. It is valid provided (31) is satisfied.

We note that a similar expression to (31) is well known for the magneto-quasistatic approximation of Maxwell’s equations,^{17,18} as well as their electro-quasistatic approximation.^{17–19} In these cases, however, instead of the traveling distance $|\mathbf{r} - \vec{r}'|$ in Eq. (22), from the most distant source point \vec{r}' to the field point \mathbf{r} , a typical or characteristic length scale of the medium is employed. The system dimensions are assumed to be all within a factor of two or so of each other.¹⁸ The typical length is then used to normalize¹⁷ or approximate¹⁸ all spatial derivatives in Maxwell’s equations. In gray matter, the typical length is approximated by the cortical thickness.^{17,19} The approaches in Refs. 17–19 are not based on the general solution of Maxwell’s equations. This is given in the magnetic case by Eq. (18). As shown above, this leads to the expression (31) and therefore, to the traveling distance $|\mathbf{r} - \vec{r}'|$ as the appropriate length to define the quasistatic dynamics.

D. The quasistatic case of the brain

In the following, we check whether the requirement (31) is fulfilled in the specific case of the brain. According to Eqs. (26) and (22), we must have

$$(2\pi f_{max})^2 |\mathbf{r} - \vec{r}'|^2 \epsilon \mu \ll 1. \quad (34)$$

As mentioned above, we typically have $f_{max} = 100$ Hz.¹ We consider the distance $|\mathbf{r} - \vec{r}'|$ mainly given by the average size of the human brain of about 15 cm. This is much larger than the cortical thickness of the order of 1 mm. The distance of about few centimeters between sensors and the scalp is also relatively small. We then take $|\mathbf{r} - \vec{r}'| \approx 0.1$ m. With regard to the permittivity, we have $\epsilon \approx 10^7 \epsilon_0$ for gray matter at frequencies below 100 Hz.^{17,20} Here $\epsilon_0 \approx 8.85 \times 10^{-12}$ C²/(N · m²) is the permittivity of free space. A magnetic permeability close to that of free space is typical for biological tissues.¹⁷ We then assume $\mu \approx \mu_0 \approx 1.26 \times 10^{-6}$ N/A², and obtain²¹

$$(2\pi f_{max})^2 |\mathbf{r} - \vec{r}'|^2 \epsilon \mu \approx 4 \times 10^{-7} \ll 1. \quad (35)$$

The condition (31) is thus very well satisfied, indicating a quasistatic dynamics. Therefore, we take (33) as a valid expression to calculate $\mathbf{B}(\mathbf{r}, t)$.

In closing this section, we discuss about a condition which is different from (31) and that has been proposed in Ref. 1 to justify the validity of the expression (33) in the quasistatic case. The analysis in Ref. 1 is based on Ampère–Maxwell equation, given here by Eq. (12). The frequency components of the displacement term $\partial \mathbf{D}(\mathbf{r}, t) / \partial t$ are compared to those of the ohmic part of the current

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density $J(\mathbf{r}, t)$. They are given, respectively, by $i\omega\epsilon\tilde{E}(\mathbf{r}, \omega)$ and $\sigma\tilde{E}(\mathbf{r}, \omega)$, where $\tilde{E}(\mathbf{r}, \omega)$ is the frequency component of the electric field, and σ is the electrical conductivity of the medium. The ratio between the absolute values of those frequency components leads then to the condition

$$w_{\max}\tau_{MW} \ll 1, \tag{36}$$

where $\tau_{MW} = \epsilon/\sigma$ is the Maxwell–Wagner reaction time.

According to (36), the shortest oscillation period $2\pi/w_{\max} = 1/f_{\max}$ must be large enough compared to the Maxwell–Wagner time. If that is the case, then the displacement term can be neglected and the Ampère–Maxwell equation can be replaced by $\nabla \times \mathbf{B}(\mathbf{r}, t) \approx \mu \mathbf{J}(\mathbf{r}, t)$. This is similar to the Ampère’s law (9) describing the static case and results in the expression (33), as can be seen from Eq. (17).

The expressions (36) and (31) are clearly different from one another. In the first place, one is linear while the other is quadratic and second, they define the quasistatic dynamics in terms of essentially different times. These are the Maxwell–Wagner reaction time and the traveling time which also differ quantitatively. Using $\epsilon \approx 10^7\epsilon_0$ as before,^{17,20} and $\sigma \approx 0.3\Omega^{-1} \cdot \text{m}^{-1}$ for gray matter at frequencies below 100 Hz,^{1,17} we have $\tau_{MW} \approx 3 \times 10^{-4}$ s. Taking $|\mathbf{r} - \mathbf{r}'| \approx 0.1$ m, $\epsilon \approx 10^7\epsilon_0$, and $\mu \approx \mu_0$, we have $t_{\text{trav}}(\mathbf{r}, \mathbf{r}') = |\mathbf{r} - \mathbf{r}'|/\sqrt{\epsilon\mu} \approx 1\mu\text{s}$. The expression (36) thus requires a relatively slower dynamics than in the case of the expression (31).

Now, we evaluate the fulfillment of the requirement (36). We use the typical value $f_{\max} = 100\text{ Hz}$ ¹ and $\tau_{MW} \approx 3 \times 10^{-4}$ s which was obtained before. This yields

$$w_{\max}\tau_{MW} = 2\pi f_{\max}\tau_{MW} \approx 0.2. \tag{37}$$

The condition (36) is thus minimally satisfied. We note that a permittivity $\epsilon \approx 10^5\epsilon_0$ was considered in Ref. 1 giving $w_{\max}\tau_{MW} \approx 2 \times 10^{-3} \ll 1$. However, according to Refs. 17 and 20, $10^5\epsilon_0$ is an underestimated value of the permittivity for gray matter at frequencies below 100 Hz.

The fact that the condition (36) is minimally fulfilled indicates that it is not a very good approximation to neglect the displacement term in the Ampère–Maxwell equation. Similar conclusions were also reached in previous modeling studies of neuronal electric and magnetic signals.^{6,22} That is the case according to typical values of frequency, permittivity and conductivity in the brain. Neglecting the displacement term is not a safe option and the general solution of Maxwell’s equations, given above by Eq. (18), must be considered.

The expression (36), which corresponds to neglecting the displacement term in the Ampère–Maxwell equation, is not a necessary condition to justify the validity of the expression (33) in the quasistatic case. As pointed out after Eq. (17), the electric displacement $\mathbf{D} = \epsilon\mathbf{E}$ depends on \mathbf{B} itself through Faraday’s law of induction, $\nabla \times \mathbf{E}(\mathbf{r}, t) = -\partial\mathbf{B}(\mathbf{r}, t)/\partial t$. Therefore, the displacement current density $\partial\mathbf{D}/\partial t$ must not be considered as a source of \mathbf{B} in the conventional sense.¹⁵ The time-dependent generalization of the Biot–Savart law, Eq. (18), does not contain indeed the displacement current density but only \mathbf{J} . Equation (18) introduces the signal

traveling time $|\mathbf{r} - \mathbf{r}'|/v$ in terms of which the quasistatic dynamics can be defined. Correspondingly, the Maxwell–Wagner time is suitable for a quasistatic approximation of the Ampère–Maxwell equation but not of the generalized Biot–Savart law (18).

IV. MAGNETIC FIELD OF THE DIFFERENT CURRENTS

In this section, we first give an overview of the magnetic field of the different neuronal current sources. A formalism is then developed for the magnetic field of synaptic and action potential currents at a relatively distant point.

According to the previous section, we take (33) as a valid expression to calculate $\mathbf{B}(\mathbf{r}, t)$. Considering the typical value $\mu \approx \mu_0$,¹⁷ it becomes

$$\mathbf{B}(\mathbf{r}, t) \approx \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}', t) \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3r'. \tag{38}$$

The current density \mathbf{J} can be written as the sum of two components.⁴ First, the conduction current density

$$\mathbf{J}^c(\mathbf{r}, t) = \sigma(\mathbf{r}, t)\mathbf{E}(\mathbf{r}, t), \tag{39}$$

which corresponds to the current caused by the electric field \mathbf{E} . This is produced by the charge carriers in the conducting medium. In the electro-quasistatic approximation¹⁹ we have $\nabla \times \mathbf{E}(\mathbf{r}, t) = -\partial\mathbf{B}(\mathbf{r}, t)/\partial t \approx \mathbf{0}$. This allows to use the scalar potential $\Phi(\mathbf{r}, t)$ so that $\mathbf{E}(\mathbf{r}, t) \approx -\nabla\Phi(\mathbf{r}, t)$. Therefore, we have

$$\mathbf{J}^c(\mathbf{r}, t) \approx -\sigma(\mathbf{r}, t)\nabla\Phi(\mathbf{r}, t). \tag{40}$$

Here, the medium is described as a nonhomogeneous conductor with conductivity $\sigma(\mathbf{r}, t)$ which is modeled on a cellular scale. On the same microscopic, micrometer scale, we have a second component of the current density corresponding to the impressed currents⁵ and represented by $\mathbf{J}^i(\mathbf{r}, t)$. This stands for the current density which is not produced by the electric field \mathbf{E} but mainly, by gradients of concentration in diffusion processes and electromotive chemical reactions in the cell pump mechanism. Then,

$$\mathbf{J}(\mathbf{r}, t) = \mathbf{J}^c(\mathbf{r}, t) + \mathbf{J}^i(\mathbf{r}, t), \tag{41}$$

with the conduction current density $\mathbf{J}^c(\mathbf{r}, t)$ given by (40).

We assume the impressed current density mainly given by a current density $\mathbf{J}^s(\mathbf{r}, t)$ corresponding to synaptic currents, and a current density $\mathbf{J}^a(\mathbf{r}, t)$ corresponding to action potential currents. That is, we have

$$\mathbf{J}^i(\mathbf{r}, t) \approx \mathbf{J}^s(\mathbf{r}, t) + \mathbf{J}^a(\mathbf{r}, t). \tag{42}$$

Thus,

$$\mathbf{J}(\mathbf{r}, t) \approx \mathbf{J}^c(\mathbf{r}, t) + \mathbf{J}^s(\mathbf{r}, t) + \mathbf{J}^a(\mathbf{r}, t), \tag{43}$$

and the expression (38) can be written in the form

$$\mathbf{B}(\mathbf{r}, t) \approx \mathbf{B}^c(\mathbf{r}, t) + \mathbf{B}^s(\mathbf{r}, t) + \mathbf{B}^a(\mathbf{r}, t), \tag{44}$$

where

$$\mathbf{B}^c(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \mathbf{J}^c(\mathbf{r}', t) \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r', \quad (45)$$

$$\mathbf{B}^s(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \mathbf{J}^s(\mathbf{r}', t) \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r', \quad (46)$$

and

$$\mathbf{B}^a(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \mathbf{J}^a(\mathbf{r}', t) \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r', \quad (47)$$

correspond, respectively, to the conduction, synaptic, and action potential currents.

We note that the case of conduction currents, given by the expressions (45) and (40), has been solved exactly.²³ Next, we look at the synaptic and action potential cases.

A. Long-distance magnetic field of synaptic currents

We estimate the synaptic contribution (46) by considering several relatively small regions in the medium, so that

$$\mathbf{B}^s(\mathbf{r}, t) = \sum_{reg} \mathbf{B}_{reg}^s(\mathbf{r}, t), \quad (48)$$

where

$$\mathbf{B}_{reg}^s(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}_{reg}} \mathbf{J}^s(\mathbf{r}', t) \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r', \quad (49)$$

corresponds to a region of volume \mathcal{V}_{reg} . The dimensions of each region must be small compared to the average distance from the region to the field point \mathbf{r} . Taking any point \mathbf{r}_{reg} in the region, we must have

$$|\mathbf{r}' - \mathbf{r}_{reg}| \ll |\mathbf{r} - \mathbf{r}_{reg}|, \quad (50)$$

for every region point \mathbf{r}' .

Equation (49) can be written in the form

$$\mathbf{B}_{reg}^s(\mathbf{r}, t) = \nabla \times \mathbf{A}_{reg}^s(\mathbf{r}, t), \quad (51)$$

with the vector potential

$$\mathbf{A}_{reg}^s(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}_{reg}} \frac{\mathbf{J}^s(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d^3 r', \quad (52)$$

We expand the factor $1/|\mathbf{r} - \mathbf{r}'|$ around $\mathbf{r}' = \mathbf{r}_{reg}$. Up to the first order term, we have

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} \approx \frac{1}{|\mathbf{r} - \mathbf{r}_{reg}|} + \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \Big|_{\mathbf{r}'=\mathbf{r}_{reg}} \cdot (\mathbf{r}' - \mathbf{r}_{reg}), \quad (53)$$

and therefore

$$\begin{aligned} \mathbf{A}_{reg}^s(\mathbf{r}, t) &\approx \frac{\mu_0}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}_{reg}|} \int_{\mathcal{V}_{reg}} \mathbf{J}^s(\mathbf{r}', t) d^3 r' + \frac{\mu_0}{4\pi |\mathbf{r} - \mathbf{r}_{reg}|^3} \int_{\mathcal{V}_{reg}} \mathbf{J}^s(\mathbf{r}', t) \\ &\quad \times [(\mathbf{r} - \mathbf{r}_{reg}) \cdot (\mathbf{r}' - \mathbf{r}_{reg})] d^3 r'. \end{aligned} \quad (54)$$

Similarly to Eq. (7), the quantity

$$\mathcal{Q}_{reg}^s(t) = \int_{\mathcal{V}_{reg}} \mathbf{J}^s(\mathbf{r}', t) d^3 r', \quad (55)$$

in the first term of the expression (54), defines the synaptic current dipole moment. Assuming this is nonzero vector, we approximate the vector potential by

$$\mathbf{A}_{reg}^s(\mathbf{r}, t) \approx \frac{\mu_0}{4\pi} \frac{\mathcal{Q}_{reg}^s(t)}{|\mathbf{r} - \mathbf{r}_{reg}|}. \quad (56)$$

Using then Eq. (51), we have

$$\mathbf{B}_{reg}^s(\mathbf{r}, t) \approx \frac{\mu_0}{4\pi} \frac{\mathcal{Q}_{reg}^s(t) \times \hat{\mathbf{R}}(\mathbf{r}, \mathbf{r}_{reg})}{|\mathbf{R}(\mathbf{r}, \mathbf{r}_{reg})|^2}, \quad (57)$$

where

$$\hat{\mathbf{R}}(\mathbf{r}, \mathbf{r}_{reg}) \approx \frac{\mathbf{R}(\mathbf{r}, \mathbf{r}_{reg})}{|\mathbf{R}(\mathbf{r}, \mathbf{r}_{reg})|}, \quad (58)$$

with

$$\mathbf{R}(\mathbf{r}, \mathbf{r}_{reg}) = \mathbf{r} - \mathbf{r}_{reg}. \quad (59)$$

The expression (57) shows a decrease with the square of the distance for synaptic currents which is characteristic for the dipolar case.¹

B. Synaptic current dipole moment

We must find the synaptic current dipole moment, $\mathcal{Q}_{reg}^s(t)$, defined by (55). Integrating the expression (43) of the current density yields

$$\mathcal{Q}_{reg}(t) \approx \mathcal{Q}_{reg}^c(t) + \mathcal{Q}_{reg}^s(t) + \mathcal{Q}_{reg}^a(t), \quad (60)$$

where

$$\mathcal{Q}_{reg}(t) = \int_{\mathcal{V}_{reg}} \mathbf{J}(\mathbf{r}', t) d^3 r', \quad (61)$$

is the total current dipole moment of the region. Correspondingly,

$$\mathcal{Q}_{reg}^c(t) = \int_{V_{reg}} \mathbf{J}^c(\mathbf{r}', t) d^3 r', \quad (62)$$

is the conduction current dipole moment, and

$$\mathcal{Q}_{reg}^a(t) = \int_{V_{reg}} \mathbf{J}^a(\mathbf{r}', t) d^3 r', \quad (63)$$

is the current dipole moment for action potential currents. We consider an approximately cylindrical shape for the neuronal cables. Then, we have opposite transverse currents on opposite sides of the cables whose contributions, in the current dipole moment (63) for action potential currents, approximately cancel each other. We then take

$$\mathcal{Q}_{reg}^a(t) \approx \mathbf{0}. \quad (64)$$

Thus, from the expression (60), we have

$$\mathcal{Q}_{reg}^s(t) \approx \mathcal{Q}_{reg}(t) - \mathcal{Q}_{reg}^c(t). \quad (65)$$

We assume quasistatic conditions such that, based on the continuity equation, $\nabla' \cdot \mathbf{J}(\mathbf{r}', t) = -\partial \rho(\mathbf{r}', t) / \partial t \approx 0$.⁴ Then, provided the contribution of the impressed currents at the region boundary is small compared to that of the conduction currents,²⁴ the total current dipole moment (61) is given by

$$\mathcal{Q}_{reg}(t) \approx \int_{S_{reg}} \mathbf{r}' [\mathbf{J}^c(\mathbf{r}', t) \cdot d\mathbf{S}'], \quad (66)$$

as shown in the Appendix. Here, S_{reg} is the surface bounding the region. Substituting (66) into (65) gives

$$\mathcal{Q}_{reg}^s(t) \approx \int_{S_{reg}} \mathbf{r}' [\mathbf{J}^c(\mathbf{r}', t) \cdot d\mathbf{S}'] - \mathcal{Q}_{reg}^c(t). \quad (67)$$

Equivalently (see the Appendix), the expression (66) can be written in the form

$$\mathcal{Q}_{reg}(t) \approx \int_{V_{reg}} \mathbf{r}' \nabla' \cdot \mathbf{J}^c(\mathbf{r}', t) d^3 r' + \mathcal{Q}_{reg}^c(t), \quad (68)$$

which can also be obtained independently by integrating Eq. (41) of the current density (see the Appendix). Importantly, (66) and (68) are independent of the chosen coordinate origin if, and only if, the net impressed current crossing the region boundary is approximately zero (see the Appendix). Using (65) and (68), we

obtain

$$\mathcal{Q}_{reg}^s(t) \approx \int_{V_{reg}} \mathbf{r}' \nabla' \cdot \mathbf{J}^c(\mathbf{r}', t) d^3 r'. \quad (69)$$

We thus have two equivalent expressions, (67) and (69), of the synaptic current dipole moment. We observe that they are given in terms of the conduction current density. In the following, we use its expression (40) to write (67) and (69) in terms of the scalar potential.

1. First expression in terms of the scalar potential

Here we consider the expression (67). By using (62) and (40), it takes the form

$$\mathcal{Q}_{reg}^s(t) \approx - \int_{S_{reg}} \sigma(\mathbf{r}', t) \mathbf{r}' [\nabla' \Phi(\mathbf{r}', t) \cdot d\mathbf{S}'] + \int_{V_{reg}} \sigma(\mathbf{r}', t) \nabla' \Phi(\mathbf{r}', t) d^3 r'. \quad (70)$$

The second term on the right-hand side here can be transformed as follows.⁴ The nonhomogeneity in the conductivity can be taken into account by visualizing the region as composed by m homogeneous parts with uniform conductivity $\sigma_{reg}^{(i)}(t)$, where $i = 1, 2, \dots, m$. Representing the volume of the i th part and its bounding surface by $V_{reg}(i)$ and $S_{reg}(i)$, respectively, we have

$$\int_{V_{reg}} \sigma(\mathbf{r}', t) \nabla' \Phi(\mathbf{r}', t) d^3 r' = \sum_{i=1}^m \sigma_{reg}^{(i)}(t) \int_{S_{reg}(i)} \Phi(\mathbf{r}', t) d\mathbf{S}'_i, \quad (71)$$

where the identity

$$\int_{V_{reg}(i)} \nabla' \Phi(\mathbf{r}', t) d^3 r' = \int_{S_{reg}(i)} \Phi(\mathbf{r}', t) d\mathbf{S}'_i, \quad (72)$$

is utilized. Here, $d\mathbf{S}'_i$ is the outward normal vector of surface element for the i th part. Next, the surface $S_{reg}(i)$ can be divided into an internal component $S_{reg}^{int}(i)$ which lies inside the given region, and an external component $S_{reg}^{ext}(i)$ which makes part of the bounding surface S_{reg} of the considered region. We then write

$$\int_{S_{reg}(i)} \Phi(\mathbf{r}', t) d\mathbf{S}'_i = \int_{S_{reg}^{int}(i)} \Phi(\mathbf{r}', t) d\mathbf{S}'_i + \int_{S_{reg}^{ext}(i)} \Phi(\mathbf{r}', t) d\mathbf{S}'_i. \quad (73)$$

Finally, representing by $S_{reg}(i, j)$ the common bounding surface between two homogeneous parts i and j , the current dipole

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moment (70) is given by

$$\begin{aligned} \mathcal{Q}_{reg}^s(t) \approx & - \int_{S_{reg}} \sigma(\mathbf{r}', t) \mathbf{r}' [\nabla' \Phi(\mathbf{r}', t) \cdot d\mathbf{S}'] + \sum_{\langle ij \rangle} \left[\sigma_{reg}^{(i)}(t) - \sigma_{reg}^{(j)}(t) \right] \\ & \times \int_{S_{reg}^{(i,j)}} \Phi(\mathbf{r}', t) d\mathbf{S}'_{ij} + \int_{S_{reg}} \sigma(\mathbf{r}', t) \Phi(\mathbf{r}', t) d\mathbf{S}'. \end{aligned} \quad (74)$$

Here, the sum is over all different pairs of adjacent homogeneous parts in the given region. The normal vector of surface element $d\mathbf{S}'_{ij}$ is directed from the i th part to the j th part.

The above expression obtained for synaptic currents can also be considered valid for impressed currents. By integrating (42) and using (64), we see that the synaptic current dipole moment results to be approximately equal to the impressed current dipole moment. We note that an expression for this is given in Ref. 23. However, such expression does not contain the first term on the right-hand side of (74). This term corresponds to the total current dipole moment (66). In Ref. 23, an external medium with zero conductivity is assumed implying that the normal component of the conduction current density vanishes on both sides close to the region boundary. In contrast, we assume an external medium with nonzero conductivity which corresponds to a non-isolated region. As such, the last expression obtained above can be seen as a generalization of the expression presented in Ref. 23.

2. Second expression in terms of the scalar potential

Here, we consider the expression (69). Using (40), we have

$$\mathcal{Q}_{reg}^s(t) \approx - \int_{V_{reg}} \mathbf{r}' \nabla' \cdot [\sigma(\mathbf{r}', t) \nabla' \Phi(\mathbf{r}', t)] d^3 r'. \quad (75)$$

Taking into account the nonhomogeneity in the conductivity as above, by considering the region as composed by m homogeneous parts with uniform conductivity $\sigma_{reg}^{(i)}(t)$, where $i = 1, 2, \dots, m$, we get

$$\mathcal{Q}_{reg}^s(t) \approx - \sum_{i=1}^m \sigma_{reg}^{(i)}(t) \int_{V_{reg}^{(i)}} \mathbf{r}' \nabla'^2 \Phi(\mathbf{r}', t) d^3 r'. \quad (76)$$

Here, different from (74), the synaptic current dipole moment is in terms of the second spatial derivatives of the scalar potential within each homogeneous part.

The last expression can also be written in terms of the free charge density ρ . In the electro-quasistatic approximation, we have $\mathbf{E}(\mathbf{r}', t) \approx -\nabla' \Phi(\mathbf{r}', t)$. This leads to Poisson's expression

$$\nabla'^2 \Phi(\mathbf{r}', t) \approx -\nabla' \cdot \mathbf{E}(\mathbf{r}', t). \quad (77)$$

Describing the i th part by a linear and isotropic medium with uniform permittivity $\epsilon_{reg}^{(i)}(t)$, we have $\mathbf{D}(\mathbf{r}', t) = \epsilon_{reg}^{(i)}(t) \mathbf{E}(\mathbf{r}', t)$.

Considering Maxwell's equation (13), we then obtain

$$\mathcal{Q}_{reg}^s(t) \approx \sum_{i=1}^m \frac{\sigma_{reg}^{(i)}(t)}{\epsilon_{reg}^{(i)}(t)} \int_{V_{reg}^{(i)}} \mathbf{r}' \rho(\mathbf{r}', t) d^3 r'. \quad (78)$$

This can be written in a more compact manner by using

$$\tau_{reg}^{(i)}(t) = \frac{\epsilon_{reg}^{(i)}(t)}{\sigma_{reg}^{(i)}(t)}, \quad (79)$$

which is the Maxwell-Wagner time for the i th part, and

$$\mathbf{p}_{reg}^{(i)}(t) = \int_{V_{reg}^{(i)}} \mathbf{r}' \rho(\mathbf{r}', t) d^3 r', \quad (80)$$

which is the electric dipole moment of the i th part. Then,

$$\mathcal{Q}_{reg}^s(t) \approx \sum_{i=1}^m \frac{\mathbf{p}_{reg}^{(i)}(t)}{\tau_{reg}^{(i)}(t)}. \quad (81)$$

Thus, through the corresponding Maxwell-Wagner time, the synaptic current dipole moment of a region is related to the electric dipole moment of each homogeneous part.

C. Long-distance magnetic field of action potential currents

We consider the case of action potential currents, given by Eq. (47). As with the synaptic case, we think of several relatively small regions in the medium, so that

$$\mathbf{B}^a(\mathbf{r}, t) = \sum_{reg} \mathbf{B}_{reg}^a(\mathbf{r}, t), \quad (82)$$

where

$$\mathbf{B}_{reg}^a(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{V_{reg}} \mathbf{J}^a(\mathbf{r}', t) \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r', \quad (83)$$

corresponds to a region of volume V_{reg} . As above, the dimensions of each region must be small compared to the average distance from the region to the field point \mathbf{r} . Taking any point \mathbf{r}_{reg} in the region, we must have

$$|\mathbf{r}' - \mathbf{r}_{reg}| \ll |\mathbf{r} - \mathbf{r}_{reg}|, \quad (84)$$

for every region point \mathbf{r}' .

Equation (83) can be written in the form

$$\mathbf{B}_{reg}^a(\mathbf{r}, t) = \nabla \times \mathbf{A}_{reg}^a(\mathbf{r}, t), \quad (85)$$

with the vector potential

$$A_{reg}^a(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{V_{reg}} \frac{\mathbf{J}^a(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d^3 r' \quad (86)$$

Employing the expansion (53) of the factor $1/|\mathbf{r} - \mathbf{r}'|$ around $\mathbf{r}' = \mathbf{r}_{reg}$, and considering (64), we have

$$A_{reg}^a(\mathbf{r}, t) \approx \frac{\mu_0}{4\pi |\mathbf{r} - \mathbf{r}_{reg}|^3} \times \int_{V_{reg}} \mathbf{J}^a(\mathbf{r}', t) [(\mathbf{r} - \mathbf{r}_{reg}) \cdot \mathbf{r}'] d^3 r' \quad (87)$$

By using (85), we then obtain

$$\begin{aligned} \mathbf{B}_{reg}^a(\mathbf{r}, t) \approx & -\frac{3\mu_0}{4\pi} \frac{\hat{\mathbf{R}}(\mathbf{r}, \mathbf{r}_{reg})}{|\mathbf{R}(\mathbf{r}, \mathbf{r}_{reg})|^3} \times \int_{V_{reg}} \mathbf{J}^a(\mathbf{r}', t) [\hat{\mathbf{R}}(\mathbf{r}, \mathbf{r}_{reg}) \cdot \mathbf{r}'] d^3 r' \\ & + \frac{\mu_0}{2\pi} \frac{\mathbf{m}_{reg}^a(t)}{|\mathbf{R}(\mathbf{r}, \mathbf{r}_{reg})|^3}, \end{aligned} \quad (88)$$

where $\mathbf{R}(\mathbf{r}, \mathbf{r}_{reg})$ and $\mathbf{r}(\mathbf{r}, \mathbf{r}_{reg})$ are given by (58) and (59), and

$$\mathbf{m}_{reg}^a(t) = \frac{1}{2} \int_{V_{reg}} \mathbf{r}' \times \mathbf{J}^a(\mathbf{r}', t) d^3 r', \quad (89)$$

is the magnetic dipole moment for action potential currents. This, as well as the integral in (88), is independent of the chosen coordinate origin, which can be shown by taking (64) into consideration.

The expression (88) shows a decrease with the cube of the distance for action potential currents. This is characteristic for the quadrupolar case.¹ Here, we have a faster decrease with the distance than in the quadratic synaptic case given above by (57).

We note that the expression (88) may be written in a familiar form for a magnetic dipole, provided $\nabla \cdot \mathbf{J}^a \approx 0$ in the given region, and $\mathbf{J}^a(\mathbf{r}', t) = \mathbf{0}$ on its boundary.²⁵ In that case, the integral in (88) can be expressed through the magnetic dipole moment (89), by

$$\int_{V_{reg}} \mathbf{J}^a(\mathbf{r}', t) [\hat{\mathbf{R}}(\mathbf{r}, \mathbf{r}_{reg}) \cdot \mathbf{r}'] d^3 r' \approx \mathbf{m}_{reg}^a(t) \times \hat{\mathbf{R}}(\mathbf{r}, \mathbf{r}_{reg}), \quad (90)$$

and (88) then takes the form

$$\begin{aligned} \mathbf{B}_{reg}^a(\mathbf{r}, t) \approx & \frac{\mu_0}{4\pi} \left\{ 3\hat{\mathbf{R}}(\mathbf{r}, \mathbf{r}_{reg}) \left[\hat{\mathbf{R}}(\mathbf{r}, \mathbf{r}_{reg}) \cdot \mathbf{m}_{reg}^a(t) \right] \right. \\ & \left. - \mathbf{m}_{reg}^a(t) \right\} \frac{1}{|\mathbf{R}(\mathbf{r}, \mathbf{r}_{reg})|^3}. \end{aligned} \quad (91)$$

Additionally, using (90), the vector potential (87) reads

$$A_{reg}^a(\mathbf{r}, t) \approx \frac{\mu_0}{4\pi} \mathbf{m}_{reg}^a(t) \frac{\hat{\mathbf{R}}(\mathbf{r}, \mathbf{r}_{reg})}{|\mathbf{R}(\mathbf{r}, \mathbf{r}_{reg})|^2}, \quad (92)$$

which, through (85), can also be utilized to obtain (91).

V. DISCUSSION

Here, we have developed a method for calculating the long-distance magnetic field generated by neuronal populations. In the following, we first discuss the reformulation of the brain's magneto-quasistatic approximation that we proposed, in relation to previous work. We next discuss the formalism developed for the magnetic field of synaptic and action potential currents at a relatively distant point.

A. The brain's magneto-quasistatic approximation

Since MEG measurements show magnetic signals that change over time, we first considered the time-dependent generalization of the Biot-Savart law given by Eq. (18). As the main contributions in neuromagnetism come from relatively low frequencies that are below a typical value $f_{max} = 100$ Hz,¹ we then followed the approach of Griffiths and Heald¹³ to approximate Eq. (18) in the quasistatic case. This method differs from the previous ones^{1,17-19} where the treatment does not rely on the general solution of Maxwell's equations, but on the quasistatic approximation of these equations themselves.

The time dependency of the current density and that of the magnetic field can be described via the different frequency components. We found that in the quasistatic case, the general nonstatic solution (18) can be approximated by (33) provided $w_{max}^2 t_{trav}^2(\mathbf{r}, \mathbf{r}') \ll 1$. This is the expression (31) and describes the quasistatic case through the current frequency component with the shortest oscillation period $2\pi/w_{max} = 1/f_{max}$. This time must be large enough compared to the largest traveling time $t_{trav}(\mathbf{r}, \mathbf{r}')$. The latter is the time needed for an electromagnetic signal to travel from the most distant source point \mathbf{r}' to the measurement point \mathbf{r} .

Note that a similar expression to (31) is well known for the magneto-quasistatic approximation of Maxwell's equations,^{17,18} as well as their electro-quasistatic approximation.¹⁷⁻¹⁹ In these cases, however, instead of the traveling distance $|\mathbf{r} - \mathbf{r}'|$, a typical or characteristic length scale of the medium is employed. The system dimensions are assumed to be all within a factor of two or so of each other.¹⁸ The typical length in gray matter is estimated by the cortical thickness.^{17,19} On the other hand, Eq. (18) leads to the expression (31) and therefore, to the traveling distance $|\mathbf{r} - \mathbf{r}'|$ as the appropriate length to define the quasistatic dynamics.

A clearly different expression from (31) can be found in Ref. 1 where the quasistatic approximation of the Ampère-Maxwell equation was examined. It is given by $w_{max}\tau_{MW} \ll 1$, where $\tau_{MW} = \epsilon/\sigma$ is the Maxwell-Wagner reaction time. That is the expression (36) which we discussed in greater detail earlier. Here, ϵ and σ are the permittivity and the electrical conductivity of the medium, respectively. We note that a permittivity $\epsilon \approx 10^5 \epsilon_0$, where ϵ_0 is the permittivity of free space, was considered in Ref. 1. This gives $w_{max}\tau_{MW} \approx 2 \times 10^{-3} \ll 1$. Here, however, based on

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Refs. 17 and 20, the permittivity for gray matter at frequencies below 100 Hz has been underestimated by two orders of magnitude. Therefore, the condition $w_{\max} \tau_{MW} \ll 1$ is actually fulfilled in a minimal manner.

The quasistatic approximation (33) of the Biot–Savart law was justified with (31) being well satisfied in the form $(2\pi f_{\max})^2 |\mathbf{r} - \mathbf{r}'|^2 \epsilon \mu \approx 4 \times 10^{-7} \ll 1$. Here, we estimated $|\mathbf{r} - \mathbf{r}'| \approx 0.1$ m, mainly given by the average size of the human brain. Additionally, we took the permittivity $\epsilon \approx 10^7 \epsilon_0$ for gray matter at frequencies below 100 Hz.^{17,20} Finally, typical for biological tissues,¹⁷ the permeability μ was approximated by that of free space, μ_0 .

B. Formalism for the long-distance magnetic field

We then focused on the magnetic field of the various currents. Following previous work,^{4,5} the current density was split into conduction and impressed current densities. The conduction current is caused by the electric field and its density is specified in the electro-quasistatic approximation by $\mathbf{J}^c(\mathbf{r}', t) \approx -\sigma(\mathbf{r}', t) \nabla' \Phi(\mathbf{r}', t)$, where $\sigma(\mathbf{r}', t)$ represents the nonhomogeneous conductivity of the medium and $\Phi(\mathbf{r}', t)$ is the scalar potential. We assumed the impressed current density $\mathbf{J}^i(\mathbf{r}', t)$ mostly given by a synaptic current density $\mathbf{J}^s(\mathbf{r}', t)$ and an action potential current density $\mathbf{J}^a(\mathbf{r}', t)$. The magnetic field case of conduction currents has been worked out exactly.²³ We studied the long-distance magnetic field in the synaptic and action potential cases.

Each population of neurons was described as a relatively small region such that its dimensions are small compared to the average distance from the region to the field point \mathbf{r} . Taking any point \mathbf{r}_{reg} in the region, we must have $|\mathbf{r}' - \mathbf{r}_{reg}| \ll |\mathbf{r} - \mathbf{r}_{reg}|$, for every region point \mathbf{r}' . Considering the distance to the measurement point in the order of centimeters, each region volume \mathcal{V}_{reg} can be thought in the order of 1 mm³. Using known values of the cortical surface, the average cortical thickness, and the total number of neurons in the human cerebral cortex,¹² we can find a quantity of approximately 4.7×10^5 regions of 1 mm³ in the human cerebral cortex, each one with an average estimate of about 3.4×10^4 neurons. The contributions from the different regions must be added together to obtain the magnetic field at a certain point. This is indicated through Eqs. (48) and (82) for synaptic currents and action potential currents, respectively. The magnetic induction produced by synaptic currents at a relatively distant point \mathbf{r} , for a region located at \mathbf{r}_{reg} , can be approximated by the expression (57). Here $\mathbf{Q}_{reg}^s(t)$ is the (time-dependent) synaptic current dipole moment defined by Eq. (55). The expression (57) shows a decrease with the square of the distance for synaptic currents that is typical for the dipolar case.¹ We then expressed the synaptic current dipole moment through the conduction current density with the help of two key assumptions. First, we consider quasistatic conditions such that the continuity equation for the free current density \mathbf{J} and the free charge density ρ , gives $\nabla' \cdot \mathbf{J}(\mathbf{r}', t) = -\partial \rho(\mathbf{r}', t) / \partial t \approx 0$.⁴ Second, we assume a small contribution of the impressed currents at the surface \mathcal{S}_{reg} bounding the given region, compared to that of the conduction currents.²⁴ Moreover, the region can be visualized as consisting of m homogeneous parts with uniform conductivity $\sigma_{reg}^{(i)}(t)$, where $i = 1, 2, \dots, m$.⁴

We obtained two equivalent expressions of $\mathbf{Q}_{reg}^s(t)$ in terms of the scalar potential and/or its spatial derivatives, given by (74) and (76). Here, in the case of (74), $\mathcal{S}_{reg}(i, j)$ refers to the common bounding surface between two homogeneous parts i and j , while in (76), $\mathcal{V}_{reg}(i)$ represents the volume of the i th part. It can be noticed that (74) requires the scalar potential at all boundaries to be known, as well as its inner normal derivative (i.e., the inner normal component of the electric field) at the region boundary. On the other hand, in the case of (76), we must know the second spatial derivatives of the scalar potential everywhere in each homogeneous part. Alternatively, (76) may be written using the free charge density (see below).

The expression (74) of the synaptic current dipole moment can also be considered valid for impressed currents. Taking into account an approximately cylindrical shape for the neuronal cables translates into a relatively small current dipole moment for action potential currents. We note that an expression for the impressed current dipole moment is given in Ref. 23. However, it does not contain the first term on the right-hand side of (74). An external medium with zero conductivity is assumed in Ref. 23. In contrast, we consider an external medium with nonzero conductivity which corresponds to a non-isolated region. Thus, the expression (74) can be seen as generalizing the expression presented in Ref. 23.

Concerning the expression (76), we notice that it takes the form (81) when additionally describing the i th part by a linear and isotropic medium with uniform permittivity $\epsilon_{reg}^{(i)}(t)$. In (81), $\mathbf{p}_{reg}^{(i)}(t)$ and $\tau_{reg}^{(i)}(t)$ are, respectively, the electric dipole moment of the i th part and the Maxwell–Wagner time for the latter. Hence, through the relevant Maxwell–Wagner time, the synaptic current dipole moment of a region is connected to the electric dipole moment of each homogeneous part. However, (81) also points to the fact that the current dipole moment (55) and the electric dipole moment (80) are dimensionally and fundamentally different concepts.

Unlike the synaptic case, the long-distance magnetic field of action potential currents is determined by the corresponding magnetic dipole moment. In this case, the magnetic induction at a relatively distant point \mathbf{r} , for a region located at \mathbf{r}_{reg} , is given by the expression (88). Here $\mathbf{m}_{reg}^a(t)$ is the magnetic dipole moment (89) for action potential currents. The expression (88) shows a characteristic quadrupolar decrease with the cube of the distance¹ which is faster than in the quadratic synaptic case given by (57). We also observe that the expression (88) may be written in the familiar form (91) for a magnetic dipole, provided $\nabla \cdot \mathbf{J}^a \approx 0$ in the given region, and $\mathbf{J}^a(\mathbf{r}', t) = \mathbf{0}$ on its boundary.²⁵

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

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Author Contributions

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APPENDIX: TOTAL CURRENT DIPOLE MOMENT

Here, we derive the expressions (66) and (68) for the total current dipole moment. It is also checked how each of these expressions can yield the same result regardless of the choice of the coordinate origin. The following are two key assumptions. First, we consider quasistatic conditions such that the continuity equation gives

$$\nabla' \cdot \mathbf{J}(\mathbf{r}', t) = -\frac{\partial \rho(\mathbf{r}', t)}{\partial t} \approx 0. \tag{A1}$$

Second, we assume a small contribution of the impressed currents at the surface S_{reg} bounding the given region, compared to that of the conduction currents. We take

$$\left| \int_{S_{reg}} \mathbf{x}' \mathbf{J}^i(\mathbf{r}', t) \cdot d\mathbf{S}' \right| \ll \left| \int_{S_{reg}} \mathbf{x}' \mathbf{J}^c(\mathbf{r}', t) \cdot d\mathbf{S}' \right|, \tag{A2}$$

and this similarly applies to the remaining components of $\int_{S_{reg}} \mathbf{r}' [\mathbf{J}^i(\mathbf{r}', t) \cdot d\mathbf{S}']$.

1. Expression (66)

We consider the x component of the total current dipole moment (61), given by

$$Q_{reg}^x(t) = \int_{V_{reg}} J_x(\mathbf{r}', t) d^3 r', \tag{A3}$$

and write $J_x(\mathbf{r}', t)$ in the form

$$J_x(\mathbf{r}', t) = \mathbf{J}(\mathbf{r}', t) \cdot \nabla' x' = \nabla' \cdot [\mathbf{x}' \mathbf{J}(\mathbf{r}', t)] - \mathbf{x}' \nabla' \cdot \mathbf{J}(\mathbf{r}', t). \tag{A4}$$

By using (A1), this simplifies to

$$J_x(\mathbf{r}', t) \approx \nabla' \cdot [\mathbf{x}' \mathbf{J}(\mathbf{r}', t)]. \tag{A5}$$

Then,

$$Q_{reg}^x(t) \approx \int_{V_{reg}} \nabla' \cdot [\mathbf{x}' \mathbf{J}(\mathbf{r}', t)] d^3 r' = \int_{S_{reg}} \mathbf{x}' \mathbf{J}(\mathbf{r}', t) \cdot d\mathbf{S}', \tag{A6}$$

where S_{reg} is the surface bounding the region, according to the divergence theorem. Furthermore, from Eq. (41), $\mathbf{J}(\mathbf{r}', t) = \mathbf{J}^c(\mathbf{r}', t) + \mathbf{J}^i(\mathbf{r}', t)$. Taking then into account (A2), we write

$$Q_{reg}^x(t) \approx \int_{S_{reg}} \mathbf{x}' \mathbf{J}^c(\mathbf{r}', t) \cdot d\mathbf{S}'. \tag{A7}$$

Similar expressions for the y and z components of the total current

dipole moment can be obtained in the same way. Therefore, we get

$$\mathbf{Q}_{reg}(t) \approx \int_{S_{reg}} \mathbf{r}' [\mathbf{J}^c(\mathbf{r}', t) \cdot d\mathbf{S}'], \tag{A8}$$

that is, the expression (66).

2. Expression (68)

a. Derivation from the expression (66)

Considering the x component in (66) [or (A8)], given by (A7), we have

$$Q_{reg}^x(t) \approx \int_{V_{reg}} \nabla' \cdot [\mathbf{x}' \mathbf{J}^c(\mathbf{r}', t)] d^3 r', \tag{A9}$$

using the divergence theorem. Similarly to (A4), we get

$$\nabla' \cdot [\mathbf{x}' \mathbf{J}^c(\mathbf{r}', t)] = \mathbf{x}' \nabla' \cdot \mathbf{J}^c(\mathbf{r}', t) + J_x^c(\mathbf{r}', t). \tag{A10}$$

Then,

$$Q_{reg}^x(t) \approx \int_{V_{reg}} \mathbf{x}' \nabla' \cdot \mathbf{J}^c(\mathbf{r}', t) d^3 r' + \int_{V_{reg}} J_x^c(\mathbf{r}', t) d^3 r'. \tag{A11}$$

The second term on the right-hand side here is the x component of the conduction current dipole moment, $\mathbf{Q}_{reg}^c(t)$, defined by (62). Expressions of the same form as that of (A11) apply to the y and z components. Therefore,

$$\mathbf{Q}_{reg}(t) \approx \int_{V_{reg}} \mathbf{r}' \nabla' \cdot \mathbf{J}^c(\mathbf{r}', t) d^3 r' + \mathbf{Q}_{reg}^c(t), \tag{A12}$$

that is, we obtain the expression (68).

b. Independent derivation

The expression (68) can also be derived starting from Eq. (41) of the current density. Integrating it, we have

$$\mathbf{Q}_{reg}(t) = \mathbf{Q}_{reg}^c(t) + \mathbf{Q}_{reg}^i(t), \tag{A13}$$

where

$$\mathbf{Q}_{reg}^i(t) = \int_{V_{reg}} \mathbf{J}^i(\mathbf{r}', t) d^3 r', \tag{A14}$$

is the impressed current dipole moment, with x component

$$\begin{aligned} \int_{V_{reg}} J_x^i(\mathbf{r}', t) d^3 r' &= \int_{V_{reg}} \mathbf{J}^i(\mathbf{r}', t) \cdot \nabla' x' d^3 r' \\ &= \int_{V_{reg}} \nabla' \cdot [\mathbf{x}' \mathbf{J}^i(\mathbf{r}', t)] d^3 r' - \int_{V_{reg}} \mathbf{x}' \nabla' \cdot \mathbf{J}^i(\mathbf{r}', t) d^3 r'. \end{aligned} \tag{A15}$$

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Then,

$$\int_{V_{reg}} J_x^i(r', t) d^3 r' = \int_{S_{reg}} x' J^i(r', t) \cdot dS' - \int_{V_{reg}} x' \nabla' \cdot J^i(r', t) d^3 r'. \tag{A16}$$

Considering again (41), we have

$$\nabla' \cdot J(r', t) = \nabla' \cdot J^c(r', t) + \nabla' \cdot J^i(r', t), \tag{A17}$$

which through (A1), implies

$$\nabla' \cdot J^i(r', t) \approx -\nabla' \cdot J^c(r', t). \tag{A18}$$

Thus,

$$\int_{V_{reg}} J_x^i(r', t) d^3 r' \approx \int_{S_{reg}} x' J^i(r', t) \cdot dS' + \int_{V_{reg}} x' \nabla' \cdot J^c(r', t) d^3 r'. \tag{A19}$$

Here, we can substitute

$$\int_{V_{reg}} x' \nabla' \cdot J^c(r', t) d^3 r' = \int_{S_{reg}} x' J^c(r', t) \cdot dS' - \int_{V_{reg}} J_x^c(r', t) d^3 r', \tag{A20}$$

which is similar to (A16). Using then (A2), we get

$$\int_{V_{reg}} J_x^i(r', t) d^3 r' \approx \int_{V_{reg}} x' \nabla' \cdot J^c(r', t) d^3 r'. \tag{A21}$$

Expressions of the same form hold for the y and z components of the impressed current dipole moment. Therefore

$$\mathcal{Q}_{reg}^i(t) \approx \int_{V_{reg}} r' \nabla' \cdot J^c(r', t) d^3 r'. \tag{A22}$$

Equation (A13) then gives

$$\mathcal{Q}_{reg}(t) \approx \int_{V_{reg}} r' \nabla' \cdot J^c(r', t) d^3 r' + \mathcal{Q}_{reg}^c(t), \tag{A23}$$

thus reproducing the expression (68).

3. Independence on the coordinate origin

The total current dipole moment,

$$\mathcal{Q}_{reg}(t) = \int_{V_{reg}} J(r', t) d^3 r', \tag{A24}$$

is independent of the chosen origin of the coordinate system. The

current density value at a given point clearly does not depend on the position of the coordinate origin. The independence of $\mathcal{Q}_{reg}(t)$ on the coordinate origin can be explicitly shown as follows. Let a new origin be set at position \mathcal{R} relative to the old origin. A given point located at r' in relation to the old origin is now located at r'' in relation to the new origin, with $r'' = r' - \mathcal{R}$. Correspondingly, the current density is described by $\mathcal{J}(r'', t)$, with $\mathcal{J}(r'', t) = J(\mathcal{R} + r'', t) = J(r', t)$. The total current dipole moment when using the new origin is

$$\mathcal{Q}_{reg}(t) = \int_{V_{reg}} \mathcal{J}(r'', t) d^3 r'' = \int_{V_{reg}} J(\mathcal{R} + r'', t) d^3 r'', \tag{A25}$$

so that

$$\mathcal{Q}_{reg}(t) = \int_{V_{reg}} J(r', t) d^3 r' = \mathcal{Q}_{reg}(t). \tag{A26}$$

The mentioned independence on the coordinate origin must be verified for the expressions (66) and (68) of the total current dipole moment. Below, when using a new origin as above, the conduction current density is described by

$$\mathcal{J}^c(r'', t) = J^c(\mathcal{R} + r'', t) = J^c(r', t). \tag{A27}$$

a. First case

Regarding the expression (66) [or (A8)], we have

$$\mathcal{Q}_{reg}(t) \approx \int_{S_{reg}} r'' [\mathcal{J}^c(r'', t) \cdot dS''] = \int_{S_{reg}} (r' - \mathcal{R}) [J^c(r', t) \cdot dS'], \tag{A28}$$

so that

$$\mathcal{Q}_{reg}(t) \approx \int_{S_{reg}} r' [J^c(r', t) \cdot dS'] - \mathcal{R} \int_{S_{reg}} J^c(r', t) \cdot dS'. \tag{A29}$$

Concerning the second term on the right-hand side here, by integrating (A18) we get

$$\int_{S_{reg}} J^c(r', t) \cdot dS' \approx - \int_{S_{reg}} J^i(r', t) \cdot dS'. \tag{A30}$$

Hence, for the expression (66) to be independent of the coordinate origin, it is enough, and necessary, that the net impressed current crossing the region boundary is approximately zero. Namely, if, and only if, we have

$$\int_{S_{reg}} J^i(r', t) \cdot dS' \approx 0, \tag{A31}$$

then

$$\int_{S_{\text{reg}}} \mathbf{J}^c(\mathbf{r}', t) \cdot d\mathbf{S}' \approx 0, \quad (\text{A32})$$

and consequently

$$\mathcal{Q}_{\text{reg}}(t) \approx \int_{S_{\text{reg}}} \mathbf{r}' [\mathbf{J}^c(\mathbf{r}', t) \cdot d\mathbf{S}'], \quad (\text{A33})$$

which gives the same result as the expression (66).

b. Second case

As for the expression (68) [or (A23)], when we employ a new origin it reads

$$\mathcal{Q}_{\text{reg}}(t) \approx \int_{V_{\text{reg}}} \mathbf{r}'' \nabla'' \cdot \mathcal{J}^c(\mathbf{r}'', t) d^3 r'' + \int_{V_{\text{reg}}} \mathcal{J}^c(\mathbf{r}'', t) d^3 r'', \quad (\text{A34})$$

which, using the divergence theorem and (A30), leads to

$$\mathcal{Q}_{\text{reg}}(t) \approx \int_{V_{\text{reg}}} \mathbf{r}' \nabla' \cdot \mathbf{J}^c(\mathbf{r}', t) d^3 r' + \mathcal{Q}_{\text{reg}}^c(t) + \mathcal{R} \int_{S_{\text{reg}}} \mathbf{J}^i(\mathbf{r}', t) \cdot d\mathbf{S}'. \quad (\text{A35})$$

Thus, similar to the case of the expression (66), if, and only if, (A31) is met, we have

$$\mathcal{Q}_{\text{reg}}(t) \approx \int_{V_{\text{reg}}} \mathbf{r}' \nabla' \cdot \mathbf{J}^c(\mathbf{r}', t) d^3 r' + \mathcal{Q}_{\text{reg}}^c(t), \quad (\text{A36})$$

which produces the same result as the expression (68).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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