



HAL
open science

A Scheme Using the Wave Structure of Second-Moment Turbulent Models for Incompressible Flows

Martin Ferrand, Jean-Marc Hérard, Thomas Norddine, Simon Ruget

► **To cite this version:**

Martin Ferrand, Jean-Marc Hérard, Thomas Norddine, Simon Ruget. A Scheme Using the Wave Structure of Second-Moment Turbulent Models for Incompressible Flows. Finite Volumes for Complex Applications X-Volume 2, Hyperbolic and Related Problems, 433, Springer Nature Switzerland, pp.111-119, 2023, Springer Proceedings in Mathematics & Statistics, 10.1007/978-3-031-40860-1_12 . hal-04516669

HAL Id: hal-04516669

<https://hal.science/hal-04516669>

Submitted on 22 Mar 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A scheme using the wave structure of second-moment turbulent models for incompressible flows

Martin Ferrand^{1,2,3}, Jean-Marc Hérard^{2,3}, Thomas Norddine^{2,3}, and Simon Ruget^{2,3}

¹ `martin.ferrand@edf.fr`

² CEREAs, École des Ponts, EDF R & D, Chatou, France

³ EDF R&D, 6, quai Watier, 78400 Chatou, France

Abstract. We focus herein on the analysis of the one-dimensional Riemann problem arising from the convective subset of a second-moment turbulent non-conservative model for incompressible flows. The sketch of proof of existence and uniqueness of the solution is given, assuming a set of approximate jump conditions. Some first numerical simulations applying for the Finite Volume method are given and compared with another scheme classically used in CFD codes. This suggests to implement standard projection schemes to cope with the complete model.

Keywords: second-moment turbulent closure, incompressible turbulent flows, hyperbolic systems, Riemann problem

1 Introduction

Unsteady Reynolds-averaged Navier–Stokes models are widely used in the industrial framework in order to predict incompressible or compressible flows (see e.g. [8]). Second-moment tensors also arise when modelling shallow-water models (see [4, 5]). All these models contain first-order non-conservative terms which require special attention (see e.g. [2]). This work is highly motivated by the occurrence of non-physical oscillations in many practical situations (see e.g. [9]).

2 Second-moment turbulence models

The system of equations is composed of mass conservation (1a), momentum conservation (1b) and second-moment transport equation (1c):

$$\operatorname{div}(\bar{\mathbf{u}}) = 0, \quad (1a)$$

$$\partial_t \bar{\mathbf{u}} + (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} + \operatorname{div}(\mathbf{R}) - \operatorname{div}(\bar{\Sigma}) = \mathbf{0}, \quad (1b)$$

$$\partial_t \mathbf{R} + (\bar{\mathbf{u}} \cdot \nabla) \mathbf{R} + \mathbf{R} \cdot \nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T \cdot \mathbf{R} + \operatorname{div}(\mathbf{G}_{\mathbf{R}}) - \Phi = \mathbf{0}, \quad (1c)$$

with $\bar{\Sigma} := -\frac{\bar{p}}{\rho_0} \mathbf{1} + \nu_0 (\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T)$ the mean stress tensor per mass unit, \bar{p} the mechanical pressure. The Reynolds stress tensor $\mathbf{R} := \bar{\mathbf{u}}' \otimes \bar{\mathbf{u}}'$ is symmetric and

it is also expected to be positive and half-definite (called realizability constraint), owing to its intrinsic form (see [8]). Models for $\mathbf{G}_R := \overline{\mathbf{u}' \otimes \mathbf{u}' \otimes \mathbf{u}'}$ usually rely on a classical first gradient assumption $\mathbf{G}_R = -C_R \nu_t(\mathbf{R}, T) \nabla \mathbf{R}$, where C_R is a positive constant, ν_t is the turbulent viscosity, and T is the time scale related to turbulent dissipation. Moreover $\Phi := \overline{\mathbf{u}' \otimes \text{div}(\Sigma') + \text{div}(\Sigma') \otimes \mathbf{u}'}$ is classically split into two parts:

$$\Phi = \Phi^r(\nabla \bar{\mathbf{u}}, \mathbf{R}) + \Phi^s(\mathbf{R}, T), \quad (2)$$

where the so-called rapid contribution verifies: $\Phi^r(\mathbf{0}, \mathbf{R}) = \mathbf{0}$ and $\Phi^r(\nabla \bar{\mathbf{u}}, \mathbf{0}) = \mathbf{0}$. Moreover Φ^r is linear with respect to $\nabla \bar{\mathbf{u}}$ (see e.g. [8] for a review of the turbulence models). The following result is useful to examine the realizability of the Reynolds stress tensor.

Lemma 1. *Assume that the governing equation of a symmetric second-rank tensor $\mathbf{R} \in \mathbb{R}^3 \times \mathbb{R}^3$ is:*

$$\partial_t \mathbf{R} + (\bar{\mathbf{u}} \cdot \nabla) \mathbf{R} + \mathbf{R} \cdot \mathbf{H} + \mathbf{H}^T \cdot \mathbf{R} = \mathbf{0}.$$

Let δ_R be the determinant of tensor \mathbf{R} . Then the governing equation of δ_R is:

$$\partial_t \delta_R + (\bar{\mathbf{u}} \cdot \nabla) \delta_R + 2\text{tr}(\mathbf{H}) \delta_R = 0.$$

A proof can be found in appendix 1 of [7]. \mathbf{H} is deduced from the terms $\mathbf{R} \cdot \nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T \cdot \mathbf{R}$, Φ^r and Φ^s , and the boundedness of the trace of \mathbf{H} must be examined to ensure positive values of δ_R .

3 A first-order time scheme

Starting with initial conditions $(\bar{\mathbf{u}}^n, \mathbf{R}^n)$ and using suitable boundary conditions, $(\bar{\mathbf{u}}^{n+1}, \mathbf{R}^{n+1})$ arises from the following first-order time scheme:

$$\text{div}(\bar{\mathbf{u}}^{n+1}) = 0, \quad (3a)$$

$$\frac{\bar{\mathbf{u}}^{n+1} - \bar{\mathbf{u}}^n}{\Delta t} + \nabla \frac{\bar{p}^{n+1}}{\rho_0} - \text{div}(\nu_0 \nabla \bar{\mathbf{u}}^{n+1}) = -((\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} + \text{div}(\mathbf{R}))^n, \quad (3b)$$

$$\begin{aligned} \frac{\mathbf{R}^{n+1} - \mathbf{R}^n}{\Delta t} - \text{div}(C_R \nu_t \nabla \mathbf{R}^{n+1}) - \Phi^s(\mathbf{R}^{n+1}, T) &= -((\bar{\mathbf{u}} \cdot \nabla) \mathbf{R})^n \\ &\quad - (\mathbf{R} \cdot \nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T \cdot \mathbf{R})^n \\ &\quad + \Phi^r(\nabla \bar{\mathbf{u}}, \mathbf{R})^n. \end{aligned} \quad (3c)$$

A Stokes-like step has been applied in the scheme (3a) and (3b) (see, among others, [12, 1]). The space discretization associated with the right-hand-side of (3b), (3c) is derived from the analysis of the following evolution system (4):

$$\partial_t \bar{\mathbf{u}} + (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} + \text{div}(\mathbf{R}) = \mathbf{0}, \quad (4a)$$

$$\partial_t \mathbf{R} + (\bar{\mathbf{u}} \cdot \nabla) \mathbf{R} + \mathbf{R} \cdot \nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T \cdot \mathbf{R} - \Phi^r(\nabla \bar{\mathbf{u}}, \mathbf{R}) = \mathbf{0}. \quad (4b)$$

4 Analysis of the evolution step

In the following, we restrict to the simple Lumley's model which does not contain so-called rapid terms: $\Phi^{\mathbf{r}} = \mathbf{0}$. We also restrict from now on to the two-dimensional framework to ease the presentation. We first recall that the latter system (4) is hyperbolic if the Reynolds stress tensor is realizable (which is guaranteed, owing to lemma 1, since in that case we have: $tr\mathbf{H} = \text{div}\bar{\mathbf{u}} = 0$). Eq. (4) is invariant under frame rotation, so we can rewrite it in the $(\mathbf{n}, \boldsymbol{\tau})$ reference frame (where $\mathbf{n}, \boldsymbol{\tau}$ are some unit orthogonal vectors), defining $x_n := \mathbf{x} \cdot \mathbf{n}$, $u_n := \bar{\mathbf{u}} \cdot \mathbf{n}$, $u_\tau := \bar{\mathbf{u}} \cdot \boldsymbol{\tau}$, $R_{nn} := \mathbf{n}^T \mathbf{R} \mathbf{n}$, $R_{n\tau} := \mathbf{n}^T \mathbf{R} \boldsymbol{\tau}$ and $R_{\tau\tau} := \boldsymbol{\tau}^T \mathbf{R} \boldsymbol{\tau}$. We now define the evolution step in the \mathbf{n} -direction, by neglecting the transverse derivatives, which yields:

$$\partial_t u_n + u_n \partial_n u_n + \partial_n R_{nn} = 0, \quad (5a)$$

$$\partial_t u_\tau + u_n \partial_n u_\tau + \partial_n R_{n\tau} = 0, \quad (5b)$$

$$\partial_t R_{nn} + u_n \partial_n R_{nn} + 2R_{nn} \partial_n u_n = 0, \quad (5c)$$

$$\partial_t R_{n\tau} + u_n \partial_n R_{n\tau} + R_{nn} \partial_n u_\tau + R_{n\tau} \partial_n u_n = 0, \quad (5d)$$

$$\partial_t R_{\tau\tau} + u_n \partial_n R_{\tau\tau} + 2R_{n\tau} \partial_n u_\tau = 0. \quad (5e)$$

We set $\mathbf{W} := (u_n, u_\tau, R_{nn}, R_{\tau\tau}, R_{n\tau})$. We assume from now on that initial and boundary conditions comply with the strictly realizability constraint, which will imply that \mathbf{R} remains strictly realizable. Once more, system (5) preserves the realizability, which implies that $R_{nn} > 0$ whatever the unit vector \mathbf{n} is.

Proposition 1 (Hyperbolic evolution step). *System (5) is hyperbolic. It admits five real eigenvalues which are:*

$$\begin{aligned} \lambda_1 &= u_n - \sqrt{2R_{nn}}, & \lambda_2 &= u_n - \sqrt{R_{nn}}, & \lambda_3 &= u_n, \\ \lambda_4 &= u_n + \sqrt{R_{nn}}, & \lambda_5 &= u_n + \sqrt{2R_{nn}}. \end{aligned} \quad (6)$$

and the associated right eigenvectors span \mathbb{R}^5 . Fields associated with eigenvalues $\lambda_{2,3,4}$ are linearly degenerate (LD). The 1-field and 5-field are genuinely non linear (GNL).

We refer to [7] for a proof. Note that this can be extended to the three-dimensional framework. Right eigenvectors are:

$$\mathbf{r}_1 = \left(1, \frac{R_{n\tau}}{R_{nn}}, -\sqrt{2R_{nn}}, -\frac{\sqrt{2}R_{n\tau}^2}{(R_{nn})^{\frac{3}{2}}}, -\frac{\sqrt{2}R_{n\tau}}{\sqrt{R_{nn}}} \right)^T, \quad (7a)$$

$$\mathbf{r}_2 = \left(0, 1, 0, -\frac{2R_{n\tau}}{\sqrt{R_{nn}}}, -\sqrt{R_{nn}} \right)^T, \quad (7b)$$

$$\mathbf{r}_3 = (0, 0, 0, 1, 0)^T, \quad (7c)$$

$$\mathbf{r}_4 = \left(0, 1, 0, \frac{2R_{n\tau}}{\sqrt{R_{nn}}}, \sqrt{R_{nn}} \right)^T, \quad (7d)$$

$$\mathbf{r}_5 = \left(1, \frac{R_{n\tau}}{R_{nn}}, \sqrt{2R_{nn}}, \frac{\sqrt{2}R_{n\tau}^2}{(R_{nn})^{\frac{3}{2}}}, \frac{\sqrt{2}R_{n\tau}}{\sqrt{R_{nn}}} \right)^T. \quad (7e)$$

Riemann invariants I_R^i of the i -th field ($f \in I_R^i$, $\nabla_{\mathbf{W}} f \cdot \mathbf{r}_i(\mathbf{W}) = 0$) are:

$$I_R^1 = \left\{ u_n + \sqrt{2R_{nn}}, \frac{R_{n\tau}}{R_{nn}}, R_{\tau\tau} - \frac{R_{n\tau}^2}{R_{nn}}, u_\tau + R_{n\tau} \sqrt{\frac{2}{R_{nn}}} \right\}, \quad (8a)$$

$$I_R^2 = \left\{ u_n, R_{nn}, u_\tau + \frac{R_{n\tau}}{\sqrt{R_{nn}}}, R_{\tau\tau} R_{nn} - R_{n\tau}^2 \right\}, \quad (8b)$$

$$I_R^3 = \left\{ u_n, u_\tau, R_{nn}, R_{n\tau} \right\}, \quad (8c)$$

$$I_R^4 = \left\{ u_n, R_{nn}, u_\tau - \frac{R_{n\tau}}{\sqrt{R_{nn}}}, R_{\tau\tau} R_{nn} - R_{n\tau}^2 \right\}, \quad (8d)$$

$$I_R^5 = \left\{ u_n - \sqrt{2R_{nn}}, \frac{R_{n\tau}}{R_{nn}}, R_{\tau\tau} - \frac{R_{n\tau}^2}{R_{nn}}, u_\tau - R_{n\tau} \sqrt{\frac{2}{R_{nn}}} \right\}. \quad (8e)$$

Due to non-conservative terms in (5), shock relations cannot be derived in a classical way. Following [2] **approximate** shock relations can nevertheless be proposed. These relations are valid in the limit of weak shocks. Assuming a linear path from left to right with respect to the variable \mathbf{W} , these relations write:

$$\sigma[u_n] = \widehat{u}_n[u_n] + [R_{nn}], \quad (9a)$$

$$\sigma[u_\tau] = \widehat{u}_n[u_\tau] + [R_{n\tau}], \quad (9b)$$

$$\sigma[R_{nn}] = 2\widehat{R}_{nn}[u_n] + \widehat{u}_n[R_{nn}], \quad (9c)$$

$$\sigma[R_{n\tau}] = \widehat{R}_{n\tau}[u_n] + \widehat{R}_{nn}[u_\tau] + \widehat{u}_n[R_{n\tau}], \quad (9d)$$

$$\sigma[R_{\tau\tau}] = 2\widehat{R}_{n\tau}[u_\tau] + \widehat{u}_n[R_{\tau\tau}], \quad (9e)$$

where $[x] := x_r - x_l$ denotes jump and $\widehat{x} := \frac{x_r + x_l}{2}$ is the arithmetic mean between left and right states of the discontinuity travelling at speed σ .

Theorem 1 (Existence and uniqueness of the solution). *Considering approximate shock relations (9), there exists a unique self-similar realizable solution ($\mathbf{W}(x_n, t) = \mathbf{w}(\frac{x_n}{t})$, with $\mathbf{W} = (u_n, u_\tau, R_{nn}, R_{\tau\tau}, R_{n\tau})$) to the Riemann problem associated with system (5) supplemented with strictly realizable initial conditions $\mathbf{W}(x_n, 0) = \mathbf{W}^L$ for $x_n < 0$ and $\mathbf{W}(x_n, 0) = \mathbf{W}^R$ for $x_n > 0$, if and only if the following condition holds:*

$$u_n^R - u_n^L < \left(\sqrt{2R_{nn}^L} + \sqrt{2R_{nn}^R} \right). \quad (10)$$

As in the compressible framework, the Riemann problem structure of system (4) allows to specify wall, symmetry and inlet/outlet boundary conditions.

Proof. The proof follows the main guidelines recalled in [11] and detailed in [7]. We consider two initial realizable states \mathbf{W}^R , \mathbf{W}^L . We look for the four

intermediate states labelled I , II , III , IV from left to right separating the 5 waves.

Step 0: Preliminary calculation. To connect left and right states $\mathbf{W}^{l,r}$ across the 1st and 5th waves we set $z := \frac{R_{nn}^r}{R_{nn}^l} > 0$. To select the physical solution for shocks (9), we use $[u_n] = u_n^r - u_n^l \leq 0$ (see [7]).

For the 1-shock connection ($z > 1$):

$$\begin{aligned} R_{nn}^r &= zR_{nn}^l; & u_n^r &= u_n^l + \frac{1-z}{\sqrt{z+1}}\sqrt{R_{nn}^l}; & u_\tau^r &= u_\tau^l + \frac{1-z}{\sqrt{z+1}}\frac{R_{n\tau}^l}{\sqrt{R_{nn}^l}}; \\ R_{n\tau}^r &= zR_{n\tau}^l; & R_{\tau\tau}^r &= R_{\tau\tau}^l + (z-1)\frac{(R_{n\tau}^l)^2}{R_{nn}^l}. \end{aligned} \quad (11)$$

For the 5-shock connection ($0 < z < 1$):

$$\begin{aligned} R_{nn}^r &= zR_{nn}^l; & u_n^r &= u_n^l + \frac{z-1}{\sqrt{z+1}}\sqrt{R_{nn}^l}; & u_\tau^r &= u_\tau^l + \frac{z-1}{\sqrt{z+1}}\frac{R_{n\tau}^l}{\sqrt{R_{nn}^l}}; \\ R_{n\tau}^r &= zR_{n\tau}^l; & R_{\tau\tau}^r &= R_{\tau\tau}^l + (z-1)\frac{(R_{n\tau}^l)^2}{R_{nn}^l}. \end{aligned} \quad (12)$$

To connect the left and right states through the 1-rarefaction wave (respectively 5-rarefaction wave) we use the Riemann invariants (8a) (respectively (8e)).

Step 1: Solution in terms of u_n and R_{nn} variables. A glance at (8b), (8c), (8d) shows that u_n and R_{nn} are constant across the three LD waves. First we can focus only on u_n and R_{nn} variables. We denote their intermediate values by $u_n^\# = u_n^I = u_n^{II} = u_n^{III} = u_n^{IV}$ and $R_{nn}^\# = R_{nn}^I = R_{nn}^{II} = R_{nn}^{III} = R_{nn}^{IV}$, and link these to initial states $\mathbf{W}^{R,L}$:

$$\begin{cases} u_n^L = u_n^\# - h_1(z_1)\sqrt{2R_{nn}^L}, & \text{with } z_1 := \frac{R_{nn}^\#}{R_{nn}^L} > 0, \\ u_n^R = u_n^\# + h_5(z_5)\sqrt{2R_{nn}^R}, & \text{with } z_5 := \frac{R_{nn}^R}{R_{nn}^\#} > 0, \end{cases} \quad (13)$$

$h_{1,5}(z)$ are defined using (8a), (8e), (11), (12), following [11]. Note that $z_1 z_5 = \frac{R_{nn}^R}{R_{nn}^L}$. We set:

$$\Psi(z_5) := u_n^R - u_n^L - \sqrt{2R_{nn}^R}h_5(z_5) - \sqrt{2R_{nn}^L}h_1(z_1(z_5)) = 0. \quad (14)$$

Solving (14) gives z_5 , hence $R_{nn}^\#$ and $u_n^\#$. The solution is unique since Ψ is a strictly monotonic function. Its lower and upper bounds enable to conclude that $u_n^\#$ and $R_{nn}^\#$ exist if (10) holds. z_1 and z_5 enable to compute the remaining components (u_τ , $R_{\tau\tau}$, $R_{n\tau}$) of intermediate states I and IV .

Step 2: Solution in terms of u_τ , $R_{n\tau}$, and $R_{\tau\tau}$ variables. Let $u_\tau^* = u_\tau^{II} = u_\tau^{III}$ and $R_{n\tau}^* = R_{n\tau}^{II} = R_{n\tau}^{III}$, given by:

$$\begin{cases} u_\tau^* + \frac{R_{n\tau}^*}{\sqrt{R_{nn}^\#}} = u_\tau^I + \frac{R_{n\tau}^I}{\sqrt{R_{nn}^\#}}, \\ u_\tau^* - \frac{R_{n\tau}^*}{\sqrt{R_{nn}^\#}} = u_\tau^{IV} - \frac{R_{n\tau}^{IV}}{\sqrt{R_{nn}^\#}}. \end{cases} \quad (15)$$

Then, deduce $R_{\tau\tau}^{II}$ and $R_{\tau\tau}^{III}$ thanks to $(R_{nn}^\# R_{\tau\tau}^{II} - R_{n\tau}^{*,2}) = (R_{nn} R_{\tau\tau} - R_{n\tau}^2)^I$ and $(R_{nn}^\# R_{\tau\tau}^{III} - R_{n\tau}^{*,2}) = (R_{nn} R_{\tau\tau} - R_{n\tau}^2)^{IV}$.

5 Numerical scheme for the evolution system (4)

The objective of the present work is to obtain a stable numerical scheme (unlike those usually used, which may use partial upwinding which respect to the material mean velocity). Therefore we restrict the presentation to low order schemes such as Rusanov scheme. Other approximate Godunov schemes and MUSL techniques might also be considered.

For the sake of brevity the scheme is presented in a one dimensional framework. The evolution step (4) can be written (using the incompressibility constraint):

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial (u_n \mathbf{W})}{\partial x_n} + \mathbf{C}(\mathbf{W}) \frac{\partial \mathbf{W}}{\partial x_n} = \mathbf{0}, \quad (16)$$

\mathbf{W}_i^{n+1} will be computed using the following Rusanov-like [10] scheme:

$$\Delta x_i (\mathbf{W}_i^{n+1} - \mathbf{W}_i^n) + \Delta t^n \left(\mathcal{F}_{i+\frac{1}{2}}^n - \mathcal{F}_{i-\frac{1}{2}}^n \right) + \Delta t^n \mathcal{N}_i^n = \mathbf{0}, \quad (17)$$

where the time step Δt^n satisfies the CFL condition $\max_i \left(|\lambda_{i+\frac{1}{2}}^n| \right) \cdot \frac{\Delta t^n}{\Delta x_i} \leq \frac{1}{2}$, Δx_i denotes the size of cell i . The numerical flux $\mathcal{F}_{i+\frac{1}{2}}^n$ and the non-conservative contribution \mathcal{N}_i^n are:

$$\mathcal{F}_{i+\frac{1}{2}}^n := (u_n)_{i+\frac{1}{2}}^n \frac{\mathbf{W}_i^n + \mathbf{W}_{i+1}^n}{2} - \frac{\lambda_{i+\frac{1}{2}}^n}{2} (\mathbf{W}_{i+1}^n - \mathbf{W}_i^n), \quad (18a)$$

$$\mathcal{N}_i^n := \mathbf{C}(\mathbf{W}_i^n) \left(\frac{\mathbf{W}_{i+1}^n - \mathbf{W}_{i-1}^n}{2} \right), \quad (18b)$$

with $\lambda_{i+\frac{1}{2}}^n = \max_k \max_{l=i, i+1} (|\lambda_k(u_n \mathbf{I} + \mathbf{C})(\mathbf{W}_l^n)|)$, and where the discrete divergence free condition on $(u_n)_{i+\frac{1}{2}}^n$ holds.

zone	L	I	II	III	IV	R
u_n	$u_n (= 0)$	u_n	u_n	u_n	u_n	$u_n (= 0)$
u_τ	$u_\tau^L (= 1)$	u_τ^L	$\frac{u_\tau^L + u_\tau^R}{2} + \frac{R_{n\tau}^L - R_{n\tau}^R}{2\sqrt{R_{nn}}}$	$\frac{u_\tau^L + u_\tau^R}{2} + \frac{R_{n\tau}^L - R_{n\tau}^R}{2\sqrt{R_{nn}}}$	u_τ^R	$u_\tau^R (= 1)$
R_{nn}	$R_{nn} (= 0.5)$	R_{nn}	R_{nn}	R_{nn}	R_{nn}	$R_{nn} (= 0.5)$
$R_{n\tau}$	$R_{n\tau}^L (= 0.4)$	$R_{n\tau}^L$	$\frac{(u_\tau^L - u_\tau^R)\sqrt{R_{nn}} + R_{n\tau}^L + R_{n\tau}^R}{2}$	$\frac{(u_\tau^L - u_\tau^R)\sqrt{R_{nn}} + R_{n\tau}^L + R_{n\tau}^R}{2}$	$R_{n\tau}^R$	$R_{n\tau}^R (= 0.5)$
$R_{\tau\tau}$	$R_{\tau\tau}^L (= 0.3)$	$R_{\tau\tau}^L$	$R_{\tau\tau}^L + \frac{(R_{n\tau}^L)^2 - (R_{n\tau}^R)^2}{R_{nn}}$	$R_{\tau\tau}^R + \frac{(R_{n\tau}^L)^2 - (R_{n\tau}^R)^2}{R_{nn}}$	$R_{\tau\tau}^R$	$R_{\tau\tau}^R (= 0.4)$

Table 1. Analytical solution for the tangential flow in the vicinity of a wall.

Unlike scheme (17) and (18), standard schemes only consider material convection upwinding for stabilization: $\lambda_{i+\frac{1}{2}}^n = |(u_n)_{i+\frac{1}{2}}^n|$.

The following test case describes a flow in the vicinity of a wall, while prescribing shear stress. The initial states $\mathbf{W}^{L,R}$ and the intermediate states (*I*, *II*, *III*, *IV*) arising in the 1-dimensional Riemann problem associated with (5) are given in Table 1. Computations have been performed using code_saturne finite volume platform. The CFL parameter is set to 0.5 and the meshes contain from 250 to 32000 cells. The L1-error and the behaviour of \mathbf{W} are shown in Fig. 1, using material-upwind scheme and scheme (17) and (18) accounting for all convective effects.

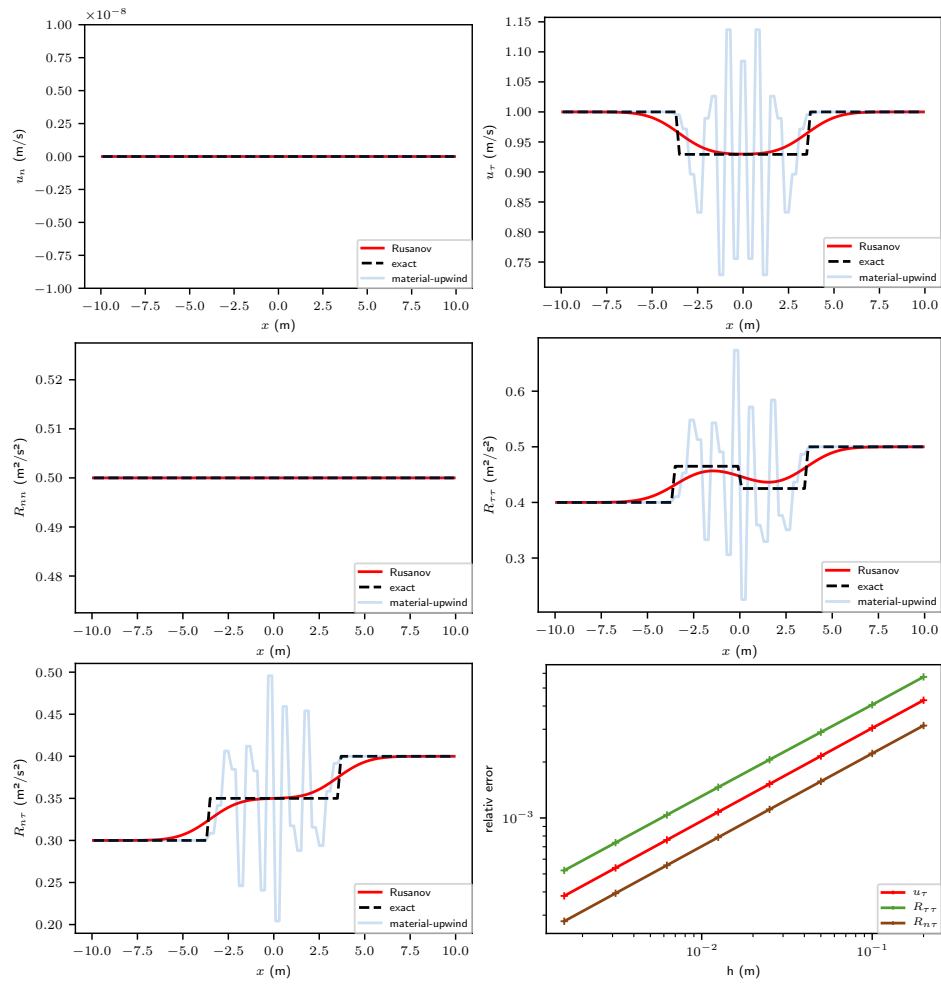


Fig. 1. Profiles of the variables u_n , u_τ , R_{nn} , $R_{\tau\tau}$, and $R_{n\tau}$ along with convergence plots obtained by applying material-upwind scheme and Rusanov-like scheme (500 cells, CFL= 0.5, convergence slop of Rusanov-like scheme is 1).

Of course, more accurate Riemann solvers can be investigated. This methodology can be applied to projection step methods or Uzawa algorithms. This strategy is also suitable for anisothermal flows or mean species transport equations with second-moment turbulence closures (see [3]). Eventually, more complex models involving non-zero contribution $\Phi^r(\nabla\bar{\mathbf{u}}, \mathbf{R})$ may be considered, using the same approach (see for instance appendix 2 in [7]).

References

1. Chorin, A. J.: Numerical solution of the Navier-Stokes equations. IN: Mathematics of computation, vol. 22, n. 104, pp 745–762, (1968)
2. Dal Maso, G., Lefloch, P., Murat, F.: Definition and weak stability of nonconservative products, *Journal de mathématiques pures et appliquées*, vol. 74, n. 6, pp 483–548, (1995)
3. Ferrand, M., Hérard, J.-M., Norddine, T., Ruget, S.: Stable schemes for second-moment turbulent models for incompressible flows. In preparation
4. Gavriluk, S., Gouin, H.: Geometric evolution of the Reynolds stress tensor, *International Journal of Engineering Science*, vol. 59, pp 65–73, (2012)
5. Gavriluk, S., Gouin, H.: Variational formulation for models of shear shallow water flows and ideal turbulence, *International Journal of Non-Linear Mechanics*, vol. 119, pp 103312, (2020)
6. Hérard, J.-M.: Basic analysis of some second moment closures part I: Incompressible isothermal turbulent flows, *Theoretical and Computational Fluid Dynamics*, vol. 6, n. 4, pp 213–233, (1994)
7. Hérard, J.-M.: Basic analysis of some second moment closures part II : incompressible turbulent flows including buoyant effects, <https://hal.archives-ouvertes.fr/hal-02007060>, (1994)
8. Pope, S. B.: *Turbulent flows*, Cambridge university press (2000)
9. Richards, P.J., Norris, S.E.: Appropriate boundary conditions for computational wind engineering models revisited, *Journal of Wind Engineering and Industrial Aerodynamics*, vol. 99, n. 4, pp 257–266, (2011)
10. Rusanov, V. V.: The calculation of the interaction of non-stationary shock waves with barriers, *Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki*, Russian Academy of Sciences, Branch of Mathematical Sciences vol. 1, n. 2, pp 267–279, (1961)
11. Smoller, J.: *Shock waves and reaction—diffusion equations*, vol. 258, Springer Science & Business Media (2012)
12. Temam, R.: Une méthode d'approximation de la solution des équations de Navier-Stokes, *Bulletin de la Société Mathématique de France*, vol. 96, pp 115–152, (1968)