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Holey graphs: very large Betti numbers are testable

Dániel Szabó¹ and Simon Apers¹

¹Université Paris Cité, CNRS, IRIF, Paris, France

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Abstract

We show that the graph property of having a (very) large k -th Betti number β_k for constant k is testable with a constant number of queries in the dense graph model. More specifically, we consider a clique complex defined by an underlying graph and prove that for any $\varepsilon > 0$, there exists $\delta(\varepsilon, k) > 0$ such that testing whether $\beta_k \geq (1 - \delta)d_k$ for $\delta \leq \delta(\varepsilon, k)$ reduces to tolerantly testing $(k + 2)$ -clique-freeness, which is known to be testable. This complements a result by Elek (2010) showing that Betti numbers are testable in the bounded-degree model. Our result combines the Euler characteristic, matroid theory and the graph removal lemma.

1 Introduction

In graph property testing we wish to decide whether a graph has a certain property, or whether it is “far” from having that property [Gol10]. In the *dense* graph model, an n -vertex graph is ε -far from having a property if we have to add or remove more than εn^2 edges for the graph to have the property. A *tester* for a given property is a randomized algorithm that, given query access to the adjacency matrix of a graph G , can distinguish with constant success probability whether G has that property or is ε -far from having it. A graph property is said to be *testable* if there exists a tester that makes a number of queries that is a function only of ε , and so independent of the graph size. Examples of testable properties are bipartiteness, triangle-freeness and, more generally, monotone (closed under removing edges) and hereditary (closed under removing vertices) graph properties [AS05, AS08].

In this work we prove the following theorem (for a more formal statement see Theorem 2).

Theorem 1 (Informal). *The property of having a (very) large k -th Betti number is testable for constant k .*

Strictly speaking, we consider the k -th Betti number of the *clique complex* associated to G . This is the simplicial complex defined by the family of all vertex subsets that induce a clique in G . On an intuitive level, the k -th Betti number β_k of this complex counts the number of independent k -dimensional “holes” in the complex, which is bounded by the number of k -cliques d_k in G . More formally, β_k equals the rank of the k -th *homology group*. We prove Theorem 1 by showing that, for any constant k and $\varepsilon > 0$, there exists $\delta(\varepsilon, k) > 0$ such that testing whether $\beta_k \geq (1 - \delta)d_k$ for some fixed $\delta \leq \delta(\varepsilon, k)$ reduces to tolerantly testing $(k + 2)$ -clique-freeness. We prove the result for

$$\delta(\varepsilon, 0) = \sqrt{2\varepsilon}, \quad \delta(\varepsilon, 1) = \varepsilon/3, \quad \delta(\varepsilon, k) = 1/\text{tower}(k^4 \log(1/\varepsilon)) \quad (k > 1).$$

Here the tower(ℓ)-function denotes a height- ℓ tower of powers of 2's – this explains the extra quantifier in “(very) large Betti numbers”. Nonetheless, this property is neither trivial for constant k and ε nor monotone or hereditary. To see this, consider the $(k + 1)$ -partite graph which has $d_k = (\frac{n}{k+1})^{k+1}$ and $\beta_k = (\frac{n}{k+1} - 1)^{k+1}$ [BSG⁺22, Proposition 1], and so $\beta_k/d_k = 1 - O(k^2/n)$. This shows that for any k and $\delta > 0$ there exist graphs with the property $\beta_k \geq (1 - \delta)d_k$. Moreover, the quantity β_k/d_k increases as a function of n , so the property cannot be monotone or hereditary.

Betti numbers in the bounded-degree model. This work was partly motivated by the result of Elek [Ele10] in the *bounded-degree model*. In this model, the graph is assumed to have a constant bound d on the vertex degrees, and a query reveals the (at most d) neighbours of a vertex. Elek showed that, for any $\varepsilon > 0$ and with a number of queries only dependent on ε , it is possible to return an estimate $\hat{\beta}_k$ satisfying $\hat{\beta}_k = \beta_k \pm \varepsilon n$ for $k < d$ (for $k \geq d$ necessarily $d_k = \beta_k = 0$). The proof is based on (sparse) graph limits, and is very different from our approach.

Unfortunately, such a result in the dense graph model is not possible: returning an estimate $\hat{\beta}_k = \beta_k \pm \varepsilon n$ (or even $\hat{\beta}_k = \beta_k \pm \varepsilon d_k$) requires $\Omega(n)$ many queries in the dense graph model. To see this, consider the case $k = 0$ for which $d_0 = n$ and β_0 equals the number of connected components of the graph. The cycle graph has $\beta_0 = 1$ while any graph with $\leq n/2$ edges has $\beta_0 \geq n/2$. However, it takes $\Omega(n)$ queries to distinguish these graphs in the dense model. This motivates the weaker formulation of large Betti number testability that we use in our work.¹

Quantum algorithms for large Betti numbers. Another motivation came from a recent stream of works on quantum algorithms for estimating Betti numbers (see e.g. [LGZ16, GCD22, Hay22, CK22]). Under certain conditions (e.g., a form of well-conditionedness), these works suggest an exponential quantum speedup over classical algorithms, returning an estimate $\hat{\beta}_k = \beta_k \pm \varepsilon n^k$ in time $\text{poly}(n, k, 1/\varepsilon)$. Such an estimate is only relevant for very large Betti numbers (β_k scaling with n^k), and a stringent question is whether “typical” graphs can have such large Betti numbers. In a recent work [AGSS23] a classical benchmark algorithm was proposed based on path-integral Monte Carlo. The algorithm has a polynomial runtime in certain regimes, narrowing down the conditions for a potential exponential quantum speedup.

The current work investigates this question from a new (property testing) perspective, and it yields a tool to investigate whether typical graphs can have a (very) large Betti number.

Open questions. Our work raises a number of open questions. The most obvious one is whether our results can be pushed further. E.g., it might be possible to test more moderately sized Betti numbers, or Betti numbers for non-constant k (the case of interest for quantum algorithms).

Another open direction is to generalize the framework of graph property testing to simplicial complexes more generally. By limiting ourselves to clique complexes, we could phrase our results in the graph property testing language, but this might not be the most natural approach.

Outline of proof and paper. In Section 2 we formally introduce property testing, simplicial complexes, and the necessary matroid theory.

The first contribution appears in Section 3. We use the matroid notion of independence to relate the Betti number β_k to the number of “independent” K_{k+2} cliques in the graph, and we bound the total number of cliques as a function of the number of independent cliques. Using the Euler

¹Note also that the contrapositive, having a small Betti number, is trivial to test. E.g., a graph cannot be far from having small Betti number β_0 since we can always add a cycle, thereby setting $\beta_0 = 1$.

characteristic we show that having a large Betti number is equivalent to having few independent K_{k+2} subgraphs.

Finally, in Section 4, we build on these tools to reduce the problem of testing large Betti numbers to that of (tolerantly) testing clique-freeness, which is known to be testable. In particular, we show that having a large Betti number implies that the graph is close to being K_{k+2} -free, while being far from having a large Betti number implies that the graph is far from being K_{k+2} -free.

2 Preliminaries

In this section we introduce necessary but well-known preliminaries on graph property testing, simplicial complexes and matroid theory.

2.1 Property testing and subgraph freeness

In graph property testing we want to decide if an n -vertex input graph G has a property P or if it is ε -far from any graph satisfying P . In the dense graph model we have query access to elements of the adjacency matrix of G , and we want to minimise the number of queries we make. This model was introduced in [GGR98].

Definition 1 (ε -far). *The distance of two graphs G and G' is defined as*

$$D(G, G') = \frac{\min_{\pi} \{G \Delta \pi(G')\}}{n^2},$$

where π is any permutation of the vertices and Δ denotes the symmetric difference of the two edge sets. We say that G is ε -far from property P if $D(G, G') > \varepsilon$ for all G' satisfying P .

Definition 2 (Property tester). *In the dense graph model, a randomised algorithm A is a property testing algorithm for property P if given query access to the adjacency matrix of the input graph G , it satisfies the following.*

- If G satisfies P then A returns “YES” with probability $\geq 2/3$.
- If G is ε -far from P then A returns “NO” with probability $\geq 2/3$.

A tolerant property testing algorithm satisfies a stronger constraint: for some $\varepsilon_1 < \varepsilon_2$, it distinguishes (i) G being ε_1 -close to P , from (ii) G being ε_2 -far from P .

We call a property P (tolerantly) testable if there is a (tolerant) property testing algorithm such that the number of queries it makes is independent of the input length.

The following well-known lemma (that can be proved using the Szemerédi regularity lemma [Sze78]) has been central to proving many testability results, and we will also use it.

Lemma 1 (Graph removal lemma, [Für95]). *For any graph H and any $\varepsilon > 0$ there exists a $\delta > 0$ such that the following holds: any n -vertex graph G ($|V(H)| < n$) that contains at most $\delta n^{|V(H)|}$ copies of H as subgraphs can be made H -free by removing at most εn^2 edges (i.e. G is ε -close to being H -free).*

It follows almost directly from this result that, for any constant-sized graph H , the property of being H -free is testable [ADL⁺94]. Combined with the fact that every testable property in the dense graph model is also *tolerantly* testable [FN07]², we get the following lemma which we are going to use later.

²More precisely, they prove that for every testable property there is a distance approximation algorithm. This implies tolerant testability.

Lemma 2. *For any graph H the property of H -freeness is tolerantly testable in the dense graph model. The number of queries depends only on the distance parameters $\varepsilon_1, \varepsilon_2$ and on $|H|$.*

2.2 Simplicial complexes

A simplicial complex is a downward closed set family over a set V of vertices. As such, it can be thought of as a higher-dimensional generalisation of graphs (albeit more restrictive than hypergraphs).

Definition 3 (Simplicial complex). *An (abstract) simplicial complex Δ is a set of subsets of the vertex set V , such that if $S \in \Delta$ and $S' \subset S$ then also $S' \in \Delta$. The sets in Δ with cardinality $k + 1$ are called the k -faces of Δ .*

We are going to denote the set of k -faces of a complex by $F_k(\Delta) = \{S \in \Delta, |S| = k + 1\}$ and its size by $d_k(\Delta) = |F_k(\Delta)|$. When it is clear from the context which simplicial complex is being considered, we will write only F_k and d_k . If the largest subset of V that is in the complex is of cardinality $D + 1$ then we say that the complex is D -dimensional.

A *clique complex* is a special case of a simplicial complex, and is defined by some underlying graph G . The sets in the clique complex associated to G are exactly the cliques of G . This implies for instance that a size- $(k + 1)$ subset $S \subseteq V$ is in the complex if (and only if) all the size- k subsets of S are in the complex.

For our analysis we wish to consider unoriented faces. This is in contrast to most of the works on simplicial homology, but for our combinatorial arguments it is sufficient and makes more intuitive sense. The k -chain group C_k of a simplicial complex Δ over an Abelian group \mathbf{G} is defined as $C_k = \{\sum_{i=1}^{d_k} \alpha_i S_i\}$ where $S_i \in F_k(\Delta)$ and $\alpha_i \in \mathbf{G}$. Most sources consider integer coefficients ($\mathbf{G} = \mathbb{Z}$), but for our unoriented case we are going to consider only binary coefficients ($\mathbf{G} = \mathbb{Z}_2$). This corresponds to taking $C_k = 2^{F_k}$, the set of all the subsets of F_k . This way, we will refer to the elements of C_k either as a sum of k -faces or as a set of k -faces – the two are equivalent.

For each $k > 0$, the k -th boundary operator δ_k is a homomorphism that maps a k -face to the sum of the $(k - 1)$ -faces that “surround” the k -face.

Definition 4 (Boundary operator). *For any $k \geq 1$ the k -th boundary operator is a homomorphism $\delta_k : C_k \rightarrow C_{k-1}$. For $S \in F_k(\Delta)$ and $S = \{v_1, v_2, \dots, v_{k+1}\}$ it is defined by $\delta_k(S) = \sum_{i=1}^{k+1} S \setminus \{v_i\}$.*

It follows from this definition that the boundary of a boundary is always zero, i.e., $\delta_k(\delta_{k+1}(\cdot)) = 0$. Sometimes we are going to use the same notation for *boundary vectors*: for $S \in F_k$, $\delta_k(S) \in \{0, 1\}^{d_{k-1}}$, where a coordinate is 1 iff the corresponding $(k - 1)$ -face appears in the boundary of S .

Now we can define the Betti numbers, that are at the center of interest in this article.

Definition 5 (Betti number). *The k -th Betti number β_k of a simplicial complex Δ is the rank of the k -th homology group:*

$$\beta_k(\Delta) = \text{rk}(\ker(\delta_k) / \text{im}(\delta_{k+1})).$$

More intuitively, define a k -dimensional hole as a subset $H \subseteq F_k(\Delta)$ that *has* no boundary and *is* no boundary. Equivalently, $\delta_k(H) = 0$ (and so $H \in \ker(\delta_k)$) and there exists no $H' \subseteq F_{k+1}(\Delta)$ such that $\delta_{k+1}(H') = H$ (and so $H \notin \text{im}(\delta_{k+1})$). Then β_k counts the number of “independent” k -dimensional holes in the complex – where we formalize the notion of independence in the next section.

Our result strongly relies on the so-called Euler characteristic. This is a topological invariant that connects the number of faces in a complex to its Betti numbers.

Definition 6 (Euler characteristic). *The Euler characteristic of a simplicial complex Δ is*

$$\chi(\Delta) = \sum_{k=0}^{\infty} (-1)^k d_k(\Delta) = \sum_{k=0}^{\infty} (-1)^k \beta_k(\Delta).$$

2.3 Matroids

The appropriate notion of independence of faces and of holes comes from matroid theory. A matroid is a downward closed set family with an additional property³ called the exchange property.

Definition 7 (Matroid). *A matroid M over ground set E is a family of subsets $I \subseteq 2^E$ called the independent subsets of E , and which satisfies the following properties.*

1. $\emptyset \in I$.
2. If $A \in I$ and $B \subseteq A$ then $B \in I$.
3. If $A, B \in I$ and $|B| < |A|$ then $\exists v \in A \setminus B$ such that $B \cup \{v\} \in I$.

The easiest example of a matroid is a graph. In this case, the ground set E in the matroid is the edge set of the graph, and we call a subset of edges independent if it is cycle-free. Matroids that can be defined this way by a graph are called *graphic matroids* or cycle matroids.

Another important example is linear independence of vectors. The elements of E are vectors from a vector space, and a subset of them is called independent if the vectors are linearly independent (over a field F). Matroids that can be defined in this way are called *linear matroids* (or representable over F).

The *simplicial matroid* (or simplicial geometry) $M_k(\Delta)$ associated to a simplicial complex Δ is a linear matroid defined as follows. It appears in e.g. [CR70, CL87].

Definition 8 (Simplicial matroid). *The k -simplicial matroid $M_k(\Delta)$ associated to a simplicial complex Δ is the linear matroid whose ground set is the set of boundary vectors $\delta_k(S) \in \{0, 1\}^{d_k-1}$ for $S \in F_k(\Delta)$.*

Motivated by this, we call a subset of k -faces independent if the corresponding boundary vectors are linearly independent (over the field $\{0, 1\}$).

A maximal independent set of a matroid M is called a *basis*. It is well known that all the bases of a matroid have the same size, equal to the *rank* $\text{rk}(M)$ of the matroid. The full k -simplicial matroid $M_k(\Delta_k^{\text{full}})$ is the k -simplicial matroid associated to the full complex $\Delta_k^{\text{full}} = \{S \subseteq V, |S| \leq k+1\}$ that contains all the $k+1$ -subsets as k -faces, but it does not have any higher dimensional face.

Claim 1 (e.g. [CL87], Proposition 6.1.5). $\text{rk}(M_k(\Delta_k^{\text{full}})) = \binom{n-1}{k}$.

For a construction, fix a vertex u of Δ_k^{full} and take the set of k -faces that contain u . It is easy to see that this set of size $\binom{n-1}{k}$ is a basis of the matroid.

³In this sense matroids are a specialisation of simplicial complexes, although we are going to use them in a different way.

3 Betti numbers via independent faces

In this section we connect the number of independent k -faces with the total number of k -faces, and connect the Betti number β_k to the number of independent k - and $(k + 1)$ -faces in the complex. While we could find in the literature variants of some of the claims in this section (see e.g. Remark 1), the key claims were unknown to us before.

The notion of independence of faces in Definition 8 leads to the following useful observation. It is a direct consequence of the fact that a set of k -faces has zero boundary if and only if the sum of the corresponding boundary vectors is the zero vector.

Claim 2. *In a k -dimensional simplicial complex (i.e., $|F_{k+1}| = 0$), a set of k -faces is independent iff no subset of them forms a k -dimensional hole.*

The independence of holes is defined similarly. A k -dimensional hole is a set of k -faces (an element of C_k), and associated to it is a characteristic vector over $\{0, 1\}^{d_k}$. We associated the same kind of (boundary) vectors to $(k + 1)$ -faces: in this sense a k -dimensional hole can be seen as the boundary of a virtual $(k + 1)$ -dimensional object. This way, a set of holes is independent if the corresponding vectors are linearly independent (over field $\{0, 1\}$). An analogue of Claim 2 tells us that a set of k -dimensional holes is independent iff no subset of them (as virtual $(k + 1)$ -faces) forms a $(k + 1)$ -dimensional hole.

Let us denote the rank $\text{rk}(M_k(\Delta))$ of the k -simplicial matroid (i.e. the size of a maximal independent set of k -faces in Δ) by $r_k(\Delta)$. We are going to need a lower bound on this value in terms of the total number of k -faces $d_k(\Delta)$. For the sake of completeness, we also include an upper bound in the statement.

Lemma 3. *For any $0 \leq k < n$ and any simplicial complex Δ*

$$\frac{k+1}{n}d_k(\Delta) \leq r_k(\Delta) \leq \min \left\{ d_k(\Delta), \binom{n-1}{k} \right\}.$$

Proof. Trivially, $r_k(\Delta) \leq d_k(\Delta)$. Moreover, the set of k -faces $F_k(\Delta)$ of any complex Δ can be obtained from that of the full complex $F_k(\Delta_k^{\text{full}})$ by removing faces, and this can only decrease the rank. Combined with Claim 1 we hence get $r_k(\Delta) \leq \binom{n-1}{k}$.

Now let us prove the main part of the claim, which is the lower bound. We use a similar argument to the one below Claim 1. Let u be a vertex in Δ that is included in a maximum number of k -faces (i.e., the vertex with the highest “ k -face-degree”). These k -faces that contain u are independent because each contains a $(k - 1)$ -face that the others do not (the one without u), and this is a non-zero element in their boundary vector. As there are d_k many k -faces in Δ , each incident to $k + 1$ vertices, the average “ k -face-degree” of a vertex is $(k + 1)d_k/n$. Thus the independent set of k -faces defined by u has at least this many k -faces, and so $r_k \geq (k + 1)d_k/n$. \square

The next lemma shows a nice connection between the rank, the number of faces and the Betti number. For $k = 0$ the formula gives the well-known graph formula $c = n - t$, where t is the number of edges in a spanning forest, n is the number of vertices, and c is the number of connected components. When $k = 1$ and the underlying graph is connected and planar, it gives the Euler formula $n + f = e + 2$ (with n the number of vertices, f the number of faces surrounded by edges and e the number of edges) because $\beta_1 = f - 1$, $d_1 = e$, $r_1 = n - 1$ and $r_2 = 0$.

Lemma 4. *For any simplicial complex, $\beta_k = d_k - r_k - r_{k+1}$.*

Proof. The proof goes by induction. Let Δ denote the simplicial complex being considered and let us take a basis of the k -simplicial matroid over Δ . For the base case, we consider the subcomplex where this is the set of all k -faces and all the higher dimensional faces are removed, in which case $r_k = d_k$ and $\beta_k = r_{k+1} = 0$ and so the formula holds. In the inductive step we will put back all of the removed faces. We start by adding the rest of the k -faces one by one, and we argue that each added face creates exactly one new independent hole.

First, note that adding a dependent k -face S to the complex creates at least one hole (otherwise we could have added it to the basis by Claim 2). Moreover, the hole is independent of the previous ones because it contains the face S , which no other hole contains so far.

Then, we prove that adding a k -face creates at most one hole. By contradiction, assume that there is a k -face S such that when added to the set, more than one new independent holes are created. We consider two of them, $\{S, R_1, \dots, R_p\}$ and $\{S, T_1, \dots, T_q\}$, which we call the “ R -hole” and the “ T -hole”. Necessarily they have zero boundary (we denote the boundary vectors the same way as the k -faces):

$$\begin{aligned} S + R_1 + \dots + R_p &= 0 \\ S + T_1 + \dots + T_q &= 0. \end{aligned}$$

Adding the equations shows that $\{R_1, \dots, R_p, T_1, \dots, T_q\}$ must also be a hole, call it the “ RT -hole”. It does not contain S , and so must have been present before adding S . However, by construction, the R -, T - and RT -holes are not independent, and so we get a contradiction.

Let Δ_k denote the complex we have now: it contains exactly the faces of Δ up to dimension k , and no faces of higher dimension. So far we proved that $r_k = d_k - \beta_k(\Delta_k)$. Let us consider the set of “potential k -holes” in Δ_k , i.e. sets H of cardinality $k+2$ where all the $(k+1)$ -subsets of H are in Δ_k . These are those holes of Δ_k that may be filled by $(k+1)$ -faces in Δ .

Now we continue the induction by adding to Δ_k the $(k+1)$ -faces of Δ one by one (and in the end the higher dimensional faces as well) to get back Δ . Each $(k+1)$ -face fills a potential hole, and it is independent of the previously added ones if and only if the hole being filled is independent of the previously filled ones (as they are the same subset). Thus, every time Δ gains an independent $(k+1)$ -face it loses an independent k -hole. This finishes the proof, as adding faces of dimension larger than $k+1$ does not change any parameter in the claim. \square

In the special case where $k = 0$ and the graph defined by the vertices and edges of Δ is connected, we have $\beta_0 = 0$, $d_0 = n$ and $r_{k+1} = n-1$ (a spanning tree of the graph is a maximal independent edge set). Thus, r_0 has to be defined as 0, which makes sense if we think about r_k as $r_k = \dim(\text{im}(\delta_k))$.

Remark 1. Let $T_\ell = \{S \subseteq V, |S| = \ell + 1\}$ and $A \subseteq T_k$. In previous works [CR70, Cor78] only complexes of the form $\Delta_{k-1}^{\text{full}} \cup A$ are analysed in detail. This family of complexes is not enough to express the k -th Betti number of an arbitrary simplicial complex. For example, for this restricted class of complexes Cordovil [Cor78, Proposition 1.2] showed that $r_k = d_k - \beta_k$, which is only a special case of Lemma 4 (with $r_{k+1} = 0$).

An easy consequence of Lemma 4 is the following statement.

Claim 3. For any simplicial complex Δ and $k \geq 1$, $\beta_k(\Delta) \leq \binom{n-1}{k+1}$.

Proof. In Δ_k^{full} we have $\beta_k = d_k - r_k - 0 = \binom{n}{k+1} - \binom{n-1}{k} = \binom{n-1}{k+1}$, and removing k -faces or adding $(k+1)$ -faces cannot increase this value. \square

4 Testing large Betti numbers

In this section we turn to our main result, proving that we can test whether a Betti number is large. We now state our main theorem again.

Theorem 2 (formal version of Theorem 1). *Consider a clique complex Δ . For any constant k and $\varepsilon > 0$, there exists $\delta(\varepsilon, k) > 0$ such that the property of having k -th Betti number $\beta_k(\Delta) \geq (1 - \delta)d_k$ is testable for any $\delta \leq \delta(\varepsilon, k)$ (with distance parameter ε).*

Even though our results in Section 3 hold for general simplicial complexes, the main theorem is restricted to clique complexes. The reason for this is that we wish to phrase our results in the well-established setting of graph property testing. By constraining ourselves to clique complexes, having a large Betti number becomes a graph property (of the underlying graph) rather than a property of an abstract simplicial complex. Also, this way we can use some previous results from graph property testing, like the tolerant testability of subgraph freeness (Lemma 2).

4.1 Warm-up: testing many components

The 0-th Betti number β_0 of a clique complex Δ equals the number of connected components of the underlying graph G . As an informal warm-up and a blueprint for the general case, we show how to test whether $\beta_0 \geq (1 - \delta)n$. The argument involves two reductions.

First, we argue that having a large 0-th Betti number is equivalent to having few independent edges. From Lemma 4 we get that $\beta_0 = n - r_1 - r_0$ where $r_0 = 0$. Thus, $\beta_0 \geq (1 - \delta)n$ is equivalent to $r_1 \leq \delta n$, and so testing large β_0 reduces to testing whether G has a small number of independent edges.

Now comes the second reduction, in which we argue that testing whether G has few independent edges can be reduced to *tolerantly* testing edge-freeness. For this, note that if G has $r_1 \leq \delta n$ independent edges then the total number of edges $|E| \leq \binom{\delta n + 1}{2} < \delta^2 n^2 / 2 + O(n)$,⁴ and so G must be $1.1\delta^2/2$ -close to being edge-free. On the other hand, if G is ε -far from having $r_1 \leq \delta n$, then G must also be ε -far from having $r_1 = 0$, i.e. from being edge-free. So we reduced the problem of testing $\beta_0 \geq (1 - \delta)n$ to that of tolerantly testing edge-freeness (with parameters $\varepsilon_1 = 1.1\delta^2/2$ and $\varepsilon_2 = \varepsilon$). It remains to note that edge-freeness is tolerantly testable by Lemma 2.

4.2 General case

We now turn to proving our general result (Theorem 2), that having a Betti number $\beta_k \geq (1 - \delta)d_k$ is testable for constant k . Following the blueprint from the previous section, we first reduce the problem to testing whether there are few independent $(k + 1)$ -faces, and then reduce testing few independent $(k + 1)$ -faces to tolerantly testing $(k + 2)$ -clique freeness.

We consider a clique complex Δ with underlying graph G . From Lemma 4 we get that

$$d_k - r_{k+1} \geq \beta_k = d_k - r_{k+1} - r_k, \tag{1}$$

which directly implies the following lemma.

Lemma 5 (Large Betti number \preceq few independent cliques). *In a clique complex Δ with underlying graph G , if $\beta_k \geq (1 - \delta)d_k$ then $r_{k+1} \leq \delta d_k$. If G is ε -far from having $\beta_k \geq (1 - \delta)d_k$ in Δ then it is also ε -far from having $r_{k+1} = 0$, i.e., G is ε -far from K_{k+2} -freeness.⁵*

⁴From Lemma 3 we would get $|E| \leq r_1 n / 2 \leq \delta n^2 / 2$. We get the better bound by noticing that if there are δn independent edges, then we have a maximum number of edges if all the independent edges are in the same connected component and this component is a $K_{\delta n + 1}$.

For our second reduction, we use Lemma 3, which tells us that if $r_{k+1} \leq \delta d_k$ then

$$d_{k+1} \leq \frac{\delta}{k+2} n d_k \leq \frac{\delta}{k+2} n \binom{n}{k+1} \leq \frac{\delta}{(k+2)!} n^{k+2}.$$

Combined with Lemma 5 we get that $\beta_k \geq (1-\delta)d_k$ implies $d_{k+1} \leq \frac{\delta}{(k+2)!} n^{k+2}$. We see that a large Betti number implies a small number of K_{k+2} 's in the graph, while being far from having a large Betti number implies being far from K_{k+2} -freeness (by Lemma 5).

In fact, by the graph removal lemma (Lemma 1), a small number of K_{k+2} 's implies that the graph is close to being K_{k+2} -free. More specifically, for any $\varepsilon' > 0$ there exists $\delta = \delta(k, \varepsilon') > 0$ such that if G has at most $\frac{\delta}{(k+2)!} n^{k+2}$ many K_{k+2} 's then G is ε' -close to being K_{k+2} -free. By picking (say) $\varepsilon' = \varepsilon/2$, it follows that we can test whether $\beta_k \geq (1-\delta)d_k$ by tolerantly testing whether G is $\varepsilon/2$ -close or ε -far from K_{k+2} -freeness. By Lemma 2 we know that K_{k+2} -freeness is indeed tolerantly testable, and this proves our main Theorem 2.

To finish, we comment on the scaling of $\delta(k, \varepsilon)$. The current best upper bound in the graph removal lemma requires $\delta(k, \varepsilon) \leq 1/\text{tower}(5(k+2)^4 \log(1/\varepsilon))$ [Fox11], where $\text{tower}(i)$ is a tower of twos of height i (e.g., $\text{tower}(3) = 2^{2^2}$). For the case of $k = 0$ we could avoid this: recall from Section 4.1 that $r_1 \leq \delta n$ implies that G is $\delta^2/2$ -close to being edge-free. Similarly, for $k = 1$ we can get a better bound: $r_2 \leq \delta n^2$ implies that G is 3δ -close to being triangle-free. Indeed, if we remove all the edges of a maximal independent triangle set (at most $3\delta n^2$ edges), then any remaining triangle in the graph would contradict the maximality of the chosen set. We leave the extension of similar arguments to higher k for future work. In conclusion, we get a tester that distinguishes $\beta_k \geq (1-\delta)d_k$ from being ε -far under the constraints

$$\delta < \sqrt{2\varepsilon} \quad (k = 0), \quad \delta < \varepsilon/3 \quad (k = 1), \quad \delta < 1/\text{tower}(5(k+2)^4 \log(1/\varepsilon)) \quad (k > 1).$$

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References

- [ADL⁺94] Noga Alon, Richard A Duke, Hanno Lefmann, Vojtech Rodl, and Raphael Yuster. The algorithmic aspects of the regularity lemma. *Journal of Algorithms*, 16(1):80–109, 1994.
- [AGSS23] Simon Apers, Sander Gribling, Sayantan Sen, and Dániel Szabó. A (simple) classical algorithm for estimating betti numbers. *Quantum*, 7:1202, 2023.
- [AS05] Noga Alon and Asaf Shapira. Every monotone graph property is testable. In *Proceedings of the thirty-seventh annual ACM Symposium on Theory of Computing*, pages 128–137, 2005.
- [AS08] Noga Alon and Asaf Shapira. A characterization of the (natural) graph properties testable with one-sided error. *SIAM Journal on Computing*, 37(6):1703–1727, 2008.

⁵For this last point, note that by Lemma 4 we have that $r_{k+1} \leq \delta d_k - r_k$ is equivalent to $\beta_k \geq (1-\delta)d_k$. Hence, a graph that is ε -far from $\beta_k \geq (1-\delta)d_k$ must also be ε -far from $r_{k+1} \leq \delta d_k - r_k$ (and so certainly from being K_{k+2} -free).

- [BSG⁺22] Dominic W Berry, Yuan Su, Casper Gyurik, Robbie King, Joao Basso, Alexander Del Toro Barba, Abhishek Rajput, Nathan Wiebe, Vedran Dunjko, and Ryan Babbush. Quantifying quantum advantage in topological data analysis. *arXiv:2209.13581*, 2022.
- [CK22] Marcos Crichigno and Tamara Kohler. Clique homology is QMA1-hard. *arXiv:2209.11793*, 2022.
- [CL87] Raul Cordovil and Bernt Lindström. Simplicial matroids. In *Combinatorial Geometries*. Cambridge University Press, 1987.
- [Cor78] Raul Cordovil. Sur les géométries simpliciales. *Comptes Rendus hebdomadaires des séances de l’Académie des Sciences*, 286(25):1219–1222, 1978.
- [CR70] Henry H. Crapo and Gian-Carlo Rota. On the Foundations of Combinatorial Theory II. Combinatorial Geometries. *Studies in Applied Mathematics*, 49(2):109–133, 1970.
- [Ele10] Gábor Elek. Betti numbers are testable. *Fete of combinatorics and computer science*, pages 139–149, 2010.
- [FN07] Eldar Fischer and Ilan Newman. Testing versus estimation of graph properties. *SIAM Journal on Computing*, 37(2):482–501, 05 2007.
- [Fox11] Jacob Fox. A new proof of the graph removal lemma. *Annals of Mathematics*, pages 561–579, 2011.
- [Für95] Zoltán Füredi. Extremal hypergraphs and combinatorial geometry. In *Proceedings of the International Congress of Mathematicians*, pages 1343–1352. Birkhäuser Basel, 1995.
- [GCD22] Casper Gyurik, Chris Cade, and Vedran Dunjko. Towards quantum advantage via topological data analysis. *Quantum*, 6:855, 2022.
- [GGR98] Oded Goldreich, Shari Goldwasser, and Dana Ron. Property testing and its connection to learning and approximation. *Journal of the ACM*, 45(4):653–750, jul 1998.
- [Gol10] Oded Goldreich. Introduction to testing graph properties. *Property testing: current research and surveys*, pages 105–141, 2010.
- [Hay22] Ryu Hayakawa. Quantum algorithm for persistent Betti numbers and topological data analysis. *Quantum*, 6:873, 2022.
- [LGZ16] Seth Lloyd, Silvano Garnerone, and Paolo Zanardi. Quantum algorithms for topological and geometric analysis of data. *Nature communications*, 7(1):1–7, 2016.
- [Sze78] Endre Szemerédi. Regular partitions of graphs. In *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, volume 260 of *Colloq. Internat. CNRS*, pages 399–401. CNRS, Paris, 1978.