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Deep learning scheme for forward utilities using ergodic BSDEs*

Guillaume Broux-Quemerais[†] Sarah Kaakaï[†] Anis Matoussi[†] Wissal Sabbagh[†]

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Abstract

In this paper, we present a probabilistic numerical method for a class of forward utilities in a stochastic factor model. For this purpose, we use the representation of dynamic consistent utilities with mean of ergodic Backward Stochastic Differential Equations (eBSDEs) introduced by Liang and Zariphopoulou in [27]. We establish a connection between the solution of the ergodic BSDE and the solution of an associated BSDE with random terminal time τ , defined as the hitting time of the positive recurrent stochastic factor V . The viewpoint based on BSDEs with random horizon yields a new characterization of the ergodic cost λ which is a part of the solution of the eBSDEs. In particular, for a certain class of eBSDEs with quadratic generator, the Cole-Hopf transform leads to a semi-explicit representation of the solution as well as a new expression of the ergodic cost λ . The latter can be estimated with Monte Carlo methods. We also propose two new deep learning numerical schemes for eBSDEs, where the ergodic cost λ is optimized according to a loss function at the random horizon τ or taking into account the whole trajectory. Finally, we present numerical results for different examples of eBSDEs and forward utilities along with the associated investment strategies.

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Introduction

In this paper, we are interested in the numerical approximation of certain classes of forward performance processes (or consistent dynamic utilities), and their associated optimal decision criterion. Introduced by [31], consistent dynamic utilities offer an interesting alternative to the classical setting of expected utility maximization at a terminal time. This forward-looking approach enables the dynamic adjustment of decision criteria, starting from preferences which are known at an initial time, rather than imposing a potentially distant and arbitrary time horizon. This approach also maintains time consistency within the given investment or decision-making context. More precisely, time consistency is ensured through the consistency condition, which states that the (random) utility $U(t, X_t^\pi)$ taken along the observable process X_t^π (typically the wealth) resulting from any admissible decision/strategy process π should be a supermartingale, and that there exists an optimal strategy such that the preference process is a martingale.

Since their introduction, there has been tremendous theoretical developments in the field. In a general setting, [15] established a sufficient condition for time-consistency when the dynamic utility is an Itô random field. The consistent dynamic utility verifies a non linear SPDE of HJB type. This work has been extended to consistent utility of investment and consumption in [13], and has been applied, for instance, to derive consistent utilities for stochastic factor market models (see e.g. [32], [1]), from a PDE viewpoint. Consistent dynamic utilities have found diverse applications over recent years, including but not limited to option valuation, insurance, mean field games ([26], [11]), long term interest rate modeling ([14]), risk measures ([9] or more recently pension design ([21],[33])). Surprisingly, the subject of numerical methods for dynamic utilities remains largely unexplored, despite its critical importance for practical applications.

In [18], a general approach is proposed using strong approximations of compounds of random maps. In this paper, we take a different approach to introduce new numerical schemes for the class of so-called homothetic dynamic utilities, taking advantage of the representation of these processes using ergodic BSDEs, introduced in [27].

We investigate the representation of an agent's preferences investing in an incomplete financial market, where stock price dynamics are driven by a stochastic factor $(V_t)_{t \geq 0}$. Homothetic dynamic utilities are expressed as separable functionals, denoted by $U(t, x) = u(x)e^{f(t, V_t)}$, where u is a standard exponential or power utility function (the expression is additive in the logarithmic case). The main result of [27] provides a representation of the function f with mean of the unique Markovian solution of a related ergodic BSDE.

Ergodic BSDEs have first been introduced in [16], with the aim to study an optimal ergodic control problem, expressed as the minimization of an averaged cost function over an infinite time horizon. Formally, the solution of an ergodic BSDE, which is an infinite horizon BSDE, is a triplet (Y, Z, λ) , where Y and Z are adapted processes and λ is a real number, which solves:

$$Y_t = Y_T + \int_t^T (F(V_s^v, Z_s) - \lambda) ds - \int_t^T Z_t^\top dW_t, \quad \forall 0 \leq t \leq T < +\infty.$$

Those equations are usually studied under Lipschitz assumptions on the driver, for a stochastic factor V with constant volatility and drift that satisfy a dissipative condition as discussed in [16]. The

dissipativity assumption has been relaxed in [10] for stochastic factors with constant volatility in general Hilbert spaces. Within this framework, the component Z of the ergodic BSDE solution is bounded, facilitating the extension of existence and uniqueness results to drivers that are locally Lipschitz in z . This broader application is detailed in [27] for ergodic BSDEs related to exponential and power forward utilities. In [22], the authors study ergodic BSDEs under weak dissipativity assumption on V , with a non constant and possibly unbounded volatility. However, for this last framework, the Markovian solution z may not be bounded, so that results are stated under uniform Lipschitz assumption on the generator.

The unknown ergodic cost λ can be interpreted in several ways. First, it is the long term growth rate of an associated risk sensitive control problem, as mentioned in [27]. It can also be represented as the linear growth rate of the initial value of the solution of an analogue finite horizon BSDE with respect to the terminal time T when the latter goes to infinity, see [22]. However, numerical schemes for the simulation of BSDEs become unstable for large horizon, so that this representation cannot be used for the approximation of the ergodic cost.

Over the past few years, machine learning algorithms have been extensively studied for their ability to solve high dimensional non-linear PDEs, based on the BSDE representation of their solution (see e.g. [19], [7], [24], [17], [25]). Two main types of neural network algorithm have been developed. The first relies on a global loss function for solving BSDEs and was initially proposed in [19]. The Deep BSDE solver consists in the training of as many neural networks as time steps to approximate the component Z of the solution. Y is computed with a forward discretisation starting from Y_0 , which is considered as a training parameter of the model. Then, Y_0 and the neural networks parameters are optimized according to a loss function on the terminal value of the discretized scheme. A convergence study of the Deep BSDE is developed in [20] and [7] shows that sharing one neural network across all time steps is more efficient. The second class of algorithms relies on a local approach and consists in solving local optimization problems at each time steps. First introduced in [17], [25], those methods use two neural networks to approximate both processes Y and Z . Local loss functions are constructed based on the iteration of time discretization of BSDEs with the terminal condition.

There are two main challenges in the simulation of ergodic BSDEs:

1. There is an additional real unknown λ . This makes the usual backward discretisation equation for Y coupled with λ .
2. This is an infinite horizon BSDE, which has to stand for all $T > 0$, for all $0 \leq t \leq T$. Thus, there is no 'terminal condition' as for the simulation of finite horizon BSDE.

In this paper, we take advantage of the recurrence property of the stochastic factor V in order to provide an horizon as well as a terminal condition to the problem of simulating the solution of an ergodic BSDE. In the case of ergodic BSDEs derived from forward utilities, an initial condition is naturally given since the initial agent's utility u_0 is known. We will thus assume that the value $y(V_0) = y(v_0) = y_0$ is fixed, which ensures uniqueness of the Markovian solution. Define τ as the first return time of the diffusion V to v_0 . This stopping time is almost surely finite and the value $y(V_\tau^{v_0}) = y(v_0) = y_0$ is thus also known. Then the solution of the ergodic BSDE above is also solution of the BSDE with random terminal time τ and generator $F(v, z) - \lambda$. The unknown λ is uniquely

determined by the fixed initial condition $Y_0 = Y_\tau = y_0$. Under additional exponential integrability assumptions on τ and using uniqueness result from [34], we show that those solutions in fact coincides on $[0, \tau]$.

In some cases, the representation result for linear BSDE with random terminal time leads to a semi-explicit representation of the solution as well as a new characterization of the ergodic cost λ , depending on the random horizon τ . More precisely, this result holds for linear BSDE as the one associated to exponential utility when assuming there is no constraint on the portfolio. More generally, this last result also stands for generators purely quadratic in z by first applying the Cole-Hopf transform, as it is the case for power utility with no constraints on the portfolio. For these examples, regression and Monte Carlo methods can be used to simulate the solution of the associated ergodic BSDEs.

We also present two deep-learning based methods for the simulation of ergodic BSDEs which allow to tackle simultaneously the approximation of the ergodic cost λ and the usual unknown processes Y and Z . In the context of ergodic BSDE, the initial value $Y_0 = y_0$ is known and will thus not be learned. We use a forward a discretisation starting from $Y_0 = y_0$, and instead approximate λ as a trainable parameter of the model. We first investigate a global solver denoted GeBSDE, approximating Z with one neural network common across all time steps. The optimization is performed according to a loss function at the random horizon τ , the output aiming to match the terminal value $Y_\tau = y_0$. We present a second algorithm denoted LAeBSDE, based on a local approach, approximating Y and Z with two distinct neural networks. The optimization is then performed according to the aggregation of local loss functions at each time steps. We provide some numerical tests to evaluate the performance of both algorithms. We investigate two examples with explicit solutions, where the driver does not depend on z presented in [22], and two examples with a driver that is quadratic in z , associated with power forward utilities adapted from [27]. In the latter case, our algorithm also facilitates the simulation of the optimal portfolio for the optimization problem given by the time consistency condition.

The paper is organized as follows. In Section 1, we introduce the class of consistent dynamic utilities we study and their link with ergodic BSDEs. In Section 2, after stating the recurrence properties of the stochastic factor V , we present the viewpoint of BSDE with random terminal time τ , with fixed initial and terminal condition. We show the solution of such equation coincides with the solution of the ergodic BSDE on $[0, \tau]$. We study a backward discretisation and the associated error estimate in Section 3. Finally we present the deep-learning algorithms for ergodic BSDE and gather our numerical results in Section 4.

Notations:

All stochastic processes in the sequel are defined on a standard probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is the natural filtration generated by a d -dimensional Brownian motion W , and is assumed right continuous and complete. For $x \in \mathbb{R}^d$, we denote x^\top the transpose of vector x , $\|\cdot\|$ the usual norm $\|x\| = \text{Tr}(xx^\top)^{\frac{1}{2}}$ and $\text{dist}(x, \Pi)$ the distance function of x to a closed convex subset $\Pi \subset \mathbb{R}^d$. We denote L^2 the space of square integrable random variables and also introduce the usual

space of solution for $\gamma \in \mathbb{R}$ and τ a \mathbb{F} stopping time:

$$\mathcal{S}^2(\gamma, \tau) = \left\{ (\varphi_t)_{t \geq 0}, \text{ real valued progressively measurable process s.t. } \mathbb{E} \left[\sup_{0 \leq s \leq \tau} e^{\gamma s} |\varphi_s|^2 \right] < \infty. \right\}$$

1 Forward utilities and ergodic BSDEs

The aim of this paper is to investigate the numerical approximation of different classes of time consistent homothetic forward (or dynamic) utilities, as introduced in [27], which allow the modelling of an agent's dynamic preferences as she invests in a stochastic factor financial market.

We start this section by introducing the setting and some useful results of [27]. In particular, we are interested in the representation of consistent homothetic forward utilities involving the unique Markovian solution of some ergodic BSDEs. This representation motivates our study of numerical approximations for ergodic BSDEs.

1.1 Consistent dynamic utilities and link with ergodic BSDEs

Forward utilities generalize the notion of utility function. Formally, a dynamic utility $U = (t, x, \omega) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}$ is a collection of random utility functions such that:

- For all $t \geq 0$, for all $x \in \mathbb{R}_+$, $U(t, x)$ is \mathcal{F}_t -measurable.
- The functions $x \in \mathbb{R}^+ \mapsto U(t, x, \omega)$ are nonnegative, strictly concave increasing functions of class \mathcal{C}^2 on $]0, \infty[$, (ω, t) a.s.
- $u_0 := U(0, \cdot)$ is a standard (deterministic) utility function.

Homothetic forward utilities, introduced in [27], are functions of the agent's wealth x and are characterized by one of the following forward utilities:

$$\text{- Logarithmic case } U(t, x) = \ln(x) + f(V_t, t), \quad (1.1)$$

$$\text{- Exponential case } U(t, x) = -e^{-\gamma x + f(V_t, t)}, \quad \gamma \in (0, 1), \quad (1.2)$$

$$\text{- Power case } U(t, x) = \frac{x^\delta}{\delta} e^{f(V_t, t)}, \quad \delta \in (0, 1), \quad (1.3)$$

where f is a deterministic function that will be specified hereafter, and V is a d' -dimensional diffusion process with local characteristics $\mu : \mathbb{R}^{d'} \rightarrow \mathbb{R}^{d'}$ and constant volatility matrix κ defined as:

$$dV_t^i = \mu^i(V_t)dt + \sum_{j=1}^d \kappa^{ij} dW_t^j, \quad V_0^i \in \mathbb{R}. \quad (1.4)$$

The agent invests in an incomplete market consisting in one riskless bond and n stocks. Assuming the numeraire to be the bond, the stock prices dynamics discounted by the interest rate are given for $i = 1, \dots, n$, by:

$$dS_t^i = S_t^i \left(b^i(V_t)dt + \sum_{j=1}^d \sigma^{ij}(V_t) dW_t^j \right), \quad (1.5)$$

where the local characteristics are driven by the d' -dimensional stochastic factor V given by (1.4), and verify the following assumption:

Assumption 1.1. 1. The functions $b = (b^i)_{1 \leq i \leq n}$ and $\sigma = (\sigma^{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq d}}$ are uniformly bounded and for all $v \in \mathbb{R}^d$, the matrix $\sigma(v)$ has full row rank n .

2. The risk premium vector $\theta = \sigma^\top (\sigma \sigma^\top)^{-1} b$ is a uniformly bounded and Lipschitz continuous function.

The agent invests a proportion $\bar{\pi} = (\bar{\pi}^1, \dots, \bar{\pi}^n)^\top$ of her wealth X^π in the the n risky assets. For an initial value $X_0^\pi = x_0 \in \mathbb{R}^+$, assuming the self-financing condition holds and rescaling the strategy vector by the volatility, the dynamic of the wealth process X can be written as:

$$dX_t^\pi = X_t^\pi \pi_t \cdot (\theta(V_t)dt + dW_t), \quad \pi_t = \sigma(V_t)^\top \bar{\pi}_t \in \mathbb{R}^d. \quad (1.6)$$

For each $t \geq 0$, the strategy $(\pi_t)_{t \geq 0}$ is assumed to be in a closed and convex set $\Pi \subset \mathbb{R}^d$. Admissible strategies are also required to be BMO. We refer to [27] for further details.

Remark 1.1. For exponential performance process, it is more convenient to use the discounted amount of wealth invested in the stock $\alpha_t = X_t^\pi \pi_t$ as control variable, leading to the following wealth process dynamics:

$$dX_t^\alpha = \alpha_t^\top (\theta(V_t)dt + dW_t). \quad (1.7)$$

A forward utility is said to be consistent if it is a supermartingale along the wealth process for any admissible control π and a martingale along the optimal wealth process. The optimal strategy thus gives maximal satisfaction to the agent, which is preserved at all times in the future. This additional time consistency property makes the notion of forward utilities coherent with the dynamic programming principle.

Definition 1.1 (*Consistent dynamic utility / forward utility*). A forward utility is a dynamic utility U satisfying the time consistency property:

- For any admissible strategy π , $U(t, X_t^\pi)$ is a supermartingale.
- There exists an admissible strategy π^* such that $U(t, X_t^{\pi^*})$ is a martingale.

When U is an Ito-random field with sufficient regularity conditions on its local characteristics, El Karoui and Mrad obtained in [15] a sufficient consistency condition of HJB type characterizing the drift of consistent utilities, as well as the optimal strategy under this condition. In particular, U is solution of a non-linear HJB-SPDE under this sufficient assumption for consistency. In the case of the homothetic forward utilities (1.1)-(1.3) with stochastic factor, the HJB-SPDE on U is equivalent to a PDE for the deterministic function f (see [27]). As pointed out by the authors, the problem is ill posed. However, a characterisation of f with mean of the Markovian solution of a related ergodic BSDE is given in [27], which allows us to develop numerical schemes for homothetic dynamic utilities.

Homothetic forward utility and ergodic BSDE Informally, an ergodic BSDE with generator F is a backward stochastic differential equation on an infinite horizon, whose solution is a triplet (Y, Z, λ) where Y, Z are adapted processes and $\lambda \in \mathbb{R}$, and satisfies for any $T > 0$, \mathbb{P} -a.s for any $0 \leq t \leq T$:

$$Y_t = Y_T + \int_t^T F(V_s, Z_s) ds - \lambda(T-t) - \int_t^T Z_s^\top dW_s \quad (1.8)$$

$$dV_t^i = \mu^i(V_t) dt + \sum_{j=1}^d \kappa^{ij} dW_t^j, \quad V_0^i \in \mathbb{R}, \quad 1 \leq i \leq d'. \quad (1.9)$$

This class of ergodic BSDE was first introduced in [16]. Existence and uniqueness results in our framework are recalled in Section 1.2 below. Note that the infinite horizon is coherent with the willingness to adapt dynamically the utility as time passes.

Let us introduce the generators associated with the different homothetic dynamic utilities defined in (1.1), (1.2), (1.3).

- *Logarithmic case*: for $v \in \mathbb{R}^{d'}$:

$$F_{\log}(v) = -\frac{1}{2} \text{dist}^2(\Pi, \theta(v)) + \frac{1}{2} \|\theta(v)\|^2. \quad (1.10)$$

- *Exponential case*: for $(v, z) \in \mathbb{R}^{d'} \times \mathbb{R}^d$:

$$F_{\exp}(v, z) = \frac{1}{2} \gamma^2 \text{dist}^2\left(\Pi, \frac{z + \theta(v)}{\gamma}\right) - \frac{1}{2} \|z + \theta(v)\|^2 + \frac{1}{2} \|z\|^2. \quad (1.11)$$

- *Power case*: for $(v, z) \in \mathbb{R}^{d'} \times \mathbb{R}^d$:

$$F^\delta(v, z) = \frac{\delta(\delta-1)}{2} \text{dist}^2\left(\Pi, \frac{\theta(v) + z}{1-\delta}\right) + \frac{\delta}{2(1-\delta)} \|\theta(v) + z\|^2 + \frac{1}{2} \|z\|^2, \quad (1.12)$$

Provided that equation (1.8) with one of the above generator admits a Markovian solution $(y(V_t), z(V_t), \lambda)_{t \geq 0}$, one can show using Ito's formula that the homothetic forward utilities U defined by (1.1)-(1.3), with

$$f(v, t) = y(v) - \lambda t,$$

are time consistent homothetic dynamic utilities, as defined in Definition 1.1. We summarize these results in the following proposition, while existence and uniqueness results of Markovian solutions to (1.8) are recalled the next section.

Proposition 1.1 (Theorem 3.2 and 4.2, [27]). *Let $(y(V_t), z(V_t), \lambda)_{t \geq 0}$ be a Markovian solution of the ergodic BSDE (1.8) with driver F given by (1.10) (resp. (1.11), (1.12)).*

Then, the associated logarithmic (resp. exponential, power) forward utility U (1.1) (resp. (1.2), (1.3)) with $f(\cdot, t) = y(\cdot) - \lambda t$ is consistent. Furthermore, the optimal strategy is given by:

$$\text{Logarithmic case } \pi_t^* = \text{Proj}_\Pi(\theta(V_t)). \quad (1.13)$$

$$\text{Exponential case } \alpha_t^* = \text{Proj}_\Pi\left(\frac{z(V_t) + \theta(V_t)}{\gamma}\right). \quad (1.14)$$

$$\text{Power case } \pi_t^* = \text{Proj}_\Pi\left(\frac{z(V_t) + \theta(V_t)}{1-\delta}\right). \quad (1.15)$$

Motivated by this representation, the aim of this paper is to propose numerical methods for the simulation of Markovian solutions of ergodic BSDEs that allow us to approximate utilities ((1.1)-(1.3)) and their optimal strategies. We actually study a larger class of ergodic BSDE, introduced below.

1.2 Markovian solution of ergodic BSDEs

Ergodic BSDEs have first been studied in [16] under a dissipativity assumption on the stochastic factor V to solve an ergodic stochastic control problem. The assumption on the stochastic factor have been relaxed in [10] with a weak dissipative condition and in [22] for non constant and possibly unbounded volatility. Ergodic BSDEs are usually studied under Lipschitz condition on the generator F . When the stochastic factor's volatility is constant, the component Z of the solution to the eBSDE is bounded, which allows the driver to only be locally Lipschitz in z . We will work within the framework of [27] with a stochastic factor satisfying a strong dissipativity assumption and constant volatility. This framework leads to the existence of a Markovian solution to the ergodic BSDE (1.8) such that Z is bounded, and thus allows the generator to have quadratic growth in z .

Assumption 1.2. *There exists a constant $C_\mu > 0$ such that for any $v, \bar{v} \in \mathbb{R}^d$:*

$$(\mu(v) - \mu(\bar{v}))^\top (v - \bar{v}) \leq -C_\mu \|v - \bar{v}\|^2. \quad (1.16)$$

The volatility matrix $\kappa = (\kappa_{ij})_{\substack{1 \leq i \leq d \\ 1 \leq j \leq d}}$ is such that $\kappa \kappa^\top$ is positive definite.

Under Assumption 1.2, from a direct application of Gronwall's lemma, the diffusion V is exponentially ergodic. The authors in [22] generalized this result under a weak dissipative assumption. This properties are essential for the correspondence with random time horizon BSDE and the algorithm we present in the sequel.

Assumption 1.3. • *There exists a positive constant K such that $\forall v \in \mathbb{R}^d, |F(v, 0)| \leq K$.*

• *There exists positive constants C_v and C_z such that $\forall v, \bar{v} \in \mathbb{R}^d, \forall z, \bar{z} \in \mathbb{R}^d$:*

$$|F(v, z) - F(\bar{v}, z)| \leq C_v(1 + \|z\|)\|v - \bar{v}\|, \quad (1.17)$$

$$|F(v, z) - F(v, \bar{z})| \leq C_z(1 + \|z\| + \|\bar{z}\|)\|z - \bar{z}\|. \quad (1.18)$$

Moreover, we require that $C_v < C_\mu$.

Note that under Assumption 1.1 indicating that the market price of risk θ is bounded and Lipschitz, every generator (1.10), (1.11) and (1.12) introduced in the previous section satisfy the above assumption. We recall the existence result for eBSDE studied in [27].

Proposition 1.2 (Existence - [27]). *Under Assumption 1.2 and 1.3, the ergodic BSDE (1.8) admits a Markovian solution $(y(V_t), z(V_t), \lambda)_{t \geq 0}$ such that $y(\cdot)$ is sub-linear and $z(\cdot)$ is bounded by $Z_{\max} = \|\kappa\| \frac{C_v}{C_\mu - C_v}$.*

Assumptions 1.2 and 1.3 thus provide the existence of a Markovian solution $(y(V_t), z(V_t), \lambda)_{t \geq 0}$ to (1.8) where $z(\cdot)$ is bounded, which is particularly convenient to apply our work to ergodic BSDE with quadratic driver F as for example to simulate exponential and power dynamic utilities. In fact, working with the truncated driver $F \circ \varphi_{Z_{\max}}$ where $\varphi_{Z_{\max}}$ is the projection on the centered ball of \mathbb{R}^d of radius Z_{\max} , the application $F \circ \varphi_{Z_{\max}}$ is then Lipschitz in v and z , namely:

$$|F \circ \varphi_{Z_{\max}}(v, z) - F \circ \varphi_{Z_{\max}}(\bar{v}, z)| \leq C_v(1 + Z_{\max})\|v - \bar{v}\|, \quad (1.19)$$

$$|F \circ \varphi_{Z_{\max}}(v, z) - F \circ \varphi_{Z_{\max}}(v, \bar{z})| \leq C_z(1 + 2Z_{\max})\|z - \bar{z}\|. \quad (1.20)$$

The uniqueness of the Markovian solution to (1.8) is usually stated up to a constant, by fixing one point of the solution, typically $y(0) = 0$. The proof follows the arguments from [10] and [16] when the driver is Lipschitz.

Theorem 1.3. [Uniqueness - [27]] Assume that Assumptions 1.2 and 1.3 hold true. Let (y, z) , (\tilde{y}, \tilde{z}) , two couple of functions such that:

- y and $\tilde{y} : \mathbb{R}^d \rightarrow \mathbb{R}$ are continuous, sub-linear and $y(0) = \tilde{y}(0)$.
- z and $\tilde{z} : \mathbb{R}^d \rightarrow (\mathbb{R}^d)^*$ are measurable and bounded by Z_{max} .

Also assume that for some constants $\lambda, \tilde{\lambda}$ and for all $v \in \mathbb{R}^{d'}$, the triplets $(y(V_t^v), z(V_t^v), \lambda)_{t \geq 0}$ and $(\tilde{y}(V_t^v), \tilde{z}(V_t^v), \tilde{\lambda})_{t \geq 0}$ satisfy the ergodic BSDE (1.8).

Then $\lambda = \tilde{\lambda}$, $y(V_t^v) = \tilde{y}(V_t^v)$ and $z(V_t^v) = \tilde{z}(V_t^v)$ \mathbb{P} -a.s and for a.e $t \geq 0$.

Initial condition In the case of ergodic BSDEs derived from forward utilities, an initial condition $Y_0 = y_0$ for the process Y is naturally fixed since the initial agent's utility and wealth $u_0(x_0)$ is known. For instance in the case of power dynamic utilities (1.3), we have

$$y_0 = \log(\delta u_0(x_0)) - \delta \log(x_0).$$

We are thus interested in solutions of the ergodic BSDE (1.8) with *fixed* initial condition:

$$\begin{aligned} Y_t &= Y_T + \int_t^T F(V_s, Z_s) ds - \lambda(T-t) - \int_t^T Z_s^\top dW_s, \quad \forall 0 \leq t \leq T. \\ Y_0 &= y_0. \end{aligned} \tag{1.21}$$

Using notations of Theorem 1.3, let $(y(V_t), z(V_t), \lambda)_{t \geq 0}$ be the unique solution of (1.8), such that $y(0) = 0$. Then, (Y, Z, λ) , with

$$Y_t = y(V_t) + y_0 - y(V_0), \quad Z_t = z(V_t) \quad \forall t \geq 0, \tag{1.22}$$

is a solution of (1.21). Note that the solution is not Markovian anymore, since Y depend on the stochastic factor initial condition V_0 . In the following, we define the unique solution of the ergodic BSDE (1.21) with fixed initial condition as the triplet $(Y, Z, \lambda)_{t \geq 0}$, with (Y, Z) verifying (1.22). With some abuse of language, with sometimes refer to this solution as the unique "Markovian" solution.

2 Connection with BSDE with random terminal time

Simulating the ergodic BSDE (1.21) presents several additional challenges, in comparison with the simulation of standard BSDEs with finite time horizons. In particular:

1. There is an additional unknown $\lambda \in \mathbb{R}$, which controls the time growth of the component Y .
2. It is an infinite horizon backward stochastic differential equation, which means that the equality

$$Y_t = Y_T + \int_t^T (F(V_s, Z_s) - \lambda) ds - \int_t^T Z_s^\top dW_s,$$

for any $T > 0$, for all $0 \leq t \leq T$. Thus, there is no known terminal condition Y_{T_H} as in usual numerical schemes for BSDEs with backward time discretization.

However, we may take advantage of recurrence properties of the stochastic factor V to establish a connection between the ergodic BSDE and a BSDE with random terminal time, thus introducing a terminal condition. More precisely, when the stochastic factor V is one dimensional ($d' = 1$), knowing the initial condition Y_0 of the ergodic BSDE allows us to introduce an analogous “ergodic BSDE with random time horizon” τ . The infinite time horizon is replaced by τ , the first return time after a minimal horizon T_H of the diffusion V to the initial point V_0 . With this choice of random terminal time, the associated BSDE with random horizon has a known terminal condition $Y_\tau = Y_0$, and hence can be approximated numerically.

2.1 First return time of the stochastic factor V

Unless stated otherwise, we assume in the following that the stochastic factor V is one dimensional ($d' = 1$). Let us fix a minimal horizon T_H and start with some properties of the first return time after T_H of the diffusion V to its initial value denoted by $v_0 \in \mathbb{R}$:

$$\tau = \inf \{t > T_H, V_t = v_0\}. \quad (2.1)$$

We require that the hitting time τ to have some exponential integrability properties in order to study the “ergodic BSDE with random horizon” introduced in the next section. Theorem 1.1 in [28] provides sufficient conditions for exponential integrability of hitting time for continuous Markov processes. More precisely, the authors obtain a lower bound for the greatest order of exponential moment of τ in terms of the scale function and the speed measure, uniformly with respect to the initial condition. We adapt this result to our framework:

Lemma 2.1. *Let $s(x) = \exp\left(-2 \int_0^x \frac{\mu(u)}{\|\kappa\|^2} du\right)$, and*

$$B_{v_0}^+ := \sup_{x \geq v_0} \left(\int_{v_0}^x s(u) du \int_x^{+\infty} \frac{2}{\|\kappa\|^2 s(u)} du \right), \quad (2.2)$$

$$B_{v_0}^- := \sup_{x \leq v_0} \left(\int_x^{v_0} s(u) du \int_{-\infty}^x \frac{2}{\|\kappa\|^2 s(u)} du \right). \quad (2.3)$$

Denoting $K_z = C_z(1 + 2Z_{\max})$ the Lipschitz constant with respect to z of the truncated driver $F \circ \phi_{Z_{\max}}$ given in (1.20), assume that $K_z^2 < \frac{1}{4 \max(B_{v_0}^-, B_{v_0}^+)}$. Then, $\exists \gamma > K_z^2$ such that

$$\mathbb{E}[\exp(\gamma\tau)] < \infty. \quad (2.4)$$

Proof. Let $\lambda \leq \frac{1}{4 \max(B_{v_0}^-, B_{v_0}^+)}$, and μ the unique invariant measure of V . Recall that $\tau \geq T_H$, and let $W(v) = \mathbb{E}[\exp(\lambda\tau) | V_{T_H} = v]$. By the Markov property,

$$\mathbb{E}[\exp(\lambda\tau)] = \mathbb{E}[W(V_{T_H})].$$

First, by Theorem 1.1 in [28], we have

$$W(v) < \infty, \quad \forall v \in \mathbb{R},$$

and equivalently, $\mu(W) \int_{-\infty}^{+\infty} W(v) \mu(dv) < \infty$ (see e.g. Proposition 1.2 in [28]).

It remains to prove that $\mathbb{E}[W(V_{T_H})] < \infty$. Since the diffusion coefficient of the stochastic factor V

is constant, V is a uniformly elliptic diffusion, and it follows that W is Lyapunov function for the diffusion V (see e.g. Theorem 2.3 in [6]).

Using similar arguments as in the proof of Theorem 7 in [22], the hypothesis of Theorem A.2 in [37] are verified, with $V = W$. By applying the result to $f = W$, we obtain that:

$$\begin{aligned}\mathbb{E}[W(V_{T_H})] &\leq |\mathbb{E}[W(V_{T_H})] - \mu(W)| + \mu(W) \\ &\leq 2Be^{-rT_H} + \mu(W) < \infty.\end{aligned}$$

□

Remark 2.1. In higher dimensions, results on positive recurrence of continuous time Markov chain associated to a stochastic differential equation stands for hitting time of any non empty open set in \mathbb{R}^d . Working with τ^ϵ the first hitting time of the Euclidean ball of \mathbb{R}^d of center v_0 and radius ϵ , we may only expect the value $y(V_{\tau^\epsilon})$ to be close to $y(v_0)$, by continuity of the Markovian solution y . We can thus only hope to obtain an approximation of the solution of the ergodic BSDE, by investigating stability results for ergodic BSDE with perturbed terminal condition. This is left for future work.

2.2 BSDE with random terminal time

Under Assumptions 1.2 and 1.3, there exists a unique solution $(y(V_t) + y_0 - y(V_0), z(V_t), \lambda)_{t \geq 0}$ to the ergodic BSDE (1.21) such that $Y_0 = y_0$, y is sub-linear with respect to v , and z is bounded by Z_{max} . By construction, $(Y_t, Z_t, \lambda)_{t \geq 0}$ is also solution of the following ‘‘ergodic’’ BSDE with random time horizon and fixed initial condition:

$$\begin{aligned}Y_t^r &= Y_\tau^r + \int_t^\tau F(V_s, Z_s^r) ds - \lambda(\tau - t) - \int_t^\tau Z_s^{r\top} dW_s, \\ Y_\tau^r &= Y_0^r = y_0,\end{aligned}\tag{2.5}$$

with τ the return time defined in (2.1).

Reciprocally, solutions (Y^r, Z^r, λ) of (2.5) can be studied directly. Theorem 2.2 provides sufficient conditions, under which solutions of (2.5) coincides with the solution of the ergodic BSDE (1.21). The setting is slightly different from usual BSDEs with random time horizon, since we have an additional unknown $\lambda \in \mathbb{R}$ and both the terminal and initial conditions are fixed. However, we can show that this constant is uniquely determined by the fixed initial and terminal conditions $Y_\tau^r = Y_0^r = y_0$. Once λ is known, the uniqueness of the solution to this BSDE with generator $F(v, z) - \lambda$ can be obtained as a consequence of standard results for BSDEs with random terminal time. We apply here the general result Theorem 3.2 from [34], which requires integrability conditions for the stopping time τ .

Theorem 2.2. *Assume that Assumption 1.2 and 1.3 are verified and that $K_z^2 < \frac{1}{4 \max(B_{v_0}^-, B_{v_0}^+)}$. Then, the ergodic BSDE with random time horizon (2.5) and fixed initial condition admits a unique solution (Y, Z, λ) , such that $Y \in \mathcal{S}^2(\gamma, \tau)$ for all $K_z^2 < \gamma \leq \frac{1}{4 \max(B_{v_0}^-, B_{v_0}^+)}$ and Z is bounded. In particular, (Y, Z, λ) coincides on $[0, \tau]$ with the unique solution $(y(V_t) + y_0 - y(V_0), z(V_t), \lambda)_{t \geq 0}$ of the ergodic BSDE (1.21) such that $Y_0 = y_0$, y is sub-linear, and z is bounded.*

Proof. The proof is done in three steps. First we show that the existence of such a solution to (2.5) is obtained straightforwardly from the existence of a solution to the eBSDE (1.21). Secondly, we show the uniqueness of the parameter λ , using a linearisation technique. The uniqueness of (Y, Z) is then obtained by applying Theorem 3.2 from [34].

Existence - By construction, the unique solution $(y(V_t) - y_0 + y(V_0), z(V_t), \lambda)_{t \geq 0}$ of the ergodic BSDE (1.21) is also solution of the BSDE with random terminal time (2.5). By construction, Z is bounded by Z_{max} and thanks to Theorem 3.2 in [34] we deduce that $Y \in \mathcal{S}^2(\gamma, \tau)$.

Uniqueness of λ - Let (Y^r, Z^r, λ) and $(\bar{Y}^r, \bar{Z}^r, \bar{\lambda})$ be two solutions of the ergodic BSDE with random terminal time (2.5) such that Z^r and \bar{Z}^r are bounded. Denote $\Delta Y_t = Y_t^r - \bar{Y}_t^r$, $\Delta Z_t = Z_t^r - \bar{Z}_t^r$ and $\Delta \lambda = \lambda - \bar{\lambda}$. The initial and terminal values of those two solution being equal to y_0 , $\Delta Y_0^r = \Delta Y_\tau^r = 0$. Let $T > 0$. The difference between those two equations between 0 and $T \wedge \tau$ thus leads:

$$(T \wedge \tau)\Delta \lambda = \Delta Y_{T \wedge \tau} + \int_0^{T \wedge \tau} \Delta Z_s^\top (\gamma_s ds - dW_s), \quad (2.6)$$

$$\text{where } \gamma_s = \begin{cases} \frac{F(V_s, Z_s^r) - F(V_s, \bar{Z}_s^r)}{|\bar{Z}_s^r - Z_s^r|^2} (\bar{Z}_s^r - Z_s^r) & \text{if } |\bar{Z}_s^r - Z_s^r| \neq 0 \\ 0 & \text{otherwise,} \end{cases} \quad (2.7)$$

Since Z^r and \bar{Z}^r are bounded and by Assumption (1.3), the process γ is bounded. Then according to the Girsanov Theorem, there exists a probability measure $\tilde{\mathbb{P}}$ under which the process $\bar{W}_t = -\int_0^t \gamma_s ds + W_t$, is a Brownian motion. The stopped process $M_{t \wedge T} := \int_0^{t \wedge T} \Delta Z_s^\top d\bar{W}_s$ is a martingale under $\tilde{\mathbb{P}}$ so that taking expectation of (2.6) leads:

$$\tilde{\mathbb{E}}[(T \wedge \tau)\Delta \lambda] = \tilde{\mathbb{E}}[\Delta Y_{T \wedge \tau}] = \tilde{\mathbb{E}}[\Delta Y_\tau \mathbf{1}_{\tau \leq T}] + \tilde{\mathbb{E}}[\Delta Y_T \mathbf{1}_{\tau > T}].$$

The first expectation on the right is zero since $\Delta Y_\tau = 0$, by definition of the stopping time τ . For the second term, using the sub-linearity property of Y and \bar{Y} :

$$\begin{aligned} \tilde{\mathbb{E}}[\Delta Y_T \mathbf{1}_{\tau > T}] &\leq \tilde{\mathbb{E}}[\Delta Y_T^2] \tilde{\mathbb{P}}(\tau > T) \\ &\leq C(1 + \tilde{\mathbb{E}}[|V_T|^2]) \tilde{\mathbb{P}}(\tau > T). \end{aligned} \quad (2.8)$$

By Proposition 5 in [22], $\sup_{T \geq 0} \tilde{\mathbb{E}}[|V_T|^2] < \infty$. Moreover, τ is almost surely finite under \mathbb{P} so that is also almost surely finite under the equivalent probability measure $\tilde{\mathbb{P}}$. Then, taking the limit of (2.8) as T goes to infinity leads $\Delta \lambda = 0$. Hence, $\Delta \lambda = 0$ and the component λ of the solution of (2.5) is thus necessarily equal to the λ solution of the ergodic BSDE (1.21).

Uniqueness - Consider the BSDE with random terminal time τ , terminal condition $Y_\tau^r = y_0$ and generator $F \circ \phi_{Z_{max}}(v, z) - \lambda$, where λ is fixed. The uniqueness can be obtained applying Theorem 3.1 in [34]. □

The solution of the ergodic BSDE (1.21) thus coincides with the solution (Y^r, Z^r, λ) of the ergodic BSDE with random time horizon and fixed initial condition (2.5) on $[0, \tau]$. We will omit the subscript r in the sequel. This point of view provides our simulation problem with a random horizon τ and a terminal condition $Y_\tau = y_0$. This allows us to adapt to our framework numerical schemes for the

simulation of BSDEs with random time horizon, such as those introduced in [3] and [5] (see Section 3). Furthermore, using this representation of the ergodic BSDE, a new representation of the ergodic cost λ can be obtained under some additional assumption of the driver F .

2.3 Characterization of the ergodic cost λ for a class of eBSDEs

The ergodic cost λ can be interpreted in several ways. As mentioned in [27], it is the long term growth rate of a risk sensitive control problem. It can also be estimated by $\frac{Y_0^T}{T}$ when $T \rightarrow \infty$, with Y_0^T the initial value of the solution of a BSDE with finite horizon T (Theorem 21 in [22]). However, numerical schemes for the simulation of BSDEs are known to be unstable for large horizon, which can lead to a significant error when using this approach to approximate λ .

The viewpoint of ergodic BSDE up to a random horizon, with fixed initial and terminal values offers a new characterization of the ergodic cost λ . In fact, considering it as a parameter in the generator of a classical BSDE with random terminal time, it can be understood as the solution of an optimization problem on the initial value y_0 . This is also the idea behind the deep-learning algorithms we present in Section 4. The adaptation of Proposition 1.3 from [12] for linear BSDEs leads to a semi-explicit expression of the solution of (2.5) as well as an expression of the ergodic cost λ as the ratio of two expectations that can be computed numerically. We investigate separately the case of linear ergodic BSDEs, and the one of ergodic BSDE with purely quadratic generator which first requires to apply the Cole-Hopf transform.

Let's first consider a random time horizon ergodic BSDE as (2.5) with driver F only depending on the stochastic factor V . This is the framework for the representation of logarithmic dynamic utilities (1.1), associated with driver F defined by (1.10). When F does not depend on z , taking the conditional expectation of (2.5) with respect to \mathcal{F}_t , and evaluating this expression at time 0, together with the fixed initial condition $Y_0 = y_0$ leads to the following characterization of the ergodic cost.

Lemma 2.3. *Let (Y, Z, λ) be the unique solution of equation (2.5), as defined in Theorem 2.2, and with a driver F only depending on V . Then the ergodic cost λ admits the following representation:*

$$\lambda = \frac{\mathbb{E}\left[\int_0^\tau F(V_s) ds\right]}{\mathbb{E}[\tau]}. \quad (2.9)$$

Exponential forward utility without constraints When there are no constraints on the portfolio that is $\Pi = \mathbb{R}^d$, the driver F_{exp} defined by (1.11) and associated with exponential utilities, is linear in z and given by:

$$F_{\text{exp}}(v, z) = -\theta(v)^\top z - \frac{1}{2}\|\theta(v)\|^2. \quad (2.10)$$

The representation result for linear BSDEs can be adapted to our framework in order to obtain a representation of the ergodic cost in this case. Note that the following result can be generalized to any linear generator F .

Proposition 2.4. *Assuming there are no constraint on the portfolio, the unique solution of the ergodic BSDE with random terminal time (2.5), as defined in Theorem 2.2, and with generator F_{exp} given by*

(1.11) is given for all $t \geq 0$ by:

$$Y_t = \mathbb{E} \left[y_0 \Gamma_{t,\tau} - \int_t^\tau \Gamma_{t,s} \left(\frac{1}{2} \|\theta(V_s)\|^2 + \lambda \right) ds \middle| \mathcal{F}_t \right], \quad a.s., \quad (2.11)$$

where:

$$\begin{aligned} d\Gamma_{t,s} &= -\Gamma_{t,s} \theta(V_s)^\top dW_s \\ \Gamma_{t,t} &= 1. \end{aligned} \quad (2.12)$$

The ergodic cost λ satisfies:

$$\lambda = \frac{1}{\mathbb{E} \left[\int_0^\tau \Gamma_{0,s} ds \right]} \left(\mathbb{E} \left[y_0 \Gamma_{0,\tau} - \frac{1}{2} \int_0^\tau \Gamma_{0,s} \|\theta(V_s)\|^2 ds \right] - y_0 \right). \quad (2.13)$$

Proof. The representation theorem for linear BSDEs ([12]) can be extended to BSDE with random terminal time. Consider (Y, Z) the unique Markovian solution to (2.5) in the sense of Theorem 2.2 and define the stopped process $(M_s^\tau)_{s \geq t}$ as:

$$M_s^\tau = Y_{s \wedge \tau} \Gamma_{t,s \wedge \tau} - \frac{1}{2} \int_t^{s \wedge \tau} \Gamma_{t,u} \left(\|\theta(V_u)\|^2 + \lambda \right) du. \quad (2.14)$$

An application of Ito's formula to the product $Y_{s \wedge \tau} \Gamma_{t,s \wedge \tau}$ shows that $(M_s^\tau)_{s \geq t}$ is a local martingale. Moreover, $\sup_{0 \leq s \leq \tau} Y_{s \wedge \tau}$ and $\sup_{0 \leq s \leq \tau} \Gamma_{t,s \wedge \tau}$ belong to L^2 so that the product $\sup_{0 \leq s \leq \tau} Y_{s \wedge \tau} \times \sup_{0 \leq s \leq \tau} \Gamma_{t,s \wedge \tau}$ is integrable. The martingale $(M_s^\tau)_{s \geq t}$ is thus uniformly integrable, and its value at time t equals the conditional expectation of its terminal value with respect to \mathcal{F}_t , from which follows (2.11).

The true value of λ to recover the solution to the ergodic BSDE with random time horizon (2.5) with generator F_{exp} is then uniquely determined from the initial condition $Y_0 = y_0$. Evaluating (2.11) at time $t = 0$ leads to the ergodic cost (2.13), which concludes the proof. \square

Power forward utility without constraints For ergodic BSDEs with random terminal time and quadratic generator of the following form:

$$F(v, z) = l(v) + a(v)^\top z + \frac{\beta}{2} \|z\|^2, \quad (2.15)$$

the Cole-Hopf transform can be used to come back to the linear case to provide a representation of the ergodic cost.

Proposition 2.5. *Consider the ergodic BSDE with random terminal time (2.5) with generator F given by (2.15) such that l and a are bounded. This equation admits a unique Markovian solution (y, z, λ) such that y is sublinear, z is bounded and $Y_0 = y_0$, which is given for all $t \geq 0$, by:*

$$Y_t = \frac{1}{\beta} \ln \mathbb{E} \left[e^{\beta y_0} \Gamma_{t,\tau} \middle| \mathcal{F}_t \right], \quad a.s., \quad (2.16)$$

where:

$$\begin{aligned} d\Gamma_{t,s} &= \Gamma_{t,s} \left(\beta l(V_s) - \lambda \right) ds + a(V_s)^\top dW_s \\ \Gamma_{t,t} &= 1. \end{aligned}$$

Moreover, the ergodic cost λ is characterized as:

$$\lambda = \underset{\lambda \in \mathbb{R}}{\operatorname{argmin}} |\mathbb{E}[\Gamma_{0,\tau}] - 1|. \quad (2.17)$$

Proof. Let's consider (Y, Z, λ) the solution of the ergodic BSDE with generator F given by (2.15) such that $Y_t = y(V_t) + y_0 - y(V_0)$, with y is sublinear and $y(0) = 0$, and Z is bounded. Let $P_s = e^{\beta Y_s}$ and $Q_s = \beta P_s Z_s$. An application of Ito formula leads to:

$$P_t = P_\tau + \int_t^\tau \left[\beta P_s (l(V_s) - \lambda) + a(V_s)^\top Q_s \right] ds - \int_t^\tau Q_s^\top dW_s. \quad (2.18)$$

For any fixed $\lambda \in \mathbb{R}$, similarly as in the proof of Proposition 2.4, the representation theorem for linear BSDEs ensures that the above equation with terminal condition $P_\tau = e^{\beta y_0}$ admits a unique solution $(P, Q) \in \mathcal{S}^2(\gamma, \tau) \times \mathcal{H}^2(\gamma, \tau)$, such that for all $t \geq 0$,

$$P_t = \mathbb{E} \left[e^{\beta y_0} \Gamma_{t,\tau} | \mathcal{F}_t \right], \quad \text{a.s.} \quad (2.19)$$

where:

$$\begin{aligned} d\Gamma_{t,s} &= \Gamma_{t,s} \left(\beta (l(V_s) - \lambda) ds + a(V_s)^\top dW_s \right) \\ \Gamma_{t,t} &= 1. \end{aligned}$$

Equation (2.19) at time zero leads $\mathbb{E}[\Gamma_{0,\tau}] = 1$. The map $\lambda \mapsto \mathbb{E}[\Gamma_{0,\tau}]$ being strictly monotonic, the ergodic cost λ is characterized as:

$$\lambda = \underset{\lambda \in \mathbb{R}}{\operatorname{argmin}} |\mathbb{E}[\Gamma_{0,\tau}] - 1|. \quad (2.20)$$

□

By construction, the ergodic cost verifies $|\lambda| \leq K$ with K given in Assumption 1.3 (see the proof of Proposition 3.1 in [27]). Hence, the minimum in the above result must be reached in the interval $[-K, K]$. Proposition 2.5 can be applied in order to obtain a representation for the ergodic cost λ in the case of power forward utilities, with generator F given by (1.12) when there are no constraints on the portfolio.

Corollary 2.6. *The unique Markovian solution to the ergodic BSDE with random time horizon (2.5) with generator F given by:*

$$F(V_t, Z_t) = \frac{1}{2} \frac{\delta}{1 - \delta} \|Z_t + \theta(V_t)\|^2 + \frac{1}{2} \|Z_t\|^2. \quad (2.21)$$

is given for all $t \geq 0$ by:

$$Y_t = y_0 + (1 - \delta) \ln \mathbb{E}[\Gamma_{t,\tau} | \mathcal{F}_t], \quad \text{a.s.} \quad (2.22)$$

where $\Gamma_{t,\tau}$ is the value in τ of the unique solution of the forward equation:

$$\begin{aligned} d\Gamma_{t,s} &= \Gamma_{t,s} \left(\frac{1}{1 - \delta} \left(\frac{\delta}{2(1 - \delta)} \|\theta(V_s)\|^2 - \lambda \right) ds + \frac{\delta}{1 - \delta} \theta(V_s)^\top dW_s \right) \\ \Gamma_{t,t} &= 1. \end{aligned} \quad (2.23)$$

Moreover, the ergodic cost λ satisfies:

$$\lambda = \underset{\lambda \in [-K, K]}{\operatorname{argmin}} |\mathbb{E}[\Gamma_{0,\tau}(\lambda)] - 1| \quad (2.24)$$

3 Numerical approximation

In this section, we present the Euler approximation of the stochastic factor V in order to control the discretisation error for a backward scheme for ergodic BSDE with random time horizon.

3.1 Approximation of the forward SDE

First, the integral form of the stochastic factor process (1.4) reads as:

$$V_t = v_0 + \int_0^t \mu(V_s) ds + \int_0^t \kappa dW_s, \quad t \geq 0. \quad (3.1)$$

Consider a discretisation of \mathbb{R}^+ with constant time step h , generating a grid $\pi = \{t_0 = 0, t_1, \dots\}$. Denoting $\Delta W_i = W_{t_{i+1}} - W_{t_i}$ the Brownian increments between times t_i and t_{i+1} , the Euler discretisation of V on the time grid π is given for all $i \geq 0$:

$$\begin{aligned} \bar{V}_{t_{i+1}} &= \bar{V}_{t_i} + \mu(\bar{V}_{t_i})h + \kappa \Delta W_i, \\ \bar{V}_0 &= v_0. \end{aligned} \quad (3.2)$$

In the sequel, we will consider the continuous Euler scheme associated to (3.2) on grid π , defined by:

$$\bar{V}_t = v_0 + \int_0^t \mu(\bar{V}_{s^-}) ds + \int_0^t \kappa dW_s, \quad t \geq 0, \quad (3.3)$$

where $s^- = \max\{t_i \in \pi, t_i \leq s\}$.

We approximate the stopping time τ from the continuous Euler approximation of V as:

$$\bar{\tau} = \inf \{t \geq T_H, \bar{V}_t = v_0\}. \quad (3.4)$$

This stopping time is well defined since the Euler discretization $(\bar{V}_t)_{t \geq 0}$ is also ergodic, see [36], [30]. The horizon $\bar{\tau}$ defined by (3.4) we consider can be written as an exit time of the diffusion of a smooth domain. In fact:

$$\bar{\tau} = \inf \{t \geq T_H, \bar{V}_t \notin \mathbb{R} \setminus \{v_0\}\}. \quad (3.5)$$

The exit domain $\mathbb{R} \setminus \{v_0\}$ is an open set with C^2 boundary, so that results from [3] and [29] related to the estimation of τ with an Euler scheme can be applied in our setting. We recall the following result from Theorem 3.9 [4].

Proposition 3.1. *Let Assumption 1.2 hold and assume that there exists $4 \leq q < \infty$ such that under the notation of Lemma 2.1, $\frac{q}{q-1}6C_\mu < \frac{1}{4B^+} \wedge \frac{1}{4B^-}$. Then there exists a constant $C > 0$ such that the error in L^1 of the approximation of the return time τ with the continuous Euler scheme is:*

$$\mathbb{E}[|\tau - \bar{\tau}|] \leq Ch^{1/2}. \quad (3.6)$$

Proof. Let $\beta \in \mathbb{R}$ be such that $\frac{q}{q-1}6C_\mu < \beta < \frac{1}{4B^+} \wedge \frac{1}{4B^-}$. Under this assumption, Lemma 2.1 ensures that the random time τ admits exponential moment of order β . An application of Markov's inequality then gives, for any $k \in \mathbb{N}$:

$$\mathbb{P}(\tau \geq k) \leq \mathbb{E}\left[e^{\beta\tau}\right] e^{-\beta k}. \quad (3.7)$$

Theorem 3.9 from [4] applies and leads the desired result. \square

3.2 Discrete-time approximation error

In the following, we recall the usual time discretisation of BSDEs, applied to the ergodic BSDE with random time horizon (2.5). Consider a time discretisation $\pi = \{0 = t_0, t_1, \dots\}$ of \mathbb{R}^+ , with time step h and the forward Euler scheme \bar{V} for the stochastic factor V defined by (3.3). Let $\bar{\lambda}$ denote an approximation of the ergodic cost λ . It can be estimated either by a Monte Carlo approximation or as a trainable parameter of the deep learning algorithm developed in the next section. Then starting from $\bar{Y}_{\bar{\tau}} = y_0$, we define the discrete time process (\bar{Y}, \bar{Z}) on π , for $i < \frac{\bar{\tau}}{h}$:

$$\bar{Z}_{t_i} = \frac{1}{h} \mathbb{E}[\bar{Y}_{t_{i+1}} \Delta W_{t_i} | \mathcal{F}_{t_i}] \quad (3.8)$$

$$\bar{Y}_{t_i} = \mathbb{E}[\bar{Y}_{t_{i+1}} | \mathcal{F}_{t_i}] + \mathbf{1}_{t_i \leq \bar{\tau}} h [F(\bar{V}_{t_i}, \bar{Z}_{t_i}) - \bar{\lambda}], \quad (3.9)$$

One can check under Assumption 1.2 that for all i , $(\bar{Y}_{t_i}, \bar{Z}_{t_i}) \in L^2$. Moreover, as the process Z is bounded, we will equivalently work with a Lipschitz driver $F \circ \varphi_{Z_{\max}}$, where $\varphi_{Z_{\max}}$ is the projection on the centered ball of \mathbb{R}^d of radius Z_{\max} . As mentioned in [8], [2], this can lead to numerical difficulties if this bound Z_{\max} is too large.

For the following, it will be convenient to work with a continuous extension of \bar{Y} in \mathcal{S}^2 . This is possible since from the martingale representation theorem, there exists a process $\tilde{Z} \in \mathcal{H}^2$ such that:

$$\bar{Y}_{t_{i+1}} - \mathbb{E}[\bar{Y}_{t_{i+1}} | \mathcal{F}_{t_i}] = \int_{t_i}^{t_{i+1}} \tilde{Z}_s^\top dW_s, \quad (3.10)$$

which allows to consider the continuous extension of $(Y_t)_{i < \frac{\bar{\tau}}{h}}$ on $[0, \bar{\tau}]$:

$$\bar{Y}_t = \bar{Y}_{\bar{\tau}} + \int_t^{\bar{\tau}} F(\bar{V}_{s-}, \bar{Z}_{s-}) ds - (\bar{\tau} - t) \bar{\lambda} - \int_t^{\bar{\tau}} \tilde{Z}_s^\top dW_s. \quad (3.11)$$

Finally, we will also consider the best approximation in L^2 of Z by a process constant on each time interval $[t_i, t_{i+1}]$, defined by:

$$\hat{Z}_{t_i} = \frac{1}{h} \mathbb{E} \left[\int_{t_i}^{t_{i+1}} Z_s ds | \mathcal{F}_{t_i} \right]. \quad (3.12)$$

Remark 3.1. From Ito's isometry and (3.10) we have for all $i \geq 0$:

$$\bar{Z}_{t_i} = \frac{1}{h} \mathbb{E} \left[\int_{t_i}^{t_{i+1}} \tilde{Z}_s ds | \mathcal{F}_{t_i} \right]. \quad (3.13)$$

We will provide a bound for the square of the discrete time approximation error, up to a stopping time θ :

$$\text{Err}(h)_\theta^2 = \max_i \mathbb{E} \left[\sup_{t \in [t_i, t_{i+1}]} \mathbf{1}_{t \leq \theta} |Y_t - \bar{Y}_{t_i}|^2 \right] + \mathbb{E} \left[\int_0^\theta \|Z_t - \bar{Z}_{t-}\|^2 dt \right], \quad (3.14)$$

where $t^- = \sup \{s \in \pi, s \leq t\}$. We will control this error through the usual error quantity:

$$R(Z)_{\mathcal{H}^2}^\pi = \mathbb{E} \left[\int_0^\tau \|Z_t - \hat{Z}_{t-}\|^2 dt \right]. \quad (3.15)$$

Remark 3.2. Let θ be an \mathcal{F}_t stopping time in the grid π . We will control the error term $\mathbb{E} \left[\int_0^\theta \|Z_s - \tilde{Z}_s\|^2 ds \right]$ which will provide the desired bound on $\text{Err}(h)$. In fact, using Jensen's inequality, one can show (see [5]) that for any stopping time θ in the time grid π :

$$\mathbb{E} \left[\int_0^\theta \|Z_s - \bar{Z}_{s-}\|^2 ds \right] \leq C \left(\mathbb{E} \left[\int_0^\theta \|Z_s - \tilde{Z}_s\|^2 ds \right] + \mathbb{E} \left[\int_0^\theta \|Z_s - \hat{Z}_{s-}\|^2 ds \right] \right). \quad (3.16)$$

Controlling the above error implies that we control the error in $\mathcal{S}^2 \times \mathcal{H}^2$ of the discrete time approximation $(\bar{Y}_{t_i}, \bar{Z}_{t_i})$ for all i . We provide a bound for the discretisation error of a backward scheme for random time horizon ergodic BSDE in terms of the quantities $R(Z)_{\mathcal{H}^2}^\pi$, $|\lambda - \bar{\lambda}|$ and $|\tau - \bar{\tau}|$. We refer to [5], whose results apply for our stopping time τ .

Proposition 3.2. *Suppose that Assumption 1.2 and 1.3 are verified and that $K_z^2 < \frac{1}{4 \max(B_{v_0}^-, B_{v_0}^+)}$, there exist a constant $C > 0$ such that:*

$$\text{Err}(h)_{\tau^+ \vee \bar{\tau}^+}^2 \leq C \left(h + \mathcal{R}(Z)_{\mathcal{H}^2} + |\lambda - \bar{\lambda}|^2 \mathbb{E}[\tau^+ \vee \bar{\tau}^+] + \mathbb{E}[|\bar{\tau} - \tau|] \right). \quad (3.17)$$

Additionally to the usual spatial error for discretisation scheme for finite horizon BSDE, we get here a term related to the estimation of the return time τ with the Euler approximation \bar{V} as well as an error term related to the estimation of the ergodic constant λ .

Proof. We follow the arguments described in [3], and add the error term specific to our ergodic equation $|\lambda - \bar{\lambda}|$ in the calculation. Let θ be an \mathcal{F}_t stopping time in the grid π . Applying Ito's lemma to $(Y - \bar{Y})^2$ between $t \wedge \theta$ and $t_{i+1} \wedge \theta$, for a time $t \in [t_i, t_{i+1}]$:

$$\begin{aligned} \text{Err}_{t \wedge \theta}^{t_{i+1} \wedge \theta} &:= \mathbb{E} \left[|Y_{t \wedge \theta} - \bar{Y}_{t \wedge \theta}|^2 + \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \|Z_s - \tilde{Z}_s\|^2 ds \right] \\ &= \mathbb{E} \left[|Y_{t_{i+1} \wedge \theta} - \bar{Y}_{t_{i+1} \wedge \theta}|^2 \right] + \mathbb{E} \left[2 \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} (Y_s - \bar{Y}_s) (\mathbf{1}_{s < \tau} (F(V_s, Z_s) - \lambda) - \mathbf{1}_{s < \bar{\tau}} (F(\bar{V}_{s-}, \bar{Z}_{s-}) + \bar{\lambda})) ds \right] \\ &= \mathbb{E} \left[|Y_{t_{i+1} \wedge \theta} - \bar{Y}_{t_{i+1} \wedge \theta}|^2 \right] + \mathbb{E} \left[2 \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} (Y_s - \bar{Y}_s) \mathbf{1}_{s \leq \bar{\tau}} (F(V_s, Z_s) - \lambda - F(\bar{V}_{s-}, \bar{Z}_{s-}) + \bar{\lambda}) ds \right] \\ &\quad + \mathbb{E} \left[2 \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} (Y_s - \bar{Y}_s) (\mathbf{1}_{s \leq \tau} - \mathbf{1}_{s \leq \bar{\tau}}) (F(V_s, Z_s) - \lambda) ds \right] \end{aligned} \quad (3.18)$$

Using the inequality $2ab \leq \alpha a^2 + \frac{1}{\alpha} b^2$ for a $\alpha > 0$ to be chosen later, we get:

$$\begin{aligned} \text{Err}_{t \wedge \theta}^{t_{i+1} \wedge \theta} &\leq \mathbb{E} \left[|Y_{t_{i+1} \wedge \theta} - \bar{Y}_{t_{i+1} \wedge \theta}|^2 \right] + \alpha \mathbb{E} \left[\int_{t \wedge \theta}^{t_{i+1} \wedge \theta} |Y_s - \bar{Y}_s|^2 ds \right] \\ &\quad + \frac{2}{\alpha} \mathbb{E} \left[\int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \mathbf{1}_{s < \bar{\tau}} (F(V_s, Z_s) - F(\bar{V}_{s-}, \bar{Z}_{s-}))^2 ds \right] + \frac{2}{\alpha} \mathbb{E} \left[\int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \mathbf{1}_{s < \bar{\tau}} |\lambda - \bar{\lambda}|^2 ds \right] \\ &\quad + \frac{2}{\alpha} \mathbb{E} \left[\int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \mathbf{1}_{\tau \leq s < \bar{\tau}} (F(V_s, Z_s) - \lambda)^2 ds + \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \mathbf{1}_{\bar{\tau} \leq s < \tau} (F(V_s, Z_s) - \lambda)^2 ds \right]. \end{aligned}$$

On the event $\{s > \tau\}$, we have $Y_s = Y_\tau$ so that $Z_s = 0$. Then, using the Lipschitz properties of the driver F (1.19), (1.20), the boundness of Z , Remark 3.2 and result on the Euler approximation of V

we obtain:

$$\begin{aligned}
\text{Err}_{t \wedge \theta}^{t_{i+1} \wedge \theta} &\leq \mathbb{E} \left[|Y_{t_{i+1} \wedge \theta} - \bar{Y}_{t_{i+1} \wedge \theta}|^2 \right] + \alpha \mathbb{E} \left[\int_{t \wedge \theta}^{t_{i+1} \wedge \theta} |Y_s - \bar{Y}_s|^2 ds \right] \\
&+ \frac{C}{\alpha} \mathbb{E} \left[\int_{t \wedge \theta}^{t_{i+1} \wedge \theta \wedge \bar{\tau}} \left(h + \|Z_s - \hat{Z}_{s-}\|^2 + \|Z_s - \tilde{Z}_s\|^2 \right) ds \right] \\
&+ \frac{C}{\alpha} \mathbb{E} \left[\int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \mathbf{1}_{\tau \leq s \leq \bar{\tau}} K^2 + \mathbf{1}_{\bar{\tau} \leq s \leq \tau} Z_{\max}^2 + \mathbf{1}_{\tau \wedge \bar{\tau} \leq s \leq \tau \vee \bar{\tau}} \lambda^2 ds \right] \\
&+ \frac{2}{\alpha} \mathbb{E} \left[\int_{t \wedge \theta}^{t_{i+1} \wedge \theta \wedge \bar{\tau}} |\lambda - \bar{\lambda}|^2 ds \right].
\end{aligned}$$

Then Gronwall's lemma leads to

$$\begin{aligned}
\mathbb{E} \left[|Y_{t \wedge \theta} - \bar{Y}_{t \wedge \theta}|^2 \right] &\leq \text{Err}_{t \wedge \theta}^{t_{i+1} \wedge \theta} \\
&\leq (1 + C_\alpha h) \mathbb{E} \left[|Y_{t_{i+1} \wedge \theta} - \bar{Y}_{t_{i+1} \wedge \theta}|^2 \right] \\
&+ \left(C_\alpha h + \frac{C}{\alpha} \right) \mathbb{E} \left[\int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \left(h + \|Z_s - \hat{Z}_{s-}\|^2 + \|Z_s - \tilde{Z}_s\|^2 \right) ds \right] \quad (3.19) \\
&+ \left(C_\alpha h + \frac{C}{\alpha} \right) \mathbb{E} \left[\int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \mathbf{1}_{\tau \wedge \bar{\tau} \leq s \leq \tau \vee \bar{\tau}} \max(K^2, Z_{\max}^2, \lambda^2) ds \right] \\
&+ \left(C_\alpha h + \frac{2}{\alpha} \right) \mathbb{E} \left[\int_{t \wedge \theta}^{t_{i+1} \wedge \theta \wedge \bar{\tau}} |\lambda - \bar{\lambda}|^2 ds \right]
\end{aligned}$$

Recall that the ergodic constant λ is bounded, so that the maximum in the third line above is finite. Substituting $t = t_i$ in (3.19), taking $\alpha > 0$ sufficiently large and the time step h small enough, we get:

$$\begin{aligned}
\mathbb{E} \left[|Y_{t_i \wedge \theta} - \bar{Y}_{t_i \wedge \theta}|^2 \right] + \mathbb{E} \left[\int_{t_i \wedge \theta}^{t_{i+1} \wedge \theta} \|Z_s - \tilde{Z}_s\|^2 ds \right] &\leq (1 + Ch) \mathbb{E} \left[|Y_{t_{i+1} \wedge \theta} - \bar{Y}_{t_{i+1} \wedge \theta}|^2 \right] \\
&+ C \mathbb{E} \left[\int_{t_i \wedge \theta}^{t_{i+1} \wedge \theta} \left(h + \|Z_s - \hat{Z}_{s-}\|^2 + \|Z_s - \tilde{Z}_s\|^2 \right) ds \right] \\
&+ C \mathbb{E} \left[\int_{t_i \wedge \theta}^{t_{i+1} \wedge \theta} \mathbf{1}_{\tau \wedge \bar{\tau} \leq s \leq \tau \vee \bar{\tau}} ds \right] + C \mathbb{E} \left[\int_{t_i \wedge \theta}^{t_{i+1} \wedge \theta \wedge \bar{\tau}} |\lambda - \bar{\lambda}|^2 ds \right].
\end{aligned}$$

Summing on i we have

$$\begin{aligned}
\max_i \mathbb{E} \left[|Y_{t_i \wedge \theta} - \bar{Y}_{t_i \wedge \theta}|^2 \right] + \mathbb{E} \left[\int_0^\theta \|Z_s - \tilde{Z}_s\|^2 ds \right] &\leq C \left(\mathbb{E} \left[|Y_\theta - \bar{Y}_\theta|^2 \right] + h \mathbb{E}[\theta] + \mathcal{R}(Z) \frac{\pi}{\mathcal{H}^2} \right. \\
&\left. + \mathbb{E}[\theta \wedge (\bar{\tau} \vee \tau) - \tau \wedge \bar{\tau}] + |\lambda - \bar{\lambda}|^2 \mathbb{E}[\theta \wedge \bar{\tau}] \right). \quad (3.20)
\end{aligned}$$

Then, it follows, by again using Remark 3.2 that

$$\text{Err}(h)_\theta^2 \leq C \left(\mathbb{E} \left[|Y_\theta - \bar{Y}_\theta|^2 \right] + h \mathbb{E}[\theta] + \mathcal{R}(Z) \frac{\pi}{\mathcal{H}^2} + \mathbb{E}[\theta \wedge (\bar{\tau} \vee \tau) - \tau \wedge \bar{\tau}] + |\lambda - \bar{\lambda}|^2 \mathbb{E}[\theta \wedge \bar{\tau}] \right). \quad (3.21)$$

For the stopping time $\theta = \tau^+ \vee \bar{\tau}^+$ in the time grid π , one observe that $Y_{\tau^+ \vee \bar{\tau}^+} = \bar{Y}_{\tau^+ \vee \bar{\tau}^+} = y_0$, so that the first term on the right side above vanishes and we conclude that

$$\text{Err}(h)_{\tau^+ \vee \bar{\tau}^+}^2 \leq C \left(h \mathbb{E}[\tau^+ \vee \bar{\tau}^+] + \mathcal{R}(Z) \frac{\pi}{\mathcal{H}^2} + \mathbb{E}[\tau - \bar{\tau}] + |\lambda - \bar{\lambda}|^2 \mathbb{E}[(\tau^+ \vee \bar{\tau}^+) \wedge \bar{\tau}] \right).$$

□

4 Deep learning algorithms for the simulation of ergodic BSDEs and forward utilities

In this section, we introduce new deep learning algorithms for the simulation of ergodic BSDEs, based on the representation (2.5) of markovian solutions using BSDEs with random time horizon. The first neural network based algorithm for solving BSDEs was initially proposed in [19]. Since then, there has been a growing interest in developing deep learning algorithms solving BSDEs with finite horizons. We introduce here two algorithms solving the ergodic BSDE (1.21), which can be seen as ergodic counterpart of the neural networks algorithms introduced in [19] and [25]. The algorithms introduced below approximate the ‘‘Markovian’’ solution $(y(V_t) + y_0 - y(V_0), z(V_t), \lambda)_{t \geq 0}$ to the ergodic BSDE with random time horizon (2.5):

$$\begin{aligned} dV_t &= \mu(V_t) + \kappa dW_t, \quad V_0 = v_0, \\ Y_t &= Y_\tau + \int_t^\tau F(V_s, Z_s) ds - \lambda(\tau - t) - \int_t^\tau Z_s^\top dW_s, \\ Y_\tau &= Y_0 = y_0, \end{aligned}$$

taking advantage of the specific characteristics of (2.5), namely:

1. The initial value Y_0 is a known quantity, equal to $y(v_0) = y_0$. This allows us to use a forward scheme starting from y_0 .
2. The recurrence property of the stochastic factor V also provides a known terminal condition to (2.5). Indeed, by definition of the return time τ , $Y_\tau = Y_0$. This allows us to define the loss functions.
3. The functions y and z are only functions of the stochastic factor V , and do not depend on time, as it is the case for standard BSDEs with fixed time horizon.

4.1 Deep-learning algorithm for the simulation of ergodic BSDE

In this section, we detail our two main algorithms for the simulation of ergodic BSDEs, called GeBSDE and LAeBSDE. The first neural network based algorithm for solving BSDEs was initially proposed in [19]. In the context of ergodic BSDEs with random terminal time of type (2.5), the initial value $Y_0 = y_0$ is known and will thus not be learned. We instead approximate the ergodic cost λ as a trainable parameter of the model. We present two algorithms:

- A global solver GeBSDE which consists in the minimization of a square loss function at the random horizon τ and which is the counterpart for ergodic BSDEs to the Deep BSDE solver of [19].
- A locally additive solver LAeBSDE optimized according to the aggregation of local loss functions up to the random horizon, counterpart to the deep backward multi-step introduced in [17] and the LaBSDE solver from [25].

Using the same notations as in the previous section, the forward stochastic factor V is approximated by a Euler discretization on the time grid π . Denoting for all $i \geq 0$, $\Delta W_{t_i} = W_{t_{i+1}} - W_{t_i}$ the Brownian increment at time t_i :

$$\begin{aligned}\bar{V}_{t_{i+1}} &= \bar{V}_{t_i} + \mu(\bar{V}_{t_i})h + \kappa\Delta W_{t_i}, \\ \bar{V}_0 &= v_0.\end{aligned}$$

We denote by $\tilde{\tau}$ the first hitting time in the time grid of \bar{V} to v_0 after T_H :

$$\tilde{\tau}_j = \inf \{t_i > T_H, t_i \in \pi; (\bar{V}_{T_H} - v_0)(\bar{V}_{t_i} - v_0) \leq 0\}, \quad (4.1)$$

assuming for ease of notations that $T_H \in \pi$.

GeBSDE solver Starting from the initial value $Y_0 = y_0$, we consider a forward discretization of the equation on the time grid π with constant time step h . The process $Z_t = z(V_t)$ at time t_i is represented by a neural network $\mathcal{Z}^\theta : \mathbb{R} \rightarrow \mathbb{R}^d$, function of \bar{V}_{t_i} and with parameters θ . The approximation $\bar{Y}_{t_i}^{\theta, \bar{\lambda}}$ of Y_{t_i} depends on the optimisation parameter θ as well as the trainable parameter $\bar{\lambda}$ through the following forward discretisation of the ergodic BSDE:

$$\bar{Y}_{t_{i+1}}^{\theta, \bar{\lambda}} = \bar{Y}_{t_i}^{\theta, \bar{\lambda}} - F(\bar{V}_{t_i}, \mathcal{Z}^\theta(\bar{V}_{t_i}))h + \bar{\lambda}h + \mathcal{Z}^\theta(\bar{V}_{t_i})\Delta W_{t_i}. \quad (4.2)$$

The output $\bar{Y}_{\tilde{\tau}}^{\theta, \bar{\lambda}}$ aims to match the terminal value $Y_\tau = y(V_0) = y_0$, by minimizing over parameters $(\theta, \bar{\lambda})$ the expected square loss function:

$$L_g(\theta, \bar{\lambda}) = \mathbb{E} \left[\left| y_0 - \bar{Y}_{\tilde{\tau}}^{\theta, \bar{\lambda}} \right|^2 \right]. \quad (4.3)$$

The loss function (4.3) is approximated by the empirical loss function over a batchsize B :

$$L_g^B(\theta, \bar{\lambda}) = \frac{1}{B} \sum_{j=1}^B \left| y_0 - \bar{Y}_{\tilde{\tau}_j}^{\theta, \bar{\lambda}, j} \right|^2. \quad (4.4)$$

Finally, we denote M the number of gradient descent performed in the optimization.

Algorithm 1: Global eBSDE Algorithm - (GeBSDE)

Let \mathcal{Z}^θ be a neural network defined on \mathbb{R} , valued in \mathbb{R}^d , with parameters θ . Let $\bar{\lambda}^0 \in \mathbb{R}$ be the initialisation of the trainable parameter representing the ergodic cost. Define $N_{T_H} = \lfloor \frac{T_H}{h} \rfloor + 1$.

for $j = 1, \dots, B$ **do**

for $k \in \{0, \dots, N_{T_H} + 1\}$, starting from $\bar{V}_0^j = v_0$ **do**

 Sample $\Delta W_{t_k}^j$ from a Gaussian vector.

$\bar{V}_{t_{k+1}}^j = \bar{V}_{t_k}^j + \mu(\bar{V}_{t_k}^j)h + \kappa^\top \Delta W_{t_k}^j$,

 Let $N_j = N_{T_H} + 1$.

while $(\bar{V}_{t_{N_{T_H}}}^j - v_0)(\bar{V}_{t_{N_j}}^j - v_0) > 0$ **do**

 Sample $\Delta W_{t_k}^j$ from a Gaussian vector.

$\bar{V}_{t_{N_j+1}}^j = \bar{V}_{t_{N_j}}^j + \mu(\bar{V}_{t_k}^j)h + \kappa^\top \Delta W_{t_k}^j$,

$N_j = N_j + 1$

 Set, $hN_j = \tilde{\tau}_j$.

for $m = 0, \dots, M$ **do**

for $j = 1, \dots, B$ **do**

for $k \in \{0, \dots, N_j - 1\}$, starting from $\bar{Y}_0^j = y_0$ **do**

$\bar{Y}_{t_{k+1}}^{\theta^m, \bar{\lambda}^m, j} = \bar{Y}_{t_k}^{\theta^m, \bar{\lambda}^m, j} - hF(\bar{V}_{t_k}^j, \mathcal{Z}^{\theta^m}(\bar{V}_{t_k}^j)) + \bar{\lambda}^m h + \mathcal{Z}^{\theta^m}(\bar{V}_{t_k}^j)^\top \Delta W_{t_k}$,

 Compute $L^B(\theta^m, \bar{\lambda}^m) = \frac{1}{B} \sum_{j=1}^B \left| y_0 - \bar{Y}_{\tilde{\tau}_j}^{\theta^m, \bar{\lambda}^m, j} \right|^2$.

 Update $\theta^{m+1} = \theta^m - \rho_m \nabla_\theta L^B(\theta^m, \bar{\lambda}^m)$ and $\bar{\lambda}^{m+1} = \bar{\lambda}^m - \rho_m \nabla_{\bar{\lambda}} L^B(\theta^m, \bar{\lambda}^m)$.

Based on [19], [7], we use a neural network consisting in 2 hidden layers of $20 + d$ neurons each, where d is the dimension of the Brownian motion. For the simulation, we use the *tanh* activation function and the Adam optimizer with a learning rate ρ_0 to update both parameters θ^m and $\bar{\lambda}^m$. The learning rate parameter can be optimized depending on the example as investigated in [7]. However, choosing a large enough initial learning rate, as well as the Adam optimizer reduce the risk the algorithm gets stuck in a local minimum. Finally, we use a Glorot normal initialization for the parameters of the neural network and the trainable parameter $\bar{\lambda}$, the latter being constrained to be in the interval $[-K, K]$.

LAeBSDE solver - Some other deep learning algorithms relies on a global optimization involving local loss function at each time step as studied in [24], [25]. Such algorithms approximate Y with a neural network, while Z can either be computed with automatic differentiation or with another neural network. Numerical results in [24] reveal that automatic differentiation may lead some additional errors, so that we will rather use a neural network to approach Z . In the context of ergodic BSDE, the ergodic cost is again approached as a trainable parameter of the model $\bar{\lambda}$ on which will depend the loss function.

The construction of local loss functions relies on the time discretization (4.2). In fact, iterating this equation with the initial condition $Y_0 = y_0$ leads for all $i \geq 1$:

$$\bar{Y}_{t_i}^{\theta, \bar{\lambda}} = y_0 - \sum_{k=0}^{i-1} F(\bar{V}_{t_k}, \mathcal{Z}^\theta(\bar{V}_{t_k}))h + \bar{\lambda}h + \mathcal{Z}^\theta(\bar{V}_{t_k})^\top \Delta W_{t_k}. \quad (4.5)$$

Then, introducing two neural networks \mathcal{Y}^{θ_1} defined on \mathbb{R} and valued in \mathbb{R} and \mathcal{Z}^{θ_2} defined on \mathbb{R} valued in \mathbb{R}^d , the local loss function at time t_i can be defined as the expected square distance between $\mathcal{Y}^{\theta_1}(\bar{V}_{t_i})$ and (4.5), that is:

$$L_{\text{loc},t_i}(\theta_1, \theta_2, \bar{\lambda}) = \mathbb{E} \left[\left| \mathcal{Y}^{\theta_1}(\bar{V}_{t_i}) + \sum_{k=0}^{i-1} F(\bar{V}_{t_k}, \mathcal{Z}^{\theta_2}(\bar{V}_{t_k}))h - \bar{\lambda}h - \mathcal{Z}^{\theta_2}(\bar{V}_{t_k})\Delta W_{t_k} - y_0 \right|^2 \right] \quad (4.6)$$

The final loss function is then constructed by summing those local loss functions over i . In our framework, τ is not necessarily bounded so that this sum could have infinitely many terms. However, when approximating expectations over a batchsize B , one can express this empirical loss function as a sum up to the time $\max_{j \in B} \tilde{\tau}_j$. Note that since the terminal time is random, the local loss function at time t_i (4.6) is computed on the set of trajectories $\mathbb{T}_i = \{j \in \{1, \dots, B\}; \tilde{\tau}_j \geq t_i\}$, for which the approximated return time is larger than t_i . The empirical version of (4.6) is:

$$L_{\text{loc},t_i}^B(\theta_1, \theta_2, \bar{\lambda}) = \frac{1}{|\mathbb{T}_i|} \sum_{j \in \mathbb{T}_i} \left| \mathcal{Y}^{\theta_1}(\bar{V}_{t_i}^j) + \sum_{k=0}^{i-1} F(\bar{V}_{t_k}^j, \mathcal{Z}^{\theta_2}(\bar{V}_{t_k}^j))h - \bar{\lambda}h - \mathcal{Z}^{\theta_2}(\bar{V}_{t_k}^j)\Delta W_{t_k}^j - y_0 \right|^2, \quad (4.7)$$

Denoting $N_{\text{max}}^B = \frac{1}{h} \max_{j \in B} \tilde{\tau}_j$, the index in the grid of the greater return time over the fixed amount of samples B , the empirical locally additive general loss is:

$$L_{\text{loc}}^B(\theta_1, \theta_2, \bar{\lambda}) = \sum_{i=1}^{N_{\text{max}}^B} L_{\text{loc},t_i}^B(\theta_1, \theta_2, \bar{\lambda}). \quad (4.8)$$

Algorithm 2: Locally additive eBSDE Algorithm - (LAeBSDE)

Let \mathcal{Y}^{θ_1} be a neural network defined on \mathbb{R} , valued in \mathbb{R} with parameters θ_1 and \mathcal{Z}^{θ_2} be a neural network defined on \mathbb{R} , valued in \mathbb{R}^d , with parameters θ_2 . Let $\bar{\lambda}^0 \in \mathbb{R}$ be the initialisation of the trainable parameter representing the ergodic cost. Define $N_{T_H} = \lfloor \frac{T_H}{h} \rfloor + 1$.

for $j = 1, \dots, B$ **do**

for $k \in \{0, \dots, N_{T_H} + 1\}$, starting from $\bar{V}_0^j = v_0$ **do**

 Sample $\Delta W_{t_k}^j$ from a Gaussian vector.

$\bar{V}_{t_{k+1}}^j = \bar{V}_{t_k}^j + \mu(\bar{V}_{t_k}^j)h + \kappa^\top \Delta W_{t_k}^j$,

 Let $N_j = N_{T_H} + 1$.

while $(\bar{V}_{t_{N_{T_H}}}^j - v_0)(\bar{V}_{t_{N_j}}^j - v_0) > 0$ **do**

 Sample $\Delta W_{t_k}^j$ from a Gaussian vector.

$\bar{V}_{t_{N_j+1}}^j = \bar{V}_{t_{N_j}}^j + \mu(\bar{V}_{t_k}^j)h + \kappa^\top \Delta W_{t_k}^j$,

$N_j = N_j + 1$

 Set, $hN_j = \tilde{\tau}_j$.

for $m = 0, \dots, M$ **do**

for $j = 1, \dots, B$ **do**

 Set, $\phi_{t_{-1}} = 0$.

for $k \in \{0, \dots, N_j - 1\}$, starting from $\bar{Y}_0^j = y_0$ **do**

$\psi_{t_k}^{\theta_2^m, \bar{\lambda}^m, j} = hF(\bar{V}_{t_k}^j, \mathcal{Z}^{\theta_2^m}(\bar{V}_{t_k}^j)) - \bar{\lambda}^m h - \mathcal{Z}^{\theta_2^m}(\bar{V}_{t_k}^j)^\top \Delta W_{t_k}$,

$\phi_{t_k}^{\theta_2^m, \bar{\lambda}^m, j} = \phi_{t_{k-1}}^{\theta_2^m, \bar{\lambda}^m, j} + \psi_{t_k}^{\theta_2^m, \bar{\lambda}^m, j}$

for $k \in \{0, \dots, N_j - 1\}$ **do**

 Define the set $\mathbb{T}_k = \{j \in \{1, \dots, B\}; \tilde{\tau}_j^+ \geq t_k\}$.

 Compute $L_{\text{loc}, t_k}^B(\theta_1^m, \theta_2^m, \bar{\lambda}^m) = \frac{1}{|\mathbb{T}_k|} \sum_{j \in \mathbb{T}_k} \left| \mathcal{Y}^{\theta_1^m}(\bar{V}_{t_k}^j) + \phi_{t_k}^{\theta_2^m, \bar{\lambda}^m, j} - y_0 \right|^2$.

 Compute $L_{\text{loc}}^B(\theta_1^m, \theta_2^m, \bar{\lambda}^m) = \sum_{k=1}^{\max_{j=1, \dots, B} N_j - 1} L_{\text{loc}, t_k}^B(\theta_1^m, \theta_2^m, \bar{\lambda}^m)$.

 Denoting $\theta = (\theta_1, \theta_2, \bar{\lambda})$, update $\theta^{m+1} = \theta^m - \rho_m \nabla_\theta L_{\text{loc}}^B(\theta^m)$.

For the simulation, we consider as before neural networks with 2 hidden layers of $20 + d$ neurons each. We check that increasing the number of layers or neurons does not improve accuracy in our numerical tests. The code of both solvers for numerical experiments is available on github : <https://github.com/gubrx/Deep-learning-eBSDE>.

Remark 4.1. Note that this algorithm can also be used by first approximating the ergodic cost λ with Monte Carlo methods based on Section 2.3, and plugging this estimator $\hat{\lambda}$ in the forward discretisation above. The optimization is then only performed on the parameters of the neural network.

4.2 Toy examples

In this section, we present the numerical results obtained with Algorithm 1 and 2 for two examples of ergodic BSDE with explicit solutions. We also investigate the approximation of the ergodic cost λ with Monte Carlo methods, within the framework of Propositions 2.3 and 2.6. For the numerical tests,

we will consider a stochastic factor V of type Ornstein-Uhlenbeck with dynamics:

$$dV_t = -\mu V_t + \kappa^\top dW_t, \quad V_0 = v_0. \quad (4.9)$$

Example 4.1. The ergodic BSDE (1.8) with driver $F(v, z) = C_v v e^{-v^2/2}$ admits a unique Markovian solution such that $y(0) = \frac{C_v}{\mu} \frac{\sqrt{2\pi}}{2}$ and z is bounded, given by:

$$(y(v), z(v), \lambda) = \left(\frac{C_v}{\mu + \frac{1}{2}\kappa^2} \int_{-\infty}^v e^{-\frac{y^2}{2}} dy, \frac{C_v}{\mu + \frac{1}{2}\kappa^2} e^{-\frac{v^2}{2}}, 0 \right). \quad (4.10)$$

Example 4.1 - Consider a discretisation with step $h = 0.01$ and a batchsize $B = 64$. For the simulation, we choose $v_0 = 0$, $C_v = 1$, $\kappa = 0.8$, $\mu = 1.5$ and $T_H = 1$. In Figure 1 and 2, we plot the evolution of the empirical loss function L^{B_ϵ} given in (4.4) over $B_\epsilon = 100B$ samples as well as the absolute error on $\bar{\lambda}$ through the number M of gradient descent performed in the algorithm.

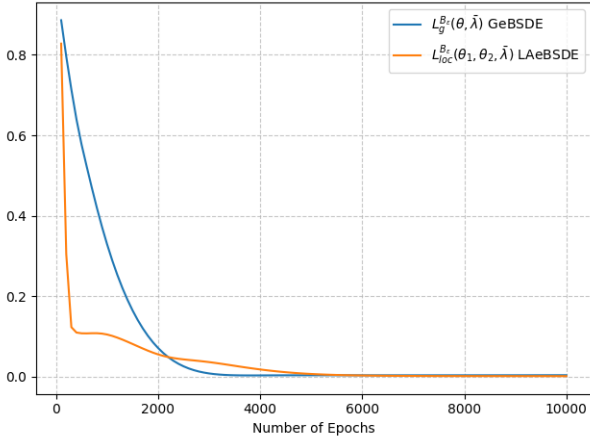


FIGURE 1. Empirical loss function L^{B_ϵ}

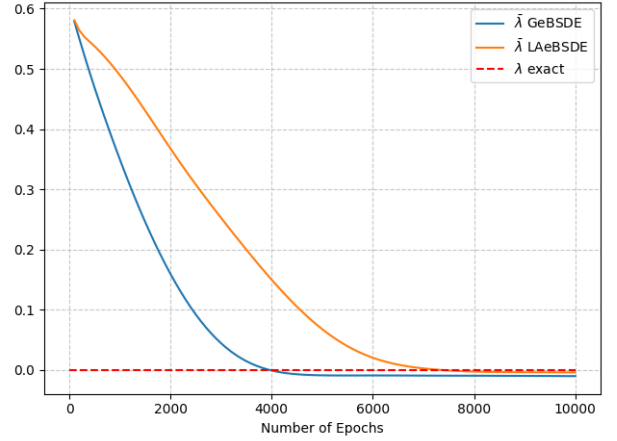


FIGURE 2. Absolute error on λ

Both algorithms converge in the sense that the loss functions as well as the absolute errors on λ go to zero as the number of training steps grows. The GeBSDE converges faster, in 4000 gradient descent to the true value of λ leading to an error of order 10^{-2} . The LAeBSDE estimation of the ergodic cost converges around 7000 epochs with a final error of order 10^{-3} . In Figure 3, we plot the the mean relative absolute error on Y at each time steps that is for a sample of size $B_\epsilon = 100B$ of realizations of the diffusion and $t_i \in \pi \cap [0, T_H]$:

$$\epsilon_{t_i}(Y) = \frac{1}{B_\epsilon} \sum_{j=1}^{B_\epsilon} \left| \frac{y(\bar{V}_{t_i}^j) - \bar{Y}_{t_i}^j}{y(\bar{V}_{t_i}^j)} \right|. \quad (4.11)$$

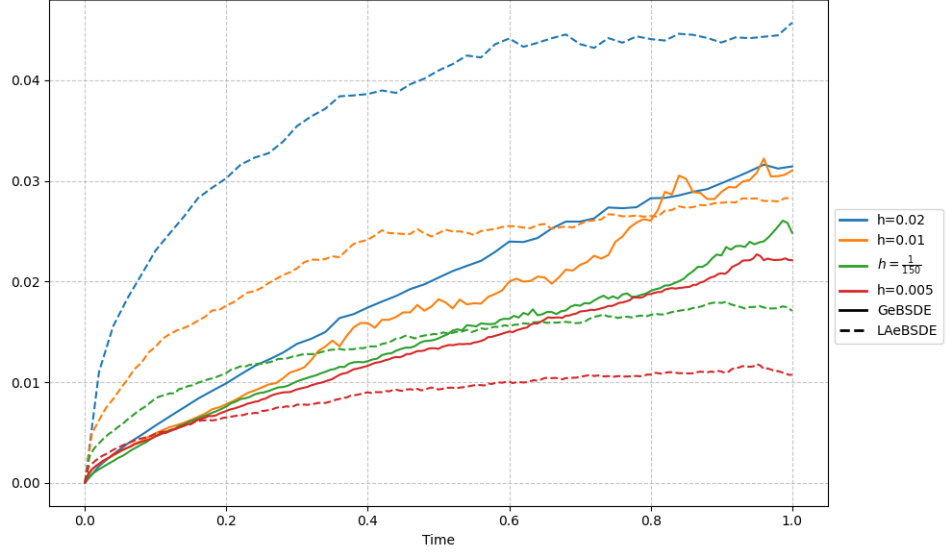


FIGURE 3. Mean relative error on Y for different time steps

The error is zero at time 0 and increases on $[0, T_H]$, which is coherent with both schemes, starting from the known initial value $Y_0 = y_0$. The mean relative error also decreases as the time step gets smaller for both algorithms. Note that the GeBSDE leads a better mean relative error for $h = 0.02$, $h = 0.01$. For smaller time steps, the LAeBSDE algorithm outperforms the GeBSDE and leads a relative error of 1% at time $T_H = 1$ for $h = 0.005$.

We also evaluate the error on Y and Z along the trajectories on $[0, T_H]$ through the integral errors:

$$I_\epsilon^h(Y) = \mathbb{E} \left[\sum_{i=1}^{N_{T_H}} h \left| y(\overline{V}_{t_i}^j) - \overline{Y}_{t_i}^j \right| \right] \quad \text{and} \quad I_\epsilon^h(Z) = \mathbb{E} \left[\sum_{i=1}^{N_{T_H}} h \left\| z(\overline{V}_{t_i}^j) - \mathcal{Z}^\theta(\overline{V}_{t_i}^j) \right\|^2 \right]. \quad (4.12)$$

The expectations above are computed on a sample of size B_ϵ . Moreover, we represent the mean and 95% confidence interval over 5 independent training procedures. The errors are computed for the same values of time step as in Figure 3.

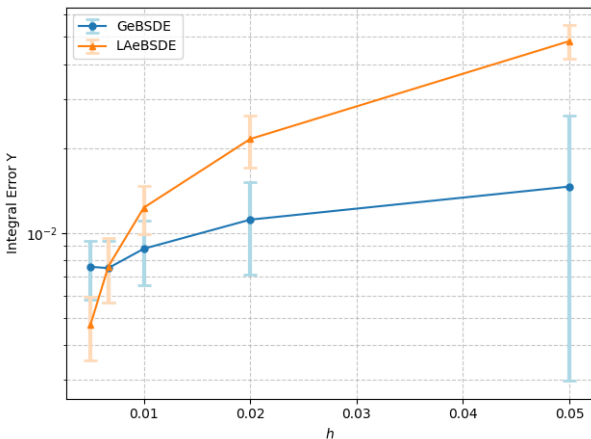


FIGURE 4. Integral error on Y over $[0, T_H]$.

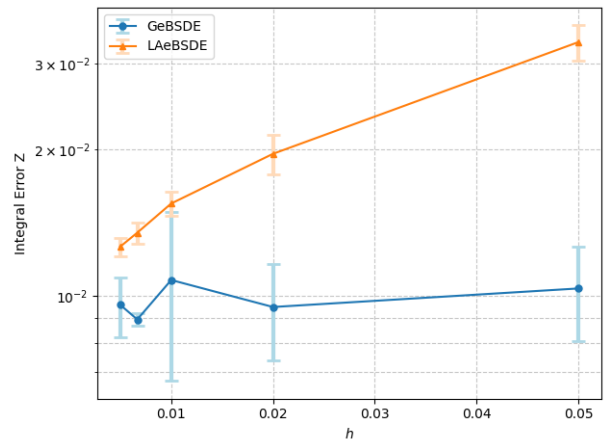


FIGURE 5. Integral error on Z over $[0, T_H]$.

The GeBSDE algorithm leads to a smaller integral error on Z for every time steps. However, we again observe that the integral error on Y for the LAeBSDE decreases rapidly with the time step and

outperforms the global algorithm for $h = 0.005$. Both algorithm provides a good approximation of the trajectory of the solution over the random interval $[0, \tau]$, as displayed in Figure 6.

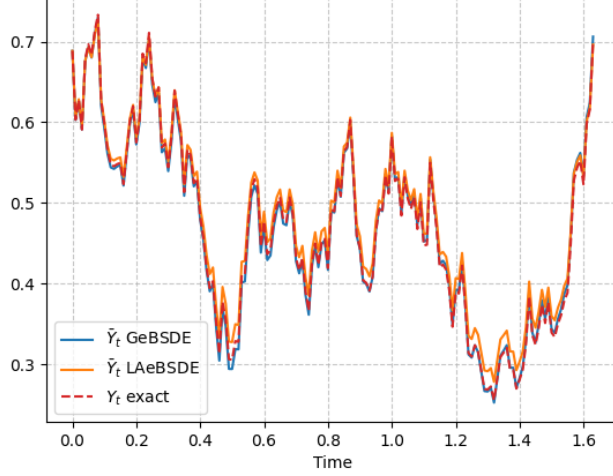


FIGURE 6. Example of trajectory of Y over $[0, \tau]$ with $h = 0.01$.

Example 4.2 (with non-zero ergodic cost λ). We generalize a second example presented in [23]. Consider the ergodic BSDE (1.8) with driver $F(v, z) = C_v|v|e^{-v^2/2}$. For the following, let's denote Φ the cumulative distribution function of the standard normal distribution: $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$.

The proof of the following lemma can be found in Annex A.

Lemma 4.3. *The eBSDE (1.8) with generator $F(v, z) = C_v|v|e^{-v^2/2}$ admits unique Markovian solution satisfying $y(0) = 0$ and such that z is bounded, given by the following triplet $(y(\cdot), z(\cdot), \lambda)$:*

$$\begin{aligned} y(v) &= \mathbb{1}_{\{v \geq 0\}} \int_0^v e^{\frac{y^2}{2}} \left(\frac{C_v}{\kappa^2} e^{-y^2} + 2 \frac{C_v}{\kappa^2} (\Phi(y) - 1) \right) dy + \mathbb{1}_{\{v < 0\}} \int_0^v e^{\frac{y^2}{2}} \left(-\frac{C_v}{\kappa^2} e^{-y^2} + 2 \frac{C_v}{\kappa^2} \Phi(y) \right) dy \\ z(v) &= \mathbb{1}_{\{v \geq 0\}} \kappa e^{\frac{v^2}{2}} \left(\frac{C_v}{\kappa^2} e^{-v^2} + 2 \frac{C_v}{\kappa^2} (\Phi(v) - 1) \right) + \mathbb{1}_{\{v < 0\}} \kappa e^{\frac{v^2}{2}} \left(-\frac{C_v}{\kappa^2} e^{-v^2} + 2 \frac{C_v}{\kappa^2} \Phi(v) \right) \\ \lambda &= \frac{C_v}{\sqrt{2\pi}}. \end{aligned} \tag{4.13}$$

Remark 4.2. In [23], the example is presented with the following parameters:

$$C_\mu = \mu = C_v = A_1 = \frac{1}{2}, \quad \kappa = 1, \quad \text{and } A_2 = 1.$$

However, for this choice of parameters, the bound $Z_{max} = \frac{C_v}{C_\mu - C_v}$ on Z for the constant volatility framework of [27] is not defined. In fact, we want to find an explicit solution such that $C_\mu > C_v$. The proof of Lemma 4.3 is postponed in Annex A.

Example 4.2 - Let $v_0 = 0$, $T_H = 1$, $C_v = 0.75$, $\mu = 1$, a time step $h = 0.01$ and the same parameters of the neural network as in the first example. In this setting, the ergodic cost λ given by (4.13) is 0.299206 and the trainable parameter $\bar{\lambda}$ for both algorithms converge towards this value in around

6000 gradient steps. Training the model with 10000 gradient descent, the absolute error on λ is of order 10^{-3} . We illustrate the convergence of the empirical loss functions as well as the convergence of the ergodic cost estimators.

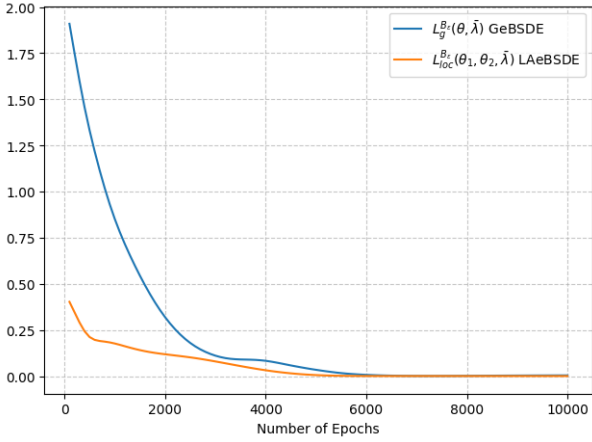


FIGURE 7. Empirical loss function over B_ϵ samples as a function of the number of epochs

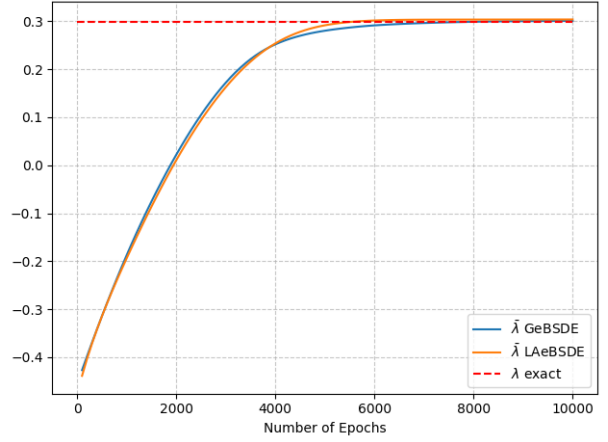


FIGURE 8. Convergence of $\bar{\lambda}$

The shape of the mean absolute error on Y for this example is quite different for the two algorithms. In fact for the GeBSDE, the solution Y being constructed with a forward iterative scheme (4.2) relying on the trained neural network \mathcal{Z}^θ , the error starts from zero and grows then almost linearly according to Figure 9. On the other hand for the LAeBSDE, the solution Y is the output of the neural network \mathcal{Y}^{θ_1} , optimized according to the aggregation of local loss functions L_{loc,t_i} given in (4.6). We observe that the mean error is almost constant on the interval $[0.2, T_H]$ for this example.

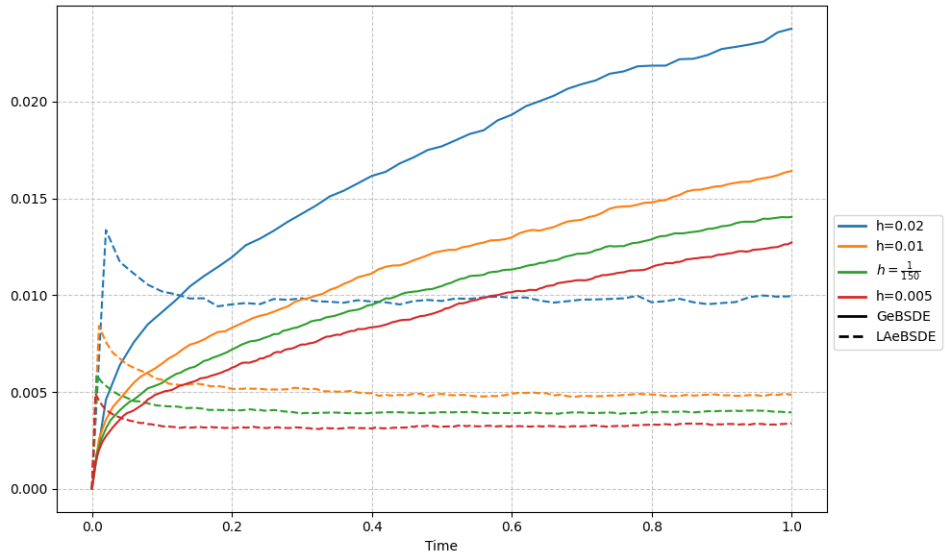


FIGURE 9. Mean relative error on Y for different time steps

As shown in Figure 10, the integral error on Y is lower and decreases faster for the LAeBSDE, which is coherent with the mean error result above.

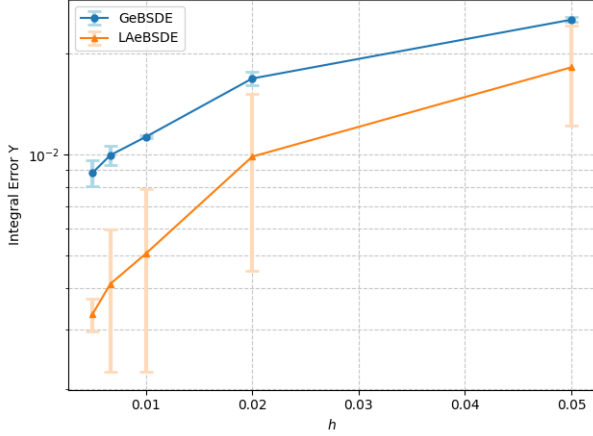


FIGURE 10. Integral error on Y over $[0, T_H]$.

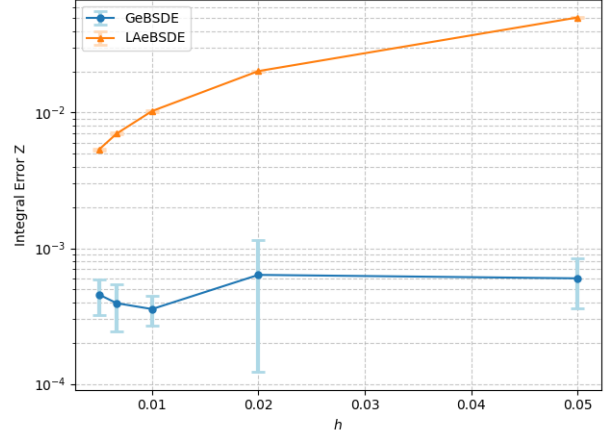


FIGURE 11. Integral error on Z over $[0, T_H]$.

Estimation of λ with Monte Carlo methods - Since for the two above examples, the generator only depends on v , Proposition 2.3 applies and we recall the characterization of the ergodic cost λ given in (2.9) :

$$\lambda = \frac{\mathbb{E} \left[\int_0^\tau F(V_s) ds \right]}{\mathbb{E}[\tau]}.$$

We use Monte Carlo methods over M samples to compute both expectations in the above formula. Approximating V and Γ with an Euler scheme on the time grid π with time step h and the integral with a Riemann sum on the time grid π , we consider the estimate:

$$\hat{\lambda} = \frac{1}{M} \sum_{m=1}^M \sum_{i=0}^{\bar{\tau}_m-1} h F(\bar{V}_{t_i}^m). \quad (4.14)$$

We gather the approximation results of the mean absolute error on 100 simulations for the estimation of the ergodic cost $\hat{\lambda}$ for Examples 4.1 and 4.2 in Table 1 and 2 respectively. The simulations are performed with the following set of parameters : $T_H = 1$, $\mu = 2$, $\kappa = 2$, $C_v = 1$ and $v_0 = 0.5$.

h	M=1000		M=10000		M=100000	
0.05	0.009743	(4.87e-05)	0.007224	(7.48e-06)	0.005691	(1.37e-06)
0.02	0.009107	(4.66e-05)	0.006597	(4.63e-06)	0.004895	(1.69e-06)
0.01	0.008833	(3.73e-05)	0.005778	(7.85e-06)	0.004374	(9.73e-07)

Table 1: Mean absolute error (variance) on $\hat{\lambda}$ for Example 4.1. The exact value of λ is 0.

h	M=1000		M=10000		M=100000	
0.05	0.002988	(2.86e-06)	0.002955	(2.50e-07)	0.002904	(1.99e-08)
0.02	0.002136	(2.30e-06)	0.001617	(4.98e-07)	0.001528	(1.67e-08)
0.01	0.001637	(1.29e-06)	0.000780	(2.62e-07)	0.000939	(3.55e-08)

Table 2: Mean absolute error (variance) on $\hat{\lambda}$ for Example 4.2. The exact value of λ given in (4.13) is 0.398942.

Finally, we display the mean and variance of lambda estimations obtained with Monte Carlo methods (4.14) using $M = 100000$ samples with the output of Algorithm 1 and 2 for $B = 64$ and 10000 gradient descents in Table 3. Statistics are computed over 100 values for the Monte Carlo estimator and on 10 independent training of our neural network algorithms.

	Exact	MC		GeBSDE		LAeBSDE	
Example 4.1	0	-0.004374	(9.73e-07)	-0.003782	(4.28e-05)	-0.004280	(3.07e-05)
Example 4.2	0.398942	0.399882	(3.55e-08)	0.400130	(1.53e-05)	0.397600	(4.63e-05)

Table 3: Comparison of λ approximations for parameters $v_0 = 0.5$, $T_H = 1$, $h = 0.01$, $\kappa = 2$, $\mu = 2$, $C_v = 1$.

4.3 Power utility examples

Let's turn our attention to ergodic BSDEs associated with power forward utilities (1.3). When there are no constraints on the portfolio, the generator (1.12) rewrites for $(v, z) \in \mathbb{R} \times \mathbb{R}^d$ as:

$$F^\delta(v, z) = \frac{\delta}{2(1-\delta)} \|\theta(v) + z\|^2 + \frac{1}{2} \|z\|^2,$$

Corollary 2.6 gives a characterization of the ergodic cost λ as the solution of the minimization problem (2.24) that we will use as a benchmark for the ergodic cost. Approximating V and Γ with an Euler scheme with time step h and the expectation with Monte Carlo method, the map:

$$\lambda \mapsto \left| \frac{1}{M} \sum_{m=1}^M \bar{\Gamma}_{0, \bar{\tau}} - 1 \right|. \quad (4.15)$$

admits a global minimum on $[-K, K]$, denoted $\hat{\lambda}$ which can be determined using Newton's method.

For the simulation, we consider a truncated linear price of risk vector $\theta(v) = \varphi_b(\theta v)$, as performed in [35], where φ_b denotes the projection on the Euclidean ball of \mathbb{R} with radius b . We use a stochastic factor V of type Ornstein-Uhlenbeck and set the parameters $\mu = 3$, $\kappa = 1.3$, $\delta = 0.5$, $\theta = 0.8$, $b = 3$ and $T_H = 1$. The number of Monte Carlo samples used for the estimation of the ergodic cost with (4.14) is $M = 100000$ and we use a time step $h = 0.01$. Finally, we observe that the bound K is larger for the following than for the examples from the previous section. Thus we use a higher learning rate of $\rho_0 = 0.0007$ to ensure a sufficient speed of convergence for $\bar{\lambda}$.

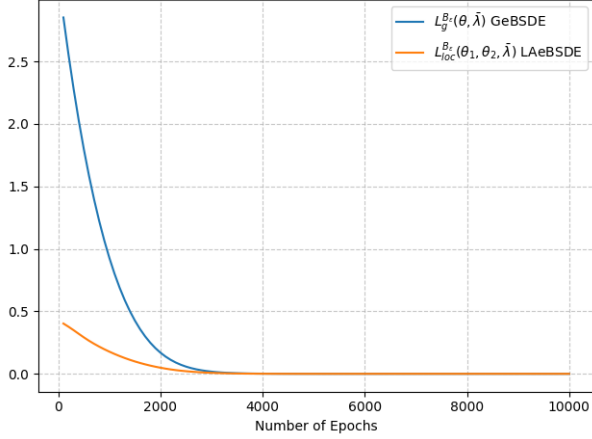


FIGURE 12. Loss function

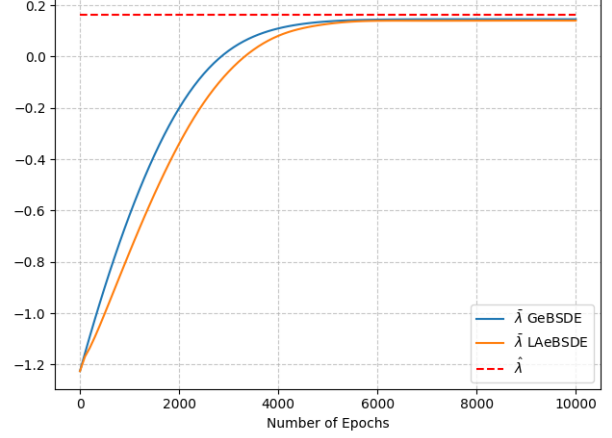


FIGURE 13. Convergence of $\bar{\lambda}$

The ergodic cost $\bar{\lambda}$ computed with Algorithm 1 and 2 and the Monte Carlo approximation $\hat{\lambda}$ converge towards the same value. The absolute error between the two types of estimators is of order 10^{-2} . Finally we illustrate the convergence of the estimator given by (4.15) depending on the time step h and the number of Monte Carlo samples M .

h	M=1000		M=10000		M=100000	
0.10	0.192683	(3.48e-04)	0.201704	(5.5e-05)	0.206999	(3.9e-05)
0.05	0.187675	(8.03e-04)	0.183987	(2.17e-04)	0.182469	(2.6e-05)
0.02	0.159276	(1.55e-03)	0.173734	(4.17e-04)	0.173557	(1.31e-04)
0.01	0.159119	(1.16e-03)	0.174493	(1.61e-03)	0.169749	(1.8e-04)

Table 4: Mean (variance) on $\bar{\lambda}$ on 10 independent runs.

Those Monte Carlo approximations of λ allow to use the semi-explicit representation of Section 2 in order to simulate the solution of the ergodic BSDE (1.8) on $[0, \tau]$. However, in the general case of generator with quadratic growth for which the Cole-Hopf transform does not help to reduce to a linear BSDE, one need another approximation procedure of the ergodic cost λ ,

A two dimensional example - Consider a financial market consisting in one stock, whose price dynamics is given by :

$$dS_t = S_t(b(V_t)dt + \sigma(V_t)dW_t^1),$$

and where the stochastic factor is given by :

$$dV_t^1 = \mu(V_t)dt + \kappa_1 dW_t^2 + \kappa_2 dW_t^3, \quad dV_t^2 = 0.$$

The admissible set of strategies is thus restricted to $\Pi = \mathbb{R} \times \{0\}$, so that $\pi_t^2 = 0$ and the wealth equation (1.6) then reduces to :

$$dX_t^\pi = X_t^\pi \pi_t^1 (\theta(V_t) + dW_t^1), \quad \text{where} \quad \theta(V_t) = \frac{b(V_t)}{\sigma(V_t)}.$$

The generator (1.12) is then given by:

$$F(V_t, Z_t) = \frac{1}{2} \frac{\delta}{1-\delta} |Z_t^1 + \theta(V_t)|^2 + \frac{1}{2} \|Z_t\|^2. \quad (4.16)$$

Denoting $\hat{\delta} = \frac{1-\delta+\delta(\frac{\kappa^1}{\|\kappa\|})^2}{1-\delta}$ and $\tilde{Y}_t = e^{\hat{\delta}(Y_t - \lambda t)}$, the authors in [27] show that the function \tilde{y} must satisfy:

$$\tilde{y}_t(v, t) + \frac{1}{2}(\kappa_1^2 + \kappa_2^2)\tilde{y}_{vv}(v, t) + \left(\mu(v) + \frac{\delta\kappa^1}{1-\delta}\theta(v)\right)\tilde{y}_v(v, t) + \frac{\hat{\delta}\delta}{2(1-\delta)}\theta^2(v)\tilde{y}(v, t) = 0. \quad (4.17)$$

Assuming a linear market price of risk $\theta(v) = \theta v$ and an Ornstein-Uhlenbeck stochastic factor with $\mu(v) = -\mu v$ and $\mu > 0$, following the methodology of [32], several solutions to this PDE can be derived.

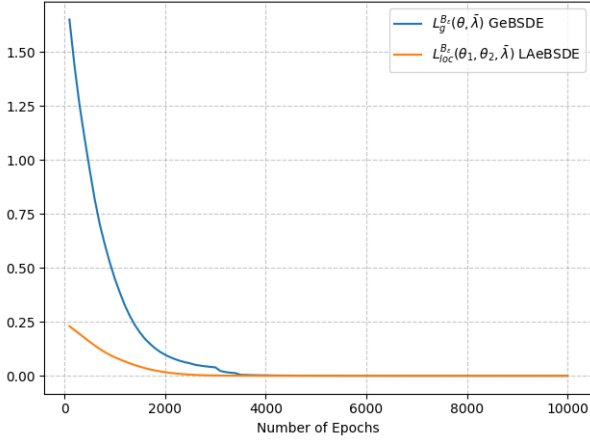


FIGURE 14. Loss function

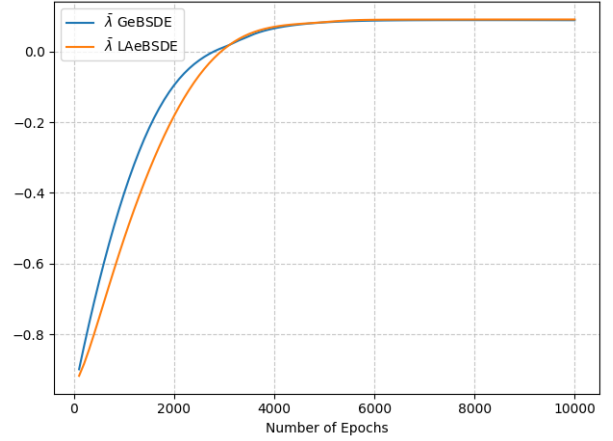


FIGURE 15. Convergence of $\bar{\lambda}$

Validation loss functions for both algorithms converge to zero and the trainable parameters $\bar{\lambda}$ also converge towards the same value.

Finally, let's come back to our initial objective to simulate homothetic forward progressive utilities of Section 1. Using one of the two algorithms GeBSDE or LAeBSDE to simulate the solution of eBSDE (1.8) on $[0, T_H]$, we are now able to plot the corresponding forward utilities. We display the shape of the approximated random field U given by (1.3) for one realization of the diffusion V . One can also access the rescaled optimal portfolio π_t^* given by (1.15). We plot the example of trajectory associated to the same realization of this power utility in Figure 19.

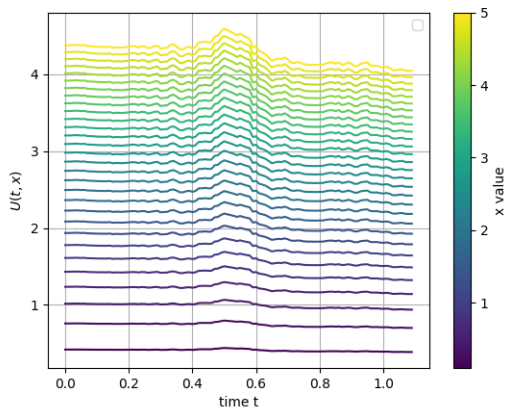


FIGURE 16. Dynamics of approximated utility $U(t, x)$

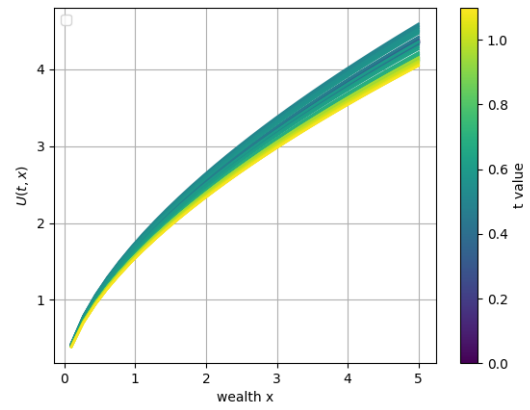


FIGURE 17. Monotonicity and concavity of approximated utility $U(t, x)$

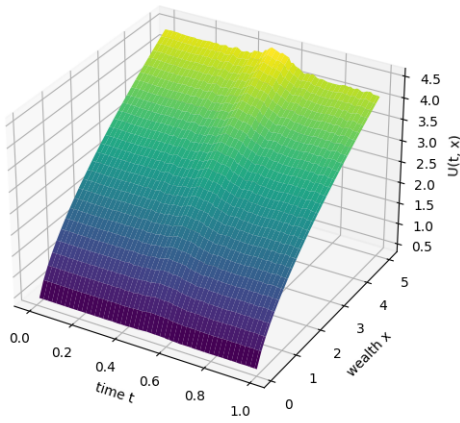


FIGURE 18. Random field $U(t, x)$

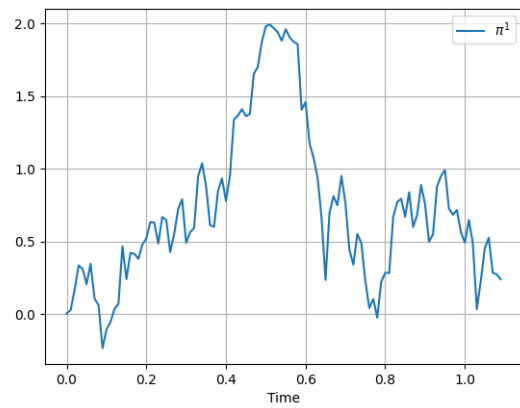


FIGURE 19. Rescaled optimal strategy π_t^1 .

Acknowledgements The authors thank Z. Bensaid (LMM - Le Mans University) for helpful discussions on deep learning methods for the simulation of BSDEs.

A Proof of existence of a solution for example 4.2

Proof. In this section, we look for a triplet $(y(\cdot), z(\cdot), \lambda)$ solution of the ergodic BSDE (1.8) with generator $F(v, z) = C_v|v|e^{-v^2/2}$, of the form:

$$y(v) = \mathbb{1}_{\{v \geq 0\}} \int_0^v e^{\frac{y^2}{2}} \left(A_1 e^{-y^2} + A_2 (\Phi(y) - 1) \right) dy + \mathbb{1}_{\{v < 0\}} \int_0^v e^{\frac{y^2}{2}} \left(-A_1 e^{-y^2} + A_2 \Phi(y) \right) dy \quad (\text{A.1})$$

$$z(v) = \mathbb{1}_{\{v \geq 0\}} \kappa e^{\frac{v^2}{2}} \left(A_1 e^{-v^2} + A_2 (\Phi(v) - 1) \right) + \mathbb{1}_{\{v < 0\}} \kappa e^{\frac{v^2}{2}} \left(-A_1 e^{-v^2} + A_2 \Phi(v) \right) \quad (\text{A.2})$$

$$\lambda = \frac{\kappa^2 A_2}{2\sqrt{2\pi}}, \quad (\text{A.3})$$

where A_1, A_2 are real parameters to be determined later so that the solution satisfies the uniqueness requirements. Let's apply Ito-formula to get the dynamics of $dy(V_t)$.

$$\begin{aligned} dy(V_t) &= e^{V_t^2/2} \left[A_1 e^{-V_t^2} + A_2 (\Phi(V_t) - 1) \right] (-\mu V_t dt + \kappa dW_t) \\ &\quad + \frac{1}{2} \kappa^2 \left(V_t e^{V_t^2/2} \left[A_1 e^{-V_t^2} + A_2 (\Phi(V_t) - 1) \right] + e^{V_t^2/2} \left[-2V_t A_1 e^{-V_t^2} + \frac{A_2}{\sqrt{2\pi}} e^{-V_t^2/2} \right] \right) dt \\ &= V_t e^{V_t^2/2} \left[A_1 e^{-V_t^2} + A_2 (\Phi(V_t) - 1) \right] \left(\frac{1}{2} \kappa^2 - \mu \right) dt \\ &\quad - \kappa^2 A_1 V_t e^{-V_t^2/2} dt + \frac{A_2 \kappa^2}{2\sqrt{2\pi}} dt + \kappa e^{V_t^2/2} \left[A_1 e^{-V_t^2} + A_2 (\Phi(V_t) - 1) \right] dW_t. \end{aligned}$$

In fact imposing $A_1 = \frac{C_v}{\kappa^2}$ and $C_\mu = \mu = \frac{1}{2} \kappa^2$, we see that the triple $(y, z, \frac{\kappa^2 A_2}{2\sqrt{2\pi}})$ given by (A.1) and (A.2) satisfies equation (1.8).

Markovian properties of the solution - One also has to ensure the following Markovian property of the solution, which can be proven PDE arguments (see Proposition 3.4 in [27]), namely that y is C^2 and that $z(v) = \kappa \nabla y(v)$ so that z is C^1 . In particular, z is continuous in 0, which from (A.2) leads $A_2 = 2A_1 = 2\frac{C_v}{\kappa^2}$.

Verification of conditions of Theorem 1.3 - To ensure uniqueness of the Markovian solution above, we need to check that z is bounded, which from the above Markovian representation of the solution also ensures that y is sub-linear with respect to v . The derivative of y given by (A.1) with respect to v is bounded so that y is in fact sub-linear. We look for sufficient conditions on the parameters so that:

$$-Z_{max} \leq z(v) = \kappa A_1 e^{-v^2/2} + \kappa A_2 e^{v^2/2} (\Phi(v) - 1) \leq Z_{max} = \frac{C_v}{\mu - C_v}. \quad (\text{A.4})$$

In the above expression of z , the first term is bounded between 0 and $\frac{C_v}{\kappa}$, while the second one is negative, increasing and goes to 0 as v goes to infinity. Then, denoting $h(v) = \kappa A_2 e^{v^2/2} (\Phi(v) - 1)$, the solution z given by (A.2) is bounded by Z_{max} if:

1. $-Z_{max} \leq h(v)$,
2. $h(v) \leq Z_{max} - \frac{C_v}{\kappa}$.

First, as h is increasing, the first inequality is satisfied imposing $-Z_{max} \leq h(0) = -\frac{\kappa A_2}{2}$. Taking $A_2 = 2\frac{C_v}{\kappa^2}$ leads to the condition:

$$\mu - C_v \leq \kappa \quad (\text{A.5})$$

This inequation together with the condition $\kappa = \sqrt{2\mu}$ leads to:

$$\mu - \sqrt{2}\sqrt{\mu} - C_v \leq 0,$$

which is equivalent to:

$$\frac{1}{2}\kappa^2 - \kappa - C_v \leq 0.$$

This leads to a sufficient condition so that 2 is satisfied:

$$\max(0, 1 - \sqrt{1 + 2C_v}) \leq \kappa \leq 1 + \sqrt{1 + 2C_v}. \quad (\text{A.6})$$

Secondly, as h is negative and going to 0, the second inequality above is satisfied if $Z_{max} - \frac{C_v}{\kappa} = \frac{C_v}{\mu - C_v} - \frac{C_v}{\kappa} \geq 0$, which is equivalent to condition (A.6) above. \square

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