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Vibrations and Potential Flow - Linear Vibrations of Structures Coupled with an Internal Fluid

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Vibrations and Potential Flow

**Linear Vibrations of Structures Coupled
with an Internal Fluid**

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Introduction of the lecture

Fluid-structure vibrations occur in various situations, in aerospace, automotive, civil engineering areas as well as in biomechanics. For a general overview of aerospace interior fluid-structure problems, we refer for instance the reader to Abramson, 1966.

The computational aspects concerning the linear vibratory response of fluid-structure systems to prescribed loads may lead, for complex structures, to a prohibitive number of degrees of freedom. In order to quantify the weak or strong interactions of the coupled fluid-structure system, in order to carry out sensitivity analysis, in order to introduce interface appropriate active/passive damping treatment (intelligent adaptive fluid-structure systems), reduced order procedures are required. That is why concepts which have been introduced for structural dynamics, such as component mode synthesis, are presently revisited and adapted to some multiphysics problems.

We review in this paper reduced order models for modal analysis of elastic structures containing an inviscid fluid (gas or liquid). These methods, based on Ritz-Galerkin projection using appropriate Ritz vectors, allow us to construct *reduced models* expressed in terms of physical displacement vector field u in the structure, and generalized displacement vector r describing the behaviour of the fluid. Those reduced models lead to unsymmetric (Craggs and Stead, 1976; Sung and Nefske, 1986) or symmetric generalized eigenvalue matrix system (Morand and Ohayon, 1979, 1995; Ohayon, 2001) involving a reduced number of degrees of freedom for the fluid. For this purpose, we construct symmetric matrix models of the fluid considered as a subsystem, by considering the response of the fluid to a prescribed normal displacement of the fluid-structure interface.

Two distinct situations are analyzed. On one hand, we consider linear vibrations of an elastic structure completely filled with a compressible gas or liquid and on the other hand, we consider the case of an elastic structure containing an incompressible liquid with free surface effects due to gravity.

The first case is a *structural acoustic* problem. In the case of a structure containing a gas, we consider a modal interaction between *structural modes in vacuo* and *acoustic modes in rigid motionless cavity*. For a structure containing a compressible liquid, we consider a modal interaction between *hydroelastic modes including "static" inertial and potential compressibility effects* and *acoustic modes in rigid motionless cavity*. Interface local fluid-structure dissipation through a local wall impedance can also be introduced easily in the formulations.

The second case is a *hydroelastic-sloshing* problem with a modal interaction between *incompressible hydroelastic structural modes* with *incompressible liquid sloshing modes* in rigid motionless cavity, involving an *elastogravity operator* related to the wall normal displacement of

the fluid-structure interface, introduced initially, under a simplified approximate expression by Tong, 1966, then analyzed through various derivations by Morand and Ohayon, chapter 6, 1995 and recently deeply analyzed theoretically and numerically by Schotté in his PhD dissertation (Schotté and Ohayon, 2003, 2005).

For the construction of reduced models, the static behavior at zero frequency play an important role. Therefore, we review “regularized” variational formulations of the problem, in the sense that the static behaviour must also be in taken into account in the boundary value problem. Those “quasi-static” potential and inertial contributions plays a fundamental role in the Ritz-Galerkin procedure (error truncation).

The general methodology corresponds to dynamic substructuring procedures adapted to fluid-structure modal analysis. For general presentations of computational methods using appropriate finite element and dynamic substructuring procedures applied to modal analysis of elastic structures containing inviscid fluids (sloshing, hydroelasticity and structural-acoustics), we refer the reader for instance to Morand and Ohayon 1995.

Structural-acoustic problem

Let us consider the linear vibrations of an elastic structure completely filled with an homogeneous, inviscid and compressible fluid. We also consider the particular case of a compressible liquid with a free surface, neglecting gravity effects.

After the derivation of the linearized equations of the fluid-structure coupled system, we introduce a linear constraint in order to obtain a regularized problem at zero frequency, and we then construct a reduced model of the fluid subsystem. Acoustic modes in rigid motionless cavity are introduced as Ritz projection vector basis, including the static solution of the coupled system. As this fluid-structure system has a resonant behaviour, a finite element computation of the unreduced model may lead to prohibitive time costs. That is why, starting from one of the possible variational formulations of the problem, convenient *reduced symmetric matrix models* are reviewed.

1 Structural-acoustic equations

1.1 Structure subjected to a fluid pressure loading

We consider an elastic structure occupying the domain Ω_S at equilibrium. The interior fluid domain is denoted Ω_F and the fluid-structure interface is denoted Σ (see Figure 1).

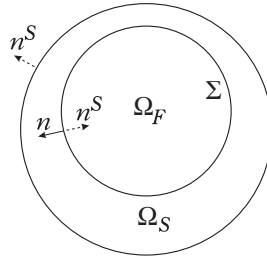


Figure 1: Elastic structure containing a gas

The angular frequency of vibration is denoted as ω . The chosen unknown field in the structure domain Ω_S is the displacement field u . The linearized strain tensor is denoted as $\epsilon_{ij}(u)$ and the corresponding stress tensor is denoted as $\sigma_{ij}(u)$. We denote by ρ_S the constant mass density at equilibrium and by n the unit normal, external to the structure domain Ω_S . Let δu be the test function, associated to u , belonging to the admissible space \mathcal{C}_u .

The weak variational formulation describing the undamped response $u \cos \omega t$ of the structure Ω_S to given harmonic forces of amplitude F^d on the external structure boundary $\partial\Omega_S \setminus \Sigma$, and to fluid pressure field p acting on the internal fluid-structure interface Σ is written as follows.

For all ω and $\forall \delta u \in \mathcal{C}_u$, find $u \in \mathcal{C}_u$ such that

$$\tilde{k}(u, \delta u) - \omega^2 \int_{\Omega_S} \rho_S u \cdot \delta u \, dx - \int_{\Sigma} p n \cdot \delta u \, d\sigma = \int_{\partial\Omega_S \setminus \Sigma} F^d \cdot \delta u \, d\sigma \quad (1)$$

in which

$$\tilde{k}(u, \delta u) = k(u, \delta u) + k_G(u, \delta u) + k_{P_0}(u, \delta u) \quad (2)$$

and where $k(u, \delta u)$ is the mechanical elastic stiffness such that

$$k(u, \delta u) = \int_{\Omega_S} \sigma_{ij}(u) \epsilon_{ij}(\delta u) \, dx \quad (3)$$

and where $k_G(u, \delta u)$ and $k_{P_0}(u, \delta u)$ are such that

$$k_G(u, \delta u) = \int_{\Omega_S} \sigma_{ij}^0 u_{l,i} \delta u_{l,j} \, dx \quad , \quad k_{P_0} = \int_{\Sigma} P_0 n_1(u) \cdot \delta u \, d\sigma \quad (4)$$

In equations (4) and (5), $k_G(u, \delta u)$ represents the initial stress or geometric stiffness in symmetric bilinear form in which σ_{ij}^0 denotes the stress tensor in an equilibrium state, and $k_{P_0}(u, \delta u)$ represents an additional load stiffness in symmetric bilinear form due to rotation of normal n , in which P_0 denotes the initial pressure existing in the reference equilibrium configuration. Finally, $n_1(u)$ represents the variation of normal n between the reference configuration and the actual configuration.

1.2 Fluid subjected to a wall normal displacement

Since the fluid is inviscid, instead of describing the small motion of the fluid by a fluid displacement vector field u_F which requires an appropriate discretization of the fluid irrotationality constraint $\text{curl } u_F = 0$ (see for instance Bermudez and Rodriguez, 1994), we will use the pressure scalar field p . It should be noted that the displacement formulation is particularly convenient for beam-type systems as the irrotationality condition is easily satisfied in the one-dimensional case (Ohayon, 1986). that, the small movements corresponding to $\omega \neq 0$ are obviously irrotational, but, in the static limit case, i.e. at zero frequency, we consider only fluids which exhibit a physical irrotational behavior.

Let us denote by c the (constant) sound speed in the fluid, and by ρ_F , the (constant) mass density of the fluid at rest ($c^2 = \beta/\rho_F$, where β denotes the bulk modulus). We denote as

Ω_F the domain occupied by the fluid at rest (which is taken as the equilibrium state). The local equations describing the harmonic response of the fluid to a prescribed arbitrary normal displacement $u.n$ of the fluid-structure interface Σ are such that

$$\nabla p - \rho_F \omega^2 u_F = 0 \quad |_{\Omega_F} \quad (5)$$

$$p = -\rho_F c^2 \nabla . u_F \quad |_{\Omega_F} \quad (6)$$

$$u_F . n = u.n \quad |_{\Sigma} \quad (7)$$

$$\text{curl } u_F = 0 \quad |_{\Omega_F} \quad (8)$$

Equation (5) corresponds to the linearized Euler equation in the fluid. Equation (6) corresponds to the constitutive equation of the fluid (we consider here a barotropic fluid which means that p is only a function of ρ_F). Equation (7) corresponds to the wall slipping condition. Equation (8) corresponds to the irrotationality condition, only necessary in order to ensure that when $\omega \rightarrow 0$, u_F tends to static irrotational motion, which corresponds to the hypothesis that for $\omega = 0$, we only consider irrotational motions (for simply connected fluid domain).

A displacement potential φ defined up to an additive constant chosen for instance as follows $\int_{\Omega_F} \varphi dx = 0$ can be therefore introduced in order to recast the system defined by equations (5-8) into a scalar one. These aspects will be discussed below.

Relation between static pressure p^s and $u.n$

For $\omega = 0$, equations (6) and (7) lead to a constant static pressure field p^s which is related to the normal wall displacement by the relation

$$p^s = -\frac{\rho_F c^2}{|\Omega_F|} \int_{\Sigma} u.n d\sigma \quad (9)$$

in which $|\Omega_F|$ denotes the measure of the volume occupied by domain Ω_F .

This constant pressure field have been used as an additional unknown field in direct variational symmetric formulation using either a velocity potential formulation (Everstine, 1981) with $j\omega$ or ω^4 additional terms (Olson and Vendini, 1989), or in direct symmetric formulations of classical generalized eigenvalue leading to finite element discretized system of the type $(AX = \omega^2 BX)$ with symmetric real matrices (Felippa and Ohayon, 1990a, 1990b; Morand and Ohayon, 1995a, 1995b).

1.3 Equations in terms of p or φ and $u.n$

The elimination of u_F between equations (5), (6), (7) and (8) leads to

$$\nabla^2 p + \frac{\omega^2}{c^2} p = 0 \quad |_{\Omega_F} \quad (10)$$

$$\frac{\partial p}{\partial n} = \rho_F \omega^2 u.n \quad |_{\Sigma} \quad (11)$$

with the constraint

$$\frac{1}{\rho_F c^2} \int_{\Omega_F} p \, dx + \int_{\Sigma} u.n \, d\sigma = 0 \quad (12)$$

Equation (10) is the classical Helmholtz equation expressed in terms of p . Equation (11) corresponds to the kinematic condition defined by equation (7):

$$\partial p / \partial n = -\rho_F \ddot{u}_F.n = \rho_F \omega^2 u.n \quad |_{\Sigma}$$

The linear constraint defined by equation (12) corresponds to the global mass conservation which ensures that the boundary problem defined by equations (10) to (11) is equivalent to the problem defined by equations (5) to (8). In the absence of the condition defined by equation (12), we would obtain a boundary value problem in terms of p which is not valid for $\omega = 0$ and which does not allow us to retrieve the value of p^s given by equation (9).

Using equations (8-9), the boundary value problem defined by equations (10-12) can be recasted into the following equivalent one using the displacement potential field φ introduced above such that $p = \rho_F \omega^2 \varphi + p^s(u.n)$ with $\int_{\Omega_F} \varphi \, dx = 0$

$$\nabla^2 \varphi + \frac{\omega^2}{c^2} \varphi - \frac{1}{|\Omega_F|} \int_{\Sigma} u.n \, d\sigma = 0 \quad |_{\Omega_F} \quad (10a)$$

$$\frac{\partial \varphi}{\partial n} = u.n \quad |_{\Sigma} \quad (11a)$$

with the constraint

$$\int_{\Omega_F} \varphi \, dx = 0 \quad (12a)$$

The two boundary value problems expressed in terms of p or in terms of φ are well-posed in the static case ($\omega = 0$). They have been used, with further transformation, leading to appropriate so-called (u, p, φ) symmetric formulations with mass coupling (leading to a final (u, φ) formulation as described by Morand and Ohayon, chapter 8, 1995) or with stiffness coupling (Sandberg and Goransson, 1988; Morand and Ohayon, chapter 8, 1995).

1.4 Variational formulation in terms of (u, p)

Let δp be the test function, associated to p , belonging to the admissible space \mathcal{C}_p . The weak variational formulation corresponding to equations (10) to (12) is obtained by the usual test-function method using Green's formula. The weak variational formulation corresponding to the structural acoustic problem is then stated as follows. Given ω and F^d , find $u \in \mathcal{C}_u$ and $p \in \mathcal{C}_p$, such that for all $\delta u \in \mathcal{C}_u$ and $\delta p \in \mathcal{C}_p$, we have

$$\tilde{k}(u, \delta u) - \omega^2 \int_{\Omega_S} \rho_S u \cdot \delta u \, dx - \int_{\Sigma} p n \cdot \delta u \, d\sigma = \int_{\partial\Omega_S \setminus \Sigma} F^d \cdot \delta u \, d\sigma \quad (13)$$

$$\frac{1}{\rho_F} \int_{\Omega_F} \nabla p \cdot \nabla \delta p \, dx - \frac{\omega^2}{\rho_F c^2} \int_{\Omega_F} p \delta p \, dx - \omega^2 \int_{\Sigma} u \cdot n \delta p \, d\sigma = 0 \quad (14)$$

with the constraint

$$\frac{1}{\rho_F c^2} \int_{\Omega_F} p \, dx + \int_{\Sigma} u \cdot n \, d\sigma = 0 \quad (15)$$

The variational formulation defined by equations (13), (14) and (15), due to the presence of the constraint defined by equation (15) which regularizes the (u, p) formulation, is therefore valid in the static case. In effect, usually, only equations (13) and (14) are written, and as pointed out above, are not valid for $\omega = 0$. In the case of a finite element discretization of equations (13), (14) and (15), we obtain a matrix system of the type $\mathbf{A} \mathbf{Y} - \omega^2 \mathbf{B} \mathbf{Y} = \mathbf{F}^d$, in which \mathbf{A} and \mathbf{B} are not symmetric. Some direct matrix manipulations may lead to symmetrized systems (Irons, 1970, see also Felippa *et al.*, 1990). As explained above, that is why various symmetric formulations using for the fluid pressure field p and displacement potential φ , defined up to an additive constant and such that $u_F = \nabla \varphi$, have been derived. The resulting *symmetric* formulations are then obtained by elimination of p or φ . In the present case, we are not considering a direct finite element approach of the variational formulation defined by equations (13), (14) and (15).

1.5 Symmetric reduced model

We will consider hereafter a dynamic substructuring approach through an appropriate decomposition of the admissible class into direct sum of admissible vector spaces (see Figure 2).

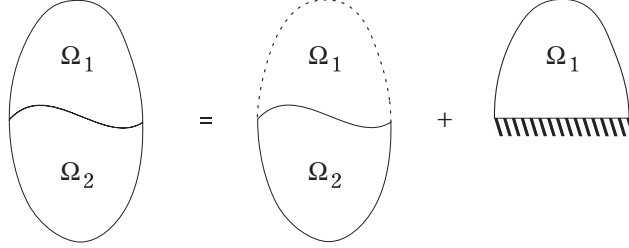


Figure 2: Dynamic substructuring scheme

Let us consider the following two basic problems. The first one corresponds to the acoustic modes in rigid motionless cavity and is obtained by setting $u = 0$ into equations (14) and (15). The calculation of these acoustic modes is generally done by using a finite element procedure. If we introduce the admissible subspace \mathcal{C}_p^* of \mathcal{C}_p

$$\mathcal{C}_p^* = \left\{ p \in \mathcal{C}_p; \int_{\Omega_F} p dx = 0 \right\} , \quad (16)$$

the variational formulation of acoustic modes is stated as follows: find $\omega^2 > 0$ and $p \in \mathcal{C}_p^*$ such that, for all $\delta p \in \mathcal{C}_p^*$, we have

$$\frac{1}{\rho_F} \int_{\Omega_F} \nabla p \cdot \nabla \delta p dx = \omega^2 \frac{1}{\rho_F c^2} \int_{\Omega_F} p \delta p dx \quad (17)$$

with the constraint

$$\int_{\Omega_F} p dx = 0 \quad (18)$$

It should be noted that, in practice, we proceed as follows: the constraint condition (18) is “omitted” which means that that we only modify the initial acoustic problem by adding a first *non physical* zero frequency constant pressure mode, the other modes corresponding to $\omega \neq 0$ remaining the same as those defined by equations (17) and (18). In this new acoustic problem without equation (18), it can be easily seen that the condition defined by equation (18) can be considered as an orthogonality condition between all the modes and the first constant non physical mode corresponding to $\omega = 0$ (Morand and Ohayon, chapter 7, 1995; see also the orthogonality conditions defined by equation (19) below). This zero frequency mode must not be retained in any Ritz-Galerkin projection analysis. In addition, we have the following orthogonality conditions

$$\frac{1}{\rho_F c^2} \int_{\Omega_F} p_\alpha p_\beta dx = \mu_\alpha \delta_{\alpha\beta} \quad , \quad \frac{1}{\rho_F} \int_{\Omega_F} \nabla p_\alpha \cdot \nabla p_\beta dx = \mu_\alpha \omega_\alpha^2 \delta_{\alpha\beta} \quad (19)$$

The second basic problem corresponds to the *static* response of the fluid to a prescribed wall normal displacement $u.n$. The solution, denoted as $p^s(u.n)$, is given by equation (9). For any deformation $u.n$ of the fluid-structure interface, $p^s(u.n)$ belongs to a subset of \mathcal{C}_p , denoted as $\mathcal{C}^{u.n}$

$$\mathcal{C}^{u.n} = \left\{ p^s \in \mathcal{C}_p ; p^s = -\frac{\rho_F c^2}{|\Omega_F|} \int_{\Sigma} u.n d\sigma \right\} \quad (20)$$

In the variational formulation defined by equations (13), (14) and (15), p is searched under the form

$$p = p^s(u.n) + \sum_{\alpha=1}^{N_p} r_\alpha p_\alpha \quad (21)$$

in which N_p denotes the number of retained acoustic modes. The decomposition (21) is unique. In addition, it should be noted that, since each eigenvector p_α corresponding to $\omega_\alpha \neq 0$, verifies the constraint defined by equation (18), then, using equation (9), we deduce that p and $u.n$ satisfy the constraint defined by equation (15). The decomposition defined by equation (21) corresponds to a decomposition of the admissible class \mathcal{C}_p into the direct sum of the admissible classes defined respectively by equations (20) and (16)

$$\mathcal{C}_p = \mathcal{C}^{u.n} \oplus \mathcal{C}_p^* \quad (22)$$

Following equation (21), the test function δp is then searched under the following form

$$\delta p = p^s(\delta u.n) + \sum_{\alpha=1}^{N_p} \delta r_\alpha p_\alpha \quad (23)$$

Variational formulation in δu defined by equation (13) and corresponding to the eigenvalue problem defined by equations (13), (14), (15) becomes

$$\tilde{k}(u, \delta u) + k^s(u, \delta u) - \sum_{\alpha=1}^{N_p} r_\alpha \int_{\Sigma} p_\alpha n \cdot \delta u d\sigma = \omega^2 \int_{\Omega_S} \rho_S u \cdot \delta u dx \quad (24)$$

in which $\tilde{k}(u, \delta u)$ is defined by equation (2) and $k^s(u, \delta u)$ is such that

$$k^s(u, \delta u) = \frac{\rho_F c^2}{|\Omega_F|} \left(\int_{\Sigma} u.n d\sigma \right) \left(\int_{\Sigma} \delta u.n d\sigma \right) \quad (25)$$

If we consider a finite element discretization of the structure, the corresponding discretized form of equation (24) can be written as

$$[\tilde{\mathbf{K}} + \mathbf{K}^s] \mathbf{U} - \omega^2 \mathbf{M} \mathbf{U} - \sum_{\alpha=1}^n \mathbf{C}_\alpha r_\alpha = \mathbf{F}^d \quad (26)$$

in which symmetric matrices $\tilde{\mathbf{K}}$ and \mathbf{K}^s correspond to finite element discretization of stiffness symmetric bilinear forms defined by equations (2), (3), (4) and (25) respectively. In equation (26), \mathbf{M} denotes the structural symmetric mass matrix and rectangular coupling matrix \mathbf{C}_α corresponds to the discretization of the coupling fluid-structure contribution $\int_\sigma p \delta u \cdot n d\sigma$. The discretized form of equation (14) in δp can then be written in generalized (acoustic) coordinates as

$$\omega_\alpha^2 \mu_\alpha r_\alpha - \omega^2 \mu_\alpha r_\alpha - \omega^2 \mathbf{C}_\alpha^T \mathbf{U} = 0 \quad (27)$$

From equations (26) and (27), we obtain the following *symmetric matrix reduced model*

$$\begin{pmatrix} \mathbf{K}^{\text{tot}} & \mathbf{0} \\ \mathbf{0} & \text{Diag } \mu_\alpha \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{r} \end{pmatrix} - \omega^2 \begin{pmatrix} \mathbf{M}^{\text{tot}} & \mathbf{D} \\ \mathbf{D}^T & \text{Diag } (\mu_\alpha / \omega_\alpha^2) \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} \mathbf{F}^d \\ \mathbf{0} \end{pmatrix} \quad (28)$$

in which \mathbf{r} denotes the vector of N generalized coordinates r_α , with $1 \leq \alpha \leq N_p$, and

$$\mathbf{K}^{\text{tot}} = \tilde{\mathbf{K}} + \mathbf{K}^s \quad (29)$$

$$\mathbf{M}^{\text{tot}} = \mathbf{M} + \sum_{\alpha=1}^{N_p} \frac{1}{\omega_\alpha^2 \mu_\alpha} \mathbf{C}_\alpha \mathbf{C}_\alpha^T \quad (30)$$

$$\mathbf{D}_\alpha = \sum_{\alpha=1}^{N_p} \frac{1}{\omega_\alpha^2} \mathbf{C}_\alpha \quad (31)$$

Further diagonalization of equation (28) implies a projection of \mathbf{U} on the solutions of the following eigenvalue problem

$$\mathbf{K}^{\text{tot}} \mathbf{U}_\beta = \lambda_\beta \mathbf{M}^{\text{tot}} \mathbf{U}_\beta \quad (32)$$

Setting

$$\mathbf{U} = \sum_{\beta=1}^{N_u} q_{\beta} \mathbf{U}_{\beta} \quad (33)$$

in which q_{β} are the generalized coordinates describing the structure. Using the orthogonality conditions associated with the solutions of equation (32), i.e. $\mathbf{U}_{\beta'}^T \mathbf{M}^{\text{tot}} \mathbf{U}_{\beta} = \mu_{\beta}^s \delta_{\beta\beta'}$ and $\mathbf{U}_{\beta'}^T \mathbf{K}^{\text{tot}} \mathbf{U}_{\beta} = \mu_{\beta}^s \lambda_{\beta} \delta_{\beta\beta'}$, equation (28) becomes

$$\begin{pmatrix} \text{Diag} \lambda_{\beta} & 0 \\ 0 & \text{Diag} \omega_{\alpha}^2 \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix} - \omega^2 \begin{pmatrix} I_{N_u} & [C_{\beta\alpha}] \\ [C_{\beta\alpha}]^T & I_{N_p} \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} \mathcal{F}^{\text{d}} \\ \mathbf{0} \end{pmatrix} \quad (34)$$

Remark on substructuring procedure. In literature, various methods of component mode synthesis are discussed (fixed interface, free interface with residual attachment modes procedures, etc). We present here, for sake of brevity, only a natural one which comes naturally from the continuous case by considering the admissible class decomposition defined by equation (22). This decomposition is the key of component mode synthesis developments (see Figure 2). Of course, further considerations involving interface deformations by solving from an eigenvalue problem posed only on the interface using fluid and structure mass and stiffness interface operators, could improve the convergence of the procedure. But this remains still an open problem.

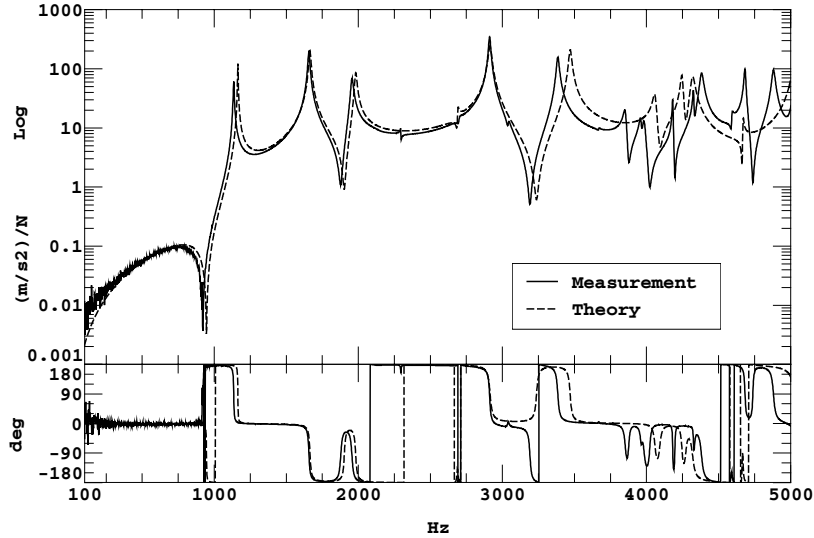


Figure 3: Experimental validation

It should be noted that two different situations are treated here.

For a heavy liquid filling the enclosure, one must mandatory use the eigenmodes defined by equation (32), i.e. *hydroelastic modes including "static" inertial and potential compressibility effects* . The effects of proper static behavior calculation on the convergence of the system relative to the number of acoustic modes have been analyzed in Menelle and Ohayon, 2003 and an experimental validation carried out in the case of parallelepipedical cavity (with one elastic face) filled with liquid is presented in Figure 3).

For a light fluid such as a gas filling the enclosure, one may use instead *in vacuo structural modes* but the resulting matrix system would not be diagonal with respect to \mathbf{U} . In effect, looking at the eigenvalue problem corresponding to equation (28), the diagonalization is obtained by solving the 'structural' problem involving additional stiffness and mass due to static effects of the internal fluid. The *in vacuo* structural modes are orthogonal with respect to \mathbf{K} and \mathbf{M} but not with respect to \mathbf{K}^{tot} and \mathbf{M}^{tot} .

Wall Impedance Condition

Wall impedance condition corresponds to a particular fluid-structure interface modeling. This interface is considered as a third medium with infinitesimal thickness, without mass, and with the following constitutive equation

$$p = j\omega Z(\omega)(u.n - u_F.n) \quad (35)$$

in which $Z(\omega)$ denotes a complex impedance. Equations (7) and (11) must be replaced by equation (35), using $\partial p / \partial n = \rho_F \omega^2 u_F.n$.

Case of a liquid with a free surface

Let us consider a liquid with a free surface at rest denoted as Γ , If we neglect gravity effects, the boundary condition on Γ is such that

$$p = 0 \mid_{\Gamma} \quad (36)$$

In this case, constraint condition (12) (or (15)) is replaced by equation (36). Equation (9) is replaced by $p^s = 0$. Admissible space defined by equation (16) becomes $\mathcal{C}_p^* = \{p \in \mathcal{C}_p ; p = 0\}$.

In this case, the static problem defined Section 3, leads to a zero pressure field.

Let us remark that in this case, the “structural” modal basis may be constituted by the hydroelastic incompressible modes using the classical *added mass* operator (Morand and Ohayon, chapter 5, 1995, Belytschko and Hughes, 1983).

The reduced modal matrix models has been extended to the dissipative case using a wall local homogeneous impedance condition (Kehr-Candille and Ohayon, 1992) or introducing a dissipative internal fluid with nonhomogeneous local impedance wall condition (Ohayon and Soize, 1998).

Incompressible hydroelastic-sloshing problem

We consider the linear vibrations of an elastic structure partially filled with an homogeneous, inviscid and incompressible liquid, taking into account gravity effects on the free surface Γ (Tong, 1966; Liu and Uras, 1992; Morand and Ohayon, 1995). We neglect in the present analysis compressibility effects of the liquid and we refer to Andrianarison PhD dissertation for those aspects (Andrianarison and Ohayon, 2006). After a derivation of the linearized equations of the fluid-structure coupled problem, introducing an appropriate linear constraint in order to obtain a “regularized” problem at zero frequency, we construct a reduced model of the “liquid subsystem”. For this analysis, sloshing modes in rigid motionless cavity are introduced as Ritz projection vector basis, including the static solution of the coupled system.

2 Hydroelastic-sloshing equations

2.1 Structure subjected to a fluid pressure loading

The notations are the same that those defined in Section 3 adapted to liquid with a free surface at rest denoted Γ (see Figure 4).

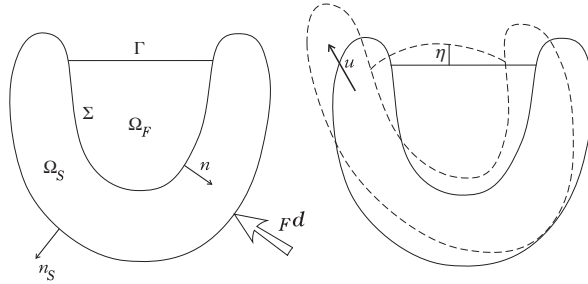


Figure 4: Structure containing a liquid with a free surface

The weak variational formulation describing the response of the structure Ω_S to given variation F^d of the applied forces with respect to the equilibrium state on the external structure boundary $\partial\Omega_S \setminus \Sigma$, and to fluid pressure field p acting on the internal fluid-structure interface Σ is written as follows.

For all real ω and $\forall \delta u \in \mathcal{C}_u$, find $u \in \mathcal{C}_u$ such that

$$\widehat{k}(u, \delta u) - \omega^2 \int_{\Omega_S} \rho_S u \cdot \delta u \, dx - \int_{\Sigma} p n \cdot \delta u \, d\sigma = \int_{\partial\Omega_S \setminus \Sigma} F^d \cdot \delta u \, d\sigma \quad (37)$$

in which

$$\widehat{k} = \widetilde{k} + k_{\Sigma} \quad (38)$$

In equation (38), $\widetilde{k}(u, \delta u)$ is defined by equation (2), and k_{Σ} is the *elastogravity* stiffness in symmetric bilinear form such that (Morand and Ohayon, chapter 6, 1995; Schotté and Ohayon, 2003, 2005)

$$k_{\Sigma}(u, \delta u) = -\frac{1}{2}\rho_F g \left\{ \int_{\Sigma} [zn_1(u) \cdot \delta u + u_z \delta u \cdot n] d\sigma + \int_{\Sigma} [zn_1(\delta u) \cdot u + \delta u_z u \cdot n] d\sigma \right\} \quad (39)$$

2.2 Fluid subjected to a wall normal displacement

We assume that the liquid is homogeneous, inviscid and incompressible. Free surface Γ is horizontal at equilibrium. We denote by z the external unit normal to Γ , and by g the gravity. The notations are similar to those of Section 3. The local equations describing the response of the fluid to a prescribed arbitrary normal displacement $u \cdot n$ of the fluid-structure interface Σ are such that

$$\nabla p - \rho_F \omega^2 u_F = 0 \quad |_{\Omega_F} \quad (40)$$

$$\nabla \cdot u_F = 0 \quad |_{\Omega_F} \quad (41)$$

$$u_F \cdot n = u \cdot n \quad |_{\Sigma} \quad (42)$$

$$p = \rho_F g u_F \cdot n \quad |_{\Gamma} \quad (43)$$

$$\text{curl } u_F = 0 \quad |_{\Omega_F} \quad (44)$$

Equation (41) corresponds to the incompressibility condition. Equation (43) is the constitutive equation on the free surface Γ due to gravity effects.

A displacement potential φ defined up to an additive constant chosen for instance as follows $\int_{\Gamma} \varphi dx = 0$ can be therefore introduced in order to recast the system defined by equations (46-49) into a scalar one. These aspects will be discussed below.

Relation between static pressure p^s and $u.n$

For $\omega = 0$, equations (41), (42) and (43) lead to the constant static pressure field which is related to the normal wall displacement by the relation

$$p^s = -\frac{\rho_F g}{|\Gamma|} \int_{\Sigma} u.n d\sigma \quad (45)$$

in which $|\Gamma|$ denotes the measure of the area of free surface Γ .

2.3 Equations in terms of p or φ and $u.n$

The elimination of u_F between equations (40) to (44) leads to

$$\nabla^2 p = 0 \quad |_{\Omega_F} \quad (46)$$

$$\frac{\partial p}{\partial n} = \rho_F \omega^2 u.n \quad |_{\Sigma} \quad (47)$$

$$\frac{\partial p}{\partial z} = \frac{\omega^2}{g} p \quad |_{\Gamma} \quad (48)$$

with the constraint

$$\frac{1}{\rho_F g} \int_{\Gamma} p d\sigma + \int_{\Sigma} u.n d\sigma = 0 \quad (49)$$

The linear constraint defined by equation (49) ensures that the boundary problem defined by equations (46) to (49) is equivalent to the problem defined by equations (40) to (44). This condition is usually omitted in literature.

Using equations (44-45), the boundary value problem defined by equations (46-49) can be recasted into the following equivalent one using the displacement potential field φ introduced above such that $p = \rho_F \omega^2 \varphi + p^s(u.n)$ with $\int_{\Gamma} \varphi dx = 0$

$$\nabla^2 \varphi = 0 \quad |_{\Omega_F} \quad (46a)$$

$$\frac{\partial \varphi}{\partial n} = u.n \quad |_{\Sigma} \quad (47a)$$

$$\frac{\partial \varphi}{\partial z} = \frac{\omega^2}{g} \varphi - \frac{1}{|\Gamma|} \int_{\Sigma} u.n d\sigma \quad |_{\Gamma} \quad (48a)$$

with the constraint

$$\int_{\Gamma} \varphi d\sigma = 0 \quad (49a)$$

The two boundary value problems expressed in terms of p or in terms of φ are well-posed in the static case ($\omega = 0$). The equations (46a,47a, 48a) have been used, using a different constraint relationship for φ , after the introduction of the elevation η of the free surface, to appropriate so-called (u, φ, η) symmetric formulations with mass coupling (leading to a final (u, η) formulation (Morand and Ohayon, chapter 6, 1995; Schotté and Ohayon, 2002, 2003).

2.4 Variational formulation in terms of (u, p)

Let δp be the test function, associated to p , belonging to the admissible space \mathcal{C}_p . The weak variational formulation corresponding to equations (46) to (49) is obtained by the usual test-function method using Green's formula. Recalling equation (37), the variational formulation of the *hydroelastic-sloshing* problem is then stated as follows. Find $u \in \mathcal{C}_u$ and $p \in \mathcal{C}_p$, such that for all $\delta u \in \mathcal{C}_u$ and $\delta p \in \mathcal{C}_p$, we have

$$\widehat{k}(u, \delta u) - \omega^2 \int_{\Omega_S} \rho_S u \cdot \delta u dx - \int_{\Sigma} p n \cdot \delta u d\sigma = \int_{\partial\Omega_S \setminus \Sigma} F^d \cdot \delta u d\sigma \quad (50)$$

$$\frac{1}{\rho_F} \int_{\Omega_F} \nabla p \cdot \nabla \delta p dx = \frac{\omega^2}{\rho_F g} \int_{\Gamma} p \delta p dx + \omega^2 \int_{\Sigma} u \cdot n \delta p d\sigma \quad (51)$$

with the constraint

$$\frac{1}{\rho_F g} \int_{\Gamma} p d\sigma + \int_{\Sigma} u \cdot n d\sigma = 0 \quad (52)$$

2.5 Symmetric Reduced Matrix Model

Let us consider the following two basic problems. The first one corresponds to the sloshing modes in rigid motionless cavity and is obtained by setting $u = 0$ into equations (47) and (49). The calculation of these acoustic modes is generally done by using a finite element procedure. If we introduce the admissible subspace \mathcal{C}_p^* of \mathcal{C}_p

$$\mathcal{C}_p^* = \left\{ p \in \mathcal{C}_p; \int_{\Gamma} p d\sigma = 0 \right\} \quad , \quad (53)$$

the variational formulation of acoustic modes is stated as follows: find $\omega^2 > 0$ and $p \in \mathcal{C}_p^*$ such that, for all $\delta p \in \mathcal{C}_p^*$, we have

$$\frac{1}{\rho_F} \int_{\Omega_F} \nabla p \cdot \nabla \delta p \, dx = \omega^2 \frac{1}{\rho_F g} \int_{\Gamma} p \delta p \, d\sigma \quad (54)$$

with the constraint

$$\int_{\Gamma} p \, d\sigma = 0 \quad (55)$$

It should be noted that, in practice, if the constraint condition (55) is “omitted”, we only add a first *non physical* zero frequency constant pressure mode, the other modes corresponding to $\omega \neq 0$ remaining the same as those defined by equations (54) and (55). This zero frequency mode must not be retained in any Ritz-Galerkin projection analysis. In addition, we have the following orthogonality conditions

$$\frac{1}{\rho_F g} \int_{\Gamma} p_\alpha p_\beta \, d\sigma = \mu_\alpha \delta_{\alpha\beta} \quad , \quad \frac{1}{\rho_F} \int_{\Omega_F} \nabla p_\alpha \cdot \nabla p_\beta \, dx = \mu_\alpha \omega_\alpha^2 \delta_{\alpha\beta} \quad (56)$$

The second basic problem corresponds to the *static* response of the fluid to a prescribed wall normal displacement $u.n$. The solution, denoted as $p^s(u.n)$, is given by equation (45). For any deformation $u.n$ of the fluid-structure interface, $p^s(u.n)$ belongs to a subset of \mathcal{C}_p , denoted as $\mathcal{C}^{u.n}$

$$\mathcal{C}^{u.n} = \left\{ p^s \in \mathcal{C}_p ; p^s = -\frac{\rho_F g}{|\Gamma|} \int_{\Sigma} u.n \, d\sigma \right\} \quad (57)$$

In the variational formulation defined by equations (50) to (52), p is searched under the form

$$p = p^s(u.n) + \sum_{\alpha=1}^{N_p} r_\alpha p_\alpha \quad (58)$$

in which N_p denotes the number of retained sloshing modes. The decomposition (58) is unique. In addition, it should be noted that, since each eigenvector p_α corresponding to $\omega_\alpha \neq 0$, verifies the constraint defined by equation (55), then, using equation (45), we deduce that p and $u.n$ satisfy the constraint defined by equation (52). The decomposition defined by equation (58) corresponds to a decomposition of the admissible class \mathcal{C}_p into the direct sum of the admissible classes defined respectively by equations (56) and (57), $\mathcal{C}_p = \mathcal{C}^{u.n} \oplus \mathcal{C}_p^*$.

The variational formulation in u defined by equation (50) becomes

$$\widehat{k}(u, \delta u) + k^s(u, \delta u) - \omega^2 \int_{\Omega_S} \rho_S u \cdot \delta u \, dx - \sum_{\alpha=1}^{N_p} r_\alpha \int_{\Sigma} p_\alpha n \cdot \delta u \, d\sigma = \int_{\partial\Omega_S \setminus \Sigma} F^d \cdot \delta u \, d\sigma \quad (59)$$

in which $\widehat{k}(u, \delta u)$ is defined by equation (38) and $k^s(u, \delta u)$ is such that

$$k^s(u, \delta u) = \frac{\rho_F g}{|\Gamma|} \left(\int_{\Sigma} u \cdot n \, d\sigma \right) \left(\int_{\Sigma} \delta u \cdot n \, d\sigma \right) \quad (60)$$

If we consider a finite element discretization of the structure, the corresponding discretized form of equation (60) can be written as

$$[\widehat{\mathbf{K}} + \mathbf{K}^s] \mathbf{U} - \sum_{\alpha=1}^n \mathbf{C}_\alpha r_\alpha - \omega^2 \mathbf{M} \mathbf{U} = \mathbf{F}^d \quad (61)$$

in which symmetric matrices $\widehat{\mathbf{K}}$ and \mathbf{K}^s correspond to finite element discretization of stiffness symmetric bilinear forms defined by equations (38) and (60) respectively. The discretized form of equation (51) in δp can then be written as

$$\omega_\alpha^2 \mu_\alpha r_\alpha = \omega^2 \mu_\alpha r_\alpha + \omega^2 \mathbf{C}_\alpha^T \mathbf{U} \quad (62)$$

From equations (61) and (62), we obtain a *symmetric matrix reduced model* whose expression is similar to the one given by expression (28).

Similarly to Section 3.6, further diagonalization can be obtained by setting

$$\mathbf{U} = \sum_{\beta=1}^{N_u} q_\beta \mathbf{U}_\beta \quad (63)$$

in which q_β are the generalized coordinates describing the structure and \mathbf{U}_β are the eigenmodes of an eigenvalue problem similar to the one described by equation (32). We then obtain a similar matrix system than the one described by equation (34)

$$\begin{pmatrix} \text{Diag} \lambda_\beta & 0 \\ 0 & \text{Diag} \omega_\alpha^2 \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix} - \omega^2 \begin{pmatrix} I_{N_u} & [C_{\beta\alpha}] \\ [C_{\beta\alpha}]^T & I_{N_p} \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} \mathcal{F}^d \\ \mathbf{0} \end{pmatrix} \quad (64)$$

It should be noted that we can also use the *incompressible hydroelastic modes*, i.e. the modes of the coupled system constituted by the elastic structure containing an incompressible

liquid, with $p = 0$ on Γ (through an *added mass operator*). In this case, the resulting matrix system is not completely diagonal with respect to \mathbf{U} variables.

Figure 5 and Figure 6 illustrate liquid motions in reservoirs.

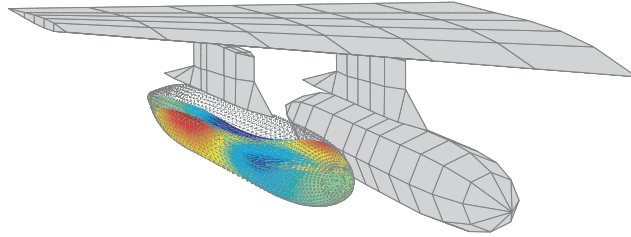


Figure 5: Wing with stores containing liquids

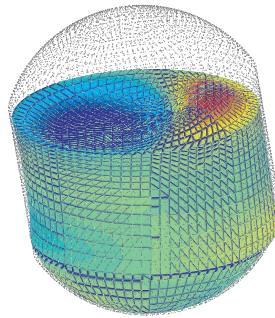


Figure 6: Tank partially filled with liquid

Conclusion

We have reviewed appropriate formulations for low modal density frequency computations of the eigenmodes of elastic structures containing linear inviscid homogeneous fluids for structural-acoustics problems, using *structural modes in vacuo* for structure containing a gas or *hydroelastic modes including "static" inertial and potential compressibility effects* for structure containing liquids, with *acoustic modes in rigid motionless cavity*, and incompressible hydroelastic-sloshing problems. Those formulations, using modal interaction schemes, with dynamic substructuring techniques lead to symmetric reduced matrix systems expressed in terms of generalized coordinates for the fluid-structure system.

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