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# SMOOTH DECOMPOSITION ANALYSIS AND ORDER REDUCTION OF NONLINEAR MECHANICAL SYSTEMS UNDER RANDOM EXCITATION

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## ABSTRACT

*This paper presents a possible alternative procedure to the Karhunen-Loève approach to construct reduced order models which capture accurately the dynamics of nonlinear discrete mechanical systems under random excitation. This procedure combines the Smooth Decomposition method and the Petrov-Galerkin approximation. The smooth decomposition method is a multivariate-data analysis method characterizing coherent structures (the smooth modes) as the eigenvectors of the generalized eigenproblem defined from the covariance matrix of the displacement field and the covariance matrix of the velocity field. The Petrov-Galerkin approximation is used to project the dynamics in a subspace generated by a set of the smooth modes. The Petrov-Galerkin approximation preserves the second order structure of the equations of motion. The procedure is considered for a mechanical system including a strongly nonlinear end-attachment. The efficiency of the approach is analyzed comparing the power spectral density functions of the reduced-order model and of the original system.*

## INTRODUCTION

The Karhunen-Loève Decomposition (KLD) also named Proper Orthogonal Decomposition (POD) or Principal Component Analysis (PCA) has been extensively used as a tool for analyzing random fields. The KLD method as a multivariate data analysis reveals some coherent structures which have been ad-

vantageously used in different domains as the stochastic-finite-elements method [1], the simulation of random fields [2], the modal analysis of linear and nonlinear systems [3–5, 7, 8], the construction of reduced-order models [9–12]. In structural vibration, the KLD has been principally applied to the displacement field but it can be applied to either the displacement, the velocity or the acceleration field and also to the displacement-velocity field [13].

Another multivariate data analysis method called Smooth Orthogonal Decomposition (SOD) has been proposed in [14]. The SOD is defined from a maximization problem associated to a scalar time series of measurements subject to a minimization constraint acting on the associated time derivative of the time series. The SOD can be used to extract normal modes and natural frequencies of multi-degree-of-freedom vibration systems. Free and forced sinusoidal responses have been considered in [14] and randomly excited systems have been analyzed in [15]. The SOD has been formulated to analyze time continuous stationary [16] and non stationary [17] vector-valued random processes. The general random field case has been considered in [18]. In these three papers, the decomposition was called Smooth Decomposition (SD) since it does not have the properties of a Karhunen-Loève decomposition. The SD is obtained solving a generalized eigenproblem defined from the covariance matrix of the random field and the covariance matrix of its associated time derivative.

The problem of constructing reduced-order models for linear and nonlinear discrete mechanical systems based on SD which preserve the second order structure has been considered in [19]

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where a Petrov-Galerkin approximation has been used to project the  $n$ -dimensional dynamics into a  $m$ -dimensional subspace. The first results show that the efficiency of the SD approach is comparable to the efficiency of the classical modal reduction techniques based on the Normal Modes (NM) or on the Karhunen-Loève modes.

In this paper the SD approach of model reduction is applied to a chain of  $M$  strongly coupled linear oscillators with a strongly nonlinear end-attachment. The nonlinear end-attachment (also named Nonlinear Energy Sink (NES)) can passively absorb and locally dissipate energy from the linear chain. The energy interactions occur due to internal resonances making possible irreversible nonlinear energy transfers from the primary system to the NES component (also named Targeted Energy Transfer (TET) concept). The purely nonlinearity of the NES enables it to resonate with any modes of the primary structure. A description of the TET can be found in [20]. The TET concept was principally analyzed in the literature in a deterministic framework. We propose here to analyze the TET when the excitation is a white-noise random process. This kind of excitation differs significantly from the deterministic case but in terms of frequency contents, a white-noise excitation is similar to an impulsive excitation in the deterministic case. It permits to analyze the system without privileging a frequency band.

## TOOLS AND METHODS

In this section, the definition and some properties of the SD are briefly recalled. The principle of Petrov-Galerkin approach is also reported. For all these points, more details can be found in [14, 16, 21].

### Smooth Decomposition

Let  $\{\mathbf{U}(t), t \in \mathbb{R}\}$  be a  $\mathbb{R}^n$ -valued zero mean second-order stationary random process indexed by  $\mathbb{R}$ . We assume that  $\{\mathbf{U}(t), t \in \mathbb{R}\}$  has a time-derivative process  $\{\dot{\mathbf{U}}(t), t \in \mathbb{R}\}$  which is also a second-order stationary process. The covariance matrices of  $\{\mathbf{U}(t), t \in \mathbb{R}\}$  and  $\{\dot{\mathbf{U}}(t), t \in \mathbb{R}\}$  are denoted  $\mathbf{R}_U = \mathbb{E}(\mathbf{U}(t)^T \mathbf{U}(t))$  and  $\mathbf{R}_{\dot{U}} = \mathbb{E}(\dot{\mathbf{U}}(t)^T \dot{\mathbf{U}}(t))$  respectively.

The SD of  $\{\mathbf{U}(t), t \in \mathbb{R}\}$  is defined by

$$\mathbf{U}(t) = \sum_{k=1}^n a_k^S(t) \Phi_k^S \quad (1)$$

where the scalar time processes  $a_k^S(t)$  named Smooth Components (SCs) are given by

$$a_k^S(t) = \frac{\Phi_k^{S^T} \mathbf{R}_U \mathbf{U}(t)}{\Phi_k^{S^T} \mathbf{R}_U \Phi_k^S} = \frac{\Phi_k^{S^T} \mathbf{R}_{\dot{U}} \mathbf{U}(t)}{\Phi_k^{S^T} \mathbf{R}_{\dot{U}} \Phi_k^S} \quad (2)$$

and the constant vectors  $\Phi_k^S$  named Smooth Modes (SMs) are characterized from the optimization problem

$$\max_{\Phi \in \mathbb{R}^n} J_{SD}(\Phi) \text{ with } J_{SD}(\Phi) = \frac{\mathbb{E}(\langle \mathbf{U}(t), \Phi \rangle^2)}{\mathbb{E}(\langle \dot{\mathbf{U}}(t), \Phi \rangle^2)} = \frac{\Phi^T \mathbf{R}_U \Phi}{\Phi^T \mathbf{R}_{\dot{U}} \Phi}$$

and solve the eigenproblem

$$\mathbf{R}_U \Phi_k^S = \mu_k^S \mathbf{R}_{\dot{U}} \Phi_k^S. \quad (3)$$

Assuming that  $\mathbf{R}_U$  and  $\mathbf{R}_{\dot{U}}$  are positive definite, the SMs constitute a  $\mathbf{R}_U$ -orthogonal and  $\mathbf{R}_{\dot{U}}$ -orthogonal basis of  $\mathbb{R}^n$  and all the eigenvalues  $\mu_k^S$  named Smooth Values (SVs) are greater than zero.

A physical interpretation of the SD can be obtained for linear vibration systems. Let  $\mathbf{R}_U$  and  $\mathbf{R}_{\dot{U}}$  be the covariance matrices of the steady state solution of a discrete linear mechanical system under zero-mean white-noise excitation. If the damping is proportional and if the modal-excitation terms are uncorrelated then the following relations hold:

- the SMs are related to the normal modes by

$$\Phi^S = \Phi^{L^{-T}} \quad (4)$$

where  $\Phi^S = [\Phi_1^S \Phi_2^S \dots \Phi_n^S]$  and  $\Phi^L = [\Phi_1^L \Phi_2^L \dots \Phi_n^L]$  where the  $\Phi_k^L$  denote the normal modes of the associated undamped linear system;

- the SVs are related to the natural resonance frequencies by

$$\mu^S = (\Omega^2)^{-1} \quad (5)$$

where  $\mu^S = \text{diag}(\mu_k^S)$  and  $\Omega^2 = \text{diag}(\omega_k^2)$  where the  $\omega_k$  denote the natural resonance frequencies of the associated undamped linear system.

The relations (4) and (5) can be used to perform modal analysis from SD.

The last point to discuss is how to order the smooth modes. Two ordering can be used, one considering the SVs ordered by decreasing value (i.e.  $\mu_1^S \geq \mu_2^S \geq \dots \geq \mu_n^S$ ) named SV-ordering and the other considering the energies of the SCs ordered by decreasing value (i.e.  $\mathbb{E}(a_1^{S^2}(t)) \geq \mathbb{E}(a_2^{S^2}(t)) \geq \dots \geq \mathbb{E}(a_n^{S^2}(t))$ ) named energy-ordering. Due to Eq. (5), the SV-ordering is in line with the classical ordering of the resonance frequencies (in increasing order) whereas the energy-ordering is in line with ordering used in the POD approach. The SV-ordering and the energy-ordering will be compared in terms of model reduction.

## Model Reduction Method

The model-reduction method considered here is based on Petrov-Galerkin approach [21]. This method has been retained in this study because, firstly, it preserves the second-order structure of the original system and, secondly, it is well adapted when the decomposition basis are not orthogonal.

Consider the equations of motion of a general  $n$ -degree-of-freedom dynamical system in the form

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{K}\mathbf{U}(t) + \mathbf{G}(\mathbf{U}(t), \dot{\mathbf{U}}(t)) = \mathbf{F}(t) \quad (6)$$

where  $\mathbf{M}$ ,  $\mathbf{C}$ , and  $\mathbf{K}$  are symmetric square matrices with dimensions  $n \times n$ ,  $\mathbf{G}$  is a non-linear  $n$ -vector function and  $\{\mathbf{F}(t), t \in \mathbb{R}\}$  is a  $n$ -vector random process.

Petrov-Galerkin approach may be viewed as a projection of the  $n$ -dimensional displacement field  $\mathbf{U}(t)$  onto a  $m$ -dimensional subspace  $\mathcal{E}_m$  with  $m < n$  along (or parallel) to the subspace  $\mathcal{S}_{n-m}$  where  $\mathbb{R}^n = \mathcal{E}_m \oplus \mathcal{S}_{n-m}$ .

Let  $(\mathbf{E}_1, \dots, \mathbf{E}_m)$  be a basis of  $\mathcal{E}_m$ , and  $(\mathbf{W}_1, \dots, \mathbf{W}_m)$  be a basis of  $\mathcal{S}_{n-m}^\perp$  such that  $\mathbf{W}^T \mathbf{E} = \mathbf{I}$  where  $\mathbf{E} = [\mathbf{E}_1 \mathbf{E}_2 \dots \mathbf{E}_n]$  and  $\mathbf{W} = [\mathbf{W}_1 \mathbf{W}_2 \dots \mathbf{W}_n]$ . An approximation of  $\mathbf{U}(t)$  is sought in the subspace  $\mathcal{E}_m$  as

$$\mathbf{U}(t) \approx \mathbf{U}^m(t) = \mathbf{E}\mathbf{Q}(t). \quad (7)$$

Substituting Eq. (7) into Eq. (6) and imposing that the residue  $\mathcal{R}(\mathbf{U}^m(t))$  defined by

$$\mathcal{R}(\mathbf{U}^m(t)) = \mathbf{M}\ddot{\mathbf{E}}\mathbf{Q}(t) + \mathbf{C}\dot{\mathbf{E}}\mathbf{Q}(t) + \mathbf{K}\mathbf{E}\mathbf{Q}(t) + \mathbf{G}(\mathbf{E}\mathbf{Q}(t), \dot{\mathbf{E}}\mathbf{Q}(t)) - \mathbf{F}(t) \quad (8)$$

is orthogonal to  $\mathcal{S}_{n-m}^\perp$ , the following reduced-order system is deduced

$$\tilde{\mathbf{M}}\ddot{\mathbf{Q}}(t) + \tilde{\mathbf{C}}\dot{\mathbf{Q}}(t) + \tilde{\mathbf{K}}\mathbf{Q}(t) + \tilde{\mathbf{G}}(\mathbf{Q}(t), \dot{\mathbf{Q}}(t)) = \mathbf{W}^T \mathbf{F}(t) \quad (9)$$

where  $\tilde{\mathbf{M}} = \mathbf{W}^T \mathbf{M} \mathbf{E}$ ,  $\tilde{\mathbf{C}} = \mathbf{W}^T \mathbf{C} \mathbf{E}$ ,  $\tilde{\mathbf{K}} = \mathbf{W}^T \mathbf{K} \mathbf{E}$  and  $\tilde{\mathbf{G}}(\mathbf{Q}, \dot{\mathbf{Q}}) = \mathbf{W}^T \mathbf{G}(\mathbf{E}\mathbf{Q}, \dot{\mathbf{E}}\mathbf{Q})$ .

In this method, two important points are the selection of the projection subspace (i.e. the projection basis  $(\mathbf{E}_1, \dots, \mathbf{E}_m)$ ) and the computation of the associated basis  $(\mathbf{W}_1, \dots, \mathbf{W}_m)$  (for the subspace  $\mathcal{S}_{n-m}^\perp$ ). In the case of the SD method, there are two ordering of interest and hence two different projection basis  $(\mathbf{E}_1, \dots, \mathbf{E}_m)$  giving two different reduced models. Independently of the selected ordering and due to the orthogonality property of the SMs,  $\mathbf{W}$  has been chosen as  $\mathbf{W} = \mathbf{R}_U \mathbf{E}$  (or in an equivalent way as  $\mathbf{W} = \mathbf{R}_U \mathbf{E}$ ).

## THE SYSTEM UNDER STUDY

### Description of the System

The system is composed of a chain of  $M$  linear oscillators with spring ( $k_c$ ) (named linear chain) strongly coupled with a nonlinear end-attachment (named NES). Each mass of the linear chain is also connected to the ground by a linear spring ( $k_g$ ) and a linear dashpot ( $\lambda_g$ ). The NES is constituted of a mass ( $m_a$ ), a linear damper ( $\lambda_a$ ) and a spring including a linear part ( $k_a$ ) and a cubic part ( $C_a$ ).  $m_a$  is assumed to be small compared to the total mass of the linear chain and the linear spring is assumed to be small compared to cubic spring.

The equations of motion are given by

$$m_a \ddot{v} + \lambda_a (\dot{v} - \dot{u}_1) + k_a (v - u_1) + C_a (v - u_1)^3 = 0, \quad (10)$$

$$\begin{aligned} \ddot{u}_1 + \lambda_g \dot{u}_1 + k_g u_1 - \lambda_a (\dot{v} - \dot{u}_1) - k_a (v - u_1) \\ - C_a (v - u_1)^3 + k_c (u_1 - u_2) = 0, \end{aligned} \quad (11)$$

$$\ddot{u}_m + \lambda_g \dot{u}_m + k_g u_m + k_c (2u_m - u_{m-1} - u_{m+1}) = 0, \quad (12)$$

$$\ddot{u}_M + \lambda_g \dot{u}_M + (k_g + k_c) u_M + k_c (u_M - u_{M-1}) = f(t) \quad (13)$$

with  $m = 1, \dots, M-1$  and where  $v$  (respectively  $u_m$ ) denotes the displacement of the NES (respectively the  $m$ th mass of the linear chain) and  $f(t)$  denotes the external excitation applied to the mass number  $M$ . It is assumed that the system possesses a weak viscous damping ( $\lambda_g$  is small).

This system was considered in [22] under impulsive excitation. We assume here that the excitation is of the form

$$f(t) = s_0 W(t) \quad (14)$$

where  $\{W(t), t \in \mathbb{R}\}$  is a Gaussian white-noise scalar process with intensity one and  $s_0$  denotes the excitation level.

Introducing the variable

$$\mathbf{U}(t) = [v(t) u_1(t) u_2(t) \dots u_M(t)]^T,$$

the equations of motion (10-13) can be easily re-written in the matrix form (6).

### Reduced Models

Three reduced models will be considered, two from the SD methods and one from the KL method:

-  $\mathbf{E} = \mathbf{E}_{SM_{SV}} = [\Phi_1^S \dots \Phi_m^S]$  where  $\Phi_k^S$  denotes the smooth modes obtained from the steady-state response and ordered according to the SV-ordering (and  $\mathbf{W} = \mathbf{W}_{SM_\mu} = \mathbf{R}_U \mathbf{E}_{SM_\mu}$ );

-  $\mathbf{E} = \mathbf{E}_{SM_{energy}} = [\Phi_1^S \dots \Phi_m^S]$  where  $\Phi_k^S$  denotes the smooth modes obtained from the steady-state response and ordered according to the energy-ordering (and  $\mathbf{W} = \mathbf{W}_{SM_{energy}} = \mathbf{R}_U \mathbf{E}_{SM_{energy}}$ );

-  $\mathbf{E} = \mathbf{E}_{KLM} = [\Psi_1 \cdots \Psi_m]$  where  $\Psi_i$  denotes the KL modes obtained from the steady-state response (and  $\mathbf{W} = \mathbf{W}_{KLM} = \mathbf{E}_{KLM}$ ).

Moreover, we will also consider the reduced model given by the classical modal truncation method. In this case, the approximation is defined from (7) and the reduced model (9) with

$$\mathbf{E} = \mathbf{E}_{NM} = [\Phi_1 \cdots \Phi_m] \text{ and } \mathbf{W} = \mathbf{E}_{NM} \quad (15)$$

where  $\Phi_i$  denotes the normal modes of the underlying linear system of (6). Note that this approach coincides with the Petrov-Galerkin projection only when the normal modes are orthogonal ( $\mathbf{M} = \mathbf{I}$ ).

## SOME RESULTS

The reduced-order models have to reproduce the steady-state responses of the original system. The Power Spectral Density Matrix (PSDM) function of the steady-state response will be considered to compare how close the reduced-order models is from the original system.

## Simulation Methodology

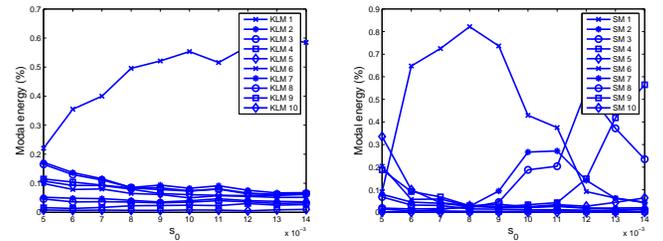
The steady-state responses of the full model (Eq. (10-13)) were obtained using Monte-Carlo method. The differential equations were solved numerically using the Newmark method with null initial conditions and discretized white-noise trajectory obtained using the method discussed in [23]. The differential equations were integrated with the sampling frequency  $f_e$  over a long time  $T$ . From the sampling trajectories, the covariance matrices were obtained using the classical time-average estimates (which are valid under the ergodic assumption) and the PSDM function using Welch's method (averaging modified periodogram) with a Hamming window of 4096 points length and without overlap. From the covariance matrices, SD approach was carry out solving the eigenproblems (3) giving access to the reduced models. KLD approach was carry out using the same data. The PSDM functions of the stationary responses of the associated reduced model (Eq. (9)) were obtained using the same procedure. Moreover, the same excitation trajectories were also used.

In all the simulations, the following numerical values were used:  $M = 9$  (corresponding to  $n = 10$  degrees of freedom),  $\lambda_g = 0.001$ ,  $k_g = 1$ ,  $k_c = 1$ ,  $m_a = 0.05$ ,  $\lambda_g = 0.001$ ,  $k_a = 0.0001$ ,  $C_a = 1$ ,  $f_e = 7$  and  $T = 524286/f_e$ .

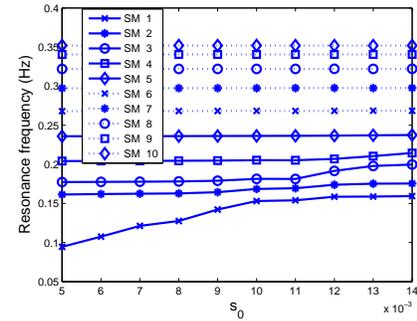
The excitation level  $s_0$  was used as the parameter of analysis. The following values were considered:  $s_0 \in \{.005, 0.006, 0.007, 0.008, 0.009, 0.010, 0.011, 0.012, 0.013, 0.014\}$ .

## Preliminary Analysis

Figure 1 shows the percentage of energy captured by the KL modes (left) and by the SMs (right) versus the excitation level  $s_0$  (the SV-ordering has been used to numbering the SMs). These curves were obtained applying the KLD and the SD procedures to the steady-state responses of the full system (Eq. (10-13)) varying the excitation level  $s_0$ . We have also reported in Fig. 2, the resonance frequencies estimated from SD analysis using Eq. (5) versus the excitation level  $s_0$  (here also the SV-ordering has been used to numbering the SMs).



**FIGURE 1.** PERCENTAGE OF ENERGY CAPTURED BY THE KLMs (left) AND THE SMs (right) VERSUS EXCITATION LEVEL  $s_0$ .



**FIGURE 2.** RESONANCE FREQUENCIES ESTIMATED FROM SD ANALYSIS USING EQ. (5) VERSUS EXCITATION LEVEL  $s_0$ .

From these results, we can make the following observations:

- the percentage of energy captured by the first KL mode rapidly increases (see the curve with cross markers in Fig. 1(left)) and this mode becomes dominant in the response when the excitation level increases;
- in case of SD analysis, the energy is first concentrated on the first SM (see the curve with cross markers in

Fig. 1(right)), next on the second SM (see the curve with star markers in Fig. 1(right)) and so on;  
- as shown in Fig. 2, the resonance frequencies increase with the excitation level. Resonance interactions (interpreted as resonance captures) appear between the first and the second resonances (see the curves with cross and star markers in Fig. 2) when the energy captured by the second SM is maximum, between the second and the third resonances (see the curves with star and circle markers in Fig. 2) when the energy captured by the first SM is maximum, and so on;  
- in the two cases, more than 90% of energy, is captured by the five KL modes and the five SMs.

Following this analysis, the reduced-order models have been considered using the KL modes and the SMs obtained from the steady state response associated to the excitation level  $s_0 = 0.012$ . Moreover we will focus on the reduced models draw up with  $m = 5$  modes. Figure 3 shows the first six KLMs (circle markers), SMs using SV-ordering (star markers), SMs using energy-ordering (square markers), normal modes estimated using Eq.(4) from the SMs using SV-ordering (cross markers) and energy-ordering (diamond markers) for the reference level excitation  $s_0 = 0.012$ . The first six normal modes of the associated linear system (red curves) are also reported. As expected (due to the high excitation level), the modes shapes significantly differ from the shapes of normal modes of the associated linear system. Interesting also is that the SMs obtained using SV-ordering differ from the SMs obtained using energy-ordering but they arise from the same set of vectors. For example, the first SM using SV-ordering (see cross markers in Mode 1 Fig. 3) coincides with the fourth SM using energy-ordering (see diamond markers in Mode 4 Fig. 3).

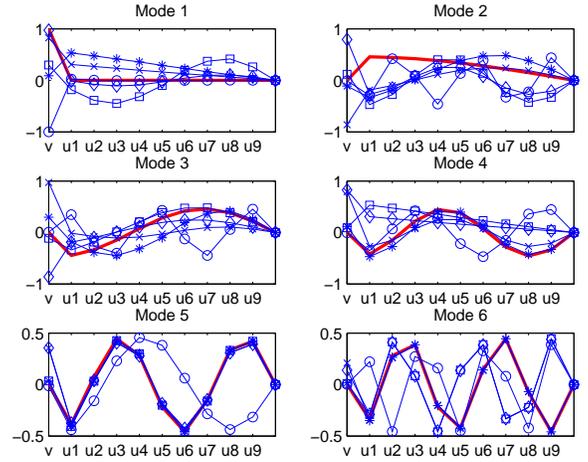
### Discussion

The objective of this section is now to compare the four reduced models defined from  $\mathbf{E} = \mathbf{E}_{NM}$  (denoted RM-NM) ,  $\mathbf{E} = \mathbf{E}_{KLM}$  (denoted RM-KLM),  $\mathbf{E} = \mathbf{E}_{SM_{SV}}$  (denoted RM-SM<sub>SV</sub>) and  $\mathbf{E} = \mathbf{E}_{SM_{energy}}$  (denoted RM-SM<sub>energy</sub>).

Note that following the last comment in the previous section, the reduced models RM-SM<sub>SV</sub> and RM-SM<sub>energy</sub> are equivalent.

The resonance frequencies of the underlying linear systems associated to the four reduced models are reported Table 1 and compared to the first five resonance frequencies of the underlying linear system of the full system. As expected the resonance frequencies of RM-NM and the full model coincide and also the resonance frequencies of RM-SM<sub>SV</sub> and RM-SM<sub>energy</sub> coincide. Note also that except the first one, the resonance frequencies of RM-SM<sub>SV</sub> and RM-NM coincide. Finally the resonance frequencies of RM-KLM differ significantly from the others.

The Frobenius norm of the PSDM of the linear chain displacements  $[u_1(t)u_2(t)\cdots u_M(t)]^T$  and the PSD function of the NES displacement  $v(t)$  of the full system have been compared



**FIGURE 3.** SHAPES OF THE MODES USED TO BUILD THE REDUCED MODELS.

**TABLE 1.** RESONANCE FREQUENCIES OF THE LINEAR SYSTEMS ASSOCIATED TO THE FULL MODEL AND TO THE REDUCED ONES.

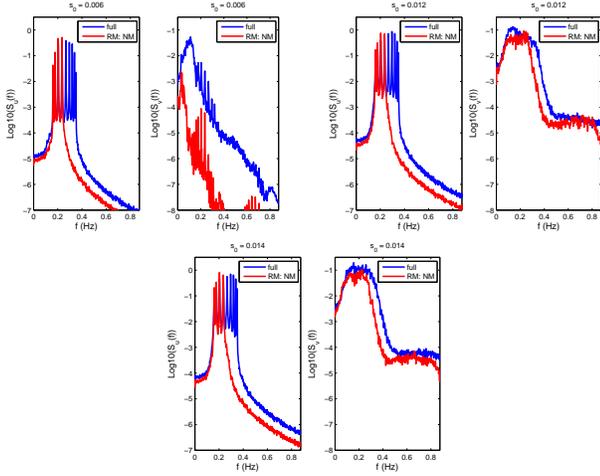
	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$	$\omega_5$
Full model	0.007	0.161	0.177	0.204	0.236
RM: NM	0.007	0.161	0.177	0.204	0.236
RM: KLM	0.023	0.236	0.268	0.297	0.322
RM: SM <sub>SV</sub>	0.091	0.161	0.177	0.204	0.236
RM: SM <sub>energy</sub>	0.091	0.162	0.177	0.204	0.236

with those obtained with the four reduced-order models. Three excitation levels have been considered: the level  $s_0 = 0.012$  used to build the reduced models , a lower level  $s_0 = 0.006$  and a upper level  $s_0 = 0.014$ . The results obtained with the model RM-NM (respectively RM-KLM, RM-SM<sub>SV</sub> and RM-SM<sub>energy</sub>) are shown Fig. 4 (respectively Fig. 5, Fig. 6 and Fig. 7).

First of all, using response data associated to white-noise excitation, it is not expected for the reduced models to reproduce the distribution of energy per frequency band over all the frequency band of the full system. Hence, we will restrict the discussion comparing the effective frequency bands of the reduced models and also pointing out the ability for each reduced model to reproduce correctly the responses corresponding the lower and upper excitation levels in its effective band.

In terms of effective frequency bands, the three reduced

models: RM-NM, RM-SM<sub>SV</sub> and RM-SM<sub>energy</sub> are localized on a frequency band including the four first resonance frequencies associated to the linear chain. A difference appears between RM-NM and RM-SM<sub>SV</sub> at low excitation level due to a spurious resonance frequency around  $f = 0.091$  in agreement with resonance frequencies of the linear system associated to the reduced model (see Table 1). The reduced model RM-KLM gives satisfactory results in a frequency band which include the four resonance frequencies after the third one associated to the linear chain.

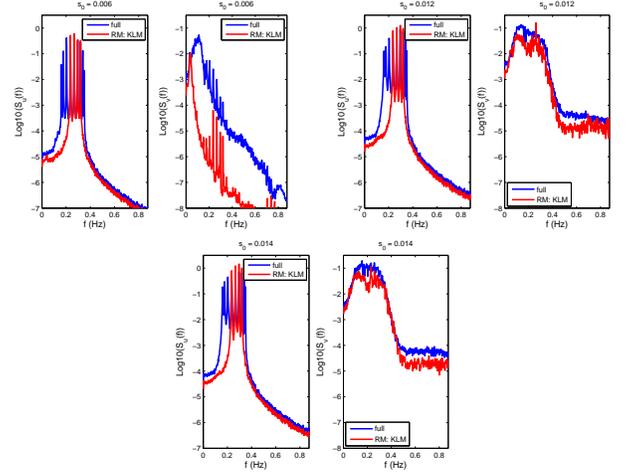


**FIGURE 4.** COMPARISON BETWEEN THE FULL MODEL AND THE RM-NM MODEL: FROBENIUS NORM OF THE PSDM OF  $[u_1(t)u_2(t)\cdots u_M(t)]^T$  (left) AND PSD OF  $v(t)$  (right) FOR THREE EXCITATION LEVELS.

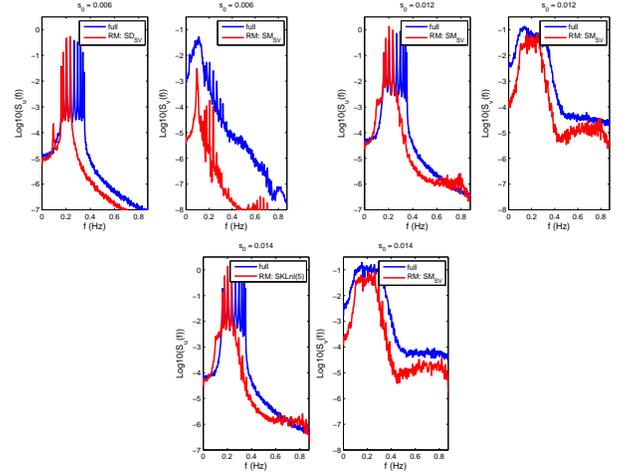
Concerning the ability of the reduced-order models to approximate the PSDM of the response of the full system, all the approaches give good results for the variable  $[u_1(t)u_2(t)\cdots u_M(t)]^T$  independently to the excitation level. For the nonlinear component  $v(t)$ , some differences appear. At low excitation level, and independently to the reduced model used, the PSD function of  $v(t)$  is not correctly reproduced. The amplitude is always under-estimated. At intermediate and high excitation level, the results obtained with the model RM-KLM seems to be best.

## CONCLUSION

The ability of reduced-order models based on smooth modes to approximate the PSDM of the response of a second-order discrete mechanical systems was studied here. The Petrov-Galerkin approach was used to draw up the reduced-order models since the projection basis used are not orthogonal with respect to the standard inner product. Comparisons were made between the



**FIGURE 5.** COMPARISON BETWEEN THE FULL MODEL AND THE RM-KLM MODEL: FROBENIUS NORM OF THE PSDM OF  $[u_1(t)u_2(t)\cdots u_M(t)]^T$  (left) AND PSD OF  $v(t)$  (right) FOR THREE EXCITATION LEVELS.

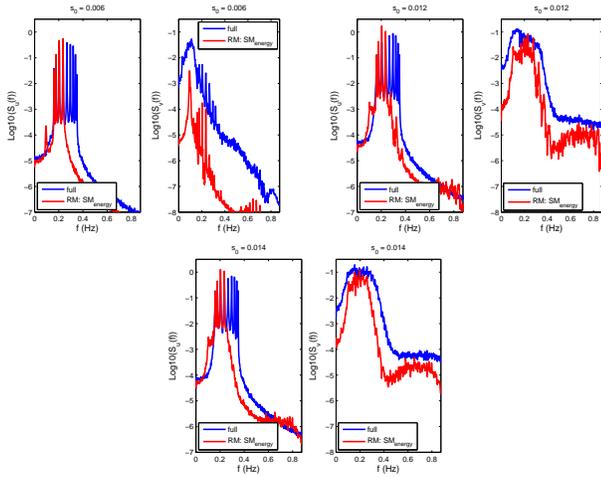


**FIGURE 6.** COMPARISON BETWEEN THE FULL MODEL AND THE RM-SM<sub>SV</sub> MODEL: FROBENIUS NORM OF THE PSDM OF  $[u_1(t)u_2(t)\cdots u_M(t)]^T$  (left) AND PSD OF  $v(t)$  (right) FOR THREE EXCITATION LEVELS.

reduced-order models obtained using the smooth modes, the Karhunen-Loeve modes, and the normal modes of the underlying linear system.

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**FIGURE 7.** COMPARISON BETWEEN THE FULL MODEL AND THE RM-SM<sub>energy</sub> MODEL: FROBENIUS NORM OF THE PSDM OF  $[u_1(t)u_2(t)\cdots u_M(t)]^T$  (left) AND PSD OF  $v(t)$  (right) FOR THREE EXCITATION LEVELS.

## REFERENCES

[1] Ghanen, R., and Spanos, P., 2003. *Stochastic Finite Elements: A Spectral Approach*. Dover, New York.

[2] Sakamoto, S., and Ghanem, R., 2002. “Simulation of multi-dimensional non-gaussian non-stationary random fields”. *Probabilistic Engineering Mechanics*, **17**, pp. 167–176.

[3] Feeny, B., and Kappagantu, R., 1998. “On the physical interpretation of proper orthogonal modes in vibrations”. *Journal of Sound and Vibration*, **211**(4), pp. 607–616.

[4] Azeez, M. F. A., and Vakakis, A. F., 2001. “Proper Orthogonal Decomposition (POD) of a class of vibroimpact oscillations”. *Journal of Sound and Vibration*, **240**, pp. 859–889.

[5] Wolter, C., Trindade, M. A., and Sampaio, R., 2002. “Obtaining mode shapes through the Karhunen-Loève expansion for distributed-parameter linear systems”. *Shock and Vibration*, **9**(4–5), pp. 177–192.

[6] Han, S., and Feeny, B., 2003. “Application of proper orthogonal decomposition to structural vibration analysis”. *Mechanical System and Signal Processing*, **17**(3), pp. 989–1001.

[7] Quaranta, G., Masarati, P., and Mantegazza, P., 2008. “Continuous-time covariance approaches for modal analysis”. *Journal of Sound and Vibration*, **310**, pp. 287–312.

[8] Meyer, M., and Matthies, H. G., 2003. “Efficient model reduction in non-linear dynamics using the Karhunen-Loève expansion and dual-weighted-residual methods”. *Computational Mechanics*, **31**, pp. 179–191.

[9] Kerschen, G., Golinval, J.-C., Vakakis, A., and Bergman, L., 2005. “The method of proper orthogonal decomposition for dynamical characterization and order reduction of

mechanical systems: an overview”. *Non-linear Dynamics, Special issue on “Reduced Order Models: Methods and Applications” Applications*, **41**, pp. 147–169.

- [10] Sampaio, R., and Soize, C., 2007. “Remarks on the efficiency of pod for model reduction in non-linear dynamics of continuous elastic systems”. *International Journal of Numerical Methods in Engineering*, **72**, pp. 22–45.
- [11] Placzeka, A., Tran, D.-M., and Ohayon, R., 2008. “Hybrid proper orthogonal decomposition formulation for linear structural dynamics”. *Journal of Sound and Vibration*, **318**, pp. 943–964.
- [12] Bellizzi, S., and Sampaio, R., 2009. “Karhunen-Loève modes obtained from displacement and velocity fields: evaluation and comparison”. *Mechanical System and Signal Processing and Signal Processing*, **23**, pp. 1218–1222.
- [13] Chelidze, D., and Zhou, W., 2006. “Smooth orthogonal decomposition-based vibration mode identification”. *Journal of Sound and Vibration*, **292**, pp. 461–473.
- [14] Feeny, B., and Farooq, U., 2008. “A nonsymmetric state-variable decomposition for modal analysis”. *Journal of Sound and Vibration*, **310**, pp. 792–800.
- [15] Bellizzi, S., and Sampaio, R., 2009. “Smooth Karhunen-Loève decomposition to analyze randomly vibrating systems”. *Journal of Sound and Vibration*, **325**, pp. 491–498.
- [16] Sampaio, R., and Bellizzi, S., 2011. “Analysis of nonstationary random processes using smooth decomposition”. *Journal of Mechanics of Materials and Structures*, **6**(7-8), pp. 1137–1152.
- [17] Bellizzi, S., and Sampaio, R., 2012. “Smooth decomposition of random fields”. *Journal of Sound and Vibration*, **331**, pp. 3509–3520.
- [18] Bellizzi, S., and Sampaio, R., 2012. “Reduced models based on smooth decomposition for random mechanical systems”. *International Review of Mechanical Engineering*, **6**(1), pp. 74–85.
- [19] Vakakis, A., Gendelman, O., Bergman, L., McFarland, D., Kerschen, G., and Lee, Y., 2008. *Nonlinear Targeted Energy Transfer in Mechanical and Structural Systems I & II*. Springer.
- [20] Antoulas, C. A., 2005. *Approximation of large-scale dynamical systems*. Advances in Design and Control SIAM.
- [21] Ma, X., Vakakis, A. F., and Bergman, L., 2008. “KarhunenLoève analysis and order reduction of the transient dynamics of linear coupled oscillators with strongly nonlinear end attachments”. *Journal of Sound and Vibration*, **309**, pp. 569–587.
- [22] Poirion, F., and Soize, C., 1989. “Simulation numérique des champs stochastiques gaussiens homogènes et non homogènes”. *La recherche Aérospatiale*, **1**, pp. 41–61.