Reduced Models for Computational Structural Acoustics in the Medium-Frequency Range

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REDUCED MODELS FOR COMPUTATIONAL STRUCTURAL ACOUSTICS IN THE MEDIUM-FREQUENCY RANGE

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ABSTRACT

This paper deals with a method in computational structural acoustics adapted for constructing reduced models in the medium-frequency range to a general three-dimensional dissipative structure consisting of an anisotropic, inhomogeneous, viscoelastic bounded medium coupled with an internal acoustic cavity. The reduced models are obtained using the Ritz-Galerkin method for which the projection subspace corresponds to the dominant eigensubspace of the energy operator of the structure in the medium-frequency band of analysis. A basic example is presented to validate the method, for which the structure and the internal acoustic cavity have a medium-frequency behavior in the frequency band of analysis.

I. INTRODUCTION

In the area of computational structural acoustics, a method has been proposed for constructing reduced models in the medium-frequency (MF) range for general structural dynamics systems (structures in a vacuum) (Soize, 1998a) and external structural-acoustic systems (structure coupled with an external acoustic fluid (gas or liquid)) (Soize, 1998b). In this method, the reduced model is constructed using the Ritz-Galerkin projection of the variational formulation of the boundary value problem on the dominant eigensubspace of the energy operator of the structure over the medium-frequency band of analysis. In this paper, we apply and adapt this method for constructing a reduced model in the medium-frequency range to a general three-dimensional dissipative structure consisting of an anisotropic, inhomogeneous, viscoelastic bounded medium coupled with an internal acoustic cavity filled with a gas or a liquid.

In Section II, we present the boundary value problem to be solved in the frequency domain. In Section III, we introduce a finite dimension approximation based on the use of the finite element analysis. Section IV concerns the construction of a vector basis for the reduced model of the structure and the internal acoustic cavity in the MF range. As a consequence, we deduce a structural vector basis adapted to the MF range when the structure is coupled with an internal acoustic cavity. Section V is devoted to the construction of the reduced model while Section VI deals with the construction of the dominant eigensubspaces. In Section VII, we present the time-stationary random response using the reduced model. Finally, in Section VIII, we present an example which validates the proposed method.

II. BOUNDARY VALUE PROBLEM

We consider linear vibrations (formulated in the frequency domain ω) of a three-dimensional structural-acoustic system around a static equilibrium configuration considered as a natural state at rest (see Fig. 1).

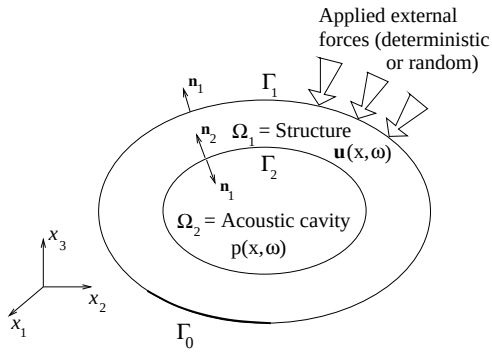

Fig. 1. Geometrical configuration of the structural-acoustic system

Let Ω_1 be the three-dimensional bounded domain occupied by the structure and made of viscoelastic material. Let $\partial\Omega_1 = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ be its boundary and $\mathbf{n}_1 = (n_{1,1}, n_{1,2}, n_{1,3})$ be its outward unit normal. Let $\mathbf{u}(\mathbf{x},\omega) = (u_1(\mathbf{x},\omega), u_2(\mathbf{x},\omega), u_3(\mathbf{x},\omega))$ be the displacement field in each point $\mathbf{x} = (x_1, x_2, x_3)$ in Cartesian coordinates and at frequency ω. On part Γ_0 of the boundary, the structure is fixed ($\mathbf{u} = \mathbf{0}$) whereas on part $\Gamma_1 \cup \Gamma_2$ it is free. The structure is coupled with an internal dissipative acoustic fluid (gas or liquid) occupying the three-dimensional bounded domain Ω_2 whose boundary $\partial\Omega_2$ is the coupling interface Γ_2.

The outward unit normal to $\partial\Omega_2$ is denoted as $\mathbf{n}_2 = (n_{2,1}, n_{2,2}, n_{2,3})$ and we have $\mathbf{n}_2 = -\mathbf{n}_1$. We denote the pressure field in Ω_2 as $p(\mathbf{x},\omega)$. We introduce a narrow MF band B such that

$$B = [\,\omega_B - \Delta\omega/2\,,\omega_B + \Delta\omega/2\,] \quad , \tag{1}$$

in which ω_B is the center frequency and $\Delta\omega$ is the bandwidth such that $\Delta\omega/\omega_B \ll 1$ and $\omega_B > \Delta\omega/2$. With B we associate interval

$$\widetilde{B} = [\,-\omega_B - \Delta\omega/2\,,-\omega_B + \Delta\omega/2\,] \quad . \tag{2}$$

The structure is submitted to a square integrable surface force field $\mathbf{x} \mapsto \eta(\omega)\,\mathbf{g}(\mathbf{x},\omega)$ from Γ_1 into $\mathbb{C}^3$, in which $\eta(\omega)$ is a function from $\mathbb{R}$ into $\mathbb{C}$, such that $\eta(\omega) = 0$ if ω is not in $B \cup \widetilde{B}$, continuous on B, verifying $|\eta(-\omega)| = |\eta(\omega)|$ and such that $|\eta(\omega)| \neq 0$ for all ω in B. For this structural-acoustic system, we use the model and the boundary value problem developed for the MF range by Ohayon and Soize (1998). Introducing components (g_1, g_2, g_3) of $\mathbf{g}$, the boundary value problem for the structure is written as follows in terms of $\mathbf{u}$ (the convention for the Fourier transform being $v(\omega) = \int_{\mathbb{R}} e^{-i\omega t} v(t)\,dt$),

$$-\omega^2 \rho_1\, u_i - \sigma_{ij,j} = 0 \quad \text{in} \quad \Omega_1 \quad , \tag{3}$$

$$\sigma_{ij} n_{1,j} = \eta\, g_i \quad \text{on} \quad \Gamma_1 \quad , \tag{4}$$

$$\sigma_{ij} n_{1,j} = -p\, n_{1,j} \quad \text{on} \quad \Gamma_2 \quad , \tag{5}$$

$$u_i = 0 \quad \text{on} \quad \Gamma_0 \quad , \tag{6}$$

in which $i = 1, 2, 3$, where the summation over index j is used, and where $\rho_1(\mathbf{x}) > 0$ is the mass density of the structure and $\sigma_{ij,j} = \sum_{j=1}^{3} \partial\sigma_{ij}/\partial x_j$. For a linear viscoelastic material, stress tensor σ_{ij} is written as

$$\sigma_{ij} = \mathbb{o}_{ijkh}(\mathbf{x},\omega)\,\varepsilon_{kh}(\mathbf{u}) + \mathbb{b}_{ijkh}(\mathbf{x},\omega)\,\varepsilon_{kh}(i\omega\,\mathbf{u}) \quad , \tag{7}$$

in which the summation over indices k and h is used and where $\varepsilon_{kh}(\mathbf{u}) = (\partial u_k/\partial x_h + \partial u_h/\partial x_k)/2$ is the linearized strain tensor. Coefficients $\mathbb{o}_{ijkh}(\mathbf{x},\omega)$ and $\mathbb{b}_{ijkh}(\mathbf{x},\omega)$ are real, depend on $\mathbf{x}$ and ω, verify the usual properties of symmetry and positiveness (Truesdell, 1984; Ciarlet, 1988; Ohayon and Soize, 1998) and are such that $\mathbb{o}_{ijkh}(\mathbf{x},-\omega) = \mathbb{o}_{ijkh}(\mathbf{x},\omega)$ and $\mathbb{b}_{ijkh}(\mathbf{x},-\omega) = \mathbb{b}_{ijkh}(\mathbf{x},\omega)$.

Concerning the internal dissipative acoustic fluid, the pressure in the fluid is written (Ohayon and Soize, 1998) as

$$p(\mathbf{x},\omega) = -i\omega\,\rho_2\,\psi(\mathbf{x},\omega) - \kappa\,\pi_2(\mathbf{u}) \quad \text{in} \quad \Omega_2 \cup \Gamma_2 \quad , \tag{8}$$

in which κ is a positive constant such that

$$\kappa = \frac{\rho_2\, c_2^2}{|\Omega_2|} \quad , \quad |\Omega_2| = \int_{\Omega_2} d\mathbf{x} \quad , \tag{9}$$

where $\rho_2 > 0$ and c_2 are the constant mass density and the constant speed of sound of the acoustic fluid at equilibrium and where $\pi_2(\mathbf{u})$ is defined by

$$\pi_2(\mathbf{u}) = \int_{\Gamma_2} \mathbf{u}(\mathbf{x},\omega)\cdot\mathbf{n}_2(\mathbf{x})\,ds(\mathbf{x}) \quad . \tag{10}$$

The new unknown field $\psi(\mathbf{x},\omega)$ is related to the velocity field $\mathbf{v}(\mathbf{x},\omega)$ of the dissipative acoustic fluid by the equation $\mathbf{v}(\mathbf{x},\omega) = (1+i\omega\tau)\,\boldsymbol{\nabla}\psi(\mathbf{x},\omega)$ in which τ is a constant coefficient (Ohayon and Soize, 1998) related to the viscosity of the acoustic fluid (τ may depend on frequency ω). The boundary value problem for the internal acoustic fluid is written as follows in terms of field ψ

$$-\omega^2 \frac{\rho_2}{c_2^2}\,\psi(\mathbf{x},\omega) - i\omega\,\tau\,\rho_2 \boldsymbol{\nabla}^2\psi(\mathbf{x},\omega) - \rho_2\,\boldsymbol{\nabla}^2\psi(\mathbf{x},\omega) = -\frac{i\omega\kappa}{c_2^2}\,\pi_2(\mathbf{u}) \quad \text{in} \quad \Omega_2 \quad , \tag{11}$$

$$\rho_2(1+i\omega\tau)\frac{\partial\psi}{\partial\mathbf{n}_2} = i\omega\,\rho_2\,\mathbf{u}\cdot\mathbf{n}_2 \quad \text{on} \quad \Gamma_2 \quad , \tag{12}$$

$$\int_{\Omega_2} \psi(\mathbf{x},\omega)\,d\mathbf{x} = 0 \quad . \tag{13}$$

The boundary value problem of the structural-acoustic problem is defined by Eqs. (3) to (13).

III. FINITE ELEMENT ANALYSIS

The finite dimension approximation is obtained by using the finite element method (Zienkiewicz and Taylor, 1989; Dautray and Lions, 1992). In order to simplify the development, we assume that the finite element meshes are compatible on coupling interface Γ_2. Let $\mathbf{U} = (U_1, \ldots, U_{n_1})$ be the vector of the DOFs which are the values of field $\mathbf{u}(\mathbf{x},\omega)$ at the nodes of the finite element mesh of domain Ω_1 and let $\boldsymbol{\Psi} = (\Psi_1, \ldots, \Psi_{n_2})$ be the vector of the DOFs which are the values of field $\psi(\mathbf{x},\omega)$ at the nodes of the finite element mesh of domain Ω_2. The finite dimension approximation of the variational formulation of the boundary value problem defined by Eqs. (3) to (13) is then written (Ohayon and Soize, 1998) as

$$\begin{bmatrix} [A_1(\omega)] + \kappa\,[J] & i\omega[C] \\ i\omega\,[C]^T & -[A_2(\omega)] \end{bmatrix} \begin{bmatrix} \mathbf{U}(\omega) \\ \boldsymbol{\Psi}(\omega) \end{bmatrix} = \begin{bmatrix} \eta(\omega)\,\mathbf{F}(\omega) \\ 0 \end{bmatrix} \quad , \tag{14}$$

with the constraint

$$\mathbf{L}^T\,\boldsymbol{\Psi} = 0 \quad . \tag{15}$$

Equation (15) corresponds to the finite element discretization of linear constraint defined by Eq. (13), in which $\mathbf{L}$ is a vector in $\mathbb{R}^{n_2}$. In Eq. (14), $\eta(\omega)\,\mathbf{F}(\omega)$ is the vector in $\mathbb{C}^{n_1}$ corresponding to the finite element discretization of the linear form $\mathbf{v} \mapsto \eta(\omega)\int_{\Gamma_1}\mathbf{g}(\mathbf{x},\omega)\cdot\mathbf{v}(\mathbf{x})\,ds(\mathbf{x})$ and $[A_1(\omega)]$ is an $(n_1\times n_1)$ symmetric complex matrix (invertible for all ω in $B\cup\widetilde{B}$) which is written as

$$[A_1(\omega)] = -\omega^2\,[M_1] + i\omega\,[D_1(\omega)] + [K_1(\omega)] \quad , \tag{16}$$

in which mass, damping and stiffness matrices $[M_1]$, $[D_1(\omega)]$ and $[K_1(\omega)]$ of the structure are positive-definite symmetric $(n_1\times n_1)$ real matrices corresponding to the finite element discretization of the mass structural bilinear form $(\mathbf{u},\mathbf{v}) \mapsto \int_{\Omega_1}\rho_1(\mathbf{x})\,\mathbf{u}\cdot\mathbf{v}\,d\mathbf{x}$, the damping structural bilinear form $(\mathbf{u},\mathbf{v}) \mapsto \int_{\Omega_1}\mathbb{b}_{ijkh}(\mathbf{x},\omega)\varepsilon_{kh}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})\,d\mathbf{x}$ and the stiffness structural bilinear form $(\mathbf{u},\mathbf{v}) \mapsto \int_{\Omega_1}\mathbb{o}_{ijkh}(\mathbf{x},\omega)\,\varepsilon_{kh}(\mathbf{u})\,\varepsilon_{ij}(\mathbf{v})\,d\mathbf{x}$. Matrix $[J]$ is an $(n_1\times n_1)$ symmetric real matrix which can be written as $[J] = \boldsymbol{\Pi}_2\,\boldsymbol{\Pi}_2^T$ in which $\boldsymbol{\Pi}_2$ is a vector in $\mathbb{R}^{n_1}$ such that $\boldsymbol{\Pi}_2^T$ corresponds to the finite element discretization of the linear form $\pi_2(\mathbf{u})$ defined by Eq. (10). In Eq. (14), $[A_2(\omega)]$ is an $(n_2\times n_2)$ symmetric complex matrix (invertible for all real ω in $B\cup\widetilde{B}$) which is written as

$$[A_2(\omega)] = -\omega^2\,[M_2] + i\omega\,[D_2(\omega)] + [K_2] \quad , \tag{17}$$

in which "mass" matrix $[M_2]$ of the fluid is a positive-definite symmetric $(n_2 \times n_2)$ real matrix corresponding to the finite element discretization of the bilinear form $(\psi, \phi) \mapsto (\rho_2/c_2^2) \int_{\Omega_2} \psi(\mathbf{x}, \omega)\phi(\mathbf{x})\, d\mathbf{x}$ and where damping and "stiffness" matrices $[D_2(\omega)]$ and $[K_2]$ of the fluid are positive-semidefinite symmetric $(n_2 \times n_2)$ real matrices corresponding to the finite element discretization of the bilinear forms $(\psi, \phi) \mapsto \tau(\omega)\, k_2(\psi, \phi)$ and $(\psi, \phi) \mapsto k_2(\psi, \phi) = \rho_2 \int_{\Omega_2} \boldsymbol{\nabla}\psi(\mathbf{x}, \omega) \cdot \boldsymbol{\nabla}\phi(\mathbf{x})\, d\mathbf{x}$. The dimension of the null space of matrices $[D_2(\omega)]$ and $[K_2]$ is equal to 1 and the solution of Eq. (14) is unique due to constraint Eq. (15). Finally, in Eq. (14), $[\,C\,]$ is an $(n_1 \times n_2)$ rectangular real matrix corresponding to the finite element discretisation of the coupling bilinear form $(\psi, \mathbf{v}) \mapsto \rho_2 \int_{\Gamma_2} \psi(\mathbf{x}, \omega)\, \mathbf{n}_2(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, ds(\mathbf{x})$. Let $\mathbf{P} = (P_1, \ldots, P_{n_2})$ be the complex vector of the DOFs which are the values of pressure field p at the nodes of the finite element mesh of domain Ω_2. Then the finite element discretization of Eq. (8) is wriiten as

$$\mathbf{P} = -i\omega\, \rho_2\, \boldsymbol{\Psi} - \kappa\, \{\boldsymbol{\Pi}_2^T\, \mathbf{U}\}\, \mathbf{W} \quad , \tag{18}$$

in which $\mathbf{W} \in \mathbb{R}^{n_2}$ is the solution of the linear equation $(c_2^2/\rho_2)[M_2]\, \mathbf{W} = \mathbf{V}$ in which $\mathbf{V} \in \mathbb{R}^{n_2}$ is the finite element discretization of linear form $\delta p \mapsto \int_{\Omega_2} \delta p(\mathbf{x})\, d\mathbf{x}$.

IV. CONSTRUCTION OF A VECTOR BASIS FOR THE REDUCED MODEL

Two vector bases adapted to MF band B can be constructed for the structure in vacuo and the internal acoustic cavity with rigid wall by applying the method presented by Soize (1998a and 1998b) (for the details, we refer the reader to these references). These two vector bases correspond to the dominant eigensubspaces of the energy operators relative to band B for the structure in vacuo and the internal acoustic fluid with rigid wall. In the context of the finite element analysis introduced in Section III, the procedure can be summarized as follows. Let $a = 1$ or $a = 2$ be the index related to the structure or the internal acoustic fluid. Let $N_a \ll n_a$ be the order of the reduced model related to the structure $(a = 1)$ or the internal acoustic fluid $(a = 2)$. Let $[P_a]$ be the $(n_a \times N_a)$ real matrix whose columns are the N_a eigenvectors $\{\mathbf{P}_a^1, \ldots, \mathbf{P}_a^{N_a}\}$ corresponding to the N_a highest eigenvalues $\lambda_a^1 \geq \ldots \geq \lambda_a^{N_a}$ of the generalized symmetric eigenvalue problem

$$[H_a]\, [P_a] = [G_a]\, [P_a]\, [\Lambda_a] \quad , \tag{19}$$

such that

$$[P_a]^T\, [G_a]\, [P_a] = [I_a] \quad , \tag{20}$$

$$[P_a]^T\, [H_a]\, [P_a] = [\Lambda_a] \quad , \tag{21}$$

in which $[I_a]$ is the $(N_a \times N_a)$ identity matrix, $[\Lambda_a]$ is the $(N_a \times N_a)$ diagonal matrix of eigenvalues $\lambda_a^1, \ldots, \lambda_a^{N_a}$, where $[G_1]$ is a positive-definite symmetric $(n_1 \times n_1)$ real matrix corresponding to the finite element discretization of the bilinear form $(\mathbf{u}, \mathbf{v}) \mapsto \int_{\Omega_1} \mathbf{u} \cdot \mathbf{v}\, d\mathbf{x}$ and $[G_2]$ is a positive-definite symmetric $(n_2 \times n_2)$ real matrix corresponding to the finite element discretization of the bilinear form $(\psi, \phi) \mapsto \int_{\Omega_2} \psi \cdot \phi\, d\mathbf{x}$, and where $[H_a]$ are positive-definite symmetric $(n_a \times n_a)$ real matrices such that

$$[H_a] = [G_a]\, [E_a]\, [G_a] \quad . \tag{22}$$

In Eq. (22), positive-definite symmetric $(n_a \times n_a)$ real matrix $[E_a]$ corresponds to the finite element discretization of the energy operator of the structure $(a = 1)$ or the internal acoustic fluid $(a = 2)$, which is such that

$$[E_a] = \int_B [e_a(\omega)]\, d\omega \quad , \tag{23}$$

$$[e_a(\omega)] = \frac{1}{\pi}\omega^2\, |\eta(\omega)|^2\, \Re e\, \{[T_a(\omega)]^*\, [M_a]\, [T_a(\omega)]\} \quad , \tag{24}$$

$$[T_a(\omega)] = [A_a(\omega)]^{-1} \quad ; \quad [T_a(\omega)]^* = \overline{[T_a(\omega)]}^T = \overline{[T_a(\omega)]} \quad , \tag{25}$$

in which $[A_a(\omega)]$ is defined by Eq. (16) for $a = 1$ and Eq. (17) for $a = 2$. It should be noted that, for $a = 2$, equation $[T_2(\omega)] = [A_2(\omega)]^{-1}$ means that matrix $[T_2(\omega)]$ has to be constructed solving the linear matrix equation $[A_2(\omega)]\, [T_2(\omega)] = [I_2]$ with the n_2 constraint equations $\mathbf{L}^T\, [T_2(\omega)] = 0^T$ in which 0 is the zero vector in $\mathbb{C}^{n_2}$.

V. CONSTRUCTION OF THE REDUCED MODEL ADAPTED TO MF BAND B

The reduced model adapted to MF band B is obtained (Soize, 1998a and 1998b) by introducing the new variable $\mathbf{q}_1(\omega) = (q_{1,1}(\omega), \ldots, q_{1,N_1}(\omega))$ for the structure and the new variable $\mathbf{q}_2(\omega) = (q_{2,1}(\omega), \ldots, q_{2,N_2}(\omega))$ for the internal acoustic fluid, such that

$$\mathbf{U}(\omega) = [P_1]\, \mathbf{q}_1(\omega) \quad ; \quad \boldsymbol{\Psi}(\omega) = [P_2]\, \mathbf{q}_2(\omega) \quad . \tag{26}$$

From Eqs. (14) and (26), we deduce that for all ω in $B \cup \widetilde{B}$, vector $\mathbf{q}_1(\omega)$ in $\mathbb{C}^{N_1}$ and vector $\mathbf{q}_2(\omega)$ in $\mathbb{C}^{N_2}$ are the unique solution of the linear equation

$$\begin{bmatrix} [\mathcal{A}_1(\omega)] + \kappa\, [\mathcal{J}\,] & i\omega [\mathcal{C}\,] \\ i\omega\, [\mathcal{C}\,]^T & -[\mathcal{A}_2(\omega)] \end{bmatrix} \begin{bmatrix} \mathbf{q}_1(\omega) \\ \mathbf{q}_2(\omega) \end{bmatrix} = \begin{bmatrix} \eta(\omega)\, \boldsymbol{\mathcal{F}}(\omega) \\ 0 \end{bmatrix} , \tag{27}$$

in which $\boldsymbol{\mathcal{F}}(\omega) \in \mathbb{C}^{N_1}$ is written as

$$\boldsymbol{\mathcal{F}}(\omega) = [P_1]^T\, \mathbf{F}(\omega) \quad , \tag{28}$$

and where

$$[\mathcal{A}_1(\omega)] = [P_1]^T\, [A_1(\omega)]\, [P_1] \ , \ [\mathcal{A}_2(\omega)] = [P_2]^T\, [A_2(\omega)]\, [P_2] \ , \tag{29}$$

$$[\,\mathcal{J}\,] = [P_1]^T\, [\,J\,]\, [P_1] = ([P_1]^T\, \boldsymbol{\Pi}_2)\, ([P_1]^T\, \boldsymbol{\Pi}_2)^T \quad , \tag{30}$$

$$[\,\mathcal{C}\,] = [P_1]^T\, [\,C\,]\, [P_2] \quad . \tag{31}$$

From Eqs. (18) and (26), and eliminating $\mathbf{q}_2(\omega)$ from the second line of Eq. (27), we deduce that

$$\mathbf{P}(\omega) = [\mathcal{P}(\omega)]^T\, \mathbf{q}_1(\omega) \quad , \tag{32}$$

in which $[\mathcal{P}(\omega)]$ is the $(N_1 \times n_2)$ complex matrix defined by

$$[\mathcal{P}(\omega)] = \omega^2\, \rho_2\, [\mathcal{C}\,]\, [\mathcal{A}_2(\omega)]^{-1}\, [P_2]^T - \kappa\, [P_1]^T\, \boldsymbol{\Pi}_2\, \mathbf{W}^T \quad . \tag{33}$$

Equations (26) to (33) constitute the reduced model adapted to MF band B.

VI. CONSTRUCTION OF DOMINANT EIGENSUBSPACES

Concerning the construction of the dominant eigensubspace of the energy operator relative to MF band B for the structure on the one hand, and for the internal acoustic fluid on the other hand, we can use the indirect procedure in the frequency domain or the procedure based on the use of the MF solving method in the time domain presented by Soize (1998a and 1998b). A detailed analysis of these procedures cannot be reproduced here. Nevertheless, in order to facilitate the understanding of Sections VII and VIII, we summarize below the main results of the MF solving method in the time domain that we have to use for the applications. For $a = 1$ and $a = 2$, the problem defined by Eqs. (19) to (21) is solved by calculating the N_a lowest eigenvalues of the following generalized symmetric eigenvalue problem

$$[G_a]\,[S_a] = [H_a]\,[S_a]\,[\Gamma_a] \quad , \tag{34}$$

$$[S_a]^T\,[H_a]\,[S_a] = [I_a] \quad , \tag{35}$$

$$[S_a]^T\,[G_a]\,[S_a] = [\Gamma_a] \quad , \tag{36}$$

for which the subspace iteration algorithm (Soize, 1998a; Bathe and Wilson, 1976; Chatelin, 1993; Golub and Van Loan, 1989) is used. The dimension m_a of the subspace used for iterations is such that $N_a < m_a \ll n_a$ with $m_a = \{2N_a\,,\, N_a+8\}$. Consequently, $[S_a]$ is an $(n_a \times m_a)$ real matrix and $[\Gamma_a]$ is a diagonal $(m_a \times m_a)$ real matrix. We have

$$[\widetilde{\Lambda}_a] = [\Gamma_a]^{-1} \quad , \tag{37}$$

$$[\widetilde{P}_a] = [S_a]\,[\Gamma_a]^{-1/2} \quad , \tag{38}$$

where $[\widetilde{P}_a]$ is the $(n_a \times m_a)$ real matrix whose first N_a columns are eigenvectors $\mathbf{P}_a^1, \ldots, \mathbf{P}_a^{N_a}$ defining matrix $[P_a]$. For each iteration of the subspace iteration algorithm, we only need to calculate an $(n_a \times m_a)$ real matrix $[W_a] = [E_a]\,[X_a]$, in which $[X_a]$ is a given $(n_a \times m_a)$ real matrix. Let $\chi_0(t)$ be the complex-valued function defined on $\mathbb{R}$ by $\chi_0(t) = e^{-i\omega_B t}\,\chi(t)$ in which $\chi(t) = (1/2\pi)\int_B e^{i\omega t}\,\widehat{\chi}(\omega)\,d\omega$ with

$$\widehat{\chi}(\omega) = \frac{1}{\pi}\,\omega^2\,|\eta(\omega)|^2 \mathbb{1}_B(\omega) \quad . \tag{39}$$

Therefore, χ_0 is an LF signal whose band is $[-\Delta\omega/2\,,\,\Delta\omega/2\,]$. Then it is proved (Soize, 1998a) that $[W_a]$ can be calculated by

$$[W_a] = 2\pi\,\Re e\,\{[Z_a(0)]\} \quad , \tag{40}$$

in which $[Z_a(t)]$ is the solution of the following LF equations in the time domain associated with the MF equations,

$$[M_a]\,[\ddot{Y}_a(t)] + [\widetilde{D}_a]\,[\dot{Y}_a(t)] + [\widetilde{K}_a]\,[Y_a(t)] = \chi_0(t)\,[X_a]\ ,\ t \in]-\infty\,,+\infty[\quad , \tag{41}$$

$$[M_a]\,[\ddot{Z}_a(t)] + [\widetilde{D}_a]\,[\dot{Z}_a(t)] + [\widetilde{K}_a]\,[Z_a(t)] = [M_a]\,\overline{[Y_a(-t)]}\ ,\ t \in]-\infty\,,0\,[\quad , \tag{42}$$

in which symmetric $(n_a \times n_a)$ complex matrices $[\widetilde{D}_a]$ and $[\widetilde{K}_a]$ are written, for $a = 1$ and $a = 2$, as

$$[\widetilde{D}_a] = [D_a(\omega_B)] + 2\,i\,\omega_B\,[M_a] \quad , \tag{43}$$

$$[\widetilde{K}_a] = -\omega_B^2\,[M_a] + i\,\omega_B\,[D_a(\omega_B)] + [K_a(\omega_B)] \quad . \tag{44}$$

It should be noted that for $a = 2$, $[K_2(\omega_B)] = [K_2]$. In addition, for $a = 2$, Eqs. (41) and (42) must be solved with the constraint (see Eq. (15))

$$\mathbf{L}^T\,[Y_2(t)] = [\,0\,] \quad ; \quad \mathbf{L}^T\,[Z_2(t)] = [\,0\,] \quad . \tag{45}$$

The LF Eqs. (41) and (42) are solved using an unconditionally stable implicit step-by-step integration method. Concerning Eq. (41), time interval $]-\infty\,,\infty[$ is replaced by the finite interval $]\,t_I\,,\,t_F]$ with the initial conditions $[Y_a(t_I)] = [\dot{Y}_a(t_I)] = [\,0\,]$. Concerning Eq. (42), time interval $]-\infty\,,0]$ is replaced by the finite interval $]-t_F\,,0]$ with the initial conditions $[Z_a(-t_F)] = [\dot{Z}_a(-t_F)] = [\,0\,]$.

VII. TIME-STATIONARY RANDOM RESPONSE USING THE REDUCED MODEL

Let us consider the case for which the structural-acoustic system is submitted to a time-stationary second-order centered random wall pressure field $\{p(\mathbf{x},t), \mathbf{x} \in \Gamma_1, t \in \mathbb{R}\}$ with values in $\mathbb{R}$. We are interested in the stationary response of the structural-acoustic system. The cross-correlation function of random field p is denoted as $R_p(\mathbf{x},\mathbf{y},\tau) = \mathbb{E}\{p(\mathbf{x},t+\tau)\,p(\mathbf{y},t)\}$ in which $\mathbb{E}$ is the mathematical expectation and is such that (see for instance Kree and Soize, 1986; Ohayon and Soize, 1998; Lin, 1967)

$$R_p(\mathbf{x},\mathbf{y},\tau) = \int_{\mathbb{R}} e^{i\omega\tau}\,S_p(\mathbf{x},\mathbf{y},\omega)\,d\omega \quad , \tag{46}$$

in which $S_p(\mathbf{x},\mathbf{y},\omega)$ is the cross-spectral density function which is written as

$$S_p(\mathbf{x},\mathbf{y},\omega) = |\eta(\omega)|^2\,s_p(\mathbf{x},\mathbf{y},\omega) \quad . \tag{47}$$

Let $\mathbb{F}(t) = (\mathbb{F}_1(t), \ldots, \mathbb{F}_{n_1}(t))$ be the stochastic process indexed by $\mathbb{R}$ with values in $\mathbb{R}^{n_1}$, corresponding to the finite element discretization of the linear form $\mathbf{v} \mapsto -\int_{\Gamma_1} p(\mathbf{x},t)\ \mathbf{n}_1(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\,ds(\mathbf{x})$. Consequently, we have

$$\mathbb{F}_\alpha(t) = -\int_{\Gamma_1} p(\mathbf{x},t)\ \mathbf{n}_1(\mathbf{x})\cdot\mathbf{b}_\alpha(\mathbf{x})\,ds(\mathbf{x}) \quad , \tag{48}$$

where $\{\mathbf{b}_1, \ldots, \mathbf{b}_{n_1}\}$ is the finite element basis associated with the finite element mesh of domain Ω_1. Therefore the $(n_1 \times n_1)$ matrix-valued spectral density function $[S_{\mathbb{F}}(\omega)]$ of stationary stochastic process $\mathbb{F}$ is such that

$$[S_{\mathbb{F}}(\omega)]_{\alpha\alpha'} = \int_{\Gamma_1}\int_{\Gamma_1} |\eta(\omega)|^2 s_p(\mathbf{x},\mathbf{y},\omega)\ \{\mathbf{n}_1(\mathbf{x})\cdot\mathbf{b}_\alpha(\mathbf{x})\} \times \{\mathbf{n}_1(\mathbf{y})\cdot\mathbf{b}_{\alpha'}(\mathbf{y})\}\ ds(\mathbf{x})\,ds(\mathbf{y}) \quad . \tag{49}$$

From Eqs. (26) and (32), we deduce that the matrix-valued spectral density functions $[S_{\mathbf{U}}(\omega)]$ and $[S_{\mathbf{P}}(\omega)]$ of $\mathbb{R}^{n_1}$-valued and $\mathbb{R}^{n_2}$-valued mean-square stationary stochastic processes $\{\mathbf{U}(t), t \in \mathbb{R}\}$ and $\{\mathbf{P}(t), t \in \mathbb{R}\}$ can be written as

$$[S_{\mathbf{U}}(\omega)] = [P_1]\,[S_{\mathbf{q}_1}(\omega)]\,[P_1]^T \quad , \tag{50}$$

$$[S_{\mathbf{P}}(\omega)] = [\mathcal{P}(\omega)]^T\,[S_{\mathbf{q}_1}(\omega)]\,\overline{[\mathcal{P}(\omega)]} \quad , \tag{51}$$

in which $[S_{\mathbf{q}_1}(\omega)]$ is the matrix-valued spectral density function of $\mathbb{R}^{N_1}$-valued mean-square stationary stochastic process $\{\mathbf{q}_1(t), t \in \mathbb{R}\}$. From Eqs. (27) and (28), and using linear filtering of stationary stochastic processes (see for instance Lin, 1967; Soize, 1994), we deduce that matrix-valued spectral density function $[S_{\mathbf{q}_1}(\omega)]$ is written as

$$[S_{\mathbf{q}_1}(\omega)] = [\mathcal{T}(\omega)]\,[P_1]^T\,[S_{\mathbb{F}}(\omega)]\,[P_1]\,\overline{[\mathcal{T}(\omega)]} \quad , \tag{52}$$

in which $(N_1 \times N_1)$ symmetric complex matrix $[\mathcal{T}(\omega)]$ is written as

$$[\mathcal{T}(\omega)] = \left([\mathcal{A}_1(\omega)] + \kappa\,[\,\mathcal{J}\,] - \omega^2\,[\mathcal{C}]\,[\mathcal{A}_2(\omega)]^{-1}\,[\mathcal{C}]^T\right)^{-1} \quad . \tag{53}$$

VIII. EXAMPLE AND VALIDATION

The example concerns the time-stationary random response in narrow MF band B of an inhomogeneous structure having an MF behavior in MF band B, coupled with an internal acoustic cavity filled with a gas and having an MF behavior in MF band B. The structural-acoustic system is referenced to an (x_1, x_2, x_3) coordinate system. The structure is a rectangular thin plate in bending mode, located in plane Ox_1x_2, to which are attached two point masses, three springs and five dashpots. Domain $\Omega_1 = \Gamma_1$ of the plate (middle surface) is rectangular and the plate is simply supported, homogeneous and isotropic, with constant thickness, width $L_1 = 0.5$ m, length $L_2 = 1.0$ m, surface-mass density $\rho_1 = 40$ kg/m^2, total mass $\mu_1 = \rho_1 L_1 L_2 = 20$ kg and constant damping rate $\xi_1 = 0.002$. We assume that the usual thin plate theory can be used (Leissa, 1993). The thickness, Young's modulus and Poisson's ratio of the plate are such that the lowest eigenfrequency of the associated conservative plate (without point masses and springs) is 5 Hz. To this plate are attached (1) two point masses having a mass of 3 kg and 4 kg located at points $(0.2, 0.4, 0)$ and $(0.35, 0.75, 0)$ respectively, (2) three springs having the same stiffness coefficient $k = \varepsilon_k\, \mu_1\, \omega_{\text{ref}}^2$ with $\omega_{\text{ref}} = 2\pi \times 550$ rad/s, $\varepsilon_k = 0.1$, located at points $(0.221, 0.278, 0)$, $(0.332, 0.537, 0)$ and $(0.443, 0.826, 0)$, and (3) five dashpots having the same damping coefficient $d = 2\varepsilon_d\, \mu_1\, \xi_1\, \omega_{\text{ref}}$ with $\varepsilon_d = 0.1$, located at points $(0.154, 0.165, 0)$, $(0.145, 0.334, 0)$, $(0.465, 0.373, 0)$, $(0.247, 0.462, 0)$ and $(0.268, 0.681, 0)$. This structure (the plate with point masses, springs and dashpots) is coupled with an acoustic cavity constituted of a rectangular room Ω_2 bounded by five rigid walls lying along the planes $x_1 = 0$, $x_1 = L_1$, $x_2 = 0$, $x_2 = L_2$, $x_3 = L_3$ with $L_3 = 9.000$ m. The sixth wall lying in plane $x_3 = 0$ is not rigid and is constituted of the plate. This bounded room is filled with a gas having a constant mass density $\rho_2 = 1$ kg/m^3, speed of sound $c_2 = 330$ m/s and dissipative coefficient $\tau = \xi_2/\omega_{\text{ref}}$ with $\xi_2 = 0.002$. The total mass of the gas is $\mu_2 = \rho_2 L_1 L_1 L_3 = 4.5$ kg. We consider the MF response of this structural-acoustic system in narrow MF band $B = 2\pi \times [500\,, 550]$ rad/s, i.e. on the $[500\,, 550]$ Hz frequency band. The excitation is time-stationary random wall pressure field $\{p(\mathbf{x}, t), \mathbf{x} \in \Gamma_1, t \in \mathbb{R}\}$ introduced in Section VII, for which the cross-spectral density function defined by Eq. (47) is such that for all ω in B, $|\eta(\omega)| = 1$ and $s_p(\mathbf{x}, \mathbf{y}, \omega) = (L_1 L_2)^{-1} \delta_0(x_1 - y_1)\, \delta_0(x_2 - y_2)$ where δ_0 is the Dirac delta function on $\mathbb{R}$ at point 0. For the validation, we are interested in calculating the power spectral density function $S_{\text{struc}}(\omega)$ relative to the structure and defined by

$$S_{\text{struc}}(\omega) = \frac{1}{|\eta(\omega)|^2\, |\Omega_1|} \int_{\Omega_1} S_{u_3}(\mathbf{x}, \mathbf{x}, \omega)\, d\mathbf{x} \quad , \qquad (54)$$

in which $S_{u_3}(\mathbf{x}, \mathbf{y}, \omega)$ is the cross-spectral density function of real-valued stochastic field $\{u_3(\mathbf{x}, t), \mathbf{x} \in \Omega_1, t \in \mathbb{R}\}$ (transverse displacement field of the plate) and the power spectral density function $S_{\text{acous}}(\omega)$ relative to the acoustic cavity and defined by

$$S_{\text{acous}}(\omega) = \frac{1}{\rho_2^2\, c_2^4} \frac{1}{|\eta(\omega)|^2\, |\Omega_2|} \int_{\Omega_2} [S_p(\mathbf{x}, \mathbf{x}, \omega)]\, d\mathbf{x} \quad , \qquad (55)$$

in which $S_p(\mathbf{x}, \mathbf{y}, \omega)$ is the cross-spectral density function of real-valued stochastic field $\{p(\mathbf{x}, t), \mathbf{x} \in \Omega_2, t \in \mathbb{R}\}$ (pressure field in the cavity).
Due to the simple geometry of the structural-acoustic system, the finite element basis for each domain can be substituted by the Ritz-Galerkin basis constituted of the normal modes of the plate without point masses and springs (Leissa, 1993) and the acoustic modes of the acoustic cavity with rigid walls (Pierce, 1989). The dynamical characteristics of this structural-acoustic system are the following. Concerning the structure in vacuo (plate with point masses and springs), the lowest eigenfrequency of the structure in vacuo is 7.74 Hz. There are 255 structural modes in frequency band $[0\,, 700]$ Hz, 179 structural modes in frequency band $[0\,, 500]$ Hz and 20 structural modes in narrow MF band $[500\,, 550]$ Hz. Then the rank of the first structural mode belonging to frequency band $[500\,, 550]$ Hz is 180 and consequently, the structure has an MF behavior in frequency band B. Concerning the acoustic cavity, the lowest acoustic eigenfrequency of the acoustic cavity with rigid walls is 18.33 Hz. There are 289 acoustic modes in frequency band $[0\,, 700]$ Hz, 128 acoustic modes in frequency band $[0\,, 500]$ Hz and 34 acoustic modes in narrow MF band $[500\,, 550]$ Hz. Then the rank of the first acoustic mode in MF band $[500\,, 550]$ Hz is 129 and consequently, the acoustic cavity has an MF behavior in MF band B.

VIII.1 Constructing the Reference Solution

The reference solution is constructed on the $[5\,, 700]$ Hz broad frequency band with a sampling frequency step $\delta\nu = 0.165$ Hz. The power spectral density functions $S_{\text{struc}}^{\text{ref}}(\omega)$ and $S_{\text{acous}}^{\text{ref}}(\omega)$ corresponding to the reference solution and defined by Eqs. (54) and (55) are directly calculated in the frequency domain using the formulation given in Section III (without using the reduced model). Figures 2 and 3 show the mappings $\nu \mapsto 10 \times \log_{10}(S_{\text{struc}}^{\text{ref}}(2\pi\nu))$ and $\nu \mapsto 10 \times \log_{10}(S_{\text{acous}}^{\text{ref}}(2\pi\nu))$ respectively over the $[5\,, 700]$ Hz broad frequency band.

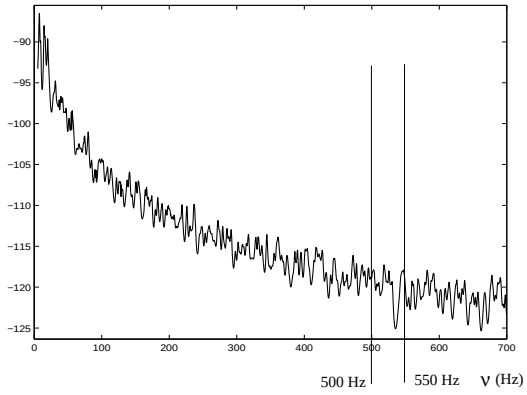

Fig. 2. Graph of function $\nu \mapsto 10 \times \log_{10}(S_{\text{struc}}^{\text{ref}}(2\pi\nu))$ corresponding to the reference solution for the MF structure coupled with the MF internal acoustic fluid

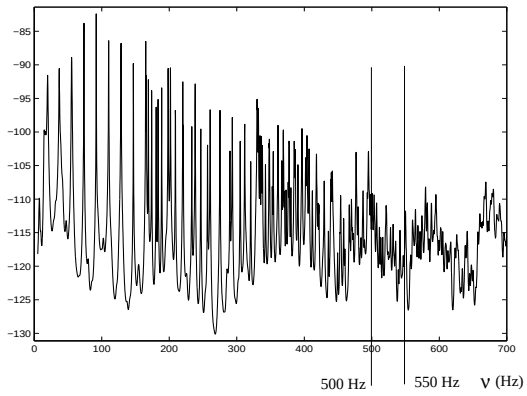

Fig. 3. Graph of function $\nu \mapsto 10 \times \log_{10}(S_{\text{acous}}^{\text{ref}}(2\pi\nu))$ corresponding to the reference solution for the MF internal acoustic fluid coupled with the MF structure

The reference solution on narrow MF band $B = [500\,,550]$ Hz is extracted from the above results.

VIII.2 Constructing the Dominant Eigensubspaces

For the structure, the dominant eigensubspace of the energy operator relative to band B for the structure in vacuo (related to matrix $[E_1]$ defined by Eqs. (23) to (25)) is constructed using the method presented in Section VI with $N_1 = 50$. Figure 4 shows the graph of the function $j \mapsto 10 \times \log_{10}(\lambda_1^j)$ for $j \in \{1, 2, \ldots, 50\}$ in which $\lambda_1^1, \ldots, \lambda_1^{50}$ are the highest eigenvalues of the generalized symmetric eigenvalue problem defined by Eq. (19) for $a = 1$. There is a strong decrease in the eigenvalues which means there exists the possibility of constructing an efficient reduced model for the structure. Figure 4 shows that the order N_1 of the reduced model is about 25 for band B.

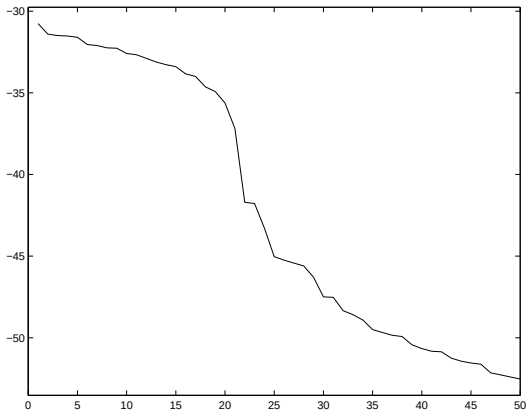

Fig. 4. Graph of function $j \mapsto 10 \times \log_{10}(\lambda_1^j)$ showing the distribution of eigenvalues λ_1^j of the energy operator of the MF structure in vacuo

For the internal acoustic fluid, the dominant eigensubspace of the energy operator relative to band B for the internal acoustic cavity with rigid walls (related to matrix $[E_2]$ defined by Eqs. (23) to (25)) is constructed using the method presented in Section VI with $N_2 = 50$. Figure 5 shows the graph of the function $j \mapsto 10 \times \log_{10}(\lambda_2^j)$ for $j \in \{1, 2, \ldots, 50\}$ in which $\lambda_2^1, \ldots, \lambda_2^{50}$ are the highest eigenvalues of the generalized symmetric eigenvalue problem defined by Eq. (19) for $a = 2$. Figure 5 shows that the order N_2 of the reduced model is about 40 for band B.

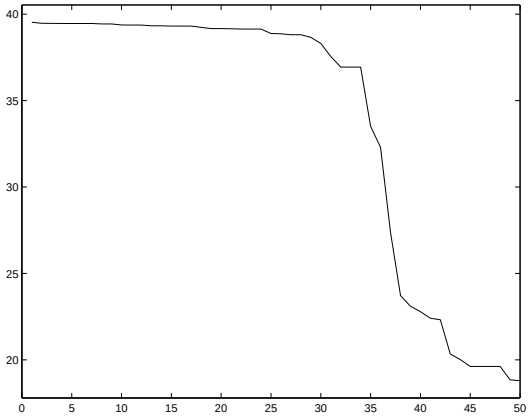

Fig. 5. Graph of function $j \mapsto 10 \times \log_{10}(\lambda_2^j)$ showing the distribution of eigenvalues λ_2^j of the energy operator of the MF internal acoustic cavity with rigid walls

VIII.3 Reduced Model Adapted to the Narrow MF Band

In this section, we present a comparison of the reference solution constructed in Section VIII.1 with the solution obtained by the reduced model constructed using the results of Sections V and VII. For the reduced model, the parameters are $N_1 = 30$ and $N_2 = 45$. Figure 6 is related to the structure and shows the comparison between function $\nu \mapsto 10 \times \log_{10}(S_{\text{struc}}^{\text{ref}}(2\pi\nu))$ (solid line corresponding to the reference solution) and function $\nu \mapsto 10 \times \log_{10}(S_{\text{struc}}(2\pi\nu))$ (dashed line corresponding to the reduced model) on narrow MF band $[500\,,550]$ Hz.

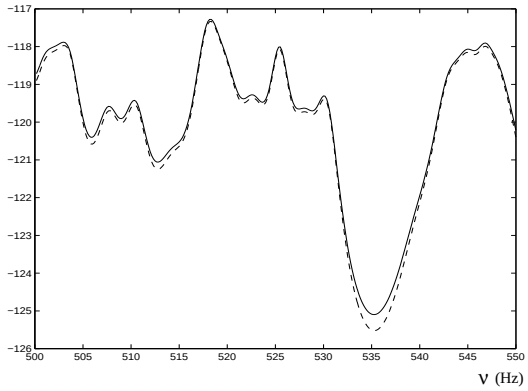

Fig. 6. Reduced model of the dynamical response of the MF structure coupled with the MF internal acoustic fluid: comparison between function $\nu \mapsto 10 \times \log_{10}(S_{\text{struc}}^{\text{ref}}(2\pi\nu))$ (reference solution (solid line)) and function $\nu \mapsto 10 \times \log_{10}(S_{\text{struc}}(2\pi\nu))$ (reduced model for $N_1 = 30$ and $N_2 = 45$ (dashed line))

Figure 7 is related to the internal acoustic fluid and shows the comparison between function $\nu \mapsto 10 \times \log_{10}(S_{\text{acous}}^{\text{ref}}(2\pi\nu))$ (solid line corresponding to the reference solution) and function $\nu \mapsto 10 \times \log_{10}(S_{\text{acous}}(2\pi\nu))$ (dashed line corresponding to the reduced model) on narrow MF band $[500\,,550]$ Hz.

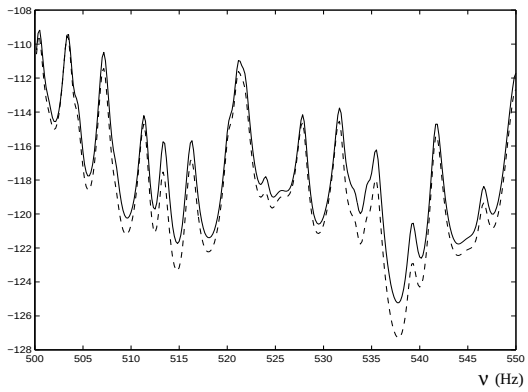

Fig. 7. Reduced model of the dynamical response of the MF internal acoustic fluid coupled with the MF structure: comparison between function $\nu \mapsto 10 \times \log_{10}(S_{\text{acous}}^{\text{ref}}(2\pi\nu))$ (reference solution (solid line)) and function $\nu \mapsto 10 \times \log_{10}(S_{\text{acous}}(2\pi\nu))$ (reduced model for $N_1 = 30$ and $N_2 = 45$ (dashed line))

Figures 6 and 7 show that the comparison is good for both the structure and the internal acoustic fluid.

IX. CONCLUSIONS

An approach is presented for constructing a reduced model in the MF range in the area of computational structural acoustics for a general three-dimensional anisotropic, inhomogeneous, viscoelastic bounded structure with an arbitrary geometry coupled with an internal acoustic fluid (gas or liquid). For a given MF band, the energy operator of the structure in vacuo and the energy operator of the internal acoustic cavity with rigid walls are positive-definite symmetric operators which have a countable set of decreasing positive eigenvalues. The eigenfunctions corresponding to the highest eigenvalues (dominant eigensubspace) of each energy operator constitute an appropriate functional basis of the corresponding admissible function space for the structure and for the internal acoustic fluid. For a structure having an MF behavior coupled with an internal acoustic fluid having an MF behavior in the MF band considered, these two functional bases allow a reduced model of the structural-acoustic system to be constructed using the Ritz-Galerkin method. The finite element method is uded to discretize the continuous case. For construction of the dominant eigensubspace of each energy operator, an efficient procedure based on the use of the subspace iteration method is proposed. It does not require explicit calculation of the energy operator. We then obtain an efficient method for constructing a reduced model in the MF range. In addition, concerning the structure, the results presented can be extended straightforwardly to a structure made of beams, plates and shells.

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