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EFFECTIVE RATIONALITY OF SECOND ORDER SYMMETRIC TENSOR SPACES

M. OLIVE AND R. DESMORAT

ABSTRACT. We consider the natural $\mathrm{SO}(3, \mathbb{k})$ linear representation, $\mathbb{k} = \mathbb{C}$ or \mathbb{R} , on the \mathbb{k} vector space $\mathbf{V} = n\mathbb{S}^2(\mathbb{R}^3)$ of n second order symmetric tensors, the associated invariant field $\mathbb{k}(\mathbf{V})^{\mathrm{SO}(3, \mathbb{k})}$ being known to be a purely transcendental extension in the complex case. We give an explicit tensorial form of a minimal generating set of the field of invariants, in both the complex and the real cases, showing that the invariant field is also a purely transcendental extension in the real case. Present results rely on some octahedral polynomial invariants obtained from Clebsch–Gordan projectors defined by a fourth order octahedral covariant. Thanks to Cartan’s map we also obtain a minimal set of generators for the $\mathrm{SL}(2, \mathbb{C})$ -rational invariant field of n binary quartics.

1. INTRODUCTION

The invariant theory, which finds its source in works as distinct as those of Boole [7] and Gauss [29], has acquired such a maturity so that its mathematical framework is both well delimited, and the application domains addressed by it is very broad: it goes from the theory of group representation [74] to the one of algebraic geometry [36, 54], cryptography via hyperelliptic curves [38, 46] and biomarkers in neuroimaging or shape recognition [55, 32, 4]. Its fields of application and investigation are not limited to mathematics, see in particular the works on Qubits [47, 43] and also in mechanics [63, 77]. For instance, let us mention specific concerns in hyper-elasticity of rubber-like materials [2, 44, 58], in the determination of distances to elasticity symmetry (isotropy) classes [69], in the modeling of porous or granular material anisotropy [13], in the description of magneto-elasticity coupling [16, 68], further applications to material instabilities [37] and wave propagation being expected.

It was in the 1960s and 1970s [60, 5] that the continuum mechanics community seized the abstract tools developed by Weyl [74] to produce notable results in *effective invariant theory*, producing *minimal integrity bases* of $\mathrm{O}(3, \mathbb{R})$ –and its closed subgroups– representations on tensor spaces of order less than 2, where $\mathrm{O}(3, \mathbb{R})$ denotes the group of orthogonal transformations in \mathbb{R}^3 . Due to the lack of communication between the mechanical and mathematical community [25], it took them a decade to obtain results in fact established in 1898 by Young [76], but difficult to access without Cartan map [15, 14] (see [3, 52, 18]): indeed, it should have been necessary to explicit the link between the $\mathrm{SL}(2, \mathbb{C})$ representations on binary forms and the $\mathrm{SO}(3, \mathbb{R})$ representations on tensor spaces.

Such a link has later been successfully exploited by Boehler–Kirilov–Onat [6] to propose, in tensorial form, an explicit integrity basis for the space of fourth order *harmonic* tensors (*i.e.* traceless and totally symmetric tensors), equivariant to the space of binary octavics, for which the associated $\mathrm{SL}(2, \mathbb{C})$ invariant algebra was already known from the works of von Gall [71] and Shioda [59].

The *effective invariant theory*, where effectiveness means the ability to display existing objects *a priori*, continues to raise questions [19, 20, 17, 10, 11, 51, 45]. The main difficulty in effectivity is particularly well illustrated by Hilbert’s finiteness theorem [36]. Indeed, after Gordan’s constructive demonstration for the finiteness of the invariant algebra of binary form [31], Hilbert obtained a theoretical and abstract result in a much more general framework (that of reductive groups). Since there was no real evidence of any construction of a finite integrity basis in Hilbert’s general proof, effectiveness was finally put aside. As emphasized by Weyl himself [74]: “Hilbert almost killed the subject”.

Nevertheless, Hilbert’s work allowed the invariant theory to fit into a very rich formal framework, and Noether’s work [33] did fix the algebraic objects specific to this theory, enriched over the years [35, 54, 21]. Nowadays’ general framework concerns some reductive group action over an algebraic variety [54], and its associated invariant algebra and field of rational invariants.

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As mentioned above, the continuum mechanics community [56, 60, 63, 77] did take up this theory in the case of real tensorial representations \mathbb{V} of the group $O(3, \mathbb{R})$. One goal was to be able to get parametrization of the associated orbit space, and thus find a finite set $\{s_1, \dots, s_p\}$ of invariants, called a *separating set*, such that for any two vectors (in fact, tensors or any family of tensors) $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{V}$

$$\forall i, \quad s_i(\mathbf{v}_1) = s_i(\mathbf{v}_2) \text{ iff } \mathbf{v}_1, \mathbf{v}_2 \text{ belong to the same orbit.}$$

In the case of first order tensors, Weyl's theorem on polarization [74, Theorem 2.9-A] was enough to obtain polynomial separating sets of $n\mathbb{R}^3 := \mathbb{R}^3 \oplus \dots \oplus \mathbb{R}^3$ (n times), so the next question was to find similar results for $n\mathbb{S}^2(\mathbb{R}^3)$, where $\mathbb{S}^2(\mathbb{R}^3)$ is the space of symmetric second order tensors. Being in the case of a real representation of a compact group, it was noticed that the algebra of polynomial invariants had this separating property [1] (while a separating set is not necessary an integrity basis). By Hilbert's theorem, it was therefore theoretically possible to obtain a finite separating set: the possibly non-minimal one coming from a finite integrity basis. A notion of *separating algebra* was then proposed in [24, 42], and Draisma *et al.* [23] obtained an algorithm to compute a polynomial separating set of a direct sum of the same representation, using polarization procedure on separating invariants.

It was therefore important to obtain an effective construction of minimal integrity bases or minimal separating sets for $n\mathbb{S}^2(\mathbb{R}^3) := \mathbb{S}^2(\mathbb{R}^3) \oplus \dots \oplus \mathbb{S}^2(\mathbb{R}^3)$ endowed with the standard $SO(3, \mathbb{R})$ diagonal representation. By using a complexification process, the problem reduces in fact to the one of determining an $SL(2, \mathbb{C})$ integrity basis of n quartic forms (see [6, 51]), first obtained by Young in 1899 [76]. Unfortunately, as recalled above, such results were unknown by the mechanical community, and they were established back over the years [56, 60, 63].

The quite high number of elements in these minimal integrity bases and polynomial separating sets was such that these results were not very useful in practice (see Table 1). There was then the idea of returning to the notion of *functional basis* (that is to say a generating set of all invariant functions [75]). It was in fact a question of taking up Weyl's own ideas [74]: finding a finite set of invariants $\{\phi_1, \dots, \phi_k\}$ such that any other invariant function f can be expressed in terms of the ϕ_i 's. As Weyl pointed out, the term *function* is here to be taken in its broadest sense.

The hard work carried out in [72, 60, 73, 5] made it possible to obtain minimal polynomial functional bases (in the sense that any subset is no longer a functional basis), in the case of $n\mathbb{S}^2(\mathbb{R}^3)$. Their cardinals are given in Table 1 (minimality was obtained in [53]).

	Cardinal	$n = 3$	$n = 4$	$n = 5$	$n = 6$
Minimal integrity basis	$3n + 4 \binom{n}{2} + 7 \binom{n}{3} + 20 \binom{n}{4} + 26 \binom{n}{5} + 10 \binom{n}{6}$	28	84	261	684
Minimal separating set	$3n + 4 \binom{n}{2} + \binom{n}{3} = \frac{n(n^2 + 17)}{6}$	22	40	65	98

TABLE 1. Cardinals of minimal integrity bases and minimal polynomial separating sets for $(n\mathbb{S}^2(\mathbb{R}^3), SO(3, \mathbb{R}))$.

If we further aim at reducing the cardinal of a separating set (*i.e.* a functional basis [75]), we have to join the standard approach in invariant theory, introducing no longer polynomial invariants but rational ones, and considering at first the complexification of the group and of its representation. In that scope, the question then becomes the one of determining a *generating set of the field of rational invariants* $\mathbb{C}(n\mathbb{S}^2(\mathbb{C}^3))^{SO(3, \mathbb{C})}$.

Now, in the complex case, and only in this case (*i.e.* of an algebraically closed field), such generating sets are obtained using a *separating property* of the orbits in general position, thanks to Popov's lemma 2.2 [54, p.155]. Indeed, for any complex linear representation (\mathbb{V}, G) of a reductive group, if one exhibits a finite set $\{r_1, \dots, r_s\}$ of rational invariants and a dense open set $X \subset \mathbb{V}$ such that, for all $\mathbf{x}_1, \mathbf{x}_2 \in X$,

$$\forall i \quad r_i(\mathbf{x}_1) = r_i(\mathbf{x}_2) \implies \mathbf{x}_1, \mathbf{x}_2 \text{ belong to the same orbit,}$$

then we can deduce that the field $\mathbb{C}(\mathbf{V})^G$ of rational invariants is generated by $\{r_1, \dots, r_s\}$. It is important here to note that such a result ceased to be true in the real case $\mathbb{R}(\mathbf{V})^G$ (see Remark 2.10).

Using this approach, we can aim at obtaining an optimal bound for the cardinal of a generating set of the field $\mathbb{C}(\mathbf{V})^G$. This can be achieved when such a field is rational, meaning that it is a purely transcendental extension of \mathbb{C} . In such a case, there exists a finite set r_1, \dots, r_d of rational invariants such that

$$\mathbb{C}(\mathbf{V})^G = \mathbb{C}(r_1, \dots, r_d), \quad r_i \in \mathbb{C}(\mathbf{V})^G,$$

where $d = \dim(\mathbf{V}) - \dim(G)$ is the *transcendence degree* of $\mathbb{C}(\mathbf{V})^G$ (see [21, Corollary 6.2] for instance). The question of the rationality of the field $\mathbb{C}(\mathbf{V})^G$ is in general a difficult one, both from a theoretical and from an effective point of view [22]. In the specific case of $G = \mathrm{SL}(2, \mathbb{C})$ representations, the rationality was theoretically obtained by Katsylo for all reducible representations [40, 39]. It was then Maeda [48] who produced an explicit generating set of 6 rational invariants for $\mathrm{SL}(2, \mathbb{C})$ representations on the space of binary octavics.

We present here effective results about the rationality of $\mathrm{SO}(3, \mathbb{C})$ representations on the space $n\mathbb{S}^2(\mathbb{C}^3)$ of n second order symmetric tensors on \mathbb{C}^3 . From such results, we also obtain effective rationality for the space $n\mathbb{S}_4$ of n quartic forms, endowed with its natural $\mathrm{SL}(2, \mathbb{C})$ representation. In fact, we propose an effective approach of the *slice lemma* [28, 40], following Maeda's strategy for binary octavics [48]:

- (1) Compute an explicit minimal integrality basis of *octahedral invariants* of $\mathbb{S}^2(\mathbb{C}^3)$;
- (2) Construct $\mathrm{SO}(3, \mathbb{C})$ rational invariants t_1, \dots, t_9 of $\mathbb{S}^2(\mathbb{C}^3) \oplus \mathbb{S}^2(\mathbb{C}^3)$ from this integrality basis;
- (3) Find a *Zariski open space* \mathcal{Z}^c in $\mathbb{S}^2(\mathbb{C}^3) \oplus \mathbb{S}^2(\mathbb{C}^3)$ so that t_1, \dots, t_9 separate all points in \mathcal{Z}^c ;
- (4) Propose a generalization to vector space $\mathbf{V} = n\mathbb{S}^2(\mathbb{C}^3)$ so to obtain a minimal generating set t_1, \dots, t_d of $\mathbb{C}(\mathbf{V})^{\mathrm{SO}(3, \mathbb{C})}$, with $d = 6n - 3$.

Thus, as a first result we will obtain (the tensor expressions for the invariants being detailed in theorem 4.2):

Theorem 1.1. *There exists an explicit set of 9 rational invariants $\{I_1, J_1, K_2, K_3, r_2, r_3, s_3, r_4, r_5\}$ separating $\mathrm{SO}(3, \mathbb{C})$ orbits of $\mathbb{S}^2(\mathbb{C}^3) \oplus \mathbb{S}^2(\mathbb{C}^3)$ in general position, so that*

$$\mathbb{C}(\mathbb{S}^2(\mathbb{C}^3) \oplus \mathbb{S}^2(\mathbb{C}^3))^{\mathrm{SO}(3, \mathbb{C})} = \mathbb{C}(I_1, J_1, K_2, K_3, r_2, r_3, s_3, r_4, r_5)$$

where the number 9 is exactly the transcendence degree of $\mathbb{C}(\mathbb{S}^2(\mathbb{C}^3) \oplus \mathbb{S}^2(\mathbb{C}^3))^{\mathrm{SO}(3, \mathbb{C})}$.

An important fact here is that all such invariants are given using integers as coefficients so that, when evaluated to *real* second order symmetric tensors, we obtain real numbers. We will have as a second result:

Theorem 1.2. *The invariant field $\mathbb{R}(\mathbb{S}^2(\mathbb{R}^3) \oplus \mathbb{S}^2(\mathbb{R}^3))^{\mathrm{SO}(3, \mathbb{R})}$ is rational and generated by the explicit minimal set of 9 rational invariants $I_1, J_1, K_2, K_3, r_2, r_3, s_3, r_4, r_5$:*

$$\mathbb{R}(\mathbb{S}^2(\mathbb{R}^3) \oplus \mathbb{S}^2(\mathbb{R}^3))^{\mathrm{SO}(3, \mathbb{R})} = \mathbb{C}(I_1, J_1, K_2, K_3, r_2, r_3, s_3, r_4, r_5)$$

These results will be generalized to the $\mathrm{SO}(3, \mathbb{k})$ representation on the space $n\mathbb{S}^2(\mathbb{k}^3)$ of n second order symmetric tensors on \mathbb{k}^3 , where $\mathbb{k} = \mathbb{R}$ or \mathbb{C} , the tensor expressions for the invariants being detailed in theorem 5.1:

Theorem 1.3. *The invariant field $\mathbb{k}(n\mathbb{S}^2(\mathbb{k}^3))^{\mathrm{SO}(3, \mathbb{k})}$ is rational and is generated by an explicit minimal set r_1, \dots, r_d of $d = 6n - 3$ rational invariants:*

$$\mathbb{k}(n\mathbb{S}^2(\mathbb{k}^3))^{\mathrm{SO}(3, \mathbb{k})} = \mathbb{k}(r_1, \dots, r_d), \quad \mathbb{k} = \mathbb{R} \text{ or } \mathbb{C}$$

where, in the case $\mathbb{k} = \mathbb{C}$, $6n - 3$ correspond to the transcendence degree of $\mathbb{C}(n\mathbb{S}^2(\mathbb{C}^3))^{\mathrm{SO}(3, \mathbb{C})}$.

Finally, we will exploit an explicit equivariant isomorphism ϕ^{-*} between the $\mathrm{SO}(3, \mathbb{C})$ space $n\mathbb{S}^2(\mathbb{C}^3)$ and the $\mathrm{SL}(2, \mathbb{C})$ space $n\mathbb{S}_4$ of n quartic forms on \mathbb{C}^2 (from the so-called Cartan's map, see section 6), so to obtain:

Theorem 1.4. *The invariant field $\mathbb{C}(n\mathbb{S}_4)^{\mathrm{SL}(2, \mathbb{C})}$ is rational and generated by a minimal set $\{\hat{r}_1, \dots, \hat{r}_d\}$ of $d = 5n - 3$ rational invariants:*

$$\mathbb{C}(n\mathbb{S}_4)^{\mathrm{SL}(2, \mathbb{C})} = \mathbb{C}(\hat{r}_1, \dots, \hat{r}_d), \quad \hat{r}_i = r_i \circ \phi^{-*},$$

where $5n - 3$ correspond to the transcendence degree of $\mathbb{C}(n\mathbb{S}_4)^{\mathrm{SL}(2, \mathbb{C})}$.

Organisation of the paper. We first recall in [section 2](#) some general results about rational invariants and *weak separating sets*, closely related in the complex case to a generating set of the field of rational invariants $\mathbb{k}(\mathbf{V})^G$, all this being done in the scope of a linear representation of a reductive group G . In [section 3](#) we focus on the standard linear representation of the octahedral group on the space $\mathbb{S}^2(\mathbb{k}^3)$ of second order symmetric tensors on \mathbb{k}^3 , where $\mathbb{k} = \mathbb{R}$ or \mathbb{C} . We then produce a minimal integrity basis for the octahedral polynomial invariants on the space $\mathbb{S}^2(\mathbb{k}^3)$, where such a space decomposes into two octahedral stable subspaces: the one Diag_2 of diagonal tensors and its complement Adiag_2 . Such an octahedral integrity basis will be given in tensorial form, using a fourth order covariant projector and contraction operations.

Thanks to the use of the *generalized cross product* ([definition 4.1](#)), we then produce a weak separating set of 9 rational invariants for the $\text{SO}(3, \mathbb{k})$ space $\mathbb{S}^2(\mathbb{k}^3) \oplus \mathbb{S}^2(\mathbb{k}^3)$ of two second order symmetric tensors, both valid in the case $\mathbb{k} = \mathbb{R}$ or \mathbb{C} . We then deduce in [corollary 4.7](#) that this weak separating set is also a generating set of the field of rational invariants $\mathbb{C}(\mathbb{S}^2(\mathbb{C}^3) \oplus \mathbb{S}^2(\mathbb{C}^3))^{\text{SO}(3, \mathbb{C})}$, and [corollary 4.8](#) provides the same result in the real case $\mathbb{k} = \mathbb{R}$. We also deduce in [section 5](#) a weak separating set of $6n - 3$ rational invariants for the $\text{SO}(3, \mathbb{k})$ space $n\mathbb{S}^2(\mathbb{k}^3)$ ([theorem 5.1](#)), so that this defines also a generating set of the associated field of invariant ([corollary 5.3](#)). Finally, all such results obtained for second order symmetric tensors are used to produce an explicit set of $5n - 3$ rational invariants of the $\text{SL}(2, \mathbb{C})$ space $n\mathbb{S}_4$ of n binary quartic forms, equivariant to the $\text{SO}(3, \mathbb{C})$ space $n\mathbb{H}^2(\mathbb{C}^3)$ of n second order *harmonic tensors* (symmetric and traceless).

2. SEPARATING SETS OF AN ORBIT SPACE

Let us consider the field \mathbb{k} to be either \mathbb{R} or \mathbb{C} and (\mathbf{V}, G, ρ) a finite dimensional \mathbb{k} linear representation of a group G , so that

$$\rho : G \longrightarrow \text{GL}(\mathbf{V})$$

is a group morphism, where $\text{GL}(\mathbf{V})$ stands for the group of linear invertible maps of \mathbf{V} . This means that for any $g \in G$, $\rho(g)$ is a linear invertible map of \mathbf{V} and that

$$\rho(g_1 g_2) \mathbf{v} = \rho(g_1) (\rho(g_2) \mathbf{v}), \quad \rho(e) \mathbf{v} = \mathbf{v}$$

for all $g_1, g_2 \in G$ and $\mathbf{v} \in \mathbf{V}$, where e is the unit element of G . For any $\mathbf{v} \in \mathbf{V}$ its associated *orbit* is the set

$$\text{Orb}(\mathbf{v}) := \{\rho(g) \mathbf{v}, \quad g \in G\}$$

and the *orbit space* is the set of all orbits.

A important question in *effective invariant theory* is to obtain a clear description of the associated orbit space [\[70\]](#), which can be done using \mathbb{k} -valued invariant functions, where such functions can be polynomials, rationals, etc.

Such invariants may be used to get either *separating sets* or *weak separating sets*, as defined below.

Definition 2.1 (Separating set). A finite set $\{s_1, \dots, s_p\}$ of \mathbb{k} -valued invariant functions is a *separating set* if for all $\mathbf{v}_1, \mathbf{v}_2$ in \mathbf{V}

$$\exists g \in G, \quad \rho(g) \mathbf{v}_1 = \mathbf{v}_2 \iff \forall k \in \{1, \dots, p\} \quad s_k(\mathbf{v}_1) = s_k(\mathbf{v}_2).$$

Invariant functions do not need to be polynomial invariants. Note that a more general definition is the one of a *separating algebra* introduced by Derksen and al [\[24, 17\]](#). Let us now recall in [Theorem 2.2](#) a theoretical result stating when a separating set is directly given by a finite generating set of the algebra $\mathbb{k}[\mathbf{V}]^G$ of polynomial invariants, where

$$\mathbb{k}[\mathbf{V}]^G := \{p \in \mathbb{k}[\mathbf{V}], \quad g \star p = p, \quad \forall g \in G, \quad \mathbf{v} \in \mathbf{V}\}, \quad (g \star p)(\mathbf{v}) := p(\rho(g^{-1}) \mathbf{v}).$$

Theorem 2.2. *Suppose that either*

- (a) G is a finite group and $\mathbb{k} = \mathbb{R}$ or \mathbb{C} ;
- (b) G is a compact Lie group and $\mathbb{k} = \mathbb{R}$.

Then any finite generating set $\{I_1, \dots, I_p\}$ of the algebra $\mathbb{k}[\mathbf{V}]^G$ of polynomial invariants is a separating set.

In case (a) the proof follows from [\[23, Lemma 2.1\]](#), while the proof of case (b) is given in [\[1, Appendix C\]](#). Note that in both cases, a finite generating set of $\mathbb{k}[\mathbf{V}]^G$, also known as an *integrity basis* [\[74\]](#), always exists from Hilbert's finiteness theorem [\[36\]](#). In the case of the real vector space $\mathbf{V} = n\mathbb{S}^2(\mathbb{R}^3)$ and the compact Lie group $G = \text{SO}(3, \mathbb{R})$, explicit computations of minimal integrity

bases were obtained in the 1960's by Smith [62], where the cardinals of such minimal integrity bases are those of table 1.

In the case of complex representations of a *linearly reductive group*, such as $\mathrm{SL}(2, \mathbb{C})$, one also defines *weak separating sets*:

Definition 2.3 (Weak separating set). Given some *dense open set* $U \subset V$, a finite set $\{s_1, \dots, s_p\}$ of \mathbb{k} -valued invariant functions is a *weak separating set* if, for all $\mathbf{v}_1, \mathbf{v}_2$ in U

$$\exists g \in G, \quad \rho(g)\mathbf{v}_1 = \mathbf{v}_2 \iff \forall k \in \{1, \dots, p\} \quad s_k(\mathbf{v}_1) = s_k(\mathbf{v}_2).$$

Remark 2.4. In the scope of invariant theory, it is classical to consider a dense open set defined in the *Zariski topology*, where closed sets are defined as

$$\mathcal{Z} := \{\mathbf{v} \in V, \quad p(\mathbf{v}) = 0, \quad \forall p \in S\}, \quad S \subset \mathbb{k}[V].$$

Every *open Zariski set* is the complementary set $\mathcal{Z}^c = V \setminus \mathcal{Z}$ of a closed set \mathcal{Z} , which is open and dense for the classical topology on V .

As a corollary of [54, Proposition 3.4] and theorem 2.5, for any complex representation V of $G = \mathrm{SO}(3, \mathbb{C})$ or $G = \mathrm{SL}(2, \mathbb{C})$, an integrity basis of the invariant algebra $\mathbb{C}[V]^G$ is also a weak separating set. Nevertheless, its cardinal can be very big: for instance, a minimal integrity basis of the $\mathrm{SL}(2, \mathbb{C})$ space of binary decimic is given by 106 invariants [10].

One question is then to obtain a weak separating set of low cardinal. An interesting approach is to consider *rational invariants* instead of polynomial ones.

Let $\mathbb{k}(V)$ be the field of quotients of the coordinate ring $\mathbb{k}[V]$,

$$\mathbb{k}(V) = \left\{ \frac{p}{q}, \quad p, q \in \mathbb{k}[V] \right\}.$$

Then we define $\mathbb{k}(V)^G$ to be the field of rational invariants

$$\mathbb{k}(V)^G := \{r \in \mathbb{k}(V), \quad g \star r = r, \quad \forall g \in G, \quad \mathbf{v} \in V\}, \quad (g \star r)(\mathbf{v}) := r(\rho(g^{-1})\mathbf{v}).$$

For the groups and fields under consideration, we have in particular:

Theorem 2.5. *Suppose (V, G, ρ) is a finite dimensional linear representation of $G = \mathrm{SL}(2, \mathbb{C})$, $\mathrm{SO}(3, \mathbb{C})$ or $\mathrm{SO}(3, \mathbb{R})$, where the ground field is either $\mathbb{k} = \mathbb{R}$ or \mathbb{C} . Then the field of rational invariants $\mathbb{k}(V)^G$ is the quotient field of $\mathbb{k}[V]^G$, meaning that for any rational invariant $r \in \mathbb{k}(V)^G$ we can write*

$$r = \frac{p}{q}, \quad p, q \in \mathbb{k}[V]^G.$$

Proof. This proof follows the one of Popov [54, Theorem 3.3] and Brion [8], initially stated in the scope of an *action* of a group on some algebraic variety, defined on an algebraically closed field.

Let us consider a given rational invariant $r \in \mathbb{k}(V)^G$ and write $r = p/q$ with $p, q \in \mathbb{k}[V]$ relatively prime. We thus have

$$g \star r = \frac{g \star p}{g \star q} = \frac{p}{q}, \quad \forall g \in G$$

so that $g \star q = \alpha(g)q$, where $\alpha : G \rightarrow \mathbb{k}^*$ is a group morphism. In case $G = \mathrm{SL}(2, \mathbb{C})$, $\mathrm{SO}(3, \mathbb{C})$ or $\mathrm{SO}(3, \mathbb{R})$ and $\mathbb{k} = \mathbb{R}$ or \mathbb{C} , α defines a one dimensional representation of G , and the only one dimensional representation is given by the trivial one (see [9] for instance), so that $\alpha \equiv 1$, which conclude the proof. \square

Remark 2.6. In the case of a finite group, the result still holds (a proof is for instance given in [8, Proposition 1], also valid in the real case).

Finally, Katsylo was able to solve the *rationality problem* [22] for $G = \mathrm{SL}(2, \mathbb{C})$, so that we get (theoretically) an optimal generating set for the field of rational invariants:

Theorem 2.7 (Katsylo [39]). *Let $(V, \mathrm{SL}(2, \mathbb{C}))$ be a finite dimensional complex linear representation of the special linear group $\mathrm{SL}(2, \mathbb{C})$. Then the field of rational invariants $\mathbb{C}(V)^{\mathrm{SL}(2, \mathbb{C})}$ is a purely transcendental extension of \mathbb{C} , so there exists rational invariants r_1, \dots, r_d such that*

$$\mathbb{C}(V)^{\mathrm{SL}(2, \mathbb{C})} = \mathbb{C}(r_1, \dots, r_d), \quad d = \dim(V) - 3.$$

Remark 2.8. The number $d = \dim(\mathbf{V}) - 3$ is here the *transcendental degree* of the field $\mathbb{C}(\mathbf{V})^{\mathrm{SL}(2, \mathbb{C})}$, which is given by (see [21, Corollary 6.2] for instance):

$$d = \dim(\mathbf{V}) - \dim(G).$$

As $\mathrm{SL}(2, \mathbb{C})$ is the universal covering space of $\mathrm{SO}(3, \mathbb{C})$ [66], we also deduce the same theoretical results for $\mathrm{SO}(3, \mathbb{C})$ reducible representations.

To conclude this section, we recall an important result from Popov [54, Lemma 2.2] which connects a finite generating set of the quotient field $\mathbb{k}(\mathbf{V})^G$ and a weak generating set:

Lemma 2.9. *Let \mathbb{k} be an algebraically closed field. If $\{r_1, \dots, r_p\}$ is a finite weak generating set of rational invariants, then $\{r_1, \dots, r_p\}$ generates the field $\mathbb{k}(\mathbf{V})^G$.*

Remark 2.10. In the real case, Lemma 2.9 is no longer true: taking $G = \{Id\}$ the trivial group and $\mathbf{V} = \mathbb{R}$, the field of invariant is simply given by $\mathbb{R}(\mathbf{V}) \simeq \mathbb{R}(x)$. The polynomial invariant $p(x) := x^3$ separates all the orbits but is not a generator of $\mathbb{R}(\mathbf{V})$.

3. OCTAHEDRAL INTEGRITY BASIS OF ONE SECOND ORDER SYMMETRIC TENSOR

Let us consider the standard octahedral linear representation on the space $\mathbb{S}^2(\mathbb{k}^3)$ of second order symmetric tensors on \mathbb{k}^3 , with $\mathbb{k} = \mathbb{R}$ or \mathbb{C} . Here, the space \mathbb{k}^3 is endowed with the standard non degenerate quadratic form

$$\mathbf{q}(\mathbf{x}) := x^2 + y^2 + z^2, \quad \mathbf{x} = (x, y, z) \in \mathbb{k}^3$$

Consider now an orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of \mathbb{k}^3 , with respect to the quadratic form \mathbf{q} . The octahedral group \mathbb{O}^+ is defined as:

$$\mathbb{O}^+ := \{g \in \mathrm{GL}(3, \mathbb{k}), \quad g\mathbf{e}_i = \pm\mathbf{e}_j, \quad \det(g) = 1\}.$$

It is a finite group of order 24 and is generated by

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The space $\mathbb{S}^2(\mathbb{k}^3)$ is the space of second order (covariant) symmetric tensors

$$\mathbb{S}^2(\mathbb{k}^3) := \{\mathbf{a} \in (\mathbb{k}^3)^* \otimes (\mathbb{k}^3)^*, \quad \mathbf{a}(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{a}(\mathbf{x}_2, \mathbf{x}_1), \quad \forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{k}^3\}.$$

In fact, using quadratic form \mathbf{q} , there is an isomorphism between $(\mathbb{k}^3)^*$ and \mathbb{k}^3 , so, from now on, we do not distinguish between covariant and contravariant tensors, and we write

$$\mathbf{a} = a_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

where Einstein convention on repeated indices has been used.

The standard \mathbb{O}^+ representation on the space $\mathbb{S}^2(\mathbb{k}^3)$ is now given by

$$(1) \quad g \star \mathbf{a} := g\mathbf{a}g^t$$

with g^t the transpose matrix of g , and where matrix product is used, so that the symmetric tensor \mathbf{a} is identified with a 3×3 matrix.

A minimal integrity basis of the invariant algebra $\mathbb{R}[\mathbb{S}^2(\mathbb{R}^3)]^{\mathbb{O}^+}$ has already been obtained in [61], but we propose here to write such an integrity basis in a tensorial intrinsic form, both in the real and in the complex case.

To do so, we next provide the Hilbert series and a homogeneous system of parameters of this algebra. From this, we deduce a degree bound for an integrity basis (see Lemma 3.3), and finally use Macaulay software [34] to check the minimality. Note that such a strategy to obtain an *explicit integrity basis* in the case of a finite group is close to the one proposed in [67] (see also [10, 11] for a similar strategy to compute integrity basis in classical invariant theory). Let us now provide the details.

For any graded algebra $\mathcal{A} = \bigoplus_k \mathcal{A}_k$, its associated Hilbert series is the formal series

$$H_{\mathcal{A}}(z) = \sum_{k \geq 0} \dim(\mathcal{A}_k)z^k.$$

In the case of invariant algebra of a finite group, there is an *a priori* way to compute such Hilbert series (see [67, Theorem 2.2.1] for instance). In our specific case, Hilbert series $H = H_{\text{octa}}$ of $\mathbb{k}[\mathbb{S}^2(\mathbb{k}^3)]^{\mathbb{O}^+}$ is obtained by direct computation:

$$(2) \quad H_{\text{octa}}(z) = \frac{1 + z^3 + z^4 + z^5 + z^6 + z^9}{(1-z)(1-z^2)^2(1-z^3)^2(1-z^4)}.$$

To go one step further, let us introduce the so-called *homogeneous system of parameters* [67] of a graded \mathbb{k} algebra $\mathcal{A} = \bigoplus_k \mathcal{A}_k$.

Definition 3.1. A family $\theta_1, \dots, \theta_s$ is a homogeneous system of parameters of a graded algebra \mathcal{A} if:

- (1) Each θ_i belongs to some homogeneous space \mathcal{A}_{k_i} of \mathcal{A} .
- (2) The family $\theta_1, \dots, \theta_s$ is algebraically free.
- (3) The algebra \mathcal{A} is a $\mathbb{k}[\theta_1, \dots, \theta_s]$ -module of finite type.

An explicit determination of a homogeneous system of parameters is in general not straightforward [19, 41, 12], but in case of a finite group, and thus the group \mathbb{O} , the following result holds [64, Proposition 5.3.7]:

Lemma 3.2. *A family $\theta_1, \dots, \theta_s$ of homogeneous invariants form a system of parameters of the invariant ring $\mathbb{k}[\mathbb{S}^2(\mathbb{k}^3)]^{\mathbb{O}^+}$ ($\mathbb{k} = \mathbb{R}$ or \mathbb{C}) if and only if $s = 6$ and*

$$\{\mathbf{a} \in \mathbb{S}^2(\mathbb{C}^3), \quad \theta_1(\mathbf{a}) = \dots = \theta_s(\mathbf{a}) = 0\} = \{\mathbf{0}\}$$

Finally, recall that a graded algebra \mathcal{A} is said to be *Cohen–Macaulay* if for any homogeneous system of parameters $\theta_1, \dots, \theta_s$, the algebra \mathcal{A} is a *free* $\mathbb{k}[\theta_1, \dots, \theta_s]$ -module, so that the following lemma holds [67, Proposition 2.3.6].

Lemma 3.3. *Let \mathcal{A} be some Cohen–Macaulay algebra with a homogeneous system of parameters $\theta_1, \dots, \theta_s$. Then the Hilbert series of \mathcal{A} is given by*

$$H(z) = \frac{z^{e_1} + \dots + z^{e_r}}{(1-z^{d_1}) \dots (1-z^{d_s})}$$

where $d_i = \deg(\theta_i)$ and $e_j \in \mathbb{N}$. Furthermore, a degree bound for a generating family of \mathcal{A} is $\max(d_i, e_j)$.

General statements having been recalled, let us now consider the invariant algebra of $\mathbb{k}[\mathbb{S}^2(\mathbb{k}^3)]^{\mathbb{O}^+}$. Following Smith and Kiral [61], first perform the decomposition:

$$\mathbb{S}^2(\mathbb{k}^3) = \text{Diag}_2 \oplus \text{Adiag}_2$$

where

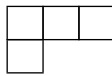
$$(3) \quad \text{Diag}_2 := \{\lambda_1 \mathbf{e}_{11} + \lambda_2 \mathbf{e}_{22} + \lambda_3 \mathbf{e}_{33}\} \subset \mathbb{S}^2(\mathbb{k}^3), \quad \mathbf{e}_{ii} := \mathbf{e}_i \otimes \mathbf{e}_i$$

and

$$(4) \quad \text{Adiag}_2 := \{\alpha_1 \mathbf{e}_{23} + \alpha_2 \mathbf{e}_{13} + \alpha_3 \mathbf{e}_{12}\} \subset \mathbb{S}^2(\mathbb{k}^3), \quad \mathbf{e}_{ij} := \mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i \quad (i \neq j)$$

are \mathbb{O}^+ stable.

Remark 3.4. Using the well-known isomorphism between \mathbb{O}^+ and the permutation group \mathfrak{S}_4 of four elements [27], we can check from \mathfrak{S}_4 character table that Adiag_2 is the irreducible (standard) representation χ^{std} corresponding to Young's table



when $\text{Diag}_2 = \chi^{\text{triv}} \oplus \chi^{(2,2)}$ is reducible, with χ^{triv} the trivial representation and $\chi^{(2,2)}$ the irreducible \mathfrak{S}_4 representation corresponding to square Young's table



Let us define the so-called *out-of-diagonal* octahedral fourth order projector tensor \mathbf{P} [57, 50, 26, 49]:

$$(5) \quad \mathbf{P} = \frac{1}{2} \sum_{i < j} \mathbf{e}_{ij} \otimes \mathbf{e}_{ij}.$$

The projection of $\mathbb{S}^2(\mathbb{k}^3)$ onto the vector space Adiag_2 parallel to Diag_2 is given by \mathbf{P} . More specifically, for any $\mathbf{c} = c_{ij}\mathbf{e}_i \otimes \mathbf{e}_j \in \mathbb{S}^2(\mathbb{k}^3)$ we have

$$(6) \quad \mathbf{P} : \mathbf{c} = \begin{pmatrix} 0 & c_{12} & c_{13} \\ c_{12} & 0 & c_{23} \\ c_{13} & c_{23} & 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{c} - \mathbf{P} : \mathbf{c} = \begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix}$$

where " : " stands for the double contraction $(\mathbf{P} : \mathbf{c})_{ij} := P_{ijpq}c_{pq}$.

A minimal integrity basis of $\mathbb{k}[\mathbb{S}^2(\mathbb{k}^3)]^{\oplus+}$ is provided in the next result.

Theorem 3.5. *Let $\mathbf{a} \in \mathbb{S}^2(\mathbb{k}^3)$ and*

$$\mathbf{a}_1 := \mathbf{P} : \mathbf{a} \in \text{Adiag}_2, \quad \mathbf{a}_2 := \mathbf{a} - \mathbf{P} : \mathbf{a} \in \text{Diag}_2.$$

The 9 polynomial invariants

$$\begin{aligned} I_1^c &:= \text{tr } \mathbf{a}, & I_2^c &:= \text{tr}(\mathbf{a}_2^2), & I_3^c &:= \text{tr}(\mathbf{a}_2^3), \\ J_2^c &:= \text{tr}(\mathbf{a}_1^2), & J_3^c &:= \text{tr}(\mathbf{a}_1^3), & J_4^c &:= \text{tr}((\mathbf{P} : \mathbf{a}_1^2)^2), \\ K_3^c &:= \text{tr}(\mathbf{a}_1^2 \mathbf{a}_2), & K_4^c &:= \text{tr}((\mathbf{a}_1 \mathbf{a}_2)^2), & I_5^c &:= \text{tr}(\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_1 (\mathbf{P} : \mathbf{a}_1^2)) \end{aligned}$$

constitute a minimal integrity basis of $\mathbb{k}[\mathbb{S}^2(\mathbb{k}^3)]^{\oplus+}$.

A proof of this theorem relies on two essential results: one about an explicit homogeneous system of parameters (corollary 3.6), and the other one related to a degree bound given by Lemma 3.3 deduced from the Cohen–Macaulayness of $\mathbb{C}[\mathbb{S}^2(\mathbb{k}^3)]^{\oplus+}$.

By a direct computation we first deduce from Lemma 3.2:

Corollary 3.6. *The family $I_1^c, I_2^c, I_3^c, J_2^c, J_3^c, J_4^c$ is a homogeneous system of parameters of the invariant algebra $\mathbb{k}[\mathbb{S}^2(\mathbb{k}^3)]^{\oplus}$.*

Proof of Theorem 3.5. From corollary 3.6, Lemma 3.3 and Hilbert series (2), we know that there exists a generating set of the invariant algebra $\mathcal{A} = \mathbb{k}[\mathbb{S}^2(\mathbb{k}^3)]^{\oplus+}$ with maximum degree 9. Taking now the following algebra,

$$\mathcal{B} := \mathbb{k}[I_1^c, I_2^c, I_3^c, J_2^c, J_3^c, J_4^c, K_3^c, K_4^c, I_5^c] = \bigoplus_k \mathcal{B}_k \subset \mathcal{A}$$

where \mathcal{B}_k are homogeneous spaces. We check by direct computation (done using Macaulay2 [34] software) that for all $k \leq 9$ we have $\dim(\mathcal{B}_k) = \dim(\mathcal{A}_k)$, where we have from Hilbert series (2)

$$H_{\text{Octa}}(z) = H_{\mathcal{A}}(z) = 1 + z + 3z^2 + 6z^3 + 11z^4 + 18z^5 + 32z^6 + 48z^7 + 75z^8 + 111z^9 + \dots$$

By induction on the degree k of the homogeneous space \mathcal{B}_k , we also check that any subfamily of $\{I_1^c, \dots, I_5^c\}$ is no longer a generating set of \mathcal{A} , so we can conclude. \square

Note here that the invariants J_2^c, J_3^c, J_4^c form a homogeneous system of parameter of the invariant algebra $\mathbb{k}[\text{Adiag}_2]^{\oplus+}$ (using [64, Proposition 5.3.7] once again), with Hilbert series given by:

$$H_{\text{Adiag}_2}(z) = \frac{1}{(1-z^2)(1-z^3)(1-z^4)}.$$

We finally directly deduce from Remark 2.6:

Corollary 3.7. *The algebra of invariant $\mathbb{k}[\text{Adiag}_2]^{\oplus}$ is generated by J_2^c, J_3^c, J_4^c , as well as the field of invariant $\mathbb{k}(\text{Adiag}_2)^{\oplus+}$, which is rational:*

$$\mathbb{k}(\text{Adiag}_2)^{\oplus} = \mathbb{k}(J_2^c, J_3^c, J_4^c).$$

4. RATIONAL INVARIANT FIELD OF TWO SYMMETRIC SECOND ORDER TENSORS

One still considers the canonical quadratic form $\mathbf{q}(\mathbf{x}) := x^2 + y^2 + z^2$ on \mathbb{k}^3 , $\mathbb{k} = \mathbb{R}$ or \mathbb{C} , and defines the associated orthogonal group of linear transformation preserving \mathbf{q} :

$$\mathrm{O}(3, \mathbb{k}) := \{g \in \mathrm{GL}(3, \mathbb{k}), \quad g^t g = I\}$$

Then $\mathrm{SO}(3, \mathbb{k})$ is the subgroup of elements $g \in \mathrm{O}(3, \mathbb{k})$ such that $\det(g) = 1$.

The diagonal representation ρ of $\mathrm{SO}(3, \mathbb{k})$ on $\mathbb{S}^2(\mathbb{k}^3) \oplus \mathbb{S}^2(\mathbb{k}^3)$ is given by

$$\rho(g)(\mathbf{a}, \mathbf{b}) := (g \star \mathbf{a}, g \star \mathbf{b}) = (g \mathbf{a} g^t, g \mathbf{b} g^t)$$

and we consider the associated invariant algebra $\mathbb{k}[\mathbb{S}^2(\mathbb{C}^3) \oplus \mathbb{S}^2(\mathbb{C}^3)]^{\mathrm{SO}(3, \mathbb{C})}$ with its quotient field corresponding to its field of rational invariants $\mathbb{C}(\mathbb{S}^2(\mathbb{C}^3) \oplus \mathbb{S}^2(\mathbb{C}^3))^{\mathrm{SO}(3, \mathbb{C})}$ (see Theorem 2.5).

From Katsylo's result (Theorem 2.7), we know that there exist rational invariants r_1, \dots, r_9 such that

$$\mathbb{C}(\mathbb{S}^2(\mathbb{C}^3) \oplus \mathbb{S}^2(\mathbb{C}^3))^{\mathrm{SO}(3, \mathbb{C})} = \mathbb{C}(r_1, \dots, r_9)$$

and we address the question of finding explicit sets of such invariants, for the complex case as well as for the real case. We already know a minimal integrity basis of $\mathbb{k}[\mathbb{S}^2(\mathbb{k}^3) \oplus \mathbb{S}^2(\mathbb{k}^3)]^{\mathrm{SO}(3, \mathbb{k})}$, given by the 10 polynomial invariants (see [65] for instance):

$$\begin{aligned} I_1 &= \mathrm{tr}(\mathbf{a}), & J_1 &= \mathrm{tr}(\mathbf{b}), & I_2 &= \mathrm{tr}(\mathbf{a}^2), & J_2 &= \mathrm{tr}(\mathbf{b}^2), & K_2 &= \mathrm{tr}(\mathbf{a}\mathbf{b}) \\ I_3 &= \mathrm{tr}(\mathbf{a}^3), & J_3 &= \mathrm{tr}(\mathbf{b}^3), & K_3 &= \mathrm{tr}(\mathbf{a}^2\mathbf{b}), & L_3 &= \mathrm{tr}(\mathbf{a}\mathbf{b}^2), & I_4 &= \mathrm{tr}(\mathbf{a}^2\mathbf{b}^2). \end{aligned}$$

Before we produce an explicit generating set of rational invariants, we need to introduce the *generalized cross product* defined for totally symmetric tensors [52]. Recall here that a totally symmetric tensor of order p is a p linear form

$$\mathbf{S} : (\mathbf{x}_1, \dots, \mathbf{x}_p) \in (\mathbb{k}^3)^p \mapsto \mathbf{S}(\mathbf{x}_1, \dots, \mathbf{x}_p)$$

which is invariant under any permutation of $(\mathbf{x}_1, \dots, \mathbf{x}_p)$.

Definition 4.1 (Generalized cross product). Let ε be the Levi-Civita symbol in \mathbb{k}^3 and $\mathbf{A} \in \mathbb{S}^p(\mathbb{k}^3)$, $\mathbf{B} \in \mathbb{S}^q(\mathbb{k}^3)$ be totally symmetric tensors. Then the totally symmetric tensor $\mathbf{A} \times \mathbf{B}$, of order $p+q-1$, is defined by

$$\mathbf{A} \times \mathbf{B} := (\mathbf{B} \cdot \varepsilon \cdot \mathbf{A})^s \in \mathbb{S}^{p+q-1}(\mathbb{k}^3)$$

where $()^s$ is the total symmetrization and \cdot means the contraction over one subscript.

In coordinates, this gives for two second order symmetric tensors $\mathbf{a}, \mathbf{b} \in \mathbb{S}^2(\mathbb{k}^3)$

$$\begin{aligned} (\mathbf{a} \times \mathbf{b})_{ijk} &= \frac{1}{6} \sum_{\sigma \in \mathfrak{S}_3} b_{\sigma(i)p} \varepsilon_{p\sigma(j)q} a_{q\sigma(k)} \\ &= \frac{1}{6} (b_{ip} \varepsilon_{pjqa} a_{qk} + b_{ip} \varepsilon_{pkqa} a_{qj} + b_{jp} \varepsilon_{piqa} a_{qk} + b_{jp} \varepsilon_{pkqa} a_{qi} + b_{kp} \varepsilon_{piqa} a_{qj} + b_{kp} \varepsilon_{pjqa} a_{qi}), \end{aligned}$$

where \mathfrak{S}_3 is the permutation group of three elements.

We obtain now:

Theorem 4.2. For a pair $(\mathbf{a}, \mathbf{b}) \in \mathbb{S}^2(\mathbb{k}^3) \oplus \mathbb{S}^2(\mathbb{k}^3)$, let $\mathbf{C}_\mathbf{a}$ be the fourth order tensor

$$(7) \quad \mathbf{C}_\mathbf{a} := (\mathbf{a}^2 \times \mathbf{a}) \cdot (\mathbf{a}^2 \times \mathbf{a})$$

and \mathbf{c}, \mathbf{d} the two second order symmetric tensors

$$(8) \quad \mathbf{c} := \mathbf{C}_\mathbf{a} : \mathbf{b} \in \mathbb{S}^2(\mathbb{k}^3), \quad \mathbf{d} := \mathbf{C}_\mathbf{a} : \mathbf{c}^2 \in \mathbb{S}^2(\mathbb{k}^3).$$

Then, the set of 9 rational invariants

$$\begin{aligned} I_1 &= \mathrm{tr} \mathbf{a}, & J_1 &= \mathrm{tr} \mathbf{b}, & K_2 &= \mathrm{tr}(\mathbf{a}\mathbf{b}), & K_3 &= \mathrm{tr}(\mathbf{a}^2\mathbf{b}), \\ r_2 &= \frac{\mathrm{tr}(\mathbf{c}^2)}{\|\mathbf{a}^2 \times \mathbf{a}\|^4}, & r_3 &= \frac{\mathrm{tr}(\mathbf{c}^3)}{\|\mathbf{a}^2 \times \mathbf{a}\|^6}, & s_3 &= \frac{\mathrm{tr}(\mathbf{a}\mathbf{c}^2)}{\|\mathbf{a}^2 \times \mathbf{a}\|^4}, & r_4 &= \frac{\mathrm{tr}(\mathbf{d}^2)}{\|\mathbf{a}^2 \times \mathbf{a}\|^{12}}, & r_5 &= \frac{\mathrm{tr}(\mathbf{a}\mathbf{d}^2)}{\|\mathbf{a}^2 \times \mathbf{a}\|^{12}} \end{aligned}$$

is a weak separating set of the $\mathrm{SO}(3, \mathbb{k})$ space $\mathbb{S}^2(\mathbb{k}^3) \oplus \mathbb{S}^2(\mathbb{k}^3)$.

Remark 4.3. In fact, the second order symmetric tensors \mathbf{c} and \mathbf{d} defined by (8) and the fourth order tensor \mathbf{C}_a given by (7) are *polynomial covariants* of $(\mathbf{a}, \mathbf{b}) \in \mathbb{S}^2(\mathbb{k}^3) \oplus \mathbb{S}^2(\mathbb{k}^3)$, meaning *i)* that their coordinates are polynomial expression in (\mathbf{a}, \mathbf{b}) coordinates and *ii)* that

$$\mathbf{c}(g\mathbf{a}g^t, g\mathbf{b}g^t) = g\mathbf{c}(\mathbf{a}, \mathbf{b})g^t, \quad \mathbf{C}_{g\mathbf{a}g^t} = g \star \mathbf{C}_a,$$

with

$$(g \star \mathbf{C}_a)(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) := \mathbf{C}_a(g^{-1}\mathbf{x}_1, g^{-1}\mathbf{x}_2, g^{-1}\mathbf{x}_3, g^{-1}\mathbf{x}_4).$$

To obtain Theorem 4.2, we first explicitly define some Zariski open set connected to the announced weak separating set. For any $(\mathbf{a}, \mathbf{b}) \in \mathbb{S}^2(\mathbb{k}^3) \oplus \mathbb{S}^2(\mathbb{k}^3)$, one defines

$$\begin{aligned} \widehat{\mathbf{c}} &:= \frac{1}{3} \|\mathbf{a}^2 \times \mathbf{a}\|^2 \mathbf{b} - \mathbf{C}_a : \mathbf{b}, \\ \widehat{\mathbf{d}} &:= \frac{1}{3} \|\mathbf{a}^2 \times \mathbf{a}\|^2 (\mathbf{C}_a : \mathbf{b})^2 - \mathbf{C}_a : ((\mathbf{C}_a : \mathbf{b})^2), \end{aligned}$$

and considers the Zariski open set

$$(9) \quad \mathcal{Z}^c := \left\{ (\mathbf{a}, \mathbf{b}) \in \mathbb{S}^2(\mathbb{k}^3) \oplus \mathbb{S}^2(\mathbb{k}^3), \quad \mathbf{a}^2 \times \mathbf{a} \neq 0 \text{ and } \widehat{\mathbf{d}}^2 \times \widehat{\mathbf{d}} \neq 0 \right\}.$$

As a first step we have:

Lemma 4.4. *Any symmetric second order tensor $\mathbf{a} \in \mathbb{S}^2(\mathbb{k}^3)$ has three distinct eigenvalues if and only if*

$$(10) \quad \mathbf{a}^2 \times \mathbf{a} \neq 0.$$

Proof. Consider the invariant given by the squared norm (which is indeed a norm in the real case)

$$\|\mathbf{a}^2 \times \mathbf{a}\|^2 := (\mathbf{a}^2 \times \mathbf{a})_{ijk} (\mathbf{a}^2 \times \mathbf{a})_{ijk}.$$

Now, taking $p_a(X)$ to be the polynomial characteristic of \mathbf{a} , we check directly that $6\|\mathbf{a}^2 \times \mathbf{a}\|^2$ is the square of the discriminant of $p_a(X)$, so we can conclude. \square

As a second step, let us point out that covariant \mathbf{C}_a given by (7) is strongly linked to the octahedral projector (5). Indeed, by direct computation we have:

Lemma 4.5. *Let $\mathbf{a} = \sum_i \lambda_i \mathbf{e}_i \otimes \mathbf{e}_i \in \text{Diag}_2$, $\mathbf{b} \in \mathbb{S}^2(\mathbb{k}^3)$ and fourth order tensor \mathbf{P} defined by (5). We have*

$$(11) \quad \|\mathbf{a}^2 \times \mathbf{a}\|^2 = \prod_{i < j} (\lambda_i - \lambda_j)^2, \quad \mathbf{C}_a = (\mathbf{a}^2 \times \mathbf{a}) \cdot (\mathbf{a}^2 \times \mathbf{a}) = \frac{1}{3} \|\mathbf{a}^2 \times \mathbf{a}\|^2 \mathbf{P}$$

and,

$$\begin{aligned} \mathbf{c} = \mathbf{C}_a : \mathbf{b} &= \frac{1}{3} \|\mathbf{a}^2 \times \mathbf{a}\|^2 \begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{12} & 0 & b_{23} \\ b_{13} & b_{23} & 0 \end{pmatrix}, \\ \widehat{\mathbf{c}} = \frac{1}{3} \|\mathbf{a}^2 \times \mathbf{a}\|^2 (\mathbf{b} - \mathbf{P} : \mathbf{b}) &= \frac{1}{3} \|\mathbf{a}^2 \times \mathbf{a}\|^2 \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix}, \\ \widehat{\mathbf{d}} = \frac{1}{27} \|\mathbf{a}^2 \times \mathbf{a}\|^6 (\mathbf{c}^2 - \mathbf{P} : \mathbf{c}^2) &= \frac{1}{27} \|\mathbf{a}^2 \times \mathbf{a}\|^6 \begin{pmatrix} b_{13}^2 + b_{12}^2 & 0 & 0 \\ 0 & b_{23}^2 + b_{12}^2 & 0 \\ 0 & 0 & b_{13}^2 + b_{23}^2 \end{pmatrix}. \end{aligned}$$

Proof of Theorem 4.2. Let $(\mathbf{a}, \mathbf{b}) \in \mathbb{S}^2(\mathbb{k}^3) \oplus \mathbb{S}^2(\mathbb{k}^3)$ and $(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}) \in \mathbb{S}^2(\mathbb{k}^3) \oplus \mathbb{S}^2(\mathbb{k}^3)$ in the Zariski open set \mathcal{Z}^c defined by (9) such that

$$I_1(\mathbf{a}, \mathbf{b}) = I_1(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}), \dots, K_2(\mathbf{a}, \mathbf{b}) = K_2(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}), \dots, r_5(\mathbf{a}, \mathbf{b}) = r_5(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}).$$

From lemma 4.4, we can always suppose that

$$\mathbf{a} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \in \text{Diag}_2, \quad \tilde{\mathbf{a}} = \begin{pmatrix} \tilde{\lambda}_1 & 0 & 0 \\ 0 & \tilde{\lambda}_2 & 0 \\ 0 & 0 & \tilde{\lambda}_3 \end{pmatrix} \text{Diag}_2, \quad \text{tr}(\mathbf{a}) = \text{tr}(\tilde{\mathbf{a}}).$$

Thus, from lemma 4.5 we have $\mathbf{P} : \mathbf{a} = \mathbf{a}$ and $\mathbf{P} : \tilde{\mathbf{a}} = \tilde{\mathbf{a}}$ with projector (5) equal to

$$\mathbf{P} = \frac{3\mathbf{C}_{\mathbf{a}}}{\|\mathbf{a}^2 \times \mathbf{a}\|^2} = \frac{3\mathbf{C}_{\tilde{\mathbf{a}}}}{\|\tilde{\mathbf{a}}^2 \times \tilde{\mathbf{a}}\|^2}.$$

We deduce that

$$r_k(\mathbf{a}, \mathbf{b}) = J_k^c(\mathbf{P} : \mathbf{b}) = J_k^c(\mathbf{P} : \tilde{\mathbf{b}}), \quad k = 1, 2, 3,$$

with octahedral invariants J_k^c defined in theorem 3.5. Thus, from theorem 2.2, we can write $\mathbf{P} : \tilde{\mathbf{b}} = \gamma \star (\mathbf{P} : \mathbf{b})$ for some $\gamma \in \mathbb{O}$.

As Diag_2 is a stable \mathbb{O} space, both $\gamma \star \mathbf{a}$ and $\gamma \star \tilde{\mathbf{a}}$ are still in Diag_2 , so we can now suppose that

$$\mathbf{b}_1 := \mathbf{P} : \mathbf{b} = \mathbf{P} : \tilde{\mathbf{b}} = \sum_{i \neq j} b_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \in \mathbb{S}^2(\mathbb{k}^3)$$

and we can write from lemma 4.5

$$\begin{aligned} s_3(\mathbf{a}, \mathbf{b}) &= \text{tr}(\mathbf{a}(\mathbf{b}_1)^2) = \text{tr}(\tilde{\mathbf{a}}(\mathbf{b}_1)^2) = \alpha, \\ r_5(\mathbf{a}, \mathbf{b}) &= \text{tr}(\mathbf{a}(\mathbf{h}_1)^2) = \text{tr}(\tilde{\mathbf{a}}(\mathbf{h}_1)^2) = \beta, \quad \mathbf{h}_1 := \mathbf{P} : (\mathbf{b}_1^2) \end{aligned}$$

so that $\lambda_i, \tilde{\lambda}_j$ are solutions of the linear system

$$(\mathcal{S}) : \begin{cases} (b_{12}^2 - b_{23}^2)x + (b_{12}^2 - b_{13}^2)y &= \alpha - (b_{12}b_{13} - b_{23}^2)t \\ b_{13}^2(b_{23}^2 - b_{12}^2)x + b_{23}^2(b_{13}^2 - b_{12}^2)y &= \beta - b_{12}^2(b_{13}^2 + b_{23}^2)t \end{cases}, \quad x + y + z = t$$

where (\mathcal{S}) has non-zero determinant $(b_{13}^2 - b_{12}^2)(b_{12}^2 - b_{23}^2)(b_{23}^2 - b_{13}^2)$. Indeed, the eigenvalues of

$$\hat{\mathbf{d}} = \frac{1}{27} \|\mathbf{a}^2 \times \mathbf{a}\|^6 (\mathbf{b}_1^2 - \mathbf{P} : (\mathbf{b}_1^2))$$

are all distincts (see Lemma 4.5). We thus deduce that $\mathbf{a} = \tilde{\mathbf{a}}$.

Finally, let

$$\mathbf{b} = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix} + \mathbf{P} : \mathbf{b}, \quad \tilde{\mathbf{b}} = \begin{pmatrix} \tilde{\mu}_1 & 0 & 0 \\ 0 & \tilde{\mu}_2 & 0 \\ 0 & 0 & \tilde{\mu}_3 \end{pmatrix} + \mathbf{P} : \mathbf{b}, \quad t' := \text{tr}(\mathbf{b}) = \text{tr}(\tilde{\mathbf{b}}).$$

From $K_2(\mathbf{a}, \mathbf{b}) = \text{tr}(\mathbf{a}\mathbf{b}) = \text{tr}(\mathbf{a}\tilde{\mathbf{b}}) = \alpha'$ and $K_3(\mathbf{a}, \mathbf{b}) = \text{tr}(\mathbf{a}^2\mathbf{b}) = \text{tr}(\mathbf{a}^2\tilde{\mathbf{b}}) = \beta'$ we deduce that $\mu_i, \tilde{\mu}_j$ are solutions of the linear system

$$(12) \quad (\mathcal{S}') : \begin{cases} (\lambda_1 - \lambda_3)x + (\lambda_2 - \lambda_3)y &= \alpha' - \lambda_3 t' \\ (\lambda_1^2 - \lambda_3^2)x + (\lambda_2^2 - \lambda_3^2)y &= \beta' - \lambda_3^2 t' \end{cases}, \quad x + y + z = t'$$

where (\mathcal{S}') has non-vanishing determinant (see (11)), so that $\mathbf{b} = \tilde{\mathbf{b}}$, which concludes the proof. \square

Remark 4.6. When taking the standard representation of $\text{SO}(3, \mathbb{k})$ on $\mathbb{S}^2(\mathbb{k}^3)$ (see (1)), we know that the vector space Diag_2 is in fact a *linear slice* [39, 48] for generic tensors (the genericity condition being of course useless for $\mathbb{k} = \mathbb{R}$): for any $\mathbf{a} \in \mathbb{S}^2(\mathbb{k}^3)$ with three distinct eigenvalues, there exists $g \in \text{SO}(3, \mathbb{k})$ such that $g \star \mathbf{a} \in \text{Diag}_2$.

From lemma 2.9 we directly have:

Corollary 4.7. *The invariant field $\mathbb{C}(\mathbb{S}^2(\mathbb{C}^3) \oplus \mathbb{S}^2(\mathbb{C}^3))^{\text{SO}(3, \mathbb{C})}$ is rational and generated by the minimal set $\{I_1, J_1, K_2, K_3, r_2, r_3, r_4, r_5, s_3\}$ of 9 rational invariants given in Theorem 4.2:*

$$\mathbb{C}(\mathbb{S}^2(\mathbb{C}^3) \oplus \mathbb{S}^2(\mathbb{C}^3))^{\text{SO}(3, \mathbb{C})} = \mathbb{C}(I_1, J_1, K_2, K_3, r_2, r_3, r_4, r_5, s_3).$$

A remarkable fact is that all the rational invariants given by Theorem 4.2 are such that

$$r(\mathbf{v}) \in \mathbb{R}(\mathbb{S}^2(\mathbb{R}^3) \oplus \mathbb{S}^2(\mathbb{R}^3)), \quad \forall \mathbf{v} \in \mathbb{S}^2(\mathbb{R}^3) \oplus \mathbb{S}^2(\mathbb{R}^3).$$

Following the proof of [51, Lemme 6.14] we thus get:

Corollary 4.8. *The invariant field $\mathbb{R}(\mathbb{S}^2(\mathbb{R}^3) \oplus \mathbb{S}^2(\mathbb{R}^3))^{\text{SO}(3, \mathbb{R})}$ is rational and generated by the minimal set $\{I_1, J_1, K_2, K_3, r_2, r_3, r_4, r_5, s_3\}$ of 9 rational invariants given in Theorem 4.2:*

$$\mathbb{R}(\mathbb{S}^2(\mathbb{R}^3) \oplus \mathbb{S}^2(\mathbb{R}^3))^{\text{SO}(3, \mathbb{R})} = \mathbb{R}(I_1, J_1, K_2, K_3, r_2, r_3, r_4, r_5, s_3).$$

5. RATIONAL INVARIANTS OF n SYMMETRIC SECOND ORDER TENSORS

Using the results of previous section leads to a $\mathrm{SO}(3, \mathbb{k})$ weak separating set of $n\mathbb{S}^2(\mathbb{k}^3)$ ($n \geq 3$), where we consider the standard diagonal representation. Recall that the cross product is given by Definition 4.1 and that the notations \cdot and $:$ stand for contractions, respectively over one subscript and over two subscripts.

Theorem 5.1. For $(\mathbf{a}, \mathbf{b}, \mathbf{c}_1, \dots, \mathbf{c}_k) \in n\mathbb{S}^2(\mathbb{k}^3)$ ($n \geq 3$), let $\mathbf{C}_\mathbf{a}$ be the fourth order tensor

$$\mathbf{C}_\mathbf{a} = (\mathbf{a}^2 \times \mathbf{a}) \cdot (\mathbf{a}^2 \times \mathbf{a})$$

and \mathbf{c}, \mathbf{d} the two second order symmetric tensors

$$\mathbf{c} := \mathbf{C}_\mathbf{a} : \mathbf{b} \in \mathbb{S}^2(\mathbb{k}^3), \quad \mathbf{d} := \mathbf{C}_\mathbf{a} : \mathbf{c}^2 \in \mathbb{S}^2(\mathbb{k}^3).$$

Then the set of $6n - 3$ rational invariants

$$\begin{aligned} I_1 &= \mathrm{tr}(\mathbf{a}), & J_1 &= \mathrm{tr}(\mathbf{b}), & K_2 &= \mathrm{tr}(\mathbf{a}\mathbf{b}), & K_3 &= \mathrm{tr}(\mathbf{a}^2\mathbf{b}), & I_{1,k} &= \mathrm{tr}(\mathbf{c}_k), \\ I_{2,k} &= \mathrm{tr}(\mathbf{a}\mathbf{c}_k), & J_{2,k} &= \mathrm{tr}(\mathbf{b}\mathbf{c}_k), & I_{3,k} &= \mathrm{tr}(\mathbf{a}^2\mathbf{c}_k), & J_{3,k} &= \mathrm{tr}(\mathbf{a}\mathbf{b}\mathbf{c}_k), & I_{5,k} &= \mathrm{tr}((\mathbf{a}\mathbf{b} - \mathbf{b}\mathbf{a})^2\mathbf{c}_k), \\ r_2 &= \frac{\mathrm{tr}(\mathbf{c}^2)}{\|\mathbf{a}^2 \times \mathbf{a}\|^4}, & r_3 &= \frac{\mathrm{tr}(\mathbf{c}^3)}{\|\mathbf{a}^2 \times \mathbf{a}\|^6}, & r_4 &= \frac{\mathrm{tr}(\mathbf{d}^2)}{\|\mathbf{a}^2 \times \mathbf{a}\|^{12}}, & s_3 &= \frac{\mathrm{tr}(\mathbf{a}\mathbf{c}^2)}{\|\mathbf{a}^2 \times \mathbf{a}\|^4}, & r_5 &= \frac{\mathrm{tr}(\mathbf{a}\mathbf{d}^2)}{\|\mathbf{a}^2 \times \mathbf{a}\|^{12}} \end{aligned}$$

is a weak separating set of the $\mathrm{SO}(3, \mathbb{k})$ space $n\mathbb{S}^2(\mathbb{k}^3)$, with associated Zariski open set $\mathcal{Z}^c \oplus (n-2)\mathbb{S}^2(\mathbb{k}^3)$, where \mathcal{Z}^c is defined by (9).

Before we get to the proof, we need:

Lemma 5.2. Let (\mathbf{a}, \mathbf{b}) in the Zariski open set \mathcal{Z}^c defined by (9). Then \mathbf{a}, \mathbf{b} do not share any common eigenvector and $(\mathbf{q}, \mathbf{a}, \mathbf{b}, (\mathbf{a}\mathbf{b})^s, [\mathbf{a}, \mathbf{b}]^2)$ is a basis of $\mathbb{S}^2(\mathbb{k}^3)$.

Proof. Let us suppose that \mathbf{a} and \mathbf{b} have one common eigenvector, so we can write

$$\mathbf{a} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & b_{23} \\ 0 & b_{23} & b_{33} \end{pmatrix}$$

and we have a non trivial element g in the isotropy group of (\mathbf{a}, \mathbf{b})

$$g := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then we obtain from Lemma 4.5

$$\widehat{\mathbf{d}} = k\|\mathbf{a}^2 \times \mathbf{a}\|^6 \begin{pmatrix} 0 & 0 & 0 \\ 0 & b_{23}^2 & 0 \\ 0 & 0 & b_{23}^2 \end{pmatrix} \Rightarrow \widehat{\mathbf{d}}^2 \times \widehat{\mathbf{d}} = 0$$

so that $(\mathbf{a}, \mathbf{b}) \notin \mathcal{Z}^c$. We deduce that any $(\mathbf{a}, \mathbf{b}) \in \mathcal{Z}^c$ has a trivial isotropy group (otherwise \mathbf{a} and \mathbf{b} have a common eigenvector), so we can conclude using [18, Lemma 4.4]. \square

Proof of Theorem 5.1. Let us consider $(\mathbf{a}, \mathbf{b}, \mathbf{c}_1, \dots, \mathbf{c}_k)$ and $(\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}, \widetilde{\mathbf{c}}_1, \dots, \widetilde{\mathbf{c}}_k)$ in the open Zariski set $\mathcal{Z}_c^2 \oplus (n-2)\mathbb{S}^2(\mathbb{k}^3)$ so that all the rational invariants from the theorem statement have equal evaluations.

First of all, from Theorem 4.2 we obtain some $g \in \mathrm{SO}(3, \mathbb{k})$ such that $(\mathbf{a}, \mathbf{b}) = (g \star \mathbf{a}, g \star \mathbf{b})$, and we can suppose from now on that $(\mathbf{a}, \mathbf{b}) = (\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}})$.

Furthermore, from corollary 4.4 we can suppose that

$$\mathbf{a} = \widetilde{\mathbf{a}} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Finally, as invariants $I_{1,k}, I_{2,k}, J_{2,k}, I_{3,k}, J_{3,k}, I_{5,k}$ give linear projections of any tensor $\mathbf{c}_k \in \mathbb{S}^2(\mathbb{k}^3)$ on $(\mathbf{q}, \mathbf{a}, \mathbf{b}, (\mathbf{a}\mathbf{b})^s, [\mathbf{a}, \mathbf{b}]^2)$, we conclude using Lemma 5.2. \square

As from the previous results with the $\mathrm{SO}(3, \mathbb{k})$ space $\mathbb{S}^2(\mathbb{k}^3) \oplus \mathbb{S}^2(\mathbb{k}^3)$ (Corollary 4.7 and 4.8), we can deduce:

Corollary 5.3. The invariant field $\mathbb{k}(n\mathbb{S}^2(\mathbb{k}^3))^{\mathrm{SO}(3, \mathbb{k})}$ is rational and generated by the minimal set of $6n - 3$ rational invariants given in Theorem 5.1, for either $\mathbb{k} = \mathbb{R}$ or $\mathbb{k} = \mathbb{C}$.

6. RATIONAL INVARIANTS OF n QUARTIC BINARY FORMS

All results previously stated concern $\mathrm{SO}(3, \mathbb{k})$ tensor spaces with \mathbb{k} either \mathbb{R} or \mathbb{C} . Concerning binary forms, Katsylo has obtained the rationality of all $\mathrm{SL}(2, \mathbb{C})$ representations [40, 39]. Maeda has explicitated generators for the $\mathrm{SL}(2, \mathbb{C})$ -invariant field of binary octavics [48].

Let us now address the question of effective rationality of $\mathrm{SL}(2, \mathbb{C})$ spaces of quartic forms. We give in corollary 6.1 a minimal set of generators for the $\mathrm{SL}(2, \mathbb{C})$ -invariant field of n binary quartics.

First define S_n to be the complex vector space of n th degree binary forms

$$\mathbf{f}(\boldsymbol{\xi}) = a_0 u^n + a_1 u^{n-1} v + \dots + a_n v^n, \quad \boldsymbol{\xi} := (u, v) \in \mathbb{C}^2, \quad a_i \in \mathbb{C}.$$

Such a space is naturally endowed with the $\mathrm{SL}(2, \mathbb{C})$ representation given by

$$(\gamma \star \mathbf{f})(\boldsymbol{\xi}) := \mathbf{f}(\gamma^{-1} \boldsymbol{\xi}), \quad \gamma \in \mathrm{SL}(2, \mathbb{C}).$$

There is a deep connection between $\mathrm{SO}(3, \mathbb{C})$ and its universal cover $\mathrm{SL}(2, \mathbb{C})$, so that the $\mathrm{SL}(2, \mathbb{C})$ representation on the space S_{2n} of $2n$ th degree binary forms is equivariant to the $\mathrm{SO}(3, \mathbb{C})$ representation on the space of n th order *harmonic polynomials* (see [18] for more details).

To obtain an explicit equivariant isomorphism, let us first consider the adjoint representation Ad of $\mathrm{SL}(2, \mathbb{C})$ on its Lie algebra $\mathfrak{sl}(2, \mathbb{C})$

$$\mathrm{Ad}_\gamma(m) := \gamma m \gamma^{-1}.$$

As it preserves the quadratic form $\det(m)$ defined on $m \in \mathfrak{sl}(2, \mathbb{C})$, we have

$$\mathrm{Ad}_\gamma \in \mathrm{SO}(3, \mathbb{C}), \quad \forall \gamma \in \mathrm{SL}(2, \mathbb{C}).$$

Take now *Cartan's map* [14, p. 48],

$$\phi : \mathbb{C}^2 \rightarrow \mathbb{C}^3, \quad (u, v) \mapsto \left(\frac{u^2 + v^2}{2}, \frac{u^2 - v^2}{2i}, iuv \right)$$

which induces an equivariant isomorphism

$$\phi^* : \mathcal{H}_2(\mathbb{C}^3) \longrightarrow S_4, \quad h \mapsto h \circ \phi$$

where $\mathcal{H}_2(\mathbb{C}^3)$ is the space of degree 2 harmonic polynomials, also isomorphic to the space $\mathbb{H}^2(\mathbb{C}^3)$ of second order symmetric and *traceless* tensors (see [30] for instance):

$$\mathbf{a} \in \mathbb{H}^2(\mathbb{C}^3) \mapsto \mathbf{p}(\mathbf{x}) := \mathbf{a}(\mathbf{x}, \mathbf{x}) \in \mathcal{H}_2(\mathbb{C}^3), \quad \Delta \mathbf{p} = 2 \mathrm{tr}(\mathbf{a}) = 0.$$

Using the inverse map ϕ^{-*} of ϕ^* , this defines an equivariant isomorphism

$$\phi^{-*} : nS_4 \longrightarrow n\mathbb{H}^2(\mathbb{C}^3).$$

In the case of a single quartic form

$$\mathbf{f} = a_0 u^4 + a_1 u^3 v + a_2 u^2 v^2 + a_3 u v^3 + a_4 v^4$$

we simply have

$$\begin{aligned} \phi^{-*}(\mathbf{f}) = & \left(\frac{a_2}{3} + a_0 + a_4 \right) \mathbf{e}_{11} + \left(\frac{a_2}{3} - a_0 - a_4 \right) \mathbf{e}_{22} - \frac{2a_2}{3} \mathbf{e}_{33} \\ & + i(a_0 - a_4) \mathbf{e}_{12} - \frac{1}{2} i(a_1 + a_3) \mathbf{e}_{13} + \frac{1}{2} (a_1 - a_3) \mathbf{e}_{23} \end{aligned}$$

where \mathbf{e}_{ij} are given by (3) and (4).

From Theorem 5.1 we finally get:

Corollary 6.1. *Let us consider the representation $(nS_4, \mathrm{SL}(2, \mathbb{C}))$ for $n \geq 2$. Then the invariant field $\mathbb{C}(nS_4)^{\mathrm{SL}(2, \mathbb{C})}$ is rational and generated by the minimal set of $5n - 3$ rational invariants given by*

$$\begin{aligned} K_2 \circ \phi^{-*}, \quad K_3 \circ \phi^{-*}, \quad I_{2,k} \circ \phi^{-*}, \quad J_{2,k} \circ \phi^{-*}, \quad I_{3,k} \circ \phi^{-*}, \quad J_{3,k} \circ \phi^{-*}, \quad I_{5,k} \circ \phi^{-*}, \\ r_2 \circ \phi^{-*}, \quad r_3 \circ \phi^{-*}, \quad r_4 \circ \phi^{-*}, \quad s_3 \circ \phi^{-*}, \quad r_5 \circ \phi^{-*} \end{aligned}$$

where K_2, \dots, r_5 are the rational invariants given in Theorem 5.1.

7. CONCLUSION

Considering both the complex $\mathbb{k} = \mathbb{C}$ and the real $\mathbb{k} = \mathbb{R}$ cases, we have obtained explicit minimal separating sets of $6n - 3$ rational invariants for the $\mathrm{SO}(3, \mathbb{k})$ linear representation on n copies of $\mathbb{S}^2(\mathbb{k}^3)$ (\mathbb{k} vector spaces second order symmetric tensors, Theorem 5.1). We have shown the rationality of the invariant field $\mathbb{k}(n\mathbb{S}^2(\mathbb{C}^3))^{\mathrm{SO}(3, \mathbb{k})}$ (Corollary 5.3). Thanks to Cartan map, we have obtained a minimal set of $5n - 3$ rational invariants for the $\mathrm{SL}(2, \mathbb{C})$ invariant field $\mathbb{C}(n\mathbb{S}_4)^{\mathrm{SL}(2, \mathbb{C})}$ of n quartic forms (Corollary 6.1), leading to an explicit version of the theoretical rationality result from Katsylo [39].

A first issue (Theorem 4.2) was that a weak separating set of the $\mathrm{SO}(3, \mathbb{k})$ space $\mathbb{S}^2(\mathbb{k}^3) \oplus \mathbb{S}^2(\mathbb{k}^3)$ consists in 9 rational invariants, the rationality of the invariant field $\mathbb{C}(\mathbb{S}^2(\mathbb{C}^3) \oplus \mathbb{S}^2(\mathbb{C}^3))^{\mathrm{SO}(3, \mathbb{C})}$ and $\mathbb{R}(\mathbb{S}^2(\mathbb{R}^3) \oplus \mathbb{S}^2(\mathbb{R}^3))^{\mathrm{SO}(3, \mathbb{R})}$ having been obtained as Corollaries 4.7 and 4.8. As a by-product, a minimal integrity basis of $\mathbb{k}[\mathbb{S}^2(\mathbb{k}^3)]^{\mathbb{O}^+}$, for orientation preserving octahedral group \mathbb{O}^+ , has been explicated in Theorem 3.5, and a homogeneous system of parameters of the invariant algebra $\mathbb{k}[\mathbb{S}^2(\mathbb{k}^3)]^{\mathbb{O}}$ has been provided.

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