HIGH FREQUENCY RESOLVENT ESTIMATES FOR THE MAGNETIC LAPLACIAN ON NON COMPACT MANIFOLDS

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Abstract. We consider the Schrödinger operator with a magnetic perturbation on non compact manifolds with infinite volume. We prove optimal estimates for the resolvent operator at high frequencies in Besov-type spaces. In the general trapping case we obtain the usual exponential blow-up, while when the resolvent is localised on the manifold end, away from possible trapped trajectories, we obtain the optimal bound by the inverse square root of the spectral parameter.

Contents

1. Introduction

In this work we prove estimates for the limiting value of the resolvent of a Laplace-Beltrami operator with lower order perturbations. This operator describes the energy of a system of two quantum particles subject to an electric, as well as a magnetic potential. These limiting resolvents are central objects in the study of the dynamics of the system thanks to the relation given by Stone's formula [15, Theorem VII.13]. Indeed, let P_m the magnetic Laplacian we will consider. We will define precisely this operator in Section 1.2. By selfadjointness of P_m, its resolvent $(P_m - z)^{-1}$ is well defined when $z \in \mathbb{C} \setminus \mathbb{R}$. The limit at the boundary of its domain is then

$$\text{s-}\lim_{\varepsilon' \to 0} (P_m - \lambda^2 + i\varepsilon')^{-1},$$

where s- lim denotes the limit in the strong operator topology. By Stone's formula

$$e^{itP_m} = \frac{1}{2i\pi} \int_0^\infty e^{it\lambda} \,\text{s-}\lim_{\varepsilon \to 0} ((P_m - \lambda^2 + i\varepsilon)^{-1} - (P_m - \lambda^2 - i\varepsilon)^{-1})\, d\lambda,$$

hence we can study the dynamics of the Schrödinger evolution via the limiting resolvents. Here, we consider a system on a non compact manifold with infinite volume which is a long range perturbation of a conical or hyperbolic end, and we study the resolvent in the high frequency regime, that is for λ large.

The main feature of this work is that we include a perturbation of order one in the Laplce-Beltrami operator, which models the presence of a mangetic field.

Since we are considering a singular limit for the resolvent, we can not study this quantity in the topology of bounded L^2 spaces, where the norm of the limit would blow up. However, we remark that to study the operator $(P_m - \lambda^2 + i\varepsilon')^{-1}$ is equivalent to solving the equation $(P_m - \lambda^2 + i\varepsilon')u = f$ when ε' tends to zero, which is a problem linked to the Fourier transform restriction properties, as one can see directly by solving for u in $(-\Delta - \lambda^2)u = f$. Since the restriction of the Fourier transform to a surface is well defined in weighted L^2 spaces, this topology represents a suitable setting in which to study limiting resolvents. In this work, by considering Besov-type spaces (defined in Section 1.3), we will be able to go beyond the usual restrictions on the weights imposed by this approach.

The aim is to obtain estimates for large λ in all generality, without assumptions on the geodesic flow, to which the resolvent is sensitive in the case of λ large. Without geodesic information we expect an exponential loss in the estimates, contrary to the the polynomial blowup coming from the low frequencies [16]. This is indeed what we will obtain: the resolvent on the entire manifold has an exponential bound and the cutoff resolvent near infinity has a polynomial decay. This is due to the fact that outside of a sufficiently large compact region there is no trapping effect and hence no loss in the estimates.

We now give some notions to state our result. The manifold M that we consider is of the form

$$M \underset{\text{diffeo}}{\simeq} K \cup (R, +\infty) \times S \tag{1.1}$$

with K a compact region and an unbounded region $(R, +\infty) \times S$ of product form with a product metric. The main examples one must keep in mind for $(R, +\infty) \times S$ are a long range perturbation of a cone or of an hyperbolic manifold, as described in Examples 1.8 and 1.9.

On the manifold M we consider P_m, the Laplace-Beltrami operator with perturbations of order one and zero decaying at infinity respectively like

$$\frac{1}{r^{2+}}, \quad \frac{1}{r^{0+}}, \quad r \in (R, +\infty). \tag{1.2}$$

The main result is the following. Precise definitions are postponed to the end of this introduction.

Theorem 1.1. *Let (M, G_M) a non compact Riemannian manifold, satisfying (1.1) and P_m the perturbed Laplace-Beltrami operator satisfying (1.2). Let $\| \cdot \|_{H^1, B^*_{>R}}, \| \cdot \|_{B_{>R}}$ weighted norms on the manifold end $(R, +\infty) \times S$ (defined in Section 1.3). Let $R < R_1 < R_2$. For any $u \in H^2(M), \lambda \gg 1$ and for any $\varepsilon' > 0$ there exists a constant $C > 0$, independent of λ and ε', such that*

$$\begin{aligned} \|u\|^2_{H^1(M \setminus (R_1, +\infty) \times S)} + \|u\|^2_{H^1, B^*_{>R}} \leq & O(e^{\lambda C}) \|(P_m - \lambda^2 + i\varepsilon')u\|^2_{L^2(M \setminus (R_2, +\infty) \times S))} \\ & + O(e^{\lambda C}) \|(P_m - \lambda^2 + i\varepsilon')u\|^2_{B_{>R}}. \end{aligned}$$

In the proof of the result we will see that the contribution of the exponential terms $O(e^{\lambda C})$ is due only to norms on compact regions of the manifold. We therefore obtain a corollary for the cutoff resolvent.

Corollary 1.2. *Under the assumptions of the previous theorem, there exists $R_3 > R$ such that for χ, smooth cutoff supported on $(R_3, +\infty) \times S$, and for any $\varepsilon' > 0$, for any $u \in H^2(M)$ it holds*

$$\|\chi u\|_{B^*_{>R}} \leq O(\lambda^{-1}) \|(P_m - \lambda^2 + i\varepsilon')u\|_{B_{>R}} .$$

In particular, for any $\mu > 0$ it holds

$$\|\chi r^{-1/2-\mu}(P_m - \lambda^2 + i\varepsilon)^{-1} r^{-1/2-\mu}\chi\|_{L^2 \to L^2} = O(\lambda^{-1}).$$

In these results, we notably include a magnetic potential and obtain optimal high frequency estimates which were previously restricted to the unperturbed case: either without geometric perturbation [7], or without the magnetic one [6]. In particular, for the cutoff resolvent we obtain in the corollary the same behavior of the free case. Moreover, we apply to the geometric case the Besov-type norms defined in Section 1.3, which were originally introduced in the euclidean case [2, 3, 7]. These norms are defined with weights decaying like $r^{-1/2}$, instead of $r^{-1/2-\mu}, \mu > 0$ as one needs to consider in weighted L^2 spaces [1, Section 4].

On top of considering a more general operator with differential perturbations and Besov-type norms with sharper weights, in this work we revisit some of the arguments in [6] trying to give a rather complete exposition of the whole strategy. We also remark that we combine the Carleman approach, which is robust enough to be adapted to our perturbed case, to the use of Besov-type norms that, as we mentioned, were introduced for more classical Hamiltonians.

In the theorem we obtain a uniform exponential bound on the resolvent of the Laplace-Beltrami operator with an order one perturbation. Having this type of resolvent estimate can be applied, as in [18] or [6], to prove that there is a strip of exponential size around the positive real axis which is free of resonances. In other words, this tells us that resonances can not accumulate exponentially fast and hence gives a lower bound on their width. For complex resonances their imaginary part gives the rate of decay of a solution associated to resonant initial data, hence the width of a resonance carries dynamical information. For example, in [5] the existence of a resonance free strip is applied to prove logarithmic time decay of solutions of the wave equation.

To treat the limiting resolvents, we will not use Mourre theory, which is more well suited for the case of low frequencies ($\lambda \ll 1$). Indeed, this technique can be used in the high frequency regime under additional assumptions on the geodesic flow, which we do not make here. The main strategy is the same as in [6] which in turn is inspired by the works of Burq ([4], [5]): we divide the manifold in two parts, a bounded region and an infinite end, and we treat the two separately. In the compact region we use are Carleman estimates, which are stable by order one perturbations and hence suitable for our operator P_m. In the unbounded region we do not use any complex theory, but we rather exploit the equation $(P_m - \lambda^2 + i\varepsilon')u = f$ and bound the solution u by the source term f. We will use simple identities like

$$Im(Bu, Cu) = (u, \frac{[B,C]}{2i}u), \quad Re(Bu, Cu) = \left(\frac{BC + CB}{2}u, u\right)$$

for B, C symmetric operators.

Remark 1.3. *For the estimates on the unbounded region $M \setminus K$ we use a slightly different approach than in [6]. In particular the strategy used to treat this region (Section 2) is frequency independent, in the sense that the proof holds for intermediate and high frequencies. In this section we replace the smallness of a semiclassical parameter $h = \lambda^{-1}$ with the decay of the radial variable. In the last section, when combining the two regions, we will need to assume $\lambda \gg 1$.*

The paper is organized as follows: we conclude the Introduction with some definitions, in Section 2 we treat the unbounded region, in Section 3 we show how to use Carleman estimates to treat the compact region, in Section 4 we use an argument presented first in [5] to conclude with the proofs of Theorem 1.1 and Corollary 1.2.

1.1. **Definition of the geometric framework.** Let S a compact $n-1$ dimensional Riemannian manifold, around a point $\omega \in S$ we will denote by $\theta \in \mathbb{R}^{n-1}$ the local coordinates. We will equip S with two different metrics $(S, \overline{g})$ and $(S, g(r))$. We will use the product manifold

$$((R, +\infty) \times S, dG) \quad dG = dr^2 + l(r)^{-2} g(r)$$

to model the infinite ends of our manifold M.

We consider a Riemannian manifold M, a compact set $K \subset M$ and a diffeomorphism Ω

$$\Omega : M \setminus K \to (R, +\infty) \times S$$

such that the metric on M is given by

$$G_M := \Omega^*(G) = \Omega^*(dr^2 + l(r)^{-2} g(r))$$

for some smooth function $l : (R, +\infty) \to \mathbb{R}^+$ verifying

$$-\frac{l'(r)}{l(r)} \geq \frac{c}{r}, \qquad -\frac{l'(r)}{l(r)} \in L^\infty((R, +\infty) \times S). \tag{1.3}$$

Remark 1.4. *Integrating the inequality in (1.3) we obtain*

$$l(r) \lesssim \frac{1}{r} \tag{1.4}$$

hence $l(r) \to 0$ as $r \to \infty$. Then l' is bounded for large enough r and the L^∞ condition in (1.3) implies $|l'(r)|r \in L^\infty((R, +\infty) \times S)$.

Remark 1.5. *With an abuse of notation we will write G for both the metric on the product $(R, +\infty) \times S$ and the manifold end $M \setminus K$.*

Remark 1.6. *Let $k^* \in \mathbb{N}$. We will denote by ϕ a smooth cutoff on $(R, +\infty)$ for which we will assume $\operatorname{supp} \phi \subset (2^{k^*+3}R, +\infty)$. Such a function can be identified via Ω with a cutoff on $M \setminus K$ of the form $\chi = \phi \circ \Omega$.*

We assume that $g(r)$, the angular metric, is a long range perturbation of a fixed metric in the following sense.

Let f a function of r with values in the space of sections $\Gamma(T^p_q(S))$:

$$f : (R, +\infty) \to \Gamma(T^p_q(S))$$
$$r \mapsto f(r),$$

represented, with respect to a basis of the tensor product $(\otimes T_\omega(S))^p \otimes (\otimes T^*_\omega(S))^q$, by the coefficients $f^{i_1,\dots i_p}_{j_1,\dots,j_q}(r,\theta)$ around a point ω. We define a topology on $\Gamma(T^p_q(S))$ given by the seminorms

$$N^{pq}_{m,J}(f) = \sum_{|\alpha|\leq m} \|\partial^\alpha f^{i_1,\dots i_p}_{j_1,\dots,j_q}\|_{L^\infty(J)}$$

with J a compact subset of the coordinate patch on $\mathbb{R}^{n-1}$.

Definition 1.7. *Let $f : (R,+\infty) \to \Gamma(T^p_q(S))$ a smooth function and $\nu \in \mathbb{R}$, then $f \in S^{-\nu}$ if*

$$N^{pq}_{m,J}(\partial^l_r f(r)) \lesssim \langle r\rangle^{-\nu-l} \quad \text{for any } m,l \in \mathbb{N},\ J\text{compact set.}$$

For fixed r, $g(r)$ is a metric on S and we have the smooth family of metrics

$$g : (R,+\infty) \to \Gamma(T^0_2(S))$$
$$r \mapsto g(r)$$

where $\Gamma(T^0_2(S))$ are the sections of the tensor bundle $T^0_2(S) = T^*(S) \otimes T^*(S)$ and $g(r,\omega)$ is a bilinear form on the tangent space $T_\omega(S)$.

Given the topology defined on $\Gamma(T^0_2(S))$, we assume

$$g(r) - \overline{g} \in S^{-\nu} \text{ for some } \nu > 0. \tag{1.5}$$

We point out two examples of infinite end the reader should keep in mind throughout this work. All the assumptions we make on l are satisfied by these two examples.

Example 1.8 (Asymptotically conical end)**.** *In the particular case of a fixed metric $g(r) = \overline{g}$ and $l(r) = r^{-1}$ we obtain on $(R,+\infty)\times S$ the following metric*

$$dr^2 + r^2\overline{g}$$

which is an exact conic end. If we replace $\overline{g}$ with a perturbation of it, $g(r)$, we obtain

$$dr^2 + r^2 g(r),$$

which is an asymptotically conical end.

When $\overline{g}$ is the metric on the sphere $\mathbb{S}^{n-1}$ we obtain a long range perturbation of the euclidean space $\mathbb{R}^n$.

Example 1.9 (Asymptotically hyperbolic end)**.** *Taking $l(r) = e^{-r}$ and again $g(r) = \overline{g}$, if we equip $(R,+\infty)\times S$ with the metric*

$$dr^2 + e^{2r}\overline{g}$$

we obtain an hyperbolic end. If we take a perturbed metric $g(r)$ we have

$$dr^2 + e^{2r} g(r)$$

an asymptotically hyperbolic end.

Remark 1.10. *The asymptotically conical end presented in Example 1.8 includes the so-called scattering metrics, terminology introduced by Melrose [14] and used in several works (for example [13], [9], [10], [17], [11]). Scattering metrics are a particular case of the metric in Example 1.8 when $g(r)$ is the Taylor expansion of a smooth metric around zero.*

Analogously, the asymptotically hyperbolic end of Example 1.9 includes the type of non compact manifolds modeled as the interior of a compact manifold via a boundary defining

function, similarly to the scattering manifolds just mentioned (see for example [8, Chapter 5]). Again, a metric such as the one defined in [8] can be obtained by Example 1.9 replacing $g(r)$ with the Taylor expansion of a smooth metric around zero.

Remark 1.11. *Let $-\Delta_{g(r)}$ the Laplace-Beltrami operator on $(S, g(r))$, where the metric is given by the matrix $(g_{j,k}(r,\theta))_{j,k}$, that is*

$$g(r) = g_{j,k}(r,\theta)d\theta_j d\theta_k.$$

Here we consider the Laplace-Beltrami operator defined from a quadratic form via the Friedrich extension. We recall its expression in local coordinates

$$-\sum_{j,k=1}^{n-1} \frac{1}{|g(r,\theta)|} D_{\theta_j}\left(|g(r,\theta)|g^{j,k}(r,\theta)D_{\theta_k}\right), \quad D_{\theta_j} = \frac{1}{i}\partial_{\theta_j}$$

where $|g(r,\theta)| := \det(g^{j,k}(r,\theta))_{j,k}^{-1/2}$ and $(g^{j,k}(r,\theta))_{j,k} = (g_{j,k}(r,\theta))_{j,k}^{-1}$. We define

$$w(r,\theta) = \frac{|g(r,\theta)|}{|\overline{g}(\theta)|}, \tag{1.6}$$

which is of the form

$$w(r,\theta) = 1 + \tilde{w}(r,\theta) \ \text{ with } \tilde{w} \in S^{-\nu}. \tag{1.7}$$

Since $dg(r) = w(r,\theta)d\overline{g}$, we can use w to conjugate $-\Delta_{g(r)}$, then $w^{\frac{1}{2}}(-\Delta_{g(r)})w^{-\frac{1}{2}}$ will be symmetric with respect to the measure induced by the fixed metric $\overline{g}$.

1.2. **The magnetic operator.** We consider A a vector field on M, thanks to the diffeomorphism Ω, we can identify its restriction to $M \setminus K$ with a map taking values in the product space $\mathbb{R} \times T_\omega(S)$. Namely

$$\begin{aligned} A : M \setminus K &\to T_0^1(M) \simeq \mathbb{R} \times T(S) \\ p &\mapsto A(p) \simeq A(\Omega(p)) = A(r,\omega) \in \mathbb{R} \times T_\omega(S). \end{aligned}$$

In a coordinate patch around (r,ω) we will denote the components of A as

$$(A_0(r,\theta), A_S(r,\theta)) = (A_0(r,\theta), A_1(r,\theta), \dots A_{n-1}(r,\theta)).$$

We assume $A_0 \in S^{-\nu}$ and

$$r^\nu l^{-2}(r)A_j, \ r^{\nu+1}l^{-2}(r)\partial_r A_j \in L^\infty((R,+\infty)\times S). \tag{1.8}$$

We also introduce the notation

$$\tilde{A}_0(r,\theta) := A_0(r,\theta), \quad \tilde{A}_i(r,\theta) := l^{-2}(r)\sum_{j=1}^{n-1} g_{i,j}(r,\theta)A_j(r,\theta), \tag{1.9}$$

hence by the previous assumption

$$r^\nu \tilde{A}_j \in L^\infty((R,+\infty)\times S)$$

for all $j = 0, \dots, n-1$.

Notation 1. Let $D_G = (D_r, \frac{1}{i} l(r)^2 \nabla_{g(r)}) = (\frac{1}{i}\partial_r, \frac{1}{i} l(r)^2 \nabla_{g(r)})$ and $\langle \cdot, \cdot \rangle_G$ the scalar product, between two vectors, induced by the metric G while $(\cdot, \cdot)_{dG}$ denotes the scalar products on $L^2((R, +\infty) \times S, dG)$.

With $(\cdot, \cdot)_{drd\overline{g}}$ we will denote the scalar products on $L^2((R, +\infty) \times S, drd\overline{g})$ and $\| \cdot \|_{L^2(drd\overline{g})}$ will be the induced L^2 norm.

We will also consider the scalar product and relative L^2 norm on a bounded region of $(R, \infty) \times S$, in doing so we will always consider the one induced by the measure $drd\overline{g}$ and we will specify the region by denoting them like $(\cdot, \cdot)_{L^2(\cdot)}$ and $\| \cdot \|_{L^2(\cdot)}$.

Let V a multiplicative potential in the symbol class

$$V \in S^{-\nu}, \nu > 0.$$

The perturbed Laplace-Beltrami operator on (M, G_M) can be defined via a quadratic form. On the manifold end $M \backslash K$, where we have an isometry with $((R, +\infty) \times S, dG)$, it agrees with the one defined by the following quadratic form on smooth compactly supported functions

$$\begin{aligned} q(u, v) =& ((D_G - A)u, (D_G - A)v)_{dG} + (u, Vv)_{dG} \\ =& \int (\langle \overline{(D_G - A)u}, (D_G - A)v \rangle_G + \overline{u} V v) l(r)^{1-n} dr dg(r). \end{aligned} \tag{1.10}$$

The operator defined by this quadratic form is symmetric with respect to the scalar product induced by the measure $dG = l(r)^{1-n} dr dg(r)$. After integration by parts and conjugation by $e^{iF} = l(r)^{\frac{1-n}{2}} w^{1/2}$ near infinity we obtain an operator

$$h^2 \tilde{P}_{\overline{g}} = h^2 (D_r - A_0)^2 + M(r) + h^2 V_m \tag{1.11}$$

with $h = \lambda^{-1}$ for some $\lambda > 0$ where V_m is a multiplicative potential including V and an effective potential and

$$M(r) := h^2 l^2(r)(1 + T(r)), \tag{1.12}$$

with $T(r)$ a differential operator of order two in the angular variables. More precisely,

$$\begin{aligned} T(r) :&= w^{1/2} (-\tilde{\Delta}_{g(r)}) w^{-1/2} \\ &= w^{1/2} \frac{1}{|g(r, \theta)|} \sum_{i,j=1}^{n-1} (D_i - \tilde{A}_i) \Big(|g(r, \theta)| g^{i,j}(r, \theta) (D_j - \tilde{A}_j) \Big) w^{-1/2}. \end{aligned} \tag{1.13}$$

We underline that the operator $-\tilde{\Delta}_{g(r)}$ is the Laplace-Beltrami operator $-\Delta_{g(r)}$ in which we have incorporated the perturbation by A.

Remark 1.12. *Conjugating by $e^{iF} = l(r)^{\frac{1-n}{2}} w^{1/2}$ has a double effect: with $l(r)^{\frac{1-n}{2}}$ we pass from an operator symmetric with respect to dG to one symmetric with respect to $drdg$; with $w^{1/2}$ we pass the symmetry from $drdg$ to $drd\overline{g}$. In conclusion, $\tilde{P}_{\overline{g}}$ is symmetric with respect to the scalar product induced by the measure $drd\overline{g}$.*

Remark 1.13. *The operator $M(r)$ includes the angular terms, so all the components act differentially only on the angular variable of S, that is θ. In particular for any function s of r*

$$[M(r), s] = 0.$$

A useful property we will use later on is

$$-M'(r) \gtrsim \frac{1}{r} M(r)(1 - O(r^{-\nu}))$$

which we can derive thanks to (1.3), see Lemma 2.4. Here $M'(r)$ denotes the differential operators obtained from M by differentiating the coefficients with respect to r.

The potential V_m is of the form

$$V_m(r,\theta) = a(r,\theta) + V_0(r,\theta)$$

where

$$a(r,\theta) = \frac{(n-1)^2}{4}\left(\frac{l'}{l}\right)^2 - \frac{(n-1)}{2}\left(\frac{l'}{l}\right)'$$

and V_0 has the properties

$$r^\nu V_0,\ r^{1+\nu}\partial_r V_0 \in L^\infty((R,+\infty)).$$

Moreover we make the following assumptions on a

$$\frac{(n-1)^2}{4}\left(\frac{l'}{l}\right)^2 =: \Lambda + a_1(r)$$

with a constant $\Lambda = \Lambda(l) \geq 0$, $a_1(r) \geq 0$ such that $r^\nu a_1(r), r^{1+\nu} a_1'(r) \in L^\infty((R,+\infty))$ and

$$\frac{(n-1)}{2}\left(\frac{l'}{l}\right)' =: a_2(r)$$

is such that $r^\nu a_2(r), r^{1+\nu} a_2'(r) \in L^\infty((R,+\infty))$.

With this notation

$$V_m = \Lambda + a_1(r) + a_2(r) + V_0(r,\theta) \tag{1.14}$$

and we can conclude that

$$r^\nu(V_m - \Lambda),\ r^{1+\nu}\partial_r V_m \in L^\infty((R,+\infty)). \tag{1.15}$$

We report the two typical cases of Examples 1.8 and 1.9.

Example 1.14. *If $l(r) = r^{-s}$ with $s \geq 1$ then*

$$\Lambda = 0, \qquad a_1(r) = \frac{(n-1)}{2}\cdot\frac{s}{r^2}, \qquad a_2(r) = \frac{(n-1)^2}{4}\cdot\frac{s}{r^2}.$$

If $l(r) = e^{-r}$ then

$$\Lambda = \frac{(n-1)^2}{4}, \qquad a_1 = a_2 = 0.$$

Notation 2. Since $V_m - \Lambda$ is a decaying potential, without loss of generalization, we define P_m the perturbed Laplace-Beltrami operator translated by Λ. In this way, near infinity the operator will be

$$P_{\overline{g}} = \tilde{P}_{\overline{g}} - \Lambda = h^2(D_r - A_0)^2 + M(r) + h^2(V_m - \Lambda),$$

which has a decaying potential $V_m - \Lambda$. We recall that Λ depends on l, that is on the choice of metric, and that in the cases $l(r) = r^{-1}$ or $l(r) = e^{-r}$ it represents the bottom of the essential spectrum.

1.3. **Besov-type norms.** Let $\psi \in C_0^\infty(\mathbb{R})$ a non negative bump function with $\|\psi\|_\infty \leq 1$ and

$$\psi(s) = \begin{cases} 1 & \text{for } |s| \leq \frac{1}{2}, \\ 0 & \text{for } |s| \geq 1. \end{cases}$$

Set $\varphi(s) := \psi(\frac{s}{2}) - \psi(s)$ which is then non negative and

$$supp\, \varphi \subset \{\frac{1}{2} < |s| < 2\}, \quad \|\varphi\|_\infty \leq 2.$$

We rescale φ dyadically in order to construct a partition of unity

$$1 = \psi(s) + \sum_{k \geq 0} \varphi(2^{-k}s).$$

Remark 1.15. *The support of $\varphi(2^{-k}s)$ is*

$$D_k := [2^{k-1}, 2^{k+1}],$$

so for fixed s there are only two non vanishing terms in the sum $\sum_{k\geq 0} \varphi(2^{-k}s)$ since the interval $[\frac{\ln s}{\ln 2} - 1, \frac{\ln s}{\ln 2} + 1]$ contains at most 2 integers.

We fix a $k^* \in \mathbb{N}$. To partition only the half line $(2^{k^*}R, +\infty)$ we start the sum at

$$k_0 := \frac{\ln R}{\ln 2} + k^* + 2,$$

for which $[2^{k-1}, 2^{k+1}] \subset (2^{k^*}R, +\infty)$ for all $k \geq k_0$. For $r > 2^{k^*}R$

$$\psi(r) + \sum_{k \geq k_0} \varphi(2^{-k}r) = \sum_{k \geq k_0} \varphi(2^{-k}r) = \begin{cases} \in (0,1) & r \in (2^{k_0-1}, 2^{k_0}], \\ 1 & r > 2^{k_0}. \end{cases}$$

We define the following norms of functions on $(2^{k^*+1}R, +\infty) \times S$

$$\|f\|_{B_{>R}} := \sum_{k \geq k_0} \|r^{1/2} f\|_{L^2(dr d\overline{g}, 2^{k-1} \leq r \leq 2^{k+1})} \tag{1.16}$$

and the dual quantity

$$\|g\|_{B^*_{>R}} := \sup_{k \geq k_0} \|r^{-1/2} g\|_{L^2(dr d\overline{g}, 2^{k-1} \leq r \leq 2^{k+1})}. \tag{1.17}$$

We also define the shorthand

$$\|g\|^2_{H^1, B^*_{>R}} := \|g\|^2_{B^*_{>R}} + \|h(D_r - A_0)g\|^2_{B^*_{>R}} + \|M(r)^{1/2} g\|^2_{B^*_{>R}}. \tag{1.18}$$

Remark 1.16. *The previous norm is well defined since $M(r)$ is a non negative operator, hence its square root exists. Despite $M(r)^{1/2}$ being a non local operator the norm is still well defined since it is an operator only in the angular variables, hence the non local action is only on the manifold S and we recall that in the definition of the norm we integrate over all S.*

We also define an adapted H^1 norm on compact regions of the manifold end, we still use the angular operator $M(r)$ to define the angular gradient but remove the weights, since we are in a compact region. For U, a bounded region in $(R, +\infty) \times S$, we write

$$\|f\|^2_{H^1(U)} = \|f\|^2_{L^2(U)} + \|h(D_r - A_0)f\|^2_{L^2(U)} + \|M(r)^{1/2} f\|^2_{L^2(U)}$$

where the L^2 norms are with respect to the measure $dr d\overline{g}$.

Remark 1.17. *The definition of the norms is independent of the choice of dyadic step, in other words the norms*

$$\|f\|_{B_{>R},m} := \sum_{k\geq(m-1)+k_0} \|r^{1/2}f\|_{L^2(drd\overline{g},2^{k-m}\leq r\leq 2^{k+m})}, \quad m\geq 1$$

are all equivalent. Given this equivalence we will drop the dependence on the index n and denote by the same symbols $\|\cdot\|_{B_{>R}}$ and $\|\cdot\|_{B^*_{>R}}$, any norm regardless of the dyadic step in the definition.

For the same reason, defining

$$D_{k,m} := 2^{k-m} \leq r \leq 2^{k+m}$$

we will drop the index m in the notation for $D_{k,m}$ and simply write D_k and $\|\cdot\|_{L^2(drd\overline{g},D_k)}$.

Remark 1.18. *We just defined norms with respect to the measure $drd\overline{g}$, while on the manifold end, the operator as defined in (1.10) is symmetric with respect to the measure $dG = l(r)^{1-n}drdg(r)$. We can do this since it is equivalent to bound the norms in which we take the measure $drd\overline{g}$ or the corresponding ones defined with respect to the measure dG. Indeed, the two are linked by the relation*

$$\|v\|_{L^2(dG)} = \|e^{iF}v\|_{L^2(drd\overline{g})} \tag{1.19}$$

with

$$e^{iF} = l(r)^{\frac{1-n}{2}}w^{1/2} = l(r)^{\frac{1-n}{2}}\frac{|g(r,\theta)|^{1/2}}{|\overline{g}(\theta)|^{1/2}}.$$

Notation 3. For the reasons presented in the above remark, all L^2 norms and scalar products considered in Section 2, which treats the manifold end $M\setminus K$, will be with respect to the measure $drd\overline{g}$. Likewise, the operator $P_{\overline{g}}$, given by,

$$P_{\overline{g}} = \tilde{P}_{\overline{g}} - \Lambda = h^2(D_r - A_0)^2 + M(r) + h^2(V_m - \Lambda),$$

will be the one used in the computations of Section 2.

Remark 1.19 ($B_{>R}$, $B^*_{>R}$ duality)**.** *In the computations of the following sections we will exploit the duality of $\|\cdot\|_{B_{>R}}$ and $\|\cdot\|_{B^*_{>R}}$ in the following way. Let $\chi\in C_0^\infty(\mathbb{R})$ a cutoff in the interval $(\frac{1}{4},4)$ and such that $\chi\equiv 1$ on $(\frac{1}{2},2)$. Define $\chi_k(r)=\chi(2^{-k}r)$, then*

$$\chi_k \equiv 1 \ \text{ on } supp\,\varphi(2^{-k}\cdot), \qquad supp\,\chi_k \subset [2^{k-2},2^{k+2}]. \tag{1.20}$$

Let ϕ supported in $(2^{k^*+3}R, +\infty) = (2^{k_0+1}, +\infty)$, which in turn implies $supp\,\varphi(2^{-k_0}\cdot) \cap supp\,\phi = \emptyset$ and

$$\begin{aligned}(\phi g, f)_{drdg} &= \sum_{k\geq k_0+1} (\phi\varphi(2^{-k}\cdot)g, f)_{drdg}\\ &= \sum_{k\geq k_0+1} (r^{-1/2}\chi_k\phi\varphi(2^{-k}\cdot)g, r^{1/2}\chi_k f)_{drdg}\\ &\leq \sum_{k\geq k_0+1} \|r^{-1/2}g\|_{L^2(drd\overline{g},D_k)}\|r^{1/2}f\|_{L^2(drd\overline{g},D_k)}\\ &\leq \|g\|_{B^*_{>R}}\|f\|_{B_{>R}}\\ &\leq \frac{\delta}{2}\|g\|^2_{B^*_{>R}} + \frac{1}{2\delta}\|f\|^2_{B_{>R}} \end{aligned} \tag{1.21}$$

for some $\delta \in (0,1)$.

2. Estimates on $M \setminus K$

In this section we bound the resolvent on the manifold end. We recall that $K \subset M$ is the compact region and the manifold end $M \setminus K$ is diffeomorphic to the product $(R, +\infty) \times S$.

From Remark 1.18, we know that in the region $M \setminus K$ we can equivalently consider the operator

$$P_{\overline{g}} = h^2(D_r - A_0)^2 + M(r) + h^2(V_m - \Lambda),$$

which is symmetric with respect to the scalar product induced by the measure $drd\overline{g}$ and with a potential $V_m - \Lambda = O(r^{-\nu})$ as $r \to \infty$. So in this section we will be interested in solutions of the equation $(P_{\overline{g}} - \lambda^2 + i\varepsilon')u = f$, where we consider $\lambda > \lambda_0$ for a fixed positive λ_0. After factorizing λ^2 we set

$$\mathcal{P} = h^2 P_{\overline{g}} - 1 + i\varepsilon. \tag{2.1}$$

The aim of this section is to bound $\|u\|_{H^1,B^*_{>R}}$ by $\|\mathcal{P}u\|_{B_{>R}}$ uniformly in λ and ε and up to some compactly supported remainder terms which will be treated in Section 3. More precisely we obtain

Proposition 2.1. *Fix $\lambda_0 > 0$. Let $u \in H^2(M)$, $\lambda > \lambda_0 > 0$ and $h = \lambda^{-1}$. For any $\delta \in (0, \lambda_0) \cap (0,1)$ there exist $c, c(\delta) > 0$ and $K(\delta) \subset (R, \infty) \times S$ bounded such that*

$$\|u\|^2_{H^1,B^*_{>R}} \leq c\delta\|u\|^2_{H^1,B^*_{>R}} + \frac{c}{\delta h^2}\|\mathcal{P}u\|^2_{B_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))}.$$

Since the $\|\cdot\|_{H^1,B^*_{>R}}$ norm contains the contributions of the L^2 norm of the function, of the radial derivative and of the angular derivatives we will proceed in the following way:

- in Section 2.1 we bound the norm $\|M(r)^{1/2}u\|_{B^*_{>R}}$,
- in Section 2.2 we bound the norm $\|h(D_r - A_0)u\|_{B^*_{>R}}$,
- finally in Section 2.3 we bound the norm $\|u\|_{B^*_{>R}}$.

Notation 4. In all of this chapter c, C and $c(\delta)$ are constants that are allowed to change from line to line. If an integral norm is denoted with the symbol $r \simeq R$ it means that the integral is supported in a compact sub interval of $(2^{k^*}R, +\infty)$ which may vary but does not depend on any parameter. On the contrary $K(\delta)$ will be a bounded region of $(R, \infty) \times S$

depending on the parameter δ. The symbols $\lesssim, \gtrsim$ will denote inequalities holding up to a positive multiplicative constant which is independent of the parameters δ, λ or ε. With the symbol $O(a)$ we mean that there exists a positive constant c such that $O(a) = c \cdot a$.

We will use the following elementary identity that holds between two symmetric operators B, C

$$Im(Bu, Cu) = (u, \frac{[B, C]}{2i}u). \tag{2.2}$$

2.1. **Estimating the angular gradient.** In this section we evaluate the norm $\|M(r)^{1/2}u\|_{B^*_{>R}}$, more precisely we obtain

Proposition 2.2. *Let $\lambda_0 > 0$ fixed. Let $\phi(r)$ a smooth cutoff on the interval $(2^{k_0+1}, +\infty)$, $\lambda > \lambda_0 > 0$ and $h = \lambda^{-1}$. For any $\delta \in (0, \lambda_0) \cap (0, 1)$ there exist $c, c(\delta) > 0$ and $K(\delta) \subset (R, \infty) \times S$ bounded such that*

$$\|M(r)^{1/2}\phi u\|^2_{B^*_{>R}} \leq \delta c\|u\|^2_{H^1, B^*_{>R}} + \frac{c'}{\delta h^2}\|\mathcal{P}(\phi u)\|^2_{B_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))}$$

with $c, c' > 0$ constants independent of δ.

Remark 2.3. *The support of the H^1 norm $\|u\|_{H^1(K(\delta))}$ is a compact interval contained in $(R, +\infty)$ with upper bound depending on δ and growing as δ approaches 0. This will not cause any particular problem since, when applying the results of this section we will fix the parameter δ and this will determine a fixed (potentially large but bounded) interval for r.*

We start by applying (2.2) to $P_{\overline{g}}$ and $(D_r - A_0)$ which are symmetric with respect to the measure $drd\overline{g}$. By definition of $\mathcal{P} = h^2 P_{\overline{g}} - 1 + i\varepsilon$

$$\begin{aligned} Im((D_r - A_0)(\phi u), \mathcal{P}(\phi u))_{drd\overline{g}} =& \varepsilon((D_r - A_0)\phi u, \phi u)_{drd\overline{g}} + Im((D_r - A_0)\phi u, h^2 P_{\overline{g}}\phi u)_{drd\overline{g}} \\ =& \varepsilon((D_r - A_0)\phi u, \phi u)_{drd\overline{g}} + (\phi u, \frac{[(D_r - A_0), h^2 P_{\overline{g}}]}{2i}\phi u)_{drd\overline{g}} \end{aligned} \tag{2.3}$$

where

$$\frac{1}{2i}[(D_r - A_0), h^2 P_{\overline{g}}] = -\frac{1}{2}M'(r) - \frac{1}{2i}[A_0, M(r)] + \frac{1}{2i}h^2 D_r(V_m). \tag{2.4}$$

From Remark 1.13 we recall that $(-M'(r)(\phi u), \phi u)_{drd\overline{g}}$ can be bounded from below by $(M(r)(\phi u), \phi u)_{drd\overline{g}} = \|M(r)^{1/2}(\phi u)\|^2_{L^2(drd\overline{g})}$ (up to some additional terms). More precisely, we have

Lemma 2.4. *Let $(\cdot, \cdot)_{d\overline{g}}$ the scalar product on $L^2(S, \overline{g})$ and $\varphi \in L^2(S, d\overline{g})$, then*

$$(\varphi, -\partial_r M(r)\varphi)_{d\overline{g}} \gtrsim \frac{1}{r}\|M(r)^{1/2}\varphi\|^2_{L^2(d\overline{g})}(1 - O(r^{-\nu})).$$

Recall the definition of $M(r)$

$$M(r) = h^2 l^2(r)(1 + T(r)), \qquad T(r) = w^{1/2}(-\tilde{\Delta}_{g(r)})w^{-1/2}.$$

We can prove Lemma 2.4 thanks to the following equivalence of norms.

Lemma 2.5. *Let $\varphi \in L^2(S, d\bar{g})$, then for $r > R$*

$$\|(1+T(r))^{1/2}\varphi\|_{L^2(d\bar{g})} \simeq \|(1-\tilde{\Delta}_{g(r)})^{1/2}\varphi\|_{L^2(d\bar{g})},$$

that is the quotient between the right and left hand sides is bounded from above and from below.

Proof. We can rewrite $T(r)$ as

$$\begin{aligned} 1+T(r) =& 1-\tilde{\Delta}_{g(r)} + w^{1/2}[-\tilde{\Delta}_{g(r)}, w^{-1/2}] \\ =& (1-\tilde{\Delta}_{g(r)})^{1/2}(1+S(r))(1-\tilde{\Delta}_{g(r)})^{1/2} \end{aligned} \tag{2.5}$$

where

$$S(r) := (1-\tilde{\Delta}_{g(r)})^{-1/2} w^{1/2}[-\tilde{\Delta}_{g(r)}, w^{-1/2}](1-\tilde{\Delta}_{g(r)})^{-1/2}. \tag{2.6}$$

We have seen in Remark 1.13 that $-\tilde{\Delta}_{g(r)}$ is a differential operator of order two in the angular variables and is indeed a perturbation of $-\Delta_{g(r)}$ by lower order terms. In particular it has an elliptic principal symbol and lower order terms with decaying coefficients. One can then construct its resolvent via a parametrix with a standard procedure, obtaining that $(1-\tilde{\Delta}_{g(r)})^{-1/2}$ is a pseudodifferential operator of order minus one in the angular variables.

Moreover, we have

$$\begin{aligned} w^{1/2}[-\tilde{\Delta}_{g(r)}, w^{-1/2}] =& w^{1/2}[-\Delta_{g(r)}, w^{-1/2}] + l^{-2}(r)\sum_{i=1}^{n-1} A_i \frac{D_i w}{w} \\ =& w^{1/2}[-\Delta_{g(r)} + \Delta_{\bar{g}}, w^{-1/2}] + w^{1/2}[-\Delta_{\bar{g}}, w^{-1/2}] + l^{-2}(r)\sum_{i=1}^{n-1} A_i \frac{D_i w}{w} \end{aligned}$$

where all the terms are differential in the angular variables, of order one or zero and with $O(r^{-\nu})$ coefficients. This is due to the fact that w^{-1} is bounded and

$$w^{1/2} D_{i,j}(w^{-1/2}) \in S^{-\nu}, \qquad w^{1/2} D_i(w^{-1/2}) \in S^{-\nu}$$

together with the boundedness of $r^\nu l^{-2}(r) A_i$ in (1.8). We have obtained

$$\|S(r)\|_{\mathcal{L}(L^2(S,d\bar{g}))} \simeq r^{-\nu}.$$

Since $S(r)$ is a bounded operator we have

$$\begin{aligned} \|(1+T(r))^{1/2}\varphi\|^2_{L^2(d\bar{g})} =& ((1+S(r))(1-\tilde{\Delta}_{g(r)})^{1/2}\varphi, (1-\tilde{\Delta}_{g(r)})^{1/2}\varphi)_{d\bar{g}} \\ \leq& \|1+S(r)\|_{\mathcal{L}(L^2(S))}\|(1-\tilde{\Delta}_{g(r)})^{1/2}\varphi\|^2_{L^2(d\bar{g})} \\ \lesssim& \|(1-\tilde{\Delta}_{g(r)})^{1/2}\varphi\|^2_{L^2(d\bar{g})}. \end{aligned}$$

We conclude using the lower bound $(S(r)v, v)_{d\bar{g}} \geq -\|S(r)\|_{\mathcal{L}(L^2(S,d\bar{g}))}\|v\|^2_{L^2(d\bar{g})}$ in

$$\begin{aligned} \|(1+T(r))^{1/2}\varphi\|^2_{L^2(d\bar{g})} =& \|(1-\tilde{\Delta}_{g(r)})^{1/2}\varphi\|^2_{L^2(d\bar{g})} + (S(r)(1-\tilde{\Delta}_{g(r)})^{1/2}\varphi, (1-\tilde{\Delta}_{g(r)})^{1/2}\varphi)_{d\bar{g}} \\ \geq& \|(1-\tilde{\Delta}_{g(r)})^{1/2}\varphi\|^2_{L^2(d\bar{g})}(1-\|S(r)\|_{\mathcal{L}(L^2(S,d\bar{g}))}) \end{aligned}$$

and recalling that for large enough r

$$1-\|S(r)\|_{\mathcal{L}(L^2(S,d\bar{g}))} > c > 0$$

with c independent of r.

□

Proof of Lemma 2.4. By definition of M

$$\begin{aligned} M'(r) =& h^2 2l'(r)l(r)(1+T(r)) + h^2 l^2(r)T'(r) \\ =& 2\frac{l'(r)}{l(r)} M(r) + h^2 l^2(r) T'(r). \end{aligned}$$

Thanks to $-l'(r)/l(r) \gtrsim r^{-1}$ then

$$\begin{aligned} (\varphi, -\partial_r M(r)\varphi)_{d\bar{g}} =& (\varphi, -2\frac{l'(r)}{l(r)} M(r)\varphi)_{d\bar{g}} + (\varphi, -h^2 l^2(r) T'(r)\varphi)_{d\bar{g}} \\ \gtrsim& \frac{2}{r} \|M(r)^{1/2}\varphi\|^2_{L^2(d\bar{g})} + (\varphi, -h^2 l^2(r) T'(r)\varphi)_{d\bar{g}}. \end{aligned} \tag{2.7}$$

Using the expression in (2.5) to compute $T'(r)$ we have

$$T'(r) = \partial_r(-\tilde{\Delta}_{g(r)}) + \partial_r(w^{1/2}[-\tilde{\Delta}_{g(r)}, w^{-1/2}]).$$

From the arguments in the proof of the previous lemma, differentiating in r the coefficients of $w^{1/2}[-\tilde{\Delta}_{g(r)}, w^{-1/2}]$ yields a differential operator of order one in the angular variables with coefficients of order $O(r^{-\nu-1})$. Moreover $(-\tilde{\Delta}_{g(r)})$ has coefficients whose expressions contain $g^{i,j}(r,\theta)$ and $\tilde{A}_i$. By definition of $\tilde{A}_i$ we compute

$$\begin{aligned} \partial_r \tilde{A}_i =& -2l^{-2}(r)\frac{l'(r)}{l(r)} \sum_{i,j} g_{i,j}(r,\theta) A_j + l^{-2}(r) \sum_{i,j} \partial_r g_{i,j}(r,\theta) A_j \\ &+ l^{-2}(r) \sum_{i,j} g_{i,j}(r,\theta) \partial_r A_j, \end{aligned}$$

where

$$r^\nu l^{-2}(r) A_j, \quad r^{1+\nu}\partial_r g_{i,j}(r,\theta), \quad r^{1+\nu} l^{-2}(r)\partial_r A_j$$

are bounded functions. Then, thanks again to $-l'(r)/l(r) \gtrsim r^{-1}$ we can bound from below the derivative of $\tilde{A}_i$

$$-\partial_r \tilde{A}_i \gtrsim -\frac{1}{r^{\nu+1}} b_0(r,\theta)$$

for some $b_0 \in L^\infty((R,+\infty)\times S)$. Since $\partial_r g^{i,j}(r,\theta) \in S^{-\nu-1}$ we can conclude that $-\partial_r(-\tilde{\Delta}_{g(r)})$ is a differential operator of order two whose coefficients can be bounded from below by $-r^{-\nu-1} b(r,\theta)$ for some bounded b. We can then lower bound the scalar product

$$(\varphi, -h^2 l^2(r) T'(r)\varphi)_{d\bar{g}} \gtrsim -\frac{1}{r^{\nu+1}} (\varphi, T_2 h^2 l^2(r)\varphi)_{d\bar{g}},$$

where T_2 is a differential operator of order two in the angular variables with bounded coefficients. It contains the contributions of $r^{\nu+1}\partial_r(-\tilde{\Delta}_{g(r)})$ and $r^{\nu+1}\partial_r(w^{1/2}[-\tilde{\Delta}_{g(r)}, w^{-1/2}])$. Then from

$$(1-\tilde{\Delta}_{g(r)})^{-1/2} T_2 (1-\tilde{\Delta}_{g(r)})^{-1/2} \in \mathcal{L}(L^2(S, d\bar{g})) \tag{2.8}$$

we obtain

$$\begin{aligned}-(\varphi, h^2l^2(r)T'(r)\varphi)_{d\bar{g}} &\gtrsim -\frac{1}{r^{\nu+1}}\|hl(r)(1-\tilde{\Delta}_{g(r)})^{1/2}\varphi\|^2_{L^2(S,d\bar{g})}\\ &\simeq -\frac{1}{r^{\nu+1}}\|M(r)^{1/2}\varphi\|^2_{L^2(d\bar{g})}\end{aligned}$$

up to some constant given by the norm of the operator in (2.8) and thanks to Lemma 2.5. The norm of (2.8) depends on r, however we can obtain a fixed constant by taking the supremum over all r, since the coefficients of T_2 are bounded in this variable. We conclude by using this lower bound in (2.7) which yields

$$(\varphi, -\partial_r M(r)\varphi)_{d\bar{g}} \gtrsim \frac{1}{r}\|M(r)^{1/2}\varphi\|^2_{L^2(d\bar{g})}(1-O(r^{-\nu})).$$

□

We recall identities (2.3) and (2.4), which yield

$$\begin{aligned}Im((D_r-A_0)(\phi u), \mathcal{P}(\phi u))_{drd\bar{g}} =&\varepsilon((D_r-A_0)\phi u, \phi u)_{drd\bar{g}} + \frac{1}{2}(\phi u, -M'(r)\phi u)_{drd\bar{g}}\\ &- \frac{1}{2i}(\phi u, [A_0, M(r)](\phi u))_{drd\bar{g}} + \frac{1}{2i}(\phi u, h^2D_r(V_m)\phi u)_{drd\bar{g}}.\end{aligned} \tag{2.9}$$

The function $M(r)^{1/2}\phi u$ is supported in $supp\,\phi \subset (2^{k_0+1}, +\infty)$, we apply Lemma 2.4 in this region

$$\begin{aligned}(\phi u, -M'(r)\phi u)_{drd\bar{g}} \gtrsim& \int_{2^{k_0+1}}^{\infty}\frac{1}{r}(1-O(r^{-\nu}))\|M(r)^{1/2}\phi u\|^2_{L^2(d\bar{g})}dr\\ =&\|r^{-1/2}M(r)^{1/2}\phi u\|^2_{L^2(drd\bar{g})} - \|r^{-1/2}O(r^{-\nu/2})M(r)^{1/2}\phi u\|^2_{L^2(drd\bar{g})}.\end{aligned}$$

For any $k \geq k_0$

$$\|r^{-1/2}M(r)^{1/2}\phi u\|_{L^2(drd\bar{g})} \geq \|r^{-1/2}M(r)^{1/2}\phi u\|_{L^2(drd\bar{g}, D_k)}$$

where we recall $D_k = [2^{k-1}, 2^{k+1}]$ (in particular $[2^{k_0-1}, 2^{k_0+1}] \cap (2^{k_0+1}, +\infty) = \emptyset$, $[2^{k_0}, 2^{k_0+2}] \cap (2^{k_0+1}, +\infty) \neq \emptyset$ and $[2^{k-1}, 2^{k+1}] \subset (2^{k_0+1}, +\infty)$ for all $k \geq k_0 + 2$). We have then obtained

$$\begin{aligned}(\phi u, -M'(r)\phi u)_{drd\bar{g}} \gtrsim& \sup_{k\geq k_0}\|r^{-1/2}M(r)^{1/2}\phi u\|^2_{L^2(drd\bar{g}, D_k)}\\ &- \|r^{-1/2}O(r^{-\nu/2})M(r)^{1/2}\phi u\|^2_{L^2(drd\bar{g})}\\ =&\|M(r)^{1/2}\phi u\|^2_{B^*_{>R}} - \|r^{-1/2}O(r^{-\nu/2})M(r)^{1/2}\phi u\|^2_{L^2(drd\bar{g})}\end{aligned}$$

which implies, thanks to (2.9),

$$\begin{aligned}\|M(r)^{1/2}\phi u\|^2_{B^*_{>R}} \lesssim& Im((D_r-A_0)(\phi u), \mathcal{P}(\phi u))_{drd\bar{g}} - \varepsilon((D_r-A_0)\phi u, \phi u)_{drd\bar{g}}\\ &+ \frac{1}{2}|(\phi u, [A_0, M(r)]\phi u)_{drd\bar{g}}| + \frac{1}{2}(\phi u, h^2D_r(V_m)\phi u)_{drd\bar{g}}\\ &+ \|r^{-1/2}O(r^{-\nu/2})M(r)^{1/2}\phi u\|^2_{L^2(drd\bar{g})}.\end{aligned} \tag{2.10}$$

In order to prove Proposition 2.1 we will need a series of intermediate lemmas to bound each term in (2.10). Notably, Proposition 2.1 is obtained directly from the following inequalities.

Lemma 2.6. *Let $\lambda_0 > 0$ fixed. Let $\phi(r)$ a smooth cutoff on the interval $(2^{k_0+1}, +\infty)$, $\lambda > \lambda_0 > 0$ and $h = \lambda^{-1}$. Let $\delta \in (0, \lambda_0) \cap (0, 1)$. The following inequalities hold*

i)
$$|Im((D_r - A_0)(\phi u), \mathcal{P}(\phi u))_{drd\bar{g}}| \leq \frac{\delta}{2}\|u\|^2_{H^1, B^*_{>R}} + \frac{1}{2\delta h^2}\|\mathcal{P}(\phi u)\|^2_{B_{>R}},$$

ii)
$$|\varepsilon((D_r - A_0)\phi u, \phi u)_{drd\bar{g}}| \leq \delta c\|u\|^2_{B^*_{>R}} + \frac{c}{\delta h^2}\|\mathcal{P}(\phi u)\|^2_{B_{>R}},$$

iii)
$$|(\phi u, [A_0, M(r)]\phi u)_{drd\bar{g}}| \leq \delta c\|u\|^2_{H^1, B^*_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))},$$

iv)
$$|(\phi u, h^2 D_r(V_m)\phi u)_{drd\bar{g}}| \leq \delta c\|u\|^2_{H^1, B^*_{>R}} + c(\delta)\|u\|^2_{L^2(K(\delta))}$$

We postpone the proofs of the present inequalities to the next section.

Proof of Proposition 2.2. Let $\delta < \lambda_0$ as in the statement of Proposition 2.2. Thanks to Lemma 2.6

$$\begin{aligned}\|M(r)^{1/2}\phi u\|^2_{B^*_{>R}} \lesssim & c\delta\|u\|^2_{H^1, B^*_{>R}} + \frac{c'}{\delta h^2}\|\mathcal{P}(\phi u)\|^2_{B_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))} \\ & + \|r^{-\nu/2} r^{-1/2} M(r)^{1/2}\phi u\|^2_{L^2(drd\bar{g})}.\end{aligned}$$

We use the partition of unity $\varphi(2^{-k}\cdot)$ to decompose the following L^2 norm

$$\|r^{-\nu/2} r^{-1/2} M(r)^{1/2}\phi u\|_{L^2(drd\bar{g})} \leq \sum_{k \geq k_0+1} \|r^{-\nu/2} r^{-1/2}\chi_k M(r)^{1/2}\phi u\|_{L^2(drd\bar{g})}$$

where we recall $\chi_k \equiv 1$ on the support of $\varphi(2^{-k}\cdot)$. We can write k in the form $k = k_0 + m + 1$ with $m \geq 0$, then on the support of χ_k we have $r = O(2^m)$ (i.e. $r = c2^m$ for a positive constant c) and hence

$$r^{-\nu/2} = O(2^{-\nu m/2}).$$

We choose $m^* = m^*(\delta) \geq 0$ such that $\sum_{m \geq m^*} 2^{-m\nu/2} < \delta$, we can then conclude the proof since

$$\begin{aligned}\|r^{-1/2} O(r^{-\nu/2}) M(r)^{1/2}\phi u\|^2_{L^2(drd\bar{g})} \lesssim & \sum_{\substack{m \geq 0 \\ k = k_0+1+m}} 2^{-m\nu}\|r^{-1/2} M(r)^{1/2}\phi u\|^2_{L^2(drd\bar{g}, D_k)} \\ \lesssim & \sum_{\substack{m=0 \\ k=k_0+1+m}}^{m^*} 2^{-m\nu}\|r^{-1/2} M(r)^{1/2}\phi u\|^2_{L^2(drd\bar{g}, D_k)} \\ & + \sum_{\substack{m \geq m^* \\ k=k_0+1+m}} 2^{-m\nu}\|r^{-1/2} M(r)^{1/2}\phi u\|^2_{L^2(drd\bar{g}, D_k)} \\ \lesssim & (\delta\|M(r)^{1/2}\phi u\|^2_{B^*_{>R}} + c(\delta)\|M(r)^{1/2}\phi u\|^2_{L^2(drd\bar{g}, K(\delta))}) \\ \lesssim & c\delta\|u\|^2_{H^1, B^*_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))},\end{aligned}$$

where the norm on the compact set $K(\delta)$ contains all the contributions of the sum for m from zero to m^*. □

The following lemma allows us to evaluate the commutator between $\mathcal{P}$ and ϕ which will yield our final bound.

Lemma 2.7. *Let $\lambda_0 > 0$ fixed. Let $\phi(r)$ a smooth cutoff on the interval $(2^{k_0+1}, +\infty)$, $\lambda > \lambda_0 > 0$ and $h = \lambda^{-1}$. Then*

$$\|\mathcal{P}(\phi u)\|_{B_{>R}} \leq \|\mathcal{P}u\|_{B_{>R}} + ch^2\|u\|_{H^1(r\simeq R)},$$

for a constant c depending on R and on the size of the interval $\{r : \ \phi(r) \in (0,1)\}$.

We recall that $M(r)^{1/2}$ commutes with functions of r. Applying the previous result and Proposition 2.2 we find that for any $\delta \in (0, \lambda_0)$ there exists $c(\delta) > 1$ such that

$$\|M(r)^{1/2}u\|^2_{B^*_{>R}} \leq \frac{c'}{\delta h^2}\|\mathcal{P}u\|^2_{B_{>R}} + \delta c\|u\|^2_{H^1,B^*_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))} \tag{2.11}$$

for constants $c, c' > 0$ independent of δ.

Proof of Lemma 2.7. Since ϕ is a function of r only

$$[\mathcal{P}, \phi] = [h^2 P_{\overline{g}}, \phi] = [h^2(D_r - A_0)^2, \phi] = h^2(D_r^2(\phi) + 2D_r(\phi)D_r - 2A_0 D_r(\phi))$$

with $D_r(\phi), D_r^2(\phi) \in C_0^\infty((2^{k_0+1}, +\infty))$. In particular in the norm

$$\begin{aligned}\|\mathcal{P}(\phi u)\|_{B_{>R}} \leq & \|\mathcal{P}u\|_{B_{>R}} + \|[\mathcal{P}, \phi]u\|_{B_{>R}} \\ \leq & \|\mathcal{P}u\|_{B_{>R}} + \| + \sum_{k\geq k_0} \|h^2 r^{1/2}(D_r^2(\phi) - 2A_0 D_r(\phi))u\|_{L^2(dr d\overline{g}, D_k)} \\ & + \sum_{k\geq k_0} \|h^2 r^{1/2} 2D_r(\phi)D_r u\|_{L^2(dr d\overline{g}, D_k)}\end{aligned}$$

the sums only have a finite number on non vanishing terms, since the supports of $D_r(\phi)$ and $D_r^2(\phi)$ intersect only a finite number of intervals $[2^{k-1}, 2^{k+1}]$. There exists $\alpha > 0$ (depending on the fixed size of the interval $\{r : \ \phi(r) \in (0,1)\}$) such that in the above L^2 norms

$$r \leq 2^{k_0+\alpha+1} = 2^{k^*+\alpha}R.$$

Hence bounding $r^{1/2}$ by $R^{1/2}$ and the functions $D_r(\phi), D_r^2(\phi), A_0$ by their L^∞ norms, we have the statement:

$$\|\mathcal{P}(\phi u)\|_{B_{>R}} \lesssim \|\mathcal{P}u\|_{B_{>R}} + h^2(\|u\|_{L^2(r\simeq R)} + \|D_r u\|_{L^2(r\simeq R)}).$$

□

2.1.1. *Auxiliary lemmas.* In this subsection we prove the inequalities stated in Lemma 2.6. We recall that everywhere in this section ϕ is a smooth cutoff on the interval $(2^{k_0+1}, \infty)$ and which is equal to one on $(2^{k_0+2}, \infty)$, where k_0 is a fixed integer (see Section 1.3).

We start with the first result which we obtain simply by duality of the norms.

Lemma 2.8. *Let $\delta \in (0,1)$,*

$$|Im((D_r - A_0)(\phi u), \mathcal{P}(\phi u))_{dr d\overline{g}}| \leq \frac{\delta}{2}\|u\|^2_{H^1,B^*_{>R}} + \frac{1}{2\delta h^2}\|\mathcal{P}(\phi u)\|^2_{B_{>R}}.$$

Proof. We multiply and divide by $h^2 r^{1/2}$ and since $supp\,(D_r - A_0)(\phi u) \subset supp\,\phi u$ we can reason by duality as in (1.21). □

For $\varepsilon((D_r-A_0)\phi u,\phi u)_{drd\bar g}$ we will bound separately $\varepsilon\|\phi u\|_{L^2(drd\bar g)}$ and $\varepsilon\|h(D_r-A_0)(\phi u)\|_{L^2(drd\bar g)}$. We consider

$$(\phi u,\mathcal{P}(\phi u))_{drd\bar g}=i\varepsilon(\phi u,\phi u)_{drd\bar g}-\|\phi u\|^2_{L^2(drd\bar g)}+(\phi u,h^2P_{\bar g}(\phi u))_{drd\bar g},$$

where $(\phi u,h^2P_{\bar g}(\phi u))_{drd\bar g}$ is real since $P_{\bar g}$ is symmetric with respect to the measure $drd\bar g$. Hence from

$$Im(\phi u,\mathcal{P}(\phi u))_{drd\bar g}=\varepsilon\|\phi u\|^2_{L^2(drd\bar g)}$$

we can prove again by duality of the norms

Lemma 2.9. *Let $\delta\in(0,1)$, then*

$$\varepsilon\|\phi u\|^2_{L^2(drd\bar g)}\le\frac{\delta}{2}\|u\|^2_{B^*_{>R}}+\frac{1}{2\delta}\|\mathcal{P}(\phi u)\|^2_{B_{>R}}.$$

Proof. The proof is analogous to Lemma 2.8. □

On the other hand, taking the real part we use the relation

$$(\phi u,h^2P_{\bar g}(\phi u))_{drd\bar g}=Re(\phi u,\mathcal{P}(\phi u))_{drd\bar g}+\|\phi u\|^2_{L^2(drd\bar g)} \tag{2.12}$$

to prove the following.

Lemma 2.10. *For any $\delta\in(0,1)$, then*

$$\varepsilon\|h(D_r-A_0)(\phi u)\|^2_{L^2(drd\bar g)}\le\delta c\|u\|^2_{B^*_{>R}}+\frac{1}{\delta}c\|\mathcal{P}(\phi u)\|^2_{B_{>R}}$$

with $c>0$ independent of δ.

Proof. By definition of $P_{\bar g}=h^2(D_r-A_0)^2+M(r)+h^2(V_m-\Lambda)$

$$\begin{aligned}(\phi u,h^2P_{\bar g}(\phi u))_{drd\bar g}=&\|h(D_r-A_0)(\phi u)\|^2_{L^2(drd\bar g)}+\|M(r)^{1/2}(\phi u)\|^2_{L^2(drd\bar g)}\\&+(\phi u,h^2(V_m-\Lambda)(\phi u))_{drd\bar g}.\end{aligned}$$

hence from (2.12)

$$\begin{aligned}\varepsilon\|h(D_r-A_0)(\phi u)\|^2_{L^2(drd\bar g)}\le&\varepsilon Re(\phi u,\mathcal{P}(\phi u))_{drd\bar g}+\varepsilon\|\phi u\|^2_{L^2(drd\bar g)}\\&-\varepsilon(\phi u,h^2(V_m-\Lambda)\phi u))_{drd\bar g}.\end{aligned} \tag{2.13}$$

Again by the duality in (1.21)

$$\varepsilon|Re(\phi u,\mathcal{P}(\phi u))_{drd\bar g}|\le\frac{\varepsilon\delta}{2}\|u\|^2_{B^*_{>R}}+\frac{\varepsilon}{2\delta}\|\mathcal{P}(\phi u)\|^2_{B_{>R}}$$

while $\varepsilon\|\phi u\|^2_{L^2(drd\bar g)}$ can be bounded via Lemma 2.9, which we also use in

$$|\varepsilon(\phi u,h^2(V_m-\Lambda)\phi u)_{drd\bar g}|\le h^2\varepsilon\|V_m-\Lambda\|_\infty\|\phi u\|^2_{L^2(drd\bar g)}.$$

□

With Lemmas 2.9 and 2.10 we bound the term $\varepsilon((D_r-A_0)\phi u,\phi u)_{drd\bar g}$.

Lemma 2.11. *For any $\mu\in(0,1)$ such that $\mu<\lambda_0$*

$$|\varepsilon((D_r-A_0)\phi u,\phi u)_{drd\bar g}|\le\mu c\|u\|^2_{B^*_{>R}}+\frac{c}{\mu h^2}\|\mathcal{P}(\phi u)\|^2_{B_{>R}}$$

for some $c>0$ independent of h and μ.

Proof. Applying Lemmas 2.9 and 2.10 with $\delta = h\mu \in (0,1)$ we have

$$\begin{aligned}|\varepsilon((D_r - A_0)\phi u, \phi u)_{drd\bar{g}}| \leq & \frac{\varepsilon}{2h}\|h(D_r - A_0)(\phi u)\|^2_{L^2(drd\bar{g})} + \frac{\varepsilon}{2h}\|\phi u\|^2_{L^2(drd\bar{g})} \\ \leq & \mu c\|u\|^2_{B^*_{>R}} + \frac{c}{\mu h^2}\|\mathcal{P}(\phi u)\|^2_{B_{>R}}\end{aligned}$$

where c, given by the previous lemmas, is bounded uniformly in h. □

We now need to bound the term

$$(\phi u, [A_0, M(r)]\phi u)_{drd\bar{g}} = (\phi u, h^2 l^2(r)[A_0, T(r)]\phi u)_{drd\bar{g}}$$

In the proof of Lemma 2.5 we remarked some useful properties on $T(r)$ in particular that

$$T(r) = -\tilde{\Delta}_{g(r)} + w^{1/2}[-\tilde{\Delta}_{g(r)}, w^{-1/2}]$$

where $-\tilde{\Delta}_{g(r)}$ is the sum of $-\Delta_{g(r)}$ with lower order perturbations, and $w^{1/2}[-\tilde{\Delta}_{g(r)}, w^{-1/2}]$ is a differential operator of order one with coefficients decaying like $O(r^{-\nu})$.

Lemma 2.12. *For any $\delta \in (0,1)$ there exist $c, c(\delta) > 0$ such that*

$$|(\phi u, [A_0, M(r)]\phi u)_{drd\bar{g}}| \leq \delta c\|u\|^2_{H^1, B^*_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))}.$$

Proof. We need to evaluate the commutator of A_0 with $-\Delta_{g(r)}$ and with differential operators of order one in the angular variables whose coefficients decay radially. First of all

$$[A_0, -\Delta_{g(r)}]$$

is a differential operator of order one with coefficients in $S^{-\nu}$, since $A_0 \in S^{-\nu}$.

As we pointed out in the proof of Lemma 2.5, $(1-\tilde{\Delta}_{g(r)})^{-1/2}$ is a pseudodifferential operator of order minus one in the angular variables. Then

$$\|r^\nu[A_0, -\Delta_{g(r)}](1 - \tilde{\Delta}_{g(r)})^{-1/2}\|_{\mathcal{L}(L^2(drd\bar{g}))} = O(1). \tag{2.14}$$

Moreover, the commutator of A_0 with a differential operator of order one with bounded coefficients is going to be a function in $S^{-\nu}$ and hence

$$\|r^\nu[A_0, -\tilde{\Delta}_{g(r)} + \Delta_{g(r)}]\|_{\mathcal{L}(L^2(drd\bar{g}))} = O(1) \tag{2.15}$$

and

$$\|r^\nu[A_0, w^{1/2}[-\tilde{\Delta}_{g(r)}, w^{-1/2}]]\|_{\mathcal{L}(L^2(drd\bar{g}))} = O(1). \tag{2.16}$$

Recalling $l(r) \lesssim r^{-1}$

$$\begin{aligned}|(\phi u, h^2 l^2(r)[A_0, -\Delta_{g(r)}]\phi u)_{drd\bar{g}}| &\leq |(r^{-\nu}r^{-1/2}\phi u, h^2 r^{-1/2} r^\nu [A_0, -\Delta_{g(r)}] l(r)\phi u)_{drd\bar{g}}| \\ &\leq \sum_{k \geq k_0+1} (|r^{-1/2-\nu}\varphi(2^{-k}r)\chi_k \phi u|, |h^2 r^{-1/2} r^\nu [A_0, -\Delta_{g(r)}] l(r)\chi_k \phi u|)_{drd\bar{g}}\end{aligned}$$

where we introduce χ_k, functions which are equal to one on the support of $\varphi(2^{-k}\cdot)$. Write $k \geq k_0$ in the form

$$k = k_0 + m + 1 \qquad m = m(k) \geq 0$$

recalling that *supp* $\chi_k = [2^{k-2}, 2^{k+2}]$, inequality

$$r^{-1} \leq 2^{-k_0} 2^{-m+1} \tag{2.17}$$

holds on the support of χ_k. Inserting $(1-\tilde{\Delta}_{g(r)})^{-1/2}$ and recalling (2.14)

$$\begin{aligned}|(\phi u, h^2 l^2(r)[A_0, -\Delta_{g(r)}]\phi u)_{dr\bar{d}\bar{g}}| \lesssim\ & h \sum_{\substack{m\geq 0\\ k=k_0+1+m}} \Big(2^{-m\nu+\nu}\|r^{-1/2}u\|_{L^2(dr\bar{d}\bar{g},D_k)} \\ & \cdot \|r^{-1/2}hl(1-\tilde{\Delta}_{g(r)})^{1/2}\phi u\|_{L^2(dr\bar{d}\bar{g},D_k)}\Big) \\ \lesssim\ & h \sum_{\substack{m\geq 0\\ k=k_0+1+m}} \Big(2^{-m\nu+\nu}\|r^{-1/2}u\|_{L^2(dr\bar{d}\bar{g},D_k)} \\ & \cdot \|r^{-1/2}M(r)^{1/2}\phi u\|_{L^2(dr\bar{d}\bar{g},D_k)}\Big)\end{aligned}$$

where we used Lemma 2.5 in the last inequality. We can choose $m^* = m^*(\delta)$ such that $\sum_{m\geq m^*} 2^{-m\nu} < \delta$ in order to bound the tail of the series by the $B^*_{>R}$ norms, while the remaining terms will be norms supported on compact intervals (depending on $m^*(\delta)$)

$$\begin{aligned}|(\phi u, h^2 l^2(r)[A_0, -\Delta_{g(r)}]\phi u)_{dr\bar{d}\bar{g}}| \lesssim & h\delta(\|u\|^2_{B^*_{>R}} + \|M(r)^{1/2}\phi u\|^2_{B^*_{>R}}) \\ & + hc(\delta)(\|u\|^2_{L^2(K(\delta))} + \|M(r)^{1/2}\phi u\|^2_{L^2(K(\delta))}) \\ \lesssim & h\delta\|u\|^2_{H^1,B^*_{>R}} \\ & + hc(\delta)(\|u\|^2_{L^2(K(\delta))} + \|M(r)^{1/2}\phi u\|^2_{L^2(K(\delta))})\end{aligned}$$

Here the constant $c(\delta)$ includes the sum $\sum_{m=0}^{m^*} 2^{-m\nu}$, hence it is a constant which grows as δ approaches 0.

Thanks to (2.15) and (2.16) we can proceed analogously to bound

$$|(\phi u, h^2 l^2(r)[A_0, -\tilde{\Delta}_{g(r)} + \Delta_{g(r)}]\phi u)_{dr\bar{d}\bar{g}}|$$

and

$$|(\phi u, h^2 l^2(r)[A_0, w^{1/2}[-\tilde{\Delta}_{g(r)}, w^{-1/2}]]\phi u)_{dr\bar{d}\bar{g}}|$$

where this time we do not need to insert $(1-\tilde{\Delta}_{g(r)})^{-1/2}$ since the commutators are of order zero. This yields

$$|(\phi u, h^2 l^2(r)[A_0, -\tilde{\Delta}_{g(r)} + \Delta_{g(r)}]\phi u)_{dr\bar{d}\bar{g}}| \lesssim hR^{-\nu}(\delta\|u\|^2_{B^*_{>R}} + c(\delta)\|u\|^2_{L^2(K(\delta))})$$

and

$$|(\phi u, h^2 l^2(r)[A_0, w^{1/2}[-\tilde{\Delta}_{g(r)}, w^{-1/2}]]\phi u)_{dr\bar{d}\bar{g}}| \lesssim hR^{-\nu}(\delta\|u\|^2_{B^*_{>R}} + c(\delta)\|u\|^2_{L^2(K(\delta))})$$

which conclude the proof.

□

We conclude the section with the proof of the last item in 2.6.

Lemma 2.13. *For any $\delta \in (0,1)$ there exists $c, c(\delta) > 0$ such that*

$$|(\phi u, h^2 D_r(V_m)\phi u)_{dr\bar{d}\bar{g}}| \leq \delta c\|u\|^2_{H^1,B^*_{>R}} + c(\delta)\|u\|^2_{L^2(K(\delta))}.$$

Proof. We proceed similarly to the previous proof. We write $k \geq k_0$ in the form $k = k_0+m+1$ for $m = m(k) \geq 0$ and since $r^{1+\nu}D_r(V_m)$ is bounded thanks to (1.15) we first have

$$\begin{aligned}|(\phi u, h^2 D_r(V_m)\phi u)_{drd\bar{g}}| \leq & h^2 \sum_{k\geq k_0+1} |(r^{-\nu}\varphi(2^{-k}\cdot)r^{-1/2}\chi_k\phi u, r^{1+\nu}D_r(V_m)r^{-1/2}\chi_k\phi u)_{drd\bar{g}}| \\ \lesssim & h^2 2^{\nu} R^{-\nu} \sum_{\substack{m\geq 0 \\ k=k_0+1+m}} 2^{-m\nu}\|r^{-1/2}u\|^2_{L^2(drd\bar{g},D_k)}\end{aligned}$$

where we have used as in (2.17) that $r \leq O(2^{-m})$. Choosing $m^* = m^*(\delta)$ such that $\sum_{m\geq m^*} 2^{-m\nu} < \delta$ we have the statement

$$|(\phi u, h^2 D_r(V_m)\phi u)_{drd\bar{g}}| \lesssim h^2 2^{\nu} R^{-\nu}(\delta\|u\|^2_{B^*_{>R}} + c(\delta)\|u\|^2_{L^2(K(\delta))}).$$

□

2.2. **Estimating the radial derivative.** In this section we give estimates on the radial part of the $H^1, B^*_{>R}$ norm, that is $\|h(D_r - A_0)u\|_{B^*_{>R}}$. More precisely we prove

Proposition 2.14. *For any $\delta \in (0,1)$ there exist $c(\delta) > 0$ and $K(\delta) \subset (R, +\infty)\times S$ bounded such that*

$$\|h(D_r - A_0)u\|^2_{B^*_{>R}} \leq c\delta\|u\|^2_{H^1,B^*_{>R}} + c'\frac{1}{\delta h^2}\|\mathcal{P}u\|^2_{B_{>R}} + c(\delta)\|u\|^2_{H^1(r\simeq R)} + \|u\|^2_{H^1(K(\delta))}.$$

We start again by considering the imaginary part of a scalar product. Let a a function of r only. We apply property (2.2) to the symmetric operators $P_{\bar{g}}$ and

$$\frac{a(D_r - A_0) + (D_r - A_0)a}{2} = a(D_r - A_0) + \frac{a'}{2i}.$$

This yields

$$\begin{aligned}Im\left(\frac{a(D_r - A_0) + (D_r - A_0)a}{2}u, \mathcal{P}u\right)_{drd\bar{g}} = & \varepsilon\left(\frac{a(D_r - A_0) + (D_r - A_0)a}{2}u, u\right)_{drd\bar{g}} \\ & + \left(u, \frac{1}{2i}[\frac{a(D_r - A_0) + (D_r - A_0)a}{2}, h^2 P_{\bar{g}}]u\right)_{drd\bar{g}},\end{aligned} \tag{2.18}$$

where we remark that

$$\left[\frac{a(D_r - A_0) + (D_r - A_0)a}{2}, h^2(D_r - A_0)^2\right] = [a, h^2(D_r - A_0)^2](D_r - A_0) + \frac{1}{2i}[a', h^2(D_r - A_0)^2]$$

is a differential operator of order two with coefficients depending on a, a', a'', a'''. In particular

$$\begin{aligned}\left[\frac{a(D_r - A_0) + (D_r - A_0)a}{2}, h^2(D_r - A_0)^2\right] &= 2a''h^2(D_r - A_0) - 2\frac{a'}{i}h^2(D_r - A_0)^2 + \frac{1}{2i}h^2 a''' \\ &= -\frac{2}{i}D_r(a')h^2(D_r - A_0) - \frac{2}{i}a'h^2(D_r - A_0)^2 + \frac{1}{2i}h^2 a''' \\ &= -\frac{2}{i}h(D_r - A_0)(a'h(D_r - A_0)) + \frac{1}{2i}h^2 a''',\end{aligned}$$

so that if we were to integrate the commutator against u, as in (2.18),

$$\begin{aligned}(u, \frac{1}{2i}\left[\frac{a(D_r-A_0)+(D_r-A_0)a}{2}, h^2(D_r-A_0)^2\right]u)_{drd\bar{g}} =&(u, h(D_r-A_0)(a'h(D_r-A_0))u)_{drd\bar{g}}\\ &-\frac{h^2}{4}(u, a'''u)_{drd\bar{g}}\\ =&\|(a')^{1/2}h(D_r-A_0)u\|^2_{L^2(drd\bar{g})}\\ &-\frac{h^2}{4}(u, a'''u)_{drd\bar{g}}, \end{aligned} \tag{2.19}$$

provided that a' is non negative.

Choosing a such that a' is a cutoff on a dyadic interval the previous identity provides us with $\|h(D_r-A_0)u\|_{L^2(drd\bar{g},D_k)}$ which are the norms that constitute $\|h(D_r-A_0)u\|_{B^*_{>R}}$.

Given this remark we consider a_k the primitive of χ_k^2, we recall that χ_k is supported on $[2^{k-2}, 2^{k+2}]$ and equal to one on the support of $\varphi(2^{-k}\cdot)$. Choosing $a = a_k$ in (2.18) and using (2.19)

$$\begin{aligned}\left(u, \frac{1}{2i}[\frac{a_k(D_r-A_0)+(D_r-A_0)a_k}{2}, h^2P_{\bar{g}}]u\right)_{drd\bar{g}} =&(u, h^2(D_r-A_0)(a_k'(D_r-A_0))u)_{drd\bar{g}}\\ &-\frac{h^2}{4}(u, a_k'''u)_{drd\bar{g}}\\ &+(u, -\frac{1}{2}a_kM'(r)u)_{drd\bar{g}}\\ &-(u, \frac{1}{2i}a_k[A_0, M(r)]u)_{drd\bar{g}}\\ &+\frac{1}{2i}(u, h^2a_kD_r(V_m)u)_{drd\bar{g}}\\ \geq&\|\chi_kh(D_r-A_0)u\|^2_{L^2(drd\bar{g})}\\ &-\frac{h^2}{4}(u, \chi_k''u)_{drd\bar{g}}-(u, \frac{1}{2i}a_k[A_0, M(r)]u)_{drd\bar{g}}\\ &+(u, \frac{1}{2i}h^2a_kD_r(V_m)u)_{drd\bar{g}} \end{aligned} \tag{2.20}$$

where $(u, -a_kM'(r)u)_{drd\bar{g}} \geq 0$ thanks to Lemma 2.4 and the non negativity of a_k. Additionally

$$0 \leq a_k(r) = \int \chi_k^2(s)ds = \int_{2^{k-2}}^{2^{k+2}} \chi_k^2(s)ds \leq \frac{15}{4}2^k \tag{2.21}$$

and without loss of generality we can assume

$$supp\ a_k \subset [2^{k-2}, +\infty),$$

since $supp\chi_k \subset [2^{k-2}, 2^{k+2}]$.

Remark 2.15. *We recall that*

$$\sum_{l\geq k_0}\varphi(2^{-l}r) = 1 \quad \text{on} \quad (2^{k_0}, +\infty)$$

and $\operatorname{supp} a_k \subset [2^{k-2}, +\infty) \subset [2^{k_0}, +\infty)$ *for any* $k \geq k_0 + 2$ *on top of the fact that* $\chi_l \equiv 1$ *on the support of* $\varphi(2^{-l}\cdot)$. *For all* $k \geq k_0 + 2$ *we can then write*

$$(a_k \cdot, \cdot)_{dr d\bar{g}} \leq \sum_{l \geq k_0} (\varphi(2^{-l} r) a_k \cdot, \cdot)_{dr d\bar{g}} \leq (\chi_{k_0} a_k \cdot, \chi_{k_0} \cdot)_{dr d\bar{g}} + \sum_{l \geq k_0 + 1} (\varphi(2^{-l} r) \chi_l a_k \cdot, \chi_l \cdot)_{dr d\bar{g}} \quad (2.22)$$

On the other hand $\operatorname{supp} a_{k_0+1} \subset [2^{k_0-1}, +\infty)$ *and*

$$\sum_{k \geq k_0} \chi_k^2 \geq 1 \quad \text{on} \quad [2^{k_0-1}, +\infty),$$

hence we also have

$$\begin{aligned}(a_{k_0+1} \cdot, \cdot)_{dr d\bar{g}} \leq & ((\chi_{k_0} + \chi_{k_0+1}) a_{k_0+1} \cdot, (\chi_{k_0} + \chi_{k_0+1}) \cdot)_{dr d\bar{g}} \\ & + \sum_{k \geq k_0 + 2} (\varphi(2^{-k} r) \chi_k a_{k_0+1} \cdot, \chi_k \cdot)_{dr d\bar{g}}\end{aligned}$$

since $\operatorname{supp} \chi_k \subset (2^{k_0}, +\infty)$ *for all* $k \geq k_0 + 2$.

Remark 2.16. *To obtain* (2.20) *we have used the following expressions for the commutators with* $M(r)$ *and* $V_m - \Lambda$*:*

$$\left[\frac{a(D_r - A_0) + (D_r - A_0)a}{2}, M(r) \right] = a \frac{M'(r)}{i} - a[A_0, M(r)],$$

$$\left[\frac{a(D_r - A_0) + (D_r - A_0)a}{2}, h^2 (V_m - \Lambda) \right] = h^2 a D_r (V_m).$$

Going back to expression (2.18) and given the contribution of (2.20) we have found

$$\begin{aligned}\|h \chi_k (D_r - A_0) u\|^2_{L^2(dr d\bar{g})} \lesssim & Im \left(a_k (D_r - A_0) u + \frac{a_k'}{2i} u, \mathcal{P} u \right)_{dr d\bar{g}} \\ & - \varepsilon \left(\frac{a_k (D_r - A_0) + (D_r - A_0) a_k}{2} u, u \right)_{dr d\bar{g}} \\ & + \frac{h^2}{4} (u, \chi_k'' u)_{dr d\bar{g}} + (u, \frac{1}{2i} a_k [A_0, M(r)] u)_{dr d\bar{g}} \\ & - (u, \frac{1}{2i} h^2 a_k D_r (V_m) u)_{dr d\bar{g}} \\ = & Im \left(a_k (D_r - A_0) u + \frac{\chi_k^2}{2i} u, \mathcal{P} u \right)_{dr d\bar{g}} - \varepsilon Re (a_k u, (D_r - A_0) u)_{dr d\bar{g}} \\ & + \frac{h^2}{4} (u, (\chi_k^2)'' u)_{dr d\bar{g}} + (u, \frac{1}{2i} a_k [A_0, M(r)] u)_{dr d\bar{g}} \\ & - (u, \frac{1}{2i} h^2 a_k D_r (V_m) u)_{dr d\bar{g}}, \end{aligned} \quad (2.23)$$

where we have used the relation

$$Re(Bu, Cu) = \left(\frac{BC + CB}{2} u, u \right) \quad (2.24)$$

for B, C symmetric.

We recall that the $B^*_{>R}$ norm is defined as a supremum over all dyadic intervals, hence we need to bound the quantity $h\chi_k(D_r - A_0)u$ uniformly in k for all $k \geq k_0 + 1$. To prove Proposition 2.14 we use a series of inequalities that we collect here and that are proven in the following subsection.

Lemma 2.17. *Let $k \geq k_0 + 1$ of the form $k = k_0 + m + 1$ for some $m = m(k) \geq 1$ then for any $\delta \in (0, 1)$ we have the following inequalities:*

i)

$$\sum_{l \geq k_0+1} Im\Big(\varphi(2^{-l}r)\chi_l|(D_r - A_0)u|,\chi_l|\mathcal{P}u|\Big)_{drd\bar{g}} + |Im\left(\frac{\chi_k^2}{2i}u, \mathcal{P}u\right)_{drd\bar{g}}| \leq \frac{c}{h^2\delta}\|\mathcal{P}u\|^2_{B_{>R}} + c\delta\|u\|^2_{H^1,B^*_{>R}},$$

ii)

$$\varepsilon|Re(a_k u, (D_r - A_0)u)_{drd\bar{g}}| \leq O(2^m)(\delta c\|u\|^2_{B^*_{>R}} + \frac{1}{\delta h}c\|\mathcal{P}u\|^2_{B_{>R}}) + O(2^m)\frac{c}{\delta}\|u\|_{H^1(r\simeq R)},$$

iii)

$$|\frac{h^2}{4}(u, (\chi_k^2)''u)_{drd\bar{g}}| \leq \delta c\|u\|^2_{B^*_{>R}} + c\|u\|^2_{L^2(K(\delta))},$$

iv)

$$\sum_{l \geq k_0+1} (\varphi(2^{-l}r)\chi_l|u|, \frac{1}{2i}\chi_l|[A_0, M(r)]u|)_{drd\bar{g}} \lesssim (c\delta\|u\|^2_{H^1,B^*_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))}),$$

v)

$$\sum_{l \geq k_0+1} (\varphi(2^{-l}r)\chi_l|u|, \frac{1}{2i}h^2\chi_l|D_r(V_m)u|)_{drd\bar{g}} \lesssim (c\delta\|u\|^2_{H^1,B^*_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))}).$$

Proof of Proposition 2.14. Given the support of χ_k, we will bound the norm $\|\cdot\|_{B^*_{>R},2}$ and hence we will need to consider the supremum over all $k \geq k_0 + 1$ (see Remark 1.17). Write

$$k = k_0 + 1 + m, \qquad m = m(k) \geq 0$$

then on the support of χ_k

$$R^{-1} \geq r^{-1}2^{m+k^*}$$

and

$$R^{-1}\|h\chi_k(D_r - A_0)u\|^2_{L^2(drd\bar{g})} \geq O(2^m)\|r^{-1/2}h\chi_k(D_r - A_0)u\|^2_{L^2(drd\bar{g})}. \tag{2.25}$$

Dividing by R the right hand side of (2.23) we have

$$\begin{aligned}O(2^m)\|r^{-1/2}h\chi_k(D_r-A_0)u\|^2_{L^2(drd\bar g)} \lesssim & |R^{-1}Im\left(a_k(D_r-A_0)u+\frac{\chi_k^2}{2i}u,\mathcal{P}u\right)_{drd\bar g}|\\ &+|R^{-1}\varepsilon Re(a_ku,(D_r-A_0)u)_{drd\bar g}|\\ &+|R^{-1}\frac{h^2}{4}(u,(\chi_k^2)''u)_{drd\bar g}|+|R^{-1}(u,\frac{1}{2i}a_k[A_0,M(r)]u)_{drd\bar g}|\\ &+|R^{-1}(u,\frac{1}{2i}h^2a_kD_r(V_m)u)_{drd\bar g}|. \end{aligned}\tag{2.26}$$

We divide the proof in two steps.

(1) Consider a_k with $k\geq k_0+2$. Thanks to Remark 2.15 for all $k\geq k_0+2$ we can use inequality (2.22). Take for example the first term in the right hand side of (2.26), inserting the partition of unity we obtain

$$\begin{aligned}|R^{-1}Im\left(a_k(D_r-A_0)u+\frac{\chi_k^2}{2i}u,\mathcal{P}u\right)_{drd\bar g}| \lesssim & Im(a_k\chi_{k_0}|(D_r-A_0)u|,\chi_{k_0}|\mathcal{P}u|)_{drd\bar g}\\ &+\sum_{l\geq k_0+1}Im\left(a_k\varphi(2^{-l}r)\chi_l|(D_r-A_0)u|,\chi_l|\mathcal{P}u|\right)_{drd\bar g}\\ &+|Im\left(\frac{\chi_k^2}{2i}u,\mathcal{P}u\right)_{drd\bar g}|\\ \lesssim & O(2^m)Im(\chi_{k_0}|(D_r-A_0)u|,\chi_{k_0}|\mathcal{P}u|)_{drd\bar g}\\ &+O(2^m)\sum_{l\geq k_0+1}Im\left(\varphi(2^{-l}r)\chi_l|(D_r-A_0)u|,\chi_l|\mathcal{P}u|\right)_{drd\bar g}\\ &+|Im\left(\frac{\chi_k^2}{2i}u,\mathcal{P}u\right)_{drd\bar g}|\\ \lesssim & O(2^m)\|u\|^2_{H^1(r\simeq R)}+O(2^m)\|\mathcal{P}u\|^2_{B_{>R}}\\ &+O(2^m)\frac{c}{h^2\delta}\|\mathcal{P}u\|^2_{B_{>R}}+O(2^m)c\delta\|u\|^2_{H^1,B^*_{>R}}, \end{aligned}$$

where we used item i) of Lemma 2.17. We can then eliminate the unbounded factor $O(2^m)$ (coming from the bound $a_k\leq O(2^k)$) with the one on the left hand side of (2.26). Doing the same with all the other terms in the right hand side of (2.26) and applying the inequalities in Lemma 2.17 we conclude that for all $k\geq k_0+2$

$$\begin{aligned}\|r^{-1/2}h\chi_k(D_r-A_0)u\|^2_{L^2(drd\bar g)} \leq & c\delta\|u\|^2_{H^1,B^*_{>R}}+c\frac{1}{\delta h^2}\|\mathcal{P}u\|^2_{B_{>R}}\\ &+c(\delta)\|u\|^2_{H^1(r\simeq R)}+\|u\|^2_{H^1(K(\delta))}. \end{aligned}\tag{2.27}$$

(2) Let $k=k_0+1$, we recall that for a_{k_0+1} we have the following bound

$$\begin{aligned}(a_{k_0+1}\cdot,\cdot)_{drd\bar g} \leq & ((\chi_{k_0}+\chi_{k_0+1})a_{k_0+1}\cdot,(\chi_{k_0}+\chi_{k_0+1})\cdot)_{drd\bar g}\\ &+\sum_{k\geq k_0+2}(\varphi(2^{-k}r)\chi_ka_{k_0+1}\cdot,\chi_k\cdot)_{drd\bar g}. \end{aligned}$$

We can then repeat the same argument as in the previous step and obtain

$$\|r^{-1/2}h\chi_{k_0+1}(D_r - A_0)u\|^2_{L^2(drd\bar{g})} \leq c\delta\|u\|^2_{H^1,B^*_{>R}} + c\frac{1}{\delta h^2}\|\mathcal{P}u\|^2_{B_{>R}} + c(\delta)\|u\|^2_{H^1(r\simeq R)} + \|u\|^2_{H^1(K(\delta))}. \tag{2.28}$$

Thanks to (2.27) and (2.28) we have the desired bound on

$$\sup_{k\geq k_0+1}\|r^{-1/2}\chi_k h(D_r - A_0)u\|^2_{L^2(drd\bar{g})}$$

and we conclude the proof.

□

2.2.1. *Auxiliary lemmas.* In this section we provide the results needed in Lemma 2.17.

Lemma 2.18. *For any $\delta \in (0,1)$ there exists $c > 0$ such that*

$$\sum_{l\geq k_0+1} Im\Big(\varphi(2^{-l}r)\chi_l|(D_r - A_0)u|, \chi_l|\mathcal{P}u|\Big)_{drd\bar{g}} \leq \frac{c}{h^2\delta}\|\mathcal{P}u\|^2_{B_{>R}} + c\delta\|u\|^2_{H^1,B^*_{>R}}$$

and

$$|Im\left(\frac{\chi_k^2}{2i}u, \mathcal{P}u\right)_{drd\bar{g}}| \leq \frac{c}{\delta}\|\mathcal{P}u\|^2_{B_{>R}} + c\delta\|u\|^2_{H^1,B^*_{>R}}.$$

Proof. Thanks to the duality of the norms $B^*_{>R}, B_{>R}$ we can insert the weights $r^{1/2}$, $r^{-1/2}$ and h, h^{-1}, then by Cauchy-Schwarz inequality

$$\begin{aligned}\sum_{l\geq k_0+1} Im\Big(\varphi(2^{-l}r)\chi_l|(D_r - A_0)u|, \chi_l|\mathcal{P}u|\Big)_{drd\bar{g}} &\leq \sum_{l\geq k_0+1}\|r^{-1/2}\chi_k h(D_r - A_0)u\|_{L^2(drd\bar{g},D_k)} \\ &\quad \frac{1}{h}\|r^{1/2}\mathcal{P}u\|_{L^2(drd\bar{g},D_k)} \\ &\lesssim \|h(D_r - A_0)u\|_{B^*_{>R}}\frac{1}{h}\|\mathcal{P}u\|_{B_{>R}} \\ &\lesssim \delta\|h(D_r - A_0)u\|^2_{B^*_{>R}} + \frac{1}{h^2\delta}\|\mathcal{P}u\|^2_{B_{>R}}.\end{aligned}$$

We do the same for the second inequality

$$|Im\left(\frac{\chi_k^2}{2i}u, \mathcal{P}u\right)_{drd\bar{g}}| = |Re\left(\frac{\chi_k}{2}r^{-1/2}u, \chi_k r^{1/2}\mathcal{P}u\right)_{drd\bar{g}}| \lesssim \frac{\delta}{4}\|u\|^2_{B^*_{>R}} + \frac{1}{4\delta}\|\mathcal{P}u\|^2_{B_{>R}}.$$

□

Lemma 2.19. *For any $\delta \in (0,1)$ there exists $c > 0$ such that*

$$\varepsilon|Re(a_k u, (D_r - A_0)u)_{drd\bar{g}}| \leq O(2^m)(\delta c\|u\|^2_{B^*_{>R}} + \frac{1}{\delta h}c\|\mathcal{P}u\|^2_{B_{>R}}) + O(2^m)\frac{c}{\delta}\|u\|_{H^1(r\simeq R)}.$$

Proof. Let $\tilde{\phi}$ a smooth function which is one on $supp\, a_k$, we can apply Lemmas 2.9 and 2.10, which yield

$$\begin{aligned}|R^{-1}\varepsilon Re(a_k u,(D_r-A_0)u)_{drd\bar{g}}| \lesssim & O(2^m)\frac{\varepsilon}{h}\|\tilde{\phi}u\|_{L^2(drd\bar{g})}\|h(D_r-A_0)(\tilde{\phi}u)\|_{L^2(drd\bar{g})}\\ &+O(2^m)\frac{\varepsilon}{h}\|\tilde{\phi}u\|_{L^2(drd\bar{g})}\|D_r(\tilde{\phi})u\|_{L^2(drd\bar{g})}\\ \lesssim & O(2^m)\frac{1}{h}(\delta c\|u\|^2_{B^*_{>R}}+\frac{1}{\delta}c'\|\mathcal{P}(\tilde{\phi}u)\|^2_{B_{>R}})\end{aligned}$$

Choosing $\delta = h\delta'$ and applying Lemma 2.7 we conclude

$$\begin{aligned}|R^{-1}\varepsilon Re(a_k u,(D_r-A_0)u)_{drd\bar{g}}| \lesssim & O(2^m)(\delta' c\|u\|^2_{B^*_{>R}}+\frac{c'}{\delta' h^2}\|\mathcal{P}(\tilde{\phi}u)\|^2_{B_{>R}})\\ \lesssim & O(2^m)(\delta' c\|u\|^2_{B^*_{>R}}+\frac{c'}{\delta' h^2}\|\mathcal{P}u\|^2_{B_{>R}}\\ &+\frac{c'}{\delta'}\|u\|^2_{H^1(r\simeq R)}).\end{aligned}$$

□

Lemma 2.20. *Let $k \geq k_0+1$ of the form $k = k_0+m+1$ for some $m = m(k) \geq 1$, then for any $\delta \in (0,1)$ there exist $c>0$ and $K(\delta) \subset (R,+\infty)\times S$ bounded such that*

$$|\frac{h^2}{4}(u,(\chi_k^2)''u)_{drd\bar{g}}| \leq \delta c\|u\|^2_{B^*_{>R}} + c\|u\|^2_{L^2(K(\delta))}.$$

Proof. By definition $\chi_k = \chi(2^{-k}\cdot)$, hence

$$(\chi_k^2)'' = 2^{1-2k}(\chi'(2^{-k}\cdot))^2 + 2^{1-2k}\chi_k\chi''(2^{-k}\cdot)$$

where $2^{-2k} = O(2^{-2m})$ and

$$|\frac{h^2}{4}(u,(\chi_k^2)''u)_{drd\bar{g}}| \leq h^2 O(2^{-2m})(\|\chi'\|_\infty + \|\chi''\|_\infty)\|u\|^2_{L^2(drd\bar{g},D_k)}.$$

Let $\delta \in (0,1)$, if m is large enough such that $O(2^{-2m}) < \delta$ then we directly have

$$|\frac{h^2}{4}(u,(\chi_k^2)''u)_{drd\bar{g}}| \leq \delta c\|u\|^2_{B^*_{>R}}.$$

Otherwise for all m such that $2^{-2m} > \delta$ the interval D_k is a bounded one, albeit depending on δ. Hence

$$|\frac{h^2}{4}(u,(\chi_k^2)''u)_{drd\bar{g}}| \leq c\|u\|^2_{L^2(K(\delta))}.$$

□

Lemma 2.21. *For any $\delta \in (0,1)$ there exist $c, c(\delta) > 0$ andK$(\delta) \subset (R,+\infty)\times S$ bounded such that*

$$\sum_{l\geq k_0+1}(\varphi(2^{-l}r)\chi_l|u|, \frac{1}{2i}\chi_l|[A_0,M(r)]u|)_{drd\bar{g}} \lesssim (c\delta\|u\|^2_{H^1,B^*_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))}).$$

Proof. The proof is analogous to the one of Lemma 2.12.

□

Lemma 2.22. *For any $\delta \in (0,1)$ there exist $c, c(\delta) > 0$ and$K(\delta) \subset (R, +\infty) \times S$ bounded such that*

$$\sum_{l \geq k_0+1} (\varphi(2^{-l}r)\chi_l|u|, \frac{1}{2i}h^2\chi_l|D_r(V_m)u|)_{drd\bar{g}} \lesssim (c\delta\|u\|^2_{H^1,B^*_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))}).$$

Proof. The proof is analogous to the one of Lemma 2.13. □

2.3. **Estimating u.** In this section we give bounds on the norm $\|u\|_{B^*_{>R}}$.

Let χ_k as defined in (1.20), taking the real part in the following scalar product we have

$$Re(\chi_k^2 u, \mathcal{P}u)_{drd\bar{g}} = -\|\chi_k u\|^2_{L^2(drd\bar{g})} + Re(\chi_k^2 u, h^2 P_{\bar{g}} u)_{drd\bar{g}}.$$

As in the previous section we write $k = k_0 + m + 1$ for a certain $m = m(k) \geq 0$. With this notation, on the support of χ_k

$$2^{k_0-1}2^m \leq r \leq 2^{k_0+3}2^m \tag{2.29}$$

where $k_0 = \frac{\ln R}{\ln 2} + k^* + 2$ and k^* is a fixed natural number. In particular

$$2^{k^*+1+m}R \leq r \leq 2^{k^*+5+m}R \tag{2.30}$$

which implies that, for some $c > 0$, it holds

$$\begin{aligned} 2^m c\|r^{-1/2}\chi_k u\|^2_{L^2(drd\bar{g})} \leq & R^{-1}\|\chi_k u\|^2_{L^2(drd\bar{g})} \\ \leq & R^{-1}|Re(\chi_k^2 u, h^2 P_{\bar{g}} u)_{drd\bar{g}}| \qquad (2.31) \\ & + R^{-1}|(r^{-1/2}\chi_k u, r^{1/2}\chi_k \mathcal{P}u)_{drd\bar{g}}|. \qquad (2.32) \end{aligned}$$

The main result of this section is:

Proposition 2.23. *For all $\delta \in (0,1)$ there exist $c, c(\delta) > 0$ and$K(\delta) \subset (R, +\infty) \times S$ bounded such that*

$$\|u\|^2_{B^*_{>R}} \leq \frac{c'}{\delta h^2}\|\mathcal{P}u\|^2_{B_{>R}} + \delta c\|u\|^2_{H^1,B^*_{>R}} + c(\delta)\|u\|^2_{H^1(r\simeq R)} + \|u\|^2_{H^1(K(\delta))}.$$

Proof. For the term in (2.32) we can plainly use the duality of the norms $B^*_{>R}$, $B_{>R}$. In (2.31) we have a term made of several parts. First we consider the one involving $h^2(D_r - A_0)^2$, using (2.24)

$$\begin{aligned} Re(\chi_k^2 u, h^2(D_r - A_0)^2 u)_{drd\bar{g}} = & (u, \frac{h^2}{2}\left(\chi_k^2(D_r - A_0)^2 + (D_r - A_0)^2\chi_k^2\right)u)_{drd\bar{g}} \\ = & (u, h^2(D_r - A_0)(\chi_k^2(D_r - A_0))u)_{drd\bar{g}} - h^2(u, (\chi_k^2)''u)_{drdg} \end{aligned}$$

Since (2.30) holds on the support of χ_k we also have for some positive constant c

$$R^{-1} \leq c2^m r^{-1}$$

and hence

$$\begin{aligned} |R^{-1}Re(\chi_k^2 u, h^2(D_r - A_0)^2 u)_{drd\bar{g}}| \lesssim & 2^m\|h(D_r - A_0)u\|^2_{B^*_{>R}} + h^2R^{-1}|(u, (\chi_k^2)''u)_{drd\bar{g}}| \\ \lesssim & 2^m(\frac{c}{\delta h^2}\|\mathcal{P}u\|^2_{B_{>R}} + \delta c\|u\|^2_{H^1,B^*_{>R}} \\ & + c(\delta)\|u\|^2_{H^1(r\simeq R)} + \|u\|^2_{H^1(K(\delta))}) \qquad (2.33) \end{aligned}$$

thanks to Proposition 2.14 and Lemma 2.20. Applying (2.11) we also obtain

$$\begin{aligned}|R^{-1}(\chi_k u, \chi_k M(r)u)_{drd\bar{g}}| \lesssim & 2^m \|r^{-1/2}M(r)^{1/2}u\|_{L^2(drd\bar{g},D_k)} \\ \leq & 2^m\Big(\frac{c}{\delta h^2}\|\mathcal{P}u\|^2_{B_{>R}} + \delta c\|u\|^2_{H^1,B^*_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))}\Big). \end{aligned} \tag{2.34}$$

Finally, we can exploit the decay of $V_m - \Lambda$ given by (1.15) and obtain

$$\begin{aligned}R^{-1}|(\chi_k u, \chi_k h^2(V_m - \Lambda)u)_{drd\bar{g}}| \lesssim & 2^m |(r^{-1/2}\chi_k u, r^{-1/2-\nu}\chi_k h^2 u)_{drd\bar{g}}| \\ \lesssim & 2^{m-m\nu}h^2\|r^{-1/2}u\|^2_{L^2(drd\bar{g},D_k)}.\end{aligned}$$

Now for any $\delta \in (0,1)$ there exists an $m^* = m^*(\delta) \geq 0$ such that for all $m \geq m^*$ we have $2^{-m\nu} < \delta$. Hence the previous inequality, if k is large enough to satisfy $m(k) \geq m^*$, renders

$$R^{-1}|(\chi_k u, \chi_k h^2(V_m - \Lambda)u)_{drd\bar{g}}| \lesssim 2^m \delta c\|u\|^2_{B^*_{>R}}. \tag{2.35}$$

Otherwise, the scalar product is bounded by the L^2 norm of u over a compact interval of r, depending on δ. Thanks to (2.33), (2.34) and (2.35) we have

$$2^m\|r^{-1/2}\chi_k u\|^2_{L^2(drd\bar{g})} \lesssim 2^m\Big(\frac{c}{\delta h^2}\|\mathcal{P}u\|^2_{B_{>R}} + \delta c\|u\|^2_{H^1,B^*_{>R}} + c(\delta)\|u\|^2_{H^1(r\simeq R)} + \|u\|^2_{H^1(K(\delta))}\Big)$$

for any $k \geq k_0 + 1$ which yields the statement.

□

3. Estimates in the compact region: unique continuation

By the result of Proposition 2.1 in the previous section we have

$$\|u\|^2_{H^1,B^*_{>R}} \leq c\delta\|u\|^2_{H^1,B^*_{>R}} + \frac{c}{\delta h^2}\|\mathcal{P}u\|^2_{B_{>R}} + c(\delta)\|u\|^2_{H^1(drd\bar{g},K(\delta))}.$$

with $K(\delta)$ a bounded region in $(R,+\infty)\times S$. Now fix δ_0 such that $c\delta_0 < 1$, then

$$\|u\|^2_{H^1,B^*_{>R}} \leq \frac{c}{\delta_0 h^2}\|\mathcal{P}u\|^2_{B_{>R}} + c(\delta_0)\|u\|^2_{H^1(K(\delta_0))}, \tag{3.1}$$

that is we can bound the $H^1, B^*_{>R}$ norm of u (which is a norm on the manifold end) by the operator $\mathcal{P} = h^2 P_{\bar{g}} - 1 + i\varepsilon$ up to a compactly supported term.

In this section we show how to bound the H^1 norm on a compact region of the manifold by applying unique continuation results.

Notation 5. We define the notation

$$X_R := (R,+\infty)\times S \simeq M \setminus K$$

and in general

$$X_a := (a,+\infty)\times S$$

so that $K(\delta_0) \subset X_R$, more precisely $K(\delta_0) \subset X_{2^{k^*}R}$ Let $a > R$ such that $K(\delta_0) \subset X_a$ and without loss of generality we can assume $K(\delta_0) \subset X_a \setminus X_{a+2}$.

The main result of this section is

Proposition 3.1. *Fix $\lambda_0 > 0$. Let $u \in H^2(M)$, $\lambda > \lambda_0 > 0$, $h = \lambda^{-1}$ and $a > R$ such that $K(\delta_0) \subset X_a \setminus X_{a+2}$. There exist $\gamma_0 \in (0,1)$ and U bounded region of $X_{2^{k^*}R} = (2^{k^*}R, +\infty) \times S$ such that*

$$\|u\|_{H^1(M\setminus X_{2^{k^*}R})} \leq O(e^{\lambda/\gamma_0})(\|(P_m - \lambda^2 + i\varepsilon')u\|_{L^2(M\setminus X_{a+3})} + \|u\|_{H^1(U)}).$$

The following proposition is a direct application of unique continuation.

Proposition 3.2. *Let (M_0, g_0) an n dimensional Riemannian manifold, T the Laplace-Beltrami operator on M_0 and $\mathcal{R}$ a differential operator of order one. Let*

$$U_0 \Subset V_0 \Subset M_0 \qquad V_0' \Subset V_0 \Subset M_0 \quad \overline{V}_0 \cap \partial M_0 = \emptyset,$$

$\alpha \in (0, 1/2)$ and $z \in \mathbb{C}$ with $Rez > z_0 > 0$, $|Imz| \neq 0$. Then there exist $c(z_0) > 0$ and $\gamma_0 \in (0,1)$ such that

$$\|u\|_{H^1(V_0')} \leq c(z_0)e^{|z|/\gamma_0}(\|(T + \mathcal{R} - z^2)u\|_{L^2(V_0)} + \|u\|_{L^2(U_0)})$$

for all $u \in H^2(V_0)$.

Proof. Define

$$\begin{aligned} M_1 = (-1,1) \times M_0, \quad & \tilde{\sigma} = (-1+2\alpha, 1-2\alpha) \times U_0, \\ & \sigma = (-1,1) \times U_0, \\ & U = (-1+\alpha, 1-\alpha) \times V_0', \\ & \tilde{V} = (-1+\alpha/2, 1-\alpha/2) \times V_0, \\ & V = (-1,1) \times V_0, \end{aligned}$$

then $\overline{U} \cap \partial M_1 = \emptyset$, $\tilde{\sigma}$ is an open subset of U and $U \Subset V \Subset M_1$. Let us also consider

$$T - \partial_t^2 + \mathcal{R}, \qquad v(t,m) = e^{tz}u(m) \in H^2(V)$$

and f such that $(T - \partial_t^2 + \mathcal{R})v = f$.

We apply [12, Theorem 9.1] to the sets $\tilde{\sigma}, U, \tilde{V}$ and the operator $T - \partial_t^2 + \mathcal{R}$. Hence there exist $c > 0$ and $\gamma_0 \in (0,1)$ for which

$$\begin{aligned} \|v\|_{H^1(U)} &\leq c\|v\|^{1-\gamma_0}_{H^1(\tilde{V})}(\|(T - \partial_t^2 + \mathcal{R})v\|_{L^2(\tilde{V})} + \|v\|_{L^2(\tilde{\sigma})})^{\gamma_0} \\ &\leq c\|v\|^{1-\gamma_0}_{H^1(V)}(\|(T - \partial_t^2 + \mathcal{R})v\|_{L^2(V)} + \|v\|_{L^2(\sigma)})^{\gamma_0} \end{aligned} \tag{3.2}$$

where

$$(T - \partial_t^2 + \mathcal{R})v = e^{tz}(T + \mathcal{R} - z^2)u(m).$$

Computing the integrals with respect to t in (3.2) yields

$$\begin{aligned} (|z|^2\|u\|^2_{L^2(V_0')} + \|\nabla_{g_0}u\|^2_{L^2(V_0')} + \|u\|^2_{L^2(V_0')})^{\gamma_0/2} \leq & c\frac{e^{Rez} + e^{-Rez}}{e^{Rez(1-\alpha)} - e^{-Rez(1-\alpha)}} \\ & (\|(T + \mathcal{R} - z^2)u\|_{L^2(V_0)} + \|u\|_{L^2(U_0)})^{\gamma_0}, \end{aligned}$$

where we used the relation $b^{1/2} - b^{-1/2} \leq (b - b^{-1})^{1/2} \leq b^{1/2} + b^{-1/2}$ which holds for $b \geq 1$. The left hand side can be bounded from below by $O(1)\|u\|^{\gamma_0}_{H^1(V_0)}$ (since $\min\{|z|^2, 1\}$ is a strictly

positive constant). On the other hand since $Rez(1-\alpha) > z_0/2 > 0$ there exists $c' = c'(z_0)$ such that $e^{Rez(1-\alpha)} - e^{-Rez(1-\alpha)} > c'$. We conclude that

$$\|u\|_{H^1(V_0')} \leq c\frac{2e^{|z|/\gamma_0}}{c'(z_0)}(\|(T+\mathcal{R}-z^2)u\|_{L^2(V_0)} + \|u\|_{L^2(U_0)}).$$

□

With these result we can now prove the main result of this section.

Proof of Proposition 3.1. We apply Proposition 3.2. Choose

$$U_0 \Subset X_{a+2} \setminus X_{a+3},\ V_0' = M \setminus X_{a+2}\ V_0 = M \setminus X_{a+3},\ M_0 = M \setminus X_{a+4}$$

and we apply Proposition 3.2 to the function $\chi_0 u$ with

$$\chi_0 = \begin{cases} 1 & \text{on } M \setminus X_{2^{k^*}R} \\ \in (0,1) & \text{on } U = X_{2^{k^*}R} \setminus X_{a+2} \\ 0 & \text{on } X_{a+2} \end{cases} \tag{3.3}$$

so that $\chi_0 u \equiv 0$ on U_0 and

$$K(\delta_0) \subset U. \tag{3.4}$$

Taking $T + \mathcal{R} - z^2 = P_m - \lambda^2 + i\varepsilon'$ results in

$$\|u\|_{H^1(M\setminus X_{2^{k^*}R})} \leq O(e^{\lambda/\gamma_0})(\|(P_m - \lambda^2 + i\varepsilon')u\|_{L^2(M\setminus X_{a+3})} + \|u\|_{H^1(U)}). \tag{3.5}$$

□

Thanks to (3.4) we can bound the perturbative term in (3.1) by $\|u\|_{H^1(U)}$, so (3.1) becomes

$$\|u\|^2_{H^1,B^*_{>R}} \leq \frac{c}{\delta_0 h^2}\|\mathcal{P}u\|^2_{B_{>R}} + c(\delta_0)\|u\|^2_{H^1(U)}$$

and a combination with Proposition 3.1 yields

$$\begin{aligned}\|u\|^2_{H^1(M\setminus X_{2^{k^*}R})} + \|u\|^2_{H^1,B^*_{>R}} \leq & O(h^{-2}e^{\lambda/\gamma_0})\|\mathcal{P}u\|^2_{L^2(M\setminus X_{a+2})} \\ & + O(h^{-2})\|\mathcal{P}u\|^2_{B_{>R}} + O(e^{\lambda/\gamma_0})\|u\|^2_{H^1(U)}.\end{aligned} \tag{3.6}$$

Notation 6. In the previous section we had actually set $\mathcal{P} = h^2 P_{\overline{g}} - 1 + i\varepsilon$. With an abuse of notation we use the same symbol to denote the corresponding quantity on the whole manifold, that is $h^2 P_m - 1 + i\varepsilon$.

The rest of the paper will be devoted to eliminate the perturbative term of exponential size $O(e^{\lambda/\gamma_0})\|u\|^2_{H^1(U)}$.

4. Estimates on the exponential remainder

In this section we consider $\lambda \gg 1$ in

$$P - \lambda^2 + i\varepsilon',$$

we recall the (slight abuse of) notation

$$\mathcal{P} = h^2 P_m - 1 + i\varepsilon$$

with $\varepsilon = O(h^2)$ in which now $h = \lambda^{-1} \ll 1$. The aim of this section is to prove the following result, which implies Theorem 1.1.

Theorem 4.1. *Let $u \in H^2(M)$, $\lambda \gg 1$, $R < a < 2^{k^*}R < a+3$, then there exists $C > 0$ such that*

$$\begin{aligned}\|u\|^2_{H^1(M\setminus X_{2^{k^*}R})} + \|u\|^2_{H^1,B^*_{>R}} \leq & O(\lambda^{-2}e^{\lambda C})\|(P_m - \lambda^2 + i\varepsilon)u\|^2_{L^2(M\setminus X_{a+3})} \\ & + O(\lambda^{-2}e^{\lambda C})\|(P_m - \lambda^2 + i\varepsilon)u\|^2_{B_{>R}}.\end{aligned}$$

More precisely, what we will be able to prove is

$$\begin{aligned}\|u\|^2_{L^2(M\setminus X_{a+3})} + O(e^{\lambda C})\|u\|^2_{B^*_{>R}} \leq & O(\lambda^{-2}e^{\lambda C})\|(P_m - \lambda^2 + i\varepsilon)u\|^2_{L^2(M\setminus X_{a+3})} \\ & + O(\lambda^{-2}e^{\lambda C})\|(P_m - \lambda^2 + i\varepsilon)u\|^2_{B_{>R}},\end{aligned} \tag{4.1}$$

see Remark 4.7 for further details. By simply considering a function supported sufficiently far at radial infinity we can then derive Corollary 1.2, that is

Corollary 4.2. *Let $u \in H^2(M)$, $\lambda \gg 1$, $R < a < 2^{k^*}R < a+3$ and χ a smooth cutoff such that $\chi \equiv 0$ on $M \setminus X_{a+3}$, $\chi \equiv 1$ on X_{a+4}. Then*

$$\|\chi u\|^2_{B^*_{>R}} \leq O(\lambda^{-2})\|(P_m - \lambda^2 + i\varepsilon)\chi u\|^2_{B_{>R}}.$$

In particular

$$\|r^{-1/2-\mu}\chi(P_m - \lambda^2 + i\varepsilon)^{-1}\chi r^{-1/2-\mu}\|_{L^2\to L^2} = O(\lambda^{-1})$$

with $\mu > 0$.

Proof. The $B_{>R} \to B^*_{>R}$ bound follows directly from inequality (4.1) thanks to the support of χ. To recover the norm in the weighted L^2 space we just remark the inclusions

$$L^2_{1/2+\mu} \hookrightarrow B_{>R}, \quad B^*_{>R} \hookrightarrow L^2_{-1/2-\mu}$$

which can be proved by direct computations. □

As we pointed out above, the main concern is now to take care of the exponentially large remainder in (3.6). To do so we can exploit the weight φ constructed in [6, Section 2]. In particular let $a > R$ such that

$$\varphi'(r) = \frac{1}{\lambda r} \quad r \geq a. \tag{4.2}$$

Moreover, $\varphi > \gamma_0^{-1}$ for all $r \geq R+2$ for a parameter $\gamma_0 > 0$ independent of λ.

Remark 4.3. *For $r \geq a$ we have*

$$\varphi(r) - \varphi(a) = \frac{1}{\lambda}\ln\left(\frac{r}{a}\right)$$

and hence the quantity

$$e^{\lambda(\varphi(r)-\varphi(a))} = \frac{r}{a} \qquad r \geq a$$

is independent of λ.

Remark 4.4. *The subset U defined in (3.3) is contained in X_{R+2}, so $\varphi > \gamma_0^{-1}$ on U and therefore*

$$e^{2\lambda/\gamma_0}\|u\|^2_{H^1(U)} = \|e^{\lambda(1/\gamma_0-\varphi)}e^{\lambda\varphi}u\|^2_{H^1(U)} \leq O(e^{-c\lambda})\|e^{\lambda\varphi}u\|_{H^1(U)}.$$

We will use the following properties of φ which are due to [6].

Lemma 4.5 (Lemma 2.1 [6]). *Let $\delta \in (0,1)$. There exists $C > 0$ such that following inequalities hold for $\lambda > \lambda(\delta) \gg 1$ and $r > R$:*

$$\begin{gathered} C\lambda^{-1}r^{-1} \leq \varphi', \\ -\varphi'\varphi'' \leq C\delta r^{-1} \\ |\varphi''| \leq C\lambda^{1/2}r^{-1}\varphi', \qquad (\varphi'')^2 \leq C\lambda^{1/2}r^{-1}\varphi' \\ |\varphi'''| \leq C\lambda r^{-1}\varphi', \qquad |\varphi'''| \leq C\lambda^{1/2}r^{-1}, \\ |\varphi^{(4)}| \leq C\lambda^{3/2}r^{-1}\varphi' \end{gathered}$$

We will conclude the proof of Theorem 4.1 thanks to the following proposition.

Proposition 4.6. *Let $v \in H^2(X_R \setminus X_{a+4})$ such that $v = \partial_r v = 0$ on ∂X_R, $\lambda \gg 1$ and $a > R$ such that $\varphi' = \lambda^{-1}r^{-1}$ for $r \geq a$. Then*

$$\begin{aligned} \|e^{\lambda(\varphi-\varphi(a))}v\|^2_{H^1(X_R\setminus X_{a+3})} \leq & O(h^{-2})\|e^{\lambda(\varphi-\varphi(a))}\mathcal{P}v\|^2_{L^2(X_R\setminus X_a)} + O(h^{-2})\|\mathcal{P}v\|_{L^2(X_a\setminus X_{a+4})} \\ & + O(h^{-2})\|\mathcal{P}v\|^2_{B_{>R}}. \end{aligned}$$

The proof, being quite technical, will be postponed to the end of this section, we first show how its application allows us to pass from (3.6) to the result in Theorem 4.1.

Let v such that the assumptions of Proposition 4.6 are satisfied and recall $K(\delta_0) \subset U$. In particular $r > a$ on $K(\delta_0)$ and $\varphi' \geq 0$ implies $\varphi - \varphi(a) > 0$ on $K(\delta_0)$. Then there is a positive constant c such that

$$\|v\|_{H^1(K(\delta_0))} \leq e^{-\lambda c}\|e^{\lambda(\varphi-\varphi(a))}v\|_{H^1(K(\delta_0))} \leq e^{-\lambda c}\|e^{\lambda(\varphi-\varphi(a))}v\|_{H^1(X_R\setminus X_{a+2})}. \tag{4.3}$$

since $K(\delta_0) \subset X_R \setminus X_{a+2}$ and hence

$$e^{2\lambda\varphi(a)}\|v\|^2_{H^1(K(\delta_0))} \leq e^{-\lambda c}\|e^{\lambda\varphi}v\|^2_{H^1(X_R\setminus X_{a+2})} \tag{4.4}$$

From Proposition 4.6 we obtain

$$\begin{aligned} \|e^{\lambda\varphi}v\|^2_{H^1(X_R\setminus X_{a+3})} \leq & O(h^{-2})\|e^{\lambda\varphi}\mathcal{P}v\|^2_{L^2(X_R\setminus X_a)} + O(h^{-2}e^{\lambda\varphi(a)})\|\mathcal{P}v\|_{L^2(X_a\setminus X_{a+4})} \\ & + O(h^{-2}e^{\lambda\varphi(a)})\|\mathcal{P}v\|^2_{B_{>R}}, \end{aligned}$$

nevertheless we can replace the left hand side with $\|e^{\lambda\varphi}v\|^2_{H^1(X_R\setminus X_{a+3})} + e^{2\lambda\varphi(a)}\|v\|^2_{H^1,B^*_{>R}}$. Indeed, applying (3.1) to v we have

$$e^{2\lambda\varphi(a)}\|v\|^2_{H^1,B^*_{>R}} \leq e^{2\lambda\varphi(a)}\frac{c}{\delta_0 h^2}\|\mathcal{P}v\|^2_{B_{>R}} + c(\delta_0)e^{2\lambda\varphi(a)}\|v\|^2_{H^1(K(\delta_0))}$$

and (4.4) implies we can absorb the remainder term in the left hand side. We have obtained

$$\begin{aligned} \|e^{\lambda\varphi}v\|^2_{H^1(X_R\setminus X_{a+3})} + e^{2\lambda\varphi(a)}\|v\|^2_{H^1,B^*_{>R}} \leq & O(h^{-2})\|e^{\lambda\varphi}\mathcal{P}v\|^2_{L^2(X_R\setminus X_a)} \\ & + O(h^{-2}e^{2\lambda\varphi(a)})\|\mathcal{P}v\|^2_{L^2(X_a\setminus X_{a+4})} \\ & + O(h^{-2}e^{2\lambda\varphi(a)})\|\mathcal{P}v\|^2_{B_{>R}}. \end{aligned} \tag{4.5}$$

Proof of Theorem 4.1. We define

$$\tilde{\chi}_1 = \begin{cases} 0 & M \setminus X_{b_1} \\ \in (0,1) & X_{b_1} \setminus X_{b_2} \\ 1 & X_{b_2} \end{cases}$$

with $R < b_1 < b_2 < R+1$ such that $\varphi < -c < 0$ on $[b_1, b_2]$ and apply (4.5) to $\tilde{\chi}_1 u$, yielding

$$\begin{aligned} \|e^{\lambda\varphi}u\|^2_{H^1(X_{b_2}\setminus X_{a+3})} + e^{2\lambda\varphi(a)}\|u\|^2_{H^1,B^*_{>R}} \leq\ & O(h^{-2})\|e^{\lambda\varphi}\mathcal{P}u\|^2_{L^2(X_R\setminus X_a)} \\ & + O(h^{-2}e^{2\lambda\varphi(a)})\|\mathcal{P}u\|^2_{L^2(X_a\setminus X_{a+4})} \\ & + O(h^{-2}e^{2\lambda\varphi(a)})\|\mathcal{P}u\|^2_{B_{>R}} \\ & + O(e^{-c\lambda})\|u\|_{H^1(X_{b_1}\setminus X_{b_2})}. \end{aligned}$$

In this inequality we have another compactly supported remainder term that we wish to absorb, since the pre-factor is a small one. To do so we need to add a term on the left hand side that is supported in a region containing $X_{b_1} \setminus X_{b_2}$, for example $M \setminus X_{2^{k^*}R}$. We can now use (3.6) and Remark 4.4 to add the contribution of this region

$$\begin{aligned} &\|e^{\lambda\varphi}u\|^2_{H^1(X_{b_2}\setminus X_{a+3})} + \|u\|^2_{H^1(M\setminus X_{2^{k^*}R})} + e^{2\lambda\varphi(a)}\|u\|^2_{H^1,B^*_{>R}} \\ &\leq O(h^{-2})\|e^{\lambda\varphi}\mathcal{P}u\|^2_{L^2(X_R\setminus X_a)} + O(h^{-2})e^{2\lambda\varphi(a)}\|\mathcal{P}u\|^2_{L^2(X_a\setminus X_{a+4})} \\ &\quad + O(h^{-2})e^{2\lambda\varphi(a)}\|\mathcal{P}u\|^2_{B_{>R}} + O(h^{-2})e^{2\lambda/\gamma_0}\|\mathcal{P}u\|^2_{L^2(M\setminus X_{a+3})} \\ &\quad + O(e^{-c\lambda})\|u\|^2_{H^1(X_{b_1}\setminus X_{b_2})} + O(e^{-c\lambda})\|e^{\lambda\varphi}u\|^2_{H^1(X_{b_2}\setminus X_{a+2})} \\ &\leq O(h^{-2})\|e^{\lambda\varphi}\mathcal{P}u\|^2_{L^2(X_R\setminus X_a)} + O(h^{-2})e^{2\lambda\varphi(a)}\|\mathcal{P}u\|^2_{B_{>R}} \\ &\quad + O(h^{-2})e^{2\lambda/\gamma_0}\|\mathcal{P}u\|^2_{L^2(M\setminus X_{a+3})} \\ &\quad + O(e^{-c\lambda})\|u\|^2_{H^1(X_{b_1}\setminus X_{b_2})} + O(e^{-c\lambda})\|e^{\lambda\varphi}u\|^2_{H^1(X_{b_2}\setminus X_{a+2})}. \end{aligned} \tag{4.6}$$

Both terms in (4.6) can be absorbed to the left hand side by $\|u\|^2_{H^1(M\setminus X_{2^{k^*}R})}$ and $\|e^{\lambda\varphi}u\|^2_{H^1(X_{b_2}\setminus X_{a+2})}$ respectively. First of all, we remark that after absorption of the remainders this last inequality implies (4.1). Then, thanks to the properties of φ we have

$$e^{\lambda(\varphi-\varphi(a))} \leq 1 \ \text{ on } X_R \setminus X_a, \qquad e^{2\lambda/\gamma_0 - 2\lambda\varphi(a)} \leq 1$$

so dividing everything by $e^{2\lambda\varphi(a)}$

$$e^{-2\lambda\varphi(a)}(\|u\|^2_{H^1(M\setminus X_{2^{k^*}R})} + \|u\|^2_{H^1,B^*_{>R}}) \leq O(h^{-2})\|\mathcal{P}u\|^2_{B_{>R}} + O(h^{-2})\|\mathcal{P}u\|^2_{L^2(M\setminus X_{a+3})}. \tag{4.7}$$

This proves the statement since we recall the rescaling

$$h^{-1}\mathcal{P} = h^{-1}(h^2 P_m - 1 + i\varepsilon) = \lambda^{-1}(P_m - \lambda^2 + i\varepsilon').$$

□

Remark 4.7. *We can obtain inequality* (4.1) *directly from the computations in the previous proof. The right hand side in inequality* (4.6) *also bounds*

$$\|u\|^2_{L^2(X_{2^{k^*}R}\setminus X_{a+3})} + \|u\|^2_{L^2(M\setminus X_{2^{k^*}R})} + e^{2\lambda\varphi(a)}\|u\|^2_{B^*_{>R}}$$

and hence we obtain

$$\begin{aligned}\|u\|^2_{L^2(M\setminus X_{a+3})} + e^{2\lambda\varphi(a)}\|u\|^2_{B^*_{>R}} \leq & O(\lambda^{-2})e^{2\lambda\varphi(a)}\|(P_m-\lambda^2+i\varepsilon')u\|^2_{B_{>R}}\\ & + O(\lambda^{-2}e^{\lambda C})\|(P_m-\lambda^2+i\varepsilon')u\|^2_{L^2(M\setminus X_{a+3})}\\ & + O(\lambda^{-2})\|e^{\lambda\varphi}(P_m-\lambda^2+i\varepsilon')u\|^2_{L^2(X_R\setminus X_a)}\end{aligned}$$

from which (4.1) *follows.*

As announced earlier, we conclude the section with the proof of Proposition 4.6. We will need first the following lemma, which is the equivalent, for a Schrödinger operator with order one perturbation,of [6, Proposition 2.3]. We nevertheless include the proof at the end of this section for the sake of clarity and completeness.

Lemma 4.8. *Let $v \in H^2(X_R\setminus\partial X_a)$ such that $v = \partial_r v = 0$ on $\partial X_R \cup \partial X_a$, $\lambda \gg 1$ and $a > R$ such that $\varphi' = \lambda^{-1}r^{-1}$ for $r \geq a$. Then*

$$\|(\varphi'/r)^{1/2}v\|_{H^1(X_R\setminus X_a)} \leq O(\lambda^{1/2})\|\mathcal{P}_\varphi v\|_{L^2(X_R\setminus X_a)}$$

where $\mathcal{P}_\varphi = e^{\lambda\varphi}\mathcal{P}e^{-\lambda\varphi}$.

Proof of Proposition 4.6. Let v as in the statement of Proposition 4.6 and define

$$\chi_1 = \begin{cases} 1 & \text{on } M\setminus X_{a+3},\\ \in(0,1) & \text{on } X_{a+3}\setminus X_{a+4},\\ 0 & \text{on } X_{a+4}.\end{cases}$$

We apply Lemma 4.8 to $e^{\lambda\varphi}\chi_1 v$ which vanishes, together with its radial derivative, on $\partial X_R \cup \partial X_{a+4}$. This yields

$$\|(\varphi'/r)^{1/2}e^{\lambda\varphi}\chi_1 v\|_{H^1(X_R\setminus X_{a+4})} \leq O(h^{-1/2})\|e^{\lambda\varphi}\mathcal{P}(\chi_1 v)\|_{L^2(X_R\setminus X_{a+4})},$$

where $(\varphi'/r)^{1/2} \geq h^{1/2}r^{-1}$ on $X_R\setminus X_{a+4}$ thanks to the inequality $\varphi' \geq C\lambda^{-1}r^{-1}$. Then

$$\begin{aligned}\|e^{\lambda\varphi}v\|_{H^1(X_R\setminus X_{a+3})} \leq & O(h^{-1})\|e^{\lambda\varphi}\mathcal{P}v\|_{L^2(X_R\setminus X_{a+4})} + O(h^{-1})\|e^{\lambda\varphi}[\mathcal{P},\chi_1]v\|_{L^2(X_{a+3}\setminus X_{a+4})}\\ \leq & O(h^{-1})\|e^{\lambda\varphi}\mathcal{P}v\|_{L^2(X_R\setminus X_{a+4})} + O(h)e^{\lambda\varphi(a+4)}\|v\|_{H^1(X_{a+3}\setminus X_{a+4})}\end{aligned}$$

since $[\mathcal{P},\chi_1] = [h^2(D_r-A_0)^2,\chi_1]$ is supported on the set $\{\chi_1 \in (0,1)\}$ and is an operator of order one in the radial variable. Dividing by $e^{\lambda\varphi(a)}$ and thanks to Remark 4.3

$$\begin{aligned}\|e^{\lambda(\varphi-\varphi(a))}v\|^2_{H^1(X_R\setminus X_{a+3})} \leq & O(h^{-2})\|e^{\lambda(\varphi-\varphi(a))}\mathcal{P}v\|^2_{L^2(X_R\setminus X_a)} + O(h^{-2})\|\mathcal{P}v\|^2_{L^2(X_a\setminus X_{a+4})}\\ & + O(1)\|v\|^2_{H^1(X_{a+3}\setminus X_{a+4})}.\end{aligned}$$

The norm of v can be bounded by the inequality (3.1) on the region at infinity and recalling (4.3) we obtain

$$\|v\|_{H^1(X_{a+3}\setminus X_{a+4})} \leq O(h^{-2})\|\mathcal{P}v\|_{B_{>R}} + e^{-\lambda c}\|e^{\lambda(\varphi-\varphi(a))}v\|_{H^1(X_R\setminus X_{a+2})}$$

from which the statement follows since $e^{-\lambda c}\|e^{\lambda(\varphi-\varphi(a))}v\|_{H^1(X_R\setminus X_{a+2})}$ is an absorbable term. □

Proof of Lemma 4.8. The conjugated operator is given by

$$\mathcal{P}_\varphi = \mathcal{P} - (\varphi')^2 + h\varphi'' + 2i\varphi' h D_r - 2iA_0\varphi'. \tag{4.8}$$

Let $\psi \in C^\infty([R,a])$ real valued, we consider the scalar product

$$Re(\psi\mathcal{P}_\varphi v, v)_{L^2(X_R\setminus X_a)} = Re\int_R^a \int_S \overline{\psi\mathcal{P}_\varphi v}\; v\; dr d\overline{g}.$$

By integration by parts

$$\begin{aligned} Re(\psi h^2(D_r - A_0)^2 v, v)_{L^2(X_R\setminus X_a)} =& Re(\psi h(D_r - A_0)v, h(D_r - A_0)v)_{L^2(X_R\setminus X_a)} \\ &+ Re(h(D_r - A_0)v, hD_r(\psi)v)_{L^2(X_R\setminus X_a)}. \end{aligned}$$

In this expression we notice

$$Re(hA_0 v, hD_r(\psi)v)_{L^2(X_R\setminus X_a)} = Re\frac{1}{i}\int_R^a\int_S h^2 A_0\psi'|v|^2\; dr d\overline{g} = 0$$

and

$$\begin{aligned} Re(hD_r v, hD_r(\psi)v)_{L^2(X_R\setminus X_a)} =& Re\; h^2\int_R^a\int_S \partial_r\overline{v}\psi' v\; dr d\overline{g} \\ =& \frac{h^2}{2}\int_R^a\int_S \partial_r|v|^2\psi'\; dr d\overline{g} = -(\frac{h^2}{2}\psi'' v, v)_{L^2(X_R\setminus X_a)}. \end{aligned}$$

Hence we have

$$\begin{aligned} Re(\psi h^2(D_r - A_0)^2 v, v)_{L^2(X_R\setminus X_a)} =& Re(\psi h(D_r - A_0)v, h(D_r - A_0)v)_{L^2(X_R\setminus X_a)} \\ &- (\frac{h^2}{2}\psi'' v, v)_{L^2(X_R\setminus X_a)}. \end{aligned} \tag{4.9}$$

Moreover

$$\begin{aligned} Re(\psi 2i\varphi' h D_r v, v)_{L^2(X_R\setminus X_a)} =& Re\; 2h\int_R^a\int_S \partial_r\overline{v} v\psi\varphi'\; dr d\overline{g} \\ =& h\int_R^a\int_S \partial_r(|v|^2)\psi\varphi'\; dr d\overline{g} \\ =& -h\int_R^a\int_S |v|^2(\psi'\varphi' + \psi\varphi'')\; dr d\overline{g} \end{aligned} \tag{4.10}$$

so if we evaluate again the scalar product $Re(\psi\mathcal{P}_\varphi v, v)_{L^2(X_R\setminus X_a)}$ we have

$$\begin{aligned} Re(\psi\mathcal{P}_\varphi v, v)_{L^2(X_R\setminus X_a)} =& Re(\psi h(D_r - A_0)v, h(D_r - A_0)v)_{L^2(X_R\setminus X_a)} \\ &+ (\psi M(r)v, v)_{L^2(X_R\setminus X_a)} \\ &- ((\psi + \psi(\varphi')^2 - \psi h^2(V_m - \Lambda) + h\varphi'\psi' + \frac{h^2}{2}\psi'')v, v)_{L^2(X_R\setminus X_a)}. \end{aligned} \tag{4.11}$$

We define

$$F(r) := -((M(r) - 1 + W)v_r, v_r)_{L^2(S)} + \|h(D_r - A_0)v_r\|^2_{L^2(S)} \tag{4.12}$$

where $v_r = v(r,\cdot)$ and

$$W := h^2(V_m - \Lambda) - (\varphi')^2 + h\varphi''.$$

By definition of $\mathcal{P}_\varphi$ we can write

$$-\mathcal{P}_\varphi = -h^2(D_r - A_0)^2 - M(r) + (1 - W) - 2ih\varphi' h(D_r - A_0) - i\varepsilon. \tag{4.13}$$

We need to compute F', so

$$\begin{aligned}
\partial_r \|h(D_r - A_0)v_r\|^2_{L^2(S)} &= 2Re(h(D_r - A_0)\partial_r v_r, h(D_r - A_0)v_r)_{L^2(S)} \\
&\quad + 2Re([\partial_r, h(D_r - A_0)]v_r, h(D_r - A_0)v_r)_{L^2(S)} \\
&= -2Re(h(D_r - A_0)hD_r v_r, (\partial_r - iA_0)v_r)_{L^2(S)} - 2Re(hA_0' v_r, h(D_r - A_0)v_r)_{L^2(S)} \\
&= -2Re(h^2(D_r - A_0)^2 v_r, (\partial_r - iA_0)v_r)_{L^2(S)} - 2Re(h^2(D_r - A_0)(A_0 v_r), (\partial_r - iA_0)v_r)_{L^2(S)} \\
&\quad - 2Re(hA_0' v_r, h(D_r - A_0)v_r)_{L^2(S)}.
\end{aligned}$$

After commuting A_0 with D_r and noticing that

$$Re(h^2 A_0(D_r - A_0)v_r, i(D_r - A_0)v_r)_{L^2(S)} = 0$$

we obtain

$$-2Re(h^2(D_r - A_0)(A_0 v_r), (\partial_r - iA_0)v_r)_{L^2(S)} = 2Re(hA_0' v_r, h(D_r - A_0)v_r)_{L^2(S)},$$

so that the last two terms in $\partial_r \|h(D_r - A_0)v_r\|^2_{L^2(S)}$ cancel. This gives us the final expression

$$\partial_r \|h(D_r - A_0)v_r\|^2_{L^2(S)} = -2Re(h^2(D_r - A_0)^2 v_r, (\partial_r - iA_0)v_r)_{L^2(S)}.$$

We can now compute F'

$$\begin{aligned}
F'(r) =& 2Re((-h^2(D_r - A_0)^2 - M(r) + (1 - W))v_r, \partial_r v_r)_{L^2(S)} \\
&- 2Re(h^2(D_r - A_0)^2 v_r, -iA_0 v_r)_{L^2(S)} \\
&- ([\partial_r, M(r)]v_r, v_r)_{L^2(S)} - (W' v_r, v_r)_{L^2(S)}
\end{aligned}$$

and adding the suitable terms

$$\begin{aligned}
F'(r) &+ 2Re(M(r)v_r, iA_0 v_r)_{L^2(S)} - 2Re((1 - W)v_r, iA_0 v_r)_{L^2(S)} \\
&= 2Re((-h^2(D_r - A_0)^2 - M(r) + (1 - W))v_r, (\partial_r - iA_0)v_r)_{L^2(S)} \\
&\quad - ([\partial_r, M(r)]v_r, v_r)_{L^2(S)} - (W' v_r, v_r)_{L^2(S)} \\
&= -2Re(\mathcal{P}_\varphi v_r, (\partial_r - iA_0)v_r)_{L^2(S)} + 4h^{-1}\varphi' \|h(D_r - A_0)v_r\|^2_{L^2(S)} \\
&\quad + 2\varepsilon Im(v_r, (\partial_r - iA_0)v_r)_{L^2(S)} \\
&\quad - ([\partial_r, M(r)]v_r, v_r)_{L^2(S)} - (W' v_r, v_r)_{L^2(S)},
\end{aligned} \tag{4.14}$$

where we used (4.13). Integrating against φ' we find

$$\begin{aligned}\int_R^a \varphi' F'\, dr =& -2Re\int_R^a (\varphi' \mathcal{P}_\varphi v_r, (\partial_r - iA_0)v_r)_{L^2(S)} + 4h^{-1}\int_R^a (\varphi')^2 \|h(D_r - A_0)v_r\|^2_{L^2(S)} dr \\ &+ 2\varepsilon Im \int_R^a (\varphi' v_r, (\partial_r - iA_0)v_r)_{L^2(S)} dr \\ &- \int_R^a \varphi'([\partial_r, M]v_r, v_r)_{L^2(S)} dr - \int_R^a (\varphi' W' v_r, v_r)_{L^2(S)} dr \\ &- 2Re\int_R^a \varphi'(M(r)v_r, iA_0 v_r)_{L^2(S)} dr + 2Re\int_R^a \varphi'((1-W)v_r, iA_0 v_r)_{L^2(S)} dr. \end{aligned} \tag{4.15}$$

Doing integration by parts, we can rewrite $\int_R^a \varphi' F'\, dr$ in terms of the integral of $\varphi'' F$ and in this regard expression (4.11) with $\psi = \varphi''$ gives us

$$\begin{aligned}\int_R^a Re(\varphi'' \mathcal{P}_\varphi v_r, v_r)_{L^2(S)} dr =& 2\int_R^a \varphi'' \|h(D_r - A_0)v_r\|^2_{L^2(S)} dr - \int_R^a \varphi'' F\, dr \\ &- \int_R^a ((h(\varphi'')^2 + h\varphi'\varphi''' + \frac{h^2}{2}\varphi^{(4)})v_r, v_r)_{L^2(S)} dr,\end{aligned}$$

where we have used $1 - h^2(V_m - \Lambda) + (\varphi')^2 = 1 - W + h\varphi''$. Using this relation we obtain

$$\begin{aligned}\int_R^a \varphi' F'\, dr =& -\int_R^a \varphi'' F\, dr \\ =& Re\int_R^a (\varphi'' \mathcal{P}_\varphi v_r, v_r)_{L^2(S)} dr - 2\int_R^a \varphi'' \|h(D_r - A_0)v_r\|^2_{L^2(S)} dr \\ &+ \int_R^a ((h(\varphi'')^2 + h\varphi'\varphi''' + \frac{h^2}{2}\varphi^{(4)})v_r, v_r)_{L^2(S)} dr. \end{aligned} \tag{4.16}$$

So finally, coupling (4.15) and (4.16)

$$\begin{aligned}2\int_R^a ((2h^{-1}(\varphi')^2 + \varphi'')h(D_r - A_0)v_r, h(D_r - A_0)v_r)_{L^2(S)} &- \int_R^a \varphi'([\partial_r, M]v_r, v_r)_{L^2(S)} dr \\ = 2Re\int_R^a (\varphi' \mathcal{P}_\varphi v_r, (\partial_r - iA_0)v_r)_{L^2(S)} dr &+ Re\int_R^a (\varphi'' \mathcal{P}_\varphi v_r, v_r)_{L^2(S)} dr \\ - 2\varepsilon Im \int_R^a (\varphi' v_r, (\partial_r - iA_0)v_r)_{L^2(S)} dr & \\ + \int_R^a ((\varphi' W' + h(\varphi'')^2 + h\varphi'\varphi''' + \frac{h^2}{2}\varphi^{(4)})v_r, v_r)_{L^2(S)} dr & \qquad (4.17) \\ + 2Re\int_R^a \varphi'(M(r)v_r, iA_0 v_r)_{L^2(S)} dr - 2Re\int_R^a \varphi'((1-W)v_r, iA_0 v_r)_{L^2(S)} dr. & \qquad (4.18)\end{aligned}$$

With the exception of the last two terms in (4.18) all the other terms can be treated as in [6, Proposition 2.3]. For the left hand side we have on one hand

$$2h^{-1}(\varphi')^2+\varphi''\geq C'\frac{\varphi'}{r} \tag{4.19}$$

thanks $\varphi'=hr^{-1}$ and Lemma 4.5, on the other hand

$$-\varphi'([\partial_r,M]v_r,v_r)_{L^2(S)}\gtrsim\frac{\varphi'}{r}(M(r)v_r,v_r)_{L^2(S)} \tag{4.20}$$

from Lemma 2.4 on $[R,a]$. Thus we have the following lower bound on the left hand side

$$\begin{aligned}2\int_R^a((2h^{-1}(\varphi')^2+\varphi'')h(D_r-A_0)v_r,h(D_r-A_0)v_r)_{L^2(S)}-\int_R^a\varphi'([\partial_r,M]v_r,v_r)_{L^2(S)}dr\\ \gtrsim 2\int_R^a\|(\varphi'/r)^{1/2}h(D_r-A_0)v\|_{L^2(S)}dr+\int_R^a\|(\varphi'/r)^{1/2}M(r)^{1/2}v\|_{L^2(S)}dr\end{aligned}$$

In the right hand side we use the inequality

$$|\varphi''|\leq C\lambda^{1/2}\frac{\varphi'}{r}$$

coming from Lemma 4.5, from which

$$\begin{aligned}|Re\int_R^a(\varphi''\mathcal{P}_\varphi v_r,v_r)_{L^2(S)}dr|\leq& O(h^{-1}\delta^{-1})\int_R^a\|\mathcal{P}_\varphi v_r\|^2_{L^2(S)}dr\\ &+O(\delta)\int_R^a\|(\varphi'/r)^{1/2}v_r\|^2_{L^2(S)}dr.\end{aligned} \tag{4.21}$$

Moreover $r^{1+\nu}W'$ is bounded from the properties of Lemma 4.5 and the fact that $r^{1+\nu}\partial_rV_m$ is also bounded. This allows to estimate (4.17) by

$$O(h^{1/2})\int_R^a\|(\varphi'/r)^{1/2}v\|^2_{L^2(S)}dr.$$

For (4.18) we have

$$\begin{aligned}&|Re\int_R^a\varphi'(M(r)v_r,iA_0v_r)_{L^2(S)}dr|\\ &\qquad\leq\int_R^a h|l(r)((\varphi'/r)^{1/2}M^{1/2}(r)v_r,[(1+T(r))^{1/2},A_0](\varphi'/r)^{1/2}v_r)_{L^2(S)}|dr\\ &\qquad\leq O(h)\int_R^a\|(\varphi'/r)^{1/2}M^{1/2}v_r\|^2_{L^2(S)}dr+O(h)\int_R^a\|(\varphi'/r)^{1/2}v_r\|^2_{L^2(S)}dr\end{aligned} \tag{4.22}$$

since $(M^{1/2}(r)v_r,iA_0M^{1/2}(r)v_r)_{L^2(S)}$ is pure complex and $[(1+T)^{1/2},A_0]$ acts as multiplication by a bounded function. Moreover in $1-W=1-h^2(V_m-\Lambda)+(\varphi')^2-h\varphi''$ the quantity $1-h^2\Lambda+(\varphi')^2-h\varphi''$ is real and V_m,A_0 are bounded, hence

$$|Re\int_R^a\varphi'((1-W)v_r,iA_0v_r)_{L^2(S)}dr|\leq h^2\int_R^a\|(\varphi'/r)^{1/2}v_r\|^2_{L^2(S)}dr. \tag{4.23}$$

The remaining terms can be bounded by the Cauchy-Schwarz inequality recalling that $\varepsilon = O(h^2)$. From (4.19)-(4.23) we have obtained

$$\int_R^a \|(\varphi'/r)^{1/2} h(D_r - A_0) v_r\|^2_{L^2(S)} dr + \int_R^a \|(\varphi'/r)^{1/2} M^{1/2} v_r\|^2_{L^2(S)} dr \\ \leq O(h^{-1}\delta^{-1}) \int_R^a \|\mathcal{P}_\varphi v_r\|^2_{L^2(S)} dr + O(\delta) \int_R^a \|(\varphi'/r)^{1/2} v_r\|^2_{L^2(S)} dr$$

for a small parameter δ.

On the other hand applying (4.11) with $\psi = r^{-1}\varphi'$, thanks to Lemma 4.5 and the fact that $r^\nu(V_m - \Lambda)$ is a bounded function we can recover

$$\int_R^a \|(\varphi'/r)^{1/2} v_r\|^2_{L^2(S)} dr \leq \int_R^a \|(\varphi'/r)^{1/2} h(D_r - A_0) v_r\|^2_{L^2(S)} dr \\ + \int_R^a \|(\varphi'/r)^{1/2} M^{1/2} v_r\|^2_{L^2(S)} dr + \int_R^a \|\mathcal{P}_\varphi v_r\|^2_{L^2(S)} dr.$$

The statement follows combining the last two inequalities.

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