



HAL
open science

AUTOMORPHISM GROUPS OF AFFINE VARIETIES AND THEIR LIE ALGEBRAS

Hanspeter Kraft, M Zaidenberg

► **To cite this version:**

Hanspeter Kraft, M Zaidenberg. AUTOMORPHISM GROUPS OF AFFINE VARIETIES AND THEIR LIE ALGEBRAS. 2024. hal-04512881

HAL Id: hal-04512881

<https://hal.science/hal-04512881v1>

Preprint submitted on 20 Mar 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

AUTOMORPHISM GROUPS OF AFFINE VARIETIES AND THEIR LIE ALGEBRAS

HANSPETER KRAFT AND MIKHAIL ZAIDENBERG

ABSTRACT. This is a brief overview of a few selected chapters on automorphism groups of affine varieties. It includes some open questions.

Mathematics Subject Classification: 14J50, 14R20, 14L30, 14E07, 22F50.
Key words: affine variety, flexible variety, automorphism group, ind-group, Lie algebra, algebraic group action, one-parameter subgroup, Tits alternative, triangulation, linearization, amalgamation.

CONTENTS

1. Introduction	2
2. Lie algebras of vector fields	3
2.1. Algebraically generated subgroups of $\text{Aut}(X)$ and Lie algebras	3
2.2. Locally finite endomorphisms and Lie subalgebras	3
3. Smoothness in ind-groups	4
4. Nested ind-groups	6
5. Algebraic subgroups and algebraic elements	7
6. Nilpotency and solvability	8
7. Aut-quasihomogeneous and flexible affine varieties	10
7.1. Aut-quasihomogeneous affine varieties	10
7.2. Flexible affine varieties	11
7.3. Flexibility of toric varieties	12
8. Rigid affine varieties	13
9. Chapters on $\text{Aut}(\mathbb{A}^n)$	15
9.1. Groups acting infinitely transitively on \mathbb{A}^n	15
9.2. Tame subgroups	16
9.3. Tamizable automorphisms	18
9.4. Stable tameness	18
9.5. Cotame automorphisms	18
9.6. Amalgamation	19
9.7. Linearization of reductive group actions	20
9.8. Solvable and unipotent ind-subgroups: triangulation	22
10. Tits' type alternative for automorphism groups	24
References	25

1. INTRODUCTION

We provide a brief overview of a few selected chapters on automorphism groups of affine varieties. It completes existing surveys on the subject, see e.g. [Arz23], [Fre17], [FuKr18], [Kal09], [KPZ17], [Kra89], [Kra96], [Kra17], [Lam24], [Miy24], [Sno89], [vdE00], [vdEKC21], [Wri17], etc.

Throughout the text, X stands for an irreducible affine variety defined over an algebraically closed field \mathbb{K} of characteristic zero, \mathbb{A}^n stands for the affine n -space over \mathbb{K} and \mathbb{G}_a resp. \mathbb{G}_m stands for the additive resp. multiplicative group of the field \mathbb{K} .

Already in the case of a smooth affine surface X , the group $\text{Aut}(X)$ is often infinite-dimensional and has a rich algebraic structure, see the survey article [KPZ17]. The present survey is focused on various aspects of $\text{Aut}(X)$ as an ind-group, on its algebraically generated subgroups and their Lie algebras. It contains a list of open problems partially borrowed in [FuKr18] and [KrZa22].

The group of regular automorphisms $\text{Aut}(X)$ is an affine ind-group, see e.g. [FuKr18, Theorem 5.1.1]. Recall that an ind-group structure on $\text{Aut}(X)$ is a filtration $\text{Aut}(X) = \bigcup_{i=1}^{\infty} A_i$ where

$$\dots \subseteq A_i \subseteq A_{i+1} \subseteq \dots$$

is an increasing sequence of affine varieties and their closed embeddings such that for each i, j and some $n(i, j)$ and $m(i)$ the action $A_i \times X \rightarrow X$, the multiplication $A_i \times A_j \rightarrow A_{n(i, j)}$ and the inversion $A_i \rightarrow A_{m(i)}$ are morphisms of affine varieties, see [FuKr18].

The ind-group $\text{Aut}(X)$ carries a Zariski ind-topology. A subset $S \subseteq \text{Aut}(X)$ is closed if $S \cap A_i$ is a Zariski closed subset of A_i for each i . A closed algebraic subset of A_i defines an algebraic family of automorphisms of X ; such families were studied in [Ram64].

A faithful action of a linear algebraic group G on X defines an embedding of G onto an algebraic subgroup of $\text{Aut}(X)$. Any such subgroup is a closed algebraic subset of some A_i .

Given an ind-group $\mathfrak{G} = \varinjlim A_i$, the tangent space $T_e \mathfrak{G} = \varinjlim T_e A_i$ carries a natural structure of a Lie algebra denoted $\text{Lie}(\mathfrak{G})$, see [Sha81, p. 189], [Kum02, Proposition 4.2.2] and [FuKr18, Sec. 1.9 and 2.1].

Let $\mathfrak{G} = \text{Aut}(X)$ and let $\text{Vec}(X)$ stand for the Lie algebra of regular vector fields on X . Recall that $\text{Vec}(X)$ is naturally isomorphic to the Lie algebra $\text{Der}(\mathcal{O}(X))$ of derivations of the \mathbb{K} -algebra $\mathcal{O}(X)$ of regular functions on X . In turn, $\text{Lie}(\text{Aut}(X))$ can be naturally identified with a subalgebra $\text{Vec}^c(\mathbb{A}^n)$ of $\text{Vec}(X)$ which consists of all vector fields on \mathbb{A}^n with constant divergence, see [Sha81, p. 191] and [FuKr18, Proposition

15.7.2]. For an algebraic subgroup $G \subseteq \text{Aut}(\mathbb{A}^n)$, $\text{Lie}(G)$ is a finite-dimensional Lie subalgebra of $\text{Vec}^c(\mathbb{A}^n)$.

Let \mathfrak{G} be a connected ind-group and $\mathfrak{H} \subseteq \mathfrak{G}$ be a closed ind-subgroup. Then $\text{Lie}(\mathfrak{H})$ is a Lie subalgebra of $\text{Lie}(\mathfrak{G})$. However, it can happen (contrary to [Sha81, Theorem 1]) that $\text{Lie}(\mathfrak{H}) = \text{Lie}(\mathfrak{G})$ while \mathfrak{H} is a proper subgroup of \mathfrak{G} , see Example 9.5. In this respect, the ind-groups are more complex than the algebraic groups.

2. LIE ALGEBRAS OF VECTOR FIELDS

2.1. Algebraically generated subgroups of $\text{Aut}(X)$ and Lie algebras. Let $G \subseteq \text{Aut}(X)$ be a subgroup. One says that G is *algebraically generated* if it is generated by a family $\{G_i \mid i \in I\}$ of connected algebraic subgroups of $\text{Aut}(X)$. Throughout this subsection we assume that G is algebraically generated. We associate to G the Lie subalgebra

$$L(G) = \langle \text{Lie}(G_i) \mid i \in I \rangle_{\text{Lie}} \subseteq \text{Vec}(X)$$

generated by the Lie subalgebras $\text{Lie}(G_i)$. In fact, $L(G)$ depends only on G and not on the choice of generating subgroups G_i , see [KrZa22, Theorem 2.3.1(4)].

For any ind-group \mathfrak{G} we also define the *canonical Lie algebra* $L_{\mathfrak{G}}$, see [KrZa22, Sect. 7.1], where

$$L_{\mathfrak{G}} = \text{Span}_{\mathbb{K}}\{T_e Y \mid Y \subseteq \mathfrak{G} \text{ a closed algebraic subset smooth in } e\}.$$

Consider also the Zariski closure \overline{G} in the ind-group $\text{Aut}(X)$. This is a closed ind-subgroup with Lie algebra $\text{Lie}(\overline{G})$. Notice that $L(G) \subseteq L_G$ and the both are ideals in $\text{Lie}(\overline{G})$, see [KrZa22, Theorem 2.3.1 and Proposition 7.1.3].

Question 2.1 ([KrZa22, Question 1]). *Do we have $L(G) = L_G = \text{Lie}(\overline{G})$?*

Then answer is affirmative provided L is finite-dimensional.

Theorem 2.2 ([CD03, Theorem 1], [KrZa22, Theorem A]). *Assume that $L(G)$ is finite dimensional. Then G is an algebraic subgroup of $\text{Aut}(X)$ and $\text{Lie}(G) = L_G = L(G)$.*

2.2. Locally finite endomorphisms and Lie subalgebras.

Definition 2.3. An endomorphism λ of a vector space V is called *locally finite* if every $v \in V$ belongs to a finite-dimensional λ -invariant subspace of V .

Every locally finite endomorphism λ has a uniquely defined *additive Jordan decomposition* $\lambda = \lambda_s + \lambda_n$ where λ_s is semisimple, λ_n locally nilpotent, and $\lambda_s \circ \lambda_n = \lambda_n \circ \lambda_s$.

Definition 2.4. A subspace $L \subseteq \text{Vec}(X) = \text{Der}(\mathcal{O}(X))$ is called *locally finite* if for any $f \in \mathcal{O}(X)$ the subspace $Lf \subseteq \mathcal{O}(X)$ is finite dimensional.

If L is locally finite, then it is finite dimensional, see [KrZa22, Lemma 1.6.2].

Let $L \subseteq \text{Vec}(X)$ be a Lie subalgebra generated by a family of locally finite Lie subalgebras $L_i \subseteq \text{Vec}(X)$, $i \in I$. There is the following question, see [KrZa22, Question 2].

Question 2.5. *Is L locally finite provided L is finite dimensional?*

According to Theorem 2.2 the answer is positive if $L_i = \text{Lie}(G_i)$, where the G_i are algebraic subgroups of $\text{Aut}(X)$. In particular, we have the following corollary.

Corollary 2.6 ([KrZa22, Corollary of Theorem E]). *Consider a family $\{\eta_i \mid i \in I\}$ of locally nilpotent vector fields on X . Let $L = \langle \eta_i \mid i \in I \rangle_{\text{Lie}}$. If L is finite dimensional, then L is locally finite.*

Recall that a vector field on X is *locally nilpotent* if so is the associated derivation $\partial \in \text{Der } \mathcal{O}(X)$, that is, if for any $f \in \mathcal{O}(X)$ we have $\partial^n f = 0$ for some $n = n(f) > 0$. A weak form of Question 2.5 is as follows.

Question 2.7. *Let $\xi, \eta \in \text{Vec}(X)$ be locally finite. Assume that the Lie subalgebra $L = \langle \xi, \eta \rangle_{\text{Lie}}$ generated by ξ and η is finite dimensional. Does it follow that L is locally finite?*

By Corollary 2.6 the latter is true if ξ and η define locally nilpotent derivations of $\mathcal{O}(X)$. Indeed, under the latter assumption we have by Theorem 2.2 $L = \text{Lie}(G)$, where $G = \langle \exp(t\xi), \exp(t\eta) \rangle$ is an algebraic group.

3. SMOOTHNESS IN IND-GROUPS

By definition, an ind-variety \mathfrak{V} comes equipped with a countable ascending filtration by algebraic varieties $\mathfrak{V} = \cup_d V_d$ where V_d is closed in V_{d+1} . Another such filtration on \mathfrak{V} is called *admissible* if the identity of \mathfrak{V} defines isomorphism of the corresponding ind-varieties. There are different notions of smoothness of a point in an ind-variety.

Definition 3.1 ([FuKr18, Definition 1.9.4]). Let \mathfrak{V} be an ind-variety, and let $x \in \mathfrak{V}$. We say that

- \mathfrak{V} is *strongly smooth* in x if there is an open neighborhood of x which has an admissible filtration consisting of smooth connected algebraic varieties.

- \mathfrak{V} is *geometrically smooth* in x if there is an admissible filtration $\mathfrak{V} = \cup_k V_d$ such that $x \in V_d$ is a smooth point for all d .

Proposition 3.2 ([FuKr18, Propositions 1.8.5, 1.9.6 and 2.4.1]). *Assume that \mathbb{K} is uncountable. Let $\varphi: \mathfrak{V} \rightarrow \mathfrak{W}$ be a bijective morphism of ind-varieties.*

- Suppose that \mathfrak{V} is connected and that \mathfrak{W} has an admissible filtration $\mathfrak{W} = \cup_k W_k$ such that all W_k are irreducible and normal. Then φ is an isomorphism.*
- Suppose that \mathfrak{V} is connected and \mathfrak{W} is strongly smooth in a point $w \in \mathfrak{W}$. Then there is an open neighborhood \mathfrak{V}' of $\varphi^{-1}(w)$ in \mathfrak{V} such that $\varphi|_{\mathfrak{V}'}: \mathfrak{V}' \rightarrow \mathfrak{W}$ is an open immersion. If \mathfrak{W} is strongly smooth in every point, then φ is an isomorphism.*

Let now $\varphi: \mathfrak{G} \rightarrow \mathfrak{H}$ be a bijective homomorphism of ind-groups. Assume that \mathfrak{G} is connected and \mathfrak{H} is strongly smooth in e . Then φ is an isomorphism.

Example 3.3 (see [FuKr18, Proposition 14.2.1]). The automorphism group $\text{Aut}(\mathbb{K}\langle x, y \rangle)$ of the rank 2 free associative algebra $\mathbb{K}\langle x, y \rangle$ has a natural structure of an ind-group. The homomorphism of abelianization $\mathbb{K}\langle x, y \rangle \rightarrow \mathbb{K}[x, y]$ induces a bijective (homo)morphism of ind-groups

$$\varphi: \text{Aut}(\mathbb{K}\langle x, y \rangle) \rightarrow \text{Aut}(\mathbb{K}[x, y]),$$

see [ML70], [Cze71-72] and [Coh85, Theorem 9.3]. The induced surjective morphism of Lie algebras

$$\text{Lie}(\text{Aut}(\mathbb{K}\langle x, y \rangle)) \rightarrow \text{Lie}(\text{Aut}(\mathbb{K}[x, y]))$$

has a non-trivial kernel, see [BW00, Sect. 11, the last paragraph] and [FuKr18, Proposition 14.2.1]. In particular, φ is not an isomorphism of ind-groups. In fact, there exists a unicuspidal curve C in $\text{Aut}(\mathbb{K}\langle x, y \rangle)$ such that $\varphi(C) \subseteq \text{Aut}(\mathbb{K}[x, y])$ has a deeper cusp and so, is not isomorphic to C , see [FuKr18, Sect. 14.3].

Using Proposition 3.2 one arrives at the following conclusion.

Corollary 3.4 ([FuKr18, Corollary 14.1.2]). *Let \mathbb{K} be uncountable. Then the ind-group $\mathfrak{G} = \text{Aut}(\mathbb{K}[x, y])$ is not strongly smooth in id . Moreover, there is no admissible filtration $\mathfrak{G} = \cup_d \mathfrak{G}_d$ with irreducible and normal algebraic varieties \mathfrak{G}_d .*

Remark 3.5. There exists as well a notion of *algebraic smoothness* of a point in an ind-variety. It has the following advantage: the point $e \in \mathfrak{G}$ is algebraically smooth in every affine ind-group \mathfrak{G} , see [Sha81, Theorem 3] and [Kum02, Theorem 4.3.7]. A geometrically smooth point of an ind-variety is algebraically smooth, see [Kum02, Lemma 4.3.4].

There are several pathological examples. For instance, there is an ind-variety \mathfrak{V} such that each point $v \in \mathfrak{V}$ is not algebraically smooth, see [Sha81, P. 188]. The ind-group $\mathfrak{G} = \mathrm{SL}(2, \mathbb{K}[t])$ has an admissible ascending filtration $\mathfrak{G} = \cup_d G_d$ such that the neutral element e of \mathfrak{G} is singular in every G_d , see [Sha04, Theorem 2] and [Kum02, Examples 4.3.5 and 4.3.8].

4. NESTED IND-GROUPS

Definition 4.1. A subgroup G of an ind-group \mathfrak{G} is *nested* if G is a union of an increasing sequence of algebraic subgroups G_i of \mathfrak{G} , see [KPZ17]. A nested ind-group \mathfrak{G} is called *unipotent* if all the G_i are unipotent algebraic groups.

The last assertion of Proposition 3.2 can be applied to nested ind-groups. Indeed, such a group is strongly smooth, see [FuKr18, Example 2.4.2]. Notice that a connected nested ind-group is algebraically generated, and any closed ind-subgroup of a nested ind-group also is nested.

Question 4.2. *Let $G \subseteq \mathrm{Aut}(X)$ be a connected nested ind-subgroup. Is it true that G is closed?*

See [Per23] for some partial results.

Conjecture 4.3 (cf. [FuKr18, 9.4.3-9.4.6]). *If every finite subset of an ind-group \mathfrak{G} (resp., of $\mathrm{Aut}(X)$) is contained in an algebraic group, then \mathfrak{G} (resp., $\mathrm{Aut}(X)$) is nested.*

Conjecture 4.4 ⁽¹⁾. *Let $\mathfrak{G} = \bigcup_i G_i$ be a connected nested ind-subgroup of $\mathrm{Aut}(X)$, let $T \subseteq \mathfrak{G}$ be a maximal torus of \mathfrak{G} , and let $C^0(T)$ be the associate Cartan subalgebra, that is, the connected component of the centralizer of T . Then for any $g \in C^0(T)$ the conjugacy class of g is closed in \mathfrak{G} .*

Recall the following definition.

Definition 4.5 ([FuKr18, Definition 1.13.2]). A subset S of an ind-group \mathfrak{G} is called *weakly closed* if for every algebraic subset $U \subseteq \mathfrak{G}$ such that $U \subseteq S$ we have $\overline{U} \subseteq S$.

Notice that a countable union of closed algebraic subsets in \mathfrak{G} is weakly closed, see [FuKr18, Proposition 1.13.6].

Question 4.6 ([FuKr18, Question 15.9.1]). *Let $g \in \mathrm{Aut}(\mathbb{A}^n)$. Are the following assertions equivalent?*

¹Cf. [KPZ17, Proposition 2.10]. However, the proof of this proposition contains a mistake, as noted J.-P. Furter.

- g is diagonalizable.
- g is semisimple.
- The conjugacy class $C(g)$ is closed in $\text{Aut}(\mathbb{A}^n)$.
- The conjugacy class $C(g)$ is weakly closed in $\text{Aut}(\mathbb{A}^n)$.

The next proposition contains a partial answer to Question 4.6.

Proposition 4.7 ([FuKr18, Corollary 15.9.8]). *Let $g \in \text{Aut}(\mathbb{A}^n)$ be semisimple. Then g is diagonalizable if and only if its conjugacy class $C(g)$ is weakly closed.*

Question 4.8 ([FuKr18, Question 15.9.11]; cf. [FuKr18, Proposition 15.9.12]). *Let $g \in \text{Aut}(\mathbb{A}^n)$ be locally finite. Does the weak closure of $C(g)$ contain the semisimple part of g ?*

5. ALGEBRAIC SUBGROUPS AND ALGEBRAIC ELEMENTS

Definition 5.1. An element g of an ind-group \mathfrak{G} is called *algebraic* if it is contained in an algebraic subgroup of \mathfrak{G} . It is called *unipotent* if it is contained in a \mathbb{G}_a -subgroup of \mathfrak{G} .

It is easily seen that g is algebraic if and only if it defines a locally finite endomorphism of $\mathcal{O}(X)$. In particular, any algebraic element g has a unique additive Jordan decomposition $g = g_s + g_n$.

Let $\text{Aut}(X)_{\text{alg}} \subseteq \text{Aut}(X)$ stand for the subgroup generated by all algebraic elements, in other words, by all algebraic subgroups of $\text{Aut}(X)$.

Theorem 5.2 ([BD15, Theorem 1.1]). *Assume that the field \mathbb{K} is uncountable. Let S be an affine surface given in $\mathbb{A}^4 = \text{Spec}(\mathbb{K}[x, y, z, u])$ by the equations*

$$yu = xP(x), \quad vx = uQ(u), \quad yv = P(x)Q(u),$$

where each of the polynomials $P, Q \in \mathbb{K}[w]$ has at least 2 distinct roots and $P(0) \neq 0$. Then the following hold.

- The normal subgroup $\text{Aut}(S)_{\text{alg}} \subseteq \text{Aut}(S)$ is not generated by a countable family of algebraic subgroups, and
- the quotient $\text{Aut}(S)/\text{Aut}(S)_{\text{alg}}$ contains a free group over an uncountable set of generators.

Conjecture 5.3. *Let $G = \langle G_1, \dots, G_n \rangle$ be a subgroup of $\text{Aut}(X)$ generated by a finite collection of algebraic subgroups G_i . Then either G is an algebraic group, or G contains non-algebraic elements.*

A weak version:

Conjecture 5.4. *Let $g, h \in \text{Aut}(X)$ be two algebraic elements. Then either g, h are contained in an algebraic subgroup of $\text{Aut}(X)$, or the group generated by g and h contains non-algebraic elements.*

Example 5.5 (see [KrZa22, Theorem 6.4.3]). Let $F = \langle g, h \rangle \subseteq \text{Aut}(\mathbb{A}^2)$ be the subgroup generated by two unipotent algebraic elements

$$g: (x, y) \mapsto (x + y^2, y) \quad \text{and} \quad h: (x, y) \mapsto (x, y + x^2).$$

Then $F \cong \mathbb{F}_2$ is a free group of rank 2. It contains non-algebraic elements, for example, gh is one of them. The closure $\mathcal{F} = \overline{F}$ is a free product $\mathcal{F} = \mathcal{J}^+ * \mathcal{J}^-$ of two nested unipotent closed subgroups \mathcal{J}^\pm of $\text{Aut}(\mathbb{A}^2)$ isomorphic to the vector group $(\mathbb{K}[t], +)$. Every algebraic subgroup of \mathcal{F} is conjugate to a subgroup of \mathcal{J}^+ or \mathcal{J}^- .

Conjecture 5.6 ([FuKr18, Question 9.1.5]). *Let $\text{Aut}(X)$ be infinite dimensional (resp., has a nontrivial connected component). Then $\text{Aut}(X)$ contains a subgroup isomorphic to either \mathbb{G}_m or \mathbb{G}_a , or in other words, $\text{Aut}(X)$ contains a non-torsion algebraic element.*

Due to Theorem 8.4 below, Conjecture 5.6 holds for affine surfaces. It holds also provided $\text{Aut}(X)$ contains a nontrivial connected set of commuting elements. Indeed, we have the following facts.

Theorem 5.7 ([CRX19, Theorem B]). *Let Y be an irreducible algebraic subvariety of $\text{Aut}(X)$ such that Y contains the identity and any $g_1, g_2 \in Y$ commute. Then the subgroup of $\text{Aut}(X)$ generated by Y is an abelian connected algebraic subgroup of $\text{Aut}(X)$.*

Corollary 5.8. *Let G be a connected abelian subgroup of $\text{Aut}(X)$.*

- *If G is nontrivial, then it contains a subgroup isomorphic to either \mathbb{G}_m or \mathbb{G}_a .*
- *If G is a closed ind-subgroup, then G is nested.*

As a consequence, an eventual counterexample to Conjecture 5.6 has the property that every commutative subgroup of $\text{Aut}(X)$ is discrete, or, in other terms, every algebraic element of $\text{Aut}(X)$ is a torsion element. See also [RvS21, Theorem A] for a description of maximal abelian subgroups of $\text{Aut}(X)$ which consist of unipotent elements.

Question 5.9. *Assume that the group $\text{Aut}(X)$ has no nonunit algebraic element. Is it then discrete?*

Notice that by Theorem 5.7 under the latter assumption every abelian subgroup of $\text{Aut}(X)$ is discrete and has no torsion.

6. NILPOTENCY AND SOLVABILITY

Theorem 6.1 ([KrZa22, Theorem B]). *Let $G \subseteq \text{Aut}(X)$ be an algebraically generated subgroup. If G is solvable, then the following hold.*

- (a) $G = U \rtimes T$ where T is an algebraic torus and $U = R_u(G)$ is a nested unipotent group.
- (b) G is unipotent if and only if the generating subgroups are.
- (c) If, in addition, G is generated by a finite family of connected algebraic groups, then G is a connected algebraic group.

Theorem 6.2 ([KrZa22, Theorem C]). *A nested unipotent subgroup $U \subseteq \text{Aut}(X)$ is solvable of derived length $\leq \max\{\dim Ux \mid x \in X\} \leq \dim X$.*

Remark 6.3. A unipotent algebraic group is nilpotent. By contrast, a nested unipotent group $G = \cup_i G_i$ is not necessarily nilpotent, since the nilpotency class of the unipotent algebraic subgroups G_i might not be bounded.

For example, consider the *de Jonquières subgroup* $\text{Jonq}_n \subseteq \text{Aut}(\mathbb{A}^n)$ of triangular automorphisms of the form

$$\varphi = (a_1x_1 + f_1, a_2x_2 + f_2(x_1), \dots, a_nx_n + f_n(x_1, \dots, x_{n-1}))$$

where $a_i \in \mathbb{K}^*$ and $f_i \in \mathbb{K}[x_1, \dots, x_{i-1}]$. The unipotent radical $\mathcal{J}_n = R_u(\text{Jonq}_n)$ is a closed nested unipotent subgroup of $\text{Aut}(\mathbb{A}^n)$. For $n \geq 2$ its Lie algebra $\text{Lie}(\mathcal{J}_n)$ is not nilpotent. However, the Lie algebra of a nilpotent ind-group also is nilpotent, see [KrZa22, Lemma 5.1.4(3)].

Indeed, let for instance $n = 2$ and

$$L_d = \langle \partial/\partial x_1, x_1^d \partial/\partial x_2 \rangle_{\text{Lie}} \subseteq \text{Lie}(\mathcal{J}_2).$$

It is easily seen that the d th member $(L_d)_d$ of the lower central series of L_d does not vanish, and so $(\text{Lie}(\mathcal{J}_2))_d \neq 0$ for any $d \geq 1$.

Question 6.4. *Let \mathfrak{G} be a connected ind-group. Is it true that \mathfrak{G} is solvable (resp., nilpotent) if $\text{Lie } \mathfrak{G}$ is?*

Definition 6.5. Let us say that a subgroup $G \subseteq \text{Aut}(X)$ is *a-generated* if $G = \langle Y \rangle$ where $Y \subseteq \text{Aut}(X)$ is an irreducible algebraic subset containing id_X . Replacing Y by $Y \cdot Y^{-1}$ we may assume that Y is symmetric, that is, $Y = Y^{-1}$.

Conjecture 6.6. *If G is a-generated and nilpotent (resp., solvable) then G is algebraic.*

In the case of a nilpotent (resp., a solvable) subgroup G one can proceed by induction on the nilpotency class of G (resp., on the derived length of G), the case of $n(G) = 1$, that is, G abelian, being fixed by Theorem 5.7. In the general case, the induction works provided the following conjecture is true.

Conjecture 6.7. *If G is a -generated and nilpotent (resp., solvable) then any member of its lower central series (resp., derived series) is as well a -generated.*

If $G/z(G)$ is abelian, then Conjecture 6.6 holds if the following is true:

Conjecture 6.8. *Consider an admissible filtration $\text{Aut}^0(X) = \lim A_n$ by irreducible affine subvarieties A_n . Let G be a (nilpotent, resp., solvable) subgroup of $\text{Aut}^0(X)$ such that $G = \langle G \cap A_n \rangle$ for some n . Then $z(G) = \langle z(G) \cap A_m \rangle$ for some m (resp., $(G, G) = \langle (G, G) \cap A_m \rangle$ for some m), where $z(G)$ stands for the center of G .*

For the following question, see [KrZa22, Sect. 5.2].

Question 6.9. *Let $G \subseteq \text{Aut}(X)$ be a solvable or nilpotent connected subgroup. Does there exist a closed embedding $X \hookrightarrow \mathbb{A}^n$ such that G extends to a subgroup of the de Jonquières group Jonq_n ? Is this true if G is nested?*

Denote by $\text{LND}(X) \subseteq \text{Vec}(X)$ the set of all locally nilpotent derivations of $\mathcal{O}(X)$. For a vector space V over \mathbb{K} we let $\text{LND}(V)$ be the set of locally nilpotent endomorphisms of V .

Definition 6.10 ([Dai19]). A subset Z of endomorphisms of a vector space V is called *uniformly locally nilpotent*, or *ULN* for short, if for any $v \in V$ there is $n \in \mathbb{N}$ such that $a_1 \cdots a_n(v) = 0$ whatever are $a_1, \dots, a_n \in Z$.

Clearly, if Z is ULN then $Z \subseteq \text{LND}(V)$ and $\text{span}_{\mathbb{K}}(Z) \subseteq \text{End}_{\mathbb{K}}(V)$ is ULN too.

Let $A \subseteq \text{Aut}(X)$ be an irreducible algebraic subvariety which contains id_X , and let $\langle A \rangle$ be the subgroup of $\text{Aut}(X)$ generated by A . The notation $L_A \subseteq \text{Vec}(X)$ have the same meaning as in subsection 2.1.

Question 6.11. *We wonder if the following equivalences hold:*

- (1) $L_A \subseteq \text{LND}(X) \Leftrightarrow \text{Ad}(A) \subseteq \text{Aut}_{\text{Lie}}(\text{Vec}(X))$ is ULN $\Leftrightarrow \langle A \rangle$ is a unipotent algebraic group;
- (2) A is metabelian $\Leftrightarrow \langle \text{Ad}(A) \rangle_{\text{Lie}}$ is commutative;
- (3) $\langle A \rangle$ is solvable $\Leftrightarrow \langle \text{Ad}(A) \rangle_{\text{Lie}}$ is;
- (4) $\langle A \rangle$ is nilpotent $\Leftrightarrow \langle \text{Ad}(A) \rangle_{\text{Lie}}$ is.

7. Aut-QUASIHOMOGENEOUS AND FLEXIBLE AFFINE VARIETIES

7.1. Aut-quasihomogeneous affine varieties.

Definition 7.1. We say that a variety X is *Aut-homogeneous* (resp. *Aut-quasihomogeneous*) if $\text{Aut}(X)$ acts transitively on X (resp. acts on X with an open orbit).

This notion and the following characterizations of Aut-quasihomogeneity for smooth affine surfaces are due to Gizatullin [Giz71b]. This was extended to normal affine surfaces by Dubouloz [Dub04].

Theorem 7.2. *Let X be a normal affine surface X non-isomorphic to $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})$. Then the following are equivalent:*

- X is Aut-quasihomogeneous;
- there are two effective \mathbb{G}_a -actions on X with distinct general orbits;
- $X \cong \bar{X} \setminus D$ where \bar{X} is a smooth projective surface and $D \subseteq \text{reg}(\bar{X})$ is a reduced divisor with simple normal crossings, which consists of a chain of smooth rational curves.

Notice that many, but not all, smooth Gizatullin surfaces are Aut-homogeneous. Among Aut-homogeneous normal affine surfaces, homogeneous spaces of algebraic groups constitute a narrow subset; these were completely classified by Gizatullin [Giz71a] and Popov [Pop73], see also [KPZ17, Theorem 4.8].

7.2. Flexible affine varieties. Flexible affine varieties were introduced in [AFK⁺13a], see also [KZ99] and [AKZ12]. For overviews on the subject see [Arz23], [AFK⁺13b], and [CPPZ21, Section 7].

Definition 7.3. Let X be an affine variety. We say that a point $x \in \text{reg}(X)$ is *flexible* if the tangent space $T_x X$ is spanned by the tangent vectors to the orbits Ux of \mathbb{G}_a -subgroups U of $\text{Aut}(X)$. The variety X is called *flexible* if every point $x \in \text{reg}(X)$ is. Clearly, X is flexible if one point $x \in \text{reg}(X)$ is and the group $\text{Aut}(X)$ acts transitively on $\text{reg}(X)$.

We let $\mathfrak{U}(X) \subseteq \text{Aut}^0(X)$ be the subgroup generated by all \mathbb{G}_a -subgroups of $\text{Aut}(X)$, in other words, by all unipotent elements of $\text{Aut}(X)$ ². We also let $\text{Aut}_{\text{alg}}(X)$ be the subgroup generated by all algebraic elements of $\text{Aut}(X)$. Notice that $\text{Aut}_{\text{alg}}(X)$ is not necessarily connected, and every $a \in \text{Aut}_{\text{alg}}(X) \setminus \text{Aut}^0(X)$ is a torsion element.

The following theorem (conjectured in [AKZ12, Sect. 4.2]) was proven in [AFK⁺13a] for affine varieties and extended in [APS14, Theorem 2] and [FKZ16, Theorem 2.12] to quasiprojective varieties.

²The subgroup $\mathfrak{U}(X)$ is denoted by $\text{SAut}(X)$ in [AFK⁺13a]. We prefer the notation $\mathfrak{U}(X)$ in order to keep $\text{SAut}(\mathbb{A}^n)$ for the subgroup of $\text{Aut}(\mathbb{A}^n)$ of volume-preserving automorphisms.

Theorem 7.4. *For a quasiffine variety X of dimension at least 2, the following are equivalent:*

- X is flexible;
- the group $\mathfrak{U}(X)$ acts transitively on the smooth locus $\text{reg}(X)$;
- the group $\mathfrak{U}(X)$ acts infinitely transitively on $\text{reg}(X)$.

The flexibility survives when removing a subvariety of codimension ≥ 2 .

Theorem 7.5 ([FKZ16, Theorem 0.1]). *Let X be a flexible smooth quasiffine variety of dimension ≥ 2 and $Y \subseteq X$ be a closed subset of codimension ≥ 2 . Then $X \setminus Y$ is flexible.*

As an application of Theorem 7.4, let us mention the following two interpolation results. The next theorem follows immediately from [AFK⁺13a, Theorem 4.14 and Remark 4.15].

Theorem 7.6. *Let X be a flexible affine variety of dimension $n \geq 2$. Then for every $m \geq 0$ and every finite subset $Z \subseteq \text{reg}(X)$ there exists an automorphism $g \in \mathfrak{U}(X)$ with prescribed m -jets at the points $p \in Z$ provided each jet fixes its center p .*

Corollary 7.7. *Under the assumptions of Theorem 7.6 there exists an \mathbb{A}^1 -orbit of a \mathbb{G}_a -action on X which passes through every point $p \in Z$ and interpolates a given smooth curve jet at p .*

See also [AFK⁺13a, Theorem 1.15] for an analog of Kleiman's Transversality theorem for flexible varieties.

7.3. Flexibility of toric varieties. Let $\mathbb{T} = (\mathbb{G}_m)^n$ stand for an algebraic n -torus. Recall that a \mathbb{T} -variety X of dimension n is a *toric variety* if \mathbb{T} acts effectively on X with an open orbit. Any toric affine variety X of dimension $n \geq 2$ with no torus factor can be defined by a strictly convex polyhedral cone in the integer lattice \mathbb{Z}^n . For instance, the affine plane \mathbb{A}^2 with the standard diagonal torus action is defined by the first quadrant $(\mathbb{N} \cup \{0\})^2$ of \mathbb{Z}^2 .

Definition 7.8. Let X be a toric variety. A *Demazure root subgroup* of $\text{Aut}(X)$ is a \mathbb{G}_a -subgroup $U \subseteq \text{Aut}(X)$ which is normalized by the acting torus \mathbb{T} .

Such subgroups, or rather the associated locally nilpotent derivations of $\mathcal{O}(X)$, correspond to certain lattice points of \mathbb{Z}^n called *Demazure roots*. In particular, if $X = \mathbb{A}^2$ then the Demazure roots are the lattice points $(-1, k)$ and $(l, -1)$ with $k, l \geq 0$. They correspond to the locally nilpotent derivations $y^k \partial / \partial x$ and $x^l \partial / \partial y$, respectively. It is

known that any normal toric variety of dimension ≥ 2 with no torus factor is flexible, see [AKZ12]. The following theorem says more.

Theorem 7.9 ([AKZ19, Theorem 1.3]). *Let X be a toric affine variety of dimension at least 2. If X has no torus factor and is smooth in codimension 2, then one can find a finite collection of Demazure root subgroups such that the group generated by these acts infinitely transitively on the smooth locus $\text{reg}(X)$.*

8. RIGID AFFINE VARIETIES

Definition 8.1. An affine variety X is called *rigid* if it admits no effective \mathbb{G}_a -action.

For the following conjectures see e.g. [PZ22, Conjectures 1.0.1 and 1.0.2].

Conjecture 8.2. *Let X be a rigid affine variety. Then $\text{Aut}^0(X)$ is an algebraic torus (of rank $\leq \dim(X)$).*

A weaker form of this conjecture is as follows.

Conjecture 8.3. *If an affine variety X admits no effective \mathbb{G}_a - and \mathbb{G}_m -actions, then the group $\text{Aut}^0(X)$ is trivial.*

In fact, both conjectures above hold in dimension 2.

Theorem 8.4 ([PZ22, Theorem 1.3(1)]). *Assume that the base field \mathbb{K} is uncountable. Let X be a normal affine surface over \mathbb{K} . Then $\text{Aut}^0(X)$ is an algebraic group if and only if X is rigid, if and only if $\text{Aut}^0(X)$ is an algebraic torus (of rank ≤ 2).*

We have the following geometric criterion of rigidity of a normal affine surface.

Theorem 8.5 ([PZ22, Theorem 1.3(2)]). *Assume that the base field \mathbb{K} is uncountable. Then a normal affine surface X over \mathbb{K} is rigid if and only if X admits a relatively minimal completion \bar{X} by a reduced effective normal crossing divisor D supported on $\text{reg}(\bar{X})$ such that any extremal chain of the dual graph $\Gamma(D)$ which consists of rational components C_1, \dots, C_k satisfies $C_i^2 \leq -2$ for $i = 1, \dots, k$.*

The dual graph $\Gamma(D)$ is defined as follows: its vertices correspond to components of D and the edges correspond to the pairs of components that intersect. A chain L in $\Gamma(D)$ is called *extremal* if all vertices of L are of valence ≤ 2 in $\Gamma(D)$ and at least one vertex is of valence 1.

Example 8.6 ([PZ22, Example 4.3.6]). Consider the affine surface $X = \mathbb{P}^2 \setminus \text{supp}(D)$ where D is a reduced effective divisor on \mathbb{P}^2 with only nodes as singularities. Then X is not rigid if and only if $\deg(D) \leq 2$, that is, D is either a projective line or a (possibly, singular) conic.

By definition, a nested ind-group consists of algebraic elements. The following theorem provides a partial converse.

Theorem 8.7 ([PR22, Theorem 1.1]). *For an affine variety X the following conditions are equivalent:*

- $\text{Aut}^0(X)$ consists of algebraic elements;
- $\text{Aut}^0(X)$ is a closed nested ind-subgroup of $\text{Aut}(X)$;
- $\text{Aut}^0(X) = \mathfrak{U}(X) \rtimes T$ where T is a maximal torus of $\text{Aut}(X)$ and $\mathfrak{U}(X) = R_u(\text{Aut}(X))$ is abelian and consists of all unipotent elements of $\text{Aut}(X)$.

Corollary 8.8 ([PR22, Corollary 4.3]). *Let X be a rigid affine variety. If $\text{Aut}^0(X)$ consists of algebraic elements, then $\text{Aut}^0(X)$ is an algebraic torus of dimension $\leq \dim(X)$.*

The proof exploits the following fact of independent interest.

Theorem 8.9 ([PR22, Theorem 3.1]). *Let \mathfrak{G} be a connected ind-group and $\mathfrak{H} \subseteq \mathfrak{G}$ be a closed connected nested ind-subgroup. Assume that any $g \in \mathfrak{G}$ is a torsion element modulo \mathfrak{H} , that is, $g^d \in \mathfrak{H}$ for some $d = d(g) > 0$. Then $\mathfrak{H} = \mathfrak{G}$.*

The following theorem gives a necessary condition for rigidity. For the proof see [AG17, Theorem 2.1]; cf. also [FZ05b, Theorem 3.3] in the case of affine surfaces.

Theorem 8.10. *Suppose that $\text{Aut}(X)$ contains two non-commuting \mathbb{G}_m -subgroups. Then $\text{Aut}(X)$ contains a \mathbb{G}_a -subgroup.*

This theorem implies (a) below; see [PR22, Proposition 5.1 and Remark 5.2] for the proof of (b) and [BG23] for (c). Note that (c) confirms Conjecture 8.2 in certain particular settings.

Theorem 8.11. *For a rigid affine variety X the following hold.*

- (a) *There exists an algebraic torus $\mathbb{T} \cong (\mathbb{G}_m)^k$ in $\text{Aut}(X)$ which contains any other algebraic subtorus of $\text{Aut}(X)$. In particular, \mathbb{T} is a normal subgroup of $\text{Aut}(X)$.*
- (b) *\mathbb{T} is a central subgroup of $\text{Aut}^0(X)$ and the only maximal connected abelian ind-subgroup of $\text{Aut}^0(X)$.*
- (c) *If X is toric or a rational normal affine variety with only constant invertible functions, finitely generated divisor class group*

and admits a torus action of complexity one, then $\text{Aut}^0(X)$ is an algebraic torus.

Concerning the conjugacy of tori in $\text{Aut}(X)$ we have the following general results, see e.g. [FuKr18, Proposition 10.5.4] for (a) and [Dem82], [Gub98] and [BH03] for (b).

Theorem 8.12.

- (a) Let $T \subseteq \text{Aut}(X)$ be an algebraic torus. Then $\dim(T) \leq \dim(X)$.
- (b) Assume that X is a normal toric variety. Then all algebraic tori $T \subseteq \text{Aut}(X)$ of dimension $\dim(X)$ are conjugate. Furthermore, any algebraic torus of dimension $\dim(X) - 1$ in $\text{Aut}(X)$ is contained in an algebraic torus of dimension $\dim(X)$.

The following examples show that the latter conclusion does not hold any longer for non-toric affine varieties.

Example 8.13. 1. (see [Dani89, Theorem 2].) Let D be the Danielewski surface $\{xy - z^2 + z = 0\}$ in $\mathbb{A}_{\mathbb{C}}^3$ and let $X = D \times \mathbb{A}^1$. Then $\text{Aut}(X)$ contains a sequence of pairwise non-conjugate algebraic 2-tori.

2. There exist examples of Aut-homogeneous smooth non-toric affine surfaces X such that $\text{Aut}(X)$ contains an algebraic family of pairwise non-conjugate algebraic 1-tori depending on 1 or 2 parameters, see [FKZ11, Theorem 1.0.1] and also [KPZ17, Theorem 5.5]. Cf. also Example 9.20 below of Danilov-Gizatullin surfaces, where the number of conjugate classes of 1-tori is finite.

Remark 8.14. Recall that the Koras-Russel threefold X given in $\mathbb{A}_{\mathbb{C}}^4$ by equation $x + x^2y + z^2 + t^3 = 0$ is diffeomorphic to $\mathbb{A}_{\mathbb{C}}^3$, but non-isomorphic to $\mathbb{A}_{\mathbb{C}}^3$, see e.g. [KML97]. It is neither flexible, nor rigid. The group $\text{Aut}(X)$ is infinite dimensional, acts on X with an open orbit and a unique fixed point (the origin). It coincides with the stabilizer of X in $\text{Aut}(\mathbb{A}_{\mathbb{C}}^4)$. This group is naturally isomorphic to the subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^3)$ (where $\mathbb{A}_{\mathbb{C}}^3 = \text{Spec } \mathbb{C}[x, z, t]$) of all automorphisms which leave invariant the ideals (x) and $(x^2, x + z^2 + t^3)$, see [DMJP10].

9. CHAPTERS ON $\text{Aut}(\mathbb{A}^n)$

9.1. Groups acting infinitely transitively on \mathbb{A}^n .

Theorem 9.1 ([And19], [AKZ19]). *For any $n \geq 2$ one can find three \mathbb{G}_a -subgroups of $\text{Aut}(\mathbb{A}^n)$ which generate a subgroup acting infinitely transitively on \mathbb{A}^n . The same is true for some $n + 2$ root subgroups, that is, \mathbb{G}_a -subgroups normalized by the standard n -torus of $\text{Aut}(\mathbb{A}^n)$.*

9.2. Tame subgroups. Notice that the de Jonquières subgroup $\text{Jonq}_n \subseteq \text{Aut}(\mathbb{A}^n)$ (see Remark 6.3) is generated by the standard torus \mathbb{T} of $\text{GL}(n, \mathbb{K})$ and the unipotent radical $\mathcal{J}_n = R_u(\text{Jonq}_n)$. In turn, \mathcal{J}_n is generated by the triangular *root* \mathbb{G}_a -subgroups

$$\exp(tx_1^{m_1} \cdots x_{i-1}^{m_{i-1}} \partial_i), \quad i = 1, \dots, n \quad \text{where} \quad \partial_i = \partial/\partial x_i.$$

The Lie algebra

$$\text{Lie}(\mathcal{J}_n) = \mathbb{K}\partial_1 \oplus \mathbb{K}[x_1]\partial_2 \oplus \cdots \oplus \mathbb{K}[x_1, \dots, x_{n-1}]\partial_n$$

consists of locally nilpotent derivations. This is a maximal Lie subalgebra in $\text{Vec}(\mathbb{A}^n)$ consisting of locally nilpotent derivations, see [Sku21, Theorems 1 and 7]; cf. also [BPZ22, Theorems 1.4 and 1.5] and the references therein for closely related results. For $\mathbb{K} = \mathbb{C}$, Jonq_n is a Borel subgroup of $\text{Aut}(\mathbb{A}^n)$ that is, a maximal (connected) solvable subgroup, see [FuPo18, Corollary 1.2].

Definition 9.2. An automorphism $\varphi \in \text{Aut}(\mathbb{A}^n)$ is said to be *tame* if it is a composition of affine and triangular automorphisms; otherwise it is called *wild*. The tame automorphisms form the *tame subgroup* $\text{Tame}_n = \langle \text{Aff}_n, \text{Jonq}_n \rangle$ of $\text{Aut}(\mathbb{A}^n)$.

For the following classical theorem see [Jun42], [vdK53]; cf. also [Kam75, Theorem 2] and [Miy24, Theorem 2.3.8].

Theorem 9.3 (Jung-van der Kulk). *Let k be an arbitrary field. Then $\text{Aut}(\mathbb{A}_k^2) = \text{Tame}_2$. Moreover, $\text{Aut}(\mathbb{A}_k^2)$ is a free product $\text{Aff}_2 *_C \text{Jonq}_2$ amalgamated over $C = \text{Aff}_2 \cap \text{Jonq}_2$.*

Using this amalgam, Danilov [Dan74] showed that the subgroup $\text{SAut}(\mathbb{A}_{\mathbb{K}}^2) = \{f \in \text{Aut}(\mathbb{A}_{\mathbb{K}}^2) \mid \det(f) = 1\}$ of unimodular automorphisms is not a simple group.

The things are different for $\text{Aut}(\mathbb{A}_{\mathbb{K}}^3)$. Nagata considered in [Nag72, Section 2.1] the following automorphism of $\mathbb{A}_{\mathbb{K}}^3 = \text{Spec}(\mathbb{K}[x, y, z])$:

$$\Phi: (x, y, z) \mapsto (x, y, z) + (\sigma z, y + 2\sigma x + \sigma^2 z, 0) \quad \text{where} \quad \sigma = x^2 - yz.$$

We have

$$\Phi = \exp(\sigma\partial) \quad \text{where} \quad \partial = z\partial/\partial x + 2x\partial/\partial y.$$

Indeed, $\partial(\sigma) = 0$ and so, $\Phi^*(\sigma) = \sigma$. Since also $\Phi^*(z) = z$ we have $\Phi \in \text{Aut}_{\mathbb{K}[z]} \mathbb{K}[z][x, y]$. Nagata showed that

$$\Phi \notin \langle \text{Aff}_2(\mathbb{K}[z]), \text{Jonq}_2(\mathbb{K}[z]) \rangle$$

and suggested that $\Phi \notin \text{Tame}_3$ is wild, see [Nag72, Conjecture 3.1]. The following theorem due to Shestakov and Umirbaev confirms Nagata's Conjecture.

Theorem 9.4 ([SU04a, SU04b]; see also [vdEKC21, Chapter 1]). *The Nagata automorphism Φ is wild. So, Tame_3 is a proper subgroup of $\text{Aut}(\mathbb{A}^3)$.*

Notice that Tame_3 is not a closed ind-subgroup of $\text{Aut}(\mathbb{A}^3)$, see [EP15]. It is unknown whether Tame_3 is dense in $\text{Aut}(\mathbb{A}^3)$.

Example 9.5 ([FuKr18, Sect. 17.3]). Letting $\mathbb{A}^3 = \text{Spec } \mathbb{K}[x, y, z]$ consider the closed connected ind-subgroup $\mathfrak{G} \subseteq \text{Aut}(\mathbb{A}^3)$ consisting of the automorphisms which send z to z . Then $\mathfrak{G}^t = \mathfrak{G} \cap \text{Tame}_3$ is a proper closed ind-subgroup of \mathfrak{G} . Nevertheless, their Lie algebras coincide: $\text{Lie}(\mathfrak{G}^t) = \text{Lie}(\mathfrak{G})$, contrary to the claim of [Sha81, Theorem 1].

Question 9.6. *Do we have $L_{\mathfrak{G}} = \text{Lie } \mathfrak{G}$ and $L_{\mathfrak{G}^t} = \text{Lie } \mathfrak{G}^t$? Is it true that $L_{\mathfrak{G}} = L_{\mathfrak{G}^t}$?*

Let us mention also the following result.

Theorem 9.7 ([KrSt13, Main Theorem]). *Every automorphism of the group $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$ is inner up to field automorphisms when restricted to the tame subgroup Tame_n .*

Remark 9.8. A tame subgroup was defined for some other automorphism groups, for instance,

- for $\text{Aut}(Q_3)$ where Q_3 is a smooth affine quadric threefold realized as the underlined variety of $\text{SL}(3, \mathbb{C})$, see [LV13];
- for $\text{Aut}(A_n)$ where $A_n = \mathbb{K}\langle x_1, \dots, x_n \rangle$ is the free associative algebra in $n \geq 2$ variables, see [BKY12].

A theorem due to Makar-Limanov and Czerniakiewicz says that $\text{Aut}(A_2) = \text{Tame}(\text{Aut}(A_2))$, see [ML70], [Cze71-72] and [Coh85, Theorem 4.1]. At the same time, $\text{Tame}(\text{Aut}(Q_3))$ and $\text{Tame}(\text{Aut}(A_3))$ are proper subgroups of $\text{Aut}(Q_3)$ and $\text{Aut}(A_3)$, respectively, see [AG10, Theorem 6.1] and [LV13, Section 5.1] for the former and [Umi07, Corollary 2.1] for the latter. The role of the Nagata automorphism for $\text{Aut}(A_3)$ is played by the Anick automorphism

$$(x, y, z) \mapsto (x, y, z) + (z, x, 0)\sigma \quad \text{where } \sigma = xz - zy,$$

which is wild.

Likewise, consider the quadric cone $S_3 \subseteq \mathbb{A}^4$ given by equation $\delta := x_1x_4 - x_2x_3 = 0$. There exists an automorphism τ of \mathbb{A}^4 which restricts to S_3 and lifts to the Cox ring $\mathbb{K}[x_1, x_2, x_3, x_4]$ of S_3 yielding the Anick automorphism of \mathbb{A}_4 :

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2 + x_1\delta, x_3, x_4 + x_3\delta).$$

The restriction $\tau|_{S_3}$ is a wild automorphism of S_3 , see [AG10, Theorem 6.1].

9.3. Tamizable automorphisms.

Definition 9.9 ([MP09, Definition 4.1]). An automorphism $\varphi \in \text{Aut}(\mathbb{A}^n)$ is called *tamizable* (resp., *linearizable*, *triangulable*) if it is conjugate in $\text{Aut}(\mathbb{A}^n)$ to a tame (resp. linear, triangular) automorphism.

Question 9.10 ([MP09]). *Is it true that any $\varphi \in \text{Aut}(\mathbb{A}^n)$ is tamizable? In particular, is the Nagata automorphism Φ tamizable?*

Example 9.11. It is known that the Nagata automorphism Φ is not conjugate to a triangular one, see [Bas84]. Nevertheless, it is *shifted linearizable*. More precisely, $2\text{id} \cdot \Phi$ is conjugate to 2id in $\text{Aut}(\mathbb{A}^3)$, see [MP09, Theorem 3.3]; cf. also [FuKr18, Lemma 15.11.1] for a general result of this type.

As follows from the Jung-van der Kulk Theorem 9.3, $\text{Aut}(\mathbb{A}^2)$ is algebraically generated, see subsection 2.1 for the definition. It is known that $\text{Aut}(\mathbb{A}^n)$ is connected for any n , see [Sha81, Lemma 4]. The following natural question arises:

Question 9.12. *Is $\text{Aut}(\mathbb{A}^n)$ algebraically generated for $n \geq 3$?*

9.4. Stable tameness.

Definition 9.13. A wild automorphism $\varphi \in \text{Aut}(\mathbb{A}^n)$ is called *stably tame* if $\varphi \oplus \text{id} \in \text{Aut}(\mathbb{A}^{n+\kappa})$ is tame for some $\kappa \geq 1$. The number κ of additional variables will be called an *excessive dimension*.

Theorem 9.14 ([Smi89]). *The Nagata automorphism Φ is stably tame with excessive dimension $\kappa = 1$.*

By [Spo07] the same is true for the Anick automorphism of A_3 , cf. Remark 9.8. More generally, the following holds.

Theorem 9.15 ([BvdEW12, Corollary 4.9]). *Let as before $\mathfrak{G} \subseteq \text{Aut}(\mathbb{A}^3)$ be the subgroup of automorphisms fixing z . Then any element $\varphi \in \mathfrak{G}$ is stably tame with excessive dimension $\kappa = 3$.*

A similar fact is also known for $\text{Aut}(A_3)$ with $\kappa = 1$, see [BKY12, Theorem 1.1].

9.5. Cotame automorphisms.

Definition 9.16 (*Cotame automorphisms*; cf. [EL19]). One says that $h \in \text{Aut}(\mathbb{A}^n) \setminus \text{Aff}_n$ is *cotame* if $\langle \text{Aff}_n, h \rangle \supseteq \text{Tame}_n$ and *topologically cotame* if $\overline{\langle \text{Aff}_n, h \rangle} \supseteq \text{Tame}_n$.

Theorem 9.17 ([Bod02], [Bod05, 1.4 and Theorem 1.8], [EL19]). *For $n \geq 3$ any triangular $h \in \text{Jonq}_n \setminus \text{Aff}_n$ is cotame, while there is no triangular cotame $h \in \text{Aut}(\mathbb{A}^2) \setminus \text{Aff}_2$.*

The following theorem of Edo extends and refines the earlier results of Bodnarchuk ([Bod01, Thm. 3]) and Furter ([Fur15, Thm. D]); cf. also [FuPo18, Theorem 1.4].

Theorem 9.18 ([Edo18, Thm. 1.2]). *For $n \geq 2$ any element $h \in \text{Aut}(\mathbb{A}^n) \setminus \text{Aff}_n$ is topologically cotame.*

9.6. Amalgamation. By Jung-van der Kulk's Theorem 9.3, $\text{Aut}(\mathbb{A}^2)$ is an amalgam of two subgroups along their intersection. A similar decomposition exists for the group $\text{GL}(2, \mathbb{K}[x])$, see [Nag59], and the group $\text{Aut}^0(A_2)$ of augmentation-preserving automorphisms of the free associative algebra A_2 , see [SY98, Theorem 1.1].

It is unknown, however, whether a similar fact holds for $\text{Aut}(\mathbb{A}^n)$ when $n \geq 3$. For the tame subgroup $\text{Tame}_3 \subseteq \text{Aut}(\mathbb{A}_3)$ we have the following result.

Theorem 9.19 ([Wri15], see also [Wri17]). *Tame_3 can be realized as a generalized amalgamated product, specifically, the product of three subgroups amalgamated along pairwise intersections.*

See [ANU20, Theorem 1] and the references therein for analogs of the Jung-van der Kulk's Theorem for the automorphism group of the free Lie algebra in three variables and the subgroup of tame automorphisms. Similar facts are also known to hold for some Aut-homogeneous affine surfaces.

Example 9.20 (*Danilov-Gizatullin surfaces*). These are smooth affine surfaces $X_d = F_n \setminus S_d$ where $\pi_n: F_n \rightarrow \mathbb{P}^1$ is the n th (smooth projective) Hirzebruch ruled surface and the curve $S_d \subseteq F_n$ is an ample section of π_n with $S_d^2 = d > n$. Two such affine surfaces are isomorphic if and only if they share the same invariant d , see [GD77, Theorem 5.8.1]. The group $\text{Aut}(X_d)$ has exactly $\lfloor d/2 \rfloor$ conjugacy classes of maximal tori of rank 1, see [FKZ07, Prop. 5.15]. The group $\text{Aut}(X_d)$ is an amalgam of 2 nested ind-subgroups if $d = 3$ and of 3 nested ind-subgroups if $d = 4, 5$, see [GD77, Sections 5-8]. Assume that the base field \mathbb{K} is uncountable. Then $\text{Aut}(X_6)$ is not an amalgam of two nested ind-groups. For $d \geq 7$, $\text{Aut}(X_d)$ is not an amalgam of any countable number of nested ind-groups, see [KPZ17, Corollary 5.6].

One can find similar results for other Aut-quasihomogeneous affine surfaces in [GD75, GD77], see also [KPZ17, Sect. 5].

9.7. Linearization of reductive group actions. Let G be a reductive linear algebraic group acting effectively on \mathbb{A}^n with a fixed point x . Then the tangent representation of G_x on $T_x X$ is faithful. Luna's Étale Slice Theorem guarantees a local linearization of the G_x -action near x . This justifies an expectation that any action of a reductive group on \mathbb{A}^n can be linearized, that is, is conjugate in $\text{Aut}(\mathbb{A}^n)$ to a linear action; see e.g. [Kam79, Conjecture 3.1] and [vdE00, Section 9.4]. However, the latter expectation occurs to be only partially true.

Let us start with reductive subgroups of $\text{Aut}(\mathbb{A}^2)$. Every algebraic subgroup of $\text{Aut}(\mathbb{A}^2)$ has bounded degree, hence also a bounded length with respect to the free amalgamated product structure, see Theorem 9.3. By a theorem of Serre [Ser03, Ch. I, Sect. 4.3, Theorem 8] (cf. also [KPZ17, Proposition 2.24]) a subgroup of bounded length in an amalgamated free product $A *_C B$ is conjugate to a subgroup of one of the factors A and B . This leads to the following theorem.

Theorem 9.21 ([Kam79, Theorem 4.3]). *An algebraic subgroup G of $\text{Aut}(\mathbb{A}^2)$ is conjugate to a subgroup of either Aff_2 or Jonq_2 . In particular, every reductive group action on \mathbb{A}^2 is linearizable.*

The next corollary was proved by geometric means by T. Igarashi ('75, unpublished), see also [Fur83, Theorem 2] and [Miy24, Theorem 2.4.6].

Corollary 9.22. *Any action of a finite group on \mathbb{A}^2 is conjugate to a linear action.*

In higher dimensions, we start with the actions of commutative reductive groups. The following theorem by Kraft-Schwarz generalizes a similar theorem for algebraic tori by Białyński-Birula, see [Bia66-67].

Theorem 9.23 ([KrSc92, Theorem VI.3.2(2)]). *Any effective action of a commutative reductive group of dimension $\geq n - 1$ on \mathbb{A}^n is linearizable.*

In particular, all maximal tori in $\text{Aut}(\mathbb{A}^n)$ are pairwise conjugate and the same holds for tori of dimension $n - 1$. The latter fails for affine surfaces and affine threefolds, in general, see Examples 8.13 and 9.20. For the actions of tori of dimension $n - 2$ on \mathbb{A}^n we have the following theorem. Its proof uses a classification of certain smooth affine threefolds diffeomorphic to \mathbb{R}^6 and non-isomorphic to $\mathbb{A}_{\mathbb{C}}^3$.

Theorem 9.24 (see [KKMLR97], [KML97], [KoRu97]). *For $\mathbb{K} = \mathbb{C}$, any action of \mathbb{G}_m on $\mathbb{A}_{\mathbb{C}}^3$ is conjugate to a linear diagonal action.*

Theorem 9.25 ([LP21, Corollaire 1.2]). *Any finite subgroup of the tame automorphism group Tame_3 is linearizable.*

Theorem 9.26 ([KaR82]). *Assume that an effective action of the algebraic torus $T = (\mathbb{G}_m)^{n-2}$ on \mathbb{A}^n is "unmixed," in the sense that no nontrivial character, together with its inverse, occur as weights of semi-invariant polynomials. Then this action can be linearized.*

Theorem 9.27 ([KoRu89]). *Suppose that the algebraic torus $T = (\mathbb{G}_m)^{n-2}$ acts effectively on \mathbb{A}^n . If the fixed point set of T has dimension ≥ 1 , then the T -action is linearizable.*

Next we turn to actions of non-commutative reductive groups. The following theorem is a corollary of Luna's Étale Slice Theorem; cf. also [BH85, Corollary 10.7].

Theorem 9.28 ([Lun73, Corollaire 2 and Remarque on p. 99]). *Any action of a reductive group G on \mathbb{A}^n with an isolated fixed point and a dense orbit is linearizable.*

Theorem 9.29 ([KrPo85], [Pop01] for $n = 3$, [Pan84-86] for $n = 4$). *For $n \leq 4$, any action of a non-commutative connected semisimple algebraic group on \mathbb{A}^n is conjugate to a linear action. The same holds for any connected reductive group for $n = 3$ and for any such group different from a one- or two-dimensional torus for $n = 4$.*

Theorem 9.30 ([KrRu14, Theorem A]). *Every faithful action of a non-finite reductive group on \mathbb{A}^3 is linearizable.*

The first examples of non-linearizable actions of connected reductive groups on \mathbb{A}^n are due to Schwarz [Sch89]. These are an $O_2(\mathbb{C})$ -action on $\mathbb{A}_{\mathbb{C}}^4$ and an $SL_2(\mathbb{C})$ -action on $\mathbb{A}_{\mathbb{C}}^7$. In a sense, the non-linearizability is a general phenomenon.

Theorem 9.31 ([Kno91]). *For any non-commutative connected reductive algebraic group G there exists a non-linearizable G -action on some affine space \mathbb{A}^n .*

All known examples of non-linearizable actions on \mathbb{A}^n are obtained from non-trivial algebraic G -vector bundles over G -modules, following an idea from [KaR82] and [BH85], see also [Kra89], [KrSc92], [MP91] and [MMP91]. Indeed, by the Quillen-Suslin theorem, such a bundle is trivial when ignoring the G -action. Hence, its total space is isomorphic to some \mathbb{A}^n . The induced G -action on \mathbb{A}^n is non-linearizable due to the fact that our bundle is non-trivial in the category of G -vector bundles.

However, this argument does not work for an abelian reductive group G . Indeed, in this case every G -vector bundle over a G -module is trivial, see [MMP96]. In particular, the following questions remain open, see e.g. [KrSc95] and [Mas03]:

Question 9.32. *Is there a non-linearizable action of an abelian group on \mathbb{A}^n , for instance, of a cyclic group $\mathbb{Z}/p\mathbb{Z}$ on \mathbb{A}^n , $n \geq 3$ and of \mathbb{G}_m on \mathbb{A}^n , $n \geq 4$?*

Notice that there is an example of Asanuma of a non-linearizable action of $\mathbb{G}_m(\mathbb{R})$ on $\mathbb{A}_{\mathbb{R}}^n$, see [Asa99].

There are several examples of non-linearizable actions of non-abelian finite groups on \mathbb{A}^n . In particular, we have the following theorem.

Theorem 9.33 ([MJ94, Theorem 5.1]). *There is a non-linearizable action of the dihedral group D_{10} on \mathbb{A}^4 , and there is a continuous one-parameter family of non-linearizable actions of D_{14} on \mathbb{A}^4 .*

See e.g. [MP95], [MMP96], [Mas03] and the references therein for further examples of non-linearizable actions of dihedral groups on \mathbb{A}^n and [FM15] for such examples with an action of the symmetric group S_3 on \mathbb{A}^4 . It seems that no example of this kind is known for a finite group action on \mathbb{A}^3 , see e.g. [FM15]. See also [PeRa86] for the Lefschetz fixed-point formula in the context of finite group actions on affine varieties.

The linearization problem is closely related to the Abhyankar–Sathaye problem on linearization of embeddings $\mathbb{A}^k \hookrightarrow \mathbb{A}^n$ and on variables in polynomial rings. We send the reader e.g. to [Kal09], [KrRu14], [vdEMV07] and the literature therein for further reading.

9.8. Solvable and unipotent ind-subgroups: triangulation. We start with the following result.

Proposition 9.34 ([FuKr18, Proposition 10.5.1]). *Let $U \subseteq \text{Aut}(X)$ be a commutative unipotent algebraic subgroup of dimension n . Assume that $\text{Cent}_{\text{Aut}(X)}(U) = U$. Then for any $x \in X$ the orbit map $U \rightarrow X$, $u \mapsto ux$, is an isomorphism. In particular, X is isomorphic to \mathbb{A}^n .*

The unipotent elements of the de Jonquières group Jonq_n form a solvable subgroup of derived length n . This subgroup is not nilpotent for $n > 1$; see [FuKr18, Sect. 15.1 and Lemma 3.2].

Definition 9.35. We say that a subgroup $G \subseteq \text{Aut}(\mathbb{A}^n)$ is *triangulable* if it is conjugate in $\text{Aut}(\mathbb{A}^n)$ to a subgroup of Jonq_n . It is said to be *stably triangulable* if $G \oplus \text{id}_{\mathbb{A}^k}$ is triangulable in $\text{Aut}(\mathbb{A}^{n+k})$ for some $k \geq 0$.

By the Kambayashi-Serre theorem, any unipotent subgroup of $\text{Aut}(\mathbb{A}^2)$ is triangulable, see [Ren68] and [Kam79, Theorem 4.3]. This is not any longer true in higher dimensions. However, the next theorem provides a large class of triangulable unipotent subgroups in any dimension.

Theorem 9.36 ([KrZa22, Theorem D]). *Let $U \subseteq \text{Aut}(\mathbb{A}^n)$ be a nested unipotent subgroup. If U acts transitively on \mathbb{A}^n , then U is triangulable.*

A similar result holds for Lie subalgebras of vector fields.

Theorem 9.37 ([Sku21, Theorem 6]). *Let $L \subseteq \text{Lie}(\text{Aut}(\mathbb{A}^n))$ be a Lie subalgebra that consists of locally nilpotent derivations. Assume that $\bigcap_{\partial \in L} \ker(\partial) = \mathbb{K}$. Then L is Ad-conjugate to a subalgebra of the Lie algebra $\text{Lie}(\text{Jonq}_n)$ of triangular derivations.*

The transitivity assumption in Theorem 9.36 is essential, as the following example shows.

Example 9.38. Consider the Nagata automorphism $\Phi = \exp(\sigma\partial)$, see subsection 9.2. It is contained in the \mathbb{G}_a -subgroup $U = \exp(t\sigma\partial)$ of $\text{Aut}(\mathbb{A}^3)$. According to Bass [Bas84], U is not triangulable. Indeed, the fixed points set $(\mathbb{A}^3)^{\mathbb{G}_a}$ is a hypersurface with an isolated singularity. However, the fixed point set of a triangulable \mathbb{G}_a -subgroup is isomorphic to a cylinder $X \times \mathbb{A}^1$ which cannot have an isolated singularity. Using the same idea one can construct non-triangulable \mathbb{G}_a -subgroups of any $\text{Aut}(\mathbb{A}^n)$, $n \geq 3$, see [Pop87].

According to Freudenburg [Fre17, Lemma 3.36], the Nagata-Bass \mathbb{G}_a -subgroup U is not stably triangulable in $\text{Aut}(\mathbb{A}^4)$. In fact, no \mathbb{G}_a -subgroup of Bass-Popov type is stably triangulable, whatever is the number of additional variables.

Proposition 9.39 ([KrZa22, Proposition 1.4.1]). *Consider a \mathbb{G}_a -subgroup $U \subseteq \text{Aut}(\mathbb{A}^n)$. Assume that the fixed point set $(\mathbb{A}^n)^U$ is a hypersurface with an isolated singularity. Then U is not stably triangulable.*

There is, however, the following positive result.

Theorem 9.40 ([Kal04]). *Assume that $\mathbb{K} = \mathbb{C}$, and let a \mathbb{G}_a -subgroup $U \subseteq \text{Aut}(\mathbb{A}^3)$ acts freely on \mathbb{A}^3 . Then U is conjugate in $\text{Aut}(\mathbb{A}^3)$ to a subgroup of translations. In particular, U is triangulable.*

An analog of the former assertion fails in \mathbb{A}^4 , see [Win90]. For further results in this direction, see e.g. the survey article [Kal09] and [GMM21] for \mathbb{G}_a -actions on affine varieties, [Fre17] and [vdEKC21] for \mathbb{G}_a -actions on affine spaces and Hilbert's 14th problem. See also [Pop17] and the references therein for (stably) birationally triangulable and non-triangulable actions of unipotent groups.

The following question concerns a modified version of stable triangulation.

Question 9.41 ([KrZa22, Question 6]). *Let $G \subseteq \text{Aut}(X)$ be a solvable or nilpotent connected subgroup. Does there exist a closed embedding $X \hookrightarrow \mathbb{A}^n$ such that G extends to a subgroup of the de Jonquières group Jonq_n ? Is this true if G is algebraic resp., nested?*

The answer is affirmative if $X = \mathbb{A}^k$ and $G \subseteq \text{Aut}(\mathbb{A}^k)$ is a unipotent algebraic group, see [KrZa22, Remark 1.4.2(3)]. For instance, the Nagata-Bass \mathbb{G}_a -subgroup of $\text{Aut}(\mathbb{A}^3)$ can be triangulated in \mathbb{A}^4 in the above sense, being non-stably triangulable in the usual sense, cf. Example 9.38.

10. TITS' TYPE ALTERNATIVE FOR AUTOMORPHISM GROUPS

A group G is called *virtually solvable* resp., *virtually nilpotent*, etc. if G is a finite extension of a solvable resp. nilpotent, etc. group. The classical Tits alternative says that any linear group G , that is, a subgroup of $\text{GL}(n, \mathbb{K})$ either is virtually solvable, or contains a free group \mathbb{F}_2 on two generators. For finitely generated subgroups, the Tits alternative holds without any restriction on the base field.

There are several analogs of this theorem for automorphism groups. For instance, the Tits alternative holds for $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$, see [Lam01, Theorem 2.4] or [Lam24, Proposition 20.17].

Notice that $\text{Aut}(\mathbb{A}_{\mathbb{K}}^2)$ is not isomorphic to a linear group over any transcendental extension of the base field \mathbb{K} . However, by [Mat23, Theorem A] the codimension 6 normal subgroup

$$G = \{\varphi \in \text{Aut}(\mathbb{A}_{\mathbb{K}}^2) \mid \varphi(0) = 0, \ d\varphi(0) = \text{id}\}$$

is a linear group over a suitable field extension.

For the proof of the following theorem see [Can11] and [Ure21] for (a), [BFL14, Theorem C] for (b) and [LP22, Main Theorem] for (c).

Theorem 10.1. *An analog of the Tits alternative holds for the following groups:*

- (a) *the birational automorphism group of any compact complex Kähler surface;*
- (b) $\text{Tame}(\text{Aut}(\text{SL}(2, \mathbb{K})))$ (see Remark 9.8);
- (c) $\text{Tame}_3 = \text{Tame}(\text{Aut}(\mathbb{A}^3))$.

In the case of algebraically generated groups of automorphisms, we have the following Tits' type alternatives.

Theorem 10.2 ([AZ22, Theorem 1.1] and [AZ23, Theorem 6.1]).

- (a) *Let X be an affine algebraic surface and $G \subseteq \text{Aut}(X)$ be a subgroup generated by \mathbb{G}_a -subgroups U_1, \dots, U_k . Then either G contains \mathbb{F}_2 or G is a metabelian unipotent algebraic group.*

- (b) *Let X be a toric affine variety and $G \subseteq \text{Aut}(X)$ be a subgroup generated by \mathbb{G}_a -subgroups U_1, \dots, U_k normalized by the acting torus. Then either G contains \mathbb{F}_2 or G is a unipotent algebraic group.*

Similar results also hold for subgroups of $\text{Tame}(\text{Aut}(\text{SL}(2, \mathbb{K})))$ and $\text{Tame}(\text{Aut}(\mathbb{A}^3))$, see [AZ23, Corollary 6.4].

REFERENCES

- [ANU20] Alibek A. Alimbaev, Ruslan Zh. Nauryzbaev, and Ualbai U. Umirbaev, *On the automorphisms of a free Lie algebra of rank 3 over an integral domain*, Sib. Math. J. 61:1 (2020), 1–10.
- [And19] Rafael B. Andrist, *Integrable generators of Lie algebras of vector fields on \mathbb{C}^n* , Forum Math. 31 (2019), 943–950.
- [Arz23] Ivan Arzhantsev, *Automorphisms of algebraic varieties and infinite transitivity*, St. Petersburg Math. J. 34:2 (2023), 143–178.
- [AFK⁺13a] Ivan Arzhantsev, Hubert Flenner, Shulim Kaliman, Frank Kutzschebauch, and Mikhail Zaidenberg, *Flexible varieties and automorphism groups*, Duke Math. J. 162:4 (2013), 767–823.
- [AFK⁺13b] Ivan Arzhantsev, Hubert Flenner, Shulim Kaliman, Frank Kutzschebauch, and Mikhail Zaidenberg, *Infinite transitivity on affine varieties*, In: Birational Geometry, Rational Curves, and Arithmetic. Fedor Bogomolov, Brendan Hassett, Yuri Tschinkel (eds.), 1–14. Springer-Verlag, New York e.a. 2013.
- [AG10] Ivan V. Arzhantsev and Sergei A. Gaifullin, *Cox rings, semigroups and automorphisms of affine algebraic varieties*, Sb. Math. 201:1 (2010), 1–21.
- [AG17] Ivan Arzhantsev and Sergey Gaifullin, *The automorphism group of a rigid affine variety*, Math. Nachr. 290:5–6 (2017), 662–671.
- [AKZ12] Ivan Arzhantsev, Karine Kuyumzhiyan, and Mikhail Zaidenberg, *Flag varieties, toric varieties, and suspensions: Three instances of infinite transitivity*, Sb. Math. 203 (2012), 923–949.
- [AKZ19] Ivan Arzhantsev, Karine Kuyumzhiyan, and Mikhail Zaidenberg, *Infinite transitivity, finite generation, and Demazure roots*, Adv. Math. 351 (2019), 1–32.
- [APS14] Ivan Arzhantsev, Alexander Perepechko, and Hendrik Süß, *Infinite transitivity on universal torsors*, J. London Math. Soc. (2) 89:3 (2014), 762–778.
- [AZ22] Ivan Arzhantsev and Mikhail Zaidenberg, *Tits alternative and highly transitive actions on toric varieties*, Int. Math. Res. Not. (IMRN) 11 (2022), 8162–8195.
- [AZ23] Ivan Arzhantsev and Mikhail Zaidenberg, *Tits-type alternative for certain groups acting on algebraic surfaces*, Proc. Amer. Math. Soc. 151:7 (2023), 2813–2829.
- [ArZa23] Ivan Arzhantsev and Yulia Zaitseva, *Affine homogeneous varieties and suspensions*, arXiv:2309.06170 (2023).
- [Asa99] Teruo Asanuma, *Non-linearizable algebraic k^* -actions on affine spaces*, Invent. Math., 138:2 (1999), 281–306.

- [Bas84] Hyman Bass, *A non-triangular action of \mathbb{G}_a on \mathbb{A}^3* , J. Pure Appl. Algebra 33 (1984), 1–5.
- [BH85] Hyman Bass and William J. Haboush, *Linearizing certain reductive group actions*, Trans. Amer. Math. Soc. 292 (1985), 463–482.
- [BKY12] Alexei Belov-Kanel and Jie-Tai Yu, *Stable tameness of automorphisms of $F\langle x, y, z \rangle$ fixing z* , Sel. Math. New Ser. 18 (2012), 799–802.
- [BH03] Florian Berchtold and Jürgen Hausen, *Demushkin’s Theorem in codimension one*, Math. Z. 244 (2003), 697–703.
- [BW00] Yuri Berest and George Wilson, *Automorphisms and ideals of the Weyl algebra*, Math. Ann. 318:1 (2000), 127–147.
- [BvdEW12] Joost Berson, Arno van den Essen, and David Wright, *Stable tameness of two-dimensional polynomial automorphisms over a regular ring*, Adv. Math. 230 (2012), 2176–2197.
- [BPZ22] Oksana Bezushchak, Anatoliy Petravchuk, and Efim Zelmanov, *Automorphisms and derivations of affine commutative and PI-algebras*, Trans. Amer. Math. Soc. 377:2 (2024), 1335–1356.
- [Bia66-67] Andrzej Białynicki-Birula, *Remarks on the action of an algebraic torus on k^n . I, II*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 14 (1966), 177–181; 15 (1967), 123–125.
- [BFL14] Cinzia Bisi, Jean-Philippe Furter, and Stéphane Lamy, *The tame automorphism group of an affine quadric threefold acting on a square complex*, J. Ec. polytech. Math. 1 (2014), 161–223.
- [BD15] Jérémy Blanc and Adrien Dubouloz, *Affine surfaces with a huge group of automorphisms*, Int. Math. Res. Not. (IMRN) 2 (2015), 422–459.
- [BG23] Viktoria Borovik and Sergey Gaifullin, *Isolated torus invariants and automorphism groups of rigid varieties*, arXiv:2007.07882 (2023).
- [Bod01] Yuri Bodnarchuk, *Some extreme properties of the affine group as an automorphisms group of the affine space*, Contribution to General Algebra 13 (2001), 15–29.
- [Bod02] Yuri Bodnarchuk, *Affine group as a subgroup of biregular transformation group of an affine space*, (Ukrainian). NaUKMA Zap., Kyiv 20 (2002), 6–10.
- [Bod05] Yuri Bodnarchuk, *On generators of the tame invertible polynomial maps group*, Int. J. Algebra Comput. 15 (2005), 851–867.
- [Can11] Serge Cantat, *Sur les groupes de transformations birationnelles des surfaces*, Ann. Math. (2) 174:1 (2011), 299–340.
- [CRX19] Serge Cantat, Andriy Regeta and Junyi Xie, *Families of commuting automorphisms, and a characterization of the affine space*, Amer. J. Math. 145:2 (2019), 413–434.
- [Cze71-72] Anastasia J. Czerniakiewicz, *Automorphisms of a free associative algebra of rank 2. I, II*, Trans. Amer. Math. Soc., I: 160 (1971), 393–401; II: 171 (1972), 309–315.
- [CPPZ21] Ivan Cheltsov, Jihun Park, Yuri Prokhorov, and Mikhail Zaidenberg, *Cylinders in Fano varieties*, EMS Surv. Math. Sci. 8 (2021), 39–105,
- [CD03] Arjeh M. Cohen and Jan Draisma, *From Lie algebras of vector fields to algebraic group actions*, Transform. Groups 8:1 (2003), 51–68.
- [Coh85] Paul M. Cohn, *Free rings and their relations*, 2nd ed., Academic Press, London 1985.

- [Dai19] Daniel Daigle, *Locally nilpotent sets of derivations*, 2019, in: Kuroda, S., Onoda, N., Freudenburg, G. (eds). *Polynomial Rings and Affine Algebraic Geometry*, 41–71. PRAAG 2018. Springer Proceedings in Mathematics & Statistics, vol 319. Springer, Cham.
- [Dani89] Włodzimierz Danielewski, *On the cancellation problem and automorphism group of affine algebraic varieties*, Preprint, 1989.
- [Dan74] Vladimir I. Danilov, *Non-simplicity of the group of unimodular automorphisms of an affine plane*, *Mat. Zametki* 15 (1974), 289–293.
- [Dem82] Aleksander S. Demushkin, *Combinatorial invariance of toric singularities*, *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* 2 (1982), 80–87.
- [Dub04] Adrien Dubouloz, *Completions of normal affine surfaces with a trivial Makar-Limanov invariant*, *Michigan Math. J.* 52 (2004), 289–308.
- [DMJP10] Adrien Dubouloz, Lucy Moser-Jauslin, and Pierre-Marie Poloni, *Inequivalent embeddings of the Koras–Russell cubic 3-fold*. *Michigan Math. J.* 59 (2010), 679–694.
- [Edo18] Éric Edo. *Closed subgroups of the polynomial automorphism group containing the affine subgroup*. *Transform. Groups* 23 (2018), 71–74.
- [EL19] Éric Edo and Drew Lewis, *Co-tame polynomial automorphisms*, *Internat. J. Algebra Comput.* 29 (2019), 803–825.
- [EP15] Éric Edo and Pierre-Marie Poloni, *On the closure of the tame automorphism group*, *Int. Math. Res. Not. (IMRN)* 19 (2015), 9736–9750.
- [Fie94] Karl-Heinz Fieseler, *On complex affine surfaces with \mathbb{C}^+ -action*, *Comment. Math. Helv.* 69:1 (1994), 5–27.
- [FKZ07] Hubert Flenner, Shulim Kaliman, and Mikhail Zaidenberg, *Completions of \mathbb{C}^* -surfaces*, in: *Affine algebraic geometry*, 149–200. Osaka Univ. Press, Osaka 2007.
- [FKZ09] Hubert Flenner, Shulim Kaliman, and Mikhail Zaidenberg, *On the Danilov-Gizatullin isomorphism theorem*, *Enseign. Math. (2)* 55:3-4 (2009), 275–283.
- [FKZ11] Hubert Flenner, Shulim Kaliman, and Mikhail Zaidenberg, *Smooth affine surfaces with non-unique \mathbb{C}^* -actions*, *J. Algebraic Geom.* 20 (2011), 329–398.
- [FKZ16] Hubert Flenner, Shulim Kaliman, and Mikhail Zaidenberg, *The Gromov–Winkelmann theorem for flexible varieties*, *J. Eur. Math. Soc.* 18:11 (2016), 2483–2510.
- [FZ05b] Hubert Flenner and Mikhail Zaidenberg, *On the uniqueness of \mathbb{C}^* -actions on affine surfaces*, In: *Affine Algebraic Geometry*, Contemporary Mathematics Vol. 369, 97–111. Amer. Math. Soc., Providence, RI, 2005.
- [Fre17] Gene Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*, 2nd edn. *Encyclopaedia of Mathematical Sciences* 136. *Invariant Theory and Algebraic Transformation Groups VII*. Berlin: Springer, 2017.
- [FM15] Gene Freudenburg and Lucy Moser-Jauslin, *A nonlinearizable action of S_3 on \mathbb{C}^4* , *Ann. Inst. Fourier, Grenoble*, 52:1 (2002), 133–143.
- [Fur15] Jean-Philippe Furter, *Polynomial composition rigidity and plane polynomial automorphisms*, *J. London Math. Soc. (2)* 91 (2015), 180–202.
- [FuKr18] Jean-Philippe Furter and Hanspeter Kraft, *On the geometry of automorphism groups of affine varieties*, arXiv:1809.04175 (2018).

- [FuPo18] Jean-Philippe Furter and Jean-Marie Poloni, *On the maximality of the triangular subgroup*, Ann. Inst. Fourier 68:1 (2018), 393–421.
- [Fur83] Mikio Furushima, *Finite groups of polynomial automorphisms in \mathbb{C}^n* , Tohoku Math. J. (2) 35 (1983), 415–424.
- [Giz71a] Marat Kh. Gizatullin, *Affine surfaces that are quasihomogeneous with respect to an algebraic group*, Math. USSR Izv. 5 (1971), 754–769.
- [Giz71b] Marat Kh. Gizatullin, *Quasihomogeneous affine surfaces*, Math. USSR Izv. 5 (1971), 1057–1081.
- [GD75] Marat Kh. Gizatullin and Vladimir I. Danilov, *Automorphisms of affine surfaces I*, Math. USSR Izv. 9:3 (1975), 493–534.
- [GD77] Marat Kh. Gizatullin and Vladimir I. Danilov, *Automorphisms of affine surfaces II*, Math. USSR Izv. 11:1 (1977), 51–98.
- [Gub98] Joseph Gubeladze, *The isomorphism problem for commutative monoid rings*, J. Pure Appl. Algebra 129 (1998), 35–65.
- [GMM21] Rajendra V. Gurjar, Kayo Masuda, and Masayoshi Miyanishi, *Affine space fibrations*, Studies in Mathematics vol. 79, De Gruyter, 2021.
- [Jun42] Heinrich W. E. Jung, *Über ganze birationale Transformationen der Ebene*, J. Reine Angew. Math. 184 (1942), 161–174.
- [Kal04] Shulim Kaliman, *Free \mathbb{C}^+ -actions on \mathbb{C}^3 are translations*, Invent. Math. 156:1 (2004), 163–173.
- [Kal09] Shulim Kaliman, *Actions of \mathbb{C}^* and \mathbb{C}^+ on affine algebraic varieties*, in: *Algebraic Geometry*, Seattle 2005. D. Abramovich e.a. (eds.), 629–654. Proceedings of Symposia in Pure mathematics vol. 80, Part 2. Amer. Math. Soc. 2009.
- [KML97] Shulim Kaliman and Leonid Makar-Limanov, *On Russell-Koras contractible threefolds*, J. Algebraic Geom. 6 (1997), 247–268.
- [KKMLR97] Shulim Kaliman, Mariusz Koras, Leonid Makar-Limanov, and Peter Russell, *\mathbb{C}^* -actions on \mathbb{C}^3 are linearizable*, Electron. Res. Announc. Amer. Math. Soc. 3 (1997), 63–71.
- [KZ99] Shulim Kaliman and Mikhail Zaidenberg, *Affine modifications and affine hypersurfaces with a very transitive automorphism group*, Transform. Groups 4 (1999), 53–95.
- [Kam75] Tatsuji Kambayashi, *On the absence of nontrivial separable forms of the affine plane*. J. Algebra 35 (1975), 449–456.
- [Kam79] Tatsuji Kambayashi, *Automorphism group of a polynomial ring and algebraic group action on an affine space*, J. Algebra, 60 (1979), 439–451.
- [KaR82] Tatsuji Kambayashi and Peter Russell, *On linearizing algebraic torus actions*, J. Pure Appl. Algebra 23:3 (1982), 243–250.
- [Kno91] Friedrich Knop, *Nonlinearizable actions of semisimple groups on affine spaces* (Nichtlinearisierbare Operationen halbeinfacher Gruppen auf affinen Räumen) (German), Invent. Math. 105:1 (1991), 217–220.
- [KoRu89] Mariusz Koras and Peter Russell, *Codimension 2 torus actions on affine n -space*, in: *Group actions and invariant theory, Proc. Conf., Montreal/Can. 1988*. Canad. Math. Soc. Conf. Proc. 10 (1989), 103–110.
- [KoRu97] Mariusz Koras and Peter Russell, *Contractible threefolds and \mathbb{C}^* -actions on \mathbb{C}^3* , J. Algebraic Geom. 6 (1997), 671–695.

- [KPZ17] Sergei Kovalenko, Alexander Perepechko and Mikhail Zaidenberg, *On automorphism groups of affine surfaces*, In: *Algebraic varieties and automorphism groups*, pp. 207–286. Adv. Stud. Pure Math., vol. **75**, Math. Soc. Japan, Tokyo, 2017.
- [Kra89] Hanspeter Kraft, *G-vector bundles and the Linearization Problem*, in: *Group Actions and Invariant Theory*, CMS Conference Proceedings vol. 10, 111–123. Amer. Math. Soc., Providence, RI, 1989.
- [Kra96] Hanspeter Kraft, *Challenging problems on affine n-space*, Séminaire Bourbaki, Exp. No. 802, 5, Vol. 1994/95. Ast érisque 237 (1996), 295–317.
- [Kra17] Hanspeter Kraft, *Automorphism groups of affine varieties and a characterization of affine n-space*, Trans. Moscow Math. Soc. 78 (2017), 171–186.
- [KrPo85] Hanspeter Kraft and Vladimir L. Popov, *Semisimple group actions on the three-dimensional affine space are linear*, Comment. Math. Helv. 60 (1985), 466–479.
- [KrRu14] Hanspeter Kraft and Peter Russell, *Families of group actions, generic isotriviality, and linearization*, Transform. Groups 19:3 (2014), 779–792.
- [KrSc92] Hanspeter Kraft and Gerald W. Schwarz, *Reductive group actions with one-dimensional quotient*, Inst. Hautes Etudes Sci. Publ. Math. 76 (1992), 1–97.
- [KrSc95] Hanspeter Kraft and Gerald Schwarz, *Finite automorphisms of affine N-space*, in: Essen, Arno van den (ed.), *Automorphisms of affine spaces*. Proceedings of the international conference on invertible polynomial maps, held in Curaçao, Netherlands Antilles, July 4-8, 1994. Dordrecht: Kluwer Academic Publishers. 55–66 (1995).
- [KrSt13] Hanspeter Kraft and Immanuel Stampfli, *On automorphisms of the affine Cremona group*, Ann. Inst. Fourier 63:3 (2013), 1137–1148.
- [KrZa22] Hanspeter Kraft and Mikhail Zaidenberg, *Algebraically generated groups and their Lie algebras*, J. London Math. Soc. (2) 109:e12866 (2024), 1–39.
- [Kum02] Shrawan Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.
- [Lam01] Stéphane Lamy, *L’alternative de Tits pour $\text{Aut}[\mathbb{C}^2]$* (in French), J. Algebra 239:2 (2001), 413–437.
- [Lam24] Stéphane Lamy, *The Cremona group. Preliminary version, February 15, 2024*, available at: <https://www.math.univ-toulouse.fr/slamy/blog/cremona.html>
- [LP21] Stéphane Lamy and Piotr Przytycki, *Presqu’un immeuble pour le groupe des automorphismes mod ér és*, Ann. H. Lebesgue 4 (2021), 605–651.
- [LP22] Stéphane Lamy and Piotr Przytycki, *Tits alternative for the 3-dimensional tame automorphism group*, arXiv:2206.05611 (2022).
- [LV13] Stéphane Lamy and Stéphane Vénéereau, *The tame and the wild automorphisms of an affine quadric threefold*, J. Math. Soc. Japan 65:1 (2013), 299–320.
- [Lun73] Domingo Luna, *Slices étales*, Mémoires de la S. M. F. 33 (1973), 81–105.
- [ML70] Leonid G. Makar-Limanov, *Automorphisms of a free algebra with two generators*, Funct. Anal. Appl., 4:3 (1970), 262–264.
- [Mas03] Kayo Masuda, *Nonlinearizable actions of dihedral groups on affine space*, Trans. Amer. Math. Soc. 356:9 (2003), 3545–3556.

- [MMP91] Mikiyo Masuda, Lucy Moser-Jauslin, and Ted Petrie, *Equivariant algebraic vector bundles over representations of reductive groups: Applications*, Proc. Nat. Acad. Sci. 88 (1991), 9065–9066.
- [MMP96] Mikiyo Masuda, Lucy Moser-Jauslin, and Ted Petrie, *The equivariant Serre problem for abelian groups*, Topology 35 (1996), 329–334.
- [MP91] Mikiyo Masuda and Ted Petrie, *Equivariant algebraic vector bundles over representations of reductive groups: Theory*, Proc. Nat. Acad. Sci. 88 (1991), 9061–9064.
- [MP95] Mikiyo Masuda and Ted Petrie, *Stably trivial equivariant algebraic vector bundles*, J. Amer. Math. Soc. 8 (1995), 687–714.
- [Mat23] Olivier Mathieu, *Linearity and nonlinearity of groups of polynomial automorphisms of the plane*, J. Algebra 637 (2024), 47–89.
- [MP09] Stefan Maubach and Pierre-Marie Poloni, *The Nagata automorphism is shifted linearizable*, J. Algebra 321 (2009), 879–889.
- [Miy24] Masayoshi Miyanishi, *Affine algebraic geometry: geometry of polynomial rings*. Series on University Mathematics, vol. 11. World Scientific, New Jersey, 2024.
- [MJ94] Lucy Moser-Jauslin, *Algebraic equivariant vector bundles and the linearity problem*, in: Proc. Symp. in Pure Mathem. vol. 56, Part I, 1994, 355–364.
- [Nag59] Hirosi Nagao, *On $GL(2, K[x])$* , J. Inst. Polytech. Osaka City Univ. Ser. A 10 (1959), 117–121.
- [Nag72] Masayoshi Nagata, *On automorphism group of $k[x, y]$* , Kinokuniya Book-Store Co., Ltd. Tokyo, Japan, 1972.
- [Pan84-86] Dmitri I. Panyushev, *Semisimple automorphism groups of four-dimensional affine space*, Math. USSR Izv. 23 (1984), 171–183. Correction: *ibid.* 27:3 (1986), 607–608.
- [Per23] Alexander Perepechko, *Structure of connected nested automorphism groups*, arXiv:2312.08359 (2023).
- [PR22] Alexander Perepechko and Andriy Regeta, *Automorphism groups of affine varieties consisting of algebraic elements*, arXiv:2203.08950 (2022).
- [PR23] Alexander Perepechko and Andriy Regeta, *When is the automorphism group of an affine variety nested?*, Transform. Groups 28:1 (2023), 401–412.
- [PZ22] Alexander Perepechko and Mikhail Zaidenberg, *Automorphism groups of rigid affine surfaces: the identity component*, arXiv:2208.09738 (2022).
- [PeRa86] Ted Petrie and John D. Randall, *Finite-order algebraic automorphisms of affine varieties*, Comment. Math. Helv. 61:2 (1986), 203–221.
- [Pop87] Vladimir L. Popov, *On actions of G_a on A^n* , in: *Algebraic groups. Utrecht 1986*, 237–242. Lecture Notes in Math., vol. 1271, Springer, Berlin, 1987.
- [Pop73] Vladimir L. Popov, *Classification of affine algebraic surfaces that are quasihomogeneous with respect to an algebraic group*, Math. USSR Izv. 7 (1973), 1039–1055 (1975).
- [Pop01] Vladimir L. Popov, *On polynomial automorphisms of affine spaces*, Izv. Math. 65:3 (2001), 569–587.
- [Pop17] Vladimir L. Popov, *Bass’ triangulability problem*, in: *Algebraic varieties and automorphism groups*, 425–441. Adv. Stud. Pure Math. vol. 75, Math. Soc. Japan, Tokyo, 2017.

- [Smi89] Martha K. Smith, *Stably tame automorphisms*, J. Pure Appl. Algebra **58** (1989), 209–212.
- [Ram64] Chakravarthi P. Ramanujam, *A note on automorphism groups of algebraic varieties*, Math. Ann. 156 (1964), 25–33.
- [RvS21] Andriy Regeta and Immanuel van Santen, *Maximal commutative unipotent subgroups and a characterization of affine spherical varieties*, arXiv:2112.04784 (2021).
- [Ren68] Rudolf Rentschler, *Opérations du groupe additif sur le plan affine*, C. R. Acad. Sci. Paris Ser. A 267 (1968), 384–387.
- [Sch89] Gerald W. Schwarz, *Exotic algebraic group actions*, C. R. Acad. Sci. Paris Ser. Math. 309 (1989), 89–94.
- [Ser03] Jean-Pierre Serre, *Trees*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.
- [Sha81] Igor R. Shafarevich, *On some infinite-dimensional groups. II*, Izv. Akad. Nauk SSSR Ser. Mat. 45:1 (1981), 214–226.
- [Sha04] Igor R. Shafarevich, *On the Group $GL(2, K[t])$* , Proc. Steklov Inst. Math. 246 (2004), 308–314.
- [SU04a] Ivan P. Shestakov and Ualbai U. Umirbaev, *Poisson brackets and two generated subalgebras of rings of polynomials*, J. Amer. Math. Soc. 17 (2004), 181–196.
- [SU04b] Ivan P. Shestakov and Ualbai U. Umirbaev, *Tame and wild automorphisms of rings of polynomials in three variables*, J. Amer. Math. Soc. 17 (2004), 197–227.
- [SY98] Vladimir Shpilrain and Jie-Tai Yu, *On generators of polynomial algebras in two commuting or non-commuting variables*, J. Pure Appl. Algebra 132 (1998), 309–315.
- [Sku21] Alexander A. Skutin, *Maximal Lie subalgebras among locally nilpotent derivations*, Sb. Math. 212:2 (2021), 265–271.
- [Smi89] Martha K. Smith, *Stably tame automorphisms*, J. Pure Appl. Algebra 58:2 (1989), 209–212.
- [Sno89] Denis Snow, *Unipotent actions on affine space*, in: *Topological Methods in Algebraic Transformation Groups (New Brunswick, NJ, 1988)*, 165–176. Progr. Math., vol. 80, Birkhauser Boston, Boston, MA, 1989.
- [Spo07] Stanisław Spodzieja, *On the Nagata automorphism*, Univ. Iagel. Acta Math. XIV (2007), 131–136.
- [Umi07] Ualbai U. Umirbaev, *The Anick automorphism of free associative algebras*, J. Reine Angew. Math. 605 (2007), 165–178.
- [Ure21] Christian Urech, *Subgroups of elliptic elements of the Cremona group*, J. Reine Angew. Math. 770 (2021), 27–57.
- [vdE00] Arno van den Essen, *Polynomial automorphisms and the Jacobian conjecture*, Progress in Mathematics, vol. 190, Birkhäuser, Basel, Boston, Berlin, 2000.
- [vdEKC21] Arno van den Essen, Shigeru Kuroda, and Anthony J. Crachiola, *Polynomial automorphisms and the Jacobian conjecture. New results from the beginning of the 21st Century*, in: *Frontiers in Mathematics*. Birkhäuser, Cham, Switzerland, 2021.

- [vdEMV07] Arno van den Essen, Stefan Maubach, and Stephane Vénéreau, *The special automorphism group of $R[t]/(t^m)[x_1, \dots, x_n]$ and coordinates of a subring of $R[t][x_1, \dots, x_n]$* , J. Pure Appl. Alg. 210 (2007), 141–146.
- [vdK53] Wouter van der Kulk, *On polynomial rings in two variables*, Nieuw Arch. Wisk. (3) 1 (1953), 33–41.
- [Win90] Jörg Winkelmann, *On free holomorphic \mathbb{C}_+ -actions on \mathbb{C}^n and homogeneous Stein manifolds*, Math. Ann. 286:1-3 (1990), 593–612.
- [Wri15] David Wright, *The generalized amalgamated product structure of the tame automorphism group in dimension three*, Transform. Groups 20:1 (2015), 291–304.
- [Wri17] David Wright, *Amalgamation and automorphism groups*, in: *Algebraic varieties and automorphism groups*, 465–474. Adv. Stud. Pure Math. vol. 75, Math. Soc. Japan, Tokyo, 2017.

DEPARTEMENT MATHEMATIK UND INFORMATIK,
UNIVERSITÄT BASEL, SPIEGELGASSE 1, CH-4051 BASEL, SWITZERLAND
Email address: hanspeter.kraft@unibas.ch

INSTITUT FOURIER, UMR 5582, LABORATOIRE DE MATHÉMATIQUES,
UNIVERSITÉ GRENOBLE ALPES, CS 40700, 38058 GRENOBLE CEDEX 9, FRANCE
Email address: mikhail.zaidenberg@univ-grenoble-alpes.fr