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# Invariant foliations for endomorphisms of $\mathbb{P}^2$ with a pluripotentialist product structure

Virgile Tapiero

## Abstract

Let  $f$  be a holomorphic endomorphism of  $\mathbb{P}^2$ , let  $T$  be its Green current and  $\mu = T \wedge T$  be its equilibrium measure. We prove that if  $\mu$  has a local product structure with respect to  $T$  then (an iterate of)  $f$  preserves a local foliation  $\mathcal{F}$  on a neighborhood of  $\text{Supp}(T) \setminus \mathcal{E}$ , where  $\mathcal{E}$  denotes the exceptional set of  $f$ . If the local foliation  $\mathcal{F}$  extends through  $\mathcal{E}$ , then it extends to  $\mathbb{P}^2$  and is an invariant pencil of lines.

**Keywords**— Holomorphic dynamics, Equilibrium measure, Green current, Lyapunov exponents, Holomorphic foliations. *MSC 2020* : 32H50, 32U40, 37D25, 37F75.

## 1 Introduction

This article concerns holomorphic endomorphisms of  $\mathbb{P}^2$  preserving geometric structures. Endomorphisms preserving pencils of curves, webs or foliations were classified by Dabija-Jonsson [9, 10] and Favre-Pereira [16, 17] by using algebraic geometry. Jonsson studied in [20] the dynamics of polynomial skew products on  $\mathbb{C}^2$  by using pluripotential theory. He proved that the equilibrium measure  $\mu$  satisfies a skew product property with respect to the Green current  $T$  and provided formulas for the Lyapunov exponents. In particular, the two exponents of  $\mu$  are larger than the logarithm of the algebraic degree  $d$  of the endomorphism. Recently, Dupont-Taffin [14] extended Jonsson's results to endomorphisms of  $\mathbb{P}^2$  preserving a pencil of lines.

In this article we are interested in the reverse property. Let  $f$  be an endomorphism of  $\mathbb{P}^2$  of degree  $d \geq 2$ , let  $T$  be its Green current and  $\mu := T \wedge T$  be its equilibrium measure. We prove that if  $\mu$  locally satisfies a product structure with respect to  $T$ , then there exists a holomorphic foliation near the support of  $T$  which is  $f$ -invariant. Moreover, if the foliation extends to  $\mathbb{P}^2$ , then this extension is an invariant pencil of lines. The precise statements are given below.

Let us recall the definition of  $T$  and  $\mu$ , we refer to Dinh and Sibony [11, 22] for a general account. We have  $T := \lim_n d^{-n} f^{n*} \omega_{\mathbb{P}^2}$ , where  $\omega_{\mathbb{P}^2}$  is the normalized Fubini-Study form, it is a positive closed  $(1, 1)$ -current on  $\mathbb{P}^2$  satisfying  $f^*T = d \times T$ . The product  $\mu = T \wedge T$  is well defined since  $T$  has local continuous potentials, it is a mixing invariant probability measure on  $\mathbb{P}^2$  satisfying  $f^*\mu = d^2 \times \mu$ . Its Lyapunov exponents satisfy  $\lambda_1 \geq \lambda_2 \geq \frac{1}{2} \text{Log } d$ , as shown by Briend-Duval [6].

For our purpose, let us develop the situation when  $f$  preserves a pencil of lines as above. It has the form  $f[z : w : t] = [P(z, w) : Q(z, w) : R(z, w, t)]$  for a convenient choice of coordinates. Dupont-Taffin [14, Thm. 1.1] proved that

$$\mu = T \wedge \pi^* \mu_\theta \text{ and } \pi_*(\mu) = \mu_\theta, \tag{1}$$

where  $\theta := [P(z, w) : Q(z, w)]$ ,  $\pi : [z : w : t] \mapsto [z : w]$  (undefined at  $[0 : 0 : 1]$ ) and  $\mu_\theta$  is the equilibrium measure of  $\theta$ . If moreover  $\theta$  is a Lattès map of  $\mathbb{P}^1$ , then

$$\mu = T \wedge \pi^* \mu_\theta \ll T \wedge \omega_{\mathbb{P}^2}$$

and the smallest exponent of  $\mu$  is equal to  $\lambda_\theta = \frac{1}{2} \text{Log } d$ , see [14, Cor. 1.3 and 1.4]. Note that, without assuming the existence of an invariant pencil of lines, Dujardin proved that  $\mu \ll T \wedge \omega_{\mathbb{P}^2}$  implies  $\lambda_2 = \frac{1}{2} \text{Log } d$ , see [12, Thm. 3.6] (see also [15] for another argument).

We recall that Lattès maps of  $\mathbb{P}^2$  are characterized by the three equivalent properties :  $\lambda_1 = \lambda_2 = \frac{1}{2} \text{Log } d$ ,  $\mu \ll \omega_{\mathbb{P}^2} \wedge \omega_{\mathbb{P}^2}$ , and  $T$  is equal to a positive smooth  $(1, 1)$ -form on a non empty open set. We refer here to the works by Berteloot, Dupont and Loeb [2, 4, 5]. Similar (adapted) characterizations are also valid on  $\mathbb{P}^1$ .

In the present article, we first begin by specifying the formula  $\mu = T \wedge \pi^* \mu_\theta$  near most of repelling periodic points  $a \in \text{Supp}(\mu)$  when  $\theta$  is Lattès : there exists Poincaré-Dulac coordinates  $(Z_a, W_a)$  such that

$$\mu = T \wedge dd^c |W_a|^2 \text{ on } (\mathbb{P}^2, a). \quad (2)$$

This is done in Section 3, by using results by Berteloot, Dupont, Loeb and Molino.

We are then motivated to study mappings satisfying Formula (2) for at least one repelling periodic point. We introduce the Radon-Nikodym decomposition of the trace measure of  $T$  with respect to  $\mu$  :

$$\sigma_T := T \wedge \omega_{\mathbb{P}^2} = \mu^a + \mu^s,$$

where  $\mu^a \ll \mu$  and  $\mu^s \perp \mu$ .

**Theorem 1.1.** *Let  $f$  be a holomorphic map of  $\mathbb{P}^2$  of degree  $d \geq 2$ ,  $T$  be its Green current,  $\mu = T \wedge T$  be its equilibrium measure and  $\mathcal{E}$  be its exceptional set. Assume that  $\text{Supp}(\mu) \cap \mathcal{E} = \emptyset$ . Assume moreover that*

1. *there exist a  $N$ -periodic repelling point  $a \in \text{Supp}(\mu)$  and Poincaré-Dulac coordinates  $(Z_a, W_a)$  for  $f^N$  at the point  $a$  such that*

$$\mu = T \wedge dd^c |W_a|^2 \text{ on } (\mathbb{P}^2, a)$$

2. *there exists an open set  $\Omega$  charged by  $\mu$  such that, for any open subset  $V \subset \Omega$ ,  $\mu(V) > 0$  implies  $\mu^s(V) > 0$ .*

*Then there exists an open neighborhood  $\mathcal{V}$  of  $\text{Supp}(T) \setminus \mathcal{E}$ , and there exists a holomorphic foliation  $\mathcal{F}$  defined on  $\mathcal{V}$ , such that  $\mathcal{F}$  is invariant by  $f^N$  on  $\mathcal{V}$ .*

Examples of mappings satisfying the hypothesis of this theorem are given by  $f = [P(z, w) : Q(z, w) : t^d]$ , where  $\theta = [P(z, w) : Q(z, w)]$  is Lattès. Indeed, in this case  $\mathcal{E}$  is composed of the center of the pencil and the invariant line at infinity, we refer to Proposition 3.3 and [15, §8] to check the items 1. and 2. More generally, if  $f$  preserves a pencil of lines, then  $\mathcal{E}$  contains its center. For a generic map of  $\mathbb{P}^2$ ,  $\mathcal{E}$  is empty.

The local first assumption  $\mu = T \wedge dd^c |W_a|^2$  implies that  $\mu \ll \sigma_T$  on  $\mathbb{P}^2$ , see Proposition 6.1. In particular  $f$  has a minimal Lyapunov exponent according to Dujardin's theorem [12]. Note that the second assumption implies that  $f$  is not Lattès.

The proof of the Theorem 1.1 requires several steps provided in Sections 4, 5 and 6. In Section 4 we introduce *Poincaré maps*  $\sigma : \mathbb{C}^2 \rightarrow \mathbb{P}^2$  associated to Poincaré-Dulac coordinates  $(Z_a, W_a)$ . Roughly speaking, the leaves of the foliation  $\mathcal{F}$  will be obtained by the images of the horizontal lines of  $\mathbb{C}^2$  by a Poincaré map  $\sigma$ . To implement this idea, we prove in Section 5 a Patching Theorem : it allows, under the second assumption of the Theorem 1.1, to patch holomorphic 1-forms  $dW_1$  and  $dW_2$  satisfying  $T \wedge dd^c|W_i|^2 \ll \mu$ .

In Section 6, we construct the foliation  $\mathcal{F}$  near  $\text{Supp}(T) \setminus \mathcal{E}$ , thus proving Theorem 1.1. First we explain that the local formula  $\mu = T \wedge dd^c|W_a|^2$  near  $a$  can be lifted to  $\mathbb{C}^2$  and gives a global formula  $\sigma^*\mu = \sigma^*T \wedge dd^c|w|^2$  on  $\mathbb{C}^2$ . Then in Section 6.2 we explain how to build the desired foliation  $\mathcal{F}$  using Theorem 6.2, proved in Section 6.3. The Theorem 6.2 asserts the following : we can construct a foliation  $\mathcal{F}$ , such that  $\sigma^*\mathcal{F}$  coincide with the horizontal foliation of  $\mathbb{C}^2$ , on a neighborhood of any compact set  $J \subset \text{Supp}(T) \setminus \mathcal{E}$  such that  $J \supset \text{Supp}(\mu)$  and  $J = \text{Supp}(\sigma_T|_J)$ .

We obtain the following corollary.

**Corollary 1.2.** *Let  $f$  be an endomorphism of  $\mathbb{P}^2$  satisfying the assumptions of Theorem 1.1. Let  $\mathcal{F}$  be a  $f^N$ -invariant holomorphic foliation on a neighborhood  $\mathcal{V}$  of  $\text{Supp}(T) \setminus \mathcal{E}$  given by this theorem.*

*Assume that for each point  $p$  of  $\text{Supp}(T) \cap \mathcal{E}$  there exists an open neighborhood  $U_p$  of  $p$  such that  $\mathcal{F}$  extends on  $\mathcal{V} \cup U_p$ . Then  $\mathcal{F}$  uniquely extends to  $\mathbb{P}^2$  and this extension is a pencil of lines invariant by  $f^N$ .*

Let us outline the proof, we give details in Section 2. Denote  $\mathcal{U} := \cup_{p \in \text{Supp}(T) \cap \mathcal{E}} U_p$ . We use the following extension theorem : any holomorphic foliation defined on an open neighborhood of a connected compact subset whose complementary set in  $\mathbb{P}^2$  is Stein, extends to a holomorphic foliation on  $\mathbb{P}^2$ . This result is due to Lins Neto [21], see also Canales [7]. Since the support of  $T$  is a connected compact subset of  $\mathbb{P}^2$  whose complementary set is Stein (see Forneaess-Sibony [18], Ueda [23]), we can apply Lins Neto's theorem to the compact set  $\text{Supp}(T) \subset \mathcal{V} \cup \mathcal{U}$ . We obtain that  $\mathcal{F}$  extends to  $\mathbb{P}^2$ . We can then check that the extension is also invariant. Finally, we use Favre-Pereira classification [16] of invariant foliations on  $\mathbb{P}^2$ .

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## 2 Holomorphic foliations and invariance

### 2.1 Holomorphic foliations on $\mathbb{P}^2$

Let  $U$  be an open subset of  $\mathbb{P}^2$ . We define a foliation  $\mathcal{F}$  on  $U$  by a collection  $(U_i, \omega_i)_{i \in I}$ , where  $\cup_{i \in I} U_i = U$  is an open cover, and where the  $\omega_i \in \Omega^1(U_i) \setminus \{0\}$  are

holomorphic 1-forms, such that the following compatibility condition is satisfied :

$$\omega_i \wedge \omega_j \equiv 0 \text{ on } U_i \cap U_j. \quad (3)$$

Two collections  $(U_i, \alpha_i)_i$  and  $(V_j, \beta_j)_j$  define the same foliation on  $U$  if  $\alpha_i \wedge \beta_j \equiv 0$  on  $U_i \cap V_j$  whenever  $U_i \cap V_j \neq \emptyset$ . It is an equivalence relation.

Let us assume that each  $U_i$  is equipped with holomorphic coordinates  $(Z_i, W_i)$  and that  $\omega_i = f_i dZ_i + g_i dW_i$ , where  $f_i, g_i \in \mathcal{O}(U)$ . Let us define the set  $U^0 := U \setminus (\cup_{i \in I} \{f_i = 0\} \cap \{g_i = 0\})$ . Since we work in complex dimension 2, the condition (3) ensures that the distribution of complex lines  $p \in U^0 \mapsto \text{Ker}(\omega_i)_p$  is integrable in the sense of Frobenius, and thus there exists a sub-bundle  $T\mathcal{F} \subset TU^0$  such that  $T_p\mathcal{F} = \text{Ker}(\omega_i)_p$ , for any  $p \in U^0$  and for any  $i \in I$  such that  $p \in U_i$ . The vector field

$$v_i := g_i \frac{\partial}{\partial Z_i} - f_i \frac{\partial}{\partial W_i}$$

satisfies  $\omega_i(v_i) = 0$  on  $U_i$  and so  $T_p\mathcal{F} = \text{Ker}(\omega_i)_p = \mathbb{C} \cdot v_i(p)$  for any  $p \in U_i \cap U^0$ .

For the next Lemma, we introduce holomorphic 1-forms  $\eta_i = (h_i dZ_i + k_i dW_i)$  defining a foliation  $\mathcal{G}$  on  $U$ . Let  $(w_i)_i$  be the corresponding vector fields and  $U^1 := U \setminus (\cup_{i \in I} \{h_i = 0\} \cap \{k_i = 0\})$ .

**Lemma 2.1.** *For every  $i \in I$ , we have*

1.  $\mathcal{F}|_{U_i} = \mathcal{G}|_{U_i} \iff (\omega_i \wedge \eta_i = 0 \text{ on } U_i) \iff (\omega_i(w_i) = 0 \text{ on } U_i)$ .
2.  $\mathcal{F}|_{U_i} = \mathcal{G}|_{U_i} \iff (T_p\mathcal{F} = T_p\mathcal{G}, \forall p \in U_i \cap U^0 \cap U^1)$ .
3. *Assume  $U = \cup_{i \in I} U_i$ ,  $\forall i \in I$ , and that  $U$  is connected. If there exists a non empty open subset  $V \subset U$  such that  $\mathcal{F}|_V = \mathcal{G}|_V$ , then by analytic continuation  $\mathcal{F} = \mathcal{G}$ .*

**Proof :**

1. The first equivalence is our definition of foliations. For the second equivalence, just observe that  $\omega_i \wedge \eta_i = \begin{vmatrix} f_i & h_i \\ g_i & k_i \end{vmatrix} dZ_i \wedge dW_i$  and that  $\omega_i(w_i) = f_i k_i - g_i h_i = \begin{vmatrix} f_i & h_i \\ g_i & k_i \end{vmatrix}$ .

2. We have to prove the reverse implication. Let us assume that  $T_p\mathcal{F} = T_p\mathcal{G}$  for any  $p \in U_i \cap U^0 \cap U^1$ . Since  $T_p\mathcal{F} = \text{Ker}(\omega_i)_p$  and  $T_p\mathcal{G} = \mathbb{C} \cdot w_i(p)$ , we get  $\omega_i(w_i)(p) = 0$  for any  $p \in U_i \cap U^0 \cap U^1$ . Since  $U_i$  is connected,  $\omega_i(w_i)(p) = 0$  for every  $p \in U_i$  by analytic continuation. Hence  $\mathcal{F}|_{U_i} = \mathcal{G}|_{U_i}$  according to the first item.

3. If  $\mathcal{F}|_V = \mathcal{G}|_V$  on  $V \subset U$ , then  $\omega_i \wedge \eta_i = 0$  on  $V$ . Since  $\omega_i \wedge \eta_i = \begin{vmatrix} f_i & h_i \\ g_i & k_i \end{vmatrix} dZ_i \wedge dW_i$ , we deduce that  $\begin{vmatrix} f_i & h_i \\ g_i & k_i \end{vmatrix} = 0$  on  $V$  and thus on  $U$  by analytic continuation. It means that  $\omega_i \wedge \eta_i = 0$  on  $U_i$  for every  $i \in I$ , and thus  $\mathcal{F} = \mathcal{G}$  on  $U$ .  $\square$

## 2.2 Pull-back of foliations and invariance

Let  $U$  and  $V$  be two open subsets of  $\mathbb{P}^2$  and let  $f : V \rightarrow U$  be a holomorphic surjective map which is not constant on each connected component of  $V$ . Let  $\mathcal{F}$  be a foliation on  $U$  defined by a collection  $(U_i, \omega_i)_{i \in I}$ . For each  $i \in I$ , the 1-form  $f^*\omega_i$  is not null on each connected component of  $f^{-1}(U_i)$  and  $f^*\omega_i \wedge f^*\omega_j = f^*(\omega_i \wedge \omega_j) = 0$  on  $f^{-1}(U_i) \cap f^{-1}(U_j)$ . Then the collection  $(U_i, f^*\omega_i)_{i \in I}$  defines a foliation  $f^*\mathcal{F}$  on  $V$ , called the pull-back of  $\mathcal{F}$  by  $f$ .

**Definition 2.2.** Assume  $V = U$  and that  $f^{-1}(V) \cap V \neq \emptyset$ . We say that  $\mathcal{F}$  is invariant by  $f$  on  $V$  if  $f^*\mathcal{F} = \mathcal{F}$  on  $f^{-1}(V) \cap V$ .

For  $V = U = \mathbb{P}^2$  the pairs  $(\mathcal{F}, f)$  where  $f^*\mathcal{F} = \mathcal{F}$ , have been classified by Favre-Pereira. We shall need the following version of their classification.

**Theorem 2.3** (Favre-Pereira [16]). Let  $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  be a holomorphic map of degree  $d \geq 2$ . Let  $\mathcal{F}$  be a foliation on  $\mathbb{P}^2$  invariant by  $f$ . Then in appropriate homogeneous coordinates  $[x : y : z]$  on  $\mathbb{P}^2$ , one of the following cases holds :

1.  $\mathcal{F}$  is the pencil on lines  $\pi[z : w : t] = [z : w]$ . In this case  $f = [P(x, y) : Q(x, y) : R(x, y, z)]$  with  $P, Q, R$  homogeneous polynomials of degree  $d$ .
2.  $\mathcal{F}$  is not a pencil of lines through a point in  $\mathbb{P}^2$  and  $f$  has the form :
  - i.  $f = [x^d : y^d : z^d]$  or  $f = [z^d : x^d : y^d]$  or
  - ii.  $f = [x^d : y^d : R(x, y, z)]$  or  $f = [y^d : x^d : S(x, y, z)]$ , where  $R$  and  $S$  depends on the three variables  $(x, y, z)$ .

We refer to [16] for a description of the foliations  $\mathcal{F}$ , and for the polynomials  $R, S$ .

In particular, Lattès maps on  $\mathbb{P}^2$  do not preserve any foliation, however they can preserve webs [17]. We shall use Theorem 2.3 in the following form.

**Corollary 2.4.** If  $f$  preserves a foliation  $\mathcal{F}$  on  $\mathbb{P}^2$  and if the smallest Lyapunov exponent of  $\mu$  satisfies  $\lambda_2 = \frac{1}{2} \text{Log } d$ , then in appropriate homogeneous coordinates  $\mathcal{F}$  is a pencil of lines given by the fibers of  $\pi[z : w : t] = [z : w]$ .

We then can prove the following result.

**Proposition 2.5.** Let  $f$  be a holomorphic map of  $\mathbb{P}^2$  of degree  $d \geq 2$  and let  $\mathcal{F}$  be a  $f$ -invariant foliation defined on a neighborhood of a fixed point of  $f$ . If  $\mathcal{F}$  admits an extension  $\mathcal{F}'$  on  $\mathbb{P}^2$ , then  $\mathcal{F}'$  is invariant by  $f$ . Moreover, if  $\lambda_2 = \frac{1}{2} \text{Log } d$  then, in appropriate coordinates,  $\mathcal{F}'$  is the pencil of lines given by  $\pi[z : w : t] = [z : w]$ . In this case the rational map induced by  $f$  on  $\mathbb{P}^1$  is Lattès.

**Proof :** The second part of the statement is a direct consequence of Corollary 2.4. Let  $a \in \mathbb{P}^2$  be the fixed point given by the statement.  $\mathcal{F}'$  coincide with  $\mathcal{F}$  on a neighborhood of  $a$ , and  $f^*\mathcal{F} = \mathcal{F}$  near  $a$ . So  $f^*\mathcal{F}' = f^*\mathcal{F} = \mathcal{F}$  near  $a$ . Using again that  $\mathcal{F}'$  is an extension of  $\mathcal{F}$ , we deduce that  $f^*\mathcal{F}' = \mathcal{F}'$  near  $a$  and thus on  $\mathbb{P}^2$  by analytic continuation (Lemma 2.1).  $\square$

### 3 Poincaré-Dulac coordinates

Let us consider  $f : (\mathbb{P}^2, a) \rightarrow (\mathbb{P}^2, a)$  a holomorphic germ such that the eigenvalues  $(\chi_1, \chi_2)$  of  $d_a f$  satisfy  $|\chi_1| \geq |\chi_2| > 1$ . By Poincaré-Dulac theorem, there exists a germ of biholomorphism  $\sigma_0 : (\mathbb{C}^2, 0) \rightarrow (\mathbb{P}^2, a)$  and a polynomial mapping  $D(z, w) = (\chi_1 z + cw^q, \chi_2 w)$ , with  $q \geq 2$ , such that the following commutative relation holds :

$$f \circ \sigma_0 = \sigma_0 \circ D \text{ on } (\mathbb{C}^2, 0).$$

The map  $\sigma_0^{-1}$  induces holomorphic coordinates  $\sigma_0^{-1} = (Z_a, W_a)$ , we call them *Poincaré-Dulac coordinates* for the germ  $f$  of  $(\mathbb{P}^2, a)$ .

### 3.1 Lattès mappings on $\mathbb{P}^1$ and multipliers on $\mathbb{P}^2$

We begin by recalling some results due to Berteloot-Loeb. For a rational map  $\theta$  of degree  $d \geq 2$  on  $\mathbb{P}^1$ , we use the following definitions :

- A *regular point*  $p \in \mathbb{P}^1$  is a point such that  $\mu_\theta$  is a strictly positive smooth  $(1, 1)$ -form on an open neighborhood of  $p$  in  $\mathbb{P}^1$ . Let  $\Omega_\theta$  be the set of regular points and let  $A_\theta := \mathbb{P}^1 \setminus \Omega_\theta$ .
- $R_1(\theta)$  is the set of repelling fixed points of  $\theta$ , it is contained in  $\text{Supp}(\mu_\theta)$ .

We refer to [4, Theorem 1.1 & Proposition 2.5] and [5, Proposition 4.1].

**Theorem 3.1** (Berteloot-Loeb).

1. If  $\theta$  is a Lattès map then  $A_\theta$  is a finite set, otherwise  $A_\theta = \mathbb{P}^1$ .
2. Let  $a \in R_1(\theta)$ ,  $D(w) := \theta'(a)w$  and suppose that  $a \in \Omega_\theta$ . There exists an invertible holomorphic germ  $\sigma_0 : (\mathbb{C}, 0) \rightarrow (\mathbb{P}^1, a)$  such that the following relation holds :

$$\sigma_0^{-1} =: W_0 \text{ and } W_0 \circ \theta = D \circ W_0 \text{ on } (\mathbb{C}, 0) \quad (4)$$

and such that  $\mu_\theta = dd^c |W_0|^2$  (which implies  $|\theta'(a)| = \sqrt{d}$ ). We call the coordinate  $W_0$  a Poincaré-Dulac coordinate for  $\theta$  at the point  $a$ .

Now we deal with the multipliers of repelling cycles for mappings  $f$  of  $\mathbb{P}^2$  of degree  $d$ , of equilibrium measure  $\mu$  and of Lyapunov exponents  $\lambda_1 \geq \lambda_2$ . Let  $\varepsilon > 0$  and  $R_\mu^{n,\varepsilon}$  be the set of  $n$ -periodic repelling points  $a \in \text{Supp}(\mu)$  satisfying

$$\frac{1}{n} \text{Log} |\det_{\mathbb{C}} d_a f^n| \geq \lambda_1 + \lambda_2 - 2\varepsilon.$$

**Proposition 3.2** (Berteloot-Dupont-Molino [3, Lemma 4.5]).

1. Let  $V \subset \mathbb{P}^2$  be an open set satisfying  $\mu(V) > 0$ . There exists  $n_\varepsilon \geq 1$  such that for any  $n \geq n_\varepsilon$  :

$$\text{Card}(R_\mu^{n,\varepsilon} \cap V) \geq d^{2n} (1 - \varepsilon)^3 \mu(V). \quad (5)$$

In particular  $R_\mu^\varepsilon := \bigcup_{n \geq 1} R_\mu^{n,\varepsilon}$  is dense in  $\text{Supp}(\mu)$ .

2. Let  $R_\mu^n$  denote the set of  $n$ -periodic repelling points satisfying

$$\frac{1}{n} \text{Log} |\det_{\mathbb{C}} d_a f^n| > \text{Log } d.$$

If  $\lambda_1 > \lambda_2$  then  $R_\mu := \bigcup_{n \geq 1} R_\mu^n$  is dense in  $\text{Supp}(\mu)$ .

We note that (5) is actually proved in [3, Lemma 4.5] for  $V = \mathbb{P}^2$ , but the same proof works for a general open set  $V$  charged by  $\mu$ . The second item is a direct consequence of the first item.



### 3.2 Endomorphisms of $\mathbb{P}^2$ preserving a pencil of lines

We recall some facts on mappings preserving a pencil of lines through a point in  $\mathbb{P}^2$ . Let  $f$  be a holomorphic map of  $\mathbb{P}^2$  of degree  $d \geq 2$ , and let  $T$  and  $\mu$  being the Green current of  $f$  and  $\mu$  its equilibrium measure. We denote  $\lambda_1 \geq \lambda_2$  the Lyapunov exponents of  $\mu$ . Let us assume that  $f$  preserves the pencil of lines given  $\pi : [z : w : t] \mapsto [z : w]$ . We denote  $\theta$  the rational map such that  $\pi \circ f = \theta \circ \pi$  on  $\mathbb{P}^2 \setminus \{[0 : 0 : 1]\}$ .

Recall that  $A_\theta$  is the complementary set of regular points  $p$  ( $\mu_\theta$  is smooth on a neighborhood of  $p$ ) in  $\mathbb{P}^1$ . We denote  $E_\theta := \pi^{-1}(A_\theta)$ , and we define  $R_{\mu,\theta}$  the set of periodic and repelling points  $a \in \text{Supp}(\mu) \setminus E_\theta$  such that (with  $n$  the period of  $a$ ) :

$$\frac{1}{n} \text{Log} |\det_{\mathbb{C}} d_a f^n| > \text{Log } d.$$

**Proposition 3.3.** *Assume that  $\theta$  is Lattès. Then  $E_\theta \cup \{[0 : 0 : 1]\}$  is a finite union of projectives lines which do not contain  $\text{Supp}(\mu)$ , and  $R_{\mu,\theta}$  is dense in  $\text{Supp}(\mu) \setminus E_\theta$ . For any  $a \in R_{\mu,\theta}$  of period  $n \geq 1$ , there exists  $(Z_a, W_a)$  Poincaré-Dulac coordinates for  $f^n$  near  $a$  such that :*

$$\mu = T \wedge dd^c |W_a|^2 \text{ on } (\mathbb{P}^2, a).$$

Let us give the arguments for the proof. Note that the relation  $\pi_* \mu = \mu_\theta$  (provided by Dupont-Taflin (1)) implies :

$$\pi(\text{Supp}(\mu)) = \text{Supp}(\mu_\theta). \quad (6)$$

According to Theorem 3.1, the assumption  $\theta$  Lattès implies that  $E_\theta \cup \{[0 : 0 : 1]\}$  is a union of a finite number of projective lines passing through the point  $[0 : 0 : 1]$ . We have  $[0 : 0 : 1] \notin \text{Supp}(\mu)$  (this point belongs to the Fatou set as a super attractive fixed point) and we can not have  $\text{Supp}(\mu) \subset E_\theta$  since  $\pi(\text{Supp}(\mu)) = \text{Supp}(\mu_\theta) = \mathbb{P}^1$  (by (6)) and since  $\pi(E_\theta) = A_\theta$  is finite. We deduce that  $\text{Supp}(\mu) \setminus E_\theta$  is a non empty open subset of  $\text{Supp}(\mu)$ , and therefore Proposition 3.2 implies the first assertion of Proposition 3.3. It remains to prove the second one.

Let us fix  $a \in \text{Supp}(\mu)$  a repelling  $N$ -periodic point satisfying  $\frac{1}{N} \text{Log} |\chi_1 \chi_2| > \text{Log } d$ , where  $\chi_1, \chi_2$  are the eigenvalues of  $d_a f^N$  (with  $|\chi_1| \geq |\chi_2|$ ). Let  $a_0 := \pi(a)$ , since  $a$  is a  $N$ -periodic point of  $f$ , the relation  $\pi \circ f = \theta \circ \pi$  ensures that  $a_0$  is a  $N$ -periodic point of  $\theta$ . This relation also ensures that  $(\theta^N)'(a_0)$  is an eigenvalue of  $d_a f^N$ . Finally,  $a_0$  is a repelling fixed point of  $\theta^N$ .

We recall that  $\Omega_\theta = \mathbb{P}^1 \setminus A_\theta$ . Now if we assume that  $a \in R_{\mu,\theta}$  i.e.  $a_0 \in \Omega_\theta$ , then to prove the second assertion in Proposition 3.3, it remains to prove the following :

**Lemma 3.4.** *With the preceding notations, if  $a_0 \in \Omega_\theta$  then*

1.  $|\chi_1| > |\chi_2| = \sqrt{d^N}$ ,
2. *There exists Poincaré-Dulac coordinates  $(Z_a, W_a)$  for  $f^N$  such that :*

$$\mu = T \wedge dd^c |W_a|^2 \text{ and } W_a \circ f^N = \chi_2 W_a \text{ on } (\mathbb{P}^2, a).$$

**Proof :** Since  $f$  and  $f^N$  have same equilibrium measure  $\mu$  and Green current  $T$  (similarly for  $\theta$  and  $\theta^N$ ), we can assume that  $N = 1$ . By assumption the fixed point



$a_0$  belongs to  $\Omega_\theta$ , hence by Theorem 3.1 there exists an open neighborhood  $U_0$  of  $a_0$  and there exists a Poincaré-Dulac coordinate  $W_0$  on  $U_0$  such that  $\mu_\theta = dd^c|W_0|^2$  on  $U_0$ . We denote  $W_a := W_0 \circ \pi$  on  $\pi^{-1}(U_0) \ni a$ . Then injecting the normal form  $\mu_\theta = dd^c|W_0|^2$  into Dupont-Taflin formula (1), one gets  $\mu = T \wedge dd^c|W_a|^2$  on  $\pi^{-1}(U_0)$ .

At this stage, by the commutative relation (4), we have :

$$W_a \circ f = W_0 \circ \pi \circ f = W_0 \circ \theta \circ \pi = \lambda \times W_0 \circ \pi = \lambda W_a, \quad (7)$$

where  $\lambda = \theta'(a_0)$ . According to Theorem 3.1, it satisfies  $|\lambda| = \sqrt{d}$ .

Let  $\tilde{Z}_a$  be a submersion on an open neighborhood  $U_a$  of  $a$  such that  $U_a \subset \pi^{-1}(U_0)$  and such that  $\tilde{\xi}_a := (\tilde{Z}_a, W_a)$  define holomorphic coordinates centered at  $a$ . If  $\tilde{D}_f := \tilde{\xi}_a \circ f \circ (\tilde{\xi}_a)^{-1}$  and  $p_w : (z, w) \mapsto w$ , then

$$p_w \circ \tilde{D}_f = p_w \circ \tilde{\xi}_a \circ f \circ (\tilde{\xi}_a)^{-1} = W_a \circ f \circ (\tilde{\xi}_a)^{-1} = \lambda W_a \circ (\tilde{\xi}_a)^{-1} = \lambda p_w,$$

where the third equality is given by (7). This shows that  $\tilde{D}_f$  has the form

$$\tilde{D}_f(z, w) = (h(z, w), \lambda w), \quad (8)$$

where  $h \in \mathcal{O}(\mathbb{C}^2, 0)$ . By construction  $d_0 \tilde{D}_f$  has the same eigenvalues  $\chi_1$  and  $\chi_2$  than  $d_a f$  which by assumption satisfy  $|\chi_1 \chi_2| > d$ . From (8) we deduce that  $\lambda$  is also an eigenvalue of  $d_0 \tilde{D}_f : \lambda \in \{\chi_1, \chi_2\}$ . But  $|\lambda| = \sqrt{d}$ , hence  $\chi_2 = \lambda$  and  $|\chi_1| > |\chi_2|$ . By Lemma 3.5 (proved below) there exist a chart  $\xi_a$  such that  $D_f \circ \xi_a = \xi_a \circ f$  on  $(\mathbb{C}^2, 0)$ , where  $D_f(z, w) = (\chi_1 z + cw^q, \lambda w)$  and  $p_w \circ \xi_a = W_a$ . Finally  $\xi_a = (Z_a, W_a)$  are Poincaré-Dulac coordinates for  $f$  at  $a$  and  $\mu = T \wedge dd^c|W_a|^2$  near  $a$ .  $\square$

**Lemma 3.5.** *Let  $g$  be a germ fixing 0 in  $\mathbb{C}^2$  such that there exists  $\xi = (Z, W)$  an invertible germ fixing 0 such that  $D := \xi \circ g \circ \xi^{-1}$  has the form  $D(z, w) = (h(z, w), \chi_2 w)$ . We denote  $\chi_1 := \partial_z h(0, 0)$  and we assume*

$$|\chi_1| > |\chi_2| > 1. \quad (9)$$

*Then there exists  $\tilde{\xi} = (\tilde{Z}, W)$  an invertible germ fixing 0 and there exists a germ  $\tilde{D}(z, w) = (\chi_1 z + \tilde{c}w^q, \chi_2 w)$  such that  $\tilde{\xi} \circ g \circ \tilde{\xi}^{-1} = \tilde{D}$ .*

**Proof :** Since  $|\chi_1| > |\chi_2| > 1$ , Poincaré-Dulac theorem yields the existence of  $\xi'$  an invertible germ fixing 0 such that :

$$D' \circ \xi' = \xi' \circ D \text{ on } (\mathbb{C}^2, 0) \quad (10)$$

where

$$D'(z, w) = (\chi_1 z + cw^q, \chi_2 w) \text{ and } c = 0 \text{ if } \chi_2^q \neq \chi_1.$$

Let  $W'$  be the second component of  $\xi'$ , our goal is to show that  $W'$  is given by  $W' : (z, w) \mapsto \gamma \times w$  on  $(\mathbb{C}^2, 0)$  for some constant  $\gamma \in \mathbb{C}^*$ . Taking  $\tilde{\xi} := \xi'' \circ \xi' \circ \xi$  with  $\xi''(z, w) := (z, \frac{1}{\gamma}w)$  and  $\tilde{D} := \xi'' \circ D' \circ (\xi'')^{-1}$ , the desired conclusion follows.

As a holomorphic function,  $W'$  is a power series near 0 of the form :

$$W'(z, w) = \sum_{r \geq 1} (\gamma_{r,0} z^r + \gamma_{r-1,1} z^{r-1} w + \cdots + \gamma_{0,r} w^r) = \sum_{r \geq 1} \mathcal{O}'(r).$$

The function  $h$  also expands into a power series near 0 of the form :

$$h(z, w) = \sum_{r \geq 1} (H_{r,0} z^r + H_{r-1,1} z^{r-1} w + \cdots + H_{0,r} w^r) = [H_{1,0} z + H_{0,1} w] + \mathcal{O}_{\geq 2}.$$

Of course  $H_{1,0}$  is equal to  $\chi_1$  (we recall that  $\chi_1, \chi_2$  are the eigenvalues of  $d_0 D$ ). The equality (10) provides

$$\chi_2 W'(z, w) = W'(h(z, w), \chi_2 w) \text{ on } (\mathbb{C}^2, 0). \quad (11)$$

Observe that (11) implies (looking at terms of order 1 in the power series) :

$$\chi_2 \gamma_{1,0} z + \chi_2 \gamma_{0,1} w = (\gamma_{1,0} H_{1,0} z + \gamma_{1,0} H_{0,1} w) + \gamma_{0,1} \chi_2 w.$$

Since  $H_{1,0} = \chi_1$  we deduce, looking at coefficients of the variable  $z$ , that  $\chi_2 \gamma_{1,0} = \gamma_{1,0} \chi_1$ . Since  $\chi_1 \neq \chi_2$ , we get

$$\gamma_{1,0} = 0 \text{ i.e. } \mathcal{O}'(1) = \gamma_{0,1} w. \quad (12)$$

We can now prove by induction on  $r \geq 2$  that :

$$\forall k \in \{2, \dots, r\}, \mathcal{O}'(k) = 0 \text{ i.e. } W'(z, w) = \gamma_{0,1} w + \sum_{k \geq r+1} \mathcal{O}'(k) \text{ on } (\mathbb{C}^2, 0).$$

Let us explain the base case  $r = 2$ , the induction step  $r > 2$  can be treated analogously. Using (11) and (12), we get :

$$\chi_2 \gamma_{2,0} z^2 + \chi_2 \gamma_{1,1} z w + \chi_2 \gamma_{0,2} w^2 = \gamma_{2,0} (H_{1,0} z + H_{0,1} w)^2 + \gamma_{1,1} (H_{1,0} z + H_{0,1} w) (\chi_2 w) + \gamma_{0,2} (\chi_2 w)^2.$$

Since  $\chi_1 = H_{1,0}$  we deduce, looking at the coefficients of  $z^2$ , that  $\chi_2 \gamma_{2,0} = \gamma_{2,0} \chi_1^2$ , it yields  $\gamma_{2,0} = 0$  by (9). Using again  $\chi_1 = H_{1,0}$  and looking at the coefficient of  $z w$  we obtain  $\chi_2 \gamma_{1,1} = \gamma_{1,1} \chi_1 \chi_2$ , thus  $\gamma_{1,1} = 0$  by (9). At last looking at the coefficients of  $w^2$  we obtain  $\chi_2 \gamma_{0,2} = \gamma_{0,2} \chi_2^2$ , and thus  $\gamma_{0,2} = 0$ . We have proved  $\mathcal{O}'(2) = 0$ .  $\square$

## 4 Poincaré maps for repelling fixed points

Let us fix a dilating germ  $f : (\mathbb{P}^2, a) \rightarrow (\mathbb{P}^2, a)$  and let us consider a germ  $D(z, w) = (\chi_1 z + c w^q, \chi_2 w)$ , with  $c \in \mathbb{C}$ ,  $q \geq 2$ , and Poincaré-Dulac coordinates  $\sigma_0^{-1} = (Z_a, W_a)$  such that  $f \circ \sigma_0 = \sigma_0 \circ D$  near 0 on  $\mathbb{C}^2$ . We assume that  $f$  is induced by a holomorphic map of  $\mathbb{P}^2$  of degree  $d \geq 2$ .

From the local commutative relation  $f \circ \sigma_0 = \sigma_0 \circ D$  near 0 in  $\mathbb{C}^2$ , we shall construct a global commutative relation semi-conjugating  $f$  to  $D$  via a globally defined holomorphic map  $\sigma : \mathbb{C}^2 \rightarrow \mathbb{P}^2$ . The construction of  $\sigma$  is classical, see for instance Berteloot-Loeb [5] when  $D$  is linear.

Let  $U_a$  be a connected open neighborhood of  $a$  on which  $\sigma_0^{-1}$  is defined. We denote  $U_0 := \sigma_0^{-1}(U_a)$ . Let  $\mathbb{D}_\varepsilon^2 \subset U_0 \cap D^{-1}(U_0)$  be a small bidisc such that  $D^{-1}(\mathbb{D}_\varepsilon^2) \subset \mathbb{D}_\varepsilon^2$ . In particular the following diagram commutes

$$\begin{array}{ccc} \mathbb{D}_\varepsilon^2 & \xrightarrow{D} & D(\mathbb{D}_\varepsilon^2) \\ \sigma_0 \downarrow & & \downarrow \sigma_0 \\ \sigma_0(\mathbb{D}_\varepsilon^2) & \xrightarrow{f} & f(\sigma_0(\mathbb{D}_\varepsilon^2)) \end{array} \quad (13)$$

**Proposition 4.1** (Poincaré maps).

1. *There exists an increasing sequence of integers  $(n_k)_k$  such that  $D^{-n_k}(\mathbb{D}_k^2) \subset \mathbb{D}_\varepsilon^2$ . In particular the restriction  $\sigma_k$  of  $f^{n_k} \circ \sigma_0 \circ D^{-n_k}$  to  $\mathbb{D}_k^2$  is well defined. It satisfies  $f \circ \sigma_k = \sigma_k \circ D$  on  $\mathbb{D}_k^2 \cap D^{-1}(\mathbb{D}_k^2)$ .*
2.  *$\sigma_k$  is an open mapping and  $\#\sigma_k^{-1}(p) \leq d^{2n_k}$  for every  $p \in \sigma_k(\mathbb{D}_k^2)$ .*
3. *The critical values  $E_k := \sigma_k(\text{Crit } \sigma_k)$  are included in  $f^{n_k}(\text{Crit } f^{n_k})$ , which is an algebraic subset of codimension 1 of  $\mathbb{P}^2$ .*
4. *For every positive integers  $k \leq l$ ,  $\sigma_k = \sigma_l$  on  $\mathbb{D}_k^2$ . Hence one can define the holomorphic map  $\sigma : \mathbb{C}^2 \rightarrow \mathbb{P}^2$  by  $\sigma := \lim_k \sigma_k$ .*
5. *The relation  $f \circ \sigma = \sigma \circ D$  holds on  $\mathbb{C}^2$ .*
6. *The map  $\sigma$  is open and have discrete fibers on  $\mathbb{C}^2$ .*
7. *If  $E := \sigma(\text{Crit } \sigma)$  then  $E = \bigcup_{k \geq 0} E_k$  and  $\mu(E) = \sigma_T(E) = 0$ .*

**Proof :** We explain the first and the fourth items, the others being a direct consequence of the previous ones, or could be deduced from the fact that  $f$  is a degree  $d^2$  ramifying covering map of  $\mathbb{P}^2$ , and from the fact that  $\mu$  and  $\sigma_T$  do not charge proper analytic subsets of  $\mathbb{P}^2$  (cf. [11] for details).

1. We have  $D^n = (\chi_1^n z + nc\chi_1^{n-1}w^q, \chi_2^n w)$  and  $D^{-n} = (\chi_1^{-n}z - nc\chi_1^{-(n-1)}w^q, \chi_2^{-n}w)$ . Hence, for every  $k \geq 1$ , there exists  $n_k$  such that  $D^{-n_k}(\mathbb{D}_k^2) \subset \mathbb{D}_\varepsilon^2$ . Let  $\sigma_k$  be the restriction of  $f^{n_k} \circ \sigma_0 \circ D^{-n_k}$  to  $\mathbb{D}_k^2$ . One can assume that  $n_k$  is increasing. We have  $f \circ \sigma_k = \sigma_k \circ D$  on  $\mathbb{D}_k^2 \cap D^{-1}(\mathbb{D}_k^2)$  thanks to the commutative diagram (13).

4. Let  $y \in \mathbb{C}^2$  and assume that  $D^m(y) \in \mathbb{D}_\varepsilon^2$  for some  $m \geq 1$ . From the inclusion  $D^{-1}(\mathbb{D}_\varepsilon^2) \subset \mathbb{D}_\varepsilon^2$ , we get  $\{y, D(y), \dots, D^m(y)\} \subset \mathbb{D}_\varepsilon^2$ . Now if  $p \in D^n(\mathbb{D}_\varepsilon^2) \cap D^{n+m}(\mathbb{D}_\varepsilon^2)$  with  $p = D^n(x) = D^{n+m}(y)$ , then  $D^m(y) = x \in \mathbb{D}_\varepsilon^2$  and thus, as explained above,  $D^j(y) \in \mathbb{D}_\varepsilon^2$  for any  $j \in \{0, \dots, m\}$ . By the commutative diagram (13) we get

$$f^m \circ \sigma_0(y) = f^{m-1} \circ \sigma_0 \circ D(y) = \dots = \sigma_0(D^m(y)) = \sigma_0(x).$$

Composing by  $f^n$  we obtain  $f^{n+m} \circ \sigma_0 \circ D^{-(n+m)}(D^{n+m}(y)) = f^n \circ \sigma_0 \circ D^{-n}(D^n(x))$ . If  $k \leq l$  then taking  $n = n_k$  and  $m = n_l - n_k \geq 0$ , one has  $\sigma_l = \sigma_k$  on  $D^{n_k}(\mathbb{D}_\varepsilon^2) \cap D^{n_l}(\mathbb{D}_\varepsilon^2) \supset \mathbb{D}_k^2$ . It allows to define  $\sigma : \mathbb{C}^2 \rightarrow \mathbb{P}^2$  by  $\sigma|_{\mathbb{D}_k^2} := \sigma_k$  for any  $k \geq 0$ . It also satisfies  $\sigma = \lim_k \sigma_k$ .  $\square$

It is important to know whether  $\sigma(\mathbb{C}^2)$  contains  $\text{Supp}(\mu)$  or not. Let us introduce the exceptional set  $\mathcal{E}$  of  $f$  :

$$\mathcal{E} := \mathbb{P}^2 \setminus \left\{ x \in \mathbb{P}^2 \left| \mu_{x,n} := \frac{1}{d^{2n}} \sum_{y \in f^{-n}(x)} \delta_y \xrightarrow[n \rightarrow +\infty]{\text{weak}^*} \mu \right. \right\}. \quad (14)$$

The set  $\mathcal{E}$  is the largest totally  $f$ -invariant proper algebraic subset of  $\mathbb{P}^2$  [11].

**Proposition 4.2.** *If the fixed point  $a$  belongs to the support of  $\mu$ , then*

$$\text{Supp}(\mu) \setminus \mathcal{E} \subset \mathbb{P}^2 \setminus \mathcal{E} \subset \sigma(\mathbb{C}^2).$$

**Proof :** Let  $q \in \mathbb{P}^2 \setminus \mathcal{E}$  and let  $\rho \in C^0(\mathbb{P}^2, \mathbb{R}^+)$  be a continuous function such that  $\rho \equiv 1$  on a ball  $B_a$  centered at  $a$  and such that  $\text{Supp}(\rho) \subset 2B_a$ , we can assume that  $\sigma_0^{-1}$  is defined on  $2B_a$ . By definition of  $\mathcal{E}$  one has  $\lim_n \langle \mu_{q,n}, \rho \rangle = \langle \mu, \rho \rangle \geq \mu(B_a) > 0$  because  $a$  belongs to the support of  $\mu$ . Thus there exists  $n_k \geq 1$  such that  $\langle \mu_{q,n_k}, \rho \rangle > 0$  and then, by definition of  $\mu_{q,n_k}$ , there exists a point  $q_{n_k} \in f^{-n_k}(q)$  such that  $\rho(q_{n_k}) > 0$ . Since the support of  $\rho$  is included in  $2B_a$  we have  $q_{n_k} \in 2B_a \subset \text{Dom}(\sigma_0^{-1})$ . This allows us to consider  $p_{n_k} := D^{n_k} \circ \sigma_0^{-1}(q_{n_k})$ , it satisfies  $\sigma(p_{n_k}) = f^{n_k}(q_{n_k}) = q$ .  $\square$

We conclude this section with the following lemma. This type of statement is classical and arises from Briend-Duval [6] work. We refer also to [13, Lemme 1.1.32] where the arguments can be borrowed to prove the lemma.

**Lemma 4.3.** *Let  $U$  be a bounded open subset of  $\mathbb{C}^2 \setminus (\text{Crit } \sigma)$ . There exists  $r > 0$  satisfying the following property. Let  $x \in U$  and let  $p := \sigma(x)$ , then  $B(p, r) \subset \sigma(U)$  and the inverse branche  $\sigma_{x,p}^{-1}$  of  $\sigma : U \rightarrow \sigma(U)$  sending  $p$  to  $x$  is defined on  $B(p, r)$ .*

**Corollary 4.4.** *Let  $U$  as in Lemma 4.3. For every Lipschitz path  $\gamma : [0, 1] \rightarrow \sigma(U)$  and for every  $x_0 \in U$  such that  $\sigma(x_0) = \gamma(0)$ , then there exists a (unique) continuous path  $\tilde{\gamma} : [0, 1] \rightarrow U$  such that  $\sigma \circ \tilde{\gamma} = \gamma$ .*

**Proof :** Lemma 4.3 ensures that there exists  $r > 0$  such that any inverse branches  $\sigma_{x,p}^{-1}$  of  $\sigma : U \rightarrow \sigma(U)$  (sending  $p$  to  $x$ ) is defined on  $B(p, r)$ . Let  $N > \kappa/r$  be an integer, where  $\kappa$  is the Lipschitz constant of  $\gamma$ . Put  $p_j := \gamma(j/N)$ ,  $0 \leq j \leq N - 1$ . Observe that  $\gamma([j/N, (j+1)/N]) \subset B(p_j, r)$  since  $\gamma$  is  $\kappa$ -Lipschitz. Using inverse branches, we can defined a sequence of points  $x_0, x_1, \dots, x_{N-1}$  such that  $\sigma(x_j) = p_j$ . We can then define the desire path  $\tilde{\gamma} : t \in [0, 1] \mapsto \gamma \circ \sigma_{x_j, p_j}^{-1}(t)$ ,  $t \in [j/N, (j+1)/N]$ .  $\square$

## 5 The Patching Theorem

We recall the Radon-Nikodym decomposition of  $\sigma_T = T \wedge \omega_{\mathbb{P}^2}$  with respect to  $\mu$  :

$$\sigma_T = \mu^a + \mu^s. \quad (15)$$

Here  $\mu^a$  is absolutely continuous with respect to  $\mu$  and  $\mu^s \perp \mu$ . Our aim is to prove : **Theorem 5.1** (Patching Theorem). *Let  $U$  be a connected chart of  $\mathbb{P}^2$  with two systems of holomorphic coordinates  $(Z_1, W_1)$  and  $(Z_2, W_2)$ . Let us suppose that these systems of coordinates satisfy :*

$$T \wedge dd^c |W_i|^2 \ll \mu \text{ on } U, \quad i \in \{1, 2\}. \quad (16)$$

*If moreover  $\mu^s(U) > 0$ , then there exists  $\beta \in \mathcal{O}^*(U)$  such that  $dW_2 = \beta dW_1$  on  $U$ .*

We give two proofs of the theorem. The first one exploits the hermitian properties of the Green current  $T$  which can be seen as a positive singular (not smooth) metric on  $\mathbb{P}^2$ . More precisely, we express  $T \wedge dd^c|W_2|^2$  as a sum of measures absolutely continuous with respect to the measures  $T \wedge dd^c|Z_1|^2$  and  $T \wedge dd^c|W_1|^2$ . By using some results about dimension of measures, recalled in the section just below, we conclude that  $dW_2$  and  $dW_1$  are proportional. Variations of this technique can be used to detect fine properties of the current  $T$ , for example in [15] it is used to give a new proof of Dujardin's theorem :  $\mu \ll \sigma_T$  implies  $\lambda_2 = \frac{1}{2} \text{Log } d$ .

The second proof do not use specific properties of the Green current, except that it has local continuous potentials. In particular, we mention in Remark 5.4 that this proof can be adapted for a more general setting.

## 5.1 Hausdorff dimension of measures

Let  $\nu$  be a finite Borel measure on  $\mathbb{P}^2$ . The *pointwise lower dimension* of  $\nu$  at  $x \in \mathbb{P}^2$  is defined by

$$\underline{d}_\nu(x) := \liminf_{r \rightarrow 0^+} \frac{\text{Log } \nu B(x, r)}{\text{Log } r}.$$

Since  $T$  has continuous Hölder potentials, there exists a lower estimate for the lower pointwise dimension of  $\sigma_T$  as follows.

**Proposition 5.2** (Dinh-Sibony [11, Proposition 1.18]). *For every  $x \in \mathbb{P}^2$  we have  $T = dd^c(u_x)$  on a small ball  $B_x$ , where  $u_x$  is a plurisubharmonic function on  $B_x$  which is  $\gamma$ -Hölder for any  $\gamma \in ]0, \gamma_0[$ , where*

$$\gamma_0 = \min \left\{ 1, \frac{\text{Log}(d)}{\text{Log}(d_\infty)} \right\} > 0 \quad \text{and} \quad d_\infty = \lim_{n \rightarrow +\infty} \left( \sup_{p \in \mathbb{P}^2} \|d_p f^n\| \right)^{1/n}.$$

In particular,  $\underline{d}_{\sigma_T}(x) \geq 2 + \gamma_0$  for every  $x \in \mathbb{P}^2$ .

**Corollary 5.3.** *Let  $U$  be a connected open set of  $\mathbb{P}^2$  such that  $\nu(U) > 0$ .*

1. *Assume that there exists  $\gamma > 2$  such that  $\underline{d}_\nu(x) \geq \gamma$  for  $\nu$ -almost every  $x \in \mathbb{P}^2$ . Then for any  $h \in \mathcal{O}(U)$ ,  $\nu\{x \in U : h(x) = 0\} > 0$  implies  $h \equiv 0$  on  $U$ .*
2. *Assume that  $\nu \leq C\sigma_T$  for some constant  $C > 0$ . Let  $h \in \mathcal{O}(U)$  such that  $h(p) = 0$  for  $\nu$ -almost every point  $p \in U$ . Then  $h \equiv 0$  on  $U$ .*

### **Proof :**

1. Let  $\text{HD}(A)$  denote the Hausdorff dimension of  $A \subset \mathbb{P}^2$ . Young [24, Proposition 2.1] proved that if  $\underline{d}_\nu(x) \geq \gamma$  for  $\nu$ -almost every  $x \in \mathbb{P}^2$ , then  $\text{HD}(A) \geq \gamma$  for every Borel set  $A$  of positive  $\nu$ -measure. Using the assumptions, we get

$$\text{HD}(\{x \in U : h(x) = 0\}) \geq \gamma > 2. \tag{17}$$

If  $h$  were not null on  $U$ , then the analytic set  $\{x \in U : h(x) = 0\}$  would have Hausdorff dimension 2, see for instance Chirka's book [8, Corollary 1 p.23]. But this is not compatible with (17).

2. Let  $\tilde{\nu} := C\sigma_T - \nu \geq 0$  which is a finite Borel measure, so we have for any  $x \in \mathbb{P}^2$ ,  $\underline{d}_{(\nu+\tilde{\nu})}(x) = \underline{d}_{\sigma_T}(x) \geq 2 + \gamma_0$  by Proposition 5.2. It follows that  $\underline{d}_\nu(x) \geq \underline{d}_{\sigma_T}(x)$  for any  $x \in \mathbb{P}^2$ . By applying the first item with  $\gamma := 2 + \gamma_0$ , we get the result.  $\square$

## 5.2 Proof of the Patching Theorem

*Proof using measure theory.* — For any differentiable function  $h$  on  $U$ , we use the following notations :

$$\frac{\partial h}{\partial Z_1} := \frac{\partial}{\partial z} [h \circ \xi_1^{-1}] \circ \xi_1 \quad \text{and} \quad \frac{\partial h}{\partial W_1} := \frac{\partial}{\partial w} [h \circ \xi_1^{-1}] \circ \xi_1, \quad (18)$$

where  $\xi_1 = (Z_1, W_1)$  is the chart associated to the coordinates  $(Z_1, W_1)$ . We are going to prove that  $dW_2 = \frac{\partial W_2}{\partial W_1} dW_1$ , the function  $\frac{\partial W_2}{\partial W_1}$  being defined by (18) with  $W_2 = p_w \circ \xi_2$ . It is a local problem, it is sufficient to prove the equality on each  $U'$  open with  $\overline{U'} \subset U$ . So we can assume without loss of generality that the holomorphic functions  $Z_1, W_1, Z_2$  and  $W_2$  are holomorphic on a neighborhood of  $\overline{U}$  and thus the partial derivatives  $\partial W_2 / \partial Z_1$  and  $\partial W_2 / \partial W_1$  are bounded on  $U$ .

The Green current  $T$  on  $U$  can be written in the coordinates  $(Z_1, W_1)$  as a  $(1, 1)$ -differential form with complex measures coefficients :

$$T|_U = \sigma_1 \frac{i}{2} dZ_1 \wedge d\overline{Z}_1 + \left( \Lambda \frac{i}{2} dZ_1 \wedge d\overline{W}_1 \right) + \overline{\left( \Lambda \frac{i}{2} dZ_1 \wedge d\overline{W}_1 \right)} + \sigma_2 \frac{i}{2} dW_1 \wedge d\overline{W}_1,$$

with  $\sigma_1$  and  $\sigma_2$  the positive measures given by  $\sigma_1 = T \wedge \frac{i}{2} dW_1 \wedge d\overline{W}_1$  and  $\sigma_2 = T \wedge \frac{i}{2} dZ_1 \wedge d\overline{Z}_1$ . The complex measure  $\Lambda$  is given by  $\Lambda = -T \wedge \frac{i}{2} dW_1 \wedge d\overline{Z}_1$ . By Cauchy-Schwarz inequality, we have for every Borel set  $A$  in  $U$  :

$$|\Lambda(A)| \leq \sqrt{\sigma_1(A)} \sqrt{\sigma_2(A)}. \quad (19)$$

The trace  $\sigma_U := \sigma_1 + \sigma_2$  gives a positive measure which is equivalent to the trace measure  $\sigma_T$  on  $U$  :

$$\exists C > 0 : \frac{1}{C} \sigma_T|_U \leq \sigma_U \leq C \sigma_T|_U \quad (20)$$

Let us denote  $\lambda := \sigma_2$  and  $\mu_1^a := \sigma_1$ . By (16) we have  $\mu_1^a = \psi_1 \mu|_U \ll \mu$ , with  $\psi_1 \in L^1(\mu|_U)$ . Let  $\lambda = h_1 \mu|_U + \mu_1^s$  be the Radon-Nikodym decomposition of  $\lambda$  with respect to  $\mu|_U$ , where  $h_1 \in L^1(\mu|_U)$  is a non negative function, and  $\mu_1^s$  is a positive measure on  $U$  singular with respect to  $\mu$ . We have (recall that  $\mu^s(U) > 0$ ) :

$$\frac{1}{C} \mu^s|_U \leq \mu_1^s \leq C \mu^s|_U \implies \mu_1^s(U) > 0 \quad (21)$$

with the same  $C > 0$  than in (20). Indeed, by using (20) we have  $\mu_1^s \leq C(\mu^a + \mu^s)$  and  $\mu^s|_U \leq C[(h_1 + \psi_1)\mu|_U + \mu_1^s]$ , thus  $\mu_1^s \leq C\mu^s$  and  $\mu^s|_U \leq C\mu_1^s$  since  $\mu_1^s \perp \mu^a$  and  $\mu^s \perp \mu$ .

The idea now is to decompose  $dW_2$  in terms of the 1-forms  $dZ_1$  and  $dW_1$  :

$$dW_2 = \frac{\partial W_2}{\partial Z_1} dZ_1 + \frac{\partial W_2}{\partial W_1} dW_1.$$

Denoting  $\alpha := \frac{\partial W_2}{\partial Z_1}$ ,  $\beta := \frac{\partial W_2}{\partial W_1}$  and  $\gamma := \alpha \overline{\beta}$ , we have :

$$dd^c |W_2|^2 = |\alpha|^2 dd^c |Z_1|^2 + 2\text{Re} \left[ \gamma \frac{i}{2} dZ_1 \wedge d\overline{W}_1 \right] + |\beta|^2 dd^c |W_1|^2.$$

Wedging by  $T$  this equality, we get :

$$\underbrace{T \wedge dd^c |W_2|^2}_{=:\mu_2^a} = |\alpha|^2 \underbrace{(T \wedge dd^c |Z_1|^2)}_{=\lambda} + \underbrace{2\operatorname{Re} \left[ \gamma \left( T \wedge \frac{i}{2} dZ_1 \wedge d\bar{W}_1 \right) \right]}_{=:\Lambda'} + |\beta|^2 \underbrace{(T \wedge dd^c |W_1|^2)}_{=\mu_1^a},$$

and thus we have :

$$\mu_2^a = |\alpha|^2 \lambda + \Lambda' + |\beta|^2 \mu_1^a, \quad (22)$$

with  $\Lambda'$  a signed measure which satisfies for any Borel set  $A \subset U$  :

$$|\Lambda'(A)| \leq 2 \left( \max_U |\gamma| \right) |\Lambda(A)| \leq 2 \left( \max_U |\gamma| \right) \sqrt{\mu_1^a(A)} \sqrt{\lambda(A)}, \quad (23)$$

where the second inequality comes from (19). Let  $A \subset U$  be any Borel set, then using (22) and using the fact that  $\lambda = h_1 \mu|_U + \mu_1^s$  we get :

$$\begin{aligned} 0 &\leq \int_A |\alpha|^2 d\mu_1^s = \mu_2^a(A) - \int_A |\beta|^2 d\mu_1^a - \Lambda'(A) - \int_A h_1 |\alpha|^2 d\mu \\ &\leq \mu_2^a(A) + |\Lambda'(A)| \leq \mu_2^a(A) + 2 \left( \max_U |\gamma| \right) \sqrt{\mu_1^a(A)} \sqrt{\lambda(A)} \text{ by (23)}. \end{aligned}$$

Since  $\mu_1^a$  and  $\mu_2^a$  are absolutely continuous with respect to  $\mu$  and since  $\mu_1^s \perp \mu$  on  $U$ , there exists  $\mathcal{A} \subset U$  a Borel set of measure 0 for  $\mu_1^a$  and  $\mu_2^a$  such that  $\mu_1^s = \mu_1^s(\cdot \cap \mathcal{A})$ . So using the preceding inequalities we have for any  $A \subset U$  :

$$0 \leq \int_A |\alpha|^2 d\mu_1^s = \int_{A \cap \mathcal{A}} |\alpha|^2 d\mu_1^s \leq 0 + 2 \left( \max_U |\gamma| \right) \sqrt{0} \sqrt{\lambda(A \cap \mathcal{A})} = 0,$$

and thus  $\int_A |\alpha|^2 d\mu_1^s = 0$  for any  $A \subset U$ . The measure  $\mu_1^s$  charges the open set  $U$  by (21), hence  $\alpha(p) = 0$  for  $\mu_1^s$ -almost every point  $p \in U$ . By construction  $\mu_1^s \leq C \sigma_T$  for some  $C > 0$ , so we can apply the second item of Corollary 5.3 with  $\nu := \mu_1^s$  to conclude that  $\alpha \equiv 0$  on  $U$ . Thus we obtain  $dW_2 = \beta dW_1$  on  $U$ , and finally  $\beta \in \mathcal{O}^*(U)$  since  $W_1$  and  $W_2$  are submersions on  $U$ .  $\square$

**Proof using pluripotential theory.** — If we assume that  $dW_1 \wedge dW_2$  is not null on  $U$ , then  $dW_1 \wedge dW_2 \neq 0$  on a Zariski open set  $V$  of  $U$ . In  $V$ , the coordinates  $(W_1, W_2)$  create a chart and thus we have :

$$\mu^s \leq \sigma_T \ll T \wedge dd^c |W_1|^2 + T \wedge dd^c |W_2|^2 \ll \mu \text{ by assumption.}$$

By the Chern-Levine-Nirenberg inequality, since  $T$  has local continuous potentials, the current  $T$  has no mass on  $U \setminus V$  which is an analytic subset. Therefore the property  $\mu^s \ll \mu$  on  $V$  extends on  $U$ . Hence  $\mu^s(U) = 0$  and the result follows.  $\square$

**Remark 5.4.** *This second proof does not use the properties of the Green current  $T$ . The Patching Theorem thus can be generalized as follows.*

*Let  $X$  be a complex surface (a two dimensional complex manifold) equipped with a smooth  $> 0$  closed  $(1, 1)$ -form  $\omega$ . Let  $S$  be a positive closed  $(1, 1)$ -current on  $X$  with continuous local potentials and being of finite mass. Denote  $\nu := S \wedge S$ ,  $\sigma_S := S \wedge \omega$ . These wedge products are well defined since  $S$  has continuous local potentials. Then  $\nu$*



is a positive finite Borel measure on  $X$  (possibly identically null) and so is  $\sigma_S$ , so we can write the Radon-Nikodym decomposition of  $\sigma_S$  with respect to  $\nu$  :  $\sigma_S = \nu^a + \nu^s$ , where  $\nu^a \ll \nu$  and  $\nu^s \perp \nu$ .

The same arguments than above allows to prove the following. Let  $U \subset X$  be a connected open subset of  $X$  equipped with two systems of holomorphic coordinates  $(Z_1, W_1)$  and  $(Z_2, W_2)$ . Assume that  $S \wedge dd^c|W_i|^2 \ll \nu$  for  $i \in \{1, 2\}$ . If  $\nu^s(U) > 0$ , then the holomorphic 1-forms  $dW_1$  and  $dW_2$  patch together : there exists  $\beta \in \mathcal{O}^*(U)$  such that  $dW_2 = \beta dW_1$  on  $U$ .

## 6 Construction of a foliation near $\text{Supp}(T) \setminus \mathcal{E}$

We want to prove the Theorem 1.1. Let  $a \in \text{Supp}(\mu)$  be a repelling  $N$ -periodic point of  $f$  with Poincaré-Dulac coordinates  $\sigma_0^{-1} = (Z_a, W_a)$ . Let  $D(z, w) = (\chi_1 z + cw^a, \chi_2 w)$  be the polynomial map such that  $f^N \circ \sigma = \sigma \circ D$ , with  $\sigma$  the Poincaré map of  $\sigma_0$  given by Proposition 4.1. We assume that the following formula holds

$$\mu = T \wedge dd^c|W_a|^2 \text{ on } (\mathbb{P}^2, a). \quad (24)$$

We also assume that there exists  $\Omega$  a open set charged by  $\mu$  such that for any open set  $V \subset \Omega$ , if  $\mu(V) > 0$  then  $\mu^s(V) > 0$ . In Lemma 6.3, we will in fact prove that it implies that, for any open set  $V \subset \mathbb{P}^2$ , if  $\mu(V) > 0$  then  $\mu^s(V) > 0$ . We also assume that the exceptional set  $\mathcal{E}$  of  $f$  does not intersect  $\text{Supp}(\mu)$ .

### 6.1 Pull back by the Poincaré map

Our purpose in this section is to see how (24) is lifted by  $\sigma$  on  $\mathbb{C}^2$ . Let us specify how the positive closed  $(1, 1)$ -current  $\sigma^*T$  and the Borel measure  $\sigma^*\mu$  are defined on  $\mathbb{C}^2$ . We refer to [1, 18, 22, 11] for general accounts on currents.

For every  $x \in \mathbb{C}^2$ , let us write  $T = dd^c(u)$  on an open neighborhood of  $\sigma(x)$ ,  $u$  being a bounded continuous *psh* function. Then  $\sigma^*T$  is defined near  $x$  by  $dd^c(u \circ \sigma)$ . Since  $u$  is bounded,  $\sigma^*\mu := \sigma^*T \wedge \sigma^*T$  is well defined near  $x$  in the sense of Bedford-Taylor. These currents are representable by integration (they have order 0), hence the trace of  $\sigma^*T$  and  $\sigma^*\mu$  are Borel measures on  $\mathbb{C}^2$ . These two measures are finite on bounded open subsets by Chern-Levine-Nirenberg inequality, thus they are Borel regular measures (Radon measures) on  $\mathbb{C}^2$ .

**Proposition 6.1.** *Let  $f$  be a degree  $d \geq 2$  map on  $\mathbb{P}^2$  of equilibrium measure  $\mu = T \wedge T$ . Assume the first hypothesis of Theorem 1.1 so that the formula (24) holds for a repelling  $N$ -periodic point  $a \in \text{Supp}(\mu)$ . With the preceding notations,*

1.  $|\chi_2| = \sqrt{d^N}$ .
2. The pull back  $\sigma^*\mu$  and the pull back  $\sigma^*T$  are related on  $\mathbb{C}^2$  by

$$\sigma^*\mu = \sigma^*T \wedge dd^c|w|^2 \text{ on } \mathbb{C}^2,$$

where  $w$  is the second standard coordinate on  $\mathbb{C}^2$ .

3. One has  $\mu \ll \sigma_T$  on  $\mathbb{P}^2$ .

**Proof :** Without loss of generality we can assume that  $N = 1$ . Using  $f^*\mu = d^2\mu$ ,  $f^*T = dT$  and  $W_a \circ f = \chi_2 W_a$ , we get by pulling back (24) by  $f$  :

$$\mu = \frac{|\chi_2|^2}{d} (T \wedge dd^c |W_a|^2) \text{ on } (\mathbb{P}^2, a).$$

Using again (24) one gets  $\mu = \frac{|\chi_2|^2}{d} \mu$  on  $(\mathbb{P}^2, a)$ . Finally  $|\chi_2|^2 = d$  since  $a \in \text{Supp}(\mu)$ .

Let us prove the second item. We want to show that  $\sigma^*T \wedge dd^c |w|^2$  and  $\sigma^*\mu$  are equal. Since they are Borel regular measures, it suffices to prove that they coincide on compact sets  $K$  of  $\mathbb{C}^2$ . Let us fix a compact set  $K \subset \mathbb{C}^2$ . Let  $k$  be large enough such that  $K \subset \mathbb{D}_k^2$ , in particular  $\sigma = \sigma_k = f^{n_k} \circ \sigma_0 \circ D^{-n_k}$  on  $K$ . Since  $f^{n_k^*}\mu = d^{2n_k}\mu$ , we get

$$(\sigma^*\mu)(K) = [(D^{-n_k})^* \sigma_0^*(f^{n_k^*}\mu)](K) = d^{2n_k} \times [(D^{-n_k})^* \sigma_0^*\mu](K).$$

We obtain by the formula (24) :

$$(\sigma^*\mu)(K) = d^{2n_k} \times [(D^{-n_k})^* \sigma_0^*T] \wedge (dd^c |W_a \circ \sigma_0 \circ D^{-n_k}|^2)(K).$$

Now using

$$W_a \circ \sigma_0 \circ D^{-n_k} = w \circ \sigma_0^{-1} \circ \sigma_0 \circ D^{-n_k} = w \circ D^{-n_k} = \chi_2^{-n_k} \times w,$$

we deduce that :

$$(\sigma^*\mu)(K) = d^{2n_k} |\chi_2|^{-2n_k} \times [(D^{-n_k})^* \sigma_0^*T] \wedge dd^c |w|^2(K).$$

But  $|\chi_2| = \sqrt{d}$  by the first item, hence we obtain :

$$(\sigma^*\mu)(K) = [(D^{-n_k})^* \sigma_0^*(d^{n_k}T)] \wedge dd^c |w|^2(K).$$

Using the invariant relation  $d^{n_k}T = f^{n_k^*}T$  one deduces :

$$(\sigma^*\mu)(K) = [(D^{-n_k})^* \sigma_0^* f^{n_k^*}T] \wedge dd^c |w|^2(K).$$

Finally, by recalling that  $\sigma|_K = f^{n_k} \circ \sigma_0 \circ D^{-n_k}|_K$ , we get

$$(\sigma^*\mu)(K) = [(\sigma^*T) \wedge dd^c |w|^2](K).$$

This proves that the measures  $\sigma^*\mu$  and  $\sigma^*T \wedge dd^c |w|^2$  are equal on  $\mathbb{C}^2$ .

Let us explain now why  $\mu \ll \sigma_T$  on  $\mathbb{P}^2$ . Let  $p$  be a point in  $\text{Supp}(\mu) \cap \sigma(\mathbb{C}^2)$  which is not a critical point of  $\sigma$ . Then there exists a inverse branch  $\sigma_p^{-1}$  of  $\sigma$  defined on a neighborhood of  $p$ , and the formula  $\sigma^*\mu = \sigma^*T \wedge dd^c |w|^2$  implies that we have  $\mu = T \wedge dd^c |w \circ \sigma_p^{-1}|^2 \ll \sigma_T$  near  $p$ . Since it is true for any point  $p$  in  $\text{Supp}(\mu) \cap \sigma(\mathbb{C}^2) \setminus \sigma(\text{Crit } \sigma)$ , we deduce that  $\mu \ll \sigma_T$  on  $\sigma(\mathbb{C}^2) \setminus \sigma(\text{Crit } \sigma)$ . Recall that  $\sigma(\mathbb{C}^2) \supset \mathbb{P}^2 \setminus \mathcal{E}$  (see Proposition 4.2) and that  $\mu$  does not charge the set  $\mathcal{E} \cup \sigma(\text{Crit } \sigma)$ , see Proposition 4.1. We thus obtain  $\mu \ll \sigma_T$  on  $\mathbb{P}^2$ .  $\square$

## 6.2 Proof of Theorem 1.1

In the following remaining sections, we are going to prove the following theorem :

**Theorem 6.2.** *Assume the hypothesis of Theorem 1.1 with the same notations. Let  $J$  be a compact subset of  $\text{Supp}(T) \setminus \mathcal{E}$  containing  $\text{Supp}(\mu)$  such that  $J = \text{Supp}(\sigma_T|_J)$ . Then there exists a open neighborhood  $\mathcal{V}$  of  $J$ , and a foliation  $\mathcal{F}$  defined on  $\mathcal{V}$ , such that  $\sigma^*\mathcal{F} = \mathcal{F}_w$  on  $\sigma^{-1}(\mathcal{V})$ , where  $\mathcal{F}_w$  is the horizontal foliation of  $\mathbb{C}^2$ .*

We then assert that this theorem implies Theorem 1.1, let us explain why. We start by explaining that  $\text{Supp}(\mu)$  is included in  $\text{Supp}(\mu^s) \setminus \mathcal{E}$ . Since  $\text{Supp}(\mu) \cap \mathcal{E} = \emptyset$ , it remains to prove :

**Lemma 6.3.** *Item 2. of Theorem 1.1 is equivalent to the following :*

2'. *Any open set charged by  $\mu$  is also charged by  $\mu^s$ .*

*In particular  $\text{Supp}(\mu) \subset \text{Supp}(\mu^s)$  and thus  $\text{Supp}(\mu^s) = \text{Supp}(T)$ .*

**Proof :** Indeed, this assumption implies 2. Let us give briefly the arguments for the reverse implication. Let  $\Omega$  be the open neighborhood provided by item 2. in the Theorem 1.1. By assumption,  $\mu^s$  charges every open subset of  $\Omega$  which is charged by  $\mu$ . Let  $V$  be a neighborhood of a point in  $\text{Supp}(\mu)$ . Since  $\mu$  is mixing there exists  $n_0$  such that  $\mu(f^{-n_0}\Omega \cap V) > 0$ . Let  $x_0 \in (\text{Supp}(\mu) \cap f^{-n_0}\Omega \cap V) \setminus \text{Crit}(f^{n_0})$  and let  $B \subset V$  be a small ball centered at  $x_0$  such that  $f^{n_0}(B) \subset \Omega$  and  $f^{n_0}$  is injective on  $B$ . Observe that  $\mu^s(f^{n_0}(B)) > 0$  by hypothesis. Since  $(f^{n_0})^*T = d^{n_0}T$  and since  $\omega_{\mathbb{P}^2}$  is quasi-invariant by the biholomorphism  $f^{n_0} : B \rightarrow f^{n_0}(B)$ , the measures  $\sigma_T \circ f^{n_0}$  and  $\sigma_T$  are equivalent on  $B$ . Moreover the measures  $\mu \circ f^{n_0}$  and  $\mu$  are also equivalent on  $B$  since  $(f^{n_0})^*\mu = d^{2n_0}\mu$ . Hence  $\mu^s(B) > 0$  as desired.  $\square$

Consider now a covering of  $\mathbb{P}^2 \setminus \mathcal{E}$  by compact sets  $\mathbb{P}^2 \setminus \mathcal{E} = \bigcup_{j=1}^{+\infty} K_j$ , where the  $K_j$ 's satisfy  $K_j \subset \overset{\circ}{K}_{j+1}$ . Then intersecting this covering with  $\text{Supp}(T)$  we obtain a covering  $\text{Supp}(T) \setminus \mathcal{E} = \bigcup_{j=1}^{+\infty} L_j$ , where  $L_j := \text{Supp}(T) \cap K_j$ . Up to re-index the sequence  $(K_j)_{j \geq 1}$ , we can assume that for any  $j \geq 1$  :

$$\text{Supp}(\mu) \subset \overset{\circ}{K}_j \cap \text{Supp}(T) \subset L_j \subset \text{Supp}(T) \setminus \mathcal{E}.$$

Observe that these inclusions are possible since  $\text{Supp}(\mu) \subset \text{Supp}(T)$  (by Proposition 6.1 or by Lemma 6.3), and since  $\text{Supp}(\mu) \cap \mathcal{E} = \emptyset$ .

**Lemma 6.4.** *Let  $J_j := \text{Supp}(\sigma_T|_{L_j})$  for any  $j \geq 1$ . Then  $(J_j)_{j \geq 1}$  is a increasing sequence of compact sets such that for any  $j \geq 1$  :*

1.  $J_j = \text{Supp}(\sigma_T|_{J_j})$  and  $\text{Supp}(\mu) \subset J_j \subset \text{Supp}(T) \setminus \mathcal{E}$ .
2.  $\text{Supp}(T) \setminus \mathcal{E} = \bigcup_{j=1}^{+\infty} J_j$ .

**Proof :** Recall that  $\text{Supp}(T) = \text{Supp}(\sigma_T)$  and that  $\mu \ll \sigma_T$  by Proposition 6.1.

1. First  $\text{Supp}(\sigma_T|_{J_j}) \subset J_j$ , second if  $p \in J_j$  then  $\sigma_T(J_j \cap B(p, \varepsilon)) = \sigma_T(L_j \cap B(p, \varepsilon))$  by definition of  $J_j$ , but  $p \in J_j = \text{Supp}(\sigma_T|_{L_j})$  thus  $\sigma_T(L_j \cap B(p, \varepsilon)) > 0$ , for any  $\varepsilon > 0$ . We deduce  $\text{Supp}(\sigma_T|_{J_j}) = J_j$ . To conclude, observe that because  $\mu \ll \sigma_T$  and  $\text{Supp}(\mu) \subset \overset{\circ}{K}_j$ , we must have  $\text{Supp}(\mu) \subset \text{Supp}(\sigma_T|_{L_j}) = J_j$ .

2. Let  $p \in \text{Supp}(T) \setminus \mathcal{E}$ , then there exists  $j \geq 1$  such that  $p \in L_j$ , let us assume  $j = 1$  for simplicity. Then in particular  $p \in \overset{\circ}{K}_2$  and there exists  $\varepsilon_0$  such that  $B(p, \varepsilon_0) \subset K_2$ . Then for every  $0 < \varepsilon \leq \varepsilon_0$ , we have  $\sigma_T(L_2 \cap B(p, \varepsilon)) = \sigma_T(\text{Supp}(T) \cap K_2 \cap B(p, \varepsilon)) = \sigma_T(K_2 \cap B(p, \varepsilon))$  by definition of  $L_2$ . We deduce  $\sigma_T(L_2 \cap B(p, \varepsilon)) = \sigma_T(B(p, \varepsilon)) > 0$  since  $B(p, \varepsilon) \subset K_2$  and since  $p \in L_1 \subset \text{Supp}(T)$ . It implies  $p \in \text{Supp}(\sigma_T|_{L_2}) = J_2$  and the conclusion follows.  $\square$

Theorem 6.2 and Lemma 6.4 thus imply the existence of a sequence of open sets  $(\mathcal{V}_j)_{j \geq 1}$  and of a sequence of foliations  $(\mathcal{F}_j)_{j \geq 1}$  such that for any  $j \geq 1$  :

$$J_j \subset \mathcal{V}_j \text{ and } \mathcal{F}_j \text{ is defined on } \mathcal{V}_j, \text{ and } \sigma^* \mathcal{F}_j = \mathcal{F}_w \text{ on } \sigma^{-1}(\mathcal{V}_j).$$

**Lemma 6.5.** *For any  $j \geq 2$ ,  $\mathcal{F}_j = \mathcal{F}_{j-1}$  on  $\mathcal{V}_j \cap \mathcal{V}_{j-1}$ .*

**Proof :** Let us explain the argument for  $j = 2$ . We have  $\sigma^* \mathcal{F}_2 = \mathcal{F}_w = \sigma^* \mathcal{F}_1$  on  $\sigma^{-1}(\mathcal{V}_2 \cap \mathcal{V}_1)$ . Let  $\mathcal{C}$  be a connected component of  $\mathcal{V}_2 \cap \mathcal{V}_1$ , and let  $U \neq \emptyset$  be an open set such that  $\sigma(U) \subset \mathcal{C}$  and such that  $\sigma$  is injective on  $U$ . Then observe that  $\mathcal{F}_2 = \mathcal{F}_1$  on  $\sigma(U)$  by applying  $(\sigma|_U)_*$  to the equality  $\sigma^* \mathcal{F}_2|_U = \sigma^* \mathcal{F}_1|_U$ . It implies  $\mathcal{F}_2 = \mathcal{F}_1$  on  $\mathcal{C}$  by analytic continuation (cf. Lemma 2.1), for every  $\mathcal{C}$ . The result follows.  $\square$

This lemma implies that the foliations  $\mathcal{F}_j$ ,  $j \geq 1$ , patch together and create a foliation

$$\mathcal{F} := \bigcup_{j=1}^{+\infty} \mathcal{F}_j$$

on the open set  $\mathcal{V} := \left( \bigcup_{j=1}^{+\infty} \mathcal{V}_j \right) \supset \text{Supp}(T) \setminus \mathcal{E}$ . Observe that we have  $\sigma^* \mathcal{F} = \bigcup_{j=1}^{+\infty} \sigma^* \mathcal{F}_j = \mathcal{F}_w$  on  $\sigma^{-1}(\mathcal{V})$ . So we can conclude that  $\mathcal{F}$  is invariant on  $\mathcal{V}$  by  $f^N$  (recall that  $N$  is the period of the repelling point  $a$ ) :

**Proposition 6.6.**  *$(f^N)^* \mathcal{F} = \mathcal{F}$  on  $f^{-N}(\mathcal{V}) \cap \mathcal{V}$ .*

**Proof :** We can assume  $N = 1$  without loss of generality. Let us fix arbitrary points  $p, q \in \mathcal{V}$  such that  $f(p) = q$ . Let us prove that there exists  $W \subset \mathcal{V}$  an open neighborhood of  $p$ , and  $V \subset \mathcal{V}$  an open neighborhood of  $q$ , satisfying  $f(W) \subset V$ , and such that  $(f^* \mathcal{F}|_V)|_W = \mathcal{F}|_W$ . It is enough to ensure the result.

Let  $x \in \sigma^{-1}(p)$  and let  $y := D(x)$  which satisfies  $\sigma(y) = q$ . We can consider a connected open set  $U_x$  (resp.  $U_y$ ) in  $\sigma^{-1}(\mathcal{V})$  containing  $x$  (resp.  $y$ ) such that  $W := \sigma(U_x)$  (resp.  $V := \sigma(U_y)$ ) is contained in  $\mathcal{V}$ . We can assume  $D(U_x) \subset U_y$  and  $f(W) \subset V$ . We denote  $\sigma_p := \sigma|_{U_x}$  and  $\sigma_q := \sigma|_{U_y}$ .

Let now  $\mathcal{G} := (f^* \mathcal{F}|_V)|_W$ , observe that  $\sigma_p^* \mathcal{G} = (D^* \sigma_q^* \mathcal{F})|_{U_x}$  by using that  $\sigma \circ D = \sigma_q \circ \sigma_p$  on  $U_x \subset D^{-1}(U_y)$ . Since  $\sigma^* \mathcal{F} = \mathcal{F}_w$  on  $\sigma^{-1}(\mathcal{V})$ , we have  $\sigma_q^* \mathcal{F} = \mathcal{F}_w|_{U_y}$  and thus  $\sigma_p^* \mathcal{G} = (D^* \mathcal{F}_w)|_{U_x} = \mathcal{F}_w|_{U_x}$ . The equality  $\sigma^* \mathcal{F} = \mathcal{F}_w$  on  $\sigma^{-1}(\mathcal{V})$ , also implies  $\mathcal{F}_w|_{U_x} = \sigma_p^*(\mathcal{F}|_W)$ , and we deduce that  $\sigma_p^* \mathcal{G} = \sigma_p^*(\mathcal{F}|_W)$ . It implies that  $\mathcal{G} = \mathcal{F}|_W$  since  $U_x$  is connected (pull-back by a local inverse of  $\sigma$  on  $U_x$  and conclude by analytic continuation, cf. Lemma 2.1).  $\square$

### 6.3 Proof of Theorem 6.2

Let us fixed from now on a compact set  $J \subset \text{Supp}(T) \setminus \mathcal{E}$  such that  $J \supset \text{Supp}(\mu)$ , and such that  $\text{Supp}(\sigma_T|_J) = J$ . Our goal is now to prove that there exists an open neighborhood  $\mathcal{V}$  of  $J$  and a foliation  $\mathcal{F}$  defined on  $\mathcal{V}$  which satisfies  $\sigma^* \mathcal{F} = \mathcal{F}_w$  on  $\mathcal{V}$ . We proceed in several steps.

#### 6.3.1 Step 1 : Inverse branches near regular values of $J$

Recall that  $J \supset \text{Supp}(\mu)$  and that  $J \cap \mathcal{E} = \emptyset$ . Thus we have that  $J \subset \sigma(\mathbb{C}^2)$  by Proposition 4.2.

By Proposition 4.1, the map  $\sigma$  is equal to  $\sigma_k = f^{n_k} \circ \sigma_0 \circ D^{-n_k}$  on  $\mathbb{D}_k^2$ . Let now  $p \in J \subset \sigma(\mathbb{C}^2)$  and let  $k \geq 1$  be such that  $p \in \sigma(\mathbb{D}_k^2)$ . Since  $\sigma$  is open,  $\sigma(\mathbb{D}_k^2)$  is an open neighborhood of  $p$ . By compactness we can cover  $J$  by a finite number of such neighborhoods

$$J \subset \mathcal{U} := \sigma(\mathbb{D}_{k_1}^2) \cup \dots \cup \sigma(\mathbb{D}_{k_N}^2).$$

Let  $R := \max_{1 \leq i \leq N} k_i$  and observe that  $\bigcup_{j=1}^N \mathbb{D}_{k_j}^2 = \mathbb{D}_R^2 =: D_R$ , thus we have a map :

$$\sigma_R := (f^{n_R} \circ \sigma_0 \circ D^{-n_R})|_{D_R} : D_R \longrightarrow \sigma_R(D_R) = \mathcal{U} \supset J.$$

According to Proposition 4.1, the critical values  $E_R$  of  $\sigma_R$  satisfies :

$$E_R = \sigma_R(\text{Crit } \sigma_R) \subset f^{n_R}(\text{Crit } f^{n_R}).$$

In particular  $\text{Reg}_{\sigma_R}(J) := J \setminus E_R$  has full  $\sigma_T|_J$ -measure and full  $\mu$ -measure.

Any point  $p \in \mathcal{U}$  satisfies  $1 \leq \#(\sigma_R^{-1}(p)) \leq d^{2n_R}$ . So for every  $p \in \text{Reg}_{\sigma_R}(J)$  there exists  $n_p \in \{1, \dots, d^{2n_R}\}$  such that  $\sigma_R^{-1}(p) = \{x_{1,p}, \dots, x_{n_p,p}\}$ . For every  $j \in \{1, \dots, n_p\}$  there exists an open connected neighborhood  $U_{j,p} \subset D_R$  of  $x_{j,p}$  such that

$$\sigma_{j,p} := \sigma_R|_{U_{j,p}} : U_{j,p} \longrightarrow \sigma(U_{j,p}) =: V_{j,p} \subset \mathcal{U}$$

is a biholomorphism. We denote  $\sigma_{j,p}^{-1} : V_{j,p} \rightarrow U_{j,p}$  the inverse map. We can assume that every  $V_{j,p}$  does not intersect  $E_R$ . We denote

$$V_p := \bigcap_{j=1}^{n_p} V_{j,p}.$$

We can assume that  $V_p$  is connected.

Observe that, even up to a reduction of  $V_p$ , the fiber of a point  $p' \in V_p$  by  $\sigma_R$  may not be given by the inverse branches  $\sigma_{1,p}^{-1}, \dots, \sigma_{n_p,p}^{-1}$ . Indeed, if  $p$  admits a preimage by  $\sigma$  on the boundary of  $D_R$ , the inclusion  $\{\sigma_{1,p}^{-1}(p'), \dots, \sigma_{n_p,p}^{-1}(p')\} \subset \sigma_R^{-1}(p')$  could be strict. To avoid this difficulty, we shall introduce special sets of inverse branches :

**Definition 6.7.** For every  $p \in \text{Reg}_{\sigma_R}(J)$ , we define

$$\text{IB}_R(p) := \bigcup_{j=1}^{n_p} \sigma_{j,p}^{-1}(V_p) \subset D_R. \quad (25)$$

Here the abbreviation "IB" stands for inverse branches.

### 6.3.2 Step 2 : Construction of the foliation near regular values

In this section we introduce for every  $p \in \text{Reg}_{\sigma_R}(J)$  a foliation  $\mathcal{F}_p$  on  $V_p$ . We shall patch together the foliations  $(\mathcal{F}_p)_{p \in \text{Reg}_{\sigma_R}(J)}$  (and also the foliations on neighborhoods of singular values  $q \in J \cap E_R$  constructed in Section 6.3.3). To do so, we shall use the following distribution of tangent complex lines, defined for each  $p \in \text{Reg}_{\sigma_R}(J)$  :

$$\mathcal{D}_p : \begin{cases} V_p & \longrightarrow \mathbb{P}(T\mathbb{P}^2) \\ p' & \longmapsto [d_{x'}\sigma_R \cdot (1, 0)] \text{ for any } x' \in \text{IB}_R(p) \text{ such that } \sigma_R(x') = p' \end{cases} \quad (26)$$

It will describe the directions followed by the different local foliations we will construct. The notation  $[\vec{v}]$  stands for the complex line  $\mathbb{C} \cdot \vec{v}$  of  $T_p\mathbb{P}^2$ . We verify in Proposition 6.8 that the maps  $\mathcal{D}_p$  are well defined. The proof of this proposition uses the Patching Theorem 5.1, so we need that  $\mu^s$  charges all the neighborhoods of points of  $\text{Supp}(\mu)$ . This property is ensured by Item 2. of Theorem 1.1 and Lemma 6.3.

**Proposition 6.8.** *Assume that  $f$  and the  $N$ -periodic repelling point  $a$  satisfy the hypothesis of Theorem 1.1 with the same notations.*

1. *Let  $x, y \in \mathbb{C}^2 \setminus (\text{Crit } \sigma)$  and let  $p := \sigma(x)$  and  $q := \sigma(y)$ . Let  $U_{x,p}$  be an open connected neighborhood of  $x$  such that the map  $\sigma_{x,p} := \sigma|_{U_{x,p}} : U_{x,p} \rightarrow \sigma(U_{x,p}) =: V_{x,p}$  is a biholomorphism. Similarly let  $\sigma_{y,q} := \sigma|_{U_{y,q}} : U_{y,q} \rightarrow \sigma(U_{y,q}) =: V_{y,q}$  be a biholomorphism. Assume that  $\Omega := V_{x,p} \cap V_{y,q}$  is connected and that  $\sigma_T(\Omega) > 0$ . Then  $\phi := (\sigma_{y,q}^{-1} \circ \sigma_{x,p})$  has the form  $\phi(z, w) = (A(z, w), B(w))$  on  $\sigma_{x,p}^{-1}(\Omega)$ .*
2. *Let  $p \in \text{Reg}_{\sigma_R}(J)$  and  $p' \in V_p$ . Then for every  $x', y' \in \sigma_R^{-1}(p') \cap \text{IB}_R(p)$ , the complex lines  $[d_{x'}\sigma \cdot (1, 0)]$  and  $[d_{y'}\sigma \cdot (1, 0)]$  are equal. In particular the map  $\mathcal{D}_p : V_p \rightarrow \mathbb{P}(T\mathbb{P}^2)$  is well defined.*
3. *Let  $p, q \in \text{Reg}_{\sigma_R}(J)$ . If  $\mathcal{C}$  is a connected component of  $V_p \cap V_q$  of measure  $\sigma_T(\mathcal{C}) > 0$ , then  $\mathcal{D}_p = \mathcal{D}_q$  on  $\mathcal{C}$ .*

#### **Proof :**

1. Let us write  $\phi = (A(z, w), B(z, w))$  and let us prove that  $B(z, w)$  does not depend on  $z$ . To do so we use that  $\sigma^*\mu = \sigma^*T \wedge dd^c|w|^2$  on  $\mathbb{C}^2$ , according to Proposition 6.1. Restricting this equality on  $U_{x,p}$  (resp. on  $U_{y,q}$ ) and pushing forward it by  $\sigma_{x,p}$  (resp. by  $\sigma_{y,q}$ ) we get on  $\Omega = V_{x,p} \cap V_{y,q}$  :

$$\mu = T \wedge dd^c|W_p|^2 \text{ and } \mu = T \wedge dd^c|W_q|^2,$$

with the notations  $(Z_p, W_p) := \sigma_{x,p}^{-1}$  and  $(Z_q, W_q) := \sigma_{y,q}^{-1}$ .

We recall that  $\sigma_T = \mu^a + \mu^s$  is defined by (15). By assumption  $\sigma_T(\Omega) > 0$  so by Lemma 6.3 it ensures that  $\mu^s(\Omega) > 0$ . By hypothesis,  $\Omega$  is also connected, thus we can apply the Patching Theorem 5.1 : there exists  $\beta \in \mathcal{O}^*(\Omega)$  such that  $dW_q = \beta dW_p$  which implies  $dB = \sigma_{x,p}^*dW_q = (\beta \circ \sigma_{x,p})dw$  on  $\sigma_{x,p}^{-1}(\Omega)$ . In particular  $\partial_z B \equiv 0$  on  $\sigma_{x,p}^{-1}(\Omega)$ .

2. Since  $x', y' \in \text{IB}_R(p)$  and  $\sigma(x') = \sigma(y') = p'$ , there exist  $i, j$  such that  $x' = \sigma_{i,p}^{-1}(p')$  and  $y' = \sigma_{j,p}^{-1}(p')$ . Let us assume  $i = 1$  and  $j = 2$  for simplicity. Observe that  $\sigma_{1,p}^{-1}$  and  $\sigma_{2,p}^{-1}$  are defined on the connected set  $V_p$  by construction, and  $\sigma_T(V_p) > 0$  since  $V_p$  is an open neighborhood of  $p \in J \subset \text{Supp}(T)$ . So we can apply the point 1. with



$p = q$ , to conclude that the map  $\phi := (\sigma_{2,p}^{-1} \circ \sigma_{1,p})$  satisfies  $\phi(z, w) = (A(z, w), B(w))$  on  $\sigma_{1,p}^{-1}(V_p)$ . In particular we have  $d_{x'}\phi \cdot (1, 0) = \partial_z A(x') \cdot (1, 0)$  and so we deduce :

$$d_{x'}\sigma_{1,p} \cdot (1, 0) = d_{y'}\sigma_{2,p} \cdot d_{x'}\phi \cdot (1, 0) = \partial_z A(x') \times d_{y'}\sigma_{2,p} \cdot (1, 0).$$

The coefficient  $\partial_z A(x')$  is not equal to 0 because  $\phi$  is a biholomorphism. Finally

$$[d_{x'}\sigma \cdot (1, 0)] = [d_{y'}\sigma \cdot (1, 0)] \in \mathbb{P}(T_{p'}\mathbb{P}^2),$$

which proves that  $\mathcal{D}_p$  is well defined on  $V_p$ .

3. Let  $\mathcal{C}$  be a connected component of  $V_p \cap V_q$ , and let us assume that  $\sigma_T(\mathcal{C}) > 0$ . Let us fix  $p'$  an element of  $\mathcal{C}$ , and let  $x \in \sigma_R^{-1}(p') \cap \text{IB}_R(p)$  and  $y \in \sigma_R^{-1}(p') \cap \text{IB}_R(q)$ . By definition of  $\text{IB}_R(p)$  there exists  $i \in \{1, \dots, n_p\}$  such that  $x = \sigma_{i,p}^{-1}(p')$ . Similarly there exists  $j \in \{1, \dots, n_q\}$  such that  $y = \sigma_{j,q}^{-1}(p')$ . Let now  $U_x := \sigma_{i,p}^{-1}(\mathcal{C}) \subset U_{i,p}$  be a neighborhood of  $x$ ,  $\mathcal{C} \subset V_p$  is indeed contained in the domain of definition of  $\sigma_{i,p}^{-1}$ . Similarly let us define  $U_y := \sigma_{j,q}^{-1}(\mathcal{C}) \subset U_{j,q}$  which contains  $y$ . We have  $\sigma(U_x) = \sigma(U_y) = \mathcal{C}$ , and by hypothesis  $\mathcal{C}$  is a connected open set such that  $\sigma_T(\mathcal{C}) > 0$ . The map  $\phi := (\sigma_{j,q}^{-1} \circ \sigma_{i,p}) : U_x \rightarrow U_y$  is then well defined, and according to the first item (applied with  $\Omega = \mathcal{C}$ ),  $\phi$  has the form  $\phi(z, w) = (A(z, w), B(w))$  on  $U_x$ . As in the proof of the preceding item, we can compute the matrix  $d_x\phi$  and check that  $d_x\sigma \cdot (1, 0) = \partial_z A(x) \times d_y\sigma \cdot (1, 0)$ . We also have  $\partial_z A(x) \neq 0$  and thus using again the second item we have :

$$\mathcal{D}_p(p') = [d_x\sigma \cdot (1, 0)] = [d_y\sigma \cdot (1, 0)] = \mathcal{D}_q(p').$$

The proof is then complete.  $\square$

**Lemma 6.9.** *Let  $p \in \text{Reg}_{\sigma_R}(J)$ . Then the holomorphic 1-forms*

$$\omega_{j,p} := d(w \circ \sigma_{j,p}^{-1}), \quad j \in \{1, \dots, n_p\},$$

*are equal modulo a multiplicative function in  $\mathcal{O}^*(V_p)$ .*

**Proof :** By the point 1. of Proposition 6.8, for each  $i, j$  we have on  $V_p$  that  $\sigma_{j,p}^{-1} \circ \sigma_{i,p} = (A_{ij}(z, w), B_{ij}(w))$  and thus  $\sigma_{i,p}^* \omega_{j,p} = d(w \circ \sigma_{j,p}^{-1} \circ \sigma_{i,p}) = B'_{ij}(w)dw$ . Then  $\omega_{j,p} = (B'_{ij} \circ (w \circ \sigma_{i,p}^{-1})) \times \omega_{i,p}$ . Because the function  $w \circ \sigma_{i,p}^{-1}$  is a submersion, we have  $(B'_{ij} \circ (w \circ \sigma_{i,p}^{-1})) \in \mathcal{O}^*(V_p)$ .  $\square$

Thanks to Lemma 6.9 we can put the following definition. We refer to Section 2.1 for the definition of foliations using 1-forms. Recall that  $\mathcal{F}_w$  is the horizontal foliation on  $\mathbb{C}^2$  given by the 1-form  $dw$ . We also recall that  $\text{IB}_R(p)$  is defined by (25).

**Definition 6.10.** *For every  $p \in \text{Reg}_{\sigma_R}(J)$  we denote  $\mathcal{F}_p$  the non singular foliation on  $V_p$  defined by one of the holomorphic 1-forms  $\omega_{j,p}$ . We note that  $\sigma|_{\text{IB}_R(p)}^* \mathcal{F}_p = \mathcal{F}_w$ .*

Finally one can reformulate the results of the present section as follows.

**Proposition 6.11.**

1. *If  $p \in \text{Reg}_{\sigma_R}(J)$ , then  $T_{p'}\mathcal{F}_p = \mathcal{D}_p(p')$  for every  $p' \in V_p$ .*
2.  *$\forall p, q \in \text{Reg}_{\sigma_R}(J)$ ,  $\mathcal{F}_p$  and  $\mathcal{F}_q$  coincide on every connected component  $\mathcal{C} \subset V_p \cap V_q$  such that  $\sigma_T(\mathcal{C}) > 0$  (use 1.,  $\mathcal{D}_p|_{\mathcal{C}} = \mathcal{D}_q|_{\mathcal{C}}$  by Prop. 6.8 and Lem. 2.1).*



### 6.3.3 Step 3 : Construction of the foliation near singular values

For every singular value  $q$  of  $\sigma_R$  which belongs to  $J$ , we construct an open neighborhood  $W_q \ni q$  and a foliation  $\mathcal{F}_q$  (possibly with singularities) on  $W_q$  which is tangent to directions  $(\mathcal{D}_p)_{p \in \text{Reg}_{\sigma_R}(J)}$  on neighborhoods of points of  $\text{Reg}_{\sigma_R}(J) \cap W_q$ , see Proposition 6.16. Thanks to this property  $\mathcal{F}_q$  will coincide with the regular foliations  $(\mathcal{F}_p)_{p \in \text{Reg}_{\sigma_R}(J)}$  on connected components of  $\bigcup_{p \in \text{Reg}_{\sigma_R}(J)} V_p \cap W_q$  which have positive measure by  $\sigma_T|_J$ , see Proposition 6.18. We finish the construction of the desired foliation in Section 6.3.4 using the compactness of  $J$ .

To create  $\mathcal{F}_q$  the idea is to construct, using  $d\sigma \cdot (1, 0)$ , a vector field on a neighborhood of  $q$ . Let us fix a singular value  $q \in J \cap E_R$ . We recall that  $\sigma(D_R) \supset J$  and according to Proposition 4.1, the fiber  $\sigma^{-1}(q)$  is discrete in  $\mathbb{C}^2$ . Thus we can consider a preimage  $y \in D_R$  and an open set  $U_y \subset \overline{U}_y \subset D_R$  containing  $y$  such that  $\overline{U}_y \cap \sigma^{-1}(q) = \{y\}$ . Let  $B_y$  be a centered ball at  $y$  such that  $\overline{B}_y \subset U_y$ .

#### Lemma 6.12.

1. *There exists a connected neighborhood  $U_{y,q} \subset B_y$  of  $y$ , such that  $W_q := \sigma(U_{y,q})$  is a connected neighborhood of  $q$  satisfying :*

$$\sigma^{-1}(p) \cap U_y \subset B_y, \quad \forall p \in W_q. \quad (27)$$

2. *Let  $p \in W_q \setminus E_R$  and let  $n_{p,q}$  be the cardinality of  $\sigma^{-1}(p) \cap U_y$ , it satisfies  $1 \leq n_{p,q} < +\infty$ . There exist a ball  $B_p \subset W_q \setminus E_R$  centered at  $p$  and a family of inverse branches  $\sigma_{1,p,q}^{-1}, \dots, \sigma_{n_{p,q},p,q}^{-1}$  such that for all  $p' \in B_p$  :*

$$\sigma^{-1}(p') \cap U_y = \left\{ \sigma_{1,p,q}^{-1}(p'), \dots, \sigma_{n_{p,q},p,q}^{-1}(p') \right\}. \quad (28)$$

#### Proof :

1. Let us assume to the contrary that for any neighborhood of  $q$  there exists a preimage by  $\sigma$  of a point in this neighborhood which belongs to  $U_y \setminus B_y$ . Then looking at smaller and smaller neighborhoods of  $q$  we obtain a sequence  $(p_n)_n$  of points converging to  $q$ , and a sequence  $(x_n)_n$  of  $U_y \setminus B_y$  such that  $\sigma(x_n) = p_n$ . Taking  $x \in \overline{U}_y \setminus B_y$  a cluster value of  $(x_n)_n$ , we have by continuity of  $\sigma$  that, up to a sub-sequence,  $\sigma(x) = \lim_n \sigma(x_n) = \lim_n p_n = q$ . Thus we have  $x \in \sigma^{-1}(q) \cap \overline{U}_y \setminus B_y$ , but this is a contradiction since  $\sigma^{-1}(q) \cap \overline{U}_y = \{y\} \subset B_y$ .

So we have proved the existence of a neighborhood  $W'_q$  of  $q$  such that for all  $p \in W'_q$ ,  $\sigma^{-1}(p) \cap U_y \subset B_y$ . Taking a small connected neighborhood  $U_{y,q}$  of  $y$  included in  $B_y$ , one has  $\sigma(U_{y,q}) \subset W'_q$ . We complete the proof by setting  $W_q := \sigma(U_{y,q})$ .

2. Observe first that because  $\sigma(B_y) \supset W_q$ , the fiber  $\sigma^{-1}(p) \cap U_y$  is not empty. As explained above the fibers of  $\sigma$  are discrete in  $\mathbb{C}^2$ , thus the fiber  $\sigma^{-1}(p) \cap U_y$  is finite. In particular we have  $1 \leq n_{p,q} < +\infty$ . Let  $(\sigma_{j,p,q})_{j=1, \dots, n_{p,q}}$  be a family of inverse branches of  $\sigma$  such that  $\sigma^{-1}(p) \cap U_y = \{\sigma_{j,p,q}^{-1}(p), 1 \leq j \leq n_{p,q}\}$ . Let us fix a ball  $B_p \subset W_q \setminus E_R$  centered at  $p$  such that all these inverse branches are defined on  $B_p$ . We denote  $\text{IB}(p, q) := \bigcup_{j=1}^{n_{p,q}} \sigma_{j,p,q}^{-1}(B_p)$ . Since  $\sigma^{-1}(p) \cap U_y \subset B_y$  by (27), up to a reduction of the radius of the ball  $B_p$ , we can assume that  $\text{IB}(p, q) \subset B_y$ . Observe that  $\text{IB}(p, q) \subset \sigma^{-1}(B_p) \cap U_y$  by construction. Using (27) and similar arguments involved in the previous item, we also have  $\sigma^{-1}(B_p) \cap U_y \subset \text{IB}(p, q)$ , up to reduce  $B_p$ . Finally, observe that  $\text{IB}(p, q) = \sigma^{-1}(B_p) \cap U_y$  is exactly (28).  $\square$

**Definition 6.13.** Let us fix  $q \in J \cap E_R$ . Let  $W_q$  and  $U_y$  be the open sets given by Lemma 6.12. We define a vector field  $v_q : W_q \setminus E_R \rightarrow T\mathbb{P}^2$  by :

$$\forall p \in W_q \setminus E_R, v_q(p) := \sum_{x \in \sigma^{-1}(p) \cap U_y} h(x) d_x \sigma \cdot (1, 0) \in T_p \mathbb{P}^2.$$

The function  $h \in \mathcal{O}(\mathbb{C}^2)$  is chosen to have  $v_q \neq 0$  on  $W_q \setminus E_R$ .

**Lemma 6.14.**

1. The vector field  $v_q : W_q \setminus E_R \rightarrow T\mathbb{P}^2$  is holomorphic.
2. The function  $h \in \mathcal{O}(\mathbb{C}^2)$  chosen such that  $v_q \neq 0$  exists. Moreover one can choose  $h$  equal to a polynomial function on  $\mathbb{C}^2$ .

**Proof :**

1. Let  $p \in W_q \setminus E_R$  fixed. Let  $(\sigma_{j,p,q}^{-1})_{j \in \{1, \dots, n_{p,q}\}}$  be the family of inverse branches defined on a ball  $B_p$  given by the second item of Lemma 6.12. According to (28) we have for any  $p' \in B_p$  :

$$v_q(p') = \sum_{j=1}^{n_{p,q}} (h \circ \sigma_{j,p,q}^{-1})(p') d_{\sigma_{j,p,q}^{-1}(p')} \sigma \cdot (1, 0). \quad (29)$$

This formula (29) shows that  $v_q$  is holomorphic on  $B_p$ . Since these arguments are valid for any  $p \in W_q \setminus E_R$ , the vector field  $v_q : W_q \setminus E_R \rightarrow T\mathbb{P}^2$  is holomorphic.

2. Let us fix  $p \in W_q \setminus E_R$  and let  $\Phi : B_p \rightarrow (\mathbb{C}^2)^{n_{p,q}}$  be defined by  $\Phi : p' \in B_p \mapsto (\sigma_{j,p,q}^{-1}(p'))_{1 \leq j \leq n_{p,q}}$ . Let us defined  $Z_{p,q} := \{(Z_j, W_j)_j \in (\mathbb{C}^2)^{n_{p,q}}, \exists i < j : Z_i = Z_j\}$ , there exists  $p' \in B_p \setminus \Phi^{-1}(Z_{p,q})$ . Let us write  $\Phi(p') = (x'_j)_j = (Z_j, W_j)_j$  and let  $\vec{V}_j := d_{x'_j} \sigma \cdot (1, 0)$ . Taking  $\lambda_1 := 1$  and  $\lambda_j := 0, j \geq 2$ , we have  $\sum_{j=1}^{n_{p,q}} \lambda_j \vec{V}_j = \vec{V}_1 \neq \vec{0}$ , since  $x'_1$  is not a critical point of  $\sigma$ . Since  $\Phi(p') \notin Z_{p,q}$ , there exists a polynomial function  $P(Z)$  such that  $P(Z_j) = \lambda_j$ . Let  $h(Z, W) := P(Z)$  on  $\mathbb{C}^2$ , we deduce from (29) that  $v_q(p') = \sum_{j=1}^{n_{p,q}} P(Z_j) \times \vec{V}_j = \vec{V}_1 \neq \vec{0}$ .  $\square$

We prove now that  $v_q$  extends holomorphically through  $E_R$ . We recall that  $E_R$  is included in an analytic subset of codimension 1 of  $\sigma(D_R)$  :

$$E_R \subset E'_R, \text{ with } E'_R := f^{n_R}(\text{Crit } f^{n_R}) \cap \sigma(D_R). \quad (30)$$

**Proposition 6.15.** The vector field  $v_q$  admits a unique holomorphic extension on  $W_q$ , still denoted  $v_q$ .

**Proof :** In this proof we extends  $v_q$  through the analytic subset  $E'_R$  defined by (30), it extends in particular  $v_q$  through  $E_R$ .

The content of the proposition is of local nature, thus we can assume that there exists  $\xi : W_q \rightarrow \xi(W_q) \subset \mathbb{C}^2$  a chart, holomorphic on a neighborhood of  $\overline{W}_q$ . We define  $v_* := \xi_* v_q$  on  $\xi(W_q \setminus E'_R) = \xi(W_q) \setminus F'$ , with  $F' := \xi(E'_R \cap W_q)$ . Let us write  $v_* = (\alpha, \beta)$ .

Let us fix  $p \in W_q \setminus E'_R$  and let us consider the ball  $B_p \subset W_q \setminus E_R$  we have constructed in Lemma 6.12. Up to reduce this ball we have  $B_p \subset W_q \setminus E'_R$ . For

$p' \in B_p$ ,  $v_*(\xi(p')) = \sum_{j=1}^{n_{p,q}} (h \circ \sigma_{j,p,q}^{-1}(p')) \times \left( d_{p'}\xi \circ d_{\sigma_{j,p,q}^{-1}(p')} \sigma \right) \cdot (1, 0)$  by (29), and  $n_{p,q} \leq d^{2n_R}$ . Thus we deduce that  $\|v_*\| \leq d^{2n_R} M_q$  on  $\xi(B_p) \setminus F'$ , where  $M_q := \sup_{D_R} |h| \times \sup_{W_q} \|d\xi\| \times \sup_{D_R} \|d\sigma \cdot (1, 0)\|$ .

$M_q$  is finite and independent of  $p$ , thus by Riemann extension theorem (cf. [19, Proposition 1.1.7]),  $\alpha$  and  $\beta$  admit unique extensions  $\tilde{\alpha}$  and  $\tilde{\beta}$  on  $\xi(W_q)$ . Finally, the vector field  $\xi^*(\tilde{\alpha}, \tilde{\beta})$  extends uniquely  $v_q$  on  $W_q$ .  $\square$

The interest of this vector field  $v_q$  is that for each  $p \in \text{Reg}_{\sigma_R}(J) \cap W_q$ , it is tangent to the distribution of directions  $\mathcal{D}_p$  on a neighborhood of  $p$  included in  $V_p \cap W_q$  :

**Lemma 6.16.** *Let  $p \in \text{Reg}_{\sigma_R}(J) \cap W_q$ . Let  $B_p$  be the ball constructed in Lemma 6.12. There exists a ball  $B_{p,q} \subset B_p \cap V_p \cap W_q$  centered at  $p$  such that :*

$$\forall t \in B_{p,q}, v_q(t) \in \mathcal{D}_p(t).$$

**Proof :** According to (29),  $v_q(p') = \sum_{j=1}^{n_{p,q}} (h \circ \sigma_{j,p,q}^{-1}(p')) d_{\sigma_{j,p,q}^{-1}(p')} \sigma \cdot (1, 0)$  for any  $p' \in B_p$ . Observe that, because  $p \in \text{Reg}_{\sigma_R}(J)$ , the inverse branches  $\sigma_{j,p,q}^{-1}$  which appear are some of those we have constructed in Section 6.3.1 for the regular value  $p$  of  $\sigma$ . So for any  $j \in \{1, \dots, n_{p,q}\}$  and any  $p' \in B_p$  we have  $\sigma_{j,p,q}^{-1}(p') \in \text{IB}_R(p)$ , we refer to Definition 6.7. Hence by definition of  $\mathcal{D}_p$  in (26), we have for any  $p' \in B_p \cap V_p$  and for any  $j \in \{1, \dots, n_{p,q}\}$ ,  $\left[ d_{\sigma_{j,p,q}^{-1}(p')} \sigma \cdot (1, 0) \right] = \mathcal{D}_p(p')$  and thus  $v_q(p') \in \mathcal{D}_p(p')$ . If  $B_{p,q}$  is a ball centered at  $p$  and included in  $B_p \cap V_p \cap W_q$ , the conclusion follows.  $\square$

Up to a reduction of the neighborhood  $U_{y,q}$  of  $y$  in Lemma 6.12, we can assume that  $W_q = \sigma(U_{y,q})$  is equipped with holomorphic coordinates. As explained in Section 2.1, we can use these local coordinates to create a non trivial holomorphic 1-form  $\omega_q$  on  $W_q$  satisfying  $\omega_q(v_q) \equiv 0$ .

**Definition 6.17.** *The (possibly singular) foliation induced by the 1-form  $\omega_q$  on  $W_q$  is denoted  $\mathcal{F}_q$ .*

Observe that the equation  $\omega_q(v_q) = 0$  ensures that the leaves of the foliation  $\mathcal{F}_q$  are tangent to the vector field  $v_q$ , at least where  $v_q \neq 0$ . We refer to Section 2.1 for more details on foliations and vector fields.

**Proposition 6.18.**

1. For any  $q \in J \cap E_R$  and for any  $p \in \text{Reg}_{\sigma_R}(J)$ , the foliations  $\mathcal{F}_p$  and  $\mathcal{F}_q$  coincide on each connected component  $\mathcal{C} \subset V_p \cap W_q$  such that  $\sigma_T(J \cap \mathcal{C}) > 0$ .
2. For any  $(q, q') \in (J \cap E_R)^2$ , the foliations  $\mathcal{F}_q$  and  $\mathcal{F}_{q'}$  coincide on each connected component  $\mathcal{C} \subset W_q \cap W_{q'}$  such that  $\sigma_T(J \cap \mathcal{C}) > 0$ .
3. For  $q \in J \cap E_R$ ,  $\sigma^* \mathcal{F}_q = \mathcal{F}_w$  on  $U_{y,q}$ , where  $U_{y,q}$  is defined in Lemma 6.12.

**Proof :**

1. Since  $\sigma_T(J \cap \mathcal{C}) > 0$  and  $\sigma_T(E_R) = 0$  (recall  $\sigma_T(E) = 0$ , cf. Lemma 4.1), there exists  $\tilde{p} \in (J \cap \mathcal{C}) \setminus E_R$ . Let  $\tilde{\mathcal{C}}$  be the connected component of  $\mathcal{C} \cap V_p \cap V_{\tilde{p}}$  containing  $\tilde{p}$ . Since  $\tilde{p} \in \text{Supp}(T)$  we have  $\sigma_T(\tilde{\mathcal{C}}) > 0$ . So according to Proposition 6.8, we have  $\mathcal{D}_p(t) = \mathcal{D}_{\tilde{p}}(t)$  for  $t \in \tilde{\mathcal{C}}$ . Moreover for  $t \in B_{\tilde{p},q} \setminus \{v_q = 0\}$ ,  $[v_q(t)] = \mathcal{D}_{\tilde{p}}(t)$  by Lemma

6.16. We deduce that  $T_t \mathcal{F}_q = [v_q(t)] = \mathcal{D}_{\tilde{p}}(t) = \mathcal{D}_p(t)$  for  $t \in \tilde{\mathcal{C}} \cap B_{\tilde{p},q} \setminus \{v_q = 0\}$ . But  $\mathcal{D}_p(t) = T_t \mathcal{F}_p$  for  $t \in V_p$ , thus we have  $T_t \mathcal{F}_q = T_t \mathcal{F}_p$  for all  $t \in \tilde{\mathcal{C}} \cap B_{\tilde{p},q} \setminus \{v_q = 0\} \subset \mathcal{C}$ . By using the second item and the third item of Lemma 2.1, we conclude that finally  $\mathcal{F}_q = \mathcal{F}_p$  on  $\mathcal{C}$ .

2. Let  $\mathcal{C}$  be a connected component of  $W_q \cap W_{q'}$  and let us assume that  $\sigma_T(J \cap \mathcal{C}) > 0$ . As in the preceding point, there exists  $p \in (\text{Supp}(\sigma_T|_J) \cap \mathcal{C}) \setminus E_R$ . The connected component  $\mathcal{C}'$  of  $\mathcal{C} \cap V_p$  containing  $p$  satisfies  $\sigma_T(J \cap \mathcal{C}') > 0$  since  $p \in \text{Supp}(\sigma_T|_J)$ . According to the previous item (applied first to  $q$  and  $p$  and then secondly to  $q'$  and  $p$ ) we have  $\mathcal{F}_q = \mathcal{F}_p = \mathcal{F}_{q'}$  on  $\mathcal{C}'$ . Using again Lemma 2.1 we have at last  $\mathcal{F}_q = \mathcal{F}_{q'}$  on  $\mathcal{C}$ .

3. We have  $\sigma_T(J \cap W_q) > 0$  since  $q \in J = \text{Supp}(\sigma_T|_J)$ . So there exists  $p \in \text{Reg}_{\sigma_R}(J) \cap W_q$ . Let  $B_{p,q}$  be the ball constructed in Lemma 6.16. Let  $\sigma_{j,p,q}^{-1}$  be an inverse branch of  $\sigma$  which appears in the formula (29) and such that  $x_j := \sigma_{j,p,q}^{-1}(p)$  belongs to  $\text{IB}_R(p) \cap U_{y,q}$ . Observe that  $\sigma_{j,p,q}^{-1}(p) \in \text{IB}_R(p)$  by definition of  $\text{IB}_R(p)$ . The fact that we can choose  $j$  such that  $x_j \in U_{y,q}$  comes from the fact that  $\sigma(U_{y,q}) = W_q$ , see Lemma 6.12.

According to the first item, the foliations  $\mathcal{F}_p$  and  $\mathcal{F}_q$  coincide on the ball  $B_{p,q} \subset V_p \cap W_q$  (since  $\sigma_T(J \cap B_{p,q}) > 0$  because  $p \in J = \text{Supp}(\sigma_T|_J)$ ). Thus we have  $\sigma^* \mathcal{F}_q = \sigma^* \mathcal{F}_p$  on  $\sigma^{-1}(B_{p,q})$ . But we have also  $\sigma^* \mathcal{F}_p = \mathcal{F}_w$  on  $\text{IB}_R(p)$ , see Definition 6.10. So we deduce that  $\sigma^* \mathcal{F}_q = \mathcal{F}_w$  on  $\sigma^{-1}(B_{p,q}) \cap \text{IB}_R(p)$ , and the point  $x_j$  belongs to  $\sigma^{-1}(B_{p,q}) \cap \text{IB}_R(p) \cap U_{y,q}$ . So  $\sigma^* \mathcal{F}_q = \mathcal{F}_w$  on  $U_{y,q}$  by analytic continuation (see Lemma 2.1).  $\square$

### 6.3.4 Step 4 : Finite covering of $J$ by foliated neighborhoods

We have constructed for each point  $p \in J$  a neighborhood  $\mathcal{O}_p$  of  $p$  ( $\mathcal{O}_p := V_p$  if  $p \in \text{Reg}_{\sigma_R}(J)$  or  $\mathcal{O}_p = W_p$  if  $p \in J \cap E_R$ ), equipped with a foliation  $\mathcal{F}_p$  on  $\mathcal{O}_p$ . Propositions 6.11 and 6.18 ensure that two foliations  $\mathcal{F}_p$  and  $\mathcal{F}_q$  of this collection coincide on  $\mathcal{O}_p \cap \mathcal{O}_q$  when the connected components of  $\mathcal{O}_p \cap \mathcal{O}_q$  are charged by  $\sigma_T$  or  $\sigma_T|_J$ . Let us modify the covering  $J \subset \cup_{p \in J} \mathcal{O}_p$  such that the two following properties hold : when two open sets of the covering intersect themselves, the foliations coincide on the intersection ; and every connected component of the covering contains an element of  $J$ . First, by an argument of Riemannian geometry, if we reduce the neighbourhoods  $\mathcal{O}_p$  sufficiently, they are connected and they satisfy the following property : if two sets  $\mathcal{O}_p$  and  $\mathcal{O}_q$  intersect themselves then  $\mathcal{O}_p \cap \mathcal{O}_q$  is also connected. Second, extract by compactness of  $J$  a finite covering  $J \subset \cup_{k=1}^L \mathcal{O}'_{p_k}$ , where  $\mathcal{O}'_{p_k}$  is a neighborhood of  $p_k$  such that  $\overline{\mathcal{O}'_{p_k}} \subset \mathcal{O}_{p_k}$ . Third, for each  $p_k$  define  $\tilde{\mathcal{O}}_{p_k}$  the open subset of  $\mathcal{O}'_{p_k}$  obtained by depriving the sets  $\mathcal{O}'_{p_k} \cap \mathcal{O}'_{p_l}$ ,  $1 \leq l \leq L$ , which do not intersect  $J$ . Finally, we assert that the following covering has the two desire properties :

$$J \subset \mathcal{V} := \tilde{\mathcal{V}} \setminus \{\text{connected components that do not contain elements of } J\} \quad (31)$$

where  $\tilde{\mathcal{V}} := \bigcup_{k=1}^L \widetilde{\mathcal{O}}_{p_k} = \left( \bigcup_{i=1}^N \tilde{V}_{p_i} \right) \cup \left( \bigcup_{j=1}^M \tilde{W}_{q_j} \right)$ . Indeed, if  $\tilde{\mathcal{O}}_{p_k} \cap \tilde{\mathcal{O}}_{p_l} \neq \emptyset$  it means that  $\overline{\mathcal{O}'_{p_k} \cap \mathcal{O}'_{p_l}} \cap J \neq \emptyset$ , and thus  $\sigma_T(J \cap \mathcal{O}_{p_k} \cap \mathcal{O}_{p_l}) > 0$  since  $J = \text{Supp}(\sigma_T|_J)$ . Using Proposition 6.8 or 6.18 we conclude that  $\mathcal{F}_{p_k} = \mathcal{F}_{p_l}$  on  $\mathcal{O}_{p_k} \cap \mathcal{O}_{p_l}$ , since this intersection is connected.

In particular, the foliations  $(\mathcal{F}_{p_i})_{i \in \{1, \dots, N\}}$  and  $(\mathcal{F}_{q_j})_{j \in \{1, \dots, M\}}$  patch all together on  $\tilde{\mathcal{V}}$  and form a (possibly singular) foliation :

**Definition 6.19.** *We denote*

$$\mathcal{F} := \left( \bigcup_{i=1}^N \mathcal{F}_{p_i} |_{\tilde{V}_{p_i}} \right) \cup \left( \bigcup_{j=1}^M \mathcal{F}_{q_j} |_{\tilde{W}_{q_j}} \right)$$

the foliation on the covering  $\tilde{\mathcal{V}}$  given by this patching.

To finish the proof of Theorem 6.2 it remains to show that  $\sigma^* \mathcal{F} = \mathcal{F}_w$  on  $\sigma^{-1}(\mathcal{V})$ . It is done in the next section and use the Proposition 6.20 below. Let us introduce an open set  $\mathcal{W} \subset \mathbb{C}^2$  such that  $\sigma(\mathcal{W}) = \tilde{\mathcal{V}}$  and on which we are able to compute  $\sigma^* \mathcal{F}$ . We denote for each  $p_i$  and  $q_j$  :

$$\tilde{\text{IB}}_R(p_i) := \text{IB}_R(p_i) \cap \sigma^{-1}(\tilde{V}_{p_i}) \quad \text{and} \quad \tilde{U}_{y_j, q_j} := U_{y_j, q_j} \cap \sigma^{-1}(\tilde{W}_{q_j}),$$

where  $U_{y_j, q_j}$  is defined in Lemma 6.12. The open set  $\mathcal{W}$  is then given by :

$$\mathcal{W} := \left( \bigcup_{i=1}^N \tilde{\text{IB}}_R(p_i) \right) \cup \left( \bigcup_{j=1}^M \tilde{U}_{y_j, q_j} \right).$$

For each  $i$  and  $j$  one has  $\sigma(\tilde{\text{IB}}_R(p_i)) = \tilde{V}_{p_i}$  and  $\sigma(\tilde{U}_{y_j, q_j}) = \tilde{W}_{q_j}$ , and thus

$$\sigma(\mathcal{W}) = \tilde{\mathcal{V}}. \quad (32)$$

By using Definition 6.10 and of the third item of Proposition 6.18, we observe that :

**Proposition 6.20.** *We have  $\sigma^* \mathcal{F} = \mathcal{F}_w$  on  $\mathcal{W}$ .*

### 6.3.5 Step 5 : The foliation is horizontal on $\sigma^{-1}(\mathcal{V})$

This is the last step to complete the proof of Theorem 6.2.

**Proposition 6.21.** *Let  $\mathcal{C}$  be a connected component of  $\sigma^{-1}(\mathcal{V})$ . Then there exists  $x \in \mathcal{C} \setminus (\text{Crit } \sigma)$  such that  $\sigma(x) \in J \setminus E_R$ .*

**Proof :** There exists  $R' \geq R$  large enough such that  $\mathcal{C} \cap \mathbb{D}_{R'}^2 \neq \emptyset$ . Assume  $R' = R$  for simplicity. The set  $\Sigma_R := D_R \cap \sigma^{-1}(E_R)$  is an analytic subset of  $D_R$  of codimension  $\geq 1$ , thus  $D_R \setminus \Sigma_R$  is connected and  $(\mathcal{C} \cap D_R) \setminus \Sigma_R \neq \emptyset$ . So if  $x_0 \in (\mathcal{C} \cap D_R) \setminus \Sigma_R$  is an arbitrary point, there exists  $U_R \subset D_R \setminus \Sigma_R$  an open connected set, relatively compact in  $D_R \setminus \Sigma_R$ , which contains  $x_0$ . The point  $\sigma(x_0)$  belongs to  $\mathcal{V}$ , let  $\mathcal{V}_0 \subset \mathcal{V}$  be the connected component containing  $\sigma(x_0)$ .

By construction (31) of  $\mathcal{V}$ , we have  $\mathcal{V}_0 \cap J \neq \emptyset$ . Since  $J = \text{Supp}(\sigma_T |_J)$  and since  $\sigma_T(E_R) = 0$ , we then deduce the existence of a point  $p \in \mathcal{V}_0 \cap J \setminus E_R$ . We can therefore increase  $U_R$  if necessary to ensure that  $\sigma(U_R) \cap \mathcal{V}_0$  contains a Lipschitz path  $\gamma : [0, 1] \rightarrow \sigma(U_R) \cap \mathcal{V}_0$  such that  $\gamma(0) = \sigma(x_0)$  and  $\gamma(1) = p$ . Then according to Corollary 4.4, there exists  $\tilde{\gamma} : [0, 1] \rightarrow U_R$  a continuous path such that  $\sigma \circ \tilde{\gamma} = \gamma$  and  $\tilde{\gamma}(0) = x_0$ . Since  $\tilde{\gamma}([0, 1]) \subset \sigma^{-1}(\mathcal{V})$  with  $\tilde{\gamma}(0) \in \mathcal{C}$ , and since  $\mathcal{C}$  is a connected component of  $\sigma^{-1}(\mathcal{V})$ , we must have  $\tilde{\gamma}([0, 1]) \subset \mathcal{C}$ . In particular, we have  $x := \tilde{\gamma}(1) \in \mathcal{C} \setminus (\text{Crit } \sigma)$  such that  $\sigma(x) = \gamma(1) = p \in J \setminus E_R$ .  $\square$

**Lemma 6.22.** *The foliation  $\sigma^*\mathcal{F}|_{\sigma^{-1}(\mathcal{V})}$  coincide with the horizontal foliation  $\mathcal{F}_w$ .*

**Proof :** Let  $\mathcal{C}$  be a connected component of  $\sigma^{-1}(\mathcal{V})$ . According to Proposition 6.21 there exists  $x \in \mathcal{C} \setminus (\text{Crit } \sigma)$  such that  $p := \sigma(x) \in J \setminus E_R$ . Thus there exists  $U_x \subset \mathcal{C}$  a connected open neighborhood of  $x$  such that  $\sigma_x := \sigma|_{U_x} : U_x \rightarrow \sigma(U_x) =: V_x$  is a biholomorphism. By (32) there exists  $y \in \mathcal{W}$  such that  $\sigma(y) = p$ . Since  $y \in D_R$  and since  $p$  is not a critical value of  $\sigma|_{D_R}$ ,  $y$  is not a critical value of  $\sigma$ . So let  $U_y \subset \mathcal{W}$  be an open neighborhood of  $y$  such that  $\sigma_y := \sigma|_{U_y} : U_y \rightarrow \sigma(U_y) =: V_y$  is a biholomorphism.

We can assume that  $V_x \subset V_y$  and  $\phi := \sigma_y^{-1} \circ \sigma_x : U_x \rightarrow \sigma_y^{-1}(V_x)$  is well defined. Observe that  $\sigma_T(V_x) > 0$  (since  $p \in J$ ) and that  $V_x$  is connected. So according to the first item of Proposition 6.8, the map  $\phi$  has the form  $\phi(z, w) = (A(z, w), B(w))$  and it preserves  $\mathcal{F}_w$ . Observe that  $\phi^*(\sigma_y^*\mathcal{F}|_{V_x}) = \sigma_x^*\mathcal{F}|_{V_x}$  by definition of  $\phi$ . Moreover, according to Proposition 6.20, we have  $\sigma_y^*(\mathcal{F}|_{V_y}) = \mathcal{F}_w|_{U_y}$ , thus we have  $(\sigma_x^*\mathcal{F})|_{U_x} = (\phi^*\mathcal{F}_w)|_{U_x} = \mathcal{F}_w|_{U_x}$ . We deduce that  $\sigma^*\mathcal{F} = \mathcal{F}_w$  on  $\mathcal{C}$  by analytic continuation (Lemma 2.1). Since it is true for any connected component  $\mathcal{C}$  the conclusion follows.  $\square$

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