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Séverin Benzoni, Emmanuel Roy, Thierry de la Rue

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Confined Poisson extensions

Séverin Benzoni, Emmanuel Roy, Thierry de la Rue

Abstract

This paper follows on from [3], where we introduced the notion of *confined extensions*, and our purpose is to widen the context in which such extensions appear. We do so in the setup of Poisson suspensions: we take a σ -finite measure-preserving dynamical system (X, μ, T) and a compact extension $(X \times G, \mu \otimes m_G, T_\varphi)$, then we consider the corresponding Poisson extension $((X \times G)^*, (\mu \otimes m_G)^*, (T_\varphi)_*) \longrightarrow (X^*, \mu^*, T_*)$. Our results give two different conditions under which that extension is confined. Finally, to show that those conditions are not void, we give an example of a system (X, μ, T) and a cocycle φ so that the compact extension $(X \times G, \mu \otimes m_G, T_\varphi)$ has an infinite ergodic index.

Contents

1	Introduction	2
1.1	Motivations	2
1.2	Basic notions and notation in ergodic theory	3
1.3	Joinings and confined extensions	4
1.4	Compact extensions and relative unique ergodicity	5
1.5	Poisson suspensions, splittings and extensions	6
2	A Poisson extension over a trivial cocycle	7
2.1	Confinement as a consequence of Poisson splittings	7
2.2	Marked Point processes	9
2.3	A $\mathfrak{S}(\mathbb{N})$ -valued cocycle and its action on $K^{\mathbb{N}}$	10
2.4	A non-confined Poisson extension	12
3	A Poisson suspension over a compact extension	14
3.1	Ergodicity of Cartesian products in spectral theory	14

3.2	Distinguishing points in a Poisson process	16
3.3	Relative unique ergodicity	19
3.4	Conclusion of the proof of Theorem 3.1	22
4	A compact extension of infinite ergodic index	23
4.1	Ergodic compact extensions	23
4.2	Description of the infinite Chacon transformation	25
4.3	Construction of the extension	26
4.3.1	If $\#I$ is odd	30
4.3.2	If $\#I$ is even	31

1 Introduction

1.1 Motivations

This paper investigates the concept of extensions of measure-preserving dynamical systems, specifically, extensions given by a factor map $\pi : (Z, \rho, R) \rightarrow (X, \mu, T)$. We mean that $\mathbf{Z} := (Z, \rho, R)$ and $\mathbf{X} := (X, \mu, T)$ are measure preserving dynamical systems such that \mathbf{X} is a factor of \mathbf{Z} via π , and conversely, we also view \mathbf{Z} as an extension of \mathbf{X} .

This paper is a continuation of the work done in [3]. There, we introduced the notion of confined extensions: they are extensions $(Z, \rho, R) \xrightarrow{\pi} (X, \mu, T)$ such that the only self-joining λ of (Z, ρ, R) in which the law of $\pi \times \pi$ is the product measure $\mu \otimes \mu$, is the product joining $\lambda = \rho \otimes \rho$ (see Definition 1.1).

This notion was first of interest to us in the study of dynamical filtrations, which are filtrations defined on some dynamical system (X, μ, T) of the form $\mathcal{F} := (\mathcal{F}_n)_{n \leq 0}$ such that each \mathcal{F}_n is T -invariant (see [3] for more details). But we also noticed other interesting results on confined extensions. For example, we listed properties \mathcal{P} that are lifted through confined extensions, i.e. if (X, μ, T) satisfies \mathcal{P} and $(Z, \rho, R) \xrightarrow{\pi} (X, \mu, T)$ is confined, then (Z, ρ, R) satisfies \mathcal{P} (see [3, Section 3.3]).

Since we noticed that confined extensions had many interesting properties, we look for examples in which that behavior arises. In [3], we considered extensions well known in the literature, namely, compact extensions and T, T^{-1} -transformations. In both cases, we gave necessary and sufficient conditions for those extensions to be confined.

In this paper, we give confinement results for a new kind of extension, in the setting of Poisson suspensions. Take (X, μ, T) a measure preserving dynamical

system where μ is a σ -finite measure, but assume that $\mu(X) = \infty$. Consider the probability space (X^*, μ^*) where X^* is the set of locally finite measures of the form $\sum_{i \geq 1} \delta_{x_i}$, with $(x_i)_{i \geq 1} \in X^{\mathbb{N}}$, and μ^* the law of the Poisson process of intensity μ . One can then define T_* on (X^*, μ^*) by applying T to each point of the point process. The resulting dynamical system (X^*, μ^*, T_*) is called the *Poisson suspension over (X, μ, T)* . A factor map $\pi : (Z, \rho, R) \rightarrow (X, \mu, T)$ between infinite measure systems will then yield a factor map between the Poisson suspensions: $\pi_* : (Z^*, \rho^*, R_*) \rightarrow (X^*, \mu^*, T_*)$. The resulting extension is what we call a *Poisson extension*.

We will consider the case where $\mathbf{Z} := (X \times G, \mu \otimes m_G, T_\varphi)$ is the compact extension given by a cocycle $\varphi : X \rightarrow G$, with G a compact group. Our results concern the following Poisson extension:

$$((X \times G)^*, (\mu \otimes m_G)^*, (T_\varphi)_*) \xrightarrow{\pi_*} (X^*, \mu^*, T_*), \quad (1)$$

with $\pi : (x, g) \mapsto x$.

In Section 2, we consider the case where $\varphi(x)$ acts as the identity map, for every $x \in X$. Using a splitting result from [7], we prove that in this case, if (X, μ, T) is of infinite ergodic index, the extension (1) is confined (see Theorem 2.2).

In Section 3, we deal with a more general cocycle φ . There, our argument will rely on the assumption that the compact extension $(X \times G, \mu \otimes m_G, T_\varphi)$ is of infinite ergodic index. In that case, we make use of Lemma 1.3, which is a well know result from Furstenberg. Through some intricate manipulations, we manage to reduce our problem to a relative unique ergodicity problem for products of the extension $\mathbf{Z} \xrightarrow{\pi} \mathbf{X}$, so that we can use Furstenberg's lemma (i.e. Lemma 1.3) to prove that (1) is confined (see Theorem 3.1).

Since the argument developed in Section 3 requires a compact extension $(X \times G, \mu \otimes m_G, T_\varphi)$ of infinite ergodic index, in Section 4, we give an example of such an extension, showing that Theorem 3.1 is not void.

1.2 Basic notions and notation in ergodic theory

A *dynamical system* is a quadruple $\mathbf{X} := (X, \mathcal{A}, \mu, T)$ such that (X, \mathcal{A}) is a standard Borel space, μ is a Borel measure which is σ -finite, i.e. there exist measurable sets $(X_n)_{n \geq 1}$ such that $\mu(X_n) < \infty$ and $X = \bigcup_{n \geq 1} X_n$, and T is an invertible measure-preserving transformation. Throughout the paper, we will often not specify the σ -algebra \mathcal{A} , and will write our dynamical systems as a triple of the form (X, μ, T) .

If we have two systems $\mathbf{X} := (X, \mu, T)$ and $\mathbf{Z} := (Z, \rho, R)$, a *factor map* is a measurable map $\pi : Z \rightarrow X$ such that $\pi_*\rho = \mu$ and $\pi \circ R = T \circ \pi$, ρ -almost surely. If such a map exists, we say that \mathbf{X} is a *factor* of \mathbf{Z} . Conversely, we also say that \mathbf{Z} is an extension of \mathbf{X} . Moreover, if there exist invariant sets $X_0 \subset X$ and $Z_0 \subset Z$ of full measure such that $\pi : Z_0 \rightarrow X_0$ is a bijection, then π is an *isomorphism* and we write $\mathbf{Z} \cong \mathbf{X}$.

The system (X, μ, T) is *ergodic* if $T^{-1}A = A$ implies that $\mu(A) = 0$ or $\mu(A^c) = 0$. It is *conservative* if there is no non-trivial set A such that the $\{T^n A\}_{n \in \mathbb{Z}}$ are disjoint. Let (X, μ, T) be a dynamical system with $\mu(X) = \infty$. The *ergodic index* of (X, μ, T) is the largest integer k such that $(X, \mu, T)^{\otimes k}$ is conservative and ergodic. If $(X, \mu, T)^{\otimes k}$ is ergodic for every integer k , the ergodic index is infinite.

Let (X, μ, T) be a conservative system. For $A \subset X$ measurable, we denote the restriction of μ to A by $\mu|_A := \mu(\cdot \cap A)$. Since the system is conservative, the return time $N_A(x) := \inf\{k \geq 1 \mid T^k x \in A\}$ is almost surely finite, allowing us to define the induced transformation

$$\begin{aligned} T|_A : A &\longrightarrow A \\ x &\longmapsto T^{N_A(x)} x \end{aligned} .$$

1.3 Joinings and confined extensions

Let $\mathbf{X} := (X, \mu, T)$ and $\mathbf{Y} := (Y, \nu, S)$ be two σ -finite measure preserving dynamical systems. A *joining* of \mathbf{X} and \mathbf{Y} is a $(T \times S)$ -invariant measure λ on $X \times Y$ whose marginals are μ and ν (therefore the marginals have to be σ -finite). It yields the dynamical system

$$\mathbf{X} \times_\lambda \mathbf{Y} := (X \times Y, \lambda, T \times S).$$

On this system, the coordinate projections are factor maps that project onto \mathbf{X} and \mathbf{Y} respectively. If it is not necessary to specify the measure, we will simply write $\mathbf{X} \times \mathbf{Y}$. For the product joining, we will use the notation $\mathbf{X} \otimes \mathbf{Y} := \mathbf{X} \times_{\mu \otimes \nu} \mathbf{Y}$. For the n -fold product self-joining, we will write $\mathbf{X}^{\otimes n}$.

We now give the definition of *confined extensions*, which concerns only probability measure preserving dynamical systems.

Definition 1.1. Let $\mathbf{X} := (X, \mu, T)$ and $\mathbf{Y} := (Y, \nu, S)$ be probability measure preserving dynamical systems, and $\pi : \mathbf{X} \rightarrow \mathbf{Y}$ be a factor map. The extension $\mathbf{X} \xrightarrow{\pi} \mathbf{Y}$ is said to be *confined* if it satisfies one of the following equivalent properties:

- (i) every 2-fold self-joining of \mathbf{X} in which the two copies of π are independent random variables is the product joining;
- (ii) for every system \mathbf{Z} , every joining of \mathbf{X} and \mathbf{Z} in which the copy of π and the projection on \mathbf{Z} are independent random variables is the product joining;
- (iii) for every $n \in \mathbb{N}^* \cup \{+\infty\}$, every n -fold self-joining of \mathbf{X} in which the n copies of π are mutually independent random variables is the n -fold product joining.

It was shown in [3, Proposition 3.3] that the definitions (i), (ii), and (iii) are equivalent. In this paper, we mainly use the definition (i). As we mentioned, this concerns only the case for probability measures. An adaptation to the infinite measure case would be more intricate, mainly because if we assume that a measure λ on $X \times X$ projects onto $\nu \otimes \nu$ on $Y \times Y$ and that ν is an infinite measure, then λ cannot be a joining of μ . That is because, in that case, both projections of λ on X would not be σ -finite.

1.4 Compact extensions and relative unique ergodicity

Definition 1.2. Let $(Z, \rho, R) \xrightarrow{\pi} (X, \mu, T)$ be an extension. It is relatively uniquely ergodic if the only R -invariant measure λ on Z such that $\pi_*\lambda = \mu$ is $\lambda = \rho$.

Let $\mathbf{X} := (X, \mu, T)$ be a measure preserving dynamical system, G a compact group and $\varphi : X \rightarrow G$ a cocycle. Let m_G denote the Haar probability measure on G . The compact extension of \mathbf{X} given by φ is the system \mathbf{Z} on $(X \times G, \mu \otimes m_G)$ given by the skew product

$$T_\varphi : \begin{array}{ccc} X \times G & \longrightarrow & X \times G \\ (x, g) & \longmapsto & (Tx, g \cdot \varphi(x)) \end{array} .$$

This is the most well known family of extensions. The only result we will need, is the following, due to Furstenberg:

Lemma 1.3 (Furstenberg [5]). Let $\mathbf{X} := (X, \mu, T)$ be an ergodic measure preserving dynamical system where μ is a finite or σ -finite measure. Assume that the compact extension $\mathbf{Z} = (X \times G, \mu \otimes m_G, T_\varphi)$ is ergodic. Let λ be a σ -finite T_φ -invariant measure on $X \times G$ such that $\lambda(\cdot \times G) = \mu$. Then

$$\lambda = \mu \otimes m_G.$$

This lemma is usually stated with μ a probability measure, but the infinite measure case is proven in the exact same way.

Furstenberg's lemma can be summarized by saying that an ergodic compact extension is relatively uniquely ergodic.

1.5 Poisson suspensions, splittings and extensions

Let $\mathbf{X} := (X, \mu, T)$ be a σ -finite measure preserving dynamical system. For convenience, we will assume that $X = \mathbb{R}^+$ and that μ is a locally finite measure, i.e. for any bounded set $B \subset \mathbb{R}^+$, we have $\mu(B) < \infty$. Let

$$X^* := \left\{ \text{locally finite measures of the form } \sum_{i \geq 1} \delta_{x_i} \right\}.$$

A point process is a probability measure on X^* . The Poisson point process of intensity μ , which we denote μ^* , is the point process characterized by the fact that, for $A_1, \dots, A_n \subset X$ measurable disjoint subsets such that $0 < \mu(A_i) < \infty$, the random variables $\omega(A_1), \dots, \omega(A_n)$, for $\omega \in X^*$, are independent Poisson random variables of respective parameter $\mu(A_i)$, for $i \in \llbracket 1, n \rrbracket$.

On the probability space (X^*, μ^*) , we define the transformation

$$T_* : \sum_{i \geq 1} \delta_{x_i} \mapsto \sum_{i \geq 1} \delta_{Tx_i}.$$

The resulting dynamical system $\mathbf{X}^* := (X^*, \mu^*, T_*)$ is called the *Poisson suspension over (X, μ, T)* .

It is well known that the Poisson suspension \mathbf{X}^* is ergodic if and only if there is no T -invariant measurable subset $A \subset X$ such that $0 < \mu(A) < \infty$ (see [8]). Moreover, this implies that if \mathbf{X}^* is ergodic, it is automatically weakly mixing. Also, note that it is not necessary that \mathbf{X} is ergodic for \mathbf{X}^* to be ergodic.

We use the notion of Poisson splittings from [7], but with different choices in the notations. A *splitting of order n* of the Poisson suspension (X^*, μ^*, T_*) is a family $\{\nu_i\}_{1 \leq i \leq n}$ of T_* -invariant probability measures on X^* and λ a $T_*^{\times n}$ -invariant joining of $\{\nu_i\}_{1 \leq i \leq n}$ such that $\Sigma_*^{(n)} \lambda = \mu^*$, where

$$\begin{aligned} \Sigma^{(n)} : X^* \times \dots \times X^* &\longrightarrow X^* \\ (\omega_1, \dots, \omega_n) &\longmapsto \omega_1 + \dots + \omega_n \end{aligned}$$

The splitting is said to be *ergodic* if λ is an ergodic joining. The splitting is a *Poisson splitting* if there exist $\{\mu_i\}_{1 \leq i \leq n}$, σ -finite measures on X such that, for

$i \in \llbracket 1, n \rrbracket$, $\nu_i = \mu_i^*$, and λ is the product measure $\mu_1^* \otimes \cdots \otimes \mu_n^*$. With that notation, the result [7, Theorem 2.6] becomes

Theorem 1.4. *Let $\mathbf{X} := (X, \mu, T)$ be a σ -finite measure preserving dynamical system of infinite ergodic index. Then any ergodic splitting of the Poisson suspension (X^*, μ^*, T_*) is a Poisson splitting.*

Consider two σ -finite systems $\mathbf{Z} := (Z, \rho, R)$ and $\mathbf{X} := (X, \mu, T)$ and a factor map $\pi : \mathbf{Z} \rightarrow \mathbf{X}$, which means that we have an extension $\mathbf{Z} \xrightarrow{\pi} \mathbf{X}$ of σ -finite systems. We can then define the map

$$\pi_* : \sum_{i \geq 1} \delta_{x_i} \mapsto \sum_{i \geq 1} \delta_{\pi(x_i)}.$$

One can check that this yields a factor map from \mathbf{Z}^* to \mathbf{X}^* , therefore we have defined an extension $\mathbf{Z}^* \xrightarrow{\pi_*} \mathbf{X}^*$ between Poisson suspensions. Such an extension is what we call a *Poisson suspension*.

2 A Poisson extension over a trivial cocycle

In this section, we study Poisson extensions over extensions of the form

$$\begin{aligned} T \times \text{Id} : X \times K &\longrightarrow X \times K \\ (x, \kappa) &\longmapsto (Tx, \kappa) \end{aligned}$$

on $(X \times K, \mu \otimes \rho)$, where K is a standard Borel space and ρ is a probability measure on K . We start in Section 2.1 by showing that if T has infinite ergodic index, the associated Poisson extension is confined. Then in Section 2.3, we see that marked point processes enable us to write Poisson extensions through a Rokhlin cocycle, and we give an application in probability theory by giving an alternative proof of the De Finetti theorem (see Corollary 2.4.1). Finally, in Section 2.4, we give an example of a non-confined Poisson extension.

2.1 Confinement as a consequence of Poisson splittings

We derive the content of this section as a consequence of Theorem 1.4. In [7], the authors proved Theorem 1.4 and gave an application of that splitting result (specifically, [7, Theorem 3.1]). Here, we note that it can be rephrased as a relative unique ergodicity result for the Poisson extension. In our notation, it becomes:

Theorem 2.1. *Let $\mathbf{X} := (X, \mu, T)$ be a σ -finite measure preserving dynamical system of infinite ergodic index, and K a standard Borel space. Let λ be an invariant marked point process over μ^* , i.e. a $(T \times \text{Id})_*$ -invariant probability measure on $(X \times K)^*$ such that $(\pi_*)_*\lambda = \mu^*$. If $(\lambda, (T \times \text{Id})_*)$ is ergodic, then there exists a probability measure ρ on K such that $\lambda = (\mu \otimes \rho)^*$.*

We deduce that the Poisson extension is confined:

Theorem 2.2. *Let $\mathbf{X} := (X, \mu, T)$ be a σ -finite measure preserving dynamical system of infinite ergodic index, and (K, ρ) a standard probability space. Then the Poisson extension*

$$((X \times K)^*, (\mu \otimes \rho)^*, (T \times \text{Id})_*) \longrightarrow (X^*, \mu^*, T_*),$$

is confined.

Proof. Set $\mathbf{Z} := ((X \times K)^*, (\mu \otimes \rho)^*, (T \times \text{Id})_*)$ and $\pi : (x, \kappa) \mapsto x$. Let λ be a 2-fold self joining of \mathbf{Z} such that $(\pi_* \times \pi_*)_*\lambda = \mu^* \otimes \mu^*$. Since \mathbf{Z}^* and \mathbf{X}^* are ergodic, and even weakly mixing (see Section 1.5 or [8]), up to taking an ergodic component, one can assume that λ is ergodic. Note that λ is a probability measure on

$$(X \times K)^* \times (X \times K)^*.$$

On this set, each projection on $(X \times K)^*$ yields a Poisson process. We view the realization of both of those processes simultaneously on $X \times K$ and we tag the points coming from the first coordinate with a 1, and the points coming from the second coordinate with a 2. To do that formally, we define the map

$$\Omega : (X \times K)^* \times (X \times K)^* \longrightarrow (X \times K \times \{1, 2\})^*,$$

so that

$$\Omega(\omega_1, \omega_2)(\cdot \times \{i\}) = \omega_i,$$

and consider $\eta := \Omega_*\lambda$. Then, η is $(T \times \text{Id}_{K \times \{1,2\}})$ -invariant, and, because $(\pi_* \times \pi_*)_*\lambda = \mu^* \otimes \mu^*$, we get $(\tilde{\pi}_*)_*\eta = (2\mu)^*$, where $\tilde{\pi} : (x, \kappa, i) \mapsto x$. Theorem 2.1 tells us that there exists $\chi \in \mathcal{P}(K \times \{1, 2\})$ such that $\eta = (2\mu \otimes \chi)^*$. Now we compute χ : let $A \subset X$ such that $0 < \mu(A) < \infty$, $B \subset K$ and $i \in \{1, 2\}$

$$\begin{aligned} e^{-2\mu(A)\chi(B \times \{i\})} &= \eta(\{\tilde{\omega}; \tilde{\omega}(A \times B \times \{i\}) = 0\}) \\ &= \lambda(\{(\omega_1, \omega_2); \omega_i(A \times B) = 0\}) \\ &= (\mu \otimes \rho)^*(\{\omega; \omega(A \times B) = 0\}) = e^{-\mu(A)\rho(B)}. \end{aligned}$$

So, $\chi(B \times \{i\}) = \frac{1}{2}\rho(B)$. Therefore $\chi = \rho \otimes \mathcal{B}(1/2, 1/2)$, so $\eta = (2\mu \otimes \rho \otimes \mathcal{B}(1/2, 1/2))^*$. Finally, we get

$$\begin{aligned}\lambda &= \Omega_*^{-1}\eta = \Omega_*^{-1}(2\mu \otimes \rho \otimes \mathcal{B}(1/2, 1/2))^* \\ &= (\mu \otimes \rho)^* \times (\mu \otimes \rho)^*.\end{aligned}$$

□

2.2 Marked Point processes

Let (X, μ) be a standard Borel space equipped with a σ -finite measure such that $\mu(X) = \infty$. Without loss of generality, we can assume that $X = \mathbb{R}_+$, thus enabling us to use the natural order on \mathbb{R}^+ , but any other order could be used here. We can also assume that μ is the Lebesgue measure on \mathbb{R}_+ (by doing so, we ignore the case where μ has atoms, but for the rest of our work, that is not a problem). Up to a set of μ^* -measure 0 we can assume that the elements ω of $(\mathbb{R}_+)^*$ are locally finite measures with no multiplicity, i.e. such that $\forall x \in \mathbb{R}_+, \omega(\{x\}) \leq 1$. This allows us to define a sequence $(t_n)_{n \in \mathbb{N}}$ of measurable maps from $(\mathbb{R}_+)^*$ to \mathbb{R}_+ such that

$$\omega = \sum_{n \geq 1} \delta_{t_n(\omega)},$$

and

$$0 \leq t_1(\omega) < t_2(\omega) < \dots.$$

Now consider a Polish space K . We will call a *marked point process over μ^** a probability measure λ on $(\mathbb{R}_+ \times K)^*$ such that $(\pi_*)_*\lambda = \mu^*$, where $\pi : (x, \kappa) \mapsto x$. We already manipulated marked point processes in the previous section, we are simply giving them a name now. We can describe marked point processes as follows: define the map

$$\begin{aligned}f : (\mathbb{R}_+)^* \times K^{\mathbb{N}} &\longrightarrow (\mathbb{R}_+ \times K)^* \\ (\omega, (\kappa_n)_{n \geq 1}) &\longmapsto \sum_{n \geq 1} \delta_{(t_n(\omega), \kappa_n)}.\end{aligned}$$

Since f is injective, we know that $f((\mathbb{R}_+)^* \times K^{\mathbb{N}})$ is a Borel set and f^{-1} is measurable, and we can write it as

$$\Phi := f^{-1} : \tilde{\omega} \mapsto (\pi_*(\tilde{\omega}), (\kappa_n(\tilde{\omega}))_{n \geq 1}), \quad (2)$$

where $(\kappa_n(\tilde{\omega}))_{n \geq 1}$ is called the sequence of the *marks* of $\tilde{\omega}$. For a marked point process λ , $\lambda(f((\mathbb{R}_+)^* \times K^{\mathbb{N}})) = 1$, therefore, up to a set of measure 0, f is a bijection. Moreover, we have the following result, from [4, Lemma 6.4.VI]:

Proposition 2.3. *Let ρ be a probability measure on K . The Poisson process $(\mu \otimes \rho)^*$ is a marked point process over μ^* and $f_*(\mu^* \otimes \rho^{\otimes \mathbb{N}}) = (\mu \otimes \rho)^*$.*

We can interpret this result as the fact that if $\tilde{\omega} \in (\mathbb{R}_+ \times K)^*$ is distributed according to the Poisson process of intensity $\mu \otimes \rho$, the sequence of marks $(\kappa_n(\tilde{\omega}))_{n \geq 1}$ is i.i.d. of law ρ .

2.3 A $\mathfrak{S}(\mathbb{N})$ -valued cocycle and its action on $K^{\mathbb{N}}$

In Section 2.2, we saw that, assuming that $X = \mathbb{R}_+$, a point process on $(X \times K)^*$ can be represented on $X^* \times K^{\mathbb{N}}$, via the map introduced in (2):

$$\begin{aligned} \Phi : (X \times K)^* &\longrightarrow X^* \times K^{\mathbb{N}} \\ \tilde{\omega} &\longmapsto (\pi_*(\tilde{\omega}), (\kappa_n(\tilde{\omega}))_{n \geq 1}) \end{aligned} .$$

Now we mean to determine the dynamic on $X^* \times K^{\mathbb{N}}$ that would correspond to $(T \times \text{Id})^*$ on $(X \times K)^*$. To do this, we will need a tool to track how the order of the points of $\omega \in X^*$ changes when T_* is applied.

The group $\mathfrak{S}(\mathbb{N})$ is the group of the permutations of \mathbb{N} , i.e. the bijections from \mathbb{N} onto itself. Equipped with the metric

$$d(\sigma, \tau) := \sum_{n \in \mathbb{N}} \frac{1}{2^n} \mathbb{1}_{\sigma(n) \neq \tau(n)},$$

it is a Polish group, i.e. $(\mathfrak{S}(\mathbb{N}), d)$ is a complete separable metric space and the map $(\sigma, \tau) \mapsto \sigma \circ \tau^{-1}$ is continuous. This group acts on $K^{\mathbb{N}}$ via the measurable action

$$(\sigma, (\kappa_n)_{n \geq 1}) \mapsto (\kappa_{\sigma^{-1}(n)})_{n \geq 1}. \quad (3)$$

We recall that the sequence $(t_n(\omega))_{n \geq 1}$ is the ordered sequence of the points of ω . To describe the action T_* on $(t_n(\omega))_{n \geq 1}$, define

$$\Psi : X^* \longrightarrow \mathfrak{S}(\mathbb{N}),$$

so that

$$T(t_n(\omega)) = t_{\Psi(\omega)(n)}(T_*\omega).$$

We consider the skew-product define by the cocycle Ψ :

$$\begin{aligned} (T_*)_{\Psi} : X^* \times K^{\mathbb{N}} &\longrightarrow X^* \times K^{\mathbb{N}} \\ (\omega, (\kappa_n)_{n \geq 1}) &\longmapsto (T_*\omega, (\kappa_{\Psi(\omega)^{-1}(n)})_{n \geq 1}) \end{aligned} .$$

Then we check that

$$\begin{aligned}
\Phi \circ (T \times \text{Id})_*(\tilde{\omega}) &= (\pi_*(T \times \text{Id})_*\tilde{\omega}, (\kappa_n((T \times \text{Id})_*\tilde{\omega}))_{n \geq 1}) \\
&= (T_*\pi_*\tilde{\omega}, (\kappa_{\Psi(\omega)^{-1}(n)}(\tilde{\omega}))_{n \geq 1}) \\
&= (T_*)_{\Psi}(\pi_*\tilde{\omega}, (\kappa_n(\tilde{\omega}))_{n \geq 1}) \\
&= (T_*)_{\Psi} \circ \Phi(\tilde{\omega}).
\end{aligned} \tag{4}$$

Combined with Proposition 2.3, this tells us that the extensions

$$((X \times K)^*, (\mu \otimes \rho)^*, (T \times \text{Id})_*) \longrightarrow (X^*, \mu^*, T_*),$$

and

$$(X^* \times K^{\mathbb{N}}, \mu^* \otimes \rho^{\otimes \mathbb{N}}, (T_*)_{\Psi}) \longrightarrow (X^*, \mu^*, T_*),$$

are isomorphic. Through this isomorphism, Theorem 2.1 becomes

Theorem 2.4. *Let $\mathbf{X} := (X, \mu, T)$ be a σ -finite measure preserving dynamical system of infinite ergodic index, and K a standard Borel space. Let λ be a $(T_*)_{\Psi}$ -invariant probability measure on $X^* \times K^{\mathbb{N}}$ such that $\lambda(\cdot \times K^{\mathbb{N}}) = \mu^*$. If $(\lambda, (T_*)_{\Psi})$ is ergodic, then there exists a probability measure ρ on K such that $\lambda = \mu^* \otimes \rho^{\otimes \mathbb{N}}$.*

As an unexpected corollary, we get the following result, which is the De Finetti theorem written in the language of ergodic theory:

Corollary 2.4.1 (De Finetti, Hewitt-Savage). *Let ρ_{∞} be a $\mathfrak{S}(\mathbb{N})$ -invariant (under the action defined by (3)) probability measure on $K^{\mathbb{N}}$ and ρ its marginal on the first coordinate. The action $(K^{\mathbb{N}}, \rho_{\infty}, \mathfrak{S}(\mathbb{N}))$ is ergodic if and only if $\rho_{\infty} = \rho^{\otimes \mathbb{N}}$.*

Proof. If $\rho_{\infty} = \rho^{\otimes \mathbb{N}}$, the action is ergodic because the shift transformation is in the closure of the automorphisms induced by $\mathfrak{S}(\mathbb{N})$.

Assume that $(K^{\mathbb{N}}, \rho_{\infty}, \mathfrak{S}(\mathbb{N}))$ is ergodic. Let (X, μ, T) be a dynamical system of infinite ergodic index. Since ρ_{∞} is $\mathfrak{S}(\mathbb{N})$ -invariant, it follows that $\mu^* \otimes \rho_{\infty}$ is $(T_*)_{\Psi}$ -invariant. Then Theorem 2.4 tells us that the ergodic decomposition of $\mu^* \otimes \rho_{\infty}$ is of the form

$$\mu^* \otimes \rho_{\infty} = \int_{\Gamma} \mu^* \otimes \gamma^{\otimes \mathbb{N}} d\mathbb{P}(\gamma) = \mu^* \otimes \int_{\Gamma} \gamma^{\otimes \mathbb{N}} d\mathbb{P}(\gamma),$$

so $\rho_{\infty} = \int_{\Gamma} \gamma^{\otimes \mathbb{N}} d\mathbb{P}(\gamma)$. However, each measure $\gamma^{\otimes \mathbb{N}}$ is $\mathfrak{S}(\mathbb{N})$ -invariant, and ρ_{∞} is ergodic under $\mathfrak{S}(\mathbb{N})$. Therefore, there exists $\gamma \in \Gamma$ such that $\rho_{\infty} = \gamma^{\otimes \mathbb{N}}$. \square

2.4 A non-confined Poisson extension

We give here an example of a non-trivial non-confined Poisson extension, to show that the infinite ergodic index assumption in Theorem 2.2 cannot be removed. Take $X := \mathbb{R}$, μ the Lebesgue measure on \mathbb{R} , and

$$T : x \mapsto x + 1.$$

The system (X, μ, T) is not ergodic and not conservative and its ergodic index is 0, but the Poisson suspension (X^*, μ^*, T_*) is ergodic, since it is Bernoulli. We get the following:

Proposition 2.5. *Let (K, ρ) be a standard probability space. The extension $((X \times K)^*, (\mu \otimes \rho)^*, (T \times \text{Id})_*) \rightarrow (X^*, \mu^*, T_*)$ is not confined.*

Proof. We will make use of the setup presented in the previous section for the study of marked point processes. We make some slight adjustments since now $X = \mathbb{R}$ (instead of \mathbb{R}_+): define a sequence $(t_n(\omega))_{n \in \mathbb{Z}}$ such that

$$\omega = \sum_{n \in \mathbb{Z}} \delta_{t_n(\omega)},$$

and

$$\cdots t_{-1}(\omega) < t_0(\omega) < 0 \leq t_1(\omega) < t_2(\omega) \cdots,$$

and a cocycle

$$\tilde{\Psi} : X^* \longrightarrow \mathfrak{S}(\mathbb{Z}),$$

so that

$$T(t_n(\omega)) = t_{\tilde{\Psi}(\omega)(n)}(T_*\omega).$$

In our present case, the map $\tilde{\Psi}$ can be described explicitly: denote the shift $S : k \mapsto k + 1$ and then one can check that

$$\tilde{\Psi} : \omega \mapsto \begin{cases} \text{Id} & \text{if } t_0(\omega) < -1 \\ S & \text{if } t_0(\omega) \geq -1 \end{cases}.$$

As in the previous section, we get an isomorphism in between the extensions

$$((X \times K)^*, (\mu \otimes \rho)^*, (T \times \text{Id})_*) \longrightarrow (X^*, \mu^*, T_*),$$

and

$$(X^* \times K^{\mathbb{Z}}, \mu^* \otimes \rho^{\otimes \mathbb{Z}}, (T_*)_{\tilde{\Psi}}) \longrightarrow (X^*, \mu^*, T_*).$$

We prove our proposition by showing that the second extension is not confined. We need to build a self-joining of $\mu^* \otimes \rho^{\otimes \mathbb{Z}}$. It will be more convenient to describe this joining as a measure on $X^* \times X^* \times K^{\mathbb{Z}} \times K^{\mathbb{Z}}$. We start with the marginal on $X^* \times X^*$ which has to be $\lambda(\cdot \times \cdot \times K^{\mathbb{Z}} \times K^{\mathbb{Z}}) = \mu^* \otimes \mu^*$, and for $(\omega_1, \omega_2) \in X^* \times X^*$, the conditional law $\lambda_{(\omega_1, \omega_2)}$ is as follows. The sequence of marks of ω_1 , $(\kappa_n(\omega_1))_{n \in \mathbb{Z}}$ is chosen with probability $\rho^{\otimes \mathbb{Z}}$. For the choice of $\kappa_n(\omega_2)$, we distinguish two situations:

- If $\omega_1([t_n(\omega_2), t_{n+1}(\omega_2)]) = 0$, we choose $\kappa_n(\omega_2)$ with law ρ .
- If $\omega_1([t_n(\omega_2), t_{n+1}(\omega_2)]) \geq 1$, we set

$$\ell := \min\{k \in \mathbb{Z} \mid t_k(\omega_1) \in [t_n(\omega_2), t_{n+1}(\omega_2)]\},$$

and then we choose $\kappa_n(\omega_2) := \kappa_\ell(\omega_1)$.

The construction of λ is concluded by taking

$$\lambda := \int \delta_{\omega_1} \otimes \delta_{\omega_2} \otimes \lambda_{(\omega_1, \omega_2)} d(\mu^* \otimes \mu^*)(\omega_1, \omega_2).$$

Note that our choices for $(\kappa_n(\omega_1))_{n \in \mathbb{Z}}$ and $(\kappa_n(\omega_2))_{n \in \mathbb{Z}}$ depend only on the relative positions of the points $\{t_n(\omega_1)\}_{n \in \mathbb{Z}}$ and $\{t_n(\omega_2)\}_{n \in \mathbb{Z}}$. Since those relative positions are preserved under application of $(T_* \times T_*)$, the measure λ is $(T_*)_{\tilde{\Psi}} \times (T_*)_{\tilde{\Psi}}$ -invariant (up to a permutation of coordinates).

From our construction, it is clear that λ is not a product measure and that

$$\lambda(\cdot \times X^* \times \cdot \times K^{\mathbb{Z}}) = \mu^* \otimes \rho^{\otimes \mathbb{Z}}.$$

We are left with checking that $\lambda(X^* \times \cdot \times K^{\mathbb{Z}} \times \cdot) = \mu^* \otimes \rho^{\otimes \mathbb{Z}}$. Consider that ω_1, ω_2 and $(\kappa_n(\omega_2))_{n < n_0}$ are known and compute the law of $\kappa_{n_0}(\omega_2)$: if $\omega_1([t_n(\omega_2), t_{n+1}(\omega_2)]) = 0$, it follows from our construction that the law of $\kappa_{n_0}(\omega_2)$ is ρ . If $\omega_1([t_n(\omega_2), t_{n+1}(\omega_2)]) \geq 1$, we have $\kappa_{n_0}(\omega_2) = \kappa_\ell(\omega_1)$ (see above for the definition of ℓ). One can check that $(\kappa_n(\omega_2))_{n < n_0}$ is measurable with respect to ω_1, ω_2 and $(\kappa_n(\omega_1))_{n < \ell}$ and $\kappa_\ell(\omega_1)$ is independent from ω_1, ω_2 and $(\kappa_n(\omega_1))_{n < \ell}$ and $\kappa_\ell(\omega_1)$. So, even with ω_1, ω_2 and $(\kappa_n(\omega_2))_{n < n_0}$ fixed, the law of $\kappa_\ell(\omega_1)$ is ρ , so the law of $\kappa_{n_0}(\omega_2)$ is also ρ .

To sum up, up to a permutation of coordinates, λ is a $(T_*)_{\tilde{\Psi}} \times (T_*)_{\tilde{\Psi}}$ -invariant measure on $X^* \times K^{\mathbb{Z}} \times X^* \times K^{\mathbb{Z}}$, such that

$$\lambda(\cdot \times \cdot \times X^* \times K^{\mathbb{Z}}) = \mu^* \otimes \rho^{\otimes \mathbb{Z}} \text{ and } \lambda(X^* \times K^{\mathbb{Z}} \times \cdot \times \cdot) = \mu^* \otimes \rho^{\otimes \mathbb{Z}}.$$

So it is a self-joining of $\mu^* \otimes \rho^{\otimes \mathbb{Z}}$. Moreover, it projects onto $\mu^* \otimes \mu^*$ without being equal to the product measure $\mu^* \otimes \rho^{\otimes \mathbb{Z}} \otimes \mu^* \otimes \rho^{\otimes \mathbb{Z}}$. This precisely means that the extension

$$(X^* \times K^{\mathbb{Z}}, \mu^* \otimes \rho^{\otimes \mathbb{Z}}, (T_*)_{\tilde{\Psi}}) \longrightarrow (X^*, \mu^*, T_*),$$

is not confined. □

3 A Poisson suspension over a compact extension

In this section, we are interested in the Poisson extension over a compact extension, i.e. over the system \mathbf{Z} given by

$$T_\varphi : \begin{array}{ccc} X \times G & \longrightarrow & X \times G \\ (x, g) & \longmapsto & (Tx, g \cdot \varphi(x)) \end{array},$$

for some measurable cocycle $\varphi : X \rightarrow G$. Our goal will be to show that

Theorem 3.1. *Let $\mathbf{X} := (X, \mu, T)$ be a dynamical system of infinite ergodic index. If the compact extension $(X \times G, \mu \otimes m_G, T_\varphi)$ is also of infinite ergodic index, then the Poisson extension*

$$((X \times G)^*, (\mu \otimes m_G)^*, (T_\varphi)_*) \xrightarrow{\pi_*} (X^*, \mu^*, T_*)$$

is confined.

We start with the Sections 3.1 and 3.2, where we introduce some useful notions and results from the literature. We then prove the main technical step in the proof of our theorem in Section 3.3. We conclude in Section 3.4.

3.1 Ergodicity of Cartesian products in spectral theory

We present briefly some results on the ergodicity of Cartesian products of σ -finite measure preserving dynamical systems.

We start by introducing some classic notions in spectral theory. Let (X, μ, T) be a σ -finite measure preserving dynamical system. Consider the space $L^2(X, \mu)$ and the operator

$$U_T : \begin{array}{ccc} L^2(X, \mu) & \longrightarrow & L^2(X, \mu) \\ f & \longmapsto & f \circ T \end{array}.$$

Denote $L_0^2(X, \mu) \subset L^2(X, \mu)$ the subspace orthogonal to the space of constant maps. For $f \in L^2(X, \mu)$, the spectral measure of f , σ_f , is defined as the measure on \mathbb{T} such that

$$\widehat{\sigma}_f(n) = \int_X f \overline{U_T^n f} d\mu.$$

There exists a finite measure σ_X^0 on \mathbb{T} , unique up to equivalence, such that $\forall f \in L_0^2(X, \mu)$, $\sigma_f \ll \sigma_X$ and for every finite measure σ such that $\sigma \ll \sigma_X$, there exists $f \in L_0^2(X, \mu)$ such that $\sigma = \sigma_f$. It is the *restricted maximal spectral type* of (X, μ, T) .

We define a L^∞ -eigenvalue as $\lambda \in \mathbb{C}$ such that there exist $f \in L^\infty(X, \mu)$ such that

$$f \circ T = \lambda f.$$

Such a map f is called a L^∞ -eigenfunction. Denote $e(T)$ the group of L^∞ -eigenvalues of (X, μ, T) , which is a sub-group of \mathbb{T} , provided T is conservative. The notion of L^∞ -eigenvalues is mainly useful in the infinite measure case. Indeed, if $\mu(X) < \infty$, we have $L^\infty(X, \mu) \subset L^2(X, \mu)$, so L^∞ -eigenfunctions are simply eigenvectors of the operator U_T .

We will use the following ergodicity criterion, due to Keane (see [1, Section 2.7]):

Theorem 3.2. *Let $\mathbf{X} := (X, \mu, T)$ be a conservative and ergodic dynamical system and $\mathbf{Y} := (Y, \nu, S)$ be an ergodic probability measure preserving dynamical system. The Cartesian product $\mathbf{X} \otimes \mathbf{Y}$ is ergodic if and only if $\sigma_Y^0(e(T)) = 0$.*

We use that criterion to prove

Corollary 3.2.1. *Let $\mathbf{X} := (X, \mu, T)$ be a conservative and ergodic dynamical system and $k \geq 1$. If $\mathbf{X}^{\otimes 2k}$ is ergodic, the product system*

$$(X, \mu, T)^{\otimes k} \otimes (X^*, \mu^*, T_*)$$

is ergodic.

Proof. Let $k \geq 1$. Assume that $\mathbf{X}^{\otimes 2k}$ is ergodic. We show that $e(T^{\times k}) = \{1\}$. Otherwise, take $\lambda \in e(T^{\times k}) \setminus \{1\}$ and $f \in L^\infty(X^k, \mu^{\otimes k})$ the associated eigenfunction. Define the tensor function as

$$f \otimes \bar{f}: X^k \times X^k \longrightarrow \mathbb{C} \\ (x_1, x_2) \longmapsto f(x_1) \overline{f(x_2)}.$$

We have

$$f \otimes \bar{f} \circ T^{\times 2k} = (f \circ T^{\times k}) \otimes \overline{(f \circ T^{\times k})} = \lambda \bar{\lambda} f \otimes \bar{f} = f \otimes \bar{f}.$$

Since $\mathbf{X}^{\otimes 2k}$ is ergodic, it yields that $f \otimes \bar{f}$ is constant, so that f is constant. However, this is incompatible with the fact that $\lambda \neq 1$. Therefore $e(T^{\times k}) = \{1\}$.

Moreover, since (X, μ, T) is ergodic, (X^*, μ^*, T_*) is as well, so $\sigma_{X^*}^0(\{1\}) = 0$. We have shown that $\sigma_{X^*}^0(e(T^{\times k})) = 0$, so Theorem 3.2 tells us that

$$(X, \mu, T)^{\otimes k} \otimes (X^*, \mu^*, T_*)$$

is ergodic. □

Remark 3.3. The result from Corollary 3.2.1 should be compared to the following result from Meyerovitch (see [9, Theorem 1.2]):

$$(X, \mu, T) \otimes (X^*, \mu^*, T_*) \text{ is ergodic if and only if } (X, \mu, T) \text{ is ergodic.}$$

This gives the result of Corollary 3.2.1 in the case $k = 1$, but with a weaker condition: we need (X, μ, T) to be ergodic, instead of $(X, \mu, T)^{\otimes 2}$. We conjecture that one could extend the result from Meyerovitch and get the conclusion of Corollary 3.2.1 under the weaker assumption that $(X, \mu, T)^{\otimes k}$ is ergodic.

3.2 Distinguishing points in a Poisson process

The purpose of this section is to study the map

$$\begin{aligned} \tilde{\Phi}_k : \quad X^k \times X^* &\longrightarrow X^* \\ (x_1, \dots, x_k, \omega) &\longmapsto \delta_{x_1} + \dots + \delta_{x_k} + \omega \end{aligned}$$

We view the points of $X^k \times X^*$ as a Poisson process for which the first k points are distinguished, so that we can track each of them individually. To avoid any multiplicity on the right-hand term, we will study this map on a smaller set $X^{(k)} \subset X^k \times X^*$, defined as

$$X^{(k)} := \{(x_1, \dots, x_k, \omega) \in X^k \times X^* \mid x_1 < \dots < x_k < t_1(\omega)\}.$$

From now on, Φ_k denotes the restriction of $\tilde{\Phi}_k$ to $X^{(k)}$. We start by computing the measure of $X^{(k)}$, using the fact t_1 follows an exponential law of parameter 1:

$$\begin{aligned}\mu^{\otimes k} \times \mu^*(X^{(k)}) &= \int_{X^*} \int_{(\mathbb{R}_+)^k} \mathbb{1}_{x_1 < \dots < x_k < t_1(\omega)} d\mu(x_1) \cdots d\mu(x_k) d\mu^*(\omega) \\ &= \int_{\mathbb{R}_+} \int_{(\mathbb{R}_+)^k} \mathbb{1}_{x_1 < \dots < x_k < t} d\mu(x_1) \cdots d\mu(x_k) e^{-t} dt \\ &= \int_{\mathbb{R}_+} \frac{t^k}{k!} e^{-t} dt = 1,\end{aligned}$$

the last equality being obtain through k successive integrations by parts. We complete this with the following result

Proposition 3.4. *Let $k \geq 1$. The map Φ_k sends $(\mu^{\otimes k} \otimes \mu^*)|_{X^{(k)}}$ onto μ^* . Therefore*

$$\Phi_k : (X^{(k)}, (\mu^{\otimes k} \otimes \mu^*)|_{X^{(k)}}) \longrightarrow (X^*, \mu^*),$$

is an isomorphism of probability spaces.

Proof. It is clear that Φ_k is a bijection whose inverse is

$$\omega \mapsto (t_1(\omega), \dots, t_k(\omega), \omega - (\delta_{t_1(\omega)} + \dots + \delta_{t_k(\omega)})).$$

We then need to prove that Φ_k is measure-preserving. We prove that result by induction on k . The case $k = 1$ can be found in [9, Proposition 6.1], but we give a proof for completeness. Denote Exp the law of an exponential variable of parameter 1.

To prove that $(\Phi_1)_*(\mu \otimes \mu^*)|_{X^{(1)}} = \mu^*$, we need to show that, if (x, ω) is chosen under $(\mu \otimes \mu^*)|_{X^{(1)}}$, the sequence

$$(x, t_1(\omega) - x, (t_{i+1}(\omega) - t_i(\omega))_{i \geq 1})$$

is i.i.d. of law Exp . First, we know that $(t_{i+1} - t_i)_{i \geq 1}$ is i.i.d. of law Exp . It is also clear that $(x, t_1 - x)$ is independent from $(t_{i+1} - t_i)_{i \geq 1}$. Therefore, we now only have to compute the law of $(x, t_1 - x)$ under $(\mu \otimes \mu^*)|_{X^{(1)}}$. Let $A, B \subset \mathbb{R}_+$ be measurable sets. We have:

$$\begin{aligned}\mu \otimes \mu^*(x < t_1, x \in A, t_1 - x \in B) &= \int_A \mu^*(x < t_1, t_1 - x \in B) d\mu(x) \\ &= \int_A \underbrace{\mu^*(x < t_1)}_{=e^{-x}} \underbrace{\mu^*(t_1 - x \in B | t_1 > x)}_{=\mu^*(t_1 \in B)} d\mu(x) \\ &= \int_A e^{-x} d\mu(x) \mu^*(t_1 \in B) = \text{Exp}(A) \cdot \text{Exp}(B),\end{aligned}$$

where we use the fact that the law of t_1 is Exp, and the loss of memory property of Exp.

Let $k \geq 1$, and assume that the result is true for k . We start by noting that $\Phi_{k+1} = \Phi_1 \circ (\text{Id} \times \Phi_k)$ and use the induction hypothesis to prove that

$$(\text{Id} \times \Phi_k)_*(\mu^{\otimes k+1} \otimes \mu^*)|_{X^{(k+1)}} = (\mu \otimes \mu^*)|_{X^{(1)}}. \quad (5)$$

Indeed, for a measurable map, $F : X^{(1)} \rightarrow \mathbb{R}$, we have

$$\begin{aligned} & \int_{X^{(k+1)}} F(x_1, \delta_{x_2} + \cdots + \delta_{k+1} + \omega) d\mu^{\otimes k+1}(x_1, \dots, x_{k+1}) d\mu^*(\omega) \\ &= \int_{\mathbb{R}_+} \int_{X^{(k)}} \mathbb{1}_{x_1 < x_2} F(x_1, \sum_{i=2}^{k+1} \delta_{x_i} + \omega) d\mu^{\otimes k}(x_2, \dots, x_{k+1}) d\mu^*(\omega) d\mu(x_1) \\ &= \int_{\mathbb{R}_+} \int_{X^*} \mathbb{1}_{x_1 < t_1(\omega)} F(x_1, \omega) d\mu^*(\omega) d\mu(x_1) \\ &= \int_{X^{(1)}} F d\mu \otimes \mu^*, \end{aligned}$$

where we used the induction hypothesis and the fact that $x_2 = t_1(\delta_{x_2} + \cdots + \delta_{k+1} + \omega)$. Therefore (5) is proven. We then combine it with the result for $k = 1$ to conclude that

$$(\Phi_{k+1})_*(\mu^{\otimes k+1} \otimes \mu^*)|_{X^{(k+1)}} = (\Phi_1)_*(\mu \otimes \mu^*)|_{X^{(1)}} = \mu^*.$$

□

We now want to study how Φ_k matches the dynamics on $X^{(k)}$ and X^* . We recall that we defined $\Psi : (\mathbb{R}_+)^* \rightarrow \mathfrak{S}(\mathbb{N})$ so that $T(t_n(\omega)) = t_{\Psi(\omega)(n)}(T_*\omega)$. We then iterate it to define

$$\Psi_p(\omega) := \Psi(T_*^{p-1}\omega) \circ \cdots \circ \Psi(\omega).$$

Now consider

$$N^{(k)}(\omega) := \inf\{p \geq 1 \mid \Psi_p(\omega)(1) = 1, \dots, \Psi_p(\omega)(k) = k\}.$$

This is the first time in which the first k points of ω are back to being the first k points of $T_*^p\omega$ and in their original order. If the random time $N^{(k)}$ is almost surely finite, we can define the automorphism $T^{N^{(k)}}$ on (X^*, μ^*) by

$$\left(T_*^{N^{(k)}}\right)(\omega) := T_*^{N^{(k)}(\omega)}(\omega).$$

We conclude this section with the following result:

Theorem 3.5. *Let $\mathbf{X} := (X, \mu, T)$ be a σ -finite measure preserving dynamical system. Assume that T has infinite ergodic index. Then $N^{(k)}$ is μ^* -almost surely well defined and Φ_k is an isomorphism between the systems*

$$(X^{(k)}, (\mu^{\otimes k} \otimes \mu^*)|_{X^{(k)}}), (T^{\times k} \times T_*)|_{X^{(k)}}$$

and

$$(X^*, \mu^*, T_*^{N^{(k)}}).$$

Proof. Let $k \geq 1$. Since $\mathbf{X} = (X, \mu, T)$ is of infinite ergodic index, the system $\mathbf{X}^{\otimes k} = (X^k, \mu^{\otimes k}, T^{\times k})$ is conservative and ergodic. Since μ^* is a probability measure, one can check that the system $(X^k \times X^*, \mu^{\otimes k} \otimes \mu^*, T^{\times k} \times T_*)$ is also conservative. Therefore, the induced system

$$(X^{(k)}, (\mu^{\otimes k} \otimes \mu^*)|_{X^{(k)}}), (T^{\times k} \times T_*)|_{X^{(k)}}$$

is well defined. Moreover, if $M^{(k)}$ is the first return time in $X^{(k)}$, then $M^{(k)}$ is $(\mu^{\otimes k} \otimes \mu^*)|_{X^{(k)}}$ -almost surely finite. However, since we have

$$\tilde{\Phi}_k \circ (T^{\times k} \times T_*) = T_* \circ \tilde{\Phi}_k, \quad (6)$$

one can check that on $X^{(k)}$, we have

$$N^{(k)} = M^{(k)} \circ \Phi_k. \quad (7)$$

So, because Proposition 3.4 tells us that $(\Phi_k)_*(\mu^{\otimes k} \otimes \mu^*)|_{X^{(k)}} = \mu^*$, we deduce that $N^{(k)}$ is μ^* -almost surely finite. Finally, by combining (6) and (7), one gets

$$\Phi_k \circ (T^{\times k} \times T_*)|_{X^{(k)}} = T_*^{N^{(k)}} \circ \Phi_k.$$

Since Proposition 3.4 tells us that Φ_k is a bijection for which $(\Phi_k)_*(\mu^{\otimes k} \otimes \mu^*)|_{X^{(k)}} = \mu^*$, we have shown that it yields the desired isomorphism of dynamical systems. \square

3.3 Relative unique ergodicity

The main step in proving Theorem 3.1 is the following relative unique ergodicity result:

Theorem 3.6. *Let $\mathbf{X} := (X, \mu, T)$ be a dynamical system of infinite ergodic index. If the compact extension $(X \times G, \mu \otimes m_G, T_\varphi)$ is also of infinite ergodic index, then the only $(T_\varphi)_*$ -invariant measure $\rho \in \mathcal{P}((X \times G)^*)$ such that $(\pi_*)_*\rho = \mu^*$ is $\rho = (\mu \otimes m_G)^*$.*

Similarly to what we did in Section 2.3, we represent the Poisson extension $((X \times G)^*, (\mu \otimes m_G)^*, (T_\varphi)_*) \xrightarrow{\pi_*} (X^*, \mu^*, T_*)$ through a Rokhlin cocycle. We do this using the representation of $((X \times G)^*, (\mu \otimes m_G)^*)$ as a marked point process given in Proposition 2.3.

We start by introducing the group $G^{\mathbb{N}} \times \mathfrak{S}(\mathbb{N})$ whose law we define by

$$((h_n)_{n \geq 1}, \tau) \cdot ((g_n)_{n \geq 1}, \sigma) = ((h_{\sigma(n)} \cdot g_n)_{n \geq 1}, \tau \circ \sigma).$$

This group acts on $(G^{\mathbb{N}}, m_G^{\otimes \mathbb{N}})$ via the maps

$$\chi_{(g_n)_{n \geq 1}, \sigma}((h_n)_{n \geq 1}) := (g_{\sigma^{-1}(n)} \cdot h_{\sigma^{-1}(n)})_{n \geq 1}.$$

Then we define the cocycle from X^* to $G^{\mathbb{N}} \times \mathfrak{S}(\mathbb{N})$ by

$$\bar{\varphi} : \omega \mapsto (\varphi(t_n(\omega))_{n \geq 1}, \Psi(\omega)).$$

This cocycle induces the following transformation

$$(T_*)_{\bar{\varphi}} : \begin{array}{ccc} X^* \times G^{\mathbb{N}} & \longrightarrow & X^* \times G^{\mathbb{N}} \\ (\omega, (g_n)_{n \geq 1}) & \longmapsto & (T_*\omega, \chi_{\bar{\varphi}(\omega)}((g_n)_{n \geq 1})) \end{array}.$$

Then, for a $(T_\varphi)_*$ -invariant measure ρ such that $(\pi_*)_*\rho = \mu^*$, an adaptation of the computation (4) show that the map Φ introduced in (2) gives an isomorphism between the extensions

$$((X \times G)^*, \rho, (T_\varphi)_*) \xrightarrow{\pi_*} (X^*, \mu^*, T_*),$$

and

$$(X^* \times G^{\mathbb{N}}, \Phi_*\rho, (T_*)_{\bar{\varphi}}) \longrightarrow (X^*, \mu^*, T_*).$$

Therefore, to prove Theorem 3.6, we need to take a $(T_*)_{\bar{\varphi}}$ -invariant measure λ such that $\lambda(\cdot \times G^{\mathbb{N}}) = \mu^*$ and show that $\lambda = \mu^* \otimes m_G^{\otimes \mathbb{N}}$. This is what we do below:

Proof of Theorem 3.6. Let λ be a $(T_*)_{\bar{\varphi}}$ -invariant measure on $X^* \times G^{\mathbb{N}}$ such that $\lambda(\cdot \times G^{\mathbb{N}}) = \mu^*$. Fix $k \geq 1$ and set λ_k as the image of λ via p_k , the projection on $X^* \times G^k$. The main idea of this proof will be to use Theorem 3.5 to distinguish the points $t_1(\omega), \dots, t_k(\omega)$ since they determine the action of $(T_*)_{\bar{\varphi}}$ on g_1, \dots, g_k and we mean to then view $(t_1(\omega), g_1), \dots, (t_k(\omega), g_k)$ as a compact extension of $t_1(\omega), \dots, t_k(\omega)$ to which we can apply Furstenberg's relative unique ergodicity Lemma.

We start our argument by understanding better the dynamics on g_1, \dots, g_k . Since (X, μ, T) has infinite ergodic index, Theorem 3.5 tells us that

$$\tilde{N}^{(k)}(\omega, (g_n)_{n \geq 1}) := N^{(k)}(\omega)$$

is λ -almost-surely finite. Now note that, by definition of $N^{(k)}$, we have

$$\begin{aligned} p_k \circ (T_*)_{\tilde{\varphi}}^{\tilde{N}^{(k)}}(\omega, (g_n)_{n \geq 1}) &= (T_*^{N^{(k)}(\omega)} \omega, \varphi^{(N^{(k)}(\omega))}(t_1(\omega)) \cdot g_1, \dots, \varphi^{(N^{(k)}(\omega))}(t_k(\omega)) \cdot g_k) \\ &= (T_*^{N^{(k)}(\omega)} \omega, \varphi_k(\omega) \cdot (g_1, \dots, g_k)), \end{aligned}$$

where we define φ_k as a cocycle taking values in G^k via the formula:

$$\varphi_k(\omega) := \varphi^{(N^{(k)}(\omega))}(t_1(\omega)), \dots, \varphi^{(N^{(k)}(\omega))}(t_k(\omega)),$$

and

$$\varphi^{(p)}(x) := \varphi(T^{p-1}x) \cdots \varphi(x).$$

Therefore λ_k is invariant under the transformation

$$(T_*^{N^{(k)}})_{\varphi_k} : \quad X^* \times G^k \quad \longrightarrow \quad X^* \times G^k \\ (\omega, (g_1, \dots, g_k)) \longmapsto ((T_*)^{N^{(k)}(\omega)} \omega, \varphi_k(\omega) \cdot (g_1, \dots, g_k)).$$

This map yields a compact extension of $(T_*)^{N^{(k)}}$, but to apply Furstenberg's Lemma (i.e. Lemma 1.3), we still have to prove that

$$(X^* \times G^k, \mu^* \otimes m_G^{\otimes k}, (T_*^{N^{(k)}})_{\varphi_k}) \quad (8)$$

is ergodic.

We recall that $M^{(k)}$ is defined as the return time on $X^{(k)}$ and that $M^{(k)} \circ \Phi_k = N^{(k)}$. Then, Theorem 3.5 tells us that $(X^* \times G^k, \mu^* \otimes m_G^{\otimes k}, (T_*^{N^{(k)}})_{\varphi_k})$ is isomorphic to

$$\left(X^{(k)} \times G^k, (\mu^{\otimes k} \otimes \mu^*) \Big|_{X^{(k)}} \otimes m_G^{\otimes k}, ((T^{\times k} \times T_*) \Big|_{X^{(k)}})_{\widehat{\varphi}_k} \right), \quad (9)$$

where $\widehat{\varphi}_k$ is the cocycle defined by

$$\widehat{\varphi}_k := \varphi_k \circ \Phi_k = (\varphi^{(M^{(k)})}(x_1), \dots, \varphi^{(M^{(k)})}(x_k)).$$

However, it is straightforward to check that, up to a permutation of coordinates, (9) is an induced system of

$$(X^k \times G^k \times X^*, \mu^{\otimes k} \otimes m_G^{\otimes k} \otimes \mu^*, T_\varphi^{\times k} \times T_*),$$

which can be written as

$$(X \times G, \mu \otimes m_G, T_\varphi)^{\otimes k} \otimes (X^*, \mu^*, T_*). \quad (10)$$

However, this is a factor of

$$(X \times G, \mu \otimes m_G, T_\varphi)^{\otimes k} \otimes ((X \times G)^*, (\mu \otimes m_G)^*, (T_\varphi)_*). \quad (11)$$

But, since $(X \times G, \mu \otimes m_G, T_\varphi)$ is of infinite ergodic index, we can apply Corollary 3.2.1 which tells us that (11) is ergodic, and therefore (10) is as well. Since an induced system on an ergodic system is also ergodic, this yields that (8) is ergodic. In conclusion, Furstenberg's Lemma implies that $\lambda_k = \mu^* \otimes m_G^{\otimes k}$.

This being true for every $k \geq 1$, it follows that $\lambda = \mu^* \otimes m_G^{\otimes \mathbb{N}}$. \square

3.4 Conclusion of the proof of Theorem 3.1

We now finish the proof of Theorem 3.1. This will be done by combining our relative unique ergodicity result from the previous section with Theorem 1.4. In our application of the splitting result (Theorem 1.4), the fact that marginals $\{\nu_i\}_{i \in [1, n]}$ are Poisson measures will already be known, and the important part will be the fact that the associated joining λ is the product joining.

Proof of Theorem 3.1. Let $\mathbf{X} := (X, \mu, T)$ be a dynamical system, and $\varphi : X \rightarrow G$ a cocycle so that the compact extension $\mathbf{Z} := (X \times G, \mu \otimes m_G, T_\varphi)$ has infinite ergodic index.

Let λ be a $(T_\varphi)_* \times (T_\varphi)_*$ -invariant self-joining of $(\mu \otimes m_G)^*$ such that

$$(\pi_* \times \pi_*)_* \lambda = \mu^* \otimes \mu^*. \quad (12)$$

Since, as mentioned in Section 1.5, (X^*, μ^*, T_*) is weakly mixing, up to replacing λ with one of its ergodic components, we can assume that the system

$$((X \times G)^* \times (X \times G)^*, \lambda, (T_\varphi)_* \times (T_\varphi)_*),$$

is ergodic. Now set $\rho := \Sigma_* \lambda$. We then use (12) to compute

$$(\pi_*)_* \rho = (\pi_*)_* \Sigma_* \lambda = \Sigma_* (\pi_* \times \pi_*)_* \lambda = \Sigma_* (\mu^* \otimes \mu^*) = (2\mu)^*.$$

In other words, (12) means that the projection of ρ on X^* is the sum of to independent Poisson point processes of intensity μ , and the result of this sum is a Poisson point process of intensity 2μ . Now, since \mathbf{X} is rigidity-free and T_φ has infinite ergodic index, we can apply Theorem 3.6 to conclude that $\rho = (2\mu \otimes m_G)^*$.

Using again the fact that T_φ has infinite ergodic index, we can now deduce from Theorem 1.4 that λ is the product joining

$$\lambda = (\mu \otimes G)^* \times (\mu \otimes G)^*.$$

□

4 A compact extension of infinite ergodic index

The construction in Theorem 3.1 relies on a compact extension

$$(X \times G, \mu \otimes m_G, T_\varphi),$$

which is of infinite ergodic index. In this section, we build a compact extension that has that property, which shows that Theorem 3.1 is not void. We start with Section 4.1, where we give a criterion for the ergodicity of compact extensions. Then, in Section 4.2, we choose a suitable system (X, μ, T) : the infinite Chacon transformation (see [2, Section 2]). Finally, in Section 4.3, we describe our choice for the cocycle φ and prove that the resulting transformation T_φ is of infinite ergodic index.

4.1 Ergodic compact extensions

Let $\mathbf{X} := (X, \mu, T)$ be a measure preserving dynamical system, G a compact group and $\varphi : X \rightarrow G$ a cocycle. We mean to study ergodic properties of the compact extension:

$$T_\varphi : \begin{array}{ccc} X \times G & \longrightarrow & X \times G \\ (x, g) & \longmapsto & (Tx, g \cdot \varphi(x)) \end{array} .$$

In this section, we prove the following lemma, for which a statement can be found in [10, Theorem 3] for the finite measure case and in [11, Lemma 1] for the infinite measure case. We give a proof inspired from [10] that works for both cases.

Lemma 4.1. *Let $\mathbf{X} := (X, \mu, T)$ be an ergodic measure preserving dynamical system where μ is a finite or σ -finite infinite measure. Assume that G is an abelian group. The compact extension given by $\mathbf{Z} = (X \times G, \mu \otimes m_G, T_\varphi)$ is ergodic if and only if, for every character $\chi \in \hat{G} \setminus \{1\}$, there is no measurable function $f : X \rightarrow \mathbb{T}$ such that*

$$\frac{f(Tx)}{f(x)} = \chi(\varphi(x)) \text{ almost everywhere.} \quad (13)$$

If there is a measurable map f so that (13) holds, we say that $\chi \circ \varphi$ is a co-boundary. Therefore, \mathbf{Z} is ergodic if there is no character $\chi \in \hat{G} \setminus \{1\}$ for which $\chi \circ \varphi$ is a co-boundary. In our proof, we will use results on Fourier analysis on locally compact abelian groups from [12]. This is why we need to assume that G is abelian.

Proof. Assume there is a character $\chi \neq 1$ and a map $f : X \rightarrow \mathbb{T}$ that satisfies (13). Then define

$$h(x, g) := f(x) \cdot \chi(g)^{-1}.$$

One can simply check that

$$\begin{aligned} h \circ T_\varphi(x, g) &= h(Tx, g \cdot \varphi(x)) \\ &= f(Tx) \cdot \chi(g \cdot \varphi(x))^{-1} \\ &= f(x) \cdot \chi(\varphi(x)) \cdot \chi(\varphi(x))^{-1} \cdot \chi(g)^{-1} \\ &= f(x) \cdot \chi(g)^{-1} = h(x, g). \end{aligned}$$

Since $\chi \neq 1$, h is not constant, and therefore \mathbf{Z} is not ergodic.

Conversely assume that there exists $h \in L^\infty(\mathbf{Z})$ non constant such that $h \circ T_\varphi = h$. For $\chi \in \hat{G}$, define

$$f_\chi(x) := \int_G h(x, g) \chi(g) dm_G(g).$$

We know that this integral is well defined for almost every x because $h \in L^\infty(\mathbf{Z})$.

Then we have, almost surely:

$$\begin{aligned}
f_\chi(Tx) &= \int_G h(Tx, g)\chi(g)dm_G(g) \\
&= \int_G h(Tx, g \cdot \varphi(x))\chi(g \cdot \varphi(x))dm_G(g) \\
&= \int_G h(x, g)\chi(g) \cdot \chi(\varphi(x))dm_G(g) = \chi(\varphi(x))f_\chi(x).
\end{aligned}$$

Now we simply need to find $\chi \neq 1$ such that f_χ is not almost everywhere equal to 0. First, notice that f_1 is T invariant, and therefore constant. Take $c \in \mathbb{C}$ such that $f_1 \equiv c$.

Assume that for every $\chi \neq 1$, $f_\chi(x) = 0$ for almost every x . Since G is compact, [12, Theorem 1.2.5] tells us that \hat{G} is discrete, and therefore countable. From that, we deduce that for almost every x , we have

$$\forall \chi \neq 1, f_\chi(x) = 0.$$

Therefore, using the fact that, for $\chi \neq 1$, $\int_G \chi dm_G = 0$ and $f_1 \equiv c$, we get that, for almost every x :

$$\forall \chi \in \hat{G}, \int_G (h(x, g) - c)\chi(g)dm_G(g) = 0.$$

Then, using a result from [12, Section 1.7.3], we know that for those x , the map $g \mapsto h(x, g) - c$ is almost surely 0. Therefore, for almost every (x, g) , $h(x, g) = c$. But we assumed that h was not constant, so this is absurd.

This means that there exists $\chi \in \hat{G} \setminus \{1\}$ such that $\mu(\{f_\chi \neq 0\}) > 0$. Since the set $\{f_\chi \neq 0\}$ is T invariant, this yields that $\mu(\{f_\chi = 0\}) = 0$. Finally, define $f := f_\chi/|f_\chi|$, which takes its values in \mathbb{T} and satisfies (13). \square

4.2 Description of the infinite Chacon transformation

Let $\mathbf{X} := (X, \mu, T)$ be the system given by the infinite Chacon transformation defined in [2, Section 2]. We chose this transformation because it is known that it has an infinite ergodic index (see [2, Theorem 2.2]), and because the rank one structure is convenient to define a suitable cocycle in Section 4.3. Other infinite measure preserving rank one transformations could be used here. For example

the nearly finite Chacon transformation introduced in [6] has all the properties we require in this work. All the following constructions and proofs could be applied to that transformation.

As any rank one transformation, the infinite Chacon transformation can be defined as an increasing union of towers $(\mathcal{T}_n)_{n \geq 1}$. The tower \mathcal{T}_n of order n , is composed of its levels $\{E_n^{(1)}, \dots, E_n^{(h_n)}\}$ so that

$$\mathcal{T}_n = \bigsqcup_{k=1}^{h_n} E_n^{(k)}.$$

We say that h_n is the height of \mathcal{T}_n . The transformation T acts on \mathcal{T}_n so that, for $k \in \llbracket 1, h_n - 1 \rrbracket$, we have

$$TE_n^{(k)} = E_n^{(k+1)}.$$

All the levels of \mathcal{T}_n have same measure under μ , and we denote it by $\mu_n := \mu(E_n^{(k)})$.

The construction of the sequence $(\mathcal{T}_n)_{n \geq 1}$ is done inductively. It will be done by taking intervals of \mathbb{R}^+ to be the levels of our towers. Start by taking the interval $[0, 1[$ to be \mathcal{T}_1 . Now assume that the tower \mathcal{T}_n has been built. The construction of \mathcal{T}_{n+1} goes as follows.

Decompose \mathcal{T}_n into three disjoint towers of equal measure $\mathcal{T}_n = \mathcal{T}_n^1 \sqcup \mathcal{T}_n^2 \sqcup \mathcal{T}_n^3$. Specifically, split each level of \mathcal{T}_n into three intervals of length $\mu_n/3$, then put the left-most interval in \mathcal{T}_n^1 , the middle one in \mathcal{T}_n^2 and the right one in \mathcal{T}_n^3 . We will call *spacers* a collection of intervals of length $\mu_n/3$, disjoint from \mathcal{T}_n . We put a spacer on top of \mathcal{T}_n^2 and $3h_n + 1$ spacers on top of \mathcal{T}_n^3 , and, once the spacers are in place, we stack \mathcal{T}_n^1 , \mathcal{T}_n^2 and \mathcal{T}_n^3 on top of each other, which yields \mathcal{T}_{n+1} . Therefore \mathcal{T}_{n+1} is a tower of height $2(3h_n + 1)$ whose levels are of measure $\mu_n/3$, which means that $\mu(\mathcal{T}_{n+1}) \geq 2\mu(\mathcal{T}_n)$. Finally, for $k \in \llbracket 1, h_{n+1} - 1 \rrbracket$, define T on $E_{n+1}^{(k)}$ as the translation that sends $E_{n+1}^{(k)}$ to $E_{n+1}^{(k+1)}$ (which is possible because they are both intervals of the same length). The transformation is not yet defined on $E_n^{(h_{n+1})}$, that will be done in the next step of the construction.

We end the construction of (X, μ, T) by setting $X := \bigcup_{n \geq 1} \mathcal{T}_n$. Since $\mu(\mathcal{T}_{n+1}) \geq 2\mu(\mathcal{T}_n)$, we have $\mu(X) = \infty$.

4.3 Construction of the extension

Take $\mathbf{X} := (X, \mu, T)$ the system given by the infinite Chacon transformation introduced in Section 4.2.

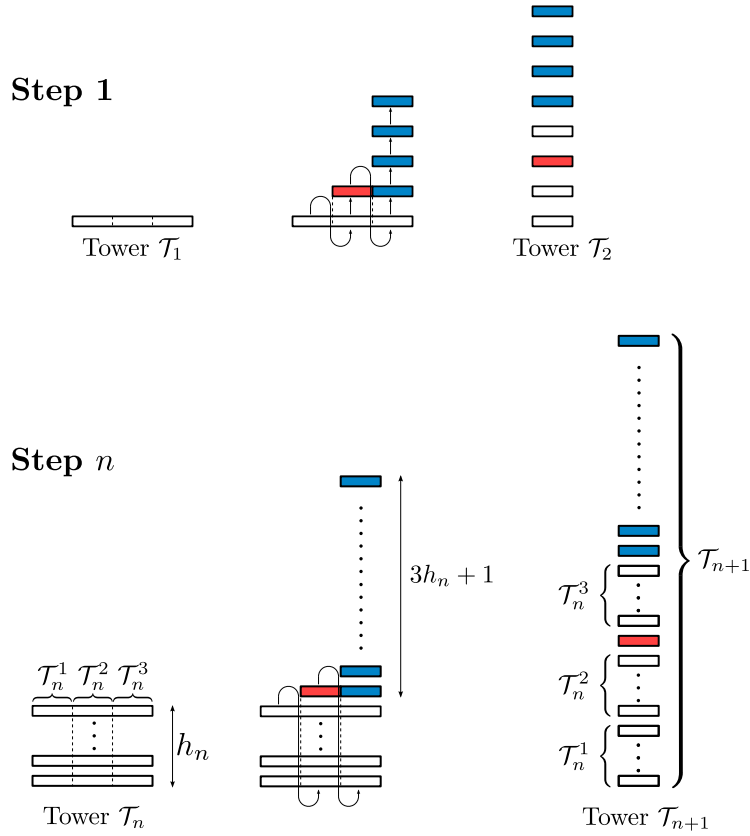


Figure 1: Construction of the infinite Chacon transformation

Let $n_0 \in \mathbb{N}$ and $i_0 \in \llbracket 1, h_{n_0} \rrbracket$. Set $A := E_{n_0}^{(i_0)}$, and consider the cocycle taking its values in $\mathbb{Z}/2\mathbb{Z}$ (identified with $\{0, 1\}$):

$$\varphi := \mathbb{1}_A.$$

We study the system \mathbf{Z} given on $(Z, \rho) := (X \times \{0, 1\}, \mu \otimes \mathcal{B}(1/2, 1/2))$ by the transformation

$$T_\varphi : \begin{array}{ccc} X \times \{0, 1\} & \longrightarrow & X \times \{0, 1\} \\ (x, \epsilon) & \longmapsto & (Tx, \epsilon + \varphi(x) \bmod 2) \end{array} .$$

This is a compact extension of \mathbf{X} .

Remark 4.2. By a simple induction, one can check that, for all $n \geq n_0$, there are 3^{n-n_0} levels in \mathcal{T}_n that belong to A .

Theorem 4.3. *The system $\mathbf{Z} = (X \times \{0, 1\}, \rho, T_\varphi)$ is of infinite ergodic index, i.e. for every $p \geq 1$, $\mathbf{Z}^{\otimes p}$ is a conservative and ergodic system.*

It is known from the work in [2, Theorem 2.2] that \mathbf{X} is of infinite ergodic index, therefore, our goal is to show that \mathbf{Z} is as well. Let $p \geq 1$. Then $\mathbf{X}^{\otimes p}$ is a conservative and ergodic system, and $\mathbf{Z}^{\otimes p}$ is the compact extension given by the cocycle

$$\begin{aligned} \varphi^{(p)} : \quad X^p &\longrightarrow \{0, 1\}^p \\ (x_1, \dots, x_p) &\mapsto (\varphi(x_1), \dots, \varphi(x_p)). \end{aligned}$$

The characters of $\{0, 1\}$ (identified to $\mathbb{Z}/2\mathbb{Z}$) are

$$\chi_1 := 1 \quad \text{and} \quad \chi_{-1} := (-1)^\bullet.$$

Therefore, the characters of $\{0, 1\}^p$ will be the tensor products of the form

$$\bigotimes_{i \notin I} \chi_1 \otimes \bigotimes_{i \in I} \chi_{-1} = \bigotimes_{i \in I} (-1)^\bullet,$$

for $I \subset \llbracket 1, p \rrbracket$. The character is then entirely determined by the choice of the set I . Then, Lemma 4.1 tells us that $\mathbf{Z}^{\otimes p}$ is non-ergodic if and only if there exist $I \subset \llbracket 1, p \rrbracket$ with $I \neq \emptyset$ and a map $f : X^p \longrightarrow \{\pm 1\}$ such that

$$\frac{f(Tx_1, \dots, Tx_p)}{f(x_1, \dots, x_p)} = (-1)^{\sum_{i \in I} \varphi(x_i)} \quad \text{almost everywhere.} \quad (14)$$

Proposition 4.5 and Corollary 4.6.1 will show that such functions cannot exist, which will complete the proof of Theorem 4.3.

We now turn our attention to the proofs of Proposition 4.5 and Corollary 4.6.1. We start by giving some setup common to both of those results, and we will then conclude each proof separately in Sections 4.3.1 and 4.3.2.

Denote $\nu_p := \mu^{\otimes p}$. We define $\mathcal{H}_n^p := \mathcal{T}_n \times \dots \times \mathcal{T}_n$. We can decompose \mathcal{H}_n^p as

$$\mathcal{H}_n^p = \bigsqcup_{k_1=1}^{h_n} \dots \bigsqcup_{k_p=1}^{h_n} E_n^{(k_1)} \times \dots \times E_n^{(k_p)}.$$

This gives a filtration on $X \times \dots \times X$:

$$\mathcal{F}_n := \sigma \left(\{E_n^{(k_1)} \times \dots \times E_n^{(k_p)}\}_{k_1, \dots, k_p \in \llbracket 1, h_n \rrbracket}, (\mathcal{H}_n^p)^c \right).$$

Each \mathcal{F}_n is not a σ -finite σ -algebra, because $\mu((\mathcal{H}_n^p)^c) = \infty$. So the conditional expectation $\mathbb{E}[\cdot | \mathcal{F}_n]$ is not well defined. However, if we fix $N \in \mathbb{N}$ and consider

the probability space $(\mathcal{H}_N^p, \frac{1}{\nu_p(\mathcal{H}_N^p)}\nu_p(\cdot \cap \mathcal{H}_N^p))$, for every $n \geq N$, \mathcal{F}_n yields a finite partition of \mathcal{H}_N^p . Moreover, we can compute that, for any measurable function f :

$$\mathbb{E}_{\mathcal{H}_N^p}[f | \mathcal{F}_n] = \frac{1}{\mu_n^p} \int_{E_n^{(k_1)} \times \dots \times E_n^{(k_p)}} f d\nu_p. \quad (15)$$

The important thing to note is that the right-hand term does not depend on N . Therefore we define the following, for $f : X^p \rightarrow \mathbb{R}$:

$$\mathbb{E}[f | \mathcal{F}_n] := \begin{cases} \frac{1}{\mu_n^p} \int_{E_n^{(k_1)} \times \dots \times E_n^{(k_p)}} f d\nu_p & \text{if } (x_1, \dots, x_p) \in E_n^{(k_1)} \times \dots \times E_n^{(k_p)} \\ 0 & \text{if } (x_1, \dots, x_p) \notin \mathcal{H}_n^p \end{cases}.$$

Despite our choice of notation, this is not a true conditional expectation. However, since we have (15), we can conclude, using the fact that $X^p = \bigcup_{N \geq 1} \mathcal{H}_N^p$ and that $\bigvee_{n \geq 1} \mathcal{F}_n$ separates the points on X^p , that by the martingale convergence theorem, we have

$$\mathbb{E}[f | \mathcal{F}_n] \xrightarrow[n \rightarrow \infty]{} f \text{ almost everywhere.}$$

Before we present the remaining details of the proof, we give a technical lemma:

Lemma 4.4. *Let $p \geq 1$. Let $i_1, \dots, i_p \in \{1, 2, 3\}$. For almost every $(x_1, \dots, x_p) \in X^p$, for every $M \geq 1$, there exists $n \geq M$ such that $(x_1, \dots, x_p) \in \mathcal{T}_n^{i_1} \times \dots \times \mathcal{T}_n^{i_p}$.*

Proof. We recall that $\mathcal{H}_n^p := \mathcal{T}_n \times \dots \times \mathcal{T}_n$ and $\nu_p := \mu^{\otimes p}$. Let $M \geq 1$. Take $M' \geq M$, and note that

$$\begin{aligned} & \nu_p(\forall n \geq M, (x_1, \dots, x_p) \in \mathcal{H}_n^p \setminus (\mathcal{T}_n^{i_1} \times \dots \times \mathcal{T}_n^{i_p})) \\ & \leq \nu_p(\forall n \in \llbracket M, M' \rrbracket, (x_1, \dots, x_p) \in \mathcal{H}_n^p \setminus (\mathcal{T}_n^{i_1} \times \dots \times \mathcal{T}_n^{i_p})). \end{aligned}$$

A straightforward induction on M' shows that

$$\begin{aligned} & \nu_p(\forall n \in \llbracket M, M' \rrbracket, (x_1, \dots, x_p) \in \mathcal{H}_n^p \setminus (\mathcal{T}_n^{i_1} \times \dots \times \mathcal{T}_n^{i_p})) \\ & = \left(\frac{3^p - 1}{3^p} \right)^{(M' - M + 1)} \nu_p(\mathcal{H}_M^p) \xrightarrow[M' \rightarrow \infty]{} 0. \end{aligned}$$

Therefore, we have

$$\nu_p(\forall n \geq M, (x_1, \dots, x_p) \in \mathcal{H}_n^p \setminus (\mathcal{T}_n^{i_1} \times \dots \times \mathcal{T}_n^{i_p})) = 0,$$

which implies

$$\nu_p(\exists M \geq 1, \forall n \geq M, (x_1, \dots, x_p) \in \mathcal{H}_n^p \setminus (\mathcal{T}_n^{i_1} \times \dots \times \mathcal{T}_n^{i_p})) = 0.$$

Combining this with the fact that $X^p = \bigcup_{n \geq 1} \mathcal{H}_n^p$ ends our proof. \square

4.3.1 If $\#I$ is odd

Let us first give a rough sketch of the argument to prove that we cannot find f satisfying (14) when $\#I$ is odd. For such a function f , we study the evolution of its values along the orbit of a point (x_1, \dots, x_p) . By (14), every time some coordinate of index $i \in I$ goes through A , it causes a change of sign for f . Now, if we choose n so that all the $\{x_i\}_{i \in [1, p]}$ start in \mathcal{T}_n^1 , to avoid hitting the spacers, after h_n applications of $T \times \dots \times T$, each coordinate is then back in the level from which it started and its orbit has gone through each level of \mathcal{T}_n exactly once. But we know from Remark 4.2 that each tower \mathcal{T}_n has 3^{n-n_0} levels that are subsets of A . So, on the piece of the orbit that we consider, the sign of f changes $\#I \cdot 3^{n-n_0}$ times, and since $\#I$ is odd, this means that the sign of f changes. But, since each x_i is back on the level from which it started, if f is close enough to being constant on cells of the form $E_n^{(k_1)} \times \dots \times E_n^{(k_p)}$, this yields a contradiction. We detail this argument in the following proposition:

Proposition 4.5. *Let $I \subset \llbracket 1, p \rrbracket$ such that $\#I$ is odd. There is no measurable map $f : X^p \rightarrow \{\pm 1\}$ that satisfies (14).*

Proof. Suppose by contradiction that there exists $f : X^p \rightarrow \{\pm 1\}$ that satisfies (14). Take $\delta > 0$ and $(x_1, \dots, x_p) \in X^p$. Up to a set of measure 0, we can assume that there exists $N \geq 1$ such that $\forall n \geq N$, we have

$$|\mathbb{E}[f \mid \mathcal{F}_n](x_1, \dots, x_p) - f(x_1, \dots, x_p)| \leq \delta. \quad (16)$$

We know from Lemma 4.4, that, up to another set of measure 0, we can assume that there is $n \geq N$ such that

$$(x_1, \dots, x_p) \in \mathcal{T}_n^1 \times \dots \times \mathcal{T}_n^1.$$

Let (k_1, \dots, k_p) such that for $i \in \llbracket 1, p \rrbracket$, $x_i \in E_n^{(k_i)}$. Using the definition of $\mathbb{E}[f \mid \mathcal{F}_n]$, we denote $E := E_n^{(k_1)} \times \dots \times E_n^{(k_p)}$, $x := (x_1, \dots, x_p)$ and compute

$$\begin{aligned} |\mathbb{E}[f \mid \mathcal{F}_n](x) - f(x)| &= \left| \frac{1}{\nu_p(E)} \int_E f d\nu_p - f(x) \right| \\ &= \left| f(x) \frac{\nu_p(E \cap \{f = f(x)\}) - \nu_p(E \cap \{f = -f(x)\})}{\nu_p(E)} - f(x) \right| \\ &= \left| \frac{\nu_p(E) - \nu_p(E \cap \{f = -f(x)\}) - \nu_p(E \cap \{f = -f(x)\})}{\nu_p(E)} - 1 \right| \\ &= 2 \frac{\nu_p(E \cap \{f \neq f(x)\})}{\nu_p(E)}, \end{aligned}$$

where we use the facts that f takes only two possible values and $|f| = 1$. This shows that (16) implies

$$\nu_p((E_n^{(k_1)} \times \cdots \times E_n^{(k_p)}) \cap \{f \neq f(x_1, \dots, x_p)\}) \leq \frac{\delta}{2} \mu_n^p. \quad (17)$$

Define

$$B := (E_n^{(k_1)} \times \cdots \times E_n^{(k_p)}) \cap (\mathcal{T}_n^1 \times \cdots \times \mathcal{T}_n^1),$$

and

$$C := (E_n^{(k_1)} \times \cdots \times E_n^{(k_p)}) \cap (\mathcal{T}_n^2 \times \cdots \times \mathcal{T}_n^2).$$

Since there is no spacer on top of \mathcal{T}_n^1 , we get that $(T \times \cdots \times T)^{h_n} B = C$. However, since $\#I$ is odd, for any $(x'_1, \dots, x'_p) \in B$, Remark 4.2 and (14) imply that

$$f(T^{h_n} x'_1, \dots, T^{h_n} x'_p) = (-1)^{\#I \cdot 3^{n-n_0}} f(x'_1, \dots, x'_p) = -f(x'_1, \dots, x'_p).$$

Therefore

$$\begin{aligned} \nu_p(C \cap \{f = -f(x_1, \dots, x_p)\}) &= \nu_p(B \cap \{f = f(x_1, \dots, x_p)\}) \\ &\geq \nu_p(B) - \frac{\delta}{2} \mu_n^p. \end{aligned}$$

But, by construction, $\nu_p(B) = \mu_n^p/3^p$. So

$$\begin{aligned} \nu_p(C \cap \{f = -f(x_1, \dots, x_p)\}) &\geq \frac{\mu_n^p}{3^p} - \frac{\delta}{2} \mu_n^p \\ &= \left(\frac{1}{3^p} - \frac{\delta}{2} \right) \mu_n^p > \frac{\delta}{2} \mu_n^p, \end{aligned}$$

if $\delta < 1/3^p$. Since $C \subset E_n^{(k_1)} \times \cdots \times E_n^{(k_p)}$, combined with (17), this implies that $f(x_1, \dots, x_p) = -f(x_1, \dots, x_p)$. However, by definition, $f \neq 0$, so this is absurd. \square

4.3.2 If $\#I$ is even

If $\#I$ is even, the argument given above no longer works: when the orbits of all the $\{x_i\}_{i \in [1, p]}$ go through the levels of \mathcal{T}_n , it causes an even number of sign changes for f , which means that f remains unchanged. Here we need to make use of the placement of the spacers in the construction of T . We will fix $i_0 \in I$ and choose n so that all the points $\{x_i\}_{i \neq i_0}$ start in \mathcal{T}_n^2 so that they can hit the spacer.

On the other hand, the choice of n enables us to assume that x_{i_0} will start in \mathcal{T}_n^1 and its orbit can go through all the levels of \mathcal{T}_n without hitting a spacer. Therefore, after $h_n + 1$ applications of $T \times \cdots \times T$, all the coordinates of index $i \neq i_0$ will be back in the level from which they started, but x_{i_0} will be one level higher, which will yield the equation we get in Proposition 4.6. Then applying our reasoning from the odd case will give a contradiction, as stated in Corollary 4.6.1.

Let us show the following:

Proposition 4.6. *Let $I \subset \llbracket 1, p \rrbracket$ such that $I \neq \emptyset$ and $\#I$ is even. Assume that there exists $f : X^p \rightarrow \{\pm 1\}$ that satisfies (14). Then, for ever $i \in I$, we have:*

$$\frac{f(x_1, \dots, Tx_i, \dots, x_p)}{f(x_1, \dots, x_p)} = (-1)^{\varphi(x_i)} \text{ almost everywhere.} \quad (18)$$

Proof. Up to a permutation of coordinates, we can assume that $1 \in I$ and deal with the case $i = 1$. Define also $\tilde{\mathcal{T}}_n := \mathcal{T}_n \cap (X \setminus E_n^{(h_n)})$, i.e. we get $\tilde{\mathcal{T}}_n$ by removing the top level from \mathcal{T}_n .

Let $(x_1, \dots, x_p) \in X^p$. Take a small $\delta > 0$. Up to a set of measure 0, we can assume that there exists $N \geq 1$ such that $\forall n \geq N$, we have

$$\begin{cases} |\mathbb{E}[f | \mathcal{F}_n](x_1, \dots, x_p) - f(x_1, \dots, x_p)| \leq \delta, \\ |\mathbb{E}[f | \mathcal{F}_n](Tx_1, \dots, x_p) - f(Tx_1, \dots, x_p)| \leq \delta. \end{cases} \quad (19)$$

Also note that in the construction of a tower, the top level is obtained with a spacer, so if $x_1 \in \mathcal{T}_m$, then for every $n > m$, $x_1 \in \mathcal{T}_n \setminus E_n^{(h_n)} = \tilde{\mathcal{T}}_n$. Since $X = \bigcup_{n \geq 1} \mathcal{T}_n$, this means that if N is large enough, we can also assume that for every $n \geq N$, $x_1 \in \tilde{\mathcal{T}}_n$. Finally, using Lemma 4.4, we know that up to another set of measure zero, we can find $n \geq N$ for which $(x_1, \dots, x_p) \in \mathcal{T}_n^1 \times \mathcal{T}_n^2 \times \cdots \times \mathcal{T}_n^2$.

Set k_1, \dots, k_p such that $x_i \in E_n^{(k_i)}$ for every $i \in \llbracket 1, p \rrbracket$. The fact that $x_1 \in \tilde{\mathcal{T}}_n$ implies that $k_1 \leq h_n - 1$. The same computation that gave (17) shows that (19) implies

$$\nu_p((E_n^{(k_1)} \times \cdots \times E_n^{(k_p)}) \cap \{f \neq f(x_1, \dots, x_p)\}) \leq \frac{\delta}{2} \mu_n^p, \quad (20)$$

and

$$\nu_p((E_n^{(k_1+1)} \times E_n^{(k_2)} \times \cdots \times E_n^{(k_p)}) \cap \{f \neq f(Tx_1, \dots, x_p)\}) \leq \frac{\delta}{2} \mu_n^p. \quad (21)$$

Denote

$$B := (E_n^{(k_1)} \times \cdots \times E_n^{(k_p)}) \cap (\mathcal{T}_n^1 \times \mathcal{T}_n^2 \times \cdots \times \mathcal{T}_n^2)$$

and

$$C := (E_n^{(k_1+1)} \times E_n^{(k_2)} \times \cdots \times E_n^{(k_p)}) \cap (\mathcal{T}_n^2 \times \mathcal{T}_n^3 \times \cdots \times \mathcal{T}_n^3).$$

One can check that, because there is a spacer on top of \mathcal{T}_n^2 and no spacer on top of \mathcal{T}_n^1 , we get

$$(T \times \cdots \times T)^{h_n+1} B = C.$$

However, using the fact that $B \subset \tilde{\mathcal{T}}_n^1 \times \mathcal{T}_n^2 \times \cdots \times \mathcal{T}_n^2$, we can track the times f changes sign between B and C . The orbits of the $\{x_i\}_{i \neq 1}$ go through each level of \mathcal{T}_n , therefore contributing $(\#I - 1)3^{n-n_0}$ sign changes. The orbit of x_1 goes through every level of \mathcal{T}_n after h_n applications of T , contributing 3^{n-n_0} sign changes, and one additional sign change from the $h_n + 1$ -th application if $E_n^{(k_1)} \subset A$, or, equivalently, if $\varphi(x_1) = 1$. This means that for every $(x'_1, \dots, x'_p) \in B$, we have

$$\begin{aligned} f(T^{h_n+1}x'_1, \dots, T^{h_n+1}x'_p) &= (-1)^{\#I 3^{n-n_0+1} E_n^{(k_1)} \subset A} f(x'_1, \dots, x'_p) \\ &= (-1)^{\varphi(x_1)} f(x'_1, \dots, x'_p) \text{ because } \#I \text{ is even.} \end{aligned}$$

So, we have

$$\begin{aligned} \nu_p(C \cap \{f = (-1)^{\varphi(x_1)} f(x_1, \dots, x_p)\}) &= \nu_p(B \cap \{f = f(x_1, \dots, x_p)\}) \\ &= \nu_p(B) - \nu_p(B \cap \{f \neq f(x_1, \dots, x_p)\}) \\ &\geq \nu_p(B) - \nu_p((E_n^{(k_1)} \times \cdots \times E_n^{(k_p)}) \cap \{f \neq f(x_1, \dots, x_p)\}) \\ &\geq \nu_p(B) - \frac{\delta}{2} \mu_n^p, \text{ because of (20).} \end{aligned}$$

Moreover, $\nu_p(B) = \mu_n^p / 3^p$, so

$$\begin{aligned} \nu_p((E_n^{(k_1+1)} \times \cdots \times E_n^{(k_p)}) \cap \{f = (-1)^{\varphi(x_1)} f(x_1, \dots, x_p)\}) \\ \geq \nu_p(C \cap \{f = (-1)^{\varphi(x_1)} f(x_1, \dots, x_p)\}) \\ \geq \left(\frac{1}{3^p} - \frac{\delta}{2}\right) \mu_n^p > \frac{\delta}{2} \mu_n^p, \end{aligned}$$

if δ is small enough ($\delta < 1/3^p$). However, this is only compatible with (21) if

$$(-1)^{\varphi(x_1)} f(x_1, \dots, x_p) = f(Tx_1, \dots, x_p).$$

□

Corollary 4.6.1. *Let $I \subset \llbracket 1, p \rrbracket$ such that $I \neq \emptyset$ and $\#I$ is even. There is no measurable map $f : X^p \rightarrow \{\pm 1\}$ that satisfies (14).*

Proof. Assume that there is a measurable map $f : X^p \rightarrow \{\pm 1\}$ that satisfies (14). Up to a permutation of coordinates, we can assume that $1 \in I$. Fix $(x_2, \dots, x_p) \in X^{p-1}$. Up to a set of measure 0 in our choice of (x_2, \dots, x_p) , we know from Proposition 4.6 that the map $g_{(x_2, \dots, x_p)} : x_1 \mapsto f(x_1, \dots, x_p)$ satisfies

$$\frac{g_{(x_2, \dots, x_p)}(Tx_1)}{g_{(x_2, \dots, x_p)}(x_1)} = (-1)^{\varphi(x_1)} \text{ almost everywhere.}$$

However, Proposition 4.5 tells us that such a map cannot exist. \square

Remark 4.7. Now that the proof of Theorem 4.3 is complete, we have an example of compact extension of an infinite measure preserving system that is of infinite ergodic index. But we only proved the infinite ergodic index of a specific extension, and we wonder if more general results can be found. In particular, a significant difference between finite and infinite ergodic theory is the fact that for probability preserving systems, an ergodic index greater or equal to 2 is automatically infinite. This is not true in the infinite measure case, but we could consider an intermediate situation: take an extension $\mathbf{Z} \xrightarrow{\pi} \mathbf{X}$ of σ -finite infinite measure systems and assume that \mathbf{X} has an infinite ergodic index. Is it possible that \mathbf{Z} have a finite ergodic index greater or equal to 2? In our example, proving that the ergodic index is at least 2 contain exactly as much difficulty as proving it is infinite, therefore suggesting that the answer could be negative.

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