



HAL
open science

A smoothing effect for the fractional Schrödinger equations on the circle and observability

Paul Alphonse, Nikolay Tzvetkov

► **To cite this version:**

Paul Alphonse, Nikolay Tzvetkov. A smoothing effect for the fractional Schrödinger equations on the circle and observability. 2024. hal-04511776

HAL Id: hal-04511776

<https://hal.science/hal-04511776>

Preprint submitted on 19 Mar 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A SMOOTHING EFFECT FOR THE FRACTIONAL SCHRÖDINGER EQUATIONS ON THE CIRCLE AND OBSERVABILITY

PAUL ALPHONSE AND NIKOLAY TZVETKOV

ABSTRACT. We show that, after a renormalisation, one can define the square of the modulus of the solution of the fractional Schrödinger equations on the circle with data in Sobolev spaces of arbitrary negative index. As an application, we obtain observability estimates with rough controls.

1. INTRODUCTION AND MAIN RESULTS

We consider the fractional Schrödinger equation on the circle

$$(S_\alpha) \quad i\partial_t u + |D_x|^\alpha u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{T},$$

where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and $\alpha > 1$ is a positive real number. In this equation, the operator $|D_x|^\alpha$ is the Fourier multiplier defined on $\mathcal{S}'(\mathbb{T}) = (C^\infty(\mathbb{T}))'$ by

$$\widehat{|D_x|^\alpha f}(n) = |n|^\alpha \widehat{f}(n), \quad n \in \mathbb{Z}, f \in \mathcal{S}'(\mathbb{T}),$$

where $\widehat{\cdot}$ denotes the Fourier transform on $\mathcal{S}'(\mathbb{T})$, given by

$$\widehat{f}(n) = (2\pi)^{-1} \langle f, e^{-inx} \rangle_{\mathcal{S}'(\mathbb{T}), C^\infty(\mathbb{T})}, \quad n \in \mathbb{Z}.$$

When α is an even integer $|D_x|^\alpha$ is simply the differential operator $(-1)^{\alpha/2} \partial_x^\alpha$. For every $u_0 \in \mathcal{S}'(\mathbb{T})$, the unique solution in $C(\mathbb{R}, \mathcal{S}'(\mathbb{T}))$ of the equation (S_α) with initial datum

$$u|_{t=0} = u_0,$$

naturally denoted $e^{it|D_x|^\alpha} u_0$, is given by the exponential sum

$$(1.1) \quad (e^{it|D_x|^\alpha} u_0)(x) = \sum_{n \in \mathbb{Z}} \widehat{u_0}(n) e^{it|n|^\alpha} e^{inx}.$$

Our goal in this work is to describe a remarkable smoothing property enjoyed by (1.1) and its application to observability estimates.

1.1. Smoothing effect. For every function $u_0 \in \mathcal{S}'(\mathbb{T})$ and every $N \geq 1$, we denote by $\Pi_N u_0$ the truncation of the function u_0 at the energy level N , that is,

$$(1.2) \quad (\Pi_N u_0)(x) = \sum_{|n| \leq N} \widehat{u_0}(n) e^{inx}.$$

Of course $\Pi_N u_0 \in C^\infty(\mathbb{T})$. For $\sigma \in \mathbb{R}$, we define the operator $\langle D_x \rangle^\sigma$ on $\mathcal{S}'(\mathbb{T})$ as

$$\widehat{\langle D_x \rangle^\sigma f}(n) = \langle n \rangle^\sigma \widehat{f}(n), \quad n \in \mathbb{Z}, f \in \mathcal{S}'(\mathbb{T}),$$

where $\langle n \rangle = (1 + n^2)^{1/2}$. We can therefore define the Sobolev spaces $H^\sigma(\mathbb{T})$ via the norm

$$\|f\|_{H^\sigma(\mathbb{T})} = \|\langle D_x \rangle^\sigma f\|_{L^2(\mathbb{T})}.$$

Let $u_0 \in L^2(\mathbb{T})$. Then, ignoring the time oscillations, it follows directly from (1.1) that

$$\langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} u_0) \in C(\mathbb{R}, H^{-\sigma}(\mathbb{T})),$$

and that

$$(1.3) \quad \langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} u_0) = \lim_{N \rightarrow +\infty} \langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} \Pi_N u_0) \quad \text{in } L^\infty(\mathbb{R}, H^{-\sigma}(\mathbb{T})).$$

2020 *Mathematics Subject Classification.* 35Q93, 35J10, 35B65, 47D06.

Key words and phrases. Fractional Schrödinger equations; smoothing effect; observability.

Suppose that $\sigma > 0$. Then, the convergence (1.3) is too weak to imply a convergence of nonlinear expressions of $\langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} \Pi_N u_0)$. For instance, there is no hope to make converge the sequence of $C^\infty(\mathbb{R} \times \mathbb{T})$ functions

$$(1.4) \quad \left| \langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} \Pi_N u_0) \right|^2,$$

simply because their zero Fourier coefficients (in x) equal the numbers

$$\|\Pi_N u_0\|_{H^\sigma(\mathbb{T})}^2,$$

which is a sequence of positive numbers which diverges as soon as $u_0 \in L^2(\mathbb{T})$ is such that $u_0 \notin H^\sigma(\mathbb{T})$. Remarkably, this is the only obstruction to the convergence of (1.4), as shown by the next statement which is the first main result in this paper.

Theorem 1.1. *Let $\alpha > 1$ be a positive real number and $u_0 \in L^2(\mathbb{T})$. For every $\sigma > 0$, the following limit exists in $\mathcal{S}'(\mathbb{R} \times \mathbb{T})$*

$$:\left| \langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} u_0) \right|^2 := \lim_{N \rightarrow +\infty} \left(\left| \langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} \Pi_N u_0) \right|^2 - \|\Pi_N u_0\|_{H^\sigma(\mathbb{T})}^2 \right).$$

Moreover,

$$:\left| \langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} u_0) \right|^2 \in W^{-s_1, \infty}(\mathbb{R}, H^{-s_2}(\mathbb{T})),$$

with

$$s_1 = \frac{2\sigma}{\alpha - 1} \quad \text{and} \quad s_2 = \begin{cases} 2\sigma & \text{when } \sigma > \frac{1}{4} - \frac{1}{4\alpha}, \\ \frac{1}{2} - \frac{2\sigma}{\alpha - 1} & \text{when } \sigma \leq \frac{1}{4} - \frac{1}{4\alpha}. \end{cases}$$

We refer to Remark 2.2 where the functional space $W^{-s_1, \infty}(\mathbb{R}, H^{-s_2}(\mathbb{T}))$ is defined.

The result of Theorem 1.1 says that the sequence (1.4) converges after a renormalization. Let us mention that a renormalization of similar spirit is used in the study of nonlinear PDE in the presence of singular randomness (see [9]) or in the classical work in quantum field theory (see [6]). The key difference between Theorem 1.1 and [6, 9] is that in the proof of Theorem 1.1, we pass into the limit in a singular nonlinear expression thanks to time oscillation effects while in [6, 9] one passes to the limit in a singular nonlinear expression thanks to random oscillation effects. Interestingly, for quadratic expressions in both cases (random or time oscillations), the only obstruction comes from the zero Fourier coefficient.

The result of Theorem 1.1 also says that the sequence $(\partial_x |\langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} \Pi_N u_0)|^2)_N$ converges in $\mathcal{S}'(\mathbb{R} \times \mathbb{T})$. In other words, for $u_0 \in L^2(\mathbb{T})$, we may give a sense of the distribution $\partial_x |\langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} u_0)|^2$ for every $\sigma \in \mathbb{R}$.

In Theorem 1.1 we only deal with a quadratic expression of (1.1). It would be interesting to understand how the result extends to higher degree polynomials. In the case of random oscillations, such an analysis is performed in [6].

1.2. Observability. The second main result in this paper is the following one, dealing with the observability properties of the equation (\mathcal{S}_α).

Theorem 1.2. *Let $\alpha > 1$, $T > 0$ be a positive time and $b \in L^1(\mathbb{T}) \setminus \{0\}$ be a non-negative function. There exists a positive constant $C_{b,T} > 0$ such that for every $u_0 \in L^2(\mathbb{T})$,*

$$(1.5) \quad \|u_0\|_{L^2(\mathbb{T})}^2 \leq C_{b,T} \int_0^T \int_{\mathbb{T}} b(x) |(e^{it|D_x|^\alpha} u_0)(x)|^2 dx dt.$$

The observability properties of type (1.5) of Schrödinger equations on tori \mathbb{T}^d have been widely studied, mostly in the non-fractional case $\alpha = 2$ when $b = 1_\omega$ and $\omega \subset \mathbb{T}^d$ is an open set. Among all this literature, the two articles [2, 4] are exceptions, since they consider more general controls $b \in L^1(\mathbb{T})$ as in the present paper. Theorem 1.2 is in particular a generalization of [2, Lemma 2.4] to the fractional case $\alpha > 1$. On the other hand, the paper [2] allows to treat the two-dimensional case. Our proof basically follows the strategy of [2]. However, the techniques used in order to prove Lemma 3.2 are different from [2].

Indeed, the article [2] relies on an approach based on results of propagation of singularities *via* the notion of semiclassical defect measure. The present paper, as for it, combines the smoothing effect stated in Theorem 1.1 (not in its full strength) with an ergodic argument, which is simplified to a projection argument when $b \in L^2(\mathbb{T})$ (including the interesting case where $b = 1_\omega$ and $\omega \subset \mathbb{T}$ is a measurable set with positive measure). Besides, we hope that our approach will allow us to deal with dispersive equations on higher dimensional tori \mathbb{T}^d .

It should also be mentioned that the observability properties of fractional Schrödinger equations have been studied in a very general geometric setting in the paper [5], with localized functions of the form $b = 1_\omega$ with ω an open subset of the ambient manifold. In the particular case $d = 1$ of the circle \mathbb{T} and when $\alpha > 1$, which is the context of the present article, [5, Theorem 1] states an observability estimate for the equation (S_α) in any positive time $T > 0$. In addition, let us mention that the assumption $\alpha > 1$ in Theorem 1.2 cannot be relaxed. Indeed, we know from [5, Theorem 1] that on the one hand, in the situation where $\alpha = 1$, the observability properties of the equation (S_1) are linked to the so-called Geometric Control Condition on $\omega \subset \mathbb{T}$. On the other hand, in the setting $0 < \alpha < 1$, the very same result [5, Theorem 1] states that the set $\omega \subset \mathbb{T}$ has to be dense so that an observability estimate holds.

Acknowledgements. We are grateful to Sebastian Herr for an interesting discussion on the smoothing properties of fractional Schrödinger equations. N.T. was partially supported by the ANR project Smooth ANR-22-CE40-0017.

2. SMOOTHING ESTIMATES

In this section, we give the proof of Theorem 1.1.

2.1. Prolegomena. We therefore consider $\alpha > 1$ a positive real number, $\sigma > 0$ and $u_0 \in L^2(\mathbb{T})$. Recall that the projectors Π_N are defined in (1.2). First notice that the conservation of mass property of the equation (S_α) implies

$$\Pi_0(|\langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} \Pi_N u_0)|^2) = \|\langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} \Pi_N u_0)\|_{L^2(\mathbb{T})}^2 = \|\Pi_N u_0\|_{H^\sigma(\mathbb{T})}^2.$$

As a consequence, the elements of the sequence

$$(2.1) \quad \left(|\langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} \Pi_N u_0)|^2 - \|\Pi_N u_0\|_{H^\sigma(\mathbb{T})}^2 \right)_{N \geq 0},$$

can be written in the following way

$$|\langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} \Pi_N u_0)|^2 - \|\Pi_N u_0\|_{H^\sigma(\mathbb{T})}^2 = \Pi_0^\perp(|\langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} \Pi_N u_0)|^2).$$

We therefore deduce that for every test function $\varphi \in \mathcal{S}'(\mathbb{R} \times \mathbb{T})$, we have

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{T}} \varphi(t, x) \left(|\langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} \Pi_N u_0)(x)|^2 - \|\Pi_N u_0\|_{H^\sigma(\mathbb{T})}^2 \right) dx dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{T}} \varphi(t, x) \Pi_0^\perp(|\langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} \Pi_N u_0)|^2)(x) dx dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{T}} (\Pi_0^\perp \varphi)(t, x) |\langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} \Pi_N u_0)(x)|^2 dx dt. \end{aligned}$$

The convergence of the sequence (2.1) in $\mathcal{S}'(\mathbb{R} \times \mathbb{T})$ and the regularity of the limit $:|\langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} u_0)|^2:$ will therefore be a consequence of the following bilinear smoothing estimates.

Proposition 2.1. *For every $\sigma > 0$, there exists a positive constant $c_\sigma > 0$ such that for every test function $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$ and for every initial data $u_0, v_0 \in H^{-\sigma}(\mathbb{T})$,*

$$\left| \int_{\mathbb{R}} \int_{\mathbb{T}} (\Pi_0^\perp \varphi)(t, x) (e^{it|D_x|^\alpha} u_0)(x) \overline{(e^{it|D_x|^\alpha} v_0)(x)} dx dt \right| \leq c_\sigma \|\varphi\|_{W^{s_1, 1}(\mathbb{R}, H^{s_2}(\mathbb{T}))} \|u_0\|_{H^{-\sigma}(\mathbb{T})} \|v_0\|_{H^{-\sigma}(\mathbb{T})},$$

with

$$s_1 = \frac{2\sigma}{\alpha - 1} \quad \text{and} \quad s_2 = \begin{cases} 2\sigma & \text{when } \sigma > \frac{1}{4} - \frac{1}{4\alpha}, \\ \frac{1}{2} - \frac{2\sigma}{\alpha - 1} & \text{when } \sigma \leq \frac{1}{4} - \frac{1}{4\alpha}. \end{cases}$$

Remark 2.2. The functional space $W^{s_1, 1}(\mathbb{R}, H^{s_2}(\mathbb{T}))$ appearing in the above statement is defined by

$$W^{s_1, 1}(\mathbb{R}, H^{s_2}(\mathbb{T})) = \{\varphi \in \mathcal{S}'(\mathbb{R} \times \mathbb{T}) : \langle D_t \rangle^{s_1} \varphi \in L^1(\mathbb{R}, H^{s_2}(\mathbb{T}))\},$$

and is endowed with the following norm

$$\|\varphi\|_{W^{s_1, 1}(\mathbb{R}, H^{s_2}(\mathbb{T}))} = \int_{\mathbb{R}} \|\langle D_t \rangle^{s_1} \varphi(t, \cdot)\|_{H^{s_2}(\mathbb{T})} dt.$$

For every $s \in \mathbb{R}$, the operator $\langle D_t \rangle^s$ is defined on $\mathcal{S}'(\mathbb{R})$ by

$$\mathcal{F}_t(\langle D_t \rangle^s f)(\tau) = \langle \tau \rangle^s \mathcal{F}_t(f)(\tau), \quad \tau \in \mathbb{R}, f \in \mathcal{S}'(\mathbb{R}),$$

where \mathcal{F}_t denotes the Fourier transform with respect to the time variable $t \in \mathbb{R}$. Moreover, let us introduce the following space

$$W^{-s_1, \infty}(\mathbb{R}, H^{-s_2}(\mathbb{T})) = \{\varphi \in \mathcal{S}'(\mathbb{R} \times \mathbb{T}) : \langle D_t \rangle^{-s_1} \varphi \in L^\infty(\mathbb{R}, H^{-s_2}(\mathbb{T}))\},$$

equipped with the norm

$$\|\varphi\|_{W^{-s_1, \infty}(\mathbb{R}, H^{-s_2}(\mathbb{T}))} = \sup_{t \in \mathbb{R}} \|\langle D_t \rangle^{-s_1} \varphi(t, \cdot)\|_{H^{-s_2}(\mathbb{T})}.$$

By classical arguments, we have the following duality in norm

$$\|\varphi\|_{W^{-s_1, \infty}(\mathbb{R}, H^{-s_2}(\mathbb{T}))} = \sup_{\|\psi\|_{W^{s_1, 1}(\mathbb{R}, H^{s_2}(\mathbb{T}))} \leq 1} \left| \int_{\mathbb{R} \times \mathbb{T}} \varphi(t, x) \psi(t, x) dx dt \right|.$$

Before proving Proposition 2.1, let us check that Theorem 1.1 is a consequence of this result. Let $u_0 \in L^2(\mathbb{T})$ and $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$. In order to simplify the writing, we set

$$u_N(t, x) = \langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} \Pi_N u_0)(x), \quad (t, x) \in \mathbb{R} \times \mathbb{T},$$

and also

$$\begin{aligned} c_N(u_0) &= \int_{\mathbb{R}} \int_{\mathbb{T}} (\Pi_0^\perp \varphi)(t, x) |\langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} \Pi_N u_0)(x)|^2 dx dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{T}} (\Pi_0^\perp \varphi)(t, x) |u_N(t, x)|^2 dx dt. \end{aligned}$$

Recall that we first aim at proving that the sequence $(c_N(u_0))_N$ converges. By combining the following decomposition

$$|u_{N_1}|^2 - |u_{N_2}|^2 = u_{N_1} \overline{u_{N_1}} - u_{N_2} \overline{u_{N_2}} = u_{N_1} (\overline{u_{N_1}} - \overline{u_{N_2}}) + \overline{u_{N_2}} (u_{N_1} - u_{N_2}),$$

and the bilinear estimate given by Proposition 2.1, we get that for every $N_1, N_2 \geq 0$,

$$\begin{aligned} |c_{N_1}(u_0) - c_{N_2}(u_0)| &\lesssim_\varphi (\|\Pi_{N_1} u_0\|_{L^2(\mathbb{T})} + \|\Pi_{N_2} u_0\|_{L^2(\mathbb{T})}) \|\Pi_{N_1} u_0 - \Pi_{N_2} u_0\|_{L^2(\mathbb{T})} \\ &\lesssim \|u_0\|_{L^2(\mathbb{T})} \|\Pi_{N_1} u_0 - \Pi_{N_2} u_0\|_{L^2(\mathbb{T})} \xrightarrow{N_1, N_2 \rightarrow +\infty} 0. \end{aligned}$$

This proves that the sequence $(c_N(u_0))_N$ is a numerical Cauchy sequence, which therefore converges. As a consequence, the sequence (2.1) converges in $\mathcal{S}'(\mathbb{R} \times \mathbb{T})$. Moreover, by

Proposition 2.1, its weak limit $|\langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} u_0)|^2$: satisfies the following bound for every test function $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$,

$$|\langle :|\langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} u_0)|^2, \varphi \rangle_{\mathcal{S}', \mathcal{S}}| \lesssim \|\varphi\|_{W^{s_1, 1}(\mathbb{R}, H^{s_2}(\mathbb{T}))}.$$

This immediately shows that

$$:|\langle D_x \rangle^\sigma (e^{it|D_x|^\alpha} u_0)|^2: \in W^{-s_1, \infty}(\mathbb{R}, H^{-s_2}(\mathbb{T})),$$

as explained in Remark 2.2.

2.2. Bilinear estimates. We now give the proof of Proposition 2.1. Let us consider $\sigma > 0$, $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$, $u_0, v_0 \in L^2(\mathbb{T})$ be fixed functions, and write

$$u_0(x) = \sum_{n \in \mathbb{Z}} a_n e^{inx}, \quad v_0(x) = \sum_{n \in \mathbb{Z}} b_n e^{inx}, \quad x \in \mathbb{T},$$

and

$$(\Pi_0^\perp \varphi)(t, x) = \sum_{n \in \mathbb{Z} \setminus \{0\}} c_n(t) e^{inx}, \quad (t, x) \in \mathbb{R} \times \mathbb{T}.$$

A straightforward integration in space first shows that

$$\int_{\mathbb{T}} (\Pi_0^\perp \varphi)(t, x) (e^{it|D_x|^\alpha} u_0)(x) \overline{(e^{it|D_x|^\alpha} v_0)(x)} dx = \sum_{\substack{n+n_1-n_2=0, \\ n \neq 0}} c_n(t) a_{n_1} \overline{b_{n_2}} e^{it(|n_1|^\alpha - |n_2|^\alpha)}.$$

The integration in time then makes the Fourier transform of the coefficients c_n appear as follows

$$\int_{\mathbb{R}} c_n(t) e^{it(|n_1|^\alpha - |n_2|^\alpha)} dt = \widehat{c}_n(|n_2|^\alpha - |n_1|^\alpha).$$

Thus, the term we aim at estimating is given by

$$\int_{\mathbb{R}} \int_{\mathbb{T}} (\Pi_0^\perp \varphi)(t, x) (e^{it|D_x|^\alpha} u_0)(x) \overline{(e^{it|D_x|^\alpha} v_0)(x)} dx = \sum_{\substack{n+n_1-n_2=0, \\ n \neq 0}} \widehat{c}_n(|n_2|^\alpha - |n_1|^\alpha) a_{n_1} \overline{b_{n_2}}.$$

Notice that since $n \neq 0$ in the above sum, we have $n_1 \neq n_2$. There are then two cases to consider for the integers n_1 and n_2 .

▷ **Resonant case.** In the case where $n_1 = -n_2$, we get $n = -2n_1$ and Cauchy-Schwarz' inequality gives the following bound

$$\begin{aligned} \sum_{\substack{n+n_1-n_2=0, \\ n \neq 0, n_1=-n_2}} |\widehat{c}_n(|n_2|^\alpha - |n_1|^\alpha)| |a_{n_1}| |b_{n_2}| &= \sum_{n_1 \in \mathbb{Z}} |\widehat{c}_{-2n_1}(0)| |a_{n_1}| |b_{-n_1}| \\ &\leq \sup_{n_1 \in \mathbb{Z}} \langle n_1 \rangle^{2\sigma} |\widehat{c}_{n_1}(0)| \sum_{n_1 \in \mathbb{Z}} \frac{|a_{n_1}| |b_{-n_1}|}{\langle 2n_1 \rangle^{2\sigma}} \\ &\lesssim \sup_{n_1 \in \mathbb{Z}} \langle n_1 \rangle^{2\sigma} |\widehat{c}_{n_1}(0)| \|u_0\|_{H^{-\sigma}(\mathbb{T})} \|v_0\|_{H^{-\sigma}(\mathbb{T})}. \end{aligned}$$

Moreover, the above supremum can be estimated by noticing that for every $n_1 \in \mathbb{Z}$,

$$\begin{aligned} \langle n_1 \rangle^{2\sigma} |\widehat{c}_{n_1}(0)| &\leq \langle n_1 \rangle^{2\sigma} \sup_{\tau \in \mathbb{R}} \langle \tau \rangle^s |\widehat{c}_{n_1}(\tau)| \leq \int_{\mathbb{R}} \langle n_1 \rangle^{2\sigma} |\langle D_t \rangle^s c_n(t)| dt \\ &\leq \int_{\mathbb{R}} \|\langle D_t \rangle^s \varphi(t, \cdot)\|_{H^{2\sigma}(\mathbb{T})} dt = \|\varphi\|_{W^{s, 1}(\mathbb{R}, H^{2\sigma}(\mathbb{T}))}, \end{aligned}$$

where we set $s = 2\sigma/(\alpha - 1) > 0$. This value for $s > 0$ will be justified later, while studying the non-resonant case. We therefore deduce that

$$\sum_{\substack{n+n_1-n_2=0, \\ n \neq 0, n_1=-n_2}} |\widehat{c}_n(|n_2|^\alpha - |n_1|^\alpha)| |a_{n_1}| |b_{n_2}| \leq \|\varphi\|_{W^{s, 1}(\mathbb{R}, H^{2\sigma}(\mathbb{T}))} \|u_0\|_{H^{-\sigma}(\mathbb{T})} \|v_0\|_{H^{-\sigma}(\mathbb{T})}.$$

▷ **Non-resonant case.** In the other situation where $n_1 \neq -n_2$, we write

$$(2.2) \quad \sum_{\substack{n+n_1-n_2=0, \\ n \neq 0, n_1 \neq -n_2}} |\widehat{c}_n(|n_2|^\alpha - |n_1|^\alpha)| |a_{n_1}| |b_{n_2}| \leq \sum_{\substack{n+n_1-n_2=0, \\ n \neq 0, n_1 \neq -n_2}} \Gamma_n \frac{|a_{n_1}| |b_{n_2}|}{\langle |n_1|^\alpha - |n_2|^\alpha \rangle^s},$$

where $s = 2\sigma/(\alpha - 1) > 0$ is the same as before, and where we set

$$(2.3) \quad \Gamma_n = \|\langle \tau \rangle^s \widehat{c}_n(\tau)\|_{L^\infty(\mathbb{R})}, \quad n \in \mathbb{Z}.$$

In order to treat the sum in the right-hand side of the estimate (2.2), we need to establish the following lemma.

Lemma 2.3. *For every real numbers $x, y \in \mathbb{R}$, we have*

$$||x|^\alpha - |y|^\alpha| \geq \frac{1}{2^{\alpha-1}} (|x| + |y|)^{\alpha-1} ||x| - |y||.$$

Proof. It sufficient to consider the case where $x, y \geq 0$. Since $\alpha > 1$, we can write

$$x^\alpha - y^\alpha = \int_0^1 \frac{d}{dt} (tx + (1-t)y)^\alpha dt = \int_0^1 \alpha (tx + (1-t)y)^{\alpha-1} (x-y) dt.$$

We therefore get that

$$\begin{aligned} |x^\alpha - y^\alpha| &= \left(\int_0^1 \alpha (tx + (1-t)y)^{\alpha-1} dt \right) |x - y| \\ &\geq \left(\int_0^1 \alpha \min(t, 1-t)^{\alpha-1} dt \right) (x+y)^{\alpha-1} |x - y| \\ &= \frac{1}{2^{\alpha-1}} (x+y)^{\alpha-1} |x - y|. \end{aligned}$$

This is the expected estimate. □

We then deduce from Lemma 2.3 that when $n_1 \neq n_2$, we have

$$||n_1|^\alpha - |n_2|^\alpha| \gtrsim (|n_1| + |n_2|)^{\alpha-1} ||n_1| - |n_2||,$$

and since $||n_1| - |n_2|| \geq 1$, we get that

$$\begin{aligned} \langle |n_1|^\alpha - |n_2|^\alpha \rangle^2 &\gtrsim ||n_1|^\alpha - |n_2|^\alpha|^2 \\ &\gtrsim (|n_1| + |n_2|)^{2(\alpha-1)} ||n_1| - |n_2||^2 \\ &\gtrsim (1 + |n_1|^2 + |n_2|^2)^{\alpha-1} \langle |n_1| - |n_2| \rangle^2 \\ &\gtrsim \langle n_1 \rangle^{\alpha-1} \langle n_2 \rangle^{\alpha-1} \langle |n_1| - |n_2| \rangle^2. \end{aligned}$$

The parameter $s > 0$ is chosen in order to satisfy $s(\alpha - 1)/2 = \sigma$. We therefore deduce that the sum we are studying is bounded in the following way

$$\sum_{\substack{n+n_1-n_2=0, \\ n \neq 0, n_1 \neq -n_2}} |\widehat{c}_n(|n_2|^\alpha - |n_1|^\alpha)| |a_{n_1}| |b_{n_2}| \leq \sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ n_1 \neq \pm n_2}} \frac{\Gamma_{n_1-n_2}}{\langle |n_1| - |n_2| \rangle^s} \frac{|a_{n_1}|}{\langle n_1 \rangle^\sigma} \frac{|b_{n_2}|}{\langle n_2 \rangle^\sigma}.$$

In order to alleviate the writing, let us set

$$K_{n_1, n_2} = \frac{\Gamma_{n_1-n_2}}{\langle |n_1| - |n_2| \rangle^s}, \quad \tilde{a}_{n_1} = \frac{|a_{n_1}|}{\langle n_1 \rangle^\sigma} \quad \text{and} \quad \tilde{b}_{n_2} = \frac{|b_{n_2}|}{\langle n_2 \rangle^\sigma}.$$

We deduce from Cauchy-Schwarz' inequality that

$$\begin{aligned} \sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ n_1 \neq \pm n_2}} K_{n_1, n_2} \tilde{a}_{n_1} \tilde{b}_{n_2} &= \sum_{n_2 \in \mathbb{Z}} \left(\sum_{\substack{n_1 \in \mathbb{Z} \\ n_1 \neq \pm n_2}} K_{n_1, n_2} \tilde{a}_{n_1} \right) \tilde{b}_{n_2} \\ &\leq \left(\sum_{n_2 \in \mathbb{Z}} \left(\sum_{\substack{n_1 \in \mathbb{Z} \\ n_1 \neq \pm n_2}} K_{n_1, n_2} \tilde{a}_{n_1} \right)^2 \right)^{1/2} \left(\sum_{n_2 \in \mathbb{Z}} \tilde{b}_{n_2}^2 \right)^{1/2} \\ &= \left(\sum_{n_2 \in \mathbb{Z}} \left(\sum_{\substack{n_1 \in \mathbb{Z} \\ n_1 \neq \pm n_2}} K_{n_1, n_2} \tilde{a}_{n_1} \right)^2 \right)^{1/2} \|v_0\|_{H^{-\sigma}(\mathbb{T})}. \end{aligned}$$

We need to introduce two new notations still in order to alleviate the writing

$$A_1 = \sup_{n_1 \in \mathbb{Z}} \sum_{\substack{n_2 \in \mathbb{Z} \\ n_2 \neq \pm n_1}} K_{n_1, n_2} \quad \text{and} \quad A_2 = \sup_{n_2 \in \mathbb{Z}} \sum_{\substack{n_1 \in \mathbb{Z} \\ n_1 \neq \pm n_2}} K_{n_1, n_2}.$$

The above double sum is also controlled with Cauchy-Schwarz' inequality as follows

$$\begin{aligned} \sum_{n_2 \in \mathbb{Z}} \left(\sum_{\substack{n_1 \in \mathbb{Z} \\ n_1 \neq \pm n_2}} K_{n_1, n_2} \tilde{a}_{n_1} \right)^2 &\leq \sum_{n_2 \in \mathbb{Z}} \left(\sum_{\substack{n_1 \in \mathbb{Z} \\ n_1 \neq \pm n_2}} K_{n_1, n_2} \right) \left(\sum_{\substack{n_1 \in \mathbb{Z} \\ n_1 \neq \pm n_2}} K_{n_1, n_2} \tilde{a}_{n_1}^2 \right) \\ &\leq A_2 \sum_{n_1 \in \mathbb{Z}} \tilde{a}_{n_1}^2 \left(\sum_{\substack{n_2 \in \mathbb{Z} \\ n_2 \neq \pm n_1}} K_{n_1, n_2} \right) \\ &\leq A_1 A_2 \|u_0\|_{H^{-\sigma}(\mathbb{T})}^2. \end{aligned}$$

It remains to study A_1 and A_2 . Since these two terms are similar, we only focus on A_1 . To that end, we consider a non-negative number $\lambda \geq 0$ such that $\lambda + s > 1/2$. Let $n_1 \in \mathbb{Z}$ be fixed. We deduce once more from Cauchy-Schwarz' inequality that

$$\begin{aligned} \sum_{\substack{n_2 \in \mathbb{Z} \\ n_2 \neq \pm n_1}} K_{n_1, n_2} &= \sum_{\substack{n_2 \in \mathbb{Z} \\ n_2 \neq \pm n_1}} \frac{\Gamma_{n_1 - n_2}}{\langle |n_1| - |n_2| \rangle^s} \leq \left(\sum_{\substack{n_2 \in \mathbb{Z} \\ n_2 \neq \pm n_1}} \langle n_1 - n_2 \rangle^{2\lambda} \Gamma_{n_1 - n_2}^2 \right)^{1/2} \\ &\quad \times \left(\sum_{\substack{n_2 \in \mathbb{Z} \\ n_2 \neq \pm n_1}} \frac{1}{\langle |n_1| - |n_2| \rangle^{2s} \langle n_1 - n_2 \rangle^{2\lambda}} \right)^{1/2}. \end{aligned}$$

There are now two terms to treat. On the one hand, it follows from the definition (2.3) of the Γ_n and Minkowski's integral inequality that

$$\begin{aligned} \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2\lambda} \Gamma_n^2 \right)^{1/2} &= \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2\lambda} \|\langle \tau \rangle^s \widehat{c}_n(\tau)\|_{L^\infty(\mathbb{R})}^2 \right)^{1/2} \\ &\leq \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2\lambda} \|\langle D_t \rangle^s c_n\|_{L^1(\mathbb{R})}^2 \right)^{1/2} \\ &= \|\langle n \rangle^{2\lambda} \langle D_t \rangle^s c_n\|_{l_n^2 L_t^1} \leq \|\langle n \rangle^{2\lambda} \langle D_t \rangle^s c_n\|_{L_t^1 l_n^2} = \|\varphi\|_{W^{s,1}(\mathbb{R}, H^\lambda(\mathbb{T}))}. \end{aligned}$$

On the other hand, the second term can be bounded by noticing that $\||n_1| - |n_2|\| \leq |n_1 - n_2|$ and writing

$$\sum_{\substack{n_2 \in \mathbb{Z} \\ n_2 \neq \pm n_1}} \frac{1}{\langle |n_1| - |n_2| \rangle^{2s} \langle n_1 - n_2 \rangle^{2\lambda}} \leq \sum_{n_2 \in \mathbb{Z}} \frac{1}{\langle |n_1| - |n_2| \rangle^{2(s+\lambda)}} \lesssim \sum_{l \in \mathbb{Z}} \frac{1}{\langle l \rangle^{2(\lambda+s)}}.$$

Since $\lambda + s > 1/2$, the above sum is finite. We therefore deduce that the term A_1 is bounded as follows

$$A_1 \lesssim_{\lambda,s} \|\varphi\|_{W^{s,1}(\mathbb{R}, H^\lambda(\mathbb{T}))}.$$

A similar estimate holds for the other term A_2 , with a similar proof. In a nutshell, we get that

$$\sum_{\substack{n+n_1-n_2=0, \\ n \neq 0, n_1 \neq -n_2}} |\widehat{c}_n(|n_2|^\alpha - |n_1|^\alpha)| |a_{n_1}| |b_{n_2}| \lesssim_{s,\lambda} \|\varphi\|_{W^{s,1}(\mathbb{R}, H^\lambda(\mathbb{T}))} \|u_0\|_{H^{-\sigma}(\mathbb{T})} \|v_0\|_{H^{-\sigma}(\mathbb{T})}.$$

The proof of Proposition 2.1 is now ended.

3. OBSERVABILITY ESTIMATES

The aim of this section is to prove Theorem 1.2. Let $\alpha > 1$ be a positive real number and $b \in L^1(\mathbb{T}) \setminus \{0\}$ be a non-negative function. The strategy consists in four steps:

1. Proof of the following $L_x^\infty L_t^2$ Strichartz estimate

$$\|e^{it|D_x|^\alpha} u_0\|_{L^\infty(\mathbb{T}, L^2(0,T))} \leq C_T \|u_0\|_{L^2(\mathbb{T})}.$$

2. Proof of the following weak observability estimate

$$\|u_0\|_{L^2(\mathbb{T})}^2 \leq C_{b,T} \left(\int_0^T \int_{\mathbb{T}} b(x) |(e^{it|D_x|^\alpha} u_0)(x)|^2 dx dt + \|u_0\|_{H^{-\alpha}(\mathbb{T})}^2 \right).$$

3. Proof of the following unique continuation result

$$(be^{it|D_x|^\alpha} u_0 = 0 \text{ on } (0, T) \times \mathbb{T}) \implies u_0 \equiv 0 \text{ on } \mathbb{T}.$$

4. Removing the $H^{-\alpha}$ -norm in the above weak observability estimate.

Steps 2,3 and 4 are now very classical and had been used to study the observability properties of Schrödinger-type equations on tori, see [3] for a review. Steps 3 and 4 are known as the Bardos–Lebeau–Rauch uniqueness-compactness argument. The $L_x^\infty L_t^2$ Strichartz estimate presented in Step 1, crucial for dealing with the case $b \in L^2(\mathbb{T})$, has been first introduced in the paper [1] in the non-fractional case $\alpha = 2$.

3.1. Step 1: Strichartz type estimate. As a preliminary, let us first prove a $L_x^\infty L_t^2$ estimate for the solutions of the equation (S_α) , which is an adaptation of [1, Proposition 2.1] to the fractional case. A key feature of this estimate is that it does not encounter derivative losses as does e.g. the $L_{t,x}^4$ Strichartz estimates for $e^{it|D_x|^\alpha}$ obtained in [8].

Lemma 3.1. *For every positive time $T > 0$, there exists a positive constant $C_T > 0$ such that for every initial datum $u_0 \in L^2(\mathbb{T})$,*

$$\|e^{it|D_x|^\alpha} u_0\|_{L^\infty(\mathbb{T}, L^2(0,T))} \leq C_T \|u_0\|_{L^2(\mathbb{T})}.$$

Proof. Setting $c_n = \widehat{u_0}(n)$, we begin by developing the $L_x^\infty L_t^2$ norm we aim at estimating

$$\begin{aligned} \|e^{it|D_x|^\alpha} u_0\|_{L^\infty(\mathbb{T}, L^2(0,T))}^2 &= \sup_{x \in \mathbb{T}} \int_0^T \left| \sum_{n \in \mathbb{Z}} c_n e^{it|n|^\alpha} e^{inx} \right|^2 dt \\ &= \sup_{x \in \mathbb{T}} \sum_{n_1, n_2 \in \mathbb{Z}} \left(\int_0^T e^{it(|n_1|^\alpha - |n_2|^\alpha)} dt \right) e^{i(n_1 - n_2)x} c_{n_1} \overline{c_{n_2}}. \end{aligned}$$

There are then two cases to consider for the integers n_1 and n_2 in the above sum.

▷ **Resonant case.** In the situation where $n_1 = \pm n_2$, we directly get that

$$\sup_{x \in \mathbb{T}} \sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ n_1 = \pm n_2}} \left(\int_0^T e^{it(|n_1|^\alpha - |n_2|^\alpha)} dt \right) e^{i(n_1 - n_2)x} c_{n_1} \overline{c_{n_2}} \lesssim_T \sum_{n_1 \in \mathbb{Z}} |c_{n_1}|^2 = \|u_0\|_{L^2(\mathbb{T})}^2.$$

▷ **Non-resonant case.** In the case where $n_1 \neq \pm n_2$, we begin by writing that

$$\sup_{x \in \mathbb{T}} \sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ n_1 \neq \pm n_2}} \left(\int_0^T e^{it(|n_1|^\alpha - |n_2|^\alpha)} dt \right) e^{i(n_1 - n_2)x} c_{n_1} \overline{c_{n_2}} \leq \sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ n_1 \neq \pm n_2}} \frac{|c_{n_1}| |c_{n_2}|}{||n_1|^\alpha - |n_2|^\alpha|}.$$

By using anew Lemma 2.3, we obtain the following bound for the above denominator

$$||n_1|^\alpha - |n_2|^\alpha| \gtrsim (|n_1| + |n_2|)^{\alpha-1} ||n_1| - |n_2|| \geq ||n_1| - |n_2||^\alpha.$$

Cauchy-Schwarz' inequality then implies that

$$\begin{aligned} \sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ n_1 \neq \pm n_2}} \frac{|c_{n_1}| |c_{n_2}|}{||n_1|^\alpha - |n_2|^\alpha|} &\lesssim \sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ n_1 \neq \pm n_2}} \frac{|c_{n_1}| |c_{n_2}|}{||n_1| - |n_2||^\alpha} \\ &= \sum_{\substack{n_1, n_2 \geq 0 \\ n_1 \neq n_2}} \frac{|c_{n_1}| |c_{n_2}| + |c_{-n_1}| |c_{n_2}| + |c_{n_1}| |c_{-n_2}| + |c_{-n_1}| |c_{-n_2}|}{|n_1 - n_2|^\alpha} \\ &\lesssim \left(\sum_{l \in \mathbb{N} \setminus \{0\}} \frac{1}{l^\alpha} \right) \|u_0\|_{L^2(\mathbb{T})}^2. \end{aligned}$$

Notice that the above sum is finite since $\alpha > 1$. This ends the proof of Lemma 3.1. \square

3.2. Step 2: estimate with error term. We now derive the following weak observability estimates from the smoothing estimates stated in Proposition 2.1.

Lemma 3.2. *For every positive time $T > 0$, there exists a positive constant $C_{b,T} > 0$ such that for every $u_0 \in L^2(\mathbb{T})$,*

$$\|u_0\|_{L^2(\mathbb{T})}^2 \leq C_{b,T} \left(\int_0^T \int_{\mathbb{T}} b(x) |(e^{it|D_x|^\alpha} u_0)(x)|^2 dx dt + \|u_0\|_{H^{-\alpha}(\mathbb{T})}^2 \right).$$

Proof. Arguing by contradiction, we consider a sequence $(u_{n,0})_n$ in $L^2(\mathbb{T})$ such that $\|u_{n,0}\|_{L^2(\mathbb{T})} = 1$ and satisfying

$$(3.1) \quad \int_0^T \int_{\mathbb{T}} b(x) |(e^{it|D_x|^\alpha} u_{n,0})(x)|^2 dx dt + \|u_{n,0}\|_{H^{-\alpha}(\mathbb{T})}^2 \leq \frac{1}{n}.$$

Setting $u_n(t) = e^{it|D_x|^\alpha} u_{n,0}$, we aim at proving that the sequence $(u_n)_n$ satisfies that for all $\psi \in C_0^\infty(0, T)$

$$(3.2) \quad \int_0^T \psi(t) \|u_n(t)\|_{L^2(\mathbb{T})}^2 dt \xrightarrow{n \rightarrow +\infty} 0.$$

This would end the proof of Lemma 3.2. Indeed, the conservation of mass of the equation (S_α) would then imply that

$$\int_0^T \psi(t) \|u_n(t)\|_{L^2(\mathbb{T})}^2 dt = \int_0^T \psi(t) \|u_{n,0}\|_{L^2(\mathbb{T})}^2 dt = \int_0^T \psi(t) dt \xrightarrow{n \rightarrow +\infty} 0.$$

This convergence of course cannot hold for all functions $\psi \in C_0^\infty(0, T)$.

▷ **Step 2.1.** First of all, let us prove that for every $x_0 \in \mathbb{T}$,

$$(3.3) \quad \int_0^T \int_{\mathbb{T}} \psi(t) (\tau_{x_0} b)(x) |u_n(t, x)|^2 dx dt \xrightarrow{n \rightarrow +\infty} 0,$$

where the translation operator τ_{x_0} is defined for every function in $f \in L^1(\mathbb{T})$ by

$$(\tau_{x_0} f)(x) = f(x - x_0), \quad x \in \mathbb{T}.$$

To that end, we will widely use the fact that for every smooth function $\varphi \in C^\infty(\mathbb{T})$, we get from the smoothing estimates stated in Proposition 2.1 applied with $\sigma = \alpha$, $u_0 = v_0 = u_{n,0}$ and the tensorized test function $\psi \otimes \varphi$ that

$$(3.4) \quad \left| \int_0^T \int_{\mathbb{T}} \psi(t)(\Pi_0^\perp \varphi)(x) |u_n(t, x)|^2 dx dt \right| \lesssim_{\psi, \varphi} \|u_{n,0}\|_{H^{-\alpha}(\mathbb{T})}^2 \xrightarrow{n \rightarrow +\infty} 0 \quad \text{by (3.1)}.$$

Let us consider a sequence $(b_j)_j$ in $C^\infty(\mathbb{T})$ that converges to the function b in $L^1(\mathbb{T})$, and write

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{T}} \psi(t)(b - \tau_{x_0} b)(x) |u_n(t, x)|^2 dx dt \right| &\leq \left| \int_0^T \int_{\mathbb{T}} \psi(t)(b_j - \tau_{x_0} b_j)(x) |u_n(t, x)|^2 dx dt \right| \\ &+ \left| \int_0^T \int_{\mathbb{T}} \psi(t)(b - b_j - \tau_{x_0}(b - b_j))(x) |u_n(t, x)|^2 dx dt \right|. \end{aligned}$$

There are now two terms to treat. On the one hand, we deduce from the convergence (3.4) with the functions $\varphi = b_j - \tau_{x_0} b_j$, which satisfy $\Pi_0^\perp \varphi = \varphi$, that the following convergence holds for every integer $j \geq 0$

$$\int_0^T \int_{\mathbb{T}} \psi(t)(b_j - \tau_{x_0} b_j)(x) |u_n(t, x)|^2 dx dt \xrightarrow{n \rightarrow +\infty} 0.$$

On the other hand, we deduce from the $L_x^\infty L_t^2$ Strichartz estimate stated in Lemma 3.1 that

$$\left| \int_0^T \int_{\mathbb{T}} \psi(t)(b - b_j - \tau_{x_0}(b - b_j))(x) |u_n(t, x)|^2 dx dt \right| \lesssim_\psi \|b - b_j\|_{L^1(\mathbb{T})},$$

since the functions $u_{n,0}$ are normalized in $L^2(\mathbb{T})$. We therefore deduce that

$$\limsup_{n \rightarrow +\infty} \left| \int_0^T \int_{\mathbb{T}} \psi(t)(b - \tau_{x_0} b)(x) |u_n(t, x)|^2 dx dt \right| \lesssim \|b - b_j\|_{L^1(\mathbb{T})} \xrightarrow{j \rightarrow +\infty} 0.$$

Recalling from (3.1) that we also have

$$\int_0^T \int_{\mathbb{T}} \psi(t)b(x) |u_n(t, x)|^2 dx dt \xrightarrow{n \rightarrow +\infty} 0,$$

this proves that the convergence (3.3) actually holds. Notice that in the very particular case where $b = 1_\omega$ and $\omega \subset \mathbb{T}$ is an open set, we directly conclude to the convergence (3.2) by covering the circle \mathbb{T} with a finite number of open sets of the form $\tau_{x_0}\omega$ with $x_0 \in \mathbb{T}$. The $L_x^\infty L_t^2$ Strichartz estimate is in fact not necessary in this case, since it suffices to apply the convergence (3.4) with the functions $\varphi = f - \tau_{x_0} f$, where $f \in C^\infty(\mathbb{T})$ is a smooth function such that $\text{supp } f \subset \omega$.

▷ **Step 2.2.** In order to derive the convergence (3.2) from (3.3) when $b \in L^1(\mathbb{T})$, we need to establish the following approximation result.

Lemma 3.3. *For every function $f \in L^1(\mathbb{T})$, we have*

$$1 \in \overline{\text{vect}\{\tau_{x_0} f : x_0 \in \mathbb{T}\}} \iff \widehat{f}(0) \neq 0,$$

the closure being taken with respect to the $L^1(\mathbb{T})$ topology.

Proof. The necessary part of this statement follows from the fact that

$$\forall x_0 \in \mathbb{T}, \quad \widehat{\tau_{x_0} f}(0) = \int_{\mathbb{T}} (\tau_{x_0} f)(x) dx = \int_{\mathbb{T}} f(x) dx = \widehat{f}(0).$$

Concerning the sufficient part, we can assume without lost of generality that $\widehat{f}(0) = 1$. Let us consider $\alpha \in \mathbb{R} \setminus 2\pi\mathbb{Q}$. In order to simplify the writing, we will use the following notation for every positive integer $n \geq 1$ and $g \in L^1(\mathbb{T})$

$$S_n(g) = \frac{1}{n} \sum_{k=0}^{n-1} \tau_{k\alpha} g \in \text{vect}\{\tau_{x_0} g : x_0 \in \mathbb{T}\}.$$

We consider a sequence $(f_j)_j$ in $L^2(\mathbb{T})$ that converges to the function f in $L^1(\mathbb{T})$, and we write

$$\begin{aligned} \|1 - S_n(f)\|_{L^1(\mathbb{T})} &\leq \|1 - S_n(f_j)\|_{L^1(\mathbb{T})} + \|S_n(f_j) - S_n(f)\|_{L^1(\mathbb{T})} \\ &\leq \|1 - S_n(f_j)\|_{L^2(\mathbb{T})} + \frac{1}{n} \sum_{k=0}^{n-1} \|\tau_{k\alpha} f_j - \tau_{k\alpha} f\|_{L^1(\mathbb{T})} \\ &\leq |1 - \widehat{f_j}(0)| + \|\widehat{f_j}(0) - S_n(f_j)\|_{L^2(\mathbb{T})} + \|f_j - f\|_{L^1(\mathbb{T})}. \end{aligned}$$

Moreover, a classical consequence of von Neumann's ergodic theorem [7, Theorem 5.1] implies that for every integer $j \geq 0$,

$$\|\widehat{f_j}(0) - S_n(f_j)\|_{L^2(\mathbb{T})} \xrightarrow{n \rightarrow +\infty} 0.$$

This implies that

$$\limsup_{n \rightarrow +\infty} \|1 - S_n(f)\|_{L^1(\mathbb{T})} \leq |1 - \widehat{f}(0)| + \|f_j - f\|_{L^1(\mathbb{T})} \leq 2\|f_j - f\|_{L^1(\mathbb{T})} \xrightarrow{j \rightarrow +\infty} 0.$$

The proof of Lemma 3.3 is therefore ended. It should be noted, however, that in the particular case where $f \in L^2(\mathbb{T})$, the above ergodic argument can be simplified to a projection argument. Indeed, let us introduce the following notation

$$F = \overline{\text{vect}\{\tau_{x_0} f : x_0 \in \mathbb{T}\}} \subset L^2(\mathbb{T}).$$

Let $\varphi \in F$ be the orthogonal projection of the constant function 1 onto the closed vector space F . Since $1 - \varphi \in F^\perp$, we get in particular that

$$\forall x_0 \in \mathbb{T}, \quad 0 = \langle 1 - \varphi, \tau_{x_0} f \rangle_{L^2(\mathbb{T})} = \sum_{n \in \mathbb{Z}} \widehat{1 - \varphi}(n) \widehat{f}(n) e^{inx_0}.$$

This formula shows that the function $x_0 \mapsto \langle 1 - \varphi, \tau_{x_0} f \rangle_{L^2(\mathbb{T})}$ belongs to the space $C(\mathbb{T})$. We therefore deduce that

$$\forall n \in \mathbb{Z}, \quad \widehat{1 - \varphi}(n) \widehat{f}(n) = 0.$$

Since $\widehat{f}(0) \neq 0$, we get that $\widehat{\varphi}(0) = 1$. Moreover, $\|\varphi\|_{L^2(\mathbb{T})} \leq 1$, and we conclude that $\varphi = 1$, implying that $1 \in F$ as expected. We were not able then to pass directly to the case where $f \in L^1(\mathbb{T})$ with an approximation argument. This is the reason why we made use of von Neumann's ergodic theorem. \square

We can now prove (3.2). Since the assumptions on the function b imply that $\widehat{b}(0) \neq 0$, we deduce from Lemma 3.3 that there exists a sequence $(b_j)_j$ with

$$b_j \in \text{vect}\{\tau_{x_0} b : x_0 \in \mathbb{T}\},$$

satisfying that

$$b_j \xrightarrow{j \rightarrow +\infty} 1 \quad \text{in } L^1(\mathbb{T}).$$

Let us now write

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{T}} \psi(t) |u_n(t, x)|^2 dx dt \right| &\leq \left| \int_0^T \int_{\mathbb{T}} \psi(t) (1 - b_j)(x) |u_n(t, x)|^2 dx dt \right| \\ &\quad + \left| \int_0^T \int_{\mathbb{T}} \psi(t) b_j(x) |u_n(t, x)|^2 dx dt \right|. \end{aligned}$$

The first term can be treated as follows by using anew Lemma 3.1

$$\left| \int_0^T \int_{\mathbb{T}} \psi(t) (1 - b_j)(x) |u_n(t, x)|^2 dx dt \right| \lesssim_\psi \|1 - b_j\|_{L^1(\mathbb{T})}.$$

Moreover, as a consequence of (3.3), we get that for every $j \geq 0$,

$$\int_0^T \int_{\mathbb{T}} \psi(t) b_j(x) |u_n(t, x)|^2 dx dt \xrightarrow{n \rightarrow +\infty} 0.$$

We therefore deduce that

$$\limsup_{n \rightarrow +\infty} \left| \int_0^T \int_{\mathbb{T}} \psi(t) |u_n(t, x)|^2 dx dt \right| \lesssim \|1 - b_j\|_{L^1(\mathbb{T})} \xrightarrow{j \rightarrow +\infty} 0.$$

This ends the proof of Lemma 3.2. \square

3.3. Step 3: unique continuation. Let us now consider the following vector space

$$\mathcal{N}_{b,T} = \{u_0 \in L^2(\mathbb{T}) : \sqrt{b} e^{it|D_x|^\alpha} u_0 = 0 \text{ on } (0, T) \times \mathbb{T}\}.$$

In this subsection, we prove the following unique continuation result.

Lemma 3.4. *For every positive time $T > 0$, we have*

$$\mathcal{N}_{b,T} = \{0\}.$$

Proof. Let us begin by checking that the vector space $\mathcal{N}_{b,T}$ is invariant under the action of the operator $|D_x|^\alpha$, with moreover

$$(3.5) \quad \exists c > 0, \forall u_0 \in \mathcal{N}_{b,T}, \quad \| |D_x|^\alpha u_0 \|_{L^2(\mathbb{T})} \leq c \|u_0\|_{L^2(\mathbb{T})}.$$

Let $u_0 \in \mathcal{N}_{b,T}$ and $\varepsilon \in (0, T)$. We also consider the function $u_\varepsilon = \varepsilon^{-1} (e^{i\varepsilon|D_x|^\alpha} u_0 - u_0)$. Since $u_\varepsilon \in \mathcal{N}_{b,T/2}$ when $0 < \varepsilon < T/2$, we deduce from Lemma 3.2 applied at time $T/2$ that

$$(3.6) \quad \|u_\varepsilon\|_{L^2(\mathbb{T})} \leq C_{b,T/2} \|u_\varepsilon\|_{H^{-\alpha}(\mathbb{T})}.$$

Moreover, Plancherel's theorem and the dominated convergence theorem imply that

$$\|u_\varepsilon\|_{H^{-\alpha}(\mathbb{T})} \rightarrow \| |D_x|^\alpha \langle D_x \rangle^{-\alpha} u_0 \|_{L^2(\mathbb{T})} \leq \|u_0\|_{L^2(\mathbb{T})} \quad \text{as } \varepsilon \rightarrow 0^+.$$

Indeed, on the one hand, the following convergence holds for every $n \in \mathbb{Z}$,

$$\langle n \rangle^{-\alpha} |\widehat{u_\varepsilon}(n)| = \left| \frac{e^{i\varepsilon|n|^\alpha} \widehat{u_0}(n) - \widehat{u_0}(n)}{\varepsilon \langle n \rangle^\alpha} \right| = \frac{|n|^\alpha}{\langle n \rangle^\alpha} \left| \frac{e^{i\varepsilon|n|^\alpha} - 1}{\varepsilon |n|^\alpha} \right| |\widehat{u_0}(n)| \xrightarrow{\varepsilon \rightarrow 0^+} \frac{|n|^\alpha}{\langle n \rangle^\alpha} |\widehat{u_0}(n)|.$$

On the other hand, we have the following domination from the mean value theorem for every $\varepsilon > 0$ and $n \in \mathbb{Z}$,

$$\langle n \rangle^{-\alpha} |\widehat{u_\varepsilon}(n)| \leq |\widehat{u_0}(n)|.$$

In particular, the sequence $(u_\varepsilon)_\varepsilon$ is bounded in $H^{-\alpha}(\mathbb{T})$. This fact combined with the estimate (3.6) implies that the sequence $(u_\varepsilon)_\varepsilon$ is also bounded in $L^2(\mathbb{T})$ and therefore weakly converges in this space up to extracting. Since the sequence $(u_\varepsilon)_\varepsilon$ also converges to $i|D_x|^\alpha u_0$ in $\mathcal{S}'(\mathbb{T})$, the aforementioned weak limit is nothing but the function $i|D_x|^\alpha u_0$, which therefore belongs to $L^2(\mathbb{T})$. Plancherel's theorem and the dominated convergence theorem then implies that

$$\|u_\varepsilon\|_{L^2(\mathbb{T})} \rightarrow \| |D_x|^\alpha u_0 \|_{L^2(\mathbb{T})} \quad \text{as } \varepsilon \rightarrow 0^+.$$

As a consequence, the sequence $(u_\varepsilon)_\varepsilon$ converges strongly to the function $i|D_x|^\alpha u_0$ in $L^2(\mathbb{T})$. The expected estimate (3.5) is then obtained by passing to the limit as $\varepsilon \rightarrow 0^+$ in (3.6). Moreover, since $u_\varepsilon \in \mathcal{N}_{b,T-\delta}$ when $0 < \varepsilon < \delta$ and that the strong convergence in $L^2(\mathbb{T})$ implies almost everywhere pointwise convergence up to extraction, we deduce that $|D_x|^\alpha u_0 \in \mathcal{N}_{b,T-\delta}$ for every $\delta \in (0, T)$. This implies that the function $|D_x|^\alpha u_0$ also belongs to $\mathcal{N}_{b,T}$. Hence, the vector space $\mathcal{N}_{b,T}$ is invariant under the action of the linear operator $|D_x|^\alpha$.

Notice that the estimate (3.5) implies that the unit ball of $\mathcal{N}_{b,T}$ is bounded in $H^\alpha(\mathbb{T})$, and is therefore relatively compact by Rellich's theorem. This implies that the vector space $\mathcal{N}_{b,T}$ is finite-dimensional by Riesz' theorem. As a consequence, since the linear operator $|D_x|^\alpha : \mathcal{N}_{b,T} \rightarrow \mathcal{N}_{b,T}$ is bounded and in the situation where $\mathcal{N}_{b,T} \neq \{0\}$, there exists $\lambda \in \mathbb{C}$ and $u \in \mathcal{N}_{b,T} \setminus \{0\}$ such that $|D_x|^\alpha u = \lambda u$. The function u not being identically equal to zero on \mathbb{T} , we deduce by passing to the Fourier side that there exists $n \in \mathbb{Z}$ such that $\lambda = |n|^\alpha$, and that the function u is actually given by

$$u(x) = c_1 e^{inx} + c_2 e^{-inx}, \quad x \in \mathbb{T},$$

with $c_1, c_2 \in \mathbb{C}$ such that $c_1 c_2 \neq 0$. Moreover, since u is an eigenfunction of the operator $|D_x|^\alpha$ and $u \in \mathcal{N}_{b,T}$, we get that the function u vanishes on $\{b > 0\}$, a measurable set with positive measure, which is absurd. As a consequence, we have $\mathcal{N}_{b,T} = \{0\}$ as expected. \square

3.4. Step 4: removing the $H^{-\alpha}$ -norm. We are now in position to end the proof of Theorem 1.2. We argue once again by contradiction by considering a sequence $(u_{n,0})_n$ in $L^2(\mathbb{T})$ such that $\|u_{n,0}\|_{L^2(\mathbb{T})} = 1$ and satisfying

$$(3.7) \quad \int_0^T \int_{\mathbb{T}} b(x) |(e^{it|D_x|^\alpha} u_{n,0})(x)|^2 dx dt \leq \frac{1}{n}.$$

The sequence $(u_{n,0})_n$ being bounded in $L^2(\mathbb{T})$, there exists a function $u_0 \in L^2(\mathbb{T})$ such that, up to extracting, the sequence $(u_{n,0})_n$ converges weakly to u_0 in $L^2(\mathbb{T})$ and strongly in $H^{-\alpha}(\mathbb{T})$, by Rellich's theorem. Notice that the weak convergence

$$\forall t \in [0, T], \quad \sqrt{b} e^{it|D_x|^\alpha} u_{n,0} \xrightarrow{n \rightarrow +\infty} \sqrt{b} e^{it|D_x|^\alpha} u_{n,0} \quad \text{in } L^2(\mathbb{T}),$$

is a consequence of the following computations, where $\varphi \in L^2(\mathbb{T})$,

$$\begin{aligned} \langle \sqrt{b} e^{it|D_x|^\alpha} u_{n,0}, \varphi \rangle_{L^2(\mathbb{T})} &= \langle u_{n,0}, e^{-it|D_x|^\alpha} \sqrt{b} \varphi \rangle_{L^2(\mathbb{T})} \\ &\xrightarrow{n \rightarrow +\infty} \langle u_0, e^{-it|D_x|^\alpha} \sqrt{b} \varphi \rangle_{L^2(\mathbb{T})} = \langle \sqrt{b} e^{it|D_x|^\alpha} u_0, \varphi \rangle_{L^2(\mathbb{T})}. \end{aligned}$$

We therefore deduce from results on weak convergence, Fatou's lemma and (3.7) that

$$\begin{aligned} \int_0^T \int_{\mathbb{T}} b(x) |(e^{it|D_x|^\alpha} u_0)(x)|^2 dx dt &\leq \int_0^T \liminf_{n \rightarrow +\infty} \int_{\mathbb{T}} b(x) |(e^{it|D_x|^\alpha} u_{n,0})(x)|^2 dx dt \\ &\leq \liminf_{n \rightarrow +\infty} \int_0^T \int_{\mathbb{T}} b(x) |(e^{it|D_x|^\alpha} u_{n,0})(x)|^2 dx dt = 0. \end{aligned}$$

Keeping the notations of Section 3.3, this proves that $u_0 \in \mathcal{N}_{b,T}$, and therefore $u_0 \equiv 0$ by Lemma 3.4. We finally deduce from Lemma 3.2 that

$$\underbrace{\|u_{n,0}\|_{L^2(\mathbb{T})}^2}_{=1} \leq C_{b,T} \left(\underbrace{\int_0^T \int_{\mathbb{T}} b(x) |(e^{it|D_x|^\alpha} u_{n,0})(x)|^2 dx dt}_{\xrightarrow{n \rightarrow +\infty} 0} + \underbrace{\|u_{n,0}\|_{H^{-\alpha}(\mathbb{T})}^2}_{\xrightarrow{n \rightarrow +\infty} 0} \right).$$

This is of course absurd, and the proof of Theorem 1.2 is ended.

REFERENCES

- [1] J. BOURGAIN, N. BURQ & M. ZWORSKI, *Control for Schrödinger operators on 2-tori: rough potentials*, J. Eur. Math. Soc. **15** (2013), 1597–1628.
- [2] N. BURQ & M. ZWORSKI, *Rough controls for Schrödinger operators on 2-tori*, Ann. H. Lebesgue **2** (2019), 331–347.
- [3] C. LAURENT, *Internal control of the Schrödinger equation*, Math. Control Relat. Fields **4** (2014), 161–186.
- [4] K. LE BALC'H & J. MARTIN, *Observability estimates for the Schrödinger equation in the plane with periodic bounded potentials from measurable sets*, preprint (2023), [arXiv:2304.08050](https://arxiv.org/abs/2304.08050).
- [5] F. MACIÀ, *Observability results related to fractional Schrödinger operators*, Vietnam J. Math. **49** (2021), 919–936.
- [6] B. SIMON, *The $P(\varphi)_2$ Euclidean (quantum) field theory*, Princeton Series in Physics. Princeton University Press, Princeton, N.J., 1974. xx+392 pp.
- [7] E. STEIN & R. SHAKARCHI, *Real analysis. Measure theory, integration, and Hilbert spaces*, Princeton Lect. Anal., vol. 3, Princeton, NJ: Princeton University Press, 2005.
- [8] C. SUN & N. TZVETKOV, *Gibbs measure dynamics for the fractional NLS*, SIAM J. Math. Anal. **52** (2020), 4638–4704.
- [9] N. TZVETKOV, *Nonlinear PDE in the presence of singular randomness*, Eur. Math. Soc. Mag. **129** (2023), 5–12.

(PAUL ALPHONSE) ENS DE LYON, UMPA, UMR CNRS 5669, LYON, FRANCE
Email address: paul.alphonse@ens-lyon.fr

(NIKOLAY TZVETKOV) ENS DE LYON, UMPA, UMR CNRS 5669, LYON, FRANCE
Email address: nikolay.tzvetkov@ens-lyon.fr