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► **To cite this version:**

| Jan Bok, Antoine Dailly, Tuomo Lehtilä. Resolving Sets in Temporal Graphs. 2024. hal-04511235

HAL Id: hal-04511235

<https://hal.science/hal-04511235v1>

Preprint submitted on 19 Mar 2024

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Resolving Sets in Temporal Graphs^{*†}

Jan Bok¹, Antoine Dailly¹, and Tuomo Lehtilä^{1,2,3}

¹Université Clermont-Auvergne, CNRS, Mines de Saint-Étienne, Clermont-Auvergne-INP, LIMOS, 63000 Clermont-Ferrand, France

²Department of Computer Science, University of Helsinki, Helsinki, Finland

³Helsinki Institute for Information Technology (HIIT), Espoo, Finland

Abstract

A *resolving set* R in a graph G is a set of vertices such that every vertex of G is uniquely identified by its distances to the vertices of R . Introduced in the 1970s, this concept has been since then extensively studied from both combinatorial and algorithmic point of view. We propose a generalization of the concept of resolving sets to temporal graphs, *i.e.*, graphs with edge sets that change over discrete time-steps. In this setting, the *temporal distance from u to v* is the earliest possible time-step at which a journey with strictly increasing time-steps on edges leaving u reaches v , *i.e.*, the first time-step at which v could receive a message broadcast from u . A *temporal resolving set* of a temporal graph \mathcal{G} is a subset R of its vertices such that every vertex of \mathcal{G} is uniquely identified by its temporal distances from vertices of R . We study the problem of finding a minimum-size temporal resolving set, and show that it is NP-complete even on very restricted graph classes and with strong constraints on the time-steps: temporal complete graphs where every edge appears in either time-step 1 or 2, temporal trees where every edge appears in at most two consecutive time-steps, and even temporal subdivided stars where every edge appears in at most two (not necessarily consecutive) time-steps. On the other hand, we give polynomial-time algorithms for temporal paths and temporal stars where every edge appears in exactly one time-step, and give a combinatorial analysis and algorithms for several temporal graph classes where the edges appear in periodic time-steps.

Temporal graphs, Resolving sets, Metric dimension, Trees, Graph algorithms, Complexity

1 Introduction

For a set R of vertices of a graph G , every vertex of G can compute a vector of its distances from the vertices of R (the distance, or number of edges, in a shortest path from u to v will be denoted by $\text{dist}(u, v)$). If all such computed vectors are unique, then we call R a *resolving set* of G . This notion was introduced in the 1970s and gave birth to the notion of *metric dimension of G* , that is, the smallest size of a resolving set of G . Metric dimension is a well-studied topic, with both combinatorial and algorithmic results, see for example surveys [30, 34].

A *temporal graph* can be defined as a graph on a given vertex set, and with an edge set that changes over discrete time-steps. Their study gained traction as a natural representation of dynamic, evolving networks [4,

^{*}This work was supported by the International Research Center "Innovation Transportation and Production Systems" of the I-SITE CAP 20-25 and by the ANR project GRALMECO (ANR-21-CE48-0004). The research of the third author was supported by Business Finland Project 6GNTF, funding decision 10769/31/2022.

[†]A shorter conference version of this manuscript was submitted to IWOCA 2024 conference.

22, 23, 32]. However, in the temporal setting, the notion of distance differs from the static setting: two vertices can be topologically close, but the *journey* (*i.e.*, a path in the underlying graph with strictly increasing time-steps¹) between them can be long, or even impossible. The shortest time-step at which a journey from u reaches v is called the *temporal distance from u to v* and is denoted by $\text{dist}_t(u, v)$. Note that in a temporal graph, there might be vertex pairs (u, v) such that there is no temporal journey from u to v , in which case we set the distance as infinite. Furthermore, the temporal distance of non-adjacent vertices is not necessarily symmetric.

The notion of temporal distance allows us to define a *temporal resolving set* as a set R of vertices of a temporal graph such that every vertex has a unique vector of temporal distances from the vertices of R . We are interested in the problem of finding a minimum-size temporal resolving set.

Our motivation is both introducing a temporal variant of the well-studied problem of resolving sets and studying its combinatorial and algorithmic properties, as well as the problem of locating in dynamic networks. Indeed, resolving sets are an analogue of geolocation in discrete structures [34], and thus temporal resolving sets are similar: if we consider that transmitters placed on vertices of the temporal resolving set emit continuously, we can locate ourselves by waiting long enough to receive the signals and constructing the temporal distance vector.

In the remainder of this section, we give an overview of the static (*i.e.*, non-temporal) version and some variants of resolving set as well as an overview of temporal graphs, before giving a formal definition of the TEMPORAL RESOLVING SET problem and an outline of our results.

Separating vertices. Standard resolving sets were introduced independently by Harary and Melter [20] and by Slater [33], and have been well-studied due to their various applications (robot navigation [28], detection in sensor networks [33], and more [34]). Their non-local nature makes them difficult to study from an algorithmic point of view: finding a minimum-size resolving set is NP-hard even on very restricted graph classes (planar graphs of bounded degree [7], bipartite graphs [8], and interval graphs of diameter 2 [14], to name a few) and W[2]- and W[1]-hard when parameterized by solution size [21] and feedback vertex set [16], respectively. On the positive side, the problem is polynomial-time solvable on trees [5, 20, 28, 33], outerplanar graphs [7] and cographs [8], to name a few. Note that there are two conditions: *reaching* (every vertex must be reached from some vertex of the resolving set) and *separating* (no two vertices may have the same distance vector). A *weak resolving set* is a set that requires only separating vertices. Minimum-size standard and weak resolving sets can thus differ in size by at most 1.

A natural variant of resolving sets consists in limiting the distance at which transmitters can emit. A first constraint is that of a robot which can only perceive its direct neighborhood: an *adjacency resolving set* [25] is a resolving set using the following distance: $\text{dist}_a(u, v) = \max(\text{dist}(u, v), 2)$. More generally, a *k -truncated resolving set* [10] is a resolving set using the following distance: $\text{dist}_k(u, v) = \min(\text{dist}(u, v), k + 1)$ (we can see that an adjacency resolving set is a 1-truncated resolving set). Note that both variants were mostly studied in their weak version. The k -truncated resolving sets have quickly attracted attention on the combinatorial side [2, 15, 17, 18]. On the algorithmic side, the corresponding decision problem is known to be NP-hard [10], even on trees (although, in this case, it becomes polynomial-time solvable when polynomial-time solvable when k is fixed) [19].

Temporal graphs. A temporal graph $\mathcal{G} = (V, E_1, \dots, E_{t_{\max}})$ is described by a sequence of edge sets representing the graph at discrete *time-steps*, which are positive integers in $\{1, \dots, t_{\max}\}$ [4] (note that the sequence might be infinite, in which case the number of time-steps is not bounded by t_{\max} , and we can adapt all definitions in consequence). An alternate, equivalent description of \mathcal{G} is $\mathcal{G} = (V, E, \lambda)$ (or (G, λ) where $G = (V, E)$) with $E = \bigcup_{i=1}^{t_{\max}} E_i$ is called the *underlying graph* and $\lambda : E \rightarrow 2^{\{1, \dots, t_{\max}\}}$ is an edge-labeling function called *time labeling* such that $\lambda(e)$ is the set of time-steps at which the edge e exists, *i.e.*, $i \in \lambda(e)$

¹Those are sometimes called *strict* journeys in the literature, but as argued in [29], strict journeys are naturally suited to applications where one cannot traverse multiple edges at the same time.

if and only if $e \in E_i$ [27]. We call a time labeling a k -labeling if $|\lambda(e)| \leq k$ for every edge e . Furthermore, we say that a temporal graph is a *temporal tree* (resp. *temporal star*, etc.) if its underlying graph, understood as the graph induced by the union of all its edge sets, is a tree (resp. star, etc.).

A specific case of temporal graphs are those with a repeating sequence of edge sets, which have been studied in contexts such as routing [12, 13, 24, 31], graph exploration [3], cops and robbers games [6, 9] and others [1, 35] due to their natural applications in *e.g.* transportation networks. Formally, a p -periodic k -labeling is a time labeling such that both $E_{i+p} = E_i$ for every $i \geq 1$ and $|\lambda(e) \cap \{1, \dots, p\}| \leq k$ for every edge e . A temporal graph with a periodic time labeling has an infinite sequence of edge sets, but can be represented with its p first time-steps, understanding that the sequences will repeat after this.

A vertex v is said to be *reachable from* another vertex u if there exists a journey from u to v . For a given vertex v , the set of vertices which can be reached from vertex v is denoted by $\mathbf{R}(v)$. For a vertex set S such that $v \in S$, we denote by $\mathbf{R}^S(v) \subseteq \mathbf{R}(v)$ the set of vertices which can be reached from v but not from any other vertex in S .

Temporal resolving sets. We extend the definition of resolving sets to the temporal setting: a resolving set in a temporal graph is a reaching and separating set. More formally, a set R of vertices of a temporal graph $\mathcal{G} = (V, E, \lambda)$ is a *temporal resolving set* if (i) for every vertex $v \in V$, there is a vertex $s \in R$ such that $v \in \mathbf{R}(s)$; (ii) for every two different vertices $u, v \in V$, there is a vertex $s \in R$ such that $\text{dist}_t(s, u) \neq \text{dist}_t(s, v)$. Note that every vertex in a temporal resolving set is trivially separated from every other vertex. The problem we are studying is the following:

TEMPORAL RESOLVING SET

Instance: A temporal graph $\mathcal{G} = (V, E, \lambda)$; and an integer k .

Question: Is there a *temporal resolving set* of size at most k ?

Due to the fact that temporal distance is not a metric in the usual sense (symmetry and the standard definition of the triangle inequality might not hold), we call the minimum size of a temporal resolving set of a graph G the *temporal resolving number of G* instead of temporal metric dimension.

Note that we can assume in the following that there is an edge e such that $1 \in \lambda(e)$ (otherwise, let m be the smallest time-step and decrease every time-step by $m - 1$). Temporal resolving set can be seen as a generalization of standard and k -truncated resolving sets: if $\lambda(e) = \{1, \dots, \text{diam}(G)\}$ for every edge e (where $\text{diam}(G) = \max\{\text{dist}(u, v) \mid u, v \in V\}$), then a temporal resolving set is a standard resolving set; and if $\lambda(e) = \{1, \dots, k\}$ for every edge e , then a temporal resolving set is a k -truncated resolving set².

Our results and outline. We focus on time labelings with few labels per edge, mostly limiting ourselves to one or two labels per edge. Although the setting is more restricted than the general case, we shall prove that these scenarios already yield NP-complete problems or non-trivial polynomial algorithms.

We present three sets of results. First, we focus in Section 2 on the computational complexity of finding a minimum-size temporal resolving set in temporal graphs with 2-labelings. In particular, we prove that the problem is NP-complete on temporal complete graphs, which contrasts heavily with other resolving set problems. The problem is also NP-complete on temporal subdivided stars, and on temporal trees even when the two time-steps are consecutive.

In Section 3, we give polynomial-time algorithms for some classes with 1-labelings. However, even for temporal paths, while the algorithm is quite natural, proving optimality is non-trivial. We also give algorithms for temporal stars, and temporal subdivided stars when $t_{\max} = 2$.

Finally, in Section 4, we take a more combinatorial approach to periodic time labelings. We find the optimal bounds for the temporal resolving number of several graph classes under this setting, namely in temporal paths, cycles, complete graphs, complete binary trees, and subdivided stars. We also prove that

²Also called a k -truncated dominating resolving set in [19].

TEMPORAL RESOLVING SET is FPT on trees with respect to number of leaves and XP on subdivided stars with p -periodic 1-labelings with respect to the period p .

2 NP-hardness of TEMPORAL RESOLVING SET

In this section, we give hardness results for TEMPORAL RESOLVING SET on very restricted graph classes, and with strong constraints on the time labeling.

In the static setting, complete graphs tend to be easy to work with: since all the vertices are twins, they are indistinguishable from one another, and hence we need to take all of them but one in order to separate them. Indeed, the metric dimension and location-domination number of K_n is $n - 1$. However, this is not the case with temporal complete graphs, since now the vertices are not necessarily twins anymore. We even prove that it is NP-hard:

Theorem 1. *TEMPORAL RESOLVING SET is NP-complete on temporal complete graphs with a 1-labeling, even when there are only two time-steps.*

Proof. We reduce from the problem of finding a minimum-size adjacency resolving set, which is NP-hard on planar graphs [11].

Let $G = (V, E)$ be a connected planar graph, and k be an integer. We construct the following temporal complete graph $\mathcal{G} = (V, E', \lambda)$:

- For every $e \in E$, $\lambda(e) = \{1\}$;
- For every pair of vertices u, v such that $uv \notin E$, $\lambda(uv) = \{2\}$.

An adjacency resolving set in G clearly is a temporal resolving set in \mathcal{G} and vice versa: the edges e such that $\lambda(e) = \{2\}$ in \mathcal{G} are exactly the paths of length at least 2 in G . \square

The next two results are both proved by reducing from 3-DIMENSIONAL MATCHING, one of the seminal NP-complete problems [26], and are inspired by the NP-completeness proof for k -truncated Metric Dimension on trees in [19], with nontrivial adaptations to constrain the setting as much as possible.

3-DIMENSIONAL MATCHING (3DM)
 Instance: A set $S \subseteq X \times Y \times Z$, where X, Y , and Z are disjoint subsets of $\{1, \dots, n\}$ of size p ; and an integer $\ell < |S|$.
 Question: Does S contain a *matching* of size at least ℓ , *i.e.*, a subset $M \subseteq S$ such that $|M| \geq \ell$ and no two elements of M agree in any coordinate?

Theorem 2. *TEMPORAL RESOLVING SET is NP-complete on temporal stars in which every edge is subdivided twice, and when each edge appears in at most two time-steps.*

Proof. We note that the problem is clearly in NP: a certificate is a set of vertices, and for each vertex, we can compute the time vectors and check that they are all different and that every vertex is reached by at least one vertex from the set in polynomial time. To prove completeness, we reduce from 3DM.

Starting from an instance (S, ℓ) of 3DM, denoting $s = |S|$ and the i -th triple in S by (x_i, y_i, z_i) with $x_i \in X, y_i \in Y, z_i \in Z$, we construct an instance $(\mathcal{G}, s + 2 - \ell)$ of TEMPORAL RESOLVING SET. This construction is detailed below, see Figure 1.

Let $V = \{u, t_1, t_2, t_3\} \cup \bigcup_{i=1}^s \{a_i, b_i, c_i\}$. We will arrange the vertices in the following way as a twice subdivided star. Center vertex is u and it is attached to vertices a_i which are adjacent to vertices b_i which

are adjacent to vertices c_i for each $1 \leq i \leq s$. Moreover, u is also adjacent to c_1 which is adjacent to c_2 and which is adjacent to c_3 . Vertices $\{a_i, b_i, c_i\}$ correspond to elements $\{x_i, y_i, z_i\}$ so that $a_i < b_i < c_i$. Edges are labeled as follows:

- For every $i \in \{1, \dots, s\}$, $\lambda(ua_i) = \{1, a_i + 2\}$;
- For every $i \in \{1, \dots, s\}$, $\lambda(a_ib_i) = \{1, b_i + 2\}$;
- For every $i \in \{1, \dots, s\}$, $\lambda(b_ic_i) = \{1, c_i + 2\}$;
- We have $\lambda(ut_1) = \{1\}$, $\lambda(t_1t_2) = \{1\}$ and $\lambda(t_2t_3) = \{2\}$;

Observe that the constructed graph is a star whose every edge is subdivided exactly twice. Moreover, every edge has at most two labels.

We shall now prove that we decide YES for 3DM on (S, ℓ) if and only if we decide YES for TEMPORAL RESOLVING SET on $((T, \lambda), s + 1 - \ell)$.

(\Rightarrow) Assume that S contains a matching M of size at least ℓ . We construct the following set: $R = \bigcup_{i \notin M} \{a_i\} \cup \{t_1\}$. Note that R contains t_1 and each a_i such that the corresponding element of S is not in M . Furthermore, every vertex of T is reached from t_1 , so we need only consider the separation part.

First observe that t_1 separates u and each t_i . Furthermore, it reaches each vertex of type a_i at moment $a_i + 4$, vertices b_i at moment $b_i + 4$ and vertices c_i at moment $c_i + 4$. Consider then some $a_h \in R$. It reaches vertex b_h at moment 3 and c_h at moment 4. Hence, together with vertex t_1 , vertex a_h separates vertices a_h, b_h and c_h from all other vertices. Furthermore, vertex a_h reaches other vertices of types a_i, b_i and c_i at the same moment as vertex t_1 . Hence, we have uniquely separated vertices u, t_1, t_2, t_3 and every a_h, b_h, c_h such that $a_h \in R$. Recall that sets X, Y and Z are disjoint. Thus, each vertex a_i is separated from vertices of type b_j for any i and j and the same is true for a_i and c_j as well as b_i and c_j . Let us next consider when we might not separate a_i from a_j (the same argument holds for pairs b_i, b_j and c_i, c_j). We may assume that $\{a_i, a_j\} \cap R = \emptyset$. Thus, corresponding elements belong to M . Therefore $a_i \neq a_j$ and hence, they are reached at different time moments from vertex t_1 , a contradiction. Therefore, R is a temporal resolving set of the claimed cardinality.

(\Leftarrow) Assume that there is a temporal resolving set R of size at most $s - \ell + 1$. Since every vertex must be reached from a vertex of R , we must have one of vertices t_1, t_2 or t_3 in R . Now, as in the previous case, only the a 's b 's and c 's must be reached and resolved. If each a_i, b_i and c_i is unique, then t_1 is a resolving set of size $1 \leq s - \ell + 1$. Moreover, if for example $a_i = a_j$ (similar argument holds for $b_i = b_j$ and $c_i = c_j$), then at most one of corresponding tuples can belong to the matching. Moreover, to separate a_i and a_j , we need a vertex in resolving set to belong to one of the branches. Hence, we may choose as our matching M the sets corresponding to branches which contain no members of the temporal resolving set. There are at least ℓ such branches and the claim follows. \square

Theorem 3. *TEMPORAL RESOLVING SET is NP-complete on temporal trees, even with only one vertex of degree at least 5, and when each edge appears in at most two, consecutive, time-steps.*

Proof. First, note that the problem is clearly in NP: a certificate is a set of vertices, and for each vertex, we can compute the time vectors, check that they are all different and check that every vertex is reached by at least one vertex from the set in polynomial time. To prove completeness, we reduce from 3-DIMENSIONAL MATCHING.

Starting from an instance (S, ℓ) of 3-DIMENSIONAL MATCHING, denoting $s = |S|$ and the i -th triple in S by (x_i, y_i, z_i) with $x_i \in X, y_i \in Y, z_i \in Z$, we will construct an instance (\mathcal{G}, ℓ') of TEMPORAL RESOLVING SET. This construction is detailed below. Note that the time-steps cover the interval $[n - 1, n^2 + 1]$ in order

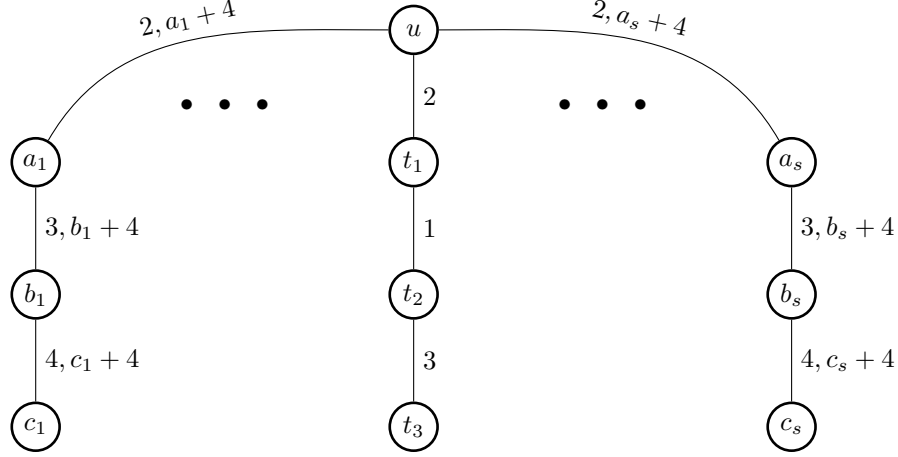


Figure 1: The construction of the proof of Theorem 2. Only the branches 1 and s are detailed together with the control branch. We have $\{a_i, b_i, c_i\} = \{x_i, y_i, z_i\}$ where $a_i < b_i < c_i$.

to simplify the notations but, as discussed in the introduction, they can be brought down to the interval $[1, n^2 - n + 2]$ instead. Let:

$$V = \{u\} \cup \bigcup_{i=1}^s \left\{ \{a_i, b_i, c_i\} \cup \left(\bigcup_{j=1}^n \{v_i^j, s_i^j, t_i^j\} \right) \cup \left(\bigcup_{j=1}^{n-1} \bigcup_{k=1}^{n-1} \{w_i^{j,j+1,k}, s_i^{j,j+1,k}, t_i^{j,j+1,k}\} \right) \right\}.$$

We will arrange the vertices in the following way, which will be formalized below: u will be connected to every v_i^1 which will be the start of the i -th branch corresponding to the i -th element of S , every t will be connected to its corresponding s and either v or w , all the $w_i^{j,j+1,k}$ will form a path linking v_i^j and v_i^{j+1} , and the vertices a_i, b_i, c_i will represent the tuple elements x_i, y_i, z_i . The edges and their labels are as follows (edges not described do not exist):

- For every $i \in \{1, \dots, s\}$, $\lambda(uv_i^1) = \{n-1, n\}$;
- For every $i \in \{1, \dots, s\}$, $\lambda(v_i^{x_i} a_i) = \{x_i n, x_i n + 1\}$, $\lambda(v_i^{y_i} b_i) = \{y_i n, y_i n + 1\}$, $\lambda(v_i^{z_i} c_i) = \{z_i n, z_i n + 1\}$;
- For every $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, n-1\}$, $\lambda(v_i^j w_i^{j,j+1,1}) = \{jn, jn + 1\}$ and $\lambda(w_i^{j,j+1,n} v_{j+1}^j) = \{(j+1)n-1, (j+1)n\}$;
- For every $i \in \{1, \dots, s\}$, $j \in \{1, \dots, n-1\}$ and $k \in \{1, \dots, n-2\}$, set

$$\lambda(w_i^{j,j+1,k} w_i^{j,j+1,k+1}) = \{jn+k, jn+k+1\};$$

- For every $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, n\}$, set

$$\lambda(s_i^j t_i^j) = \lambda(t_i^j v_i^j) = \{n^2 + 1\};$$

- For every $i \in \{1, \dots, s\}$, $j \in \{1, \dots, n-1\}$ and $k \in \{1, \dots, n-1\}$, set

$$\lambda(s_i^{j,j+1,k} t_i^{j,j+1,k}) = \lambda(t_i^{j,j+1,k} w_i^{j,j+1,k}) = \{n^2 + 1\}.$$

Note that the underlying graph T thus constructed is a tree, and that every edge has, as time labels, an interval of size at most 2. Furthermore, u is the only vertex with degree at least 5. This construction is depicted on Figure 2.

Let $\ell' = s(n(n-1) + 1) + (s - \ell)$. We prove that we decide YES for 3-DIMENSIONAL MATCHING on (S, ℓ) if and only if we decide YES for TEMPORAL RESOLVING SET on $((T, \lambda), \ell')$.

(\Rightarrow) Assume that S contains a matching M of size at least ℓ . We construct the following set: $R =$

$$\bigcup_{i \notin M} \{v_i^1\} \cup \bigcup_{i=1}^s \left\{ \bigcup_{j=1}^n \{t_i^j\} \cup \bigcup_{j=1}^{n-1} \bigcup_{k=1}^n \{t_i^{j,j+1,k}\} \right\},$$

i.e., R contains every t and all the first vertices of every branch of T such that the corresponding element of S is not in M . Note that every vertex of T is reached from an element of R , so we need to consider the separation part. First, note that u and each s , t , v , and w is uniquely separated by R (u by the v_i^1 's we selected, the other ones by the t 's). Hence, the only possible vertices not separated by R are a 's, b 's and c 's. Note that, by construction, a_i (resp. b_i , c_i) will be reached at time $x_i n + 1$ (resp. $y_i n + 1$, $z_i n + 1$) from any vertex v_j^1 such that $j \neq i$, and at time $x_i n$ (resp. $y_i n$, $z_i n$) from v_i^1 . Hence, all the a_i 's, b_i 's and c_i 's in branches i such that $i \notin M$ are separated by R . Furthermore, for every branch i , the vertices a_i , b_i and c_i are separated from each other. Assume now that two vertices are not separated by R , they have to be a_i and a_j (without loss of generality) such that $i \neq j$ and $i, j \in M$. However, this is only possible if $x_i = x_j$, in which case the elements i and j from M cannot be in the same matching, which is a contradiction. Hence, R is a temporal resolving set of size $s(n(n-1) + 1) + (s - \ell)$: it contains $(s - \ell)$ vertices v_i^1 , and there are $n(n-1) + 1$ vertices t for each of the s branches.

(\Leftarrow) Assume that there is a temporal resolving set R of size at most $\ell' = s(n(n-1) + 1) + (s - \ell)$. Since every vertex must be reached from a vertex of R , for every pair of adjacent s and t , at least one of them must be in R . Without loss of generality, assume that every t is in R (since this allows to reach and separate every v and w): this means that the number of non- t vertices in R is at most $s - \ell$ (since every branch contains $n(n-1) + 1$ pairs of adjacent s and t). Now, as in the previous case, only the a 's b 's and c 's must be reached and separated, as well as u . Selecting either u or any v_i^1 will take care of u and allow to reach the a 's, b 's and c 's. Again, the possible conflicts among those vertices are the ones such that (without loss of generality) $x_i = x_j$ for $i \neq j$. In this case, the only way to separate the pair would have been to either select a_i or a_j , or to select any vertex above them in either (or both) of the branches i and j . Since R is a temporal resolving set, all such pairs have been separated. We construct M the following way: add to M every i such that the only vertices of the branch i in R are its s 's and t 's. No two elements of M can verify $x_i = x_j$ (resp. $y_i = y_j$, $z_i = z_j$), since that would imply that the corresponding pair (a_i, a_j) (resp. (b_i, b_j) , (c_i, c_j)) would not be separated: no vertex of branch i would have been in R , and thus R would not be a temporal resolving set. Hence, M is a matching of size at least ℓ : at most $(s - \ell)$ branches contain a vertex of R that is not a t , and thus at least ℓ branches do not. \square

Remark 4. We can set $\lambda(t_i^j v_i^j) = \lambda(t_i^{j,j+1,k} w_i^{j,j+1,k}) = \{n^2 + 2, n^2 + 3\}$ and $\lambda(s_i^j t_i^j) = \lambda(s_i^{j,j+1,k} t_i^{j,j+1,k}) = \{n^2 + 1, n^2 + 2\}$, to obtain a construction with time intervals of size exactly 2.

3 Polynomial-time algorithms for subclasses of trees

In this section, we give polynomial-time algorithms for TEMPORAL RESOLVING SET. We study temporal paths and stars with one time label per edge, and temporal subdivided stars with one time label per edge and where every label is in $\{1, 2\}$. Recall that TEMPORAL RESOLVING SET is already NP-complete on temporal subdivided stars with 2-labeling (Theorem 2), so these results are a first step for bridging the gap between polynomial-time and NP-hard.

Algorithm 1 Temporal resolving set for temporal paths with 1-labeling

Input: A temporal path $\mathcal{P} = (P_n, \lambda)$.

Output: A minimum-size temporal resolving set R of \mathcal{P} .

- 1: Set $v = v_1$ and $R = \emptyset$.
 - 2: **while** true **do**
 - 3: Let $s = v_i$ where i is the largest integer such that v_i reaches v . Add s to R and set $a = i$.
 - 4: **if** Each vertex in $\mathbf{R}^R(s)$ has unique distance among vertices in $\mathbf{R}^R(s)$ to s **then**
 - 5: $w = v_j$ where j is the smallest integer satisfying $j > i$ for each $v_i \in \mathbf{R}(s)$.
 - 6: **if** $v_n \in \mathbf{R}^R(s)$ or $v_n \in R$ **then return** R .
 - 7: **end if**
 - 8: **else** Let $w = v_b$ where $v_b \in \mathbf{R}^R(s)$ is the vertex which does not have unique distance among vertices in $\mathbf{R}^R(s)$ to s and among those vertices b is minimal such that $b > a$.
 - 9: **end if**
 - 10: Let $v = w$.
 - 11: **end while**
-

for some $1 \leq a \leq n$. Let $R \subseteq V(P_n)$ (resp. $R' \subseteq V(P_m)$) be a minimum-size temporal resolving set of \mathcal{P} (resp. \mathcal{P}'). If $|R| \geq |R'|$, then the claim follows. Thus, assume by contradiction that $|R| < |R'|$. First observe that if $R \subseteq V(P_m)$, then R is a temporal resolving set in \mathcal{P}' and thus $|R| \geq |R'|$, a contradiction. Hence, we may assume that there exists some vertex $s \in R \setminus V(P_m)$. Let us consider the set $R'' = \{v_a\} \cup R \cap V(P_m)$. Note that $|R''| \leq |R|$. First of all, every vertex in \mathcal{P}' is reached by some vertex of R'' . Secondly, if two vertices of \mathcal{P}' are not separated by vertices in $R'' \setminus \{v_a\}$, then they were separated by s in \mathcal{P} . Moreover, in R'' they are separated by v_a . Indeed, if $w \in \mathbf{R}(s) \cap V(P_m)$, then $w \in \mathbf{R}(v_a) \cap V(P_m)$. Moreover, since v_a is a leaf in P_m , every vertex in $\mathbf{R}(v_a) \cap V(P_m)$ has a unique temporal distance to v_a . Therefore, R'' is a resolving set in \mathcal{P}' with cardinality $|R''| < |R'|$, a contradiction. Thus, the claim follows. \square

Theorem 6. *There is a linear-time algorithm solving TEMPORAL RESOLVING SET on temporal paths where each edge appears only once.*

Proof. In the following, we first show that Algorithm 1 returns a temporal resolving set R of $\mathcal{P} = (P_n, \lambda)$. After that, we prove that R is minimum-sized and finally, that the algorithm has linear time complexity.

First of all, consider vertices in $\mathbf{R}(s_1)$. Note that if there are vertices $u, v_b \in \mathbf{R}(s_1)$ such that $\text{dist}_t(r_1, u) = \text{dist}_t(r_1, v_b)$, then one of them is on the left side of r_1 and other one is on its right side. Let us assume, without loss of generality, that u is on the left side of r_1 . Moreover, no third vertex w can have $\text{dist}_t(r_1, u) = \text{dist}_t(r_1, v_b) = \text{dist}_t(r_1, w)$. Let us assume that v_b is the vertex with the smallest index on the right side of r_1 such that it is not separated by r_1 from some other vertex (in this case, from u).

In this case, Algorithm 1 has chosen $w = v_b$ on Step 8 and set $v = w$ after that on Step 10. Hence, in the following while-loop, we choose r_2 as the rightmost vertex which reaches $v (= v_b)$. Furthermore, r_2 cannot reach u . Indeed, since r_1 cannot separate between u and v_b , there are two edges with the same time labels on the path from u to v_b . Therefore, on the path from r_2 to u , there are two edges with the same time labels. Hence, r_2 separates v_b from u . Consequently, we may observe that if there were any other vertices in $\mathbf{R}^{\{r_1\}}(r_1)$ which were not separated by r_1 , they are separated by r_2 . A similar argument works for all pairs r_i, r_{i+1} . Note that in the end, either we choose $r_{|R|} = v_n$ or $r_{|R|}$ separates every vertex in $\mathbf{R}^R(r_{|R|})$. Hence, we eventually enter the if-clause on Step 6 and return R .

We now show that there does not exist any resolving set of smaller size than R in \mathcal{P} . We do this by induction on the number n of vertices. First of all, Algorithm 1 outputs a temporal resolving set of size 1 when $n \in \{1, 2\}$, which is optimal. Thus, we assume from now on that it outputs a minimum-size temporal resolving set for $n \leq n'$.

Let $n = n' + 1$, and R be the temporal resolving set constructed using Algorithm 1 on $\mathcal{P} = (P_n, \lambda)$. Assume first that r_1 separates every vertex in $\mathbf{R}(r_1)$. Observe that if $|R| = 1$, then it is minimum-size. Hence, we may assume that $\mathcal{P}' = \mathcal{P} \setminus \mathbf{R}(r_1)$ (where $\mathcal{G} \setminus V'$ for a temporal graph $\mathcal{G} = (V, E, \lambda)$ denotes the temporal subgraph $\mathcal{G}' = (V \setminus V', E \setminus \{uv : u \in V' \text{ or } v \in V'\}, \lambda)$) is nonempty. By induction, Algorithm 1 outputs a minimum-size temporal resolving set $R \setminus \{r_1\}$ of \mathcal{P}' . Observe that $\ell(r_1) \cap \mathbf{R}(w) = \emptyset$ for any w on the rightside of r_1 . Moreover, we require at least one vertex in set $\ell(r_1) \cup \{r_1\}$ in any resolving set of \mathcal{P} to reach vertex v_1 . Observe that Algorithm 1 returns the temporal resolving set $R \setminus \{r_1\}$ for $\mathcal{P} \setminus \mathbf{R}(r_1)$. By induction, set $R \setminus \{r_1\}$ has minimum size. By Theorem 5, we require in any temporal path containing $\mathcal{P} \setminus \mathbf{R}(r_1)$ (and having v_n as a leaf) at least $|R| - 1$ vertices. Furthermore, by our observations, we require in a set $\ell(r_1) \cup \{r_1\}$ at least one vertex to reach v_1 . Furthermore, since these vertices do not reach any vertex in $V(\mathcal{P}) \setminus \mathbf{R}(r_1)$, we require at least $|R|$ vertices in a resolving set of \mathcal{P} , as claimed.

Assume next that there are vertices v_ℓ and v_r in $\mathbf{R}(r_1)$ which are not separated by r_1 . Further assume that v_ℓ (resp. v_r) is on the left (resp. right) side of r_1 . Consequently, r_2 is the rightmost vertex which reaches v_r . We show that if $v_\ell \neq v_1$, then the claim follows. Assume that $\ell \geq 2$. Consider the temporal path $\mathcal{P}' = \mathcal{P} \setminus \{v_1\}$. Note that r_1 is the rightmost vertex which reaches v_2 . Moreover, r_1 does not separate vertices v_ℓ and v_r . Thus, Algorithm 1 outputs R as a resolving set for \mathcal{P}' . By our induction hypothesis, R is minimum-size. By Theorem 5, we know that R is at least as large as a minimum-size temporal resolving set of \mathcal{P}' . Thus, R is also a minimum-size temporal resolving set for \mathcal{P} . Note that this implies that r_1 separates every pair of vertices in $\mathbf{R}(r_1)$ except for (v_1, v_r) .

We now need to analyze several cases depending on whether $t_{r-1} \leq t_r$ and whether $t_r \leq t_{r+1}$. We distinguish four cases and for all of them, we conclude that at least $|R|$ vertices are necessary in a temporal resolving set of \mathcal{P} .

Our aim is to conclude that temporal resolving set is of size at least $|R|$ in \mathcal{P} in all of the cases.

Assume first that $t_{r-1} \leq t_r$ and $t_r \leq t_{r+1}$. Thus, $r_2 = v_{r+1}$. Furthermore, r_2 separates every vertex in $\mathbf{R}(r_2) \setminus \{v_r\}$. Indeed since r_1 reaches v_r , we have $\mathbf{R}(r_2) \setminus \{v_r\} = \mathbf{R}(r_2) \setminus \ell(r_2)$, and r_2 pairwise separates every vertex on its right side. Denote by \mathcal{P}_r the temporal subpath $\mathcal{P} \setminus (\mathbf{R}(r_1) \cup \mathbf{R}(r_2))$. Note that Algorithm 1 outputs $R \setminus \{r_1, r_2\}$ as a temporal resolving set of \mathcal{P}_r and, by induction, this has minimum cardinality in \mathcal{P}_r . Note that if $|R| = 2$, then R has the smallest possible size in \mathcal{P} . Hence, we assume that $|R| > 2$. By Theorem 5, any subpath of \mathcal{P} containing \mathcal{P}' and leaf v_n has temporal resolving number at least $|R| - 2$. Furthermore, we require at least two vertices in a temporal resolving set of \mathcal{P} among vertices $\ell(r_2) \cup \{r_2\}$. Note that none of these vertices reach any vertex in $V(\mathcal{P}')$. Thus, at least $|R|$ vertices are necessary in a temporal resolving set of \mathcal{P} as claimed.

Consider now the case with $t_{r-1} = t_r$ and $t_r > t_{r+1}$. We have $\mathbf{R}(r_1) = \{v_1, \dots, v_r\}$. Consider subpath $\mathcal{P}' = \mathcal{P} \setminus \mathbf{R}(r_1)$. Note that Algorithm 1 outputs $R \setminus \{r_1\}$ for \mathcal{P}' since $r_2 \neq v_{r+1}$ as $t_r > t_{r+1}$. Moreover, $R \setminus \{r_1\}$ is a minimum-sized temporal resolving set by induction assumption. By Theorem 5, any subpath of \mathcal{P} containing \mathcal{P}' and v_n requires at least $|R| - 1$ vertices in any temporal resolving set. Moreover, we require at least one vertex in $\ell(r_1) \cup \{r_1\}$ for any temporal resolving set of \mathcal{P} . Note that vertices in $\ell(r_1) \cup \{r_1\}$ do not reach vertices in $V(\mathcal{P}')$. Hence, we require at least $|R|$ vertices in any minimum-sized temporal resolving set of \mathcal{P} .

Consider next the case with $t_{r-1} < t_r$ and $t_r > t_{r+1}$. We have $\mathbf{R}(r_1) = \{v_1, \dots, v_{r+1}\}$. Consider \mathcal{P}' such that $V(\mathcal{P}') = \{v_{r-1}, \dots, v_n\}$, $E(\mathcal{P}') \subseteq E(\mathcal{P})$, $\lambda(v_{r-1}v_r) = t_r$ and $\lambda(v_{r+i}v_{r+i+1}) = t_{r+i}$ for each $i \geq 0$. Note that Algorithm 1 now outputs the temporal resolving set $R' = (R \cup \{v_r\}) \setminus \{r_1\}$. By induction assumption, R' has minimum cardinality in \mathcal{P}' . In particular, any temporal resolving set of \mathcal{P}' requires one of the vertices v_{r-1} or v_r as these are the only vertices which reach v_{r-1} . Consider a temporal resolving set R'' of \mathcal{P} . Let $R^* = R'' \cap \ell(v_{r+1})$. Observe that $(R'' \setminus R^*) \cup \{v_r\}$ is a temporal resolving set of \mathcal{P}' . Hence, $|R''| - |R^*| + 1 \geq |R|$. Since we require at least one vertex in R^* to reach v_1 , we have $|R''| \geq |R|$. Therefore, $|R|$ has the minimum cardinality over temporal resolving sets of \mathcal{P} , as claimed.

Next, we consider the case where $t_{r-1} > t_r$ and $t_r \neq t_{r+1}$. Observe that $v_{r+1} \notin \mathbf{R}(r_1)$. Since $v_\ell = v_1$,

r_1 separates v_{r+1} from other vertices in $\mathbf{R}(r_1)$. Assume next that all time labels have even values. This has no effect on the temporal resolving set or the algorithm (we can multiply every time label by two without changing any reachability in the temporal path). We do the following modification to the time labeling of \mathcal{P} , obtaining path \mathcal{P}_m . We change t_r into $t'_r = t_r - 1$. Note that since we assumed that every time label has even value, time label t'_r has an odd value unlike all other time labels, and $t_{r-1} > t'_r$. Moreover if $t_{r+1} > t_r$, then $t_{r+1} > t'_r$ and if $t_{r+1} < t_r$, then $t_{r+1} < t'_r$. Note that in this change we maintain v_r and v_1 unseparable by r_1 and every set $\mathbf{R}(v_i)$ remains unchanged.

Observe that Algorithm 1 returns the same temporal resolving set R for \mathcal{P}_m since r_2 is still the rightmost vertex which reaches v_r . Moreover, any temporal resolving set for \mathcal{P} remains as a temporal resolving set for \mathcal{P}_m since sets $\mathbf{R}(v_i)$ remain unchanged and, since the label t'_r is the only odd label, it cannot cause any two vertices to become unseparated. Let $\mathcal{P}'_m = \mathcal{P}_m \setminus \mathbf{R}(r_1)$. Observe that Algorithm 1 outputs a temporal resolving set $R' = R \setminus \{r_1\}$ for \mathcal{P}'_m . Furthermore, by the induction hypothesis, it is minimum-size. Consider next a temporal resolving set R'' of \mathcal{P}_m . Assume first that there are at least two vertices in $\mathbf{R}(r_1) \cap R''$ and let us denote the rightmost of them by w . Denote $\mathcal{P}_w = \mathcal{P}_m \setminus \ell(w)$. Note that $R'' \setminus \ell(w)$ is a temporal resolving set of \mathcal{P}_w . Observe that \mathcal{P}'_m is a subgraph of \mathcal{P}_w . Thus, by Theorem 5, we have $|R'' \setminus \ell(w)| \geq |R| - 1$ and $|R''| \geq |R|$. Assume then that we have $|R'' \cap \mathbf{R}(r_1)| = 1$. Let $w_1 \in R'' \cap \mathbf{R}(r_1)$. Note that if $v_r \in \mathbf{R}(w_1)$, then w_1 does not separate v_ℓ and v_r . Thus, some other vertex in $R'' \setminus \mathbf{R}(r_1)$ reaches v_r and due to the odd time label, $R'' \setminus \mathbf{R}(r_1)$ is a temporal resolving set of \mathcal{P}'_m . Hence, by Theorem 5, we have $|R''| \geq |R|$, allowing us to conclude that R is a minimum-size temporal resolving set of \mathcal{P}_m . If we have a temporal resolving set R^* of \mathcal{P} with $|R^*| < |R|$, then R^* is also a temporal resolving set of \mathcal{P}_m with $|R^*| < |R|$, a contradiction.

As the last case, we consider $t_{r-1} > t_r = t_{r+1}$. Again, observe that $v_{r+1} \notin \mathbf{R}(r_1)$. Furthermore, $v_{r+1} = r_2$. Hence, together, r_1 and r_2 separate all vertices in $\mathbf{R}(r_1) \cup \mathbf{R}(r_2)$. Furthermore, since at least one vertex in $\ell(r_1) \cup \{r_1\}$ is required, note that if $|R| = 2$, then it has minimum size in \mathcal{P} . Thus, assume that $|R| \geq 3$. Let $\mathcal{P}' = \mathcal{P} \setminus (\mathbf{R}(r_1) \cup \mathbf{R}(r_2))$. Note that set $R \setminus \{r_1, r_2\}$ is a minimum-size temporal resolving set of \mathcal{P}' since r_1 and r_2 separate all vertices in $\mathbf{R}(r_1) \cup \mathbf{R}(r_2)$. Furthermore, Algorithm 1 outputs set $R \setminus \{r_1, r_2\}$ for \mathcal{P}' . By the induction assumption, set $R \setminus \{r_1, r_2\}$ is a smallest temporal resolving set of \mathcal{P}' . Hence, by Theorem 5, any subpath of \mathcal{P} containing \mathcal{P}' and leaf v_n requires at least $|R| - 2$ vertices in any temporal resolving set. Furthermore, at least two vertices $\ell(r_2) \cup \{r_2\}$ are required in any temporal resolving set of \mathcal{P} , and these vertices do not reach any vertex in $V(\mathcal{P}')$. Thus, \mathcal{P} does not have any temporal resolving set with cardinality less than $|R|$, as claimed.

Finally, we show that Algorithm 1 has linear time complexity. First of all, the while-loop ends at some point since every temporal path has a temporal resolving set by taking every vertex in the underlying path. Secondly, Step 3 uses at most $i + 1 - a$ comparisons and in total at most $2n$ comparisons. In Step 4, observe that sets $\mathbf{R}^R(s)$ do not overlap. Thus, each vertex is considered only once. Moreover, the time labels on the left and the right side of s are ordered from small to large. Thus, checking if each time label has a unique value can be done in linear time on $|\mathbf{R}^R(s)|$. Again, in Step 5, the sets $\mathbf{R}(s) \setminus \ell(s)$ do not overlap. Thus, this step takes at most linear time in total. Finally, all the other steps take at most constant time. Hence, the algorithm has linear-time complexity. \square

The following lemma gives some structure on the minimum-size temporal resolving sets of temporal paths, with respect to the one output by Algorithm 1. In particular, it states that the constructed temporal resolving set R places each vertex in R as far away from the leaf v_1 as possible. It will be used in the case of subdivided stars, allowing us to reuse Algorithm 1 in order to find a partial solution.

Lemma 7. *Let \mathcal{P} be a temporal path on $n \geq 2$ vertices with 1-labeling λ on vertices v_1, \dots, v_n and edges $v_i v_{i+1}$ for $1 \leq i \leq n - 1$ where v_i is on the left side of v_{i+1} for each i . Let $R = \{r_1, \dots, r_{|R|}\}$ be the temporal resolving set output by Algorithm 1, where r_i is on the left side of r_{i+1} for each i . Let $R' = \{r'_1, \dots, r'_{|R|}\}$ be another temporal resolving set of \mathcal{P} . We have $r'_i \in \ell(r_i) \cup \{r_i\}$ for each i .*

Proof. Recall that by Theorem 6, Algorithm 1 returns a temporal resolving set of minimum size for path \mathcal{P} . Consider first the case with $|R| = 1$. Notice that r_1 is the rightmost vertex which reaches v_1 . Thus, the claim holds in this case. Consequently, by the same argument, we have that $r'_1 \in \ell(r_1) \cup \{r_1\}$ even when $|R| > 1$. Assume next that the claim does not hold for some \mathcal{P} , R and R' . Furthermore, let $r'_i \notin \ell(r_i) \cup \{r_i\}$ and $r'_j \in \ell(r_j) \cup \{r_j\}$ for every $j < i$. Let us assume first that $r_{i-1} = r'_{i-1}$. Denote by $R_j = \{r_1, \dots, r_j\}$ and $R'_j = \{r'_1, \dots, r'_j\}$ for any $j \leq |R|$.

Consider first the case where r_{i-1} separates all vertices in $\mathbf{R}^{R_{i-1}}(r_{i-1})$. Let w be the leftmost vertex which is not reached by R_{i-1} . In this case, Algorithm chooses r_i as the rightmost vertex which reaches w . Since also r'_i reaches w , we have $r'_i \in \ell(r_i) \cup \{r_i\}$, a contradiction. Hence, there exist vertices $u, w \in \mathbf{R}^{R_{i-1}}(r_{i-1})$ which are not separated r_{i-1} . Let $u \in \ell(r_{i-1})$ and $w \in \mathbf{R}(r_{i-1}) \setminus \ell(r_{i-1})$. Furthermore, we assume that w is the leftmost vertex with these properties. Notice that r_i is the rightmost vertex which reaches w . Moreover, if $u, w \in \mathbf{R}^{R'_{i-1}}(r'_{i-1})$ and $r'_{i-1} \neq u$, then r'_{i-1} does not separate u and w . Thus, vertex r'_i reaches w . However, since r_i was the rightmost such vertex, we have $r'_i \in \ell(r_i) \cup \{r_i\}$, a contradiction. Furthermore, if $r'_{i-1} = u$, then $w \notin \mathbf{R}(r'_{i-1})$. Indeed, there are two edges with the same time label on the path from u to w since r_{i-1} does not separate them. Therefore, to reach w , we have $w \in \mathbf{R}(r'_i)$. Again, this leads to a contradiction since r_i was the rightmost vertex reaching w . Hence, $u \notin \mathbf{R}^{R'_{i-1}}(r'_{i-1})$ and r'_{i-1} is on the rightside of u . In particular, this implies that $u \in \mathbf{R}(r'_{i-2})$. Furthermore, we have $u \notin \mathbf{R}(r_{i-2})$. Indeed, otherwise r_{i-2} or r_{i-1} would separate w and u . Moreover, u is on the rightside of r_{i-2} . Since $u \in \mathbf{R}(r'_{i-2})$ but $u \notin \mathbf{R}(r_{i-2})$, we have r'_{i-2} on the rightside of r_{i-2} . However, this is a contradiction with the minimality of i . Therefore, the claim holds. \square

3.2 Temporal stars

In this subsection, we give polynomial-time algorithms for finding minimum-size temporal resolving sets for temporal stars with 1-labeling and temporal subdivided stars with 1-labeling using only values 1 and 2.

Theorem 8. *Let S be a star, λ be a 1-labeling, and $\mathcal{S} = (S, \lambda)$. The set $R = \{c\} \cup \bigcup_{j=1}^m VL'_j$ is a minimum-size temporal resolving set of \mathcal{S} .*

Proof. Throughout this proof, we denote by S_n the star with central vertex c and n leaves v_1, \dots, v_n . We furthermore assume that the time labeling λ uses time labels from 1 to m and does not contain any gap (the underlying graph being a star, we can remove gaps without changing reachability); since $\lambda(cv_i)$ contains only one integer, we will use it to denote the integer it contains by abuse of notation. We can further assume without loss of generality that $\lambda(cv_i) \leq \lambda(cv_{i+1})$ for each $1 \leq i \leq n-1$. Let $VL_j = \{v_i : \lambda(v_i) = j\}$ and $L_j = |VL_j|$. We denote by VL'_j a set $VL_j \setminus \{v\}$ where v is an arbitrary vertex of VL_j .

We first prove that R is a temporal resolving set of \mathcal{S} . First, we have $V(S_n) \subseteq \mathbf{R}(c)$ and $c \in R$. Furthermore, by the definition of the sets VL'_j , c separates every vertex in $V(S_n) \setminus R$. Hence, R is a temporal resolving set of \mathcal{S} .

Next, let us prove the minimality of R . First, assume that for some j , we have $u, v \in VL_j \setminus R$. However, we have $\lambda(cu) = \lambda(cv) = j$. Thus, these vertices cannot be separated and we have $|VL_j \setminus R| \leq 1$ for each j . Assume then that $|VL_j \setminus R| = 1$ for each $1 \leq j \leq m$ and that $c \notin R$. However, now there is a vertex $v \in VL_1 \setminus R$ and the only vertices that can reach v are v and c since 1 is the smallest label. Thus, no vertex in R reaches v and hence, we cannot simultaneously have $|VL_j \setminus S| = 1$ for each $1 \leq j \leq m$ and $c \notin R$. Therefore, R has minimum cardinality. \square

We now consider subdivided stars together with 1-labeling λ using only values 1 and 2. In particular, we present a polynomial-time algorithm for this case. The central vertex of a subdivided star is denoted by c . By a branch of a subdivided star, we mean a path starting from vertex c without the vertex c itself. Branches are denoted by B_1, \dots, B_Δ and the leaves by $\ell_1, \dots, \ell_\Delta$. Furthermore, the vertex in B_i adjacent

to c is denoted by v_i , the vertex adjacent to by u_i and the third vertex of branch by w_i (if they exist, note that some of these vertices might also be denoted ℓ_i). We further assume that branches are ordered so that $\lambda(cv_i) \leq \lambda(cv_{i+1})$ for each $i \leq \Delta - 1$. We assume that $\lambda(cv_i) = 1$ for each $i \leq I_1$ where $0 \leq I_1 \leq \Delta$ and denote $B^1 = \bigcup_{i=1}^{I_1} B_i$ and $B^2 = \bigcup_{i=I_1+1}^{\Delta} B_i$.

The following theorem shows that Algorithm 2 returns a minimum-size temporal resolving set in polynomial time for a given temporal subdivided star using only values 1 and 2 in its 1-labeling.

Algorithm 2 Temporal metric dimension for subdivided star SS_{Δ} with 1-labels using values 1 and 2

Input: Subdivided star SS_{Δ} of degree $\Delta \geq 3$ together with time labeling λ for each edge $e \in E(SS_{\Delta})$ such that $\lambda(e) \in \{1, 2\}$.

Output: An optimal temporal resolving set R on SS_{Δ} .

```

1: For each path from  $\ell_i$  to  $c$ , we create temporal resolving set  $R_i$  using Algorithm 1.
2: Let  $R' = (\bigcup_{i=1}^{\Delta} R_i) \setminus \{c\}$ .
3: Let  $B_c = \bigcup_{c \in R_i} B_i$ .
4: if  $B_c = \emptyset$  then return  $R'$ .
5: end if
6: for  $j \in \{1, 2\}$  do
7:   if  $v \in N(c) \cap V(B_c)$  and  $v \in V(B_i)$  and  $B_i \in B^j$ , for some  $i \leq \Delta$ , and  $R_i \setminus \{c\}$  does not separate  $v$  from some other vertex in  $B_i$  but  $v \in \mathbf{R}(R_i \setminus \{c\})$  then
8:     add  $v$  to  $Q_j$ .
9:   end if
10: end for
11: for  $1 \leq j \leq \Delta$  do
12:   Remove branch  $B_j$  from  $B_c$  if  $Q_2 \cap V(B_j) \neq \emptyset$ .
13: end for
14: Let  $r = |\{i \mid Q_1 \cap V(B_i) \neq \emptyset\}|$ .
15: for  $0 \leq j \leq 2$  do
16:   for each vertex set  $R''$  such that  $R'' \subseteq \mathbf{R}(c) \cap (V(B_c) \cup \{c\})$ ,  $|R'' \cap V(B_i)| \leq 1$  for every  $B_i \in B_c$ ,  $|R'' \cap Q_1| \leq 1$ ,  $|R'' \cap \{r \mid \text{dist}(c, r) = 2\}| \leq 1$  and  $|R''| = |B_c| - 1 - r + j$  do
17:     if  $R' \cup R''$  is a temporal resolving set of  $SS_{\Delta}$  then return  $R' \cup R''$ 
18:     end if
19:   end for
20: end for

```

Theorem 9. *Given a subdivided star SS_{Δ} of maximum degree $\Delta \geq 3$ and 1-labeling of edges restricted to values 1 and 2, Algorithm 2 returns a minimum size temporal resolving set of SS_{Δ} in polynomial time.*

Proof. For brevity, we say that center c is the rightmost vertex and consider the leaves as leftmost vertices of their branches.

We first show that Algorithm 2 returns a temporal resolving set, then that the temporal resolving set has minimum possible size, and finally that the algorithm operates in polynomial time.

First of all, by Theorem 6, Algorithm 1 returns a minimum size temporal resolving set R_i for a path from ℓ_i to c in Step 1. Denote this path together with its time labeling λ by \mathcal{P}_i . Let us first consider the set R' which is returned in Step 4. If we enter the if-clause on Step 4, then we have $B_c = \emptyset$ and $c \notin \bigcup_{i=1}^{\Delta} R_i$. Thus, $R' = \bigcup_{i=1}^{\Delta} R_i$. Since each R_i is a temporal resolving set for path from ℓ_i to c , each vertex in the substar is reached by some vertex in R' . Moreover, each of these sets separates all vertices within the same path \mathcal{P}_i . Thus, if two vertices $u \in V(B_i)$ and $v \in V(B_j)$ are not separated by R' , then we have $i \neq j$. However, since

$c \notin R'$, u and v are reached by at least two different vertices in R' , from two different sets R_i . Let $u \in \mathbf{R}(r_u)$ for some $r_u \in R_i$. If $v \notin \mathbf{R}(r_u)$, then r_u separates u and v . If there is a path from r_u to u to v , then r_u separates vertices u and v . Thus, r_u is on the path from u to v . Similarly, we have $r_v \in R_j$ between vertices v and u . Thus, neither of u nor v can be vertex c . Consequently, also $c \notin R'$ is on the path from u to v . Moreover, since our labeling consists of values 1 and 2, r_v cannot reach u . We conclude that u and v are separated. Hence, if R' is returned in Step 4, then it is a temporal resolving set of the subdivided star.

In the for-clause between Steps 6 and 10, we consider each vertex $v \in N(c)$ in some branch $B_i \in B^j$ which is reached by $R_i \setminus \{c\}$ but not separated from some other vertex in B_i . Each such vertex is added to set Q_j for j corresponding to B^j . Observe that any vertex in $N[c]$ which reaches v , also separates it from other vertices.

In the for-clause between Steps 11 and 13, every branch which has non-empty intersection with Q_2 is removed from B_c . Notice that if $v \in Q_2$, then $v \in N(c)$. Hence, if we remove a branch B_j during this step from B_c , then we had $\mathbf{R}(c) \cap V(B_j) = \{v\}$.

In the for-clause between Steps 15 and 20, we construct a temporal resolving set for the substar. Observe that in particular the for-clause can always find, with $j = 3$, the set R'' which contains c and the first vertex in every branch in B_c which has empty intersection with Q_1 . Together with R' , this forms a temporal resolving set of the substar (although not always minimum-size). Indeed, recall that $R_i \cup \{c\}$ is a temporal resolving set for \mathcal{P}_i . Thus, $(R_i \cup \{v_i\}) \setminus \{c\}$ is also a temporal resolving set for $\mathcal{P}_i - c$. Furthermore, each v_i can only reach the first vertices of other branches in B^2 and no vertex in branches in B^1 . These vertices are either in $R' \cup R''$, in Q_2 , or in a branch which has temporal resolving set $R_j \setminus \{c\}$. Since we have $c \in R''$, vertices in Q_j for $j \in \{1, 2\}$ are reached and separated from other vertices. By these considerations, the only vertex pairs which might not be separated belong to two different branches and do not belong to Q_j . Let $v \in V(B_i)$ and $u \in V(B_j)$ be two vertices which are not separated. Note that if $u \notin N(c)$, then it cannot be reached by any vertex in $V(B_i)$ and vice versa. Thus, $v, u \in N(c) \setminus R''$. However, now v (u) is reached by some vertex r_v (r_u) in $R' \cap V(B_i)$ ($R' \cap V(B_j)$). Moreover, we have $u \notin \mathbf{R}(r_v)$ ($v \notin \mathbf{R}(r_u)$). Thus, u and v are separated and $R' \cup R''$ is a temporal resolving set of the substar.

Let us next show that the returned temporal resolving set has the minimum size. Consider first set R' on Step 4 and assume that it is returned. In this case, we have $B_c = \emptyset$ and thus, $c \notin \bigcup_{i=1}^{\Delta} R_i$. Consider path $\mathcal{P}_i - \mathbf{R}(c)$. Note that for this path, Algorithm 1 returns a temporal resolving set of cardinality equal to $|R_i|$. Furthermore, by Theorem 6, this set has minimum possible cardinality. By Theorem 7, there is no temporal resolving set of cardinality $|R_i|$ for path \mathcal{P}_i which contains vertex c . Moreover, we require for each path $\mathcal{P}_i - \mathbf{R}(c)$ at least $|R_i|$ vertices in a temporal resolving set. Thus, R' has the smallest possible cardinality.

Let us then show that if R' is not returned on Step 4, then $R' \cup R''$ has the minimum cardinality for a temporal resolving set in \mathcal{SS}_{Δ} . By our earlier considerations, $R' \cup R''$ is a temporal resolving set for some suitable R'' . Let us first show that we may assume that R' is a subset of some minimum-size temporal resolving set of \mathcal{SS}_{Δ} . First of all, in Step 1, Algorithm 1 returns a minimum-size temporal resolving set R_i for each path \mathcal{P}_i . Furthermore, for path $\mathcal{P}_i - \mathbf{R}(c) \setminus (R_i \setminus \{c\})$, Algorithm 1 returns the temporal resolving set $R_i \setminus \{c\}$. Indeed, $\mathcal{P}_i - \mathbf{R}(c) \setminus (R_i \setminus \{c\})$ is a path since Algorithm 1 always picks the last vertex to reach previously considered vertices and we only use labels 1 and 2. When traversing the path from leaf to center/, the algorithm does not consider how to separate/reach vertices ahead of it. Let $r_i \in R_i \setminus \{c\}$ be the vertex closest to c in $R_i \setminus \{c\}$. If $c \notin R_i$, then by Theorem 7, any temporal resolving set containing a vertex in \mathcal{P}_i closer to c than vertex r_i has at least $|R_i| + 1$ vertices. Moreover, if $c \in R_i$, then any temporal resolving set containing a vertex in \mathcal{P}_i closer to c than vertex r_i has at least $|R_i|$ vertices. By these considerations, r_i is the vertex closest to center c other than c which can be contained in any temporal resolving set of cardinality $|R_i \setminus \{c\}|$ over all subpaths of \mathcal{P}_i containing leaf ℓ_i .

Furthermore, if R is a resolving set of \mathcal{SS}_{Δ} and $R_{P_i} = R \cap V(\mathcal{P}_i^r)$ (where \mathcal{P}_i^r is the subpath of \mathcal{P}_i from ℓ_i to r_i), then if $|R_{P_i}| = |R_i \setminus \{c\}|$, then $R_P = (R \setminus R_{P_i}) \cup (R_i \setminus \{c\})$ is also a temporal resolving set of \mathcal{SS}_{Δ} and if $|R_{P_i}| > |R_i \setminus \{c\}|$, then $R'_P = \{c\} \cup (R \setminus R_{P_i}) \cup (R_i \setminus \{c\})$ is also a temporal resolving set of \mathcal{SS}_{Δ} . Consider

first the case with $|R_{P_i}| = |R_i \setminus \{c\}|$. In this case, R_i reaches and separates every vertex in \mathcal{P}_i^r and r_i is a leaf of \mathcal{P}_i^r . Assume on the contrary that R_P is not a temporal resolving set. In this case, there are (at least) two vertices in $\mathbf{R}(r_i)$ which are not separated by r_i . Note that one of them is on the right side (call this u) of r_i and one is on the left side (call this v). Moreover, if $r_i \notin R$, then we have some $r'_i \in R$ on the left side of r_i which is the rightmost vertex of R_{P_i} . If r'_i separates u and v , then $u \notin r'_i$. In this case, a vertex in $R \setminus R_{P_i}$ reaches u and separates it from v . Thus, $r_i \in R$. Let r'_{i-1} (r_{i-1}) be the first vertex on the left side of r_i in R (R_i). By Theorem 7, the vertex r'_{i-1} is not on the right side of r_{i-1} . Thus, R_{P_i} does not separate vertices u and v but set $R \setminus R_{P_i}$ does separate them. Therefore, R_P also separates u and v . Thus, R_P is a temporal resolving set of \mathcal{P} . Moreover, we have $|R_P| = |R|$.

Let us next consider the case with $|R_{P_i}| > |R_i \setminus \{c\}|$. Recall that R_i is a temporal resolving set of \mathcal{P}_i . Thus, $R'_P = R_P \cup \{c\}$ does not separate some vertices, those vertices belong to different branches of \mathcal{SS}_Δ and are only reached by c . However, in this case they are also not reached by R_{P_i} and thus, not separated by R , a contradiction. Thus, R'_P is a temporal resolving set of \mathcal{SS}_Δ with $|R'_P| \leq |R|$. Therefore, we have shown that R' or $R' \cup \{c\}$ is a subset of some minimum-size temporal resolving set of \mathcal{SS}_Δ .

In the following, we consider R'' from Step 16. Observe that there are at most $|B_c| - r$ vertices v_i in $N(c)$ which are not reached by any vertex in R' . Let us have $|B_c \cap B^1| = c_1$ and $|B_c \cap B^2| = c_2$. Out of these vertices, note that c cannot separate any two vertices $v_i, v_j \in N(c) \cap V(B_c) \cap V(B^h)$ for $h \in \{1, 2\}$. Moreover, to reach every vertex v_i in $B_c \cap B^1$, we require either c to be in R'' or some vertex from B_i to be in set R'' . Thus, we have $|R''| \geq |B_c| - r - 1$. We have earlier shown that for $|R''| = |B_c| - r + 1$, there always exists a temporal resolving set. Thus, $|B_c| - r - 1 \leq |R''| \leq |B_c| - r + 1$. Let us next show that we may assume that $R'' \subseteq \mathbf{R}(c) \cap (V(B_c) \cup \{c\})$. Assume first on the contrary, that for every minimum set R^* such that $R' \cup R^*$ is a temporal resolving set of \mathcal{SS}_Δ , we have $w \in R^*$ and $w \notin \mathbf{R}(c)$. Furthermore, assume that among such sets, R^* contains the smallest possible number of vertices outside of $\mathbf{R}(c)$. Since R' separates vertices of $\mathcal{SS}_\Delta - \mathbf{R}(c)$, we have a vertex $v \in \mathbf{R}(w) \cap \mathbf{R}(c)$. Otherwise, $R' \cup R^* \setminus \{w\}$ would be a temporal resolving set of smaller size. If $\mathbf{R}(w) \cap \mathbf{R}(c) \setminus \{c\} = v$, then $R' \cup (R^* \setminus \{w\}) \cup \{v\}$ is a temporal resolving set of \mathcal{SS}_Δ , a contradiction. Thus, $u, v \in \mathbf{R}(w) \cap \mathbf{R}(c)$. Note that u and v belong to the same branch. Assume that $\text{dist}_t(c, u) = 2$ and $\text{dist}_t(c, v) = 1$. In this case, $R' \cup (R^* \setminus \{w\}) \cup \{u\}$ is a temporal resolving set of \mathcal{SS}_Δ , a contradiction. Thus, we may assume that $R'' \subseteq \mathbf{R}(c)$.

Let us next show that we may assume that $R'' \subseteq V(B_c) \cup \{c\}$. Assume next on the contrary that for every minimum set R^* such that $R' \cup R^*$ is a temporal resolving set of \mathcal{SS}_Δ , we have $w \in R^*$ and $w \notin V(B_c) \cup \{c\}$. Furthermore, assume that among such sets, R^* contains the smallest possible number of vertices outside of $V(B_c) \cup \{c\}$. We may assume that $R^* \subseteq \mathbf{R}(c)$. Let us first assume that $w \in Q_2 \cap R^*$. Observe that $\mathbf{R}(w) \cap \mathbf{R}(c) = \{w, c\}$. Furthermore, set $R' \cup (R^* \setminus \{w\}) \cup \{c\}$ is a temporal resolving set of \mathcal{SS}_Δ , a contradiction.

Thus, we may assume that $w \in \mathbf{R}(c) \setminus (V(B_c) \cup Q_2 \cup \{c\})$. Observe that R' reaches and separates every vertex in the branch w is located. Furthermore, if $\text{dist}_t(w, c) = 2$, then the only vertex, which is possibly separated from some other vertex by w but not by $R' \cup (R^* \setminus \{w\})$, is c . Thus, $R' \cup (R^* \setminus \{w\}) \cup \{c\}$ is a temporal resolving set of \mathcal{SS}_Δ , a contradiction. Assume next that $\text{dist}_t(w, c) = 1$. Now, $\mathbf{R}(c) \cap \mathbf{R}(w) = \{c\} \cup \{w\} \cup (N(c) \cap \{u \mid \text{dist}_t(c, u) = 2\}) \cup (N(w) \cap \{u \mid \text{dist}_t(c, u) = 2\})$. Note that vertices in $\{w\} \cup (N(w) \cap \{u \mid \text{dist}_t(c, u) = 2\})$ are already separated from other vertices by R' . Furthermore, for each pair of vertices which might be separated by w but not by c , there is one, say u_c , in $\mathbf{R}(c) \setminus \mathbf{R}(w)$ and another one in $\mathbf{R}(w) \cap \mathbf{R}(c)$, say u_w . Furthermore, we have $\text{dist}_t(c, u_w) = \text{dist}_t(c, u_c) = 2$. Hence, $u_w \in N(c) \cap \{u \mid \text{dist}_t(c, u) = 2\}$ and $u_c \in \{u \mid \text{dist}_t(c, u) = 2 \text{ and } \text{dist}(c, u) = 2\}$. Furthermore, if a pair u_w, u_c is not separated by $R' \cup R^* \setminus \{w\}$, then there is still a vertex x in R^* which reaches u_c since w does not reach it and $R' \cup R^*$ is a temporal resolving set of \mathcal{SS}_Δ . Observe that there are exactly two options for x since $\text{dist}_t(x, u_w) = 2$: c and the single vertex, say v_c , on the path from c to u_c (note that we cannot have $u_c \in R^*$ since u_c would separate u_w and u_c). Assume next that there is a vertex $v_x \in N(c) \cap V(B_c) \setminus \{v_c\}$ such that $\text{dist}_t(c, v_x) = 1$. In this case, $R' \cup (R^* \setminus \{w\}) \cup \{v_x\}$ is a temporal resolving set of \mathcal{SS}_Δ . If v_x does not exist, then $R' \cup (R^* \setminus \{w\}) \cup \{u_c\}$ is

a temporal resolving set of \mathcal{SS}_Δ . Indeed, both vertices v_c and c reach vertices in $N(c)$ at temporal distance 2 from c at the same time-step as w . Furthermore, u_c separates itself from u_w . Therefore, $R'' \subseteq V(B_c) \cup \{c\}$ and hence, $R'' \subseteq \mathbf{R}(c) \cap (V(B_c) \cup \{c\})$.

Let us next show that we may assume that $|R'' \cap V(B_i)| \leq 1$ for each i . Observe first that since $R'' \subseteq \mathbf{R}(c)$, we have $|R'' \cap V(B_i)| \leq 2$. Assume on the contrary, that we have $|R'' \cap V(B_i)| = 2$ for some i . Thus, we have $v_i, u_i \in R''$, $\lambda(cv_i) = 1$ and $\lambda(v_i u_i) = 2$. Let us first assume that for every $v_j \in V(B_c)$ with $j \neq i$, we have $\lambda(cv_j) = 2$. Then, $R = R' \cup (R'' \setminus v_i) \cup \{c\}$ is a temporal resolving set of \mathcal{SS}_Δ . Indeed, u_i separates vertices v_i, u_i from other vertices in $V(B_c)$. Furthermore, c reaches exactly the same set of vertices in $\mathbf{R}(c) \cap V(B_c)$ as v_i does and the only distances which differ among these vertices are those to c and v_i themselves. Thus, R is a temporal resolving set in this case. Assume then that there exists a vertex $v_j \in V(B_c)$ with $j \neq i$ such that $\lambda(cv_j) = 1$. In this case, $R = R' \cup (R'' \setminus \{u_i, u_j\}) \cup \{v_j\}$ is a temporal resolving set of \mathcal{SS}_Δ . Indeed, the only vertices in $V(B_c) \cup \{c\}$ which were separated from other vertices only by vertices u_i and u_j were in the set $\{v_j, u_i, u_j\}$. However, u_i and u_j (if it exists) are separated from other vertices in $V(B_c) \cup \{c\}$ by v_i and v_j while v_i and v_j are separated from other vertices by themselves. Thus, R is a temporal resolving set.

Let us next show that we may assume that $|R'' \cap Q_1| \leq 1$. Recall that $Q_1 \subseteq N(c) \cap \{v \mid \text{dist}_t(c, v) = 1\}$. Assume on the contrary that we have $v_i, v_j \in R'' \cap Q_1$. Then, $R = R' \cup (R'' \setminus \{v_i\}) \cup \{c\}$ is a temporal resolving set of \mathcal{SS}_Δ . Indeed, recall that c , together with R' , separates vertices in Q_1 from all other vertices. Furthermore, the only vertex which v_i could separate which is not separated c or v_j is u_j if $\text{dist}_t(c, u_j) = 2$. However, since by the definition of Q_1 , the set R' reaches vertex $v_j \in Q_1$, R' also reaches u_j . Hence, R' separates u_j from other vertices at temporal distance 2 from c , and R is a temporal resolving set of \mathcal{SS}_Δ .

Let us next show that we may assume that $|R'' \cap \{a \mid \text{dist}(c, a) = 2\}| \leq 1$. We assume on the contrary that we have two vertices $u_i, u_j \in R'' \cap \{a \mid \text{dist}(c, a) = 2\}$. By our assumptions, we have $R'' \subseteq \mathbf{R}(c)$. Hence, $\text{dist}_t(c, u_i) = \text{dist}_t(c, u_j) = 2$. We claim that $R = R' \cup (R'' \setminus \{u_i, u_j\}) \cup \{v_i, v_j\}$ is a temporal resolving set of \mathcal{SS}_Δ . Indeed, the only vertices in $\mathbf{R}(c)$ which are reached by u_i and u_j are in the set $\{v_i, v_j, u_i, u_j, c\}$. Out of these, v_i and v_j separate themselves. Furthermore, u_i is separated from vertices in $\{u \mid \text{dist}_t(c, u) = \text{dist}_t(c, u) = 2\}$ by v_i and from vertices in $\{u \mid \text{dist}_t(c, u) = 1 \text{ and } \text{dist}_t(c, u) = 2\}$ by v_j . Similar arguments hold for u_j . Finally, c is the only vertex at temporal distance 1 from both v_i and v_j . Hence, R is a temporal resolving set of \mathcal{SS}_Δ .

Since we are getting a temporal resolving set in Steps 15 to 20 and there is a minimum-size temporal resolving set of the form $R' \cup R''$, we conclude that we have found a minimum-size temporal resolving set in Steps 15 to 20 (or in Step 4).

Let us finally show that Algorithm 2 ends in polynomial time. Recall that it takes polynomial time on the number of vertices n , especially when the time labels are in set $\{1, 2\}$, to check if a given set is a temporal resolving set of a graph. Furthermore, in Step 1, Algorithm 1 works in linear-time and we need to apply it at most n times. Steps 2 to 14 clearly operate in polynomial-time. In Steps 15 to 20, we test for a given $j \in \{0, 1, 2\}$, at most $\binom{|B_c|+1}{3-j} \cdot r \cdot |B_c| \in O(n^5)$ times if a given set is a temporal resolving set. Hence, Algorithm 2 works in polynomial time, finishing the proof. \square

4 Combinatorial results for p -periodic 1-labelings

In this section, we focus on p -periodic 1-labelings, and bound the minimum size of temporal resolving sets for several graph classes. When $p = 1$ or $\lambda(e)$ is the same for every edge e , those are exactly the usual resolving sets.

Given a temporal graph $\mathcal{G} = (G, \lambda)$ where λ is a p -periodic 1-labeling, we denote by $M_p(\mathcal{G})$ the minimum size of a temporal resolving set of \mathcal{G} . Furthermore, if $\lambda(e) = \{i, i + p, i + 2p, \dots\}$, then, by abuse of notation, we denote $\lambda(e) = i$.

Note that, in this section, reachability is trivially assured (since the time-steps repeat indefinitely and

considered graphs are connected), so to prove that a given set is a temporal resolving set, we only need to prove that it is separating.

Theorem 10. *Let P_n be a path on n vertices, λ a p -periodic 1-labeling, and $\mathcal{P} = (P_n, \lambda)$. We have $M_p(\mathcal{P}) = 1$.*

Proof. Let u_1, \dots, u_n be the vertices of P_n , with edges $u_i u_{i+1}$ for $1 \leq i \leq n-1$. Let $R = \{u_1\}$. For $i \in \{2, \dots, n-1\}$, u_i is reached from u_1 strictly before u_{i+1} . The same reasoning works for $R = \{u_n\}$. Hence, any of the leaves (clearly) forms a minimum-size temporal resolving set. \square

In particular, the proof of Theorem 10 implies that in a temporal tree \mathcal{T} with p -periodic 1-labeling if we have a path from r to u to v , then r separates vertices u and v . In this case, we say that vertices u and v are *path-separated* (by r). In the following two theorems, we introduce combinatorial results for some simple graph classes.

Theorem 11. *Let C_n be a cycle on n vertices, λ a p -periodic 1-labeling, and $\mathcal{C} = (C_n, \lambda)$. We have $1 \leq M_p(\mathcal{C}) \leq 2$.*

Proof. Let C_n be a cycle on n vertices, λ be a p -periodic 1-labeling, and $\mathcal{C} = (C_n, \lambda)$. Let $e = uv$ be a locally maximally labeled edge of \mathcal{C} , i.e., an edge with $\lambda(e)$ such that the adjacent edges have labels at most $\lambda(e)$ (such an edge has to exist). We claim that u and v form a resolving set.

Suppose for a contradiction that there are two vertices, x and y , not separated by u and v . That means that they have the same temporal distance from u and from v .

Let us first consider temporal distances from u . By e being locally maximally labeled edge, it must be that precisely one of the paths from u to x and from u to y attaining the minimal temporal distance must go through e . Otherwise, u path-separates x and y . Without loss of generality, let e be on the path from u to y . Thus, $\text{dist}_t(v, y) = \text{dist}_t(u, y) - p$. However, by the same reasoning as above, path attaining the temporal distance from v to x must now use edge e and thus, $\text{dist}_t(u, x) = \text{dist}_t(v, x) - p$. Since $\text{dist}_t(u, x) = \text{dist}_t(v, x)$. We have $\text{dist}_t(v, y) = \text{dist}_t(v, x) - 2p$, a contradiction. \square

Theorem 12. *Let K_n be a complete graph on $n = b + p^b$ vertices with $b \geq 1$, λ a p -periodic 1-labeling, and $\mathcal{K} = (K_n, \lambda)$. We have $b \leq M_p(\mathcal{K}) \leq n - 1$ and both bounds are tight.*

Proof. The upper bound is trivial. To prove its tightness, consider $\mathcal{K} = (K_n, \lambda)$ with λ assigning the same time label to all edges. All pairs of vertices in the graph are twins and therefore, we have to take at least one of the vertices in all such pairs. This results in taking $n - 1$ vertices.

Let us prove the lower bound. For a contradiction, suppose there would be less than b vertices in a temporal resolving set, say $b' < b$. There are still more than p^b vertices to separate but there are just $p^{b'}$ possible distance vectors to the vertices outside of our chosen set, less than p^b . This means that some vertices have to share a distance vector and thus, they are not separated, a contradiction.

We shall now construct a complete graph K_n with a p -periodic 1-labeling which attains the lower bound. To this end, take a subset B of b vertices in a fixed order. Then, give every vertex $v \in V(K_n) \setminus B$ a unique p -ary tuple $\ell(v)$ of length b containing values from 1 to p . We label an edge between i -th element of B and $v \in V(K_n) \setminus B$ by j if the i -th position of $\ell(v)$ is j . The remaining edges of the graph, i.e., edges running between the vertices of B and between the vertices outside of B will get label p . Clearly, the vertices of B now form a resolving set since the constructed b -tuples are precisely the vectors of distances. \square

The following lemma will help us to simplify the remaining results on trees, as we will be able to consider only temporal resolving sets composed of leaves.

Lemma 13. *Let T be a tree, λ a p -periodic 1-labeling, $\mathcal{T} = (T, \lambda)$ with $M_p(\mathcal{T}) \geq 2$. There is a temporal resolving set of \mathcal{T} of size $M_p(\mathcal{T})$ containing only leaves of T .*

Proof. Let T be a tree, λ a p -periodic 1-labeling, and $\mathcal{T} = (T, \lambda)$. Suppose we have a minimum temporal resolving set R of size at least two with the minimum number of non-leaves.

If every leaf is in the resolving set, then every two non-leaf vertices are path-separated by a leaf and thus there can be no non-leaves in R , as otherwise we can construct a smaller temporal resolving set. Hence, we may assume that not all leaves do belong to R .

Assume next that there is at least one non-leaf vertex $v \in R$. We root the tree in v . Our aim is to find a suitable leaf $\ell \notin R$, which will be exchanged with v in order to get a same-size temporal resolving set with lesser number of leaves. Let us denote for branch B the label of the edge between v and the vertex adjacent to v in $V(B)$ by $\lambda(B)$. Let $m = \min\{\lambda(B) \mid B \text{ is a branch}\}$. Denote by B^m the set of branches B which have $\lambda(B) = m$. If there is a branch $B \in B^m$ such that $R \cap V(B) = \emptyset$, then we choose ℓ as a leaf of B . Otherwise, if there exists branch B' such that $R \cap V(B') = \emptyset$, then we select ℓ as a leaf of B' and if such B' does not exist, then we select ℓ as an arbitrary leaf not in R . Denote $R' = (R \cup \{\ell\}) \setminus \{v\}$. Notice that since R is a temporal resolving set of \mathcal{T} , after this process we have $R' \cap V(B) \neq \emptyset$ for each $B \in B^m$. In the following, for $x, y \in V(T)$, we always assume that $x \in V(B_x) \cap R'$ where B_x is the branch in which x resides, $B_x \in B^m$, and $y \in R \setminus V(B_x)$.

We now consider different cases for pair of vertices $a, b \in V(T)$ that is separated by v (which is in R). In particular, we show that R' separates them as well.

Case 1. $a \in V(B), b \in V(B')$ for any $B, B' \notin B^m$: Note that we may have $B = B'$. Recall that $x \in V(B_x) \cap R'$ and $B_x \in B^m$. We have $\text{dist}_t(x, a) = \text{dist}_t(x, v) + \text{dist}_t(v, a) - m$ and $\text{dist}_t(x, b) = \text{dist}_t(x, v) + \text{dist}_t(v, b) - m$. Since v separates a and b , we have $\text{dist}_t(v, a) \neq \text{dist}_t(v, b)$. Hence, $\text{dist}_t(x, a) \neq \text{dist}_t(x, b)$.

Case 2. $a, b \in V(B)$ for $B \in B^m$: Let $y \in V(B_y) \cap R'$ and $B_y \neq B$. We have $\text{dist}_t(y, a) = \text{dist}_t(y, v) + \text{dist}_t(v, a) + (p - \lambda(B_y))$ and $\text{dist}_t(y, b) = \text{dist}_t(y, v) + \text{dist}_t(v, b) + (p - \lambda(B_y))$ since $\lambda(B) \leq \lambda(B_y)$. Since v separates a and b , we have $\text{dist}_t(v, a) \neq \text{dist}_t(v, b)$. Hence, $\text{dist}_t(y, a) \neq \text{dist}_t(y, b)$.

Case 3. $a \in V(B), b \notin V(B)$ for $B \in B^m$: Since $B \in B^m$, we have $V(B) \cap R' \neq \emptyset$. Assume without loss of generality, that $B = B_x$. We have $x \in V(B_x) \cap R'$. Let vertex c be the last vertex on the common paths from x to a and x to b (possibly $c = x$). We have $c \in V(B_x)$. Let us denote by $\lambda(a)$ ($\lambda(b)$) the time label of the first edge on the path from c to a (to b). Furthermore, denote by $\lambda(X)$ the label of the last edge on the path from x to c . We let $\lambda(X) = 0$ if $x = c$.

When we have $\lambda(X) < \min\{\lambda(a), \lambda(b)\}$ or $\lambda(X) \geq \max\{\lambda(a), \lambda(b)\}$, vertex x separates a and b if and only if c separates a and b . If $\lambda(a) \leq \lambda(X) < \lambda(b)$, then vertex x separates vertices a and b if and only if $\text{dist}_t(c, b) \neq \text{dist}_t(c, a) + p$.

Similarly, if $\lambda(b) \leq \lambda(X) < \lambda(a)$, then vertex x separates vertices a and b if and only if $\text{dist}_t(c, a) \neq \text{dist}_t(c, b) + p$. Moreover, in all three subcases we have $\text{dist}_t(c, a) \leq \text{dist}_t(v, a)$ and $\text{dist}_t(c, b) \geq \text{dist}_t(v, b)$. Observe that if x does not separate vertices a and b , then the time label of the last edge on the paths from v to a and from v to b is the same.

We conclude that $\text{dist}_t(v, a) = \text{dist}_t(v, b) + h \cdot p$ for some integer h . Again, we denote by B_b the branch in which b is located. We shall now prove a crucial lemma, saying that in all the three cases, h is a positive integer.

Claim 1. In all the three aforementioned cases we have $h \geq 1$ if x does not separate a and b .

Proof. We shall divide the proof according to the cases which might occur.

Subcase 3.a. $\lambda(X) < \min\{\lambda(a), \lambda(b)\}$ or $\lambda(X) \geq \max\{\lambda(a), \lambda(b)\}$: In this case, x separates a and b if and only if c separates a and b . We have $\text{dist}_t(c, a) \leq \text{dist}_t(v, a)$ and $\text{dist}_t(c, b) \geq \text{dist}_t(v, b)$ and at least one of these two inequalities is strict (otherwise c separates a and b). Hence, if c does not separate a and b , then $\text{dist}_t(v, a) = \text{dist}_t(v, b) + h \cdot p$ for some positive integer h . Indeed, the time label of the last edge on the paths from v to a and from v to b is identical since c does not separate these vertices.

Subcase 3.b. $\lambda(b) \leq \lambda(X) < \lambda(a)$: In this case, x separates vertices a and b if and only if $\text{dist}_t(c, a) \neq \text{dist}_t(c, b) + p$. Hence, we assume that $\text{dist}_t(c, a) = \text{dist}_t(c, b) + p$. We have $\text{dist}_t(v, a) = \text{dist}_t(v, b) + h \cdot p$ for some integer $h \neq 0$. Recall that we have $\text{dist}_t(c, b) \geq \text{dist}_t(v, b)$ and $\text{dist}_t(c, b) + p = \text{dist}_t(c, a) \leq \text{dist}_t(v, a)$. Hence, $\text{dist}_t(v, b) + p \leq \text{dist}_t(c, a) \leq \text{dist}_t(v, a)$. Thus, $h \geq 2$.

Consider next vertex $y \in R' \cap V(B_b)$. We have

$$\text{dist}_t(c', b) \leq \text{dist}_t(v, b) \leq \text{dist}_t(c, b) = \text{dist}_t(c, a) \leq \text{dist}_t(v, a) \leq \text{dist}_t(c', a) - p.$$

The last inequality is due to Equation (1). Since x does not separate a and b , at least one of the inequalities $\text{dist}_t(c, a) \leq \text{dist}_t(v, a)$ and $\text{dist}_t(c, b) \geq \text{dist}_t(v, b)$ is strict. Hence, $\text{dist}_t(c', a) > \text{dist}_t(c', b) + p$. Therefore, y separates a and b for every possible linear ordering of $\lambda(a')$, $\lambda(b')$, and $\lambda(Y)$.

Subcase 3.c. $\lambda(a) \leq \lambda(X) < \lambda(b)$: In this case, x separates vertices a and b if and only if $\text{dist}_t(c, b) \neq \text{dist}_t(c, a) + p$. Assume that this is not the case and $\text{dist}_t(c, b) = \text{dist}_t(c, a) + p$. Recall that we have $\text{dist}_t(v, a) = \text{dist}_t(v, b) + h \cdot p$ for some integer $h \neq 0$. We have $\text{dist}_t(v, a) = \text{dist}_t(v, c) + \text{dist}_t(c, a) + (p - \lambda(b))$ because $\lambda(b) > \lambda(a)$. Since $\text{dist}_t(v, c) \geq \lambda(b)$, we have $\text{dist}_t(v, a) \geq \text{dist}_t(c, a) + p$. Recall that, we have $\text{dist}_t(c, b) \geq \text{dist}_t(v, b)$. Further,

$$\text{dist}_t(v, b) + h \cdot p = \text{dist}_t(v, a) \geq \text{dist}_t(c, a) + p = \text{dist}_t(c, b) \geq \text{dist}_t(v, b).$$

Finally, since $\text{dist}_t(v, a) \neq \text{dist}_t(v, b)$, we have $h \geq 1$.

With this, we have proved the claim for all the three subcases. \square

We now proceed with the proof of Case 3 together with the assumption $\text{dist}_t(v, a) = \text{dist}_t(v, b) + h \cdot p$ for a positive integer h .

Consider now a vertex $r \in R' \cap V(B_r)$ for $B_r \notin B_x \cup B_b$ (if such a r exists). We have

$$\begin{aligned} \text{dist}_t(r, a) &= \text{dist}_t(r, v) + \text{dist}_t(v, a) + (p - \lambda(B_r)) \\ &= \text{dist}_t(r, v) + \text{dist}_t(v, b) + (h + 1) \cdot p - \lambda(B_r) \\ &\geq \text{dist}_t(r, v) + \text{dist}_t(v, b) + 2p - \lambda(B_r). \end{aligned}$$

Moreover, if $\lambda(B_b) \leq \lambda(B_r)$, then $\text{dist}_t(r, b) = \text{dist}_t(r, v) + \text{dist}_t(v, b) + (p - \lambda(B_r)) < \text{dist}_t(r, a)$. Furthermore, if $\lambda(B_b) > \lambda(B_r)$, then $\text{dist}_t(r, b) = \text{dist}_t(r, v) + \text{dist}_t(v, b) - \lambda(B_r) < \text{dist}_t(r, a)$. Hence, r separates a and b in this case.

If we have a vertex $y \in V(B_b) \cap R'$, then we denote by c' the last common vertex on the path from y to a and from y to b . Since $B_x \in B^m$, we have

$$\text{dist}_t(c', a) = \text{dist}_t(v, a) + \text{dist}_t(c', v) + (p - \lambda(B_b)) \geq \text{dist}_t(v, a) + p. \quad (1)$$

We denote by $\lambda(a')$ and $\lambda(b')$ the time label of the first edge on the path from c' to a and from c' to b , respectively. Furthermore, denote by $\lambda(Y)$ the label of the last edge on the path from y to c' . We let $\lambda(Y) = 0$ if $y = c'$. Similarly to x , when we have $\lambda(Y) < \min\{\lambda(a'), \lambda(b')\}$ or $\lambda(Y) \geq \max\{\lambda(a'), \lambda(b')\}$, vertex y separates a and b if and only if c' separates a and b . If $\lambda(a') \leq \lambda(Y) < \lambda(b')$, then vertex y separates vertices a and b if and only if $\text{dist}_t(c', b) \neq \text{dist}_t(c', a) + p$. Similarly, if $\lambda(b') \leq \lambda(Y) < \lambda(a')$, then vertex y separates vertices a and b if and only if $\text{dist}_t(c', a) \neq \text{dist}_t(c', b) + p$. In particular, if $\text{dist}_t(c', a) > \text{dist}_t(c', b) + p$, then y separates a and b .

We have

$$\text{dist}_t(c', b) \leq \text{dist}_t(v, b) \leq \text{dist}_t(v, a) - p \leq \text{dist}_t(c', a) - 2p.$$

The second inequality is due to Claim and the last inequality is due to Equation (1). Hence, $\text{dist}_t(c', a) > \text{dist}_t(c', b) + p$. Therefore, y separates a and b for every ordering of $\lambda(a')$, $\lambda(b')$ and $\lambda(Y)$.

Case 4. $a \in V(B), b = v$: Let $y \in R' \setminus V(B)$. In this case, y simply path-separates a and b .

Therefore, set R' separates all vertices which are separated by R in all cases. Since $|R| = |R'|$, this contradicts the assumption that R has the minimum possible number of non-leaves. Hence, the theorem follows. \square

Theorem 13 gives us an FPT algorithm for TEMPORAL RESOLVING SET in trees with respect to the number of leaves. It also allows us to prove a brute-force-like polynomial-time algorithm for temporal subdivided stars with a p -periodic 1-labeling for fixed p . This shows that TEMPORAL RESOLVING SET is in XP with respect to the period of the time labeling in this context.

Theorem 14. *TEMPORAL RESOLVING SET is polynomial-time solvable in temporal subdivided stars with a p -periodic labeling for fixed p .*

Proof. Let S be a subdivided star with central vertex c and ℓ leaves, λ be a p -periodic 1-labeling, and $\mathcal{S} = (S, \lambda)$. By Theorem 13, there exists a minimum-size temporal resolving set in \mathcal{S} which contains only leaves of S , if $M_p(\mathcal{S}) > 1$.

Note that if some branches, say b of them, share the same time label on their edges incident with c , then, to separate vertices in these branches, we need to select at least $b - 1$ leaves in a temporal resolving set. Since there are at most p distinct labels, we have, for any temporal resolving set R of \mathcal{S} , $|R| \geq \ell - p$. Furthermore, checking whether a given vertex set is a temporal resolving set of \mathcal{S} can be done in polynomial time.

If n is the order of S , then there are $\binom{\ell}{\ell-p} = \binom{\ell}{p} \in O(n^p)$ ways to select a vertex set containing exactly $\ell - p$ leaves. Since p is a fixed constant, we can check all these sets in polynomial time. If any of them is a temporal resolving set, then we found a minimum-size temporal resolving set of \mathcal{S} . Otherwise, we iterate the process by examining all vertex sets of size $\ell - p + 1, \ell - p + 2, \dots$, until we find a temporal resolving set. The process will eventually stop as the set containing all leaves of S is a temporal resolving set of \mathcal{S} . Thus, we need to check at most $\sum_{i=0}^p \binom{\ell}{p-i}$ vertex sets, that is, we need to do $O(n^p)$ polynomial-time operations. \square

We end this section with two combinatorial results for other two subclasses of temporal trees: subdivided stars and complete binary trees.

Theorem 15. *Let S be a subdivided star on $\ell \geq 2$ leaves, λ a p -periodic 1-labeling, and $\mathcal{S} = (S, \lambda)$. We have $\max(1, \ell - p) \leq M_p(\mathcal{S}) \leq \ell - 1$ and both bounds are tight.*

Proof. Suppose that we have a subdivided star S with a p -periodic 1-labeling. Denote by c the central vertex of S . By Theorem 13, if $M_p(\mathcal{S}) \geq 2$, then there exists a minimum-size temporal resolving set of S containing only leaves. Again, as in the proof of Theorem 14, one can argue by a simple application of the pigeonhole principle applied to the distinct labels of edges incident with c that at least $\ell - p$ leaves have to be chosen in order to separate vertices in $N(c)$. For $\ell - p < 1$, a trivial bound saying that at least one vertex has to be chosen is clearly better.

The lower bound is tight, as we can take a star on ℓ leaves such that first p edges will have different labels and the remaining $\ell - p$ leaves will all share the same label, say 1. As there are $\ell - p + 1$ vertices being mutually twins, any minimum-size temporal resolving set has to be of size at least $\ell - p$.

For the upper bound, we show that taking any $\ell - 1$ leaves is enough to form a temporal resolving set R , taking Theorem 13 into account. Indeed, a leaf $\ell \in R$ path-separates every vertex in the same branch from each other and from vertices in other branches. Furthermore, ℓ also path-separates vertices in the single branch with leaf outside of R from each other. Thus, R separates all vertices in \mathcal{S} .

The tightness can be exemplified by a subdivided star with ℓ leaves with exactly same labels on all its edges. In this case, all leaves are twins and we are forced to take all but one to separate them. \square

Theorem 16. *Let T be a complete binary tree on $2^n - 1$ vertices, λ a 2-periodic 1-labeling, and $\mathcal{T} = (T, \lambda)$. We have $2^{n-3} \leq M_2(\mathcal{T}) \leq 2^{n-2}$. Both bounds are tight.*

Proof. The *center* of a tree is a vertex v such that the maximum distance between the vertex v and any other vertex of the tree is minimal. (In this case, the center is a unique vertex.) For convenience, we consider all trees in this proof as rooted in the center. Based on the distance from the center, we say that vertices at distance i from the center are *on level i* . For a given i , *higher* levels are levels from 0 to $i - 1$, and *lower* levels are those with value at least $i + 1$. A complete binary tree on $2^n - 1$ vertices has 2^{n-1} leaves and they are on level $n - 1$; the center is on level 0. We say that a vertex u is a *descendant*, or *ancestor* of v if v is on the unique path from root to u , or u is on the unique path from root to v , respectively. Finally, we shall say that leaves are *close* if they are at distance 2 in the underlying tree. Seven vertex subtrees rooted in a vertex on level $n - 3$ and induced by all the vertices under such a vertex are called *essential*.

Again, by Theorem 13, we consider only sets of leaves as candidates for optimal temporal resolving sets if these are of size at least two.

For the lower bound, let us have $\mathcal{T} = (T, \lambda)$, where T is a complete binary tree on $2^n - 1$ vertices and λ is a 2-periodic 1-labeling. We focus on essential subtrees. If we take less than 2^{n-3} vertices into our candidate set R , then necessarily at least one of the essential subtrees will have none of its leaves in R . Let us denote its root by r . In such a subtree, the four paths from r to the leaves have to be labeled by all four possible combinations of labels 1 and 2, otherwise we easily find two vertices not separated by R . However, even if this is the case, none of the vertices from R are able to separate the leaf with labels 1 and 1 on the path from r (denote the leaf by l_1) and the leaf with labels 2 and 1 on the path from r (denote this leaf by l_2). Indeed, the temporal distance of any vertex c of R to l_1 and l_2 is $d + 3$ where d is the temporal distance of c to r . Thus at least 2^{n-3} leaves are needed in any temporal resolving set of \mathcal{T} .

To show that the lower bound is tight, consider \mathcal{T} with a 2-periodic 1-labeling λ , constructed in a top-down fashion, proceeding level by level we label one of the edges going to the lower level with 1 and the other one with 2. We shall form a temporal resolving set R of size 2^{n-3} by taking one leaf from each essential subtree: the one connected to the root of the respective essential subtree by path labeled only with 1. We have to prove that all vertices are now separated. Let us have two different vertices, say u and v , outside of R and suppose they are not path-separated. We distinguish the following cases.

- *Vertices u and v are both on levels $n - 3$ or higher:* In this case u and v are path separated and thus, this case cannot occur.
- *Vertices u and v are on levels $n - 2$ and/or $n - 1$:* Either u and v are in the same essential subtree \mathcal{T}_e and then, by a simple calculation, the temporal distances from the unique vertex from R in \mathcal{T}_e separate these two vertices. (Temporal distances in \mathcal{T}_e range from 0 to 6 and they appear uniquely.) Assume then that u and v are in different essential subtrees. Let $u \in V(\mathcal{T}_1)$ and $v \in V(\mathcal{T}_e')$. We observe that the distance from the leaf, say l_e in $R \cap V(\mathcal{T}_e)$ to v is at least 7. Thus, l_e separates vertices u and v .
- *Vertex u is on level at most $n - 3$, while v is on a level between $n - 3$ and $n - 1$, or vice versa:* In this case, u and v are path-separated unless we are in the special case of \mathcal{T} having precisely 7 vertices and being itself an essential subtree. However, we already know that separation is guaranteed in this case.

We covered all the possible cases and the tightness follows.

For the upper bound, we consider temporal tree $\mathcal{T} = (T, \lambda)$, where λ is a 2-periodic 1-labeling. We show that taking one of the leaves in each of the 2^{n-2} subtrees rooted in a vertex on level $n - 2$ suffices to separate all vertices in \mathcal{T} . Let us denote such a set by R . Again, we have to be careful only about the pairs of vertices that are not path-separated. Suppose we have such a pair of different vertices $u, v \notin R$ so that there is no vertex $c \in R$ such that either u is on the path from c to v , or v is on the path from u to c . Based on our choice of R , both u and v have to be leaves. We observe that distance from a vertex $r \in R$ close to u has distance in set $\{2, 3, 4\}$ while the distance to other leaves is at least 4. However, if $\text{dist}_t(r, v) = 4$, then $\text{dist}_t(r, u) \leq 3$. Thus, R separates all vertices and the claim is proved.

To show that the upper bound is tight, consider $\mathcal{T} = (T, \lambda)$ with a 2-periodic 1-labeling λ where all edges have the same label. Take any subset R' of leaves of size at most $2^{n-2} - 1$. In such a way, there has to be at least one subtree rooted at level $n - 2$ out of all 2^{n-2} possible ones with no leaves in R' . The leaves of such a subtree are not separated by R' and thus, the tightness is proved. \square

5 Conclusion

We extended the definition of resolving sets to temporal graphs. We have proved that TEMPORAL RESOLVING SET is hard even in very restricted graph classes for some labelings. On the other hand, it seems that the problem might be easier with p -periodic labelings. In particular, the fact that the standard resolving sets are somewhat easier (they can be solved in polynomial-time in some classes for which TEMPORAL RESOLVING SET is NP-hard, such as complete graphs or trees) suggests that having time labelings containing a large number of labels or where the highest possible time-step is low might make the problem more tractable. In particular, since minimum k -truncated resolving sets can be found in polynomial-time in trees when k is fixed, a potential direction would be to study the parameterized complexity of temporal resolving sets, with the total number of available time-steps as a natural parameter. This parameter is unbounded in our NP-hardness reductions for trees.

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