Model order reduction of random parameter-dependent linear systems

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Abstract

This paper is devoted to model reduction of linear time invariant (LTI) systems whose parameters are random variables governed by probabilistic laws. A new and original method is proposed to deal with this challenging problem. It combines the truncation balanced realization (TBR) based method well known in model reduction of LTI systems and the generalized polynomial chaos (GPC) formalism known as a powerful tool for uncertainty propagation. The GPC formalism is used to represent and compute a random parameter-dependent balancing transformation (RPD-BT) which puts the random parameters-dependent LTI system in a balanced form almost surely within the probabilistic range of random parameters. Model reduction is then performed by truncating states that are almost surely weakly controllable and observable. The truncation error is characterized by its moments. These are shown to be bounded by the Hankel singular values moments that are also estimated by using the GPC formalism. The proposed method is tested through its application on an simple mechanical system model.

Key words: Model reduction, uncertain systems, generalized polynomial chaos, random parameters, balanced transformation, controllability and observability gramians, Hankel singular values.

1. Introduction

Model reduction is of major importance in simulation, design and control theory. It includes different methodologies which help to generate, from a given complex model, a simpler one while keeping the most important properties of the original model.

For system models which do not incorporate uncertainties, there are numerous methods for model reduction, such as the Truncated Balanced Realization (TBR) (Moore, 1981; Scherpen, 1993; Fujimoto and Tsubakino, 2008; Hahn and Edgar, 2002; Wood et al, 1996), Krylov Subspace Methods (also known as Moment Matching, Grimme et al, 1995; Antoulas, 2001; 2005) or Modal Reduction Methods (Davion, 1968; Varga, 1995), Singular Perturbations based methods (Kokotovic, 1963; Djennoune and Bettayeb, 2003) or Proper Orthogonal Decomposition (POD) methods (Berkooz et al, 1993). However, model reduction when one or several systems parameters are likely to change is more complicated. Indeed, the major difficulty is to preserve the effects that changes in the parameters have on the model behaviour. So, in this framework, methods for model reduction are not numerous. Most of existing methods are issued from extensions of deterministic versions. For example, the balanced truncation method, which consists in ordering states variables with respect to their controllability and observability degrees measured by Hankel singular values, was extended to systems with structured uncertainty modeled by linear fractional transformation (Beck et al, 1996). The proposed method defines how to build a state transformation putting the solutions of linear matrix inequalities (LMIs) equal and diagonal under some constraints on minimal eigenvalues. Truncation of the states having small singular values is then performed to obtain a reduced order model. This method requires that the system be represented as a linear fractional transformation (LFT). This condition is restrictive since the LFT representation is not easy to establish in particular for high dimension systems. Sun and Hahn have extended balancing and proper orthogonal decomposition (POD) techniques to systems with uncertain parameters (Sun and Hahn, 2006). The developed method offers an important advantage which consists in retaining the effect that uncertain parameters have on the original model by including the parametric uncertainty into the procedure used for computing the state transformation in the balancing and

POD techniques. The exploited idea is to lump both the inputs and the uncertain parameters in the same vector of inputs. Empirical gramians for balancing or the correlation matrix for POD are then computed for the system with the new input vector.

The moment matching approach has been extended to deal with LTI systems with uncertain parameters. In (Weile et al, 1999), the case where the state matrix is linearly dependent on one parameter has been considered. The main result in this study was about the calculating of the projection matrix which matches some of the first moments of the transfer function $G(s,p)$ with respect to the parameter p. This result is in addition with the one established in the deterministic case where the project matrix matches some of the first moments of the transfer function with respect to s. A generalization of this result to LTI systems with multiple uncertain parameters is carried out in (Daniel et al, 2005). However, these methods require a linear dependence of the state matrix on the system parameters.

The LMI approach was exploited by Trofino and Coutinho to solve a robust order reduction. Their method is based on the solution of an LMI optimization problem in which an upper bound on the H_2 or H_∞ norms of the approximation error is minimized (Trofino and Coutinho, 2004). In another register, the Routh-Pade approximation method has been combined with interval arithmetic's to deal with model reduction of interval systems (Bandyopadhyay et al, 1997). The instability of reduced models obtained is the main drawback of this approach. In this context, other studies have proposed algorithms to reduce the possibility of losing the stability (Dolgin and Zeheb, 2003; Wang et al, 2012).

Recently, Panzer and co-authors have proposed a novel methodology for model reduction of parameter-dependent linear dynamic systems (Panzer et al, 2010). The main idea is, first, to calculate reduced order models at some given values of parameters by using suitable projection matrices. The reduced parametric model is then generated by interpolating the matrices of the local reduced order models. A similar idea was developed in (Amsallem and Farhat, 2011). In a previous work, Baur and Benner have combined the balanced truncation method with interpolation algorithms to generate an LTI parametric reduced order model (Baur and Benner, 2009). The main principle in the developed procedure is to compute local reduced order model at some points in the parameter space then to interpolate them. Crucial steps are essentially the choice of the parameter points where the local models are to be calculated and reduced and the choice of the weighting functions. The first problem is dealt with the sparse grid technique while the weighting functions are chosen so as to minimize the interpolation error. The same method has been combined with the interpolatory H_2 optimal model reduction method (Baur and al, 2009). The main issue about these methods is related to the definition of an error estimation criterion to well assess the quality of reduced order models obtained. Otherwise, other methods have been developed for particular classes of systems as polytopic uncertain linear systems (Fen, 1996) and discrete linear systems described by polygons (Dolgin and Zeheb, 2004, 2005).

All the mentioned studies did not use any information about how evolves parameters uncertainty. However, in numerous practical cases, parameters can be described by probabilistic laws. These are obtained from experimental and/or simulation data. So, it is interesting to exploit this supplementary information in order to develop more efficient methods for model reduction of LTI systems with uncertain parameters. This is the main goal of this paper which is devoted to model order reduction of random parameter dependent (RPD)- LTI systems. In fact, the taking into account of the probabilistic aspects of the uncertainty related to parameters allows to exploit the polynomial chaos (PC) formalism. This theory, proposed by Wiener (Wiener, 1932), pioneered by Ghanem and Spanos (Ghanem and Spanos, 1991) and extended to the so-called generalized polynomial chaos (GPC) (Xiu and Karniadakis, 2002), sets the possibility to develop any second order stochastic process into series of weighted orthogonal polynomials with respect to a given probability measure. The only problem with the use of the GPC is related to the calculation of the weighting coefficients, namely, the stochastic modes. For this objective intrusive or non-intrusive schemes have been developed (Babuska et al, 2004; 2007; Crestaux et al, 2009). The power of the GPC approach has been tested in numerous applications related to systems and control theories such as in, the stability analysis and the prediction of the dynamic behavior of uncertain linear and nonlinear systems (Fisher, 2009; Nechak et al, 2011; 2012, 2013a), the sensitivity analysis (Crestaux et. al, 2009; Sudret, 2007), the parameter estimation and state observer synthesis (Li and Xiu, 2009; Blanchard et. al, 2010, Smith et. al, 2007) and controller design problems, (Fisher and Bhatacharya, 2009; Hover and Triantafyllou, 2006).

In this paper, a new method is proposed to deal with model reduction of RPD-LTI systems. The main originality of this work lies in the combination of the generalized polynomial chaos formalism with the TBR method. The objective is to derive an efficient methodology for model reduction of RPD-LTI systems in which parameters are modeled by random variables with given density functions. Recently, the GPC formalism is proposed to deal with model reduction of finite element models of electromagnetic devices exhibiting statistical variability in their parameters (Sumant et al, 2012). The method proposed to represent the reduced order system matrices using polynomial chaos expansions whose coefficients are computed using a certain number of deterministic reduced order finite element models obtained at specific values of random parameters issued from the multidimensional sparse grid through the Smolyak algorithm. In this paper, the method proposed does not use any model reduction procedure before the computation of the final reduced order model. Indeed, the main idea is to use the GPC formalism to com-

pute a random parameter dependent balanced transformation which puts the original model in a balanced form almost surely within the probabilistic range of system's parameters. A random parameter dependent truncated balanced realization (RPD-TBR) is then generated by deleting states that are weakly controllable and observable almost surely. Controllability and observability degrees of state variables are determined by RPD-Hankel singular values. The latter are also characterized in a probabilistic way (mean value, standard deviation and density function) by using the GPC approach instead of the well-known prohibitive Monte Carlo (MC) method. The knowledge of Hankel singular values is crucial for the truncation of the balanced realization since they give information about how the system states are simultaneously controllable and observable. In this context, the Sum of Square (SOS) formalism has been used in a recent study to quantify robust controllability and observability degrees in LTI systems with uncertain parameters belonging to semi-algebraic sets (Sojoudi et al, 2009). So, the GPC provides in the same framework an efficient and more general solution. A preliminary study has been proposed about this idea in (Nechak et al, 2013b). This paper gives a more complete study about the feasibility of the RPD-TBR based method. Moreover, an error bound is defined for the RPD-TBR generated. Indeed, in this paper, the truncation error, which has been shown to be bounded in the case of deterministic LTI systems, is characterized by statistical moments of Hankel singular values of the deleted state variables. Another important result is that effects of changes in the parameter values in the original model are retained in the reduced order model.

This paper is organized as follows. First, the generalized polynomial chaos formalism is presented in Section 2. Model reduction of LTI models with random parameters is then developed in Section 3. Efficiency of the proposed method is studied in Section 4 through its application on a physical system. Conclusions and perspectives of this work are given at the end of this paper.

2. Polynomial chaos

Place Let $X(\omega)$ be a random variable with a probability density function denoted by f_X. The space Θ of functions which associate for each random event $\omega \in \Omega$ a value in $\mathbb{R}$ is an Hilbert space $L^2(\Omega, \beta, \Pr)$ provided with the inner scalar product $\langle X, Y \rangle = E[XY]$, $E[\bullet]$ being the expectation operator, where Ω is the sample space, β is the $\sigma-$algebra of the subsets of Ω and Pr is the probability measure.

Definition 1: Let $\{\xi(\omega)\}_{i=1}^{\infty}$ be an infinite countable set of normalized independent Gaussian random variables. The following spaces are then defined.

- $\hat{\Gamma}_p$ is the space of all polynomial functions in variables $\{\xi(\omega)\}_{i=1}^{\infty}$ of degree less than or equal to p.
- Γ_p is the set of polynomials in $\hat{\Gamma}_p$ that are orthogonal to those in the space $\hat{\Gamma}_{p-1}$
- $\tilde{\Gamma}_p$ is the space spanned by Γ_p such that $\hat{\Gamma}_p = \hat{\Gamma}_{p-1} \oplus \Gamma_p$ and $\Theta = \bigoplus_{i=0}^{\infty} \tilde{\Gamma}_i$

The p-order homogenous chaos is defined by the subspace Γ_p of Θ while Γ_p is called the p-order polynomial chaos (PC). The latter consists of all orthogonal polynomials of order p that are built from all possible combinations of the random variables in the set $\{\xi(\omega)\}_{i=1}^{\infty}$. The polynomial chaos theory asks the possibility to express any second order random variable $X(\omega) \in L^2(\Omega, \beta, \Pr)$ by a polynomial chaos representation as follows:

$$\begin{aligned} X(\omega) = {} & \bar{x}_0 \Gamma_0 + \sum_{i_1=1}^{\infty} \bar{x}_{i_1} \Gamma_1\left(\xi_{i_1}(\omega)\right) \\ & + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \bar{x}_{i_1 i_2} \Gamma_2\left(\xi_{i_1}(\omega), \xi_{i_2}(\omega)\right) \\ & + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} \bar{x}_{i_1 i_2 i_3} \Gamma_3\left(\xi_{i_1}(\omega), \xi_{i_2}(\omega), \xi_{i_3}(\omega)\right) + \cdots \end{aligned} \tag{1}$$

where $\Gamma_p(\cdot)$ denote polynomials of the p- order polynomial chaos Γ_p that are functions in variables defined within the set $\{\xi(\omega)\}_{i=1}^{\infty}$.

For practical use, terms in the previous polynomial chaos expansion are rearranged which helps to rewrite it in a more compact form as:

$$X(\xi(\omega)) = \sum_{j=0}^{\infty} \bar{x}_j \phi_j(\xi(\omega)) \tag{2}$$

where there exists a correspondence between functions $\phi_j(\cdot)$ and $\Gamma_p(\cdot)$, see (Ghanem and Spanos, 1991).

Theorem 1 (Cameron and Martin, 1947): Let $X \in L^2(\Omega, \beta, \Pr)$ a real random variable square integrable, $\{\xi(\omega)\}_{i=1}^{\infty}$ the infinite set of independent normalized Gaussian random variables and Γ_p is the p-order polynomial chaos. Then, the polynomial chaos representation of X is convergent in the least square sense as:

$$\lim_{P \to \infty} E\left[\left(\sum_{j=0}^{P} \bar{x}_j \phi_j(\xi(\omega)) - X(\omega) \right)^2 \right] = 0 \tag{3}$$

In theory, each polynomial chaos is a set of polynomial functions in the infinite set of random variables $\{\xi_i\}$. The polynomial chaos is then said to be with infinite dimension. However, as the number of uncertain parameters in a physical system is finite, we instead use the finite dimensional polynomial chaos. Otherwise, as a series expansion at infinity can't be used in practice, the sum (2) is truncated to a finite number of terms P which depends on the dimension d and the order p of the polynomial chaos as follows:

$$P + 1 = \frac{(p+d)!}{p!\,d!} \tag{4}$$

To summarize, the development in the polynomial chaos basis uses two types of approximations. The first is to consider only the polynomial chaos to some order p. The second is an approximation of the stochastic dimension consequence of the finite number of uncertainty sources in physical systems. So the dimension is truncated to the number d. Thus, the polynomial chaos expansion (2) is re-written:

$$X(\xi) \simeq \sum_{j=0}^{P} \overline{x}_j \phi_j(\xi) \tag{5}$$

with $\xi = (\xi_1 \quad \xi_2 \quad \cdots \quad \xi_d)$.

The set $\{\phi_i\}$ includes orthogonal polynomial functions with respect to the Gaussian measure of probability. They form a basis in $L^2(\Omega, \beta, \Pr)$ space provided with the scalar product $\langle \bullet \rangle$ defined as:

$$\langle \phi_i(\xi), \phi_j(\xi) \rangle = \langle \phi_i(\xi), \phi_i(\xi) \rangle \delta_{ij} \tag{6}$$

δ_{ij} being the Kronecker symbol given by:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{7}$$

and

$$\langle \phi_i(\xi), \phi_i(\xi) \rangle = \int_{\xi} \phi_i(\xi) \phi_i(\xi) f(\xi) d\xi \tag{8}$$

where $f(\xi)$ is the probability density function of ξ given by:

$$f(\xi) = \prod_{l=1}^{d} \frac{\exp(-\xi_l^2 / 2)}{\sqrt{2\pi}} \tag{9}$$

Ghanem and Spanos have proposed a method to construct the polynomial chaos basis. They have shown that with Gaussian random variables, the most suitable orthogonal functions are Hermite polynomials (Ghanem and Spanos, 1991).

The polynomial chaos expansion defines a separation of the stochastic character of the random function from its deterministic character. The latter is modeled by the so-called stochastic modes $\overline{x}_j$ defining coordinates of $X(\xi)$ in the orthogonal basis formed by polynomials $\phi_j(\xi)$ which model the random character of $X(\xi)$. Mode $\overline{x}_0$ represent the mean value of random function $X(\xi)$ while $\overline{x}_{j(j=1,\ldots,P)}$ define the dispersion of $X(\xi)$ around the mean value $\overline{x}_0$. So, the variance of $X(\xi)$ is approximated by:

$$\tau_X^2 \simeq \sum_{j=1}^{P} \overline{x}_j^2 \langle \phi_j, \phi_j \rangle \tag{10}$$

2.1. Generalized polynomial chaos

The representation in the PC basis, named also Wiener-Hermite expansion, helps to describe accurately random functions. Its convergence demonstrated by the Cameron-Martin theorem has been verified in numerous practical cases. However, it turned out that convergence properties (speed and accuracy) are unsuitable when random functions are with non-Gaussian distributions. So, Xiu and Karniadakis have shown that convergence properties depend on the probabilistic coordinates of random variables $\{\xi\}_{i=1}^{d}$ in the probability space (Xiu and Karniadakis, 2002). Indeed, they have carried out an optimal correspondence between families of probability laws and orthogonal polynomial families. So, for random variables ξ with certain distributions, the orthogonal functions ϕ_j can be chosen in such a way that its weight functions have the same form as the probability function $f(\xi)$. Table.1 summarizes the correspondence between polynomial families and density functions (Askey, 1985; Xiu and Karniadakis, 2003).

Table 1. Correspondence between probability density functions and orthogonal polynomials

Density of ξ	Polynomials ϕ_j	Support
Gaussian	Hermite	$(-\infty, \infty)$
Uniform	Legendre	$[a, b]$
Gamma	Laguerre	$[0, \infty)$
Beta	Jacobi	$[a, b]$

2.2. Computing of a PC or GPC expansion

The computing of a truncated PC or GPC expansion given by a (5)-like expression is turned into the problem of finding the stochastic modes $\overline{x}_j$, $j \in \{0, \ldots, P\}$. Non-intrusive spectral projection (NISP) or regression techniques can be used for this calculation (Babuska et al, 2004, 2007; Crestaux et al, 2009). The NISP technique used in this paper consists of performing Galerkin projections on $\{\phi_j\}_{j=0}^{P}$. So, the stochastic modes are given by:

$$\overline{x}_j = \frac{\langle X(\xi), \phi_j(\xi) \rangle}{\langle \phi_j(\xi)^2 \rangle} \tag{11}$$

where

$$\langle X(\xi), \phi_j(\xi) \rangle = \int_{\xi} X(\xi) \phi_j(\xi) f(\xi) d\xi \tag{12}$$

Integrals (12) can be approximated by using multidimensional numerical integration methods such as the multidimensional Gauss collocation method that is:

$$\langle X(\xi), \phi_j(\xi) \rangle \approx \sum_{i=1}^{Q} X(\xi^{(i)}) \phi_j(\xi^{(i)}) w^{(i)} \tag{13}$$

where $\xi^{(i)}, w^{(i)}$ are the integration points and weights, respectively, while Q is the number of integration points.

3. Model reduction of random parameter-dependent LTI systems

Consider a state-space model of a linear time invariant system whose matrices are random:

$$\begin{cases} \dot{x}(t,\xi) = A(\xi)x(t,\xi) + B(\xi)u(t) \\ y(t,\xi) = C(\xi)x(t,\xi) + D(\xi)u(t) \end{cases} \quad (14)$$

where:
$A(\xi) \in \mathbb{R}^{n\times n}, B(\xi) \in \mathbb{R}^{n\times m}, C(\xi) \in \mathbb{R}^{q\times n}, D(\xi) \in \mathbb{R}^{q\times m}$ are random parameter-dependent matrices, $u(t) \in \mathbb{R}^m, y(t,\xi) \in \mathbb{R}^q$ and $x(t,\xi) \in \mathbb{R}^n$ are, respectively, the inputs, outputs and states of the system, while $\xi \in \mathbb{R}^d$ is a vector of random variables. $x(t,\xi)$ and $y(t,\xi)$ define processes that depend on the time t and the random variable ξ. Once a fixed value is given for ξ then the state trajectory and the system output become deterministic. Otherwise, it must be noted that initial conditions don't depend on ξ that is: $x(0,\xi) = x_0$. So the set of state trajectories $\{x(t,\xi)\}_{t\geq 0}$ depends on the initial condition x_0, the random variable ξ and also the input $u(t)$

The aim of model order reduction of (14) is to search for a random parameter-dependent model defined by:

$$\begin{cases} \dot{z}^r(t,\xi) = A_r(\xi)z^r(t,\xi) + B_r(\xi)u(t) \\ y_r(t,\xi) = C_r(\xi)z^r(t,\xi) + D_r(\xi)u(t) \end{cases} \quad (15)$$

where $z^r(t,\xi) \in \mathbb{R}^r$ with $r << n$, $y_r(t,\xi) \in \mathbb{R}^q$, $A_r(\xi) \in \mathbb{R}^{r\times r}, B_r(\xi) \in \mathbb{R}^{r\times m}, C_r(\xi) \in \mathbb{R}^{q\times r}$ and $A_r(\xi) \in \mathbb{R}^{q\times m}$ such that :

$$E^h\left[y(t,\xi)\right] \approx E^h\left[y_r(t,\xi)\right] \quad (16)$$

$E^h[.]$ denoting the h^{th} order moment . It represents the mean value when h = 1 and the variance when h = 2.

For System (14) with fixed-value parameters $\left(\xi = \xi^{(k)}\right)$, the quality of the corresponding reduced model (15) is assessed by verifying that its dynamic behaviour reproduces accurately that of the original model i.e. $y_r\left(t,\xi^{(k)}\right) \approx y\left(t,\xi^{(k)}\right)$. The main goal in model reduction of System (14) with random parameters is to generate a reduced model (15) such that $y_r\left(t,\xi^{(k)}\right) \approx y\left(t,\xi^{(k)}\right)$ be fulfilled for all fixed value parameters $\left(\xi = \xi^{(k)}, k = 1,...,\infty\right)$ belonging to the probabilistic support of variable ξ. This leads to the definition of the condition (16) which means that statistical moments of $y_r(t,\xi)$ must well approximate those of the original output $y(t,\xi)$. The first and second order moments and density functions are the most common used to characterize statistically random process. So, it will be said that model (15) is a good approximation of model (14) almost surely if the mean value and the variance of its output well approximate those of the original model (14).

3.1. Truncated balanced realization of LTI systems with deterministic parameters

Consider systems (14) with fixed values for parameters in ξ and denote by $G(s)$ its transfer function.

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \quad (17)$$

Definition 2 (Controllability and observability energy functions): Let $x(0) = x_0$ be the state at time $t = 0$. Under asymptotic stability (A is Hurwitz matrix i.e all real parts of eigenvalues of matrix A are in the left half-plane), controllability and observability conditions, Scherpen has defined Controllability and observability energy functions at x_0 respectively by:

$$L_c(x_0) = \min_{\substack{u\in L^2(-\infty,0) \\ x(-\infty)=0, x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} \|u(t)\|^2 \, dt \quad (18)$$

$$L_0(x_0) = \frac{1}{2}\int_0^{\infty} \|y(t)\|^2 \, dt \; ; u(t) = 0 \; \forall t \geq 0 \quad (19)$$

as quadratic functions given as:

$$L_c(x_0) = \frac{1}{2}x_0^T W_c^{-1} x_0 \quad (20)$$

$$L_o(x_0) = \frac{1}{2}x_0^T W_o x_0 \quad (21)$$

where W_c, W_o are the so-called controllability and observability gramians defined respectively by

$$W_c = \int_0^{\infty} e^{At} BB^T e^{A^T t} dt \quad (22)$$

$$W_o = \int_0^{\infty} e^{At} C^T C e^{A^T t} dt \quad (23)$$

that are the unique symmetric positive definite solutions of the following Lyapunov equations

$$AW_c + W_c A^T = -BB^T \quad (24)$$

$$A^T W_o + W_o A = -C^T C \quad (25)$$

and the square roots of the eigenvalues of the product $W_c W_o$ are the so-called Hankel singular values $\sigma_i \, (i = 1,...,n)$ of System (17).

The controllability function $L_c(x_0)$ is the minimum amount of input energy required to drive the state from the origin at $t = -\infty$ to x_0 at $t = 0$, while the observability function $L_0(x_0)$ is the amount of output energy generated by the zero-input response from x_0.

Definition 3 (Balanced realization) (Moore, 1981): An asymptotically stable minimal realization of System (14) is said to be in a balanced form if:

$$W_c = W_o = \Sigma = \mathrm{diag}(\sigma_1,...,\sigma_n) \quad (26)$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$ and $\sum$ is the solution of the pair of Lyapunov equations for controllability and observability. There exists a linear coordinate transformation $x_b(t) = T_b^{-1} x(t)$, named balancing transformation, which puts System (14) and thus the corresponding controllability and observability energy functions in a balanced form. In this configuration, state variables $x_{b,i}, (i = 1,...,n)$ are ordered with respect to their controllability and observability degrees measured by Hankel singular values $\sigma_i, i = 1,...,n$. State variables with small Hankel singular values are said to be weakly controllable and observable. So, a reduced model is obtained by truncating states having small Hankel singular values. This result is given by the following theorem.

Theorem 2 (Pernebo and Silverman, 1982): Let $G_b(s)$ be the transfer function of the balanced realization of System (14) given by:

$$\begin{cases} \dot{x}_b(t) = A_b x_b(t) + B_b u(t) \\ y(t) = C_b x_b(t) + Du(t) \end{cases} \tag{27}$$

where $x_b(t) = T_b^{-1} x(t)$, T_b being the balancing transformation, and consequently $A_b = T_b^{-1} A T_b$, $B_b = T_b^{-1} B$, $C_b = C T_b$. So if there exists $r < n$ such that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \gg \sigma_{r+1} \geq \ldots \geq \sigma_n$ then the reduced order model

$$\begin{cases} \dot{x}_r(t) = A_r x_r(t) + B_r u(t) \\ y_r(t) = C_r x_r(t) + Du(t) \end{cases}, \tag{28}$$

obtained by the truncation of state variables $x_{r+1,b}(t),...,x_{n,b}(t)$ by setting them to zero, is minimal and asymptotically stable and satisfies

$$\left\| G(s) - G_r(s) \right\|_\infty \leq 2 \sum_{i=r+1}^{n} \sigma_i \tag{29}$$

where $G_r(s)$ is the transfer function of the reduced model (28).

The main step to derive the balanced truncated model (28) is to search for the balancing transformation T_b. The latter is given by the well-known Laub algorithm (Laub et al, 1987).

3.2. Random parameter dependent balanced realization

We propose in the sequel a generalization of the balanced realization concept described in the above subsection to random parameter-dependent LTI systems. So, some definitions, related to the proposed generalization, are provided.

Let $x_s(t,\xi)$ be the state trajectory of System (14) corresponding to a given initial condition $x_0 \neq 0$ in the neighborhood of the zero-equilibrium and the zero-input $u(t) = 0$.

Definition 4: The zero equilibrium of System (14) is said to be almost surely stochastically stable if (Ficher and Bhattacharya, 2009)

$$\Pr\left[\lim_{t \to \infty} x_s(t,\xi) \to 0 \right] = 1 \tag{30}$$

Definition 5: The zero equilibrium of System (14) is said to be almost surely stable with respect to the h^{th} moment if $\forall \varepsilon > 0, \exists \delta > 0$ such that:

$$\sup_{t \geq 0} E^h \left[x_s(t,\xi) \right] \leq \varepsilon, \ \forall x_0 : |x_0| \leq \delta \tag{31}$$

Definition 6: The zero equilibrium of System (14) is said to be almost surely asymptotically stable with respect to the h^{th} moment if it is stable in the h^{th} moment and

$$\lim_{t \to \infty} E^h \left[x_s(t,\xi) \right] = 0, \tag{32}$$

Definitions 5 and 6 state that stability of System (14) can be analyzed via the study of the stability of the moments of the random process defined by equations in (14). The almost sure stability has been shown to be equivalent to the stability in the h^{th} moment for linear autonomous systems (Chen and Hsu,1995). Fisher and Bhatacharya have used the GPC formalism to study stability of System (14) and nonlinear systems (Fisher and Bhatacharya, 2010). The proposed method has been used efficiently to analyze the stability of a mechanical system with a random friction coefficient (Nechak et al, 2011, 2013).

As the stability, almost sure controllability and observability notions are defined as follow:

Definition 7: The couple $(A(\xi), B(\xi))$ is said to be almost surely controllable if it is controllable for each fixed value of the random variable ξ, that is the controllability matrix $C_o(\xi)$ is full rank for each fixed value of ξ so,

$$\Pr\left(rank\left(C_o(\xi) \right) = n \right) = 1 \tag{33}$$

Definition 8: The couple $(A(\xi), C(\xi))$ is said to be almost surely observable if it is observable for each fixed value of the random variable ξ, that is the observability matrix $O_b(\xi)$ is full rank for each fixed value of ξ so,

$$\Pr\left(rank\left(O_b(\xi) \right) = n \right) = 1 \tag{34}$$

Suppose that System (14) is almost surely asymptotically stable and almost surely controllable and observable. Random parameters-dependent (RPD) controllability and observability gramians, denoted by $W_c(\xi)$ and $W_o(\xi)$ respectively, are then defined as follows:

$$W_c(\xi) = \int_0^\infty e^{A(\xi)t} B(\xi) B(\xi)^T e^{A(\xi)^T t} dt \tag{35}$$

$$W_o(\xi) = \int_0^\infty e^{A(\xi)t} C(\xi)^T C(\xi) e^{A(\xi)^T t} dt \tag{36}$$

RPD-Controllability and observability gramians $(W_c(\xi)$ and $W_o(\xi))$ are positive definite matrices almost surely and solutions of RPD-linear Lyapunov equations

associated to the almost sure controllability and observability properties. They are given by:

$$A(\xi)W_c(\xi)+W_c(\xi)A(\xi)^T=-B(\xi)B(\xi)^T \quad (37)$$

$$A(\xi)^T W_o(\xi)+W_o(\xi)A(\xi)=-C(\xi)^T C(\xi) \quad (38)$$

In the following, the balanced realization notion defined for LTI systems with constant parameters is generalized to the-dependent LTI systems.

Let

$$\begin{cases}\dot{x}_b(t,\xi)=A_b(\xi)x_b(t,\xi)+B_b(\xi)u(t)\\ y(t,\xi)=C_b(\xi)x_b(t,\xi)+D_b(\xi)u(t)\end{cases} \quad (39)$$

be an RPD-state space representation of (14).

Definition 9: Realization (39) is said to be an almost surely balanced realization of System (14) if it is a balanced realization of System (14) for every fixed value of the random parameter ξ, that is:

$$\Pr\left(W_c(\xi)=W_o(\xi)=\Sigma(\xi)\right)=1 \quad (40)$$

where,
$$\Sigma(\xi)=\begin{pmatrix}\sigma_1(\xi) & 0 & 0\\ 0 & \ddots & 0\\ \vdots & & \\ 0 & 0 & \sigma_n(\xi)\end{pmatrix}$$

is the solution of the pair of RPD-Lyapunov equations given by (21).

$$\begin{cases}A_b(\xi)\Sigma(\xi)+\Sigma(\xi)A_b(\xi)^T=-B_b(\xi)B_b(\xi)^T\\ A_b(\xi)\Sigma(\xi)+\Sigma(\xi)A_b(\xi)=-C_b(\xi)^T C_b(\xi)\end{cases} \quad (41)$$

The RPD-Hankel singular values $\sigma_i(\xi)$, $i\in\{1,...,n\}$ are in this case random functions in the random variable ξ and thus can be expressed by GPC expansions. So:

$$\sigma_i(\xi)\approx\sum_{j=0}^{P}\bar{\sigma}_{i,j}\phi_j(\xi) \quad (42)$$

Thus:

$$\Sigma(\xi)\approx\begin{pmatrix}\bar{\sigma}_{1,0} & 0 & 0\\ 0 & \ddots & 0\\ \vdots & & \\ 0 & 0 & \bar{\sigma}_{n,0}\end{pmatrix}+\begin{pmatrix}\sum_{j=1}^{P}\bar{\sigma}_{1,j}\phi_j(\xi) & 0 & 0\\ 0 & \ddots & 0\\ \vdots & & \\ 0 & 0 & \sum_{j=1}^{P}\bar{\sigma}_{n,j}\phi_j(\xi)\end{pmatrix} \quad (43)$$

where $\bar{\sigma}_{i,0}$ denotes mean values of the Hankel singular values while $\bar{\sigma}_{i,j}$ define the dispersion of the Hankel singular values around their mean values. Otherwise, the number P of terms in a the GPC expansion (42) depends, as said in Section 2, on the number d of random parameters and the GPC order p. The number d being known, the number of terms P depends only on the chaos order p. There is no method which helps to control, a priori, the error related to a fixed value of the chaos order p. In practice, p is fixed by a convergence study i.e. the chaos order is increased until no enhance in accuracy of the GPC expansion is obtained. The main problem is then to search for a random parameter-dependent balanced transformation $T_b(\xi)$ such that:

$$x_b(t,\xi)=T_b^{-1}(\xi)x(t,\xi), \quad (44)$$

which puts System (14) in an almost surely balanced form (39). The GPC is proposed to represent and thus to compute the RPD-balanced transformation. So, entries $t_{ij}(\xi)$ of $T_b(\xi)$ are expressed represented by :

$$t_{ij}(\xi)\approx\sum_{k=0}^{P}\bar{t}_{ij,k}\phi_k(\xi) \quad (45)$$

with

$$\bar{t}_{ij,k}=\frac{\left\langle t_{ij}(\xi),\phi_k(\xi)\right\rangle}{\left\langle\phi_k(\xi)^2\right\rangle},\ k\in\{0,...,P\},\{i,j\}\subset\{1,...,n\} \quad (46)$$

Stochastic modes $\bar{t}_{ij,k}$ given by (46) are computed by using the NISP method as described in Section 2. A (13)-like formula is used. For this aim, the balancing transformation is needed to be computed at Gauss collocation points, using the Moore-Laub algorithm.

Once the RPD- balanced transformation computed, the random parameter-matrices of System (39) can be obtained as:

$$\begin{cases}A_b(\xi)=T_b^{-1}(\xi)A(\xi)T_b(\xi)\\ B_b(\xi)=T_b^{-1}(\xi)B(\xi)\\ C_b(\xi)=C(\xi)T_b(\xi)\\ D_b(\xi)=D(\xi)\end{cases} \quad (47)$$

3.3. Random parameter-dependent truncated balanced realization (RPD-TBR)

The random parameter-dependent balanced realization (RPD-BR) defined by equations (39) can be exploited to derive a RPD-TBR. The essential idea is that in a RPD-BR, state variables are ordered with respect to their controllability and observability degrees measured by the RPD-Hankel singular values. State variables that possess small Hankel singular values almost surely within the probabilistic support of system parameters are weakly controllable and observable almost surely. Thus, suppressing them will not affect the dynamic behaviour of the original

model. The main question is then how to measure RPD-Hankel singular value. This task is accomplished by using the GPC representation (42) of RPD-Hankel singular values. Using (42), it is easier to estimate the whole distribution of RPD-Hankel singular values comparing to a Monte Carlo (MC) method which is known to be too costly. So, by exploiting (42), it is possible to know how controllability and observability degrees of state variables evolve within the probabilistic support of system parameters. The main idea is just to sample the probabilistic distribution of the random variable ξ then to compute the corresponding sum after replacing each value of the sampling data in the polynomial functions used. Based on this calculation, RPD-Hankel singular values can be characterized and then used to analyze the reducibility of the original model (14). We prose for this aim the generalization of the criterion defined for deterministic LTI systems in Pernebo-Silverman theorem. So if there exists a truncation order *r* for which the following almost sure (*a.s*) inequality,

$$\sigma_1(\xi) \overset{a.s}{\geq} \sigma_2(\xi) \overset{a.s}{\geq} \cdots \overset{a.s}{\geq} \sigma_r(\xi) \overset{a.s}{>>} \sigma_{r+1}(\xi) \overset{a.s}{\geq} \cdots \overset{a.s}{\geq} \sigma_n(\xi) \quad (48)$$

is fulfilled almost surely, then a RPD-TBR can be obtained by removing the *n-r* states variables that have small Hankel singular values almost surely.

Consider the balanced realization (19) rewritten and given by

$$\begin{cases} \begin{bmatrix} \dot{x}_1^b(t,\xi) \\ \dot{x}_2^b(t,\xi) \end{bmatrix} = \begin{bmatrix} A_{11}^b(\xi) & A_{12}^b(\xi) \\ A_{21}^b(\xi) & A_{22}^b(\xi) \end{bmatrix} \begin{bmatrix} x_1^b(t,\xi) \\ x_2^b(t,\xi) \end{bmatrix} \\ \qquad\qquad + \begin{bmatrix} B_1^b(\xi) \\ B_2^b(\xi) \end{bmatrix} u(t) \\ y_b(t,\xi) = \begin{bmatrix} C_1^b(\xi) & C_2^b(\xi) \end{bmatrix} \begin{bmatrix} x_1^b(t,\xi) \\ x_2^b(t,\xi) \end{bmatrix} \\ \qquad\qquad + \begin{bmatrix} D_1^b(\xi) \\ D_2^b(\xi) \end{bmatrix} u(t) \end{cases} \quad (49)$$

where $x_1^b = \begin{bmatrix} x_{b,1} & \cdots & x_{b,r} \end{bmatrix}^T$, $x_2^b = \begin{bmatrix} x_{b,r+1} & \cdots & x_{b,n} \end{bmatrix}^T$

The RPD-TBR (15) can be obtained through setting variables that are weakly controllable and observable almost surely (variables corresponding to small RPD- Hankel singular values) to zero;

$$x_{b,r+1}(t,\xi) = x_{b,r+2}(t,\xi) = \cdots = x_{b,n}(t,\xi) = 0 \quad (50)$$

which yields

$$\begin{cases} A_r(\xi) = A_{11}^b(\xi) \\ B_r(\xi) = B_1^b(\xi) \\ C_r(\xi) = C_1^b(\xi) \\ D_r(\xi) = D_1^b(\xi) \end{cases} \quad (51)$$

The RPD-TBR can be obtained also by setting the dynamic of state variables whose RPD-Hankel singular values are small to zero. This allows forcing the static gain of the truncated balanced realization to be the same as the original system.

$$\dot{x}_{b,r+1}(t,\xi) = \dot{x}_{b,r+2}(t,\xi) = \cdots = \dot{x}_{b,n}(t,\xi) = 0 \quad (52)$$

In this case, matrices of the RPD-TBR are given as follows:

$$\begin{cases} A_r(\xi) = A_{11}^b(\xi) - A_{12}^b(\xi)\left[A_{22}^b(\xi)\right]^{-1} A_{21}^b(\xi) \\ B_r(\xi) = B_1^b(\xi) - A_{12}^b(\xi)\left[A_{22}^b(\xi)\right]^{-1} B_2^b(\xi) \\ C_r(\xi) = C_1^b(\xi) - C_2^b(\xi)\left[A_{22}^b(\xi)\right]^{-1} A_{21}^b(\xi) \\ D_r(\xi) = D_1^b(\xi) - C_2^b(\xi)\left[A_{22}^b(\xi)\right]^{-1} B_2^b(\xi) \end{cases} \quad (53)$$

3.4. Truncation error

Ggg An obvious concern with model order reduction is the evaluation of a bound on the error reduction. For the TBR of LTI systems, Pernebo and Silverman have shown that the H_∞ error between an LTI model and its TBR is bounded with the sum of Hankel singular values corresponding to the truncated state variables. We propose in the following the generalization of this result to RPD-TBR. As the systems dealt with are random, the truncation error will be characterized statistically.

Proposition 1: Let $G(s,\xi)$ and $G_r(s,\xi)$ be the random parameter-dependent transfer functions of the original model (14) and its reduced model (15). Criterion (48) is supposed to be verified then the mean value of the H_∞ error between $G(s,\xi)$ and $G_r(s,\xi)$ is bounded by the sum of mean values of Hankel singular values of the truncated state variables, that is:

$$E\left[\left\|G(s,\xi) - G_r(s,\xi)\right\|_\infty\right] \leq 2\sum_{i=r+1}^{n} \bar{\sigma}_{i,0} \quad (54)$$

Proof:

From the LTI theory (Pernebo ans Silverman, 1982), for fixed values of ξ, it has been shown that :

$$\left\|G\left(s,\xi^{(k)}\right) - G_r\left(s,\xi^{(k)}\right)\right\|_\infty \leq 2\sum_{i=r+1}^{n} \sigma_i\left(\xi^{(k)}\right) \quad (55)$$

By applying the expectation operator on the above expression, it can be written:

$$\begin{aligned} \int_\xi \left\|G(s,\xi) - G_r(s,\xi)\right\|_\infty f(\xi)\,d\xi &\leq 2\int_\xi \left(\sum_{i=r+1}^{n} \sigma_i(\xi)\right) f(\xi)\,d\xi \\ &\leq 2\sum_{i=r+1}^{n} \int_\xi \sigma_i(\xi) f(\xi)\,d\xi \\ &\leq 2\sum_{i=r+1}^{n} E\left[\sigma_i(\xi)\right] \end{aligned}$$

Hankel singular values being approximated by the GPC, we have:

$$E\left[\sigma_i(\xi)\right] \approx \bar{\sigma}_{i,0}$$

Thus

$$E\left[\left\|G(s,\xi)-G_r(s,\xi)\right\|_\infty\right] \le 2\sum_{i=r+1}^{n}\bar{\sigma}_{i,0}$$

which completes the proof.

Proposition 2 : Let $G(s,\xi)$ and $G_r(s,\xi)$ be the random parameter-dependent transfer functions of the original model (14) and its reduced model (15). Criterion (48) is supposed to be verified then the variance of the H_∞ error between $G(s,\xi)$ and $G_r(s,\xi)$ is bounded as follows:

$$\mathrm{var}\left[\left\|G(s,\xi)-G_r(s,\xi)\right\|_\infty\right] \le 4\left(\sum_{i=r+1}^{n}\bar{\sigma}_{i,0}^2 + 2\sum_{\substack{i=r+1\\ i<j\le n}}^{n}\bar{\sigma}_{i,0}\bar{\sigma}_{j,0}\right) + 4\left(\sum_{i=r+1}^{n}\mathrm{var}\left(\sigma_i(\xi)\right) + 2\sum_{\substack{i=r+1\\ i<j\le n}}^{n}\mathrm{cov}\left(\sigma_i(\xi),\sigma_j(\xi)\right)\right) \quad (56)$$

Proof:

$$\mathrm{var}\left[\left\|G(s,\xi)-G_r(s,\xi)\right\|_\infty\right] \le E\left[\left\|G(s,\xi)-G_r(s,\xi)\right\|_\infty^2\right] - E\left[\left\|G(s,\xi)-G_r(s,\xi)\right\|_\infty\right]^2 \quad (57)$$

Applying the expectation operator to the squared expression (55) yields the following result:

$$E\left[\left\|G(s,\xi)-G_r(s,\xi)\right\|_\infty^2\right] \le 4\sum_{i=r+1}^{n}E\left[\sigma_i(\xi)^2\right] + 8\sum_{\substack{i=r+1\\ i<j\le n}}^{n}E\left[\sigma_i(\xi)\sigma_j(\xi)\right] \quad (58)$$

Consequently:

$$\mathrm{var}\left[\left\|G(s,\xi)-G_r(s,\xi)\right\|_\infty^2\right] \le 4\sum_{i=r+1}^{n}E\left[\sigma_i(\xi)^2\right] + 8\sum_{\substack{i=r+1\\ i<j\le n}}^{n}E\left[\sigma_i(\xi)\sigma_j(\xi)\right] \quad (59)$$

By using the GPC expansions of Hankel singular values, it can be verified that:

$$E\left[\sigma_i(\xi)^2\right] \approx \sum_{i=0}^{P}\bar{\sigma}_{i,k}^2\left\langle\phi_k^2(\xi)\right\rangle = \bar{\sigma}_{i,0}^2 + \mathrm{var}\left(\sigma_i(\xi)\right) \quad (60)$$

And

$$E\left[\sigma_i(\xi)\sigma_j(\xi)\right] \approx \sum_{i=0}^{P}\bar{\sigma}_{i,k}\bar{\sigma}_{j,k}\left\langle\phi_k^2(\xi)\right\rangle \approx \bar{\sigma}_{i,0}\bar{\sigma}_{j,0} + \mathrm{cov}\left(\sigma_i(\xi),\sigma_j(\xi)\right) \quad (61)$$

where $\mathrm{cov}(.)$ is the covariance operator.

Replacing $E\left[\sigma_i(\xi)^2\right]$ and $E\left[\sigma_i(\xi)\sigma_j(\xi)\right]$ in (59) by (60) and (61) yields (56) which completes the proof.

Model reduction of parameter dependent models is of major importance in practice. Indeed, system parameters in several practical cases are uncertain. Parameters often characterize geometry, materials, boundary values, initial values and control parameters. In most cases, uncertainty is inherent to those parameters which are modeled in several cases by probabilistic models (density functions). The need for RPD-reduced model can results from different requirements. In some applications, many simulations have to be performed for different values of parameters such as parameter studies for parameters optimization, inverse problem and design in general. The MC method is a typical example in this context which is the most used method in industry to characterize the dynamic behaviour of a given system with uncertain parameters. The principle is to simulate the differential equations for a given parameter set obtained following random generators and then to execute some numerical solver for each combination of parameters in the set. This procedure is known to be prohibitive due to both convergence properties of the MC method and the complexity of physical systems. So, it is, in these cases, interesting to have the possibility to generate a reduced model with random parameters closed to those of the original system in order to have a more efficient and less expansive MC method. We propose in this perspective, to apply the RPD-BTR developed in this paper.

4. Application

In order to assess the efficiency of the proposed method, a model reduction problem of a mechanical system (Fig.1) is considered. The latter is a two degrees of freedom system which consists of two cells (mass, stiffness, damping). The stiffness k_1 and damping d_1 coefficients are supposed to be random parameters driven by uniform laws within [0.9, 1.1] and [0.5, 0.8] respectively. Two standard independent random uniform variables ξ_1, ξ_2 within interval [-1, 1] are considered to represent k_1 and d_1. See Table.2 for numerical values of all system parameters.

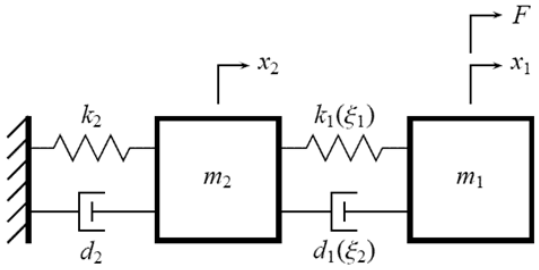

Fig. 1. Mechanical system

Matrices of (14)-like state space representation of the system in Fig.1 are given in (62) by considering the following state vector Where $\dot{x}(t,\xi) = \left[x_1 \quad x_2 \quad \dot{x}_1 \quad \dot{x}_2\right](t,\xi)$ with $\xi = \left[\xi_1 \quad \xi_2\right]^T$

$$A(\xi)=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1(\xi_1)}{m_1} & \dfrac{k_1(\xi_1)}{m_1} & -\dfrac{d_1(\xi_2)}{m_1} & \dfrac{d_1}{m_1} \\ \dfrac{k_1(\xi_1)}{m_2} & -\dfrac{k_1(\xi_1)+k_2}{m_2} & \dfrac{d_1(\xi_2)}{m_2} & -\dfrac{d_1(\xi_2)+d_2}{m_2} \end{bmatrix},$$

$$B(\xi)=B=\begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{m_1} \\ 0 \end{bmatrix}, C(\xi)=C=\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \tag{62}$$

Table 2. Parameters for system in Fig.1

$m_1=1$	$m_2=1$
$k_1(\xi_1)=1+0.1\xi_1$	$k_2=1$
$d_1(\xi_2)=0.65+0.15\xi_2$	$d_2=1$

4.1. Random parameter dependent balanced realization

From Askey scheme, Legendre polynomials $L_k(\xi)$ are the most suitable polynomial to represent the balancing transformation $T_b(\xi)$ of Model (62) as well as the parameter-dependent Hankel singular values. So:

$$t_{ij}(\xi)\approx\sum_{k=0}^{P}\bar{t}_{ij,k}L_k(\xi) \tag{63}$$

$$\sigma_i(\xi)\approx\sum_{k=0}^{P}\bar{\sigma}_{i,k}L_k(\xi) \tag{64}$$

where stochastic modes $\bar{t}_{ij}$ and $\bar{\sigma}_{i,k}$ are defined by (13)-like formula which can be expressed as follows:

$$\bar{t}_{ij,k}=\frac{\int_{-1}^{1}\int_{-1}^{1}t_{ij}(\xi_1,\xi_2)L_k(\xi_1,\xi_2)f_1(\xi_1)f_2(\xi_2)d\xi_1d\xi_2}{\int_{-1}^{1}\int_{-1}^{1}L^2{}_k(\xi_1,\xi_2)f_1(\xi_1)f_2(\xi_2)d\xi_1d\xi_2} \tag{65}$$

$$\bar{\sigma}_{i,k}=\frac{\int_{-1}^{1}\int_{-1}^{1}\sigma_i(\xi_1,\xi_2)L_k(\xi_1,\xi_2)f_1(\xi_1)f_2(\xi_2)d\xi_1d\xi_2}{\int_{-1}^{1}\int_{-1}^{1}L^2{}_k(\xi_1,\xi_2)f_1(\xi_1)f_2(\xi_2)d\xi_1d\xi_2} \tag{66}$$

The multi-dimensional Gauss collocation method is used to calculate stochastic modes (65) and (66).

The Balanced transformation is computed for several values of the Legendre polynomial chaos (LePC) order $p\in\{2,3,4\}$ in order to observe the effect of the LePC order on the obtained balanced realization. For each value of p a RPD-BR (39) is computed. The first and second order moments of their Step responses are plotted in Fig.2 and Fig.3. Moreover, the probability density function of the final value of the step response of System (62) is plotted in Fig.4. All results are compared to references obtained from the Monte Carlo method with a number N=10.000 of samples of parameters k_1 and d_1.

The balanced realization obtained for System (62) must model a dynamic behavior having theoretically the same statistical properties as System (62). This is verified as shown in Fig.2, Fig.3 and Fig.4. The RPD-BR presents the same instantaneous mean value and variance of the step response. The probability density function of the final value is suitably modeled. It can be seen also that the accuracy is good with $p = 2$ as well as with $p = 3$ and $p = 4$. This shows the good convergence property of the LePC expansion used to represent the balancing transformation.

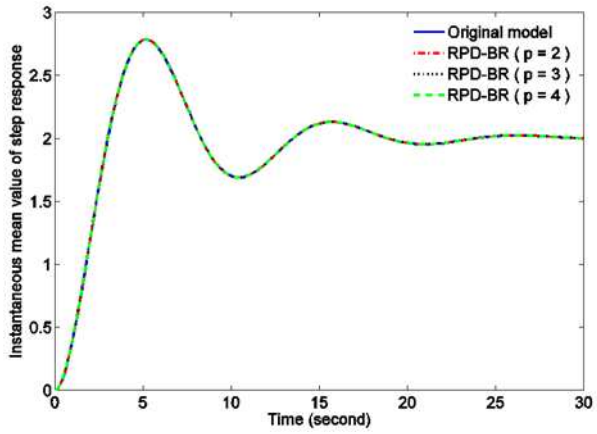

Fig. 2. Instantaneous mean value of the step response

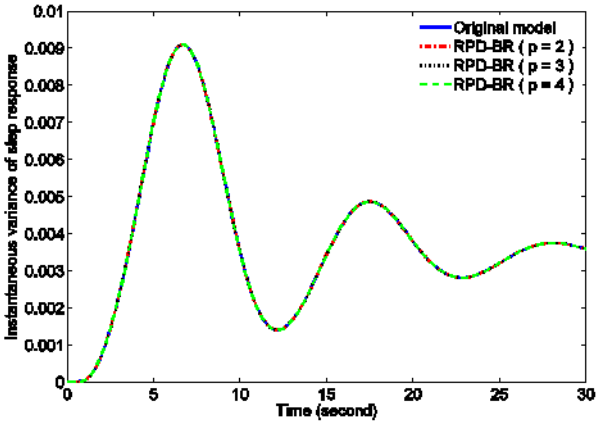

Fig. 3. Instantaneous variance of the step response

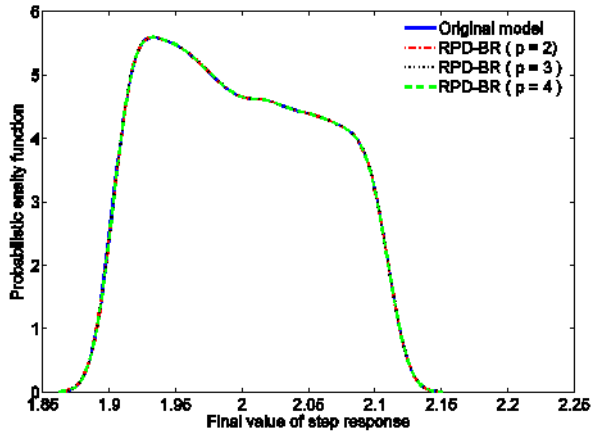

Fig. 4. The probability density function of the final value of the step response

4.2. Random parameter dependent truncated balanced realization

One of the main interests of a balanced realization is to exploit it to derive a reduced order model by truncating states that are hard to control and to observe. Hankel singular values measure controllability and observability degrees. In the case dealt with in this section, these degrees are random functions depending on the stiffness k_1 and damping d_1 parameters which are modeled by LePC expansions (64). Their probabilistic density functions are represented in Fig.5 for $p \in \{2,3,4\}$ and compared to reference results obtained by using the MC method with a number N = 10.000 of samples.

LePC expansions used allow good representations of random Hankel singular values. This is pointed out by the correct probabilistic density functions represented in Fig.5. From latter results, two groups of singular values functions can be identified. The first one includes $\sigma_1(\xi)$ and $\sigma_2(\xi)$ while the second includes $\sigma_3(\xi)$ and $\sigma_4(\xi)$. Probability density functions of $\sigma_3(\xi)$ and $\sigma_4(\xi)$ show that the corresponding states are almost surely weakly observable and weakly controllable comparing to the first and second states variables which, from the corresponding probability density functions, are strongly observable and controllable. Mean values, standard deviations, minimum and maximum of Hakel singular values estimated with LePC expansions are given in Tables (3,4,5) while Table.6 presents MC estimations. From all results, it can be said that: $\sigma_1(\xi) > \sigma_2(\xi) >> \sigma_3(\xi) > \sigma_4(\xi)$ almost surely for $\xi \in [-1,1]\times[-1,1]$ or equivalently for $k_1 \in [0.9,\ 1.1]$ and $d_1 \in [0.5, 0.8]$. So, truncating the last two states variables in the balanced realization will not affect the dynamic behavior of the original system (62). To verify the accuracy of the reduced order model, first and second order moments of its step response and the probability density function of the corresponding final value are computed and compared to those of the original model (62).

Table 3. Statistical properties of Hankel singular values estimated by LePC expansion with p = 2

LePC (p = 2)	$\sigma_1(\xi)$	$\sigma_2(\xi)$	$\sigma_3(\xi)$	$\sigma_4(\xi)$
Mean value	2.3043	1.3010	0.0204	0.0188
Variance	0.0885	0.0656	0.0037	0.0012
Minimum	2.1239	1.1698	0.0142	0.0173
Maximum	2.5441	1.4845	0.0294	0.0228

Table 3. Statistical properties of Hankel singular values estimated by LePC expansion with p = 3

LePC (p = 4)	$\sigma_1(\xi)$	$\sigma_2(\xi)$	$\sigma_3(\xi)$	$\sigma_4(\xi)$
Mean value	2.3043	1.3010	0.0204	0.0188
Variance	0.0882	0.0652	0.0036	0.0012
Minimum	2.1213	1.1674	0.0141	0.0173
Maximum	2.5471	1.4873	0.0295	0.0228

Table 4. Statistical properties of Hankel singular values estimated by LePC expansion with p = 4

LePC (p = 2)	$\sigma_1(\xi)$	$\sigma_2(\xi)$	$\sigma_3(\xi)$	$\sigma_4(\xi)$
Mean value	2.3043	1.3010	0.0204	0.0188
Variance	0.0885	0.0656	0.0037	0.0012
Minimum	2.1239	1.1698	0.01242	0.0173
Maximum	2.5441	1.4845	0.0294	0.0228

Table 5. Statistical properties of Hankel singular values estimated by Monte Carlo (MC) method with N = 10.000 samples

MC	$\sigma_1(\xi)$	$\sigma_2(\xi)$	$\sigma_3(\xi)$	$\sigma_4(\xi)$
Mean value	2.3043	1.3010	0.0204	0.0188
Variance	0.0882	0.0652	0.0036	0.0012
Minimum	2.1213	1.1674	0.0141	0.0173
Maximum	2.5471	1.4873	0.0295	0.0228

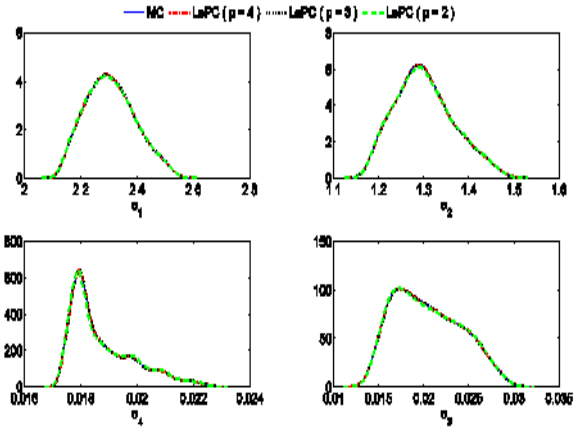

Fig. 5. Probability density functions of Hankel singular values

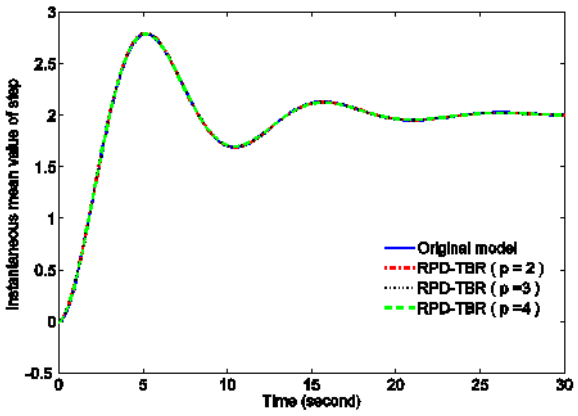

Fig. 6. Instantaneous mean value of the step response

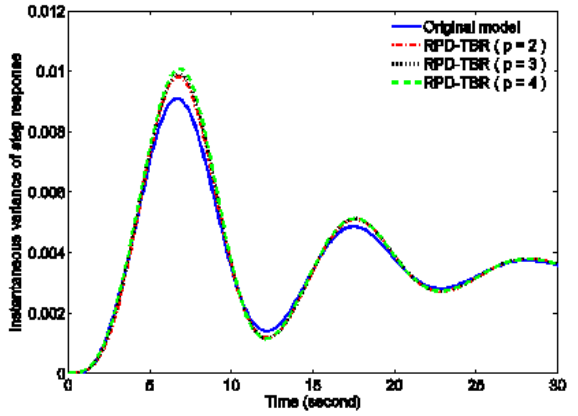

Fig. 7. Instantaneous variance of the step response

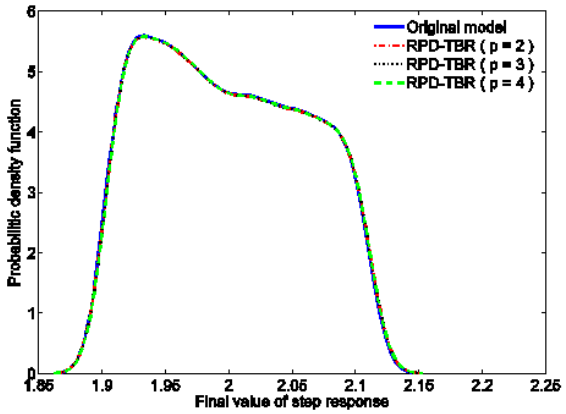

Fig. 8. The probability density function of the final value of the step response

A random parameter dependent truncated balanced realization (RPD-TBR) is derived from the corresponding balanced realization obtained with $p \in \{2,3,4\}$ by suppressing the two states that are hard to control and to observe in the RPD-BR. Eigenvalues of reduced models are plotted in Fig.9. These correspond to 10.000 samples of the couple of random variables ξ_1, ξ_2 which model the stiffness and damping coefficients k_1 and d_1 respectively. The generated RDP-TBRs are almost surely asymptotically stable since their eigenvalues are with negative real parts. All results show a suitable accuracy of the generated RPD-TBRs. Instantaneous mean values and variances of step responses of the RPD-TBR are correctly approximated. Moreover, the probability density function of the final value of the step response is also suitably approximated by the RPD-TBRs. Otherwise, the comparison between LePC expansions used show that the one with $p = 2$ is sufficient to derive a suitable reduced order model for System (62).

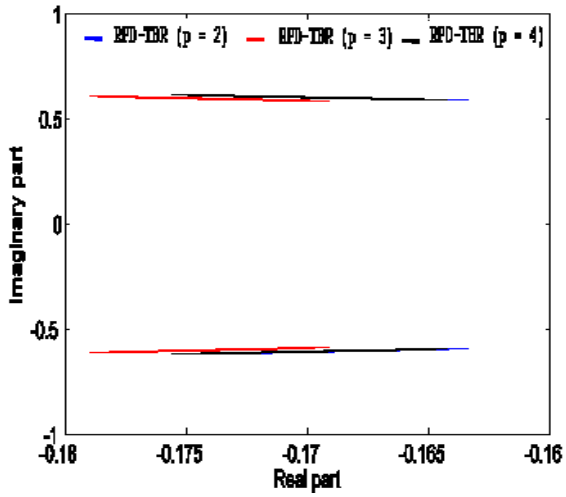

Fig. 2. Eigen values of original model and reduced order models within uncertainty interval

5. Conclusion

A new methodology for model order reduction of random parameter-dependent LTI systems has been proposed in this paper. The method is based on the random parameter-dependent balanced realization concept obtained from the generalization of the well-known balanced realization in the linear deterministic case. The main principle is to search for a random parameter-dependent balancing transformation which puts the system in a balanced form almost surely within the probabilistic support of random parameters. The generalized polynomial chaos formalism has been proposed to compute the random parameter dependent balancing transformation as well as the random Hankel singular values. Based on statistics of Hankel singular values estimated via their GPC expansions, states that have small controllability and observability degrees almost surely are truncated to derive a random parameter dependent reduced model characterized by a bounded truncation error. The proposed method has been tested on two degrees of freedom mechanical system. Its efficiency was pointed out. The main question which remains asked is about the stability of the RPD-TBR. We have no guaranty on the stability of the RPD-TBR. This problem is common to all methods of model reduction in particular in the case of models with uncertain parameters.

In this paper, the RPD-TBR has been used instead of the original RPD model and combined efficiently with a MC type method to predict accurately the dynamic behavior of the uncertain system used. Other possible exploitations of the RPD-TBR are in control and observation schemes. These perspectives are dealt with in our research in progress.

An important issue is related to the case where the GPC expansion used to compute the balancing transformation needs a high chaos order. This will lead to high numbers of terms in the GPC expansion and thus to more complex balancing transformations which, consequently, yields random parameters-dependent balanced realization difficult to be exploited in model order reduction. The use of the multi-element GPC in this context can be beneficial since this method helps to decrease the polynomial chaos order.

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