Reflected brownian motion in a wedge: sum-of-exponential absorption probability at the vertex and differential properties

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Abstract

We study a Brownian motion with drift in a wedge of angle β which is obliquely reflected on each edge along angles ε and δ. We assume that the classical parameter $\alpha = \frac{\delta+\varepsilon-\pi}{\beta}$ is greater than 1 and we focus on transient cases where the process can either be absorbed in the vertex or escape to infinity. We show that $\alpha \in \mathbb{N}^*$ is a necessary and sufficient condition for the absorption probability, seen as a function of the starting point, to be written as a finite sum of terms of exponential product form. In such cases, we give expressions for the absorption probability and its Laplace transform. When $\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$ we find an explicit D-algebraic expression for the Laplace transform. Our results rely on Tutte's invariant method and a recursive compensation approach.

1 Introduction

Context In dimension one, it is known that a standard Brownian motion with positive drift $\mu > 0$ started at $u > 0$ has probability $e^{-2\mu u}$ to reach 0. A simple way of achieving this result is to use Girsanov's theorem and the reflection principle. In dimension 2, we consider an obliquely reflected Brownian motion in a cone with drift belonging to the interior of the cone and directions of reflection strongly oriented towards the apex of the cone. A phenomenon of competition between the reflections and the drift appears and the process is either absorbed at the vertex or escapes to infinity. Lakner, Liu, and Reed [16] studied this absorption phenomenon and showed the existence and uniqueness of a solution to the absorbed process. Ernst et al. [7] were able to obtain a general formula for the probability of absorption at the vertex using Carleman's boundary value problems theory. In particular, they characterised the cases where this probability has an exponential product form, *i.e.* when the reflection vectors are opposite. Franceschi and Raschel [10] then generalised this result to higher dimensions by showing that the coplanarity of the reflection vectors was a necessary and sufficient condition for the absorption probability to have an exponential product form. In a sense, this condition can be seen as dual to the classical skew symmetry condition discovered by Harrison and Williams [12, 14, 22] which characterises cases where the stationary distribution is exponential. In dimension 2, when the process is recurrent, Dieker and Moriarty [6], preceded by Foshini [11] in the symmetric case, determined a necessary and sufficient condition for the stationary distribution to be a sum of exponentials terms of product form. It is therefore very natural to look for an analogous result to the one of Dieker and Moriarty. This article aims to find, when the process is transient, a necessary and sufficient condition for the absorption probability to be a sum-of-exponentials function of the starting point and to compute this probability. We also identify other remarkable cases where the Laplace transform of the absorption probability is D-algebraic.

Key parameter and main results To present our results in more detail, we need to introduce a few parameters usually used to define a semimartingale reflecting Brownian motion (SRBM). We define the cone $C := \{(r\cos(t), r\sin(t)) : r \geqslant 0 \text{ and } 0 \leqslant t \leqslant \beta\}$ of angle $\beta \in (0, \pi)$ and consider $\widetilde{Z}_t$ an obliquely reflected standard Brownian motion with drift $\widetilde{\mu} \in \mathbb{R}^2$ of angle $\theta \in (-\pi, \pi]$ and reflection vectors of angles $\delta \in (0, \pi)$ and $\varepsilon \in (0, \pi)$, see Figure 1 to visualize these angles. We define

$$\alpha := \frac{\delta + \varepsilon - \pi}{\beta} \tag{1}$$

which is a famous key parameter in the SRBM literature. As a general rule, such a process is most of the time studied in the literature in the case where $\alpha < 1$, *i.e.* in the case where the process is a semimartingale markov process, see the seminal work of Varadhan and Williams [19, 21]. We will not give here a precise mathematical definition of the process, which can be found in many articles, see the survey of Williams [23]. We will simply point out that it behaves like a standard Brownian motion with drift inside the cone, it is reflected in a given direction when it touches an edge (being pushed by the local time on the boundary) and it spends zero time at the vertex of the cone. The famous skew symmetric condition, where the stationary distribution has an exponential product form, corresponds to $\alpha = 0$, and Dieker and Moriarty's condition for a sum-of-exponential stationary density corresponds to $\alpha \in -\mathbb{N}$. The dual skew symmetric case [7, 10], where the escape probability has an exponential product form, correspond to $\alpha = 1$. For our purposes, in this article, we will assume that

$$\alpha \geqslant 1 \tag{2}$$

so that the process can be trapped at the vertex and we will consider transient cases where the drift $\widetilde{\mu}$ belongs to the interior of the cone C, that is when $\theta \in (0, \beta)$. We define the first hitting time of the vertex

$$T := \inf\{t > 0 : \widetilde{Z}_t = 0\}.$$

The article [16] makes a detailed study of the absorbed process, its existence, and its uniqueness in this case. As explained in the articles [7, 10, 16], by following the results from Taylor and Williams [18], when $\alpha \geqslant 1$ the process Z_t is well defined until it hits the vertex at time T, which amounts to considering the process $(\widetilde{Z}_t)_{0 \leqslant t \leqslant T}$.

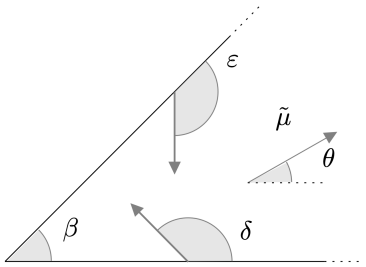

Figure 1: Configuration of the angles used to describe the model.

The main results of the article are as follows. We prove that the absorption probability at the vertex $\mathbb{P}(T < \infty)$ is a sum-of-exponential function of the starting point if and only if

$$\alpha \in \mathbb{N}^* \tag{3}$$

plus the small technical condition

$$\forall j \in \{1, \dots, 2\alpha - 2\},\ \theta - 2\delta + j\beta \not\equiv 0 \bmod(\pi) \tag{4}$$

which excludes cases where there are multiple poles in the Laplace transform. In fact, our results are much more accurate than that. Assuming that (3) and (4) hold, if (u, v) is the starting point of the process (mapped onto the quadrant, see (8)) the absorption probability is of the form

$$\mathbb{P}_{(u,v)}(T < \infty) = \sum_{k=1}^{2\alpha-1} c_k \exp\left(a_k u + b_k v\right), \tag{5}$$

where the coefficients a_k, b_k and c_k are computed explicitly in Theorem 12. In the cases where $\theta - 2\delta + j\beta \equiv 0 \bmod(\pi)$ for some $j \in \{1, \dots, 2\alpha - 2\}$, the absorption probability has the form $\mathbb{P}_{(u,v)}(T < \infty) = \sum_{k=1}^{2\alpha-1} A_k(u, v) \exp\left(a_k u + b_k v\right)$, where the A_k are affine functions of the variables u and v, see last paragraph of the article.

In Theorem 11 we state another more general and stronger result which explicitly determines the Laplace transform of the absorption probability in terms of a conformal gluing function when

$$\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}.$$

In this case, we also find the differential nature of the Laplace transform. In other words we find sufficient conditions on α for the Laplace transform to be rational, algebraic (i.e. satisfying a

polynomial equation with coefficients in the field of rational functions), D-finite (i.e. satisfying a linear differential equation with coefficients in the field of rational functions) or D-algebraic (i.e. satisfying a polynomial differential equation with coefficients in the field of rational functions). The differential nature of the Laplace transform reflects in various ways on the absorption probability itself. For example, if it is rational it implies that the absorption probability is a linear combination of exponentials multiplied by polynomials. If it is D-algebraic it will give a recurrence relation for the moments. We refer to the introduction of [3] which explains in more detail the interest of such a classification in this hierarchy of functions:

$$\text{rational} \subset \text{algebraic} \subset \text{D-finite} \subset \text{D-algebraic}. \tag{6}$$

The following table gives sufficient conditions for the Laplace transform to belong to this hierarchy.

rational	algebraic	D-finite	D-algebraic
$\alpha \in \mathbb{N}^*$	$\pi/\beta \in \mathbb{Q}$ and $\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$	$\alpha \in \mathbb{N}^* + \frac{\pi}{\beta}\mathbb{Z}$	$\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$

Plan and strategy of proof Section 2 presents the preliminaries needed to prove our results. For technical reasons, we first transfer the problem initially defined in a wedge into a quadrant thanks to a simple linear transform. The starting point of the proof is a kernel functional equation satisfied by the Laplace transform of the absorption probability as a function of the starting point. This equation is derived from a partial differential equation solved by the probability of absorption. This functional equation leads to a boundary value problem (BVP) already studied in [7]. In Section 3, we apply successfully Tutte's invariant method [20] to this BVP finding some *decoupling functions*, in a similar way to what was done in the recurrent case for the stationary distribution [3, 9]. We then compute explicitly the Laplace transform, see Theorems 9 and 11. Inverting the bivariate Laplace Transform is no easy task because of a complicated factorization of a two variable polynomial by the kernel. In Section 4, we then offer a geometrical way to construct the solutions inspired by the compensation approach developed with success in the discrete case for some queueing problems and random walks by Adan, Wessels, and Zijm [1].

Related literature and perspectives This paper develops an original way of showing these results, which is an alternative, although closely related, to the Dieker and Moriarty [6] method in the recurrent case. Another approach to show our results might have been to use an equivalence based on time reversal and developed very recently by Harrison [13] which shows that the hitting time of the vertex is inherently connected to the stationary distribution of a certain dual process, and then apply the results of [6] to a certain trapezoid described in [13].

It is also important to mention the strong links between the results of this article and the Weil chambers and reflection groups. For example, Biane, Bougerol, and O'Connell [2] express the persistence probability, that is the probability that a Brownian motion with drift stays forever in a Weyl chamber, as a sum-of-exponential. We may also mention the article by Defosseux [5] which expresses similar results for a space-time Brownian motion.

It is also possible to interpret our problem as the study of the probability of triple collisions for transient competing particle systems with asymmetric collisions. Indeed, a Brownian motion reflected in a quadrant is nothing more than the gap process of such a system made of three particles, and reaching the vertex of the quadrant is equivalent to a triple collision. A very interesting literature is devoted to the study of the absence or presence of such collisions, and as we cannot claim exhaustiveness in these few lines we will limit ourselves to mentioning the articles by Ichiba, Karatzas, and Shkolnikov [15] and Bruggeman and Sarantsev [4].

To conclude this introduction, we must emphasise that this article is an important step towards a more ambitious outcome. Indeed, we believe that the present results can be extended. More precisely, we believe, as was done in the recurrent case for the stationary distribution in the article of Bousquet-Mélou et al. [3], that it is possible to characterise the algebraic and differential nature of the Laplace transforms of the absorption probability. In a sense, this would exhaustively rank the complexity of the absorption probability in the hierarchy (6) according to the value of α. Such a result, which would provide sufficient but also necessary conditions, would require difference Galois theory which is beyond the scope of this article.

2 Preliminaries

From the cone to the quadrant The results stated in the introduction for the standard Brownian motion $\widetilde{Z}_t$ reflected in a cone C of angle $\beta \in (0, \pi)$, drift angle θ and reflection angles

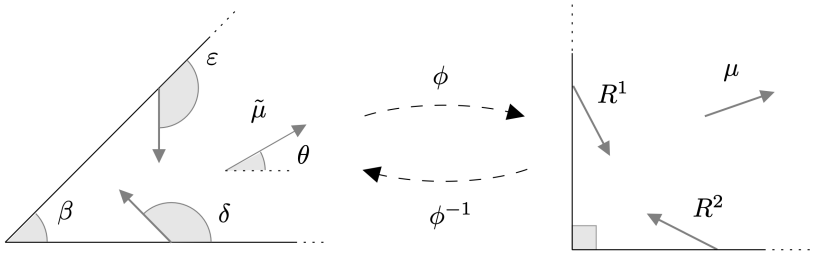

Figure 2: Equivalence between the reflected Brownian motion $\widetilde{Z}_t$ in a β-wedge with drift $\tilde{\mu}$ and reflection angles δ and ε and the reflected Brownian motion Z_t in the first quadrant with drift μ and reflection vectors R^1 and R^2.

δ and ε will be proved by considering a Brownian motion Z_t reflected in the quadrant $\mathbb{R}_+^2$ with a covariance matrix, a drift and a reflection matrix noted respectively

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}, \qquad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \qquad R = \left(R^1, R^2\right) = \begin{pmatrix} 1 & -r_2 \\ -r_1 & 1 \end{pmatrix},$$

which satisfy the following relations

$$\cos\beta = \frac{-\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}, \quad \tan\theta = \frac{\mu_2\sqrt{\det(\Sigma)}}{\sigma_{22}\mu_1 - \sigma_{12}\mu_2}, \quad \tan\delta = \frac{-\sqrt{\det(\Sigma)}}{r_2\sigma_{22} + \sigma_{12}}, \quad \tan\varepsilon = \frac{-\sqrt{\det(\Sigma)}}{\sigma_{11}r_1 + \sigma_{12}}. \tag{7}$$

The study of these two processes is equivalent by considering ϕ a simple bijective linear transform defined by

$$\phi := \begin{pmatrix} \sqrt{\sigma_{11}} & 0 \\ 0 & \sqrt{\sigma_{22}} \end{pmatrix} \begin{pmatrix} \sin(\beta) & -\cos(\beta) \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$$

which maps the cone C onto the first quadrant $\mathbb{R}_+^2 = \phi(C)$, and $\widetilde{Z}_t$ onto $Z_t = \phi(\widetilde{Z}_t)$.

We have of course $\phi(0) = 0$ and $T = \inf\{t > 0 : \widetilde{Z}_t = 0\} = \inf\{t > 0 : Z_t = 0\}$. It is then equivalent to compute the absorption probability for the process $\widetilde{Z}_t$ starting from $(\widetilde{u}, \widetilde{v}) \in C$ and for the process Z_t starting from $(u, v) = \phi(\widetilde{u}, \widetilde{v}) \in \mathbb{R}_+^2$. We denote the escape probability

$$f(u, v) := \mathbb{P}_{(u,v)}(T < \infty) \quad \text{and} \quad \widetilde{f}(\widetilde{u}, \widetilde{v}) := f(\phi(\widetilde{u}, \widetilde{v})). \tag{8}$$

This linear transform doesn't affect the form of the absorption probability. More precisely, the absorption probability $f(u, v)$ is a sum-of-exponential, given by (5), if and only if $\widetilde{f}(\widetilde{u}, \widetilde{v})$ is a sum-of-exponential, given by

$$\widetilde{f}(\widetilde{u}, \widetilde{v}) = \sum_{k=1}^{2\alpha-1} c_k \exp\left(a_k \sqrt{\frac{\sigma_{22}}{\det(\Sigma)}}\widetilde{u} + \frac{1}{\sqrt{\sigma_{22}}}\left[b_k - \frac{a_k\sigma_{12}}{\sqrt{\det(\Sigma)}}\right]\widetilde{v}\right).$$

Partial differential equation The escape probability of the process Z_t starting from (u, v) defined by

$$g(u, v) := 1 - f(u, v) = \mathbb{P}_{(u,v)}(T = \infty)$$

satisfies the following partial differential equation, see Proposition 11 in [7]. The function g is both bounded and continuous in the quarter plane and on its boundary and continuously differentiable in the quarter plane and on its boundary (except perhaps at the corner), and satisfies the elliptic partial differential equation

$$\mathcal{G}g := \left(\frac{1}{2}\nabla \cdot \Sigma\nabla + \mu \cdot \nabla\right) g = 0 \text{ on } \mathbb{R}_+^2 \tag{9}$$

with oblique Neumann boundary conditions

$$\partial_{R^1} g(0, \cdot\,) := \left(R^1 \cdot \nabla\right) g(0, \cdot\,) = 0, \; \partial_{R^2} g(\cdot\,, 0) := \left(R^2 \cdot \nabla\right) g(\cdot\,, 0) = 0 \text{ on } \mathbb{R}_+ \tag{10}$$

and the limit conditions

$$g(0, 0) = 0, \qquad \lim_{\|(u,v)\| \to \infty} g(u, v) = 1. \tag{11}$$

The absorption probability $f = 1 - g$ satisfies the same partial differential equation replacing (11) with the appropriate limit conditions.

Laplace transform and functional equation We define the Laplace transform of the escape probability $g(u,v)$ by

$$\varphi(x,y) := \iint_{\mathbb{R}_+^2} \mathbb{P}_{(u,v)}(T=\infty) e^{-xu-yv} \mathrm{d}u\mathrm{d}v$$

and the Laplace transforms of the escape probabilities $g(0,v)$ and $g(u,0)$ when the process starts from the boundaries by

$$\varphi_1(y) := \int_{\mathbb{R}_+} \mathbb{P}_{(0,v)}(T=\infty) e^{-yv} \mathrm{d}v, \qquad \varphi_2(x) := \int_{\mathbb{R}_+} \mathbb{P}_{(u,0)}(T=\infty) e^{-xu} \mathrm{d}u. \tag{12}$$

One can easily use integrations by parts to translate the partial differential equation made of the three conditions (9), (10), (11) into a functional equation for the Laplace transforms.

Proposition 1 (Prop. 12 in [7]). *The Laplace transforms φ, φ_1 and φ_2 satisfy the following kernel functional equation, for $(x,y) \in \mathbb{C}^2$ such that $\Re x > 0$ and $\Re y > 0$ we have*

$$K(x,y)\varphi(x,y) = k_1(x,y)\varphi_1(y) + k_2(x,y)\varphi_2(x) \tag{13}$$

where

$$\begin{cases} K(x,y) := \dfrac{1}{2}(x,y)^\intercal \cdot \Sigma (x,y)^\intercal + \mu \cdot (x,y)^\intercal = \dfrac{1}{2}(\sigma_{11}x^2 + \sigma_{22}y^2 + 2\sigma_{12}xy) + \mu_1 x + \mu_2 y, \\ k_1(x,y) := \dfrac{\sigma_{11}}{2}(x + r_1 y) + \sigma_{12} y + \mu_1, \\ k_2(x,y) := \dfrac{\sigma_{22}}{2}(r_2 x + y) + \sigma_{12} x + \mu_2. \end{cases} \tag{14}$$

Study of the kernel K and uniformization To solve functional equation (13), we first need to study K, and more precisely its vanishing set

$$\mathcal{S} := \{(x,y) \in \mathbb{C}^2 : K(x,y) = 0\}. \tag{15}$$

The set $\mathcal{S}$ is an elliptic curve that passes through the origin. For $x \in \mathbb{C}$, the equation $K(x,y) = 0$ in y is quadratic, and has therefore two solutions $Y^+(x)$ and $Y^-(x)$ in $\mathbb{C}$:

$$Y^\pm(x) := \frac{-\sigma_{12}x - \mu_2 \pm \sqrt{(\sigma_{12}x + \mu_2)^2 - \sigma_{22}(\sigma_{11}x^2 + 2\mu_1 x)}}{\sigma_{22}}. \tag{16}$$

Likewise, we define $X^+(y)$ and $X^-(y)$ to be the two solutions of the equation $K(x,y) = 0$ in the variable x. The curve $\mathcal{S}$ can be thought of as the image of the multivalued function Y (*resp.* X) which has two ramification points x^+ and x^- (*resp.* y^+ and y^-) given by

$$x^\pm := \frac{(\mu_2\sigma_{12} - \mu_1\sigma_{22}) \pm \sqrt{(\mu_2\sigma_{12} - \mu_1\sigma_{22})^2 + \det(\Sigma)\mu_2^2}}{\det(\Sigma)}.$$

The branches X^+ and X^- are analytic on $\mathbb{C} \setminus ((-\infty, y^-] \cup [y^+, +\infty))$, and Y^+ and Y^- are analytic on $\mathbb{C} \setminus ((-\infty, x^-] \cup [x^+, +\infty))$. It will be handy to work with the following rational uniformization of $\mathcal{S}$, first stated in [8, Proposition 5], defined by

$$(\mathbf{x}, \mathbf{y})(s) := \left(\frac{x^+ + x^-}{2} + \frac{x^+ - x^-}{4}\left(s + \frac{1}{s}\right), \frac{y^+ + y^-}{2} + \frac{y^+ - y^-}{4}\left(\frac{s}{e^{i\beta}} + \frac{e^{i\beta}}{s}\right)\right), \tag{17}$$

which is such that

$$\mathcal{S} = \{(\mathbf{x}(s), \mathbf{y}(s)),\ s \in \mathbb{C}^*\}.$$

In the following, we adopt the notation

$$\mathbf{q} := e^{2i\beta}. \tag{18}$$

The functions $\mathbf{x}$ and $\mathbf{y}$ satisfy the following invariance properties: for all $s \in \mathbb{C}^*$

$$\mathbf{x}(s) = \mathbf{x}(s^{-1}) \text{ and } \mathbf{y}(s) = \mathbf{y}(\mathbf{q}s^{-1}). \tag{19}$$

Lemma 2. *There exists $C_1, C_2 \in \mathbb{C}$ such that for all $s \in \mathbb{C}^*$ the polynomials defined in (14) satisfy*

$$k_1(\mathbf{x}(s), \mathbf{y}(s)) = C_1 \frac{(s - s_0')(s - s_1)}{s}, \qquad k_2(\mathbf{x}(s), \mathbf{y}(s)) = C_2 \frac{(s - s_0'')(s - s_2)}{s},$$

with

$$s_0' := e^{i(2\beta - \theta)}, \qquad s_0'' := e^{-i\theta}, \qquad s_1 := e^{i(\theta + 2\varepsilon)}, \qquad s_2 := e^{i(\theta - 2\delta)}. \tag{20}$$

Proof. For $i \in \{1,2\}$, $k_i(\mathbf{x}(s), \mathbf{y}(s)) = 0$ is a degree-two polynomial equation whose roots can be computed with some basic trigonometry using (7). □

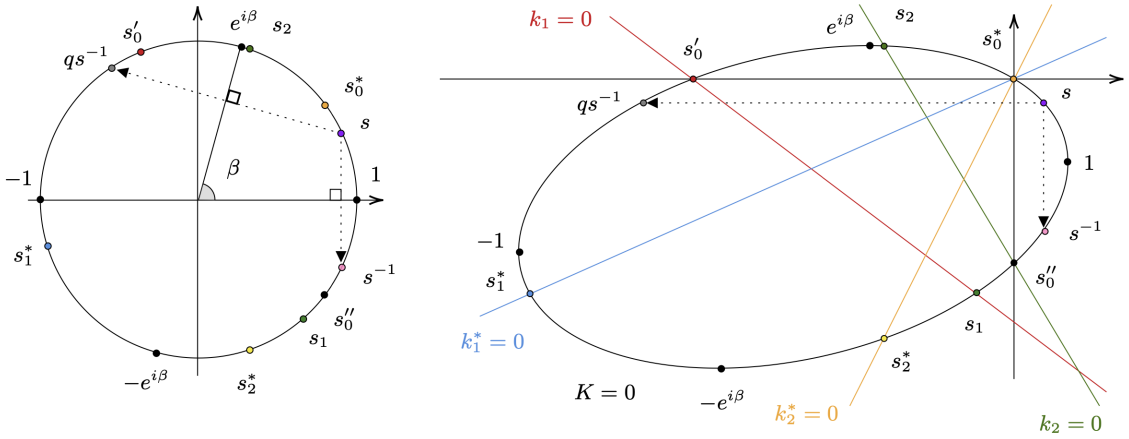

Figure 3: On the left, the points introduced in Lemmas 2 and 13 on the unit complex circle $\mathbb{U}$; on the right, their images by $(\mathbf{x}, \mathbf{y})$ on the ellipse $\mathcal{S} \cap \mathbb{R}^2$. For the sake of readability, points on the ellipse are labelled with their preimages. Invariances in (19) are represented by the dotted arrows.

Boundary Value Problem We define a hyperbola $\mathcal{H}$ deeply linked to the kernel by

$$\mathcal{H} := Y^{\pm}([x^+, \infty)) = \{y \in \mathbb{C} : K(x, y) = 0 \text{ and } x \in [x^+, \infty)\}.$$

Noticing that $\mathbf{x}(\mathbb{R}_+) = [x^+, +\infty)$ and by the invariance $\mathbf{y}(s) = \mathbf{y}(\mathbf{q}s^{-1})$ we can see that

$$\mathcal{H} = \mathbf{y}(\mathbb{R}_+) = \mathbf{y}(\mathbf{q}\mathbb{R}_+). \tag{21}$$

This hyperbola is the boundary of the Boundary Value Problem stated below. We now define $\mathcal{G}_{\mathcal{H}}$ the domain of $\mathbb{C}$ bounded by $\mathcal{H}$ and containing y^+, see Figure 4. By (21), remembering that $\mathbf{q} = e^{2i\beta}$ and $\mathbf{y}^{-1}(y^+) = e^{i\beta}$ we see that

$$\overline{\mathcal{G}}_{\mathcal{H}} = \mathbf{y}\Big(\{ae^{ib}, (a, b) \in \mathbb{R}_+ \times [0, 2\beta]\}\Big), \tag{22}$$

see Figure 5. Finally, we compute

$$\mathbf{y}(s_1) = -\frac{2(r_1\mu_1 + \mu_2)}{\sigma_{22} + \sigma_{11}r_1^2 + 2\sigma_{12}r_1}. \tag{23}$$

The following proposition is a Carleman Boundary Value Problem which characterizes the Laplace transform φ_1 and which can be easily obtained from the functional equation (13), see [7].

Proposition 3 (Proposition 22 and Lemma 32 in [7])**.** *The Laplace transform φ_1 satisfies the boundary value problem:*

1. *φ_1 is meromorphic on the open domain $\mathcal{G}_{\mathcal{H}}$ and continuous on $\overline{\mathcal{G}}_{\mathcal{H}} := \mathcal{G}_{\mathcal{H}} \cup \mathcal{H}$;*
2. *φ_1 admits one or two poles in $\overline{\mathcal{G}}_{\mathcal{H}}$, 0 is always a simple pole of φ_1 and $\mathbf{y}(s_1)$ is a simple pole of φ_1 if and only if $2\varepsilon + \theta \geqslant 2\pi$;*
3. *for some positive constant C the asymptotics of φ_1 when $y \to \infty$ is given by*

$$\varphi_1(y) \sim Cy^{-\alpha-1}; \tag{24}$$

4. *φ_1 satisfies the boundary condition*

$$\varphi_1(\overline{y}) = G(y)\varphi_1(y), \qquad \forall y \in \mathcal{H} \tag{25}$$

where

$$G(y) = \frac{k_1}{k_2}(X^+(y), y)\frac{k_2}{k_1}(X^+(y), \overline{y}). \tag{26}$$

In the next section, our strategy will be to find cases where the function G simplifies in order to find rational and D-algebraic solutions to this Boundary Value Problem.

3 Tutte's invariants and Laplace transform

3.1 Decoupling and Tutte's invariant

Heuristic of the method The method involves finding all cases where there exists decoupling in the following sense. Recall that $\mathcal{S} = \{(x, y) \in \mathbb{C}^2 : K(x, y) = 0\}$.

Definition 4 (Decoupling). *A pair of rational functions (P, Q) satisfying*

$$\frac{k_2(x, y)}{k_1(x, y)} = \lambda \frac{P(x)}{Q(y)} \text{ for all } (x, y) \in \mathcal{S} \tag{27}$$

for some constant λ is called a decoupling pair.

We shall see that the existence of a decoupling pair leads to the study of what is called an invariant, which glues together the upper and the lower branches of the hyperbola $\mathcal{H}$ in the following sense.

Definition 5 (Invariant). *A function I which is meromorphic in $\mathcal{G}_{\mathcal{H}}$, continuous on its boundary $\mathcal{H}$ and satisfying $I(\overline{y}) = I(y)$ for all $y \in \mathcal{H}$ is called an invariant.*

Under the existence of a decoupling pair (P, Q), the boundary condition (25) can be rewritten as

$$Q\varphi_1(\overline{y}) = Q\varphi_1(y), \qquad \forall y \in \mathcal{H}. \tag{28}$$

If (P, Q) is a decoupling pair, the function $Q\varphi_1$ is then called the unknown invariant (since we are looking for φ_1). We now introduce a conformal gluing function w, which we call the canonical invariant, in terms of a classical Gauss hypergeometric function which is often called *generalized Chebyshev polynomial*,

$$w(y) := {}_2F_1\left(-\frac{\pi}{\beta}, \frac{\pi}{\beta}; \frac{1}{2}; \frac{1}{2}\left(1 - \frac{2y - (y^+ + y^-)}{y^+ - y^-}\right)\right) = \cos\left(\frac{\pi}{\beta} \arccos\left(\frac{2y - (y^+ + y^-)}{y^+ - y^-}\right)\right).$$

The fact that w is a conformal invariant (in the sense of Definition 5) is proven in Lemma 5.3 of [3]. In particular, w is analytic and bijective from $\mathcal{G}_{\mathcal{H}}$ to $\mathbb{C} \setminus (-\infty, -1]$ and

$$w(\overline{y}) = w(y), \qquad \forall y \in \mathcal{H}. \tag{29}$$

The same lemma gives, for some constant $\hat{C}$, the asymptotics

$$w(y) \sim \hat{C} y^{\pi/\beta}. \tag{30}$$

It is also well known that w is a polynomial when $\pi/\beta \in \mathbb{Z}$, is algebraic when $\pi/\beta \in \mathbb{Q}$ and is always D-finite. Remark that the set of D-finite functions is stable by multiplication but not by division and $1/w$ is D-algebraic but not necessarily D-finite. See Proposition 5.2 of [3].

The key point of Tutte's invariant method is to express the unknown invariant in terms of the canonical invariant. The following crucial lemma shows that there are few invariants.

Lemma 6 (Invariant lemma). *If I is an invariant in the sense of Definition 5 which doesn't have any pole on $\overline{\mathcal{G}}_{\mathcal{H}}$ and has a finite limit at $+\infty$, then I is constant.*

Proof. Since w is conformal, and maps $\mathcal{G}_{\mathcal{H}}$ to the cut plane $\mathbb{C} \setminus (-\infty, -1]$, $I \circ w^{-1}$ is analytic on this cut plane, and continuous on the cut thanks to (29). By Morera's theorem, $I \circ w^{-1}$ (or better yet, its continuous extension) is meromorphic on $\mathbb{C}$. Since I is bounded, Liouville's theorem implies that $I \circ w^{-1}$ (and then I) is constant. □

Decoupling In the following proposition, we obtain a necessary and sufficient condition for the existence of a decoupling condition (27) and we give explicit decoupling pairs.

Proposition 7 (Decoupling condition). *There exists a rational decoupling pair (P, Q) if and only if*

$$\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}.$$

In this case let $d, r \in \mathbb{Z}$ such that $\alpha = d - 1 + r\pi/\beta$, i.e. $\delta + \varepsilon = (d-1)\beta + (r+1)\pi$. Then d cannot be equal to 1 and we distinguish two cases:

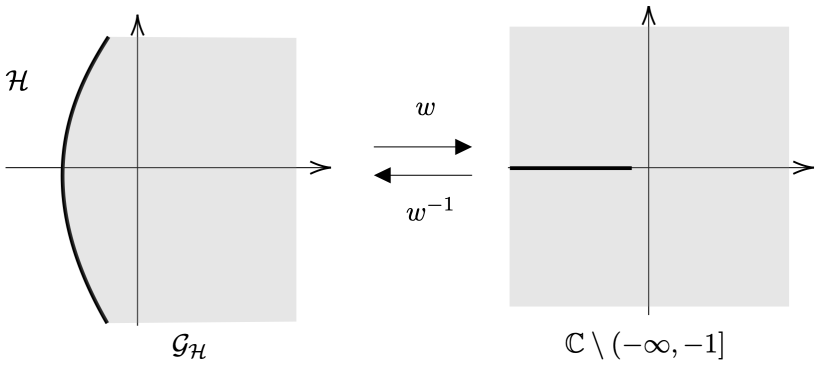

Figure 4: Domains and codomains for w and its inverse.

- *If $d \geqslant 2$, then one can choose the following polynomial decoupling pair,*

$$P(x) := x \prod_{k=0}^{d-2} \frac{x - \mathbf{x}(s_2\mathbf{q}^k)}{-\mathbf{x}(s_2\mathbf{q}^k)}, \qquad Q(y) := y \prod_{k=0}^{d-2} \frac{y - \mathbf{y}\left(s_1/\mathbf{q}^k\right)}{-\mathbf{y}\left(s_1/\mathbf{q}^k\right)}. \tag{31}$$

- *If $d \leqslant 0$, then one can choose the following rational decoupling pair,*

$$P(x) := x \prod_{k=1}^{1-d} \frac{-\mathbf{x}(s_2\mathbf{q}^k)^{-1}}{x - \mathbf{x}(s_2/\mathbf{q}^k)}, \qquad Q(y) := y \prod_{k=1}^{1-d} \frac{-\mathbf{y}\left(s_1/\mathbf{q}^k\right)^{-1}}{y - \mathbf{y}\left(s_1\mathbf{q}^k\right)}. \tag{32}$$

We have $d = \deg(P) = \deg(Q)$.

Proof. If there exists a rational decoupling (P, Q), see (27), then by the invariances of $\mathbf{x}$ and $\mathbf{y}$ established in (19) we have

$$\frac{k_2(\mathbf{x}(s), \mathbf{y}(s))k_1(\mathbf{x}(s^{-1}), \mathbf{y}(s^{-1}))}{k_1(\mathbf{x}(s), \mathbf{y}(s))k_2(\mathbf{x}(s^{-1}), \mathbf{y}(s^{-1}))} = \frac{P(\mathbf{x}(s))Q(\mathbf{y}(s^{-1}))}{Q(\mathbf{y}(s))P(\mathbf{x}(s^{-1}))} = \frac{Q(\mathbf{y}(\mathbf{q}s))}{Q(\mathbf{y}(s))}. \tag{33}$$

According to Lemma 2 we also have

$$\frac{k_2(\mathbf{x}(s), \mathbf{y}(s))k_1(\mathbf{x}(s^{-1}), \mathbf{y}(s^{-1}))}{k_1(\mathbf{x}(s), \mathbf{y}(s))k_2(\mathbf{x}(s^{-1}), \mathbf{y}(s^{-1}))} = \frac{(s - s_0'')(s - s_2)}{(s - s_0')(s - s_1)} \cdot \frac{(s^{-1} - s_0')(s^{-1} - s_1)}{(s^{-1} - s_0'')(s^{-1} - s_2)}. \tag{34}$$

Taking the limit as s goes to $+\infty$ of (33) and (34), we get

$$\frac{s_0' s_1}{s_0'' s_2} = \mathbf{q}^d, \quad \text{where } d := \deg(Q) \in \mathbb{Z}. \tag{35}$$

Plugging in the values of s_0', s_0'', s_1, s_2 obtained in Lemma 2 and remembering that $\mathbf{q} = e^{2i\beta}$ in (35) we obtain that $e^{2i(\beta+\delta+\varepsilon)} = e^{2id\beta}$ and then there exists $r \in \mathbb{Z}$ such that

$$\alpha = \frac{\delta + \varepsilon - \pi}{\beta} = d - 1 + r\frac{\pi}{\beta}. \tag{36}$$

Conversely, we assume that (36) holds. First, let us treat the case $d \geqslant 2$ and assume that P and Q are given by (31). Through the uniformization (17), one gets

$$P(\mathbf{x}(s)) = \frac{1}{s^d}(s - s_0'')\left(s - \frac{1}{s_0''}\right) \prod_{k=0}^{d-2} \frac{s - s_2\mathbf{q}^k}{-\mathbf{x}(s_2\mathbf{q}^k)} \left(s - \frac{1}{s_2\mathbf{q}^k}\right),$$

$$Q(\mathbf{y}(s)) = \frac{1}{s^d}(s - s_0')\left(s - \frac{\mathbf{q}}{s_0'}\right) \prod_{k=0}^{d-2} \frac{s - s_1/\mathbf{q}^k}{-\mathbf{y}\left(s_1/\mathbf{q}^k\right)} \left(s - \frac{\mathbf{q}^{k+1}}{s_1}\right).$$

We know that $s_0' = \mathbf{q}s_0''$. Given that $\alpha = d - 1 + r\pi/\beta$ with d and r in $\mathbb{Z}$, we can also use the fact that $s_1 = \mathbf{q}^{d-1}s_2$. When taking the ratio of $P(\mathbf{x}(s))$ and $Q(\mathbf{y}(s))$, these identities produce a telescoping which gives

$$\frac{P(\mathbf{x}(s))}{Q(\mathbf{y}(s))} = \prod_{k=0}^{d-2} \frac{\mathbf{y}(s_1/\mathbf{q}^k)}{\mathbf{x}(s_2\mathbf{q}^k)} \frac{(s - s_0'')(s - s_2)}{(s - s_0')(s - s_1)} = \frac{1}{\lambda} \frac{k_2(\mathbf{x}(s), \mathbf{y}(s))}{k_1(\mathbf{x}(s), \mathbf{y}(s))},$$

where the last equality comes from Lemma 2 and taking

$$\lambda := \frac{C_1}{C_2} \prod_{k=0}^{d-2} \frac{\mathbf{x}(s_2\mathbf{q}^k)}{\mathbf{y}(s_1/\mathbf{q}^k)}. \tag{37}$$

We deduce that (P, Q) is a decoupling pair. The proof is similar for the case $d \leqslant 0$. The fact that d cannot be equal to 1 directly derives from the fact that $\varepsilon, \delta \in (0, \pi)$, $\beta \in (0, \pi)$ and $\alpha \geqslant 1$. □

Lemma 8 (Simple root condition). *Let $\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$ and $(d, r) \in \mathbb{Z}^2$ such that $\alpha = d - 1 + r\pi/\beta$. If $\beta/\pi \in \mathbb{Q}$ then (d, r) is not unique, in this case for $\beta/\pi = p/q$ for p and q relatively prime and $p < q$, we (can) choose d such that $|d| < q$. If $d \geqslant 2$ (resp. $d \leqslant 0$) then P and Q have no multiple roots (resp. pole) if and only if for all $k \in \{1, \dots, 2d-4\}$ (resp. $k \in \{2d-1, \dots, -2\}$) we have*

$$\theta - 2\delta + k\beta \not\equiv 0 \bmod(\pi). \tag{38}$$

Proof. Let $d \geqslant 2$. The polynomial P has a double root (or more) if and only if for some $i \neq j \in \{0, \dots, d-2\}$ we have $\mathbf{x}(s_2\mathbf{q}^i) = \mathbf{x}(s_2\mathbf{q}^j)$. Using the expression of $\mathbf{x}$ given in (17) there are only two ways for this to happen. The first one is that $\mathbf{q}^i = \mathbf{q}^j$, i.e. $i\beta = j\beta \bmod(\pi)$ which is not possible even when $\beta/\pi = p/q \in \mathbb{Q}$ since $|i - j| < |d| < q$. The second one is that $s_2\mathbf{q}^i = (s_2\mathbf{q}^j)^{-1}$, using the value of s_2 in (20) it is equivalent to $\theta - 2\delta + (i + j)\beta \equiv 0 \bmod(\pi)$. Similarly, Q has a double root if and only if $\theta - 2\delta + (2d - i - j - 3)\beta \equiv 0 \bmod(\pi)$. The case $d \leqslant 0$ is similar. □

One should observe that the decoupling condition $\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$ doesn't depend on θ (and hence on the drift) while the multiple root condition does.

3.2 Explicit expression for the Laplace transforms

We now state our first main result when $\alpha \in \mathbb{N}^*$.

Theorem 9 (Laplace transforms, $\alpha \in \mathbb{N}^*$). *If $\alpha \in \mathbb{N}^*$ then the rational function defined by*

$$L(x, y) := \frac{k_1(x, y)P(x) + k_2(x, y)Q(y)}{K(x, y)}$$

is a polynomial and we have

$$\varphi_1(y) = \frac{1}{Q(y)}, \qquad \varphi_2(x) = \frac{1}{P(x)}, \qquad \varphi(x, y) = \frac{L(x, y)}{P(x)Q(y)}$$

where P and Q are given in (31).

Proof. Assuming that $\alpha = d - 1 \in \mathbb{N}^*$, the polynomials P and Q given in (31) form a decoupling pair by Proposition 7. The boundary value problem of Proposition 3 thus implies that $Q\varphi_1$ is an invariant, see (28). According to Lemma 6, we only need to prove that $Q\varphi_1$ doesn't have any pole on $\overline{\mathcal{G}}_{\mathcal{H}}$, and has a finite limit as y goes to $+\infty$. By (24) and (31), and since $\alpha = d - 1$ we have

$$Q(y)\varphi_1(y) \sim \frac{y^d}{\prod_{k=0}^{d-2} -\mathbf{y}(s_1/\mathbf{q}^k)} \cdot Cy^{-\alpha-1} = \frac{C}{\prod_{k=0}^{d-2} -\mathbf{y}(s_1/\mathbf{q}^k)}.$$

Furthermore the poles of φ_1 given in Proposition 3 *i.e.* 0 and $\mathbf{y}(s_1)$ when $2\varepsilon + \theta \geqslant 2\pi$, are compensated by the zeros of Q. Indeed 0 and $\mathbf{y}(s_1)$ are always roots of Q. By Lemma 6, there exists $\kappa \in \mathbb{C}$ such that $Q\varphi \equiv \kappa$. On the one hand, using the fact that $Q'(0) = 1$ gives

$$\lim_{y \to 0} y\varphi_1(y) = \kappa \lim_{y \to 0} \frac{y}{Q(y)} = \frac{\kappa}{Q'(0)} = \kappa.$$

On the other hand, by the final value theorem and (11) we have

$$\lim_{y \to 0} y\varphi_1(y) = \lim_{v \to +\infty} \mathbb{P}_{(0,v)}(T = +\infty) = 1. \tag{39}$$

Hence $\kappa = 1$ and $\varphi_1 = 1/Q$. The same method also works to show that $\varphi_2 = 1/P$. Replacing φ_1 and φ_2 in the functional equation (13), one can obtain

$$\frac{k_2(x, y)}{k_1(x, y)} = -\frac{P(x)}{Q(y)}, \quad \text{for all } (x, y) \in \mathcal{S}. \tag{40}$$

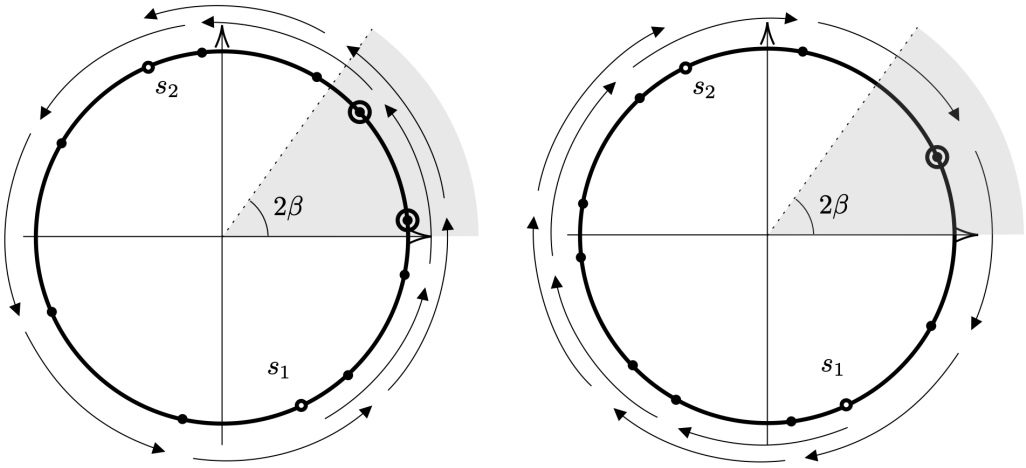

Figure 5: Enumeration of P *and* Z *for* $(d, r) = (-9, 2)$ *(on the left) and* $(d, r) = (11, -1)$ *(on the right). The shaded area corresponds to* $\mathbf{y}^{-1}(\overline{\mathcal{G}}_{\mathcal{H}})$.

Comparing with Definition 4 we can see that the constant λ given in (37) is equal to -1. The polynomial $k_2Q + k_1P$ vanishes on $\mathcal{S}$ the set of the zeros of K. By Hilbert's Nullstellensatz,

$$k_2Q + k_1P \in \sqrt{(K)}$$

where $(K) := \{LK, L \in \mathbb{C}[X, Y]\}$ and $\sqrt{(K)} := \{H \in \mathbb{C}[X, Y] : \exists m \in \mathbb{N} \text{ s.t. } H^m \in (K)\}$ are the ideal generated by K and its radical. Here, K is irreducible which implies that $\sqrt{(K)} = (K)$. This shows that there exists $L \in \mathbb{C}[X, Y]$ such that $L(x, y)K(x, y) = k_1(x, y)P(x) + k_2(x, y)Q(y)$. We see that $L(x, y) \in \mathbb{R}$ for all $(x, y) \in \mathbb{R}^2$, so the coefficients of L must be real. Substituting the values for φ_1 and φ_2 into the functional equation (13) yields $\varphi(x, y) = L(x, y)/(P(x)Q(y))$. □

We now state a lemma useful to obtain our second main result which deals with the case where $\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$. We consider d and $r \in \mathbb{Z}$ such that $\alpha = (d-1) + r\pi/\beta$. If $\beta/\pi = p/q \in \mathbb{Q}$, with p and q relatively prime, we (can) choose $|d| < q$. For further use, we need to study the number of zeros and poles of $Q\varphi_1$ which belong to $\overline{\mathcal{G}}_{\mathcal{H}}$. First of all, we can see that 0 is always a root of Q and a pole of φ_1, which therefore compensate each other considering $Q\varphi_1$.

When $d \geqslant 2$ we denote

$$\mathrm{Z} := \{\mathbf{y}(s_1/\mathbf{q}^k) \in \overline{\mathcal{G}}_{\mathcal{H}} : k = 1, \ldots, d-2\} \tag{41}$$

which is a set containing (all the) zeros of $Q\varphi_1$ in $\overline{\mathcal{G}}_{\mathcal{H}}$, see (31). Note that in this definition k cannot be taken equal to 0 since when $\mathbf{y}(s_1)$ is both a pole of φ_1 in $\overline{\mathcal{G}}_{\mathcal{H}}$ and a zero of Q they compensate each other, by item 2 of Proposition 3.

When $d \leqslant 0$ we denote

$$\mathrm{P} := \{\mathbf{y}(s_1\mathbf{q}^k) \in \overline{\mathcal{G}}_{\mathcal{H}} : k = 0, \ldots, 1-d\} \tag{42}$$

which is the set of poles of $Q\varphi_1$ in $\overline{\mathcal{G}}_{\mathcal{H}}$, see (32). Note that in this definition k can be taken equal to 0 since $\mathbf{y}(s_1)$ is a pole of φ_1 which can belongs to $\overline{\mathcal{G}}_{\mathcal{H}}$, see item 2 of Proposition 3. See Figure 5 to visualize P and Z.

Lemma 10 (Cardinal of Z and P)**.** *Let* $\alpha \geqslant 1$ *and assume that* $\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$ *and* (38) *holds. Let* d *and* $r \in \mathbb{Z}$ *such that* $\alpha = (d-1) + r\pi/\beta$*, i.e.* $(d-1)\beta + (r+1)\pi = \delta + \varepsilon$*. Then* $d \neq 1$ *and we have*

(i) If $d \geqslant 2$ *then* $r \leqslant 0$ *and we have*

$$\mathrm{Card}(\mathrm{Z}) = -r.$$

(ii) If $d \leqslant 0$ *then* $r > 0$ *and we have*

$$\mathrm{Card}(\mathrm{P}) = r.$$

Proof. Using the fact that $\varepsilon, \delta \in (0, \pi)$, $\beta \in (0, \pi)$ and $\alpha \geqslant 1$ it is easy to see that d cannot be equal to 1, that $d \geqslant 2$ implies $r \leqslant 0$ and that $d < 1$ implies $r > 0$.

(i) Assume that $d \geqslant 2$ and $r \leqslant 0$. Recalling equation (22) and noticing that for all $k \in \mathbb{Z}$, $s_1/\mathbf{q}^k \in \mathbb{U}$ (where $\mathbb{U}$ is the complex unit circle) we define

$$C_0 := \{e^{ib}, b \in [0, 2\beta]\} = \mathbf{y}^{-1}(\overline{\mathcal{G}}_{\mathcal{H}}) \cap \mathbb{U}$$

and we have

$$\mathrm{Card}(\mathrm{Z}) = \mathrm{Card}\Big(\{s_1/\mathbf{q}^k \in C_0 : k = 1, \dots, d-2\}\Big).$$

We recall that $s_1 = \mathbf{q}^{d-1}s_2$, and so we need to count the number of points $s_1/\mathbf{q}^k$ for $k = 1, \dots, d-2$ which have their argument in $(0, 2\beta)$ modulo 2π. These points can be obtained by making $d-1$ successive rotations of angle -2β, starting from s_1 to s_2 (without taking into account s_1 and s_2). By doing this, the number of complete revolutions around the unit circle in the clockwise direction is $-r$. This comes from the fact that, denoting $\arg s_1 = \theta + 2\varepsilon \in (\theta, \theta + 2\pi)$ and $\arg s_2 = \theta - 2\delta + 2\pi \in (\theta, \theta + 2\pi)$, we have

$$\arg s_2 - \arg s_1 = (d-1)(-2\beta) - r(2\pi) > 0.$$

Since $0 < \theta < \beta$, there are exactly $-r$ points $s_1/\mathbf{q}^k$ for $k = 1, \dots, d-2$ which have their argument in $(0, 2\beta)$ modulo 2π, see Figure 5. Interested readers may refer to the study of mechanical or Sturmian sequences [17] where this kind of counting problem is standard.

(ii) The case $d \leqslant 0$ and $r > 0$ is similar considering $1 - d$ successive rotations of angle 2β from s_1 to s_2 making r turn around the unit circle in the counter-clockwise direction.

□

We now state our second main result about φ_1 when $\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$. A symmetrical result holds for φ_2, and φ can thus be determined by (13).

Theorem 11 (Laplace transforms, $\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$)**.** *Assume that $\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$ and $\alpha \geqslant 1$ and the simple root condition (38) holds. Let d and $r \in \mathbb{Z}$ such that $\alpha = (d-1) + r\pi/\beta$, i.e. $(d-1)\beta + (r+1)\pi = \delta + \varepsilon$, then*

$$\varphi_1(y) = \frac{S(w(y))}{Q(y)},$$

where S is a rational function of degree $-r$ given by

$$S(z) := \prod_{q \in \mathrm{Z}} \frac{z - w(q)}{w(0) - w(q)}, \text{ if } d \geqslant 2 \quad \text{and} \quad S(z) := \prod_{p \in \mathrm{P}} \frac{w(0) - w(p)}{z - w(p)}, \text{ if } d \leqslant 0. \tag{43}$$

We deduce sufficient conditions for φ_1, φ_2 and φ to belongs to the hierarchy (6). If $\alpha \in \mathbb{N}^*$ these Laplace transforms are rational, if $\pi/\beta \in \mathbb{Q}$ and $\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$ they are algebraic, if $\alpha \in \mathbb{N}^* + \frac{\pi}{\beta}\mathbb{Z}$ they are D-finite and if $\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$ they are D-algebraic.

Proof. Recall the definitions of Z in (41), P in (42) and Q in (31) and (32). The function $(Q\varphi_1)/(S \circ w)$ is continuous on $\mathcal{H}$ and meromorphic on $\mathcal{G}_{\mathcal{H}}$. By (28) and (29), we have for all $y \in \mathcal{H}$,

$$\frac{Q\varphi_1(\overline{y})}{S(w(\overline{y}))} = \frac{Q\varphi_1(y)}{S(w(y))}.$$

The function $(Q\varphi_1)/(S \circ w)$ is then an invariant in the sense of Definition 5. Recall that $\deg(Q) = d$, $\deg(S) = -r$ by Lemma 10, $\varphi_1(y) \sim Cy^{-\alpha-1}$ by (24), $w(y) \sim \hat{C}y^{\pi/\beta}$ by (30). For a constant κ we obtain when $y \to \infty$,

$$\frac{Q\varphi_1(y)}{S(w(y))} \sim \kappa \frac{y^d y^{-\alpha-1}}{y^{-r\pi/\beta}} = \kappa$$

where the last equality comes from $\alpha = (d-1) + r\pi/\beta$. By construction, $(Q\varphi_1)/(S \circ w)$ does not have any pole on $\overline{\mathcal{G}}_{\mathcal{H}}$. Indeed if $d \geqslant 2$ the roots of Q compensate the poles of $1/(S \circ w)$ and if $d \leqslant 0$ the zeros of $1/(S \circ w)$ compensate the poles of $Q\varphi_1$. Then, the invariant Lemma 6 assures that

$$\frac{Q\varphi_1(y)}{S(w(y))} = \kappa.$$

Applying again the final value theorem (39) and using the fact that $\lim_{y \to 0} Q(y)/y = Q'(0) = 1$ and $S(w(0)) = 1$ we obtain the value of the constant: $\kappa = \frac{Q(y)}{y} \frac{y\varphi_1(y)}{S(w(y))} \underset{y \to 0}{\longrightarrow} 1 = \kappa$. The sufficient conditions given in the theorem therefore follow from the properties of w stated below (30). □

4 Absorption probability via compensation approach

This section deals with the case where $\alpha \in \mathbb{N}^*$. The aim is to show that the absorption probability is a sum of exponentials and to calculate precisely all the coefficients of this sum. To that end we invert the Laplace transforms and we explain the recursive compensation phenomenon which appears in this sum.

Inverse Laplace transform When $\alpha \in \mathbb{N}^*$ and the simple root condition (38) holds, we invert the Laplace transforms φ_1 and φ_2 obtained in Theorem 9 by performing a partial fraction decomposition. Therefore, remembering that φ_1 and φ_2 are defined in (12) as the Laplace transforms of the escape probability on the boundaries, the absorption probabilities starting from the boundaries can be written as sum-of-exponential and are explicitly given by

$$\mathbb{P}_{(u,0)}(T<\infty) = \sum_{i=0}^{\alpha-1} d_i \exp\left(\mathbf{x}(s_2\mathbf{q}^i)u\right), \quad \mathbb{P}_{(0,v)}(T<\infty) = \sum_{j=0}^{\alpha-1} e_j \exp\left(\mathbf{y}(s_1/\mathbf{q}^j)v\right), \tag{44}$$

where

$$d_i = \frac{-1}{P'(\mathbf{x}(s_2\mathbf{q}^i))}, \quad e_j = \frac{-1}{Q'(\mathbf{y}(s_1/\mathbf{q}^j))}.$$

However, inverting the bivariate Laplace transform $\varphi(x,y)$ and computing the coefficients involved is not immediately obvious. We now state the last main result of this article.

Theorem 12 (Sum-of-exponential absorption probability)**.** *Let $(Z_t)_{0\leqslant t\leqslant T}$ a reflected Brownian motion in the quadrant of drift $\mu \in \mathbb{R}_+^2$, such that $\alpha \geqslant 1$, starting from $(u,v) \in \mathbb{R}_+^2$, where T is the first hitting time of the vertex. We assume that for all $j \in \{1,\dots,2\alpha-2\}$, $\theta - 2\delta + j\beta \not\equiv 0 \bmod(\pi)$. The following statements are equivalent:*

(i) $\alpha = n$, for some integer $n \geqslant 1$;

(ii) there exist coefficients $a_1,\dots,a_{2n-1},b_1,\dots,b_{2n-1},c_1,\dots,c_{2n-1}$ such that

$$\mathbb{P}_{(u,v)}(T<\infty) = \sum_{k=1}^{2n-1} c_k \exp(a_k u + b_k v). \tag{45}$$

In this case, the constants a_k and b_k are given by

$$(a_{2k},b_{2k}) := \left(\mathbf{x}\left(s_1/\mathbf{q}^k\right), \mathbf{y}\left(s_1/\mathbf{q}^k\right)\right) \quad \text{and} \quad (a_{2k+1},b_{2k+1}) := \left(\mathbf{x}\left(s_1/\mathbf{q}^{k+1}\right), \mathbf{y}\left(s_1/\mathbf{q}^k\right)\right) \tag{46}$$

and can also be computed thanks to the recurrence relationship stated in Proposition 14. The coefficients c_k are determined by the recurrence relationship given in Proposition 15.

Proof. First we assume that $\alpha = n$ and for all $j \in \{1,\dots,2\alpha-2\}$, $\theta-2\delta+j\beta \not\equiv 0 \bmod(\pi)$. Theorem 9 gives an explicit expression of the Laplace transform φ. Performing a partial fraction decomposition of $\varphi(x,y) = \frac{L(x,y)}{P(x)Q(y)}$ it is possible to invert the Laplace transform. Recall the definition of P and Q given in (31) and remark that $\{s_2\mathbf{q}^i : i=0,\dots,\alpha-1\} = \{s_1/\mathbf{q}^i : i=1,\dots,\alpha\}$ since $s_1 = s_2\mathrm{q}^\alpha$, we obtain α^2 constants $\tilde{c}_{i,j}$ such that

$$f(u,v) = \mathbb{P}_{(u,v)}(T<\infty) = \sum_{i=1}^{\alpha}\sum_{j=0}^{\alpha-1} \tilde{c}_{i,j} \exp\Big(\mathbf{x}(s_1/\mathbf{q}^i)u + \mathbf{y}(s_1/\mathbf{q}^j)v\Big). \tag{47}$$

Actually, only $2\alpha-1$ of those constants $\tilde{c}_{i,j}$ are non-zero. More precisely, we are now going to show that if $i \notin \{j, j+1\}$ then $\tilde{c}_{i,j} = 0$.

Considering (47), the partial differential equation (9) leads to

$$0 = \mathcal{G}f(u,v) = \sum_{i=1}^{\alpha}\sum_{j=0}^{\alpha-1} \tilde{c}_{i,j} K\Big(\mathbf{x}(s_1/\mathbf{q}^i), \mathbf{y}(s_1/\mathbf{q}^j)\Big) \exp\Big(\mathbf{x}(s_1/\mathbf{q}^i)u + \mathbf{y}(s_1/\mathbf{q}^j)v\Big).$$

By linear independence of the exponential functions (the coefficients inside the exponentials are all different by (38)), this implies that $\tilde{c}_{i,j} = 0$ when $K\Big(\mathbf{x}(s_1/\mathbf{q}^i), \mathbf{y}(s_1/\mathbf{q}^j)\Big) \neq 0$. By (19) this must hold for all (i,j) such that $s_1/\mathbf{q}^i \notin \{s_1/\mathbf{q}^j, s_1/\mathbf{q}^{j+1}\}$ and then for $i \notin \{j, j+1\}$.

For $i \in \{0,\dots,\alpha\}$ we set the constants $c_{2i} = \tilde{c}_{i,i}$ and $c_{2i+1} = \tilde{c}_{i,i-1}$ and we obtain (45). Proposition 15 will give recurrence formulas satisfied by these constants.

Reciprocally, if the absorption probability $f(u,v)$ is a sum of exponentials then $f(0,v)$ is also a sum of exponentials where we denote m the number of distinct exponentials in this sum. We deduce that $\varphi_1(y)$, which is the Laplace transform of $g(0,v) = 1 - f(0,v)$, is therefore equivalent up to a multiplicative constant to y^{-m-1} when $y \to \infty$. We also know by (24) that $\varphi_1(y)$ is equivalent up to a multiplicative constant to $y^{-\alpha-1}$, which implies that $\alpha = m \in \mathbb{N}$. □

In what follows it will be convenient to denote by k_1^* and k_2^* the following functions

$$k_1^*(x,y) := (x,y) \cdot R^1 = x - r_1 y, \qquad k_2^*(x,y) := (x,y) \cdot R^2 = -r_2 x + y \tag{48}$$

as they naturally appear when applying ∂_{R^1} and ∂_{R^2}, see (10), to a function of the form e^{au+bv}.

Lemma 13. *The functions k_1, k_2, k_1^* and k_2^* satisfy the following relations*

$$k_1 \circ (\mathbf{x},\mathbf{y})(s) = -\frac{\sigma_{11}}{2} k_1^* \circ (\mathbf{x},\mathbf{y})(\mathbf{q}s^{-1}), \qquad k_2 \circ (\mathbf{x},\mathbf{y})(s) = -\frac{\sigma_{22}}{2} k_2^* \circ (\mathbf{x},\mathbf{y})(s^{-1}).$$

As a direct consequence we have, for some constants C_1^* and C_2^*, for all $s \in \mathbb{C}^*$,

$$k_1^*(\mathbf{x}(s),\mathbf{y}(s)) = C_1^* \frac{(s-s_0^*)(s-s_1^*)}{s}, \qquad k_2^*(\mathbf{x}(s),\mathbf{y}(s)) = C_2^* \frac{(s-s_0^*)(s-s_2^*)}{s}$$

where $s_0^* := s_0''^{-1} = \mathbf{q}s_0'^{-1}$, $s_1^* := \mathbf{q}s_1^{-1}$ and $s_2^* := s_2^{-1}$.

Proof. It is equivalent to prove that for all x and y, $k_1(X^+(y),y) = -\frac{\sigma_{22}}{2} k_1^*(X^-(y),y)$ and $k_2(x,Y^+(x)) = -\frac{\sigma_{11}}{2} k_2^*(x,Y^-(x))$ which can be easily verified using the definitions (16), (14) and (48). Then, Lemma 2 allows us to conclude. See Figure 3 for a geometric interpretation. □

The following proposition establishes a recurrence relationship which allows to compute (a_k,b_k). It gives a very natural geometric interpretation of this sequence of points which belongs to the ellipse $\mathcal{E} := \mathcal{S} \cap \mathbb{R}^2 = \{(x,y) \in \mathbb{R}^2 : K(x,y) = 0\}$, starts at the intersection with the line $\{k_1^* = 0\}$ and ends at the intersection with the line $\{k_2^* = 0\}$. It can be visualized in Figure 6.

Proposition 14 (Recursive relationship of the sequence (a_k,b_k))**.** *The sequence $(a_k,b_k) \in \mathcal{E}$ defined in (46) satisfies the following relations*

$$(a_{2k+1},b_{2k+1}) = \left(\frac{b_{2k}}{a_{2k}} \cdot \frac{\sigma_{22}b_{2k} + 2\mu_2}{\sigma_{11}}, b_{2k}\right), \quad (a_{2k+2},b_{2k+2}) = \left(a_{2k+1}, \frac{a_{2k+1}}{b_{2k+1}} \cdot \frac{\sigma_{11}a_{2k+1} + 2\mu_1}{\sigma_{22}}\right)$$

where $b_1 = \mathbf{y}(s_1) = \mathbf{y}(s_1^*)$ was computed in (23), $a_1 = \mathbf{x}(s_1/\mathbf{q}) = \mathbf{x}(s_1^*) = r_1 b_1$ and $k_1^*(a_1,b_1) = 0$. We also have $k_2^*(a_{2\alpha-1},b_{2\alpha-1}) = 0$ and one can easily compute

$$a_{2\alpha-1} = \mathbf{x}(s_2^*) = -\frac{2(\mu_1 + r_2\mu_2)}{\sigma_{11} + \sigma_{22}r_2^2 + 2\sigma_{12}r_2} \text{ and } b_{2\alpha-1} = \mathbf{y}(s_2^*) = r_2 a_{2\alpha-1}.$$

Proof. By definition (46) we have $b_{2k} = b_{2k+1}$ and $a_{2k} \neq a_{2k+1}$. Furthermore $K(a_{2k},b_{2k}) = K(a_{2k+1},b_{2k+1}) = 0$, then a_{2k} and a_{2k+1} must be the two distinct roots of the quadratic equation $K(x,b_{2k}) = 0$. Vieta's formula gives the value of the product of those roots in terms of the coefficients of the equation and we get the relation for (a_{2k+1},b_{2k+1}). The same method applies for the second relation about (a_{2k+2},b_{2k+2}). □

The aim is now to compute explicitly the coefficients c_k which appear in Theorem 12.

Proposition 15 (Recursive relationship of the sequence c_k)**.** *We assume $\alpha \in \mathbb{N}^*$ and condition (38). We recall that the constants a_k, b_k are defined in (46). The constants c_k introduced in Theorem 12 are determined by the recurrence relations*

$$c_{2k} = -c_{2k-1}\frac{k_2^*(a_{2k-1},b_{2k-1})}{k_2^*(a_{2k},b_{2k})} \text{ and } c_{2k+1} = -c_{2k}\frac{k_1^*(a_{2k},b_{2k})}{k_1^*(a_{2k+1},b_{2k+1})}$$

and the normalization relationship $\sum_{k=1}^{2\alpha-1} c_k = 1$.

Proof. For the first relation, let us observe that

$$\partial_{R^2} \exp(au+bv)|_{v=0} = k_2^*(a,b)\exp(au).$$

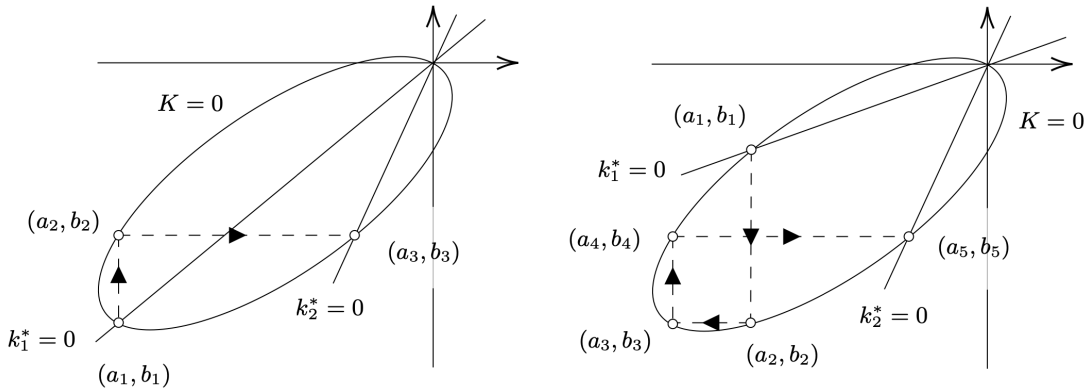

Figure 6: Construction of the finite sequences $\{(a_i,b_i) : 1 \leqslant i \leqslant 2\alpha-1\}$. *On the left* $\alpha = 2$ *while* $\alpha = 3$ *on the right.*

We denote $f(u,v) = \mathbb{P}_{(u,v)}(T<\infty)$. Using Theorem 12, noticing that $a_{2k-1} = a_{2k}$, we evaluate $\partial_{R^2} f$ at $v=0$ and the Neumann condition (10) gives

$$0 = \sum_{k=1}^{\alpha-1} \Big(c_{2k-1} k_2^*(a_{2k-1},b_{2k-1}) + c_{2k} k_2^*(a_{2k},b_{2k}) \Big) \exp(a_{2k}u) + k_2^*(a_{2\alpha-1},b_{2\alpha-1}) \exp(a_{2\alpha-1}u). \quad (49)$$

By Lemma 13 we see that $k_2^*(a_{2\alpha-1},b_{2\alpha-1}) = k_2^* \circ (\mathbf{x},\mathbf{y})\,(s_2^*) = 0$ so that the last term in (49) is zero. Under the simple roots condition (38), $a_{2i} \neq a_{2j}$ for all $i \neq j$, the family $\{u \mapsto \exp(a_{2k}u)\}$ is therefore linearly independent and for all k we obtain

$$c_{2k-1} k_2^*(a_{2k-1},b_{2k-1}) + c_{2k} k_2^*(a_{2k},b_{2k}) = 0.$$

The proof of the second relation is similar. The normalization comes from the fact that $f(0,0)=1$. □

The following paragraph aims to give a geometric interpretation to all the coefficients a_k, b_k and c_k and to explain the compensation phenomenon which appears in the sum of exponentials.

Heuristic of the compensation approach Using a recursive compensation method (with a finite number of iterations), it is possible to find a solution to the partial differential equation stated in (9) and (10) that is a candidate for being the probability of absorption at the vertex. It is interesting to remark that the positivity of this solution is by no means obvious and that the uniqueness of the solution of this kind of PDE usually requires the positivity of the solution.

In this paragraph, we explain the compensation phenomenon. By using an analytic approach, we showed in Theorem 12 that when $\alpha \in \mathbb{N}^*$ and (38) holds the absorption probability is

$$f(u,v) = \mathbb{P}_{(u,v)}(T<\infty) = \sum_{k=1}^{2\alpha-1} c_k \exp(a_k u + b_k v),$$

where the (a_k,b_k) are determined in Proposition 14 and the c_k in Proposition 15.

We define the following function vector spaces

$$E_0 := \{h \in \mathcal{C}^2(\mathbb{R}_+^2) :\ \mathcal{G}h = 0 \text{ on } \mathbb{R}_+^2\},$$

$$E_1 := \{h \in \mathcal{C}^2(\mathbb{R}_+^2) :\ \partial_{R^1} h(0,\cdot\,) = 0 \text{ on } \mathbb{R}_+\} \text{ and } \ E_2 := \{h \in \mathcal{C}^2(\mathbb{R}_+^2) :\ \partial_{R^2} h(\cdot\,,0) = 0 \text{ on } \mathbb{R}_+\}.$$

One may remark that a function h satisfies the PDE (9) if and only if $h \in E_0$ and h satisfy the Neumann boundary conditions (10) if and only if $h \in E_1 \cap E_2$. Furthermore, the function $(u,v) \mapsto e^{au+bv}$ belong to E_0 if and only if $K(a,b)=0$, belongs to E_1 if and only if $k_1^*(a,b)=0$, and belongs to E_2 if and only if $k_2^*(a,b)=0$.

By Proposition 14 all the (a_k,b_k) are on the ellipse $\mathcal{E}$ defined by $K=0$, it is then easy to understand why $f \in E_0$, i.e. why f satisfies the partial differential equation (9).

We are now seeking to understand why the coefficients c_k given in Proposition 15 ensure that $f \in E_1 \cap E_2$, i.e. why f satisfies the Neumann boundary conditions (10). In fact, the c_k have been

chosen such that by grouping the terms of the sum by pairs (except the first or the last term) they *compensate* each other to ensure the inclusions in E_1 and E_2:

$$f(u,v) = \underbrace{c_1 \exp(a_1 u + b_1 v)}_{\in E_1} + \sum_{k=1}^{\alpha-1} \underbrace{c_{2k} \exp(a_{2k} u + b_{2k} v) + c_{2k+1} \exp(a_{2k+1} u + b_{2k} v)}_{\in E_1}$$
$$= \sum_{k=1}^{\alpha-1} \underbrace{c_{2k-1} \exp(a_{2k} u + b_{2k-1} v) + c_{2k} \exp(a_{2k} u + b_{2k} v)}_{\in E_2} + \underbrace{c_{2\alpha-1} \exp(a_{2\alpha-1} u + b_{2\alpha-1} v)}_{\in E_2}$$

so that $f \in E_1 \cap E_2$. This is due to the fact that $(c_{2k} k_1^*(a_{2k}, b_{2k}) + c_{2k+1} k_1^*(a_{2k+1}, b_{2k}))e^{b_{2k} v} = 0$ and $(c_{2k-1} k_2^*(a_{2k}, b_{2k-1}) + c_{2k} k_2^*(a_{2k}, b_{2k}))e^{a_{2k} u} = 0$.

We now understand the phenomenon of compensation which explains why f is a solution of the partial differential equation (9) with Neumann boundary conditions (10). One may also verify that the limit conditions (11) are also satisfied. Let us note, on the other hand, that the positivity of this function is absolutely not obvious to check.

Double roots. This last paragraph deals with the case where P or Q have double roots, *i.e.* when for some integer $j \in \{1, \dots, 2\alpha - 2\}$, $\theta - 2\delta + j\beta \equiv 0 \bmod(\pi)$, see Lemma 8. The number of cases to handle to give a general explicit formula is too big. Nonetheless, we can give the general shape of the absorption probability: if $\alpha \in \mathbb{N}^*$, the absorption probability can be written as

$$\mathbb{P}_{(u,v)}(T < \infty) = \sum_{k=0}^{2\alpha-1} A_k(u,v) \exp(a_k u + b_k v)$$

where a_k and b_k are given in Equation (46) and the A_k are affine functions of u and v. Indeed, Theorem 9 holds even when there are multiple roots. Inverting the Laplace transform we show that the absorption probability can be written $\sum_{k=0}^{2\alpha-1} A_k(u,v) \exp(a_k u + b_k v)$ where the A_k are polynomials. A direct calculation shows that P and Q can't have triple roots. This proves that the total degree of A_k is less than 1 for all k. We can also give an intuitive explanation for the fact that there are no triple roots: the geometric interpretation tells us that the sequence (a_k, b_k) cannot visit a point thrice, otherwise it would loop indefinitely.

The case where $\alpha = 2$ is completely solved below as an example.

Example 16 (Double roots, $\alpha = 2$)**.** *For $\alpha = 2$, we distinguish two cases with double roots*

- *if $\theta - 2\delta + \beta = -\pi$ then*

$$\mathbb{P}_{(u,v)}(T < \infty) = \Big(1 + c\Big) \exp\Big(\mathbf{x}(s_2)u + \mathbf{y}(s_1)v\Big) - \Big(\mathbf{x}(s_2)u + c\Big) \exp\Big(\mathbf{x}(s_2)u + \mathbf{y}(s_1/\mathbf{q})v\Big)$$

- *if $\theta - 2\delta + 2\beta = -\pi$ then*

$$\mathbb{P}_{(u,v)}(T < \infty) = -\Big(\mathbf{y}(s_1)v + c\Big) \exp\Big(\mathbf{x}(s_2)u + \mathbf{y}(s_1)v\Big) + \Big(1 + c\Big) \exp\Big(\mathbf{x}(s_2\mathbf{q})u + \mathbf{y}(s_1)v\Big)$$

where $c = \dfrac{1}{r_1 r_2 - 1}$.

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