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► **To cite this version:**

| Daniel Barlet. Some comments on my study of period-integrals. 2024. hal-04510893

HAL Id: hal-04510893

<https://hal.science/hal-04510893>

Preprint submitted on 19 Mar 2024

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Some comments on my study of period-integrals

Daniel Barlet.*

19 Mars 2024

ABSTRACT. This text is a presentation (without proofs) of some of my recent results on the singular terms of asymptotic expansions of period-integrals using (a,b)-modules. I try to explain why this simple algebraic structure is interesting and useful.

AMS CLASSIFICATION. 32 S 25; 32 S 40 ; 34 E 05

KEY WORDS. Asymptotic Expansions ; Period-integral; Convergent (a,b)-Module; Hermitian Period; Geometric (convergent) (a,b)-Modules; (convergent) Fresco; Bernstein Polynomial ; Higher Order Bernstein Polynomials.

1 Introduction

The aim of this text is devoted to give some comments on my work since more than thirty years on the local study of the singularity of a holomorphic function in a complex manifold.

As my point of view seems not very popular and looks at least strange and often few interesting for many specialists of this subject, I shall try to explain why it is pertinent and interesting¹.

In particular I shall try to explain how the use of the structure of (a,b)-module may be seen as a tentative of having a "local Hodge Theory" even if, at first glance, such a notion looks meaning less. Note that I do not pretend that this point of view is an alternative to the use of desingularization and usual (global) Hodge Theory (or Mixte Hodge modules [29]), but I think that it may be a complementary approach

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¹But the notion of "interesting result" is not so easy to define in an objective way. Only long range vision may give a serious answer to such a question.

and can give interesting and more precise informations when we are interested in a specific period integral.

2 The context

Consider a holomorphic (non constant) germ $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ with an isolated singularity for the eigenvalue $r := \exp(-2i\pi\alpha)$ of the monodromy (we assume that α is in $]0, 1] \cap \mathbb{Q}$). Then, for a given homology class $\gamma \in H_n(F_0, \mathbb{C})_r$, where F_0 is the Milnor fiber of f at 0 and where $H_n(F_0, \mathbb{C})_r$ is the generalized eigenspace for the eigenvalue r of the monodromy of f acting on $H_n(F_0, \mathbb{C})$, we defined the period-integral

$$\Phi_{\omega, \gamma}(s) := \int_{\gamma(s)} \omega / df \quad (1)$$

where ω is any germ in Ω_0^{n+1} , where s is in H the universal cover of a small punctured disc D^* with center 0 in \mathbb{C} and where $(\gamma(s))_{s \in H}$ is the horizontal family of compact n -cycles in the fibers of f taking the value γ at the base point s_0 in H .

Such multivalued holomorphic function admits an asymptotic expansion at the origin (convergent in sectors) for α, γ, ω given, which looks like

$$\Phi_{\omega, \gamma}(s) \simeq \sum_{j \in [0, n], m \in \mathbb{N}} \varphi_{m, j}(\gamma) s^{\alpha+m-1} (\text{Log } s)^j \quad (2)$$

To understand for which γ, j, m we have $\varphi_{m, j}(\gamma) \neq 0$ in such an expansion is a difficult problem (but interesting to my point of view).

The first difficulty is to represent the compact n -cycle γ (and the horizontal family deduced from it) in order to have a "concrete" integral to compute. Our approach is slightly different and I choose to consider a family of cohomology classes in the fibers $f^{-1}(s)$ induced by a given germ $\omega' \in \Omega_0^{n+1}$, that is to say replace the (holomorphic) period-integral $\int_{\gamma(s)} \omega / df$ by the hermitian period

$$\Psi_{\omega, \omega'}(s) := \int_{f^{-1}(s)} \rho \omega / df \wedge \overline{\omega' / df}$$

where $\rho \in \mathcal{C}_c^\infty(\mathbb{C}^{n+1})$ is identically 1 near the origin and with a small enough support. Then, thanks to an old result (see [2]) the function $s \mapsto \Psi_{\omega, \omega'}(s)$ is smooth on D^* (small enough) and admits an asymptotic expansion in the space

$$|\Xi|_{\mathcal{A}}^{(n)} := \sum_{\alpha \in \mathcal{A}, j \in [0, n]} \mathbb{C}[[s, \bar{s}]] |s|^{2\alpha-2} (\text{Log } |s|^2)^j$$

where \mathcal{A} is a finite subset in $]0, 1] \cap \mathbb{Q}$.

To consider an auxiliary germ $\omega' \in \Omega_0^{n+1}$ is a way to induced an anti-holomorphic family of cohomology classes $\overline{\omega'}/df$ in $H^n(f^{-1}(s), \mathbb{C})_r^2$.

This presents the advantage to have an integral on the fibers of f (so no more "topological cycle" to find in $f^{-1}(s)$) with also the fact that the non vanishing of some $\varphi_{j,m}(\gamma)$ implies the existence of some $\omega' \in \Omega_0^{n+1}$ such that the asymptotic expansion of $\Psi_{\omega, \omega'}$ presents a non zero term³ like $|s|^{2\alpha-2} s^m \bar{s}^{m'} (\text{Log}|s|^2)^j$ for some $m' \in [0, n+1]$.

Using complex Mellin transform of this function (see [17] and [18]) this is equivalent to the existence of a pole of order at least j at the point $-\alpha - m$ for the meromorphic extension of the function

$$F_{m'}^{\omega, \omega'}[\lambda] := \frac{1}{\Gamma(\lambda)} \int_{\mathbb{C}^{n+1}} |f|^{2\lambda} \bar{f}^{-m'} \rho \omega \wedge \bar{\omega}' \quad (3)$$

holomorphic for $\text{Re}(\lambda) > 0$ and whose meromorphic extension to the complex line is consequence of the existence of the Bernstein-Sato polynomial of f (see [19] and [2] or [20]).

So, our results giving necessary or sufficient conditions for the existence of such a pole in our papers [14] and [15] are in fact results about non zero terms in the asymptotic expansion (1) of the integral period.

3 (a,b)-modules and asymptotic expansions

Now let me explain the reasons to use (a,b)-modules in this context. The first one comes from a remark due to Kyoji Saito (in [27], but see also [26] and [28]) that the Brieskorn module H_0^{n+1} of an isolated singularity germ $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ has a natural structure of $\mathbb{C}\{\{\partial_s^{-1}\}\}$ -module which is free of rank μ (the Milnor number), where $\mathbb{C}\{\{\partial_s^{-1}\}\}$ is the \mathbb{C} -algebra of Gevrey series⁴.

This structure, with also the obvious $\mathbb{C}\{s\}$ -module structure gives a way to look at the Gauss-Manin connexion, that is to say to consider the germ of the differential system at the origin in \mathbb{C} whose multivalued solutions are the period-integrals (1) for any germ $\omega \in \Omega_0^{n+1}$. Recall that in the (totally) isolated singularity case the Brieskorn module⁵ is simply given by

$$H_0^{n+1} := \Omega_0^{n+1} / df \wedge d\Omega_0^{n-1}.$$

²The use of the canonical hermitian form on $H^n(F_0, \mathbb{C})$ is needed for $\alpha = 1$ in the case of an isolated singularity for the eigenvalue r to replace the compact support condition in the usual pairing $H^n(f^{-1}(s), \mathbb{C}) \times H_c^n(f^{-1}(s), \mathbb{C}) \rightarrow \mathbb{C}$ corresponding to the hermitian intersection form for $\alpha \neq 1$ see [6] or [8].

³For $\alpha = 1$ we consider only terms with $j \geq 1$, so singular terms.

⁴microdifferential operators of order ≤ 0 with constant coefficients.

⁵For any holomorphic germ we may define for each degree $p \in [0, n]$ an (a,b)-module corresponding to the Gauss-Manin system in degree p via the $(p+1)$ -th cohomology sheaf of the de Rham complex $(\text{Ker}(\wedge df)^\bullet, d^\bullet)$ on $Y = f^{-1}(0)$. See [9], [10] or [16]. But we discuss here only the case $p = n$.

In our setting (isolated singularity for the eigenvalue $r := \exp(-2i\pi\alpha)$) the Brieskorn module is defined as the quotient by its $b := \partial_s^{-1}$ -torsion of the quotient

$$\mathcal{H}_0^{n+1} := \Omega_0^{n+1} / d(\text{Ker}(\wedge df))^n \quad \text{where} \quad \wedge df : \Omega_0^n \rightarrow \Omega_0^{n+1}.$$

A serious reason to introduce the (a,b)-module structure (which is nothing else than the simultaneous use of a $\mathbb{C}\{s\}$ and a $\mathbb{C}\{\{\partial_s^{-1}\}\}$ -module structures which are "compatible") is the fact that the Gauss-Manin-connection (so the action of ∂_s) is meromorphic and then compels to introduce denominators in s while the action of ∂_s^{-1} (the primitive without constant in s) is already well defined on the $\mathbb{C}\{s\}$ -module H_0^{n+1} and do not need any denominator in s .

But what is the benefice not to invert s and ∂_s^{-1} ??

The answer is very simple : if you look at the action of s and of ∂_s^{-1} on polynomials $\mathbb{C}[s]$ or on holomorphic germs $\mathbb{C}\{s\}$ you immediately see that the filtration by the $b^p \mathbb{C}\{s\}, p \in \mathbb{N}$, defines a natural filtration by sub-modules for both actions of a and b . It is an easy consequence of the commutation relation

$$ab - ba = b^2 \quad \text{where} \quad a := \times s \quad \text{and} \quad b = \partial_s^{-1} := \int_0^s$$

But here it is important to remark that **the filtration defined by $a^p H_0^{n+1}, p \in \mathbb{N}$, is not stable by the action of b** in general⁶. So the b -module structure has to be considered as the "primary one" which is not the usual point of view!

Moreover it has been shown by Varchenko (see [31]) that in the case of an isolated singularity this b -filtration corresponds to the Hodge filtration of the mixte Hodge structure defined by Stenbrink (see [30]) on the cohomology $H^n(F_0, \mathbb{C})$ in this context.

Since the Gauss-Manin connection is regular, every formal solution converges, I decided to explore the role of this bi-structure module in a formal setting and I choose the following definition⁷

Definition 3.0.1 *An (a,b)-module E is a free finite type $\mathbb{C}[[b]]$ -module endowed with a \mathbb{C} -linear endomorphism "a" satisfying the commutation relation $ab - ba = b^2$ and which is continuous for the topology defined by b -adic filtration $(b^p E)_{p \geq 0}$ of E .*

In particular we have $aS(b)e = S(b)ae + b^2 S'(b)e$ for any $e \in E$ and any $S \in \mathbb{C}[[b]]$.

Then it is very simple to construct an (a,b)-module: take a rank k free $\mathbb{C}[[b]]$ -module E with basis e_1, \dots, e_k and define "a" by prescribing the value of $a(e_j) \in E, j \in [1, k]$

⁶Remark that the Nullstellensatz gives an integer N such that $f^N \Omega_0^{n+1} \subset df \wedge \Omega_0^n$ and so $a^N H_0^{n+1} \subset b^N H_0^{n+1}$. So the b -filtration is finer than the a -filtration.

⁷Let me say that the "strange choice" to call $a := \times s$ and $b := \partial_s^{-1}$ was motivated by the following considerations: if you keep the usual notations you risk, at some point, to use an obvious "usual formula" in a context where it is not "obvious". And my interest was to see precisely what comes only from the commutation relation $ab - ba = b^2$ in the study of period-integrals. This choice is also useful to understand where the use of the inverse of b is interesting. It also helps me to convince myself that b is the "primary" variable, which is not so easy to accept !

as you want. Then this define uniquely an (a,b) -module structure on E .
The rank of E on $\mathbb{C}[[b]]$ is called the rank of E .

Since the most easy case for a regular connection is the case of a simple pole (this happens for the Gauss-Manin connection of a quasi-homogeneous polynomial) this leads to the following definition.

Definition 3.0.2 *An (a,b) -module has a **simple pole** if and only if $aE \subset bE$.*

Remark that under this hypothesis (in fact the existence of an integer N such that $a^N E \subset bE$ is enough) the (a,b) -module E is a -complete, so is a module over $\mathbb{C}[[a]]$.

Then the following question makes sense and is, of course, fundamental:

- Given a formal simple pole differential system $z\partial_z F(z) = A(z)F(z)$ where A is a (k, k) -matrix with entries in $\mathbb{C}[[z]]$ and F an "unknown element" in $\mathbb{C}[[z]]^k$, does there exists an (a,b) -module E representing such a differential system (in a sense to be precise) ?

The answer to this question is yes, and the relation from the matrix A to the corresponding (a,b) -module is conceptually simple : take $e := \partial_z F$ and look for X a (k, k) -matrix with entries in $\mathbb{C}[[b]]$ such that $ae = bX(b)e$. Proposition 1.1 in [7] gives the existence and uniqueness of a matrix X such that, in the formal completion "in a " of E (but remember that E is in fact already a -complete!) the equality $ae = A(a)be$ holds true in E .

But to compute explicitly the matrix X from the matrix A is very complicated in general. In some sense, this complexity may explain partly why the (a,b) -module point of view can really give interesting results and is not a simple game playing with a change of notations.

NOTATION. An (a,b) -module is in fact a left module over a non commutative algebra \mathcal{A} which contains both $\mathbb{C}[[b]]$ and $\mathbb{C}[a]$ which is defined as follows:

$$\mathcal{A} := \left\{ \sum_{q=0}^{\infty} P_q(a)b^q \right\} \quad \text{where} \quad P_q \in \mathbb{C}[a] \quad \forall q \in \mathbb{N}$$

where the commutation relation $ab - ba = b^2$ allows to write any element in \mathcal{A} (in a unique way) as $\sum_{q=0}^{\infty} b^q \Pi_q(a)$ where Π_q is also in $\mathbb{C}[a]$. Then it is clear that a left \mathcal{A} -module is both a $\mathbb{C}[[b]]$ -module and a $\mathbb{C}[a]$ -module.

Now an (a,b) -module is simply a left \mathcal{A} -modules which is free and of finite type on the sub-algebra $\mathbb{C}[[b]]$ of \mathcal{A} .

The regularity condition defined below implies that the left action of $\mathbb{C}[a]$ on a regular (a,b) -module extends to an action of $\mathbb{C}[[a]]$.

Then it is easy to define a "regular" (a,b) -module to encode regularity of a connection (recall that Gauss-Manin connection is always regular) as follows:

Definition 3.0.3 An (a,b) -module E is **regular** when it is a sub-module of a simple pole (a,b) -module.

It is easy to see that when E is regular there exists a smallest canonical simple pole (a,b) -module E^\sharp containing a given regular (a,b) -module E which is the saturation by $b^{-1}a$ of E in $K \otimes_{\mathbb{C}[[b]]} E$ where $K := \mathbb{C}[[b]][[b^{-1}]]$.

The next step is to define the Bernstein polynomial of a regular (a,b) -module.

Definition 3.0.4 For a simple pole (a,b) -module E the **Bernstein polynomial** of E is the minimal polynomial of the action of $-b^{-1}a$ on the finite dimensional vector space E/bE .

For a regular (a,b) -module, its Bernstein polynomial is, by definition, the Bernstein polynomial of E^\sharp .

EXAMPLE. Any regular rank 1 (a,b) -module has always a simple pole and is isomorphic to $E_\lambda := \mathbb{C}[[b]]e_\lambda$ for some $\lambda \in \mathbb{C}$ where $ae_\lambda = \lambda be_\lambda$.

Now to encode the famous result of Kashiwara (see [22]) on the rationality and negativity of the roots of the (usual) Bernstein polynomial (see [19] or [20]) of a holomorphic germ we define "geometric" (a,b) -modules as follows.

Definition 3.0.5 A regular (a,b) -module is **geometric** if and only if the roots of its Bernstein polynomial are negative rational numbers.

Now I shall show that with such a definition we are quite near from our initial point of view which was the study of the asymptotic expansion of a period-integral.

Definition 3.0.6 Let α be in $]0, 1] \cap \mathbb{Q}$ and define for $N \in \mathbb{N}$ the (a,b) -module :

$$\begin{aligned} \Xi_\alpha^{(N)} &:= \bigoplus_{j=0}^N \mathbb{C}[[b]]e_j \quad \text{where} \\ ae_j &= \alpha b(e_j + e_{j-1}) \quad \text{with the convention } e_{-1} = 0. \end{aligned}$$

It is easy to see that for $\alpha \neq 1$ we may identify $\Xi_\alpha^{(N)}$ with the usual asymptotic expansion $\mathbb{C}[[s]]$ -module

$$\left\{ \sum_{j \in [0, N], m \in \mathbb{N}} \varphi_{m,j} s^{\alpha+m-1} (\text{Log } s)^j \right\}$$

with the standard actions $a = \times s$ and $b := \int_0^s$. Moreover it has a simple pole and its Bernstein polynomial is equal to $(x + \alpha)^{N+1}$. So it is a geometric (a,b) -module. For any finite subset $\mathcal{A} \subset]0, 1] \cap \mathbb{Q}$ and any finite dimensional vector space V , the space

$$\Xi_{\mathcal{A}}^{(N)} \otimes_{\mathbb{C}} V := \left(\bigoplus_{\alpha \in \mathcal{A}} \Xi_\alpha^{(N)} \right) \otimes_{\mathbb{C}} V$$

where a acts by $a \otimes id_V$ and b by $b \otimes id_V$ is again a geometric (a,b) -module. Then we have the following "embedding Theorem"

Theorem 3.0.7 Any geometric (a,b) -module E admits an embedding as a sub-module of some $\Xi_{\mathcal{A}}^{(N)} \otimes_{\mathbb{C}} V$. Moreover we may choose \mathcal{A} as the image in $\mathbb{Q}/\mathbb{Z} \simeq]0, 1] \cap \mathbb{Q}$ of the opposite of the roots of the Bernstein polynomial of E , and V of dimension equal to the rank of E ⁸.

Of course, the converse holds true : any left \mathcal{A} sub-module of some $\Xi_{\mathcal{A}}^{(N)} \otimes_{\mathbb{C}} V$ is a geometric (a,b) -module.

4 The use of frescos

To go now to the study of asymptotic expansions for the periods integrals associated to one germ $\omega \in \Omega_0^{n+1}$ (either for any choice of γ or for one choice of γ respectively) the following definition will be used.

Definition 4.0.1 Any sub-module of some $\Xi_{\mathcal{A}}^{(N)} \otimes_{\mathbb{C}} V$ generated (as a left \mathcal{A} -module) by a unique element is called a **fresco**.

When, moreover, the vector space V is one dimensional such a fresco will be called a **theme**.

In the setting described at the beginning, the notion of fresco corresponds to the differential system associated to a given class $[\omega] \in H_0^{n+1}$. To such a class the associated fresco \mathcal{F}_ω is the sub- (a,b) -module of the Brieskorn module H_0^{n+1} generated by the class $[\omega]$ (so $\mathcal{F}_\omega := \mathcal{A}[\omega] \subset H_0^{n+1}$).

The notion of theme corresponds to the choice of the class $[\omega]$ with a choice of a n -cycle γ in $H^n(F_0, \mathbb{C})$.

The frescos have very nice properties :

1. The Bernstein polynomial of a fresco F is the *characteristic polynomial* of $-b^{-1}a$ acting on F^\sharp/bF^\sharp .
2. Assume that $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$ is an exact sequence of (a,b) -modules⁹. Then if E_2 is a fresco, E_1 and E_3 are frescos.
3. In the situation above assuming that E_2 is a fresco, the Bernstein polynomials of E_1, E_2, E_3 satisfy

$$B_{E_2}(x) = B_{E_1}(x+q)B_{E_3}(x) \quad \text{where} \quad q := rk(E_3).$$

The following structure theorem for frescos is very useful¹⁰

⁸In fact we have a better estimate for the minimal dimension of V which is the rank of $S_1(E)$ the semi-simple part of E (see below for the definition).

⁹Note that our definition of an (a,b) -module implies that in such an exact sequence this implies that E_1 is *normal* in E_2 which means that $bE_2 \cap E_1 = bE_1$ in order to avoid b -torsion in E_3 .

¹⁰It allows to give "normal form" for a given fresco and allows to construct a versal family (with finitely many parameters) for themes with given rank and fixed Bernstein polynomial. See [13].

Theorem 4.0.2 (see [11]) *Let \mathcal{F} be a fresco with rank k . Then there exists an element in \mathcal{A}*

$$P := (a - \lambda_1 b)S_1(a - \lambda_2 b)S_2 \dots S_{k-1}(a - \lambda_k b)$$

where S_1, \dots, S_{k-1} are invertible elements in $\mathbb{C}[[b]]$, such that \mathcal{F} is isomorphic to the quotient $\mathcal{A}/P\mathcal{A}$ as a left \mathcal{A} -module. Moreover, in this situation the numbers $-(\lambda_j + j - k)$ are negative rational numbers and the Bernstein polynomial of \mathcal{F} is equal to

$$B_{\mathcal{F}}(x) = \prod_{j=1}^k (x + \lambda_j + j - k).$$

May be it is time to explain to the reader the kind of result which can be proved using these tools.

Theorem 4.0.3 (see [14] and [15]) *Assume that in the situation described at the beginning the fresco \mathcal{F}_ω associated to $\omega \in \Omega_0^{n+1}$ has a root of order $q \geq 1$ in $-\alpha - \mathbb{N}$. Then there exists a form $\omega' \in \Omega_0^{n+1}$ such that the hermitian period*

$$\Psi_{\omega, \omega'}(s) = \int_{f^{-1}(s)} \rho\omega/df \wedge \overline{\omega'/df}$$

has a non zero term in of the form $|s|^{2\alpha-2} s^m \bar{s}^{m'} (\text{Log}|s|^2)^j$ with $j \geq q - 1$ if $\alpha \neq 1$ and with $j \geq q$ for $\alpha = 1$.

Conversely, if such a form ω' exists, then the Bernstein polynomial of \mathcal{F}_ω has at least q roots (distinct or not) in $-\alpha - \mathbb{N}$.

Note that in this situation, this implies that the (usual) reduced Bernstein polynomial $b_{f,0}$ of f at the origin has at least q roots in $-\alpha - \mathbb{N}$ thanks to [2].

5 Semi-simple filtration and Higher Bernstein polynomials

The existence of several roots in $-\alpha - \mathbb{N}$ for the reduced Bernstein polynomial of the holomorphic germ f at the origin may imply the existence of some nilpotent order for the eigenvalue $\exp(-2i\pi\alpha)$ of the monodromy of f at the origin in the case of an isolated singularity. But *this is not always the case*. This phenomena is the same for the Bernstein polynomial of the fresco \mathcal{F}_ω associated to a germ $\omega \in \Omega_0^{n+1}$.

This explains why in the theorem above the hypothesis in the direct part (existence of a root with multiplicity q ...) is stronger than the conclusion of the converse (existence of q roots distinct or not ...).

The goal is now to understand when two different roots in $-\alpha - \mathbb{N}$ for the Bernstein polynomial of the fresco \mathcal{F}_ω really produces a non zero *Log*-term in the expansion of the period integral of ω . This is, of course, an attempt to understand the same question for the roots of the (reduced) usual Bernstein polynomial of f itself.

This problem is clearly linked with the study of the nilpotent part of the action of

the *logarithm of the monodromy*, which is "naturally" defined on the saturation E^\sharp of a geometric (a,b) -module via the action of $b^{-1}a$ (see section 6 in [15]). So this leads to study the semi-simple filtration of a geometric (a,b) -module which we define as follows.

Definition 5.0.1 *An (a,b) -module E is semi-simple if it is a sub-module of a finite direct sum $\bigoplus_{j=1}^N E_{\lambda_j}$ where $E_{\lambda} := \mathbb{C}[[b]]e_{\lambda}$ for $\lambda \in \mathbb{C}$, the action of a being defined by $ae_{\lambda} = \lambda be_{\lambda}$.*

Proposition 5.0.2 *For any regular (a,b) -module E there exists a canonical strictly increasing filtration by normal¹¹ sub-modules $(S_j)_{j \in [0,d]}$ with $S_0(E) := \{0\}$ and also $S_d(E) := E$, such that the quotients $S_{j+1}(E)/S_j(E)$ are semi-simple for each j in $[0, d-1]$. The integer d is called the **nilpotent order** of E .*

Since the problem we have in mind concerns only the eigenvalue $\exp(-2i\pi\alpha)$ we shall use the following "decomposition" result associated to the following definition.

Definition 5.0.3 *A regular (a,b) -module is \mathcal{A} -**primitive** if each root of its Bernstein polynomial is in $-\mathcal{A}$ for any subset \mathcal{A} in \mathbb{C}/\mathbb{Z} .*

Proposition 5.0.4 *For any regular (a,b) -module E and any $\mathcal{A} \subset \mathbb{C}/\mathbb{Z}$ there exists a unique normal sub-module $E_{[\neq \mathcal{A}]}$ in E such that the quotient*

$$E^{[\mathcal{A}]} := E/E_{[\neq \mathcal{A}]}$$

is $[\mathcal{A}]$ -primitive and such that any $[\mathcal{A}]$ -primitive sub-module of E does not intersect $E_{[\neq \mathcal{A}]}$.

It is a key point to remark that the semi-simple filtration of a regular (a,b) -module is compatible with its $[\alpha]$ -primitive "decomposition" given by the following proposition (see [15] Lemma 4.2.5)

Proposition 5.0.5 *Let E be a regular (a,b) -module and \mathcal{A} be a subset in \mathbb{C}/\mathbb{Z} . Then for each integer j we have*

$$S_j(E_{[\mathcal{A}]}) = S_j(E)_{[\mathcal{A}]} \quad \text{and also} \quad S_j(E^{[\mathcal{A}]}) = S_j(E)^{[\mathcal{A}]}.$$

Note that in the case where E is a fresco $E^{[\alpha]}$ is also a fresco and the Bernstein polynomial of $E^{[\alpha]}$ is exactly the α -part of the Bernstein polynomial of E .

Now we can give a "nilpotent weight" to the roots of the Bernstein polynomial of a fresco with the following definition.

¹¹recall that $F \subset E$ is normal when $F \cap bE = bF$. Then the quotient E/F is an (a,b) -module.

Definition 5.0.6 Let \mathcal{F} be an α -primitive fresco with nilpotent order d and for $j \in [1, d]$ and let δ_j be the rank $\mathcal{F}/S_j(\mathcal{F})$. Define for $j \in [1, d]$ the j -th **Bernstein polynomial** $B_j(\mathcal{F})$ of \mathcal{F} by the formula

$$B_j(\mathcal{F})[x] := \tilde{B}_j(x - \delta_j)$$

where \tilde{B}_j is the Bernstein polynomial of $S_j(\mathcal{F})/S_{j-1}(\mathcal{F})$.

Then, for any fresco \mathcal{F} defined its j -th Bernstein polynomial by

$$B_j(x) := \prod_{\alpha} B_j(\mathcal{F}^{[\alpha]})(x)$$

where $B_j = 1$ when $\mathcal{F}^{[\alpha]} = \{0\}$ or when $j > d(\mathcal{F}^{[\alpha]})$, the nilpotent order of $\mathcal{F}^{[\alpha]}$.

The link with the Bernstein polynomial in the case of a fresco is given by the following result ([15] Th. 7.3.2)

Theorem 5.0.7 Let \mathcal{F} be a fresco. The Bernstein polynomial of the fresco \mathcal{F} is given by

$$B_{\mathcal{F}} = B_1(\mathcal{F}) \dots B_d(\mathcal{F})$$

where d is the nilpotent order of \mathcal{F} .

Moreover, each root of each $B_j(\mathcal{F})$ is simple. The degrees of the polynomials B_j are non increasing and are equal to the ranks of the quotients $S_j(\mathcal{F})/S_{j-1}(\mathcal{F})$.

Now the main result of our study is the following theorem.

Theorem 5.0.8 In the situation described at the beginning, assume that for some $\omega \in \Omega_0^{n+1}$ the fresco $\mathcal{F}_{\omega}^{[\alpha]}$ has nilpotent order $d \geq 1$ and that $-\alpha - m$ is a root of the d -th Bernstein polynomial of $\mathcal{F}_{\omega}^{[\alpha]}$. Then there exists $\omega' \in \Omega_0^{n+1}$ and an integer $m' \in [0, n+1]$ such that the asymptotic expansion of the hermitian period

$$\Psi_{\omega, \omega'}(s) := \int_{f^{-1}(s)} \rho\omega/df \wedge \overline{\omega'/df}$$

has a non zero term of the type $|s|^{2\alpha-2} s^m \bar{s}^{m'} (\text{Log}|s|^2)^{d-1}$ for $\alpha \neq 1$ and of the type $|s|^{2\alpha-2} s^m \bar{s}^{m'} (\text{Log}|s|^2)^d$ for $\alpha = 1$.

Conversely, if such ω' and m' exists for a given ω , then $\mathcal{F}_{\omega}^{[\alpha]}$ has nilpotent order at least equal to d and d -th Bernstein polynomial of $\mathcal{F}_{\omega}^{[\alpha]}$ has a root equal to $-\alpha - m_1$ with $m_1 \in [0, m]$.

The reader which is interested with more consequences, more details and proofs may consult [14] and [15] where he will also find some examples of computations of the Bernstein polynomial of frescos in explicite examples.

To conclude this presentation I must confess that I was not able to prove the last result above using only the "formal" (a,b)-module point of view presented here. So I was compelled to develop the **convergent theory** of (a,b)-modules which does not

reduce to an easy exercise on inequalities; for instance see the "division theorem" (in section 2.5 in [15]) in the algebra \mathcal{A}_{conv} , the convergent analog of \mathcal{A} , where the geometric condition is useful and this result allows to prove the structure theorem for convergent frescos, analog of Theorem 4.0.2 above.

6 Bibliography

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