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# Gröbner bases over polytopal affinoid algebras

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## ABSTRACT

Polyhedral affinoid algebras have been introduced by Einsiedler, Kapranov and Lind in [5] to connect rigid analytic geometry (analytic geometry over non-archimedean fields) and tropical geometry. In this article, we present a theory of Gröbner bases for polytopal affinoid algebras that extends both Caruso et al.’s theory of Gröbner bases on Tate algebras of [1] and Pauer et al.’s theory of Gröbner bases on Laurent polynomials of [9].

We provide effective algorithms to compute Gröbner bases for both ideals of Laurent polynomials and ideals in polytopal affinoid algebras. Experiments with a Sagemath implementation are provided.

## CCS CONCEPTS

• Computing methodologies → Algebraic algorithms.

## KEYWORDS

Algorithms, Gröbner bases, Tate algebra, Laurent polynomials, Tropical analytic geometry

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## 1 INTRODUCTION

Rigid geometry was born in Tate’s article [13]. He defined affinoid algebras as ideal quotients of the Tate algebras, the algebras of converging power series on the unit ball of some complete non-archimedean valued field. They are the building blocks of Tate’s rigid geometry in the same way ideal quotients in polynomial rings are the building blocks of algebraic geometry. In the past 50 years, they have found various applications, among them one can cite Raynaud’s contribution to solving Abhyankar’s conjecture [11].

Polyhedral and polytopal affinoid algebras have been defined by Einsiedler et al. in [5] using Tate algebras with convergence conditions given by a polyhedron or a polytope (respectively). They are one of the main ingredients of the development of tropical analytic geometry as in e.g. [8, 10, 6]. It has found applications with

Gubler in [7] to prove the Bogomolov conjecture for totally degenerate abelian varieties. One motivation of this article is to make progress toward an *effective counterpart* to tropical analytic geometry.

To do so, the most natural tool to implement is Gröbner bases (GB). The case of GB over Tate algebras has been studied in [1]. Modern algorithms like F5 and FGLM have been generalized to this context in [4, 2]. Overconvergence has been studied in [3, 14] culminating in the definition and computation of Universal Analytic Gröbner Bases. They allow a first step toward tropical analytic geometry in the context of polynomial ideals.

To extend these results to polytopal affinoid algebras, it is natural to work with Laurent polynomials and series. To define GB in this context, one could of course, instead of working with  $x^{\pm 1}, y^{\pm 1}, \dots$ , choose to use nonnegative monomials  $x_1, x_2, y_1, y_2, \dots$  along with relations  $x_1 x_2 - 1 = 0, y_1 y_2 - 1 = 0, \dots$ . As we believe a direct approach would be more suitable for tropical applications, we have chosen to implement an instance of Laurent polynomials and series which does not hide its monomials with negative exponents, and which has been introduced by Pauer and Unterkircher in [9]. Motivated by the study of systems of linear partial difference equations, they have developed a theory of generalized monomial orderings and Gröbner bases for Laurent polynomials. We provide a short introduction to the generalized monomial ordering part of this theory in Section 3. In addition, in Section 6, we fill in a gap that prevented their theory to be completely effective: we provide an algorithm to compute all the necessary leading monomials in this context.

Equipped with a new notion of term ordering built on those of [1, 14] and [9], we study the multivariate division in polytopal affinoid algebras in Section 4. These results enable us to provide in Section 5 a theory of Gröbner bases for ideals in polytopal affinoid domains. We make a decisive step toward effectivity in Subsection 5.3 with an adapted Buchberger algorithm. Finally, Section 6 provide the last remaining tools needed for GB computations over polytopal affinoid algebras. In addition, a short software demonstration in Sagemath [12] can be read in Annex.

## 2 SETTING

Let  $K$  be a field with a discrete valuation  $\text{val}$  making it complete, and let  $K^\circ$  be the subring of  $K$  consisting of elements of nonnegative valuation. Let  $\pi$  be a uniformizer of  $K$ , i.e. a generator of the maximal ideal  $\{x \in K^\circ, \text{val}(x) > 0\}$  of  $K^\circ$ . Typical examples of such a setting are  $p$ -adic fields like  $K = \mathbb{Q}_p$  with  $K^\circ = \mathbb{Z}_p$  and  $\pi = p$ , or Laurent series fields like  $K = \mathbb{Q}((U))$  with  $K^\circ = \mathbb{Q}[[U]]$  and  $\pi = U$ .

We fix a positive integer  $n$ . Let  $X_1, \dots, X_n$  be  $n$  variables. We use the short notations  $\mathbf{X}$  for  $(X_1, \dots, X_n)$  and  $\mathbf{X}^{\pm 1}$  for  $(X_1^{\pm 1}, \dots, X_n^{\pm 1})$ .

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If  $\mathbf{i} = (i_1, \dots, i_n) \in \mathbb{Z}^n$ , we shall write  $\mathbf{X}^{\mathbf{i}}$  for  $X_1^{i_1} \dots X_n^{i_n}$ . We define  $T_{\geq 0}$  and  $T$  to be the multiplicative monoids  $\{\mathbf{X}^{\mathbf{i}}, \mathbf{i} \in \mathbb{N}^n\}$  and  $\{\mathbf{X}^{\mathbf{i}}, \mathbf{i} \in \mathbb{Z}^n\}$  respectively. We will frequently represent a monomial in either the set  $T_{\geq 0}$  or  $T$  by utilizing the  $n$ -tuple of its exponents in  $\mathbb{N}^n$  or  $\mathbb{Z}^n$ .

Let  $\mathcal{P} \subset \mathbb{R}^n$  be a polytope with vertices in  $\mathbb{Q}^n$ . Let  $P = \mathcal{P} \cap \mathbb{Q}^n$ . For  $a, b \in \mathbb{R}^n$ ,  $a \cdot b$  denote the usual scalar product in  $\mathbb{R}^n$ . Following [10], we define the polytopal affinoid algebra  $K\{\mathbf{X}; P\}$  as the following algebra:

$$\left\{ \sum_{\mathbf{u} \in \mathbb{Z}^n} a_{\mathbf{u}} \mathbf{X}^{\mathbf{u}} : a_{\mathbf{u}} \in K, \forall r \in P, \text{val}(a_{\mathbf{u}}) - r \cdot \mathbf{u} \xrightarrow{|\mathbf{u}| \rightarrow +\infty} +\infty \right\}.$$

For  $f = \sum_{\mathbf{u} \in \mathbb{Z}^n} a_{\mathbf{u}} \mathbf{X}^{\mathbf{u}} \in K\{\mathbf{X}; P\}$ , we define  $\text{supp}(f) := \{\mathbf{X}^{\mathbf{u}}, \mathbf{u} \in \mathbb{Z}^n, a_{\mathbf{u}} \neq 0\}$ . For each  $r \in P$  we define a valuation  $\text{val}_r$  on  $K\{\mathbf{X}; P\}$  by

$$\text{val}_r \left( \sum_{\mathbf{u} \in \mathbb{Z}^n} a_{\mathbf{u}} \mathbf{X}^{\mathbf{u}} \right) = \min_{\mathbf{u} \in \mathbb{Z}^n} \text{val}(a_{\mathbf{u}}) - r \cdot \mathbf{u},$$

and for  $f$  in  $K\{\mathbf{X}; P\}$ , we define  $\text{val}_P(f) = \inf_{r \in P} \text{val}_r(f)$ . By [10],  $\text{val}_P$  is a valuation on  $K\{\mathbf{X}; P\}$  and

$$\text{val}_P(f) = \inf_{r \in P} \text{val}_r(f) = \min_{r \in P} \text{val}_r(f) = \min_{r \in \text{vert}(P)} \text{val}_r(f)$$

Moreover,  $f$  is in  $K\{\mathbf{X}; P\}$  if and only if  $\text{val}(a_{\mathbf{u}}) - r \cdot \mathbf{u} \rightarrow +\infty$  for all  $r \in \text{vert}(P)$  (the vertices of  $P$ ). In other words:

$$K\{\mathbf{X}; P\} = \bigcap_{r \in \text{vert}(P)} K\{\mathbf{X}; \{r\}\}$$

Elements of  $K\{\mathbf{X}; P\}$  are exactly the Laurent power series which converge on the set  $\text{val}^{-1}(P) \subset (K^\times)^n$  and  $\text{val}_P(f)$  is the minimum valuation reached by  $f$  on  $\text{val}^{-1}(P)$ .

A case of particular interest is when  $P = \prod_{i=1}^n [r_i, s_i]$  for some  $r_i < s_i \in \mathbb{Q}$ . The set  $\text{val}^{-1}(P)$  is then the polyannulus  $\{(\epsilon_1, \dots, \epsilon_n) \in (K^\times)^n : r_i \leq \text{val}(\epsilon_i) \leq s_i\}$  and in rigid geometry, such a  $K\{\mathbf{X}; P\}$  is called a Laurent domain.

### 3 GENERALIZED MONOMIAL ORDER

In the realm of Gröbner basis theory for ideals within the polynomial ring  $K[\mathbf{X}]$ , a fundamental component involves monomial orders defined on the monoid  $T_{\geq 0}$ . Recall that a monomial order on  $T_{\geq 0}$  is a total order that satisfies, for all  $r, s, t \in T_{\geq 0}$ :

$$(1) 1 \leq t \quad (2) r < s \implies rt < st$$

When attempting to extend this theory to Gröbner bases for ideals in the Laurent polynomial ring  $K[\mathbf{X}^{\pm 1}]$ , a natural inclination is to consider "monomial orders on  $T$ ." These should be total orders on  $T$  that adhere to conditions (1) and (2). However, such orders can not exist.

To illustrate this, take any non-trivial element  $t \in T$ . According to (1), we have  $1 < t$ , which implies  $t^{-1} < 1$  by (2). This leads to a contradiction. The problem is that, unlike  $T_{\geq 0}$ ,  $T$  contains non-trivial invertible elements.

In the article [9], the authors introduced generalized monomial orders (or g.m.o) as a workaround to the previous problem. Their approach involves representing  $T$  as a finite union  $\cup_i T_i$  of submonoids  $T_i$ , ensuring that each  $T_i$  does not contain any non-trivial invertible elements, thereby allowing the definition of a monomial order on each of them. The crucial requirement is that the order

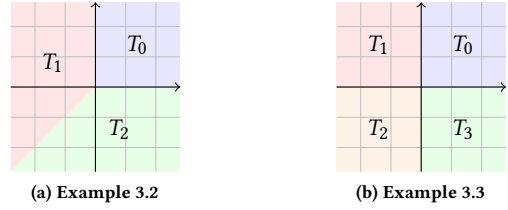


Figure 1: Conic decompositions for  $n = 2$

on  $T$  must restrict to a monomial order on each  $T_i$  in a compatible manner (refer to condition 2 in Definition 3.4). This final condition ensures that the leading term of a product, which typically isn't the product of the leading terms for a g.m.o, predominantly depends on the submonoids  $T_i$ .

In this section, we expose this notion, simplifying a little the exposition in [9] to fit our use case.

Sometimes, we will need to consider orders defined on various sets. As a general guideline, the symbols  $\leq$  and  $<$  will be employed irrespective of the set on which the order is established. The clarity of the set should always be apparent from the context. In instances where ambiguity might arise, we employ subscripts (such as  $<_G$ ,  $<_\omega$ , ...) to distinguish between different orders.

**DEFINITION 3.1** ([9, DEFINITION 2.1]). *A conic decomposition of  $T$  is a finite family  $(T_i)_{i \in I}$  of finitely generated submonoids of  $T$  such that*

- (1) *for each  $i$ , the only invertible element in the monoid  $T_i$  is 1 and the group generated by  $T_i$  is  $T$ .*
- (2) *the union of all the  $T_i$ 's equal  $T$*

Examples 3.2 and 3.3 illustrate Definition 3.1. We will use the conic decomposition of Example 3.2 in our implementation in §6.2 and in the Annex.

**EXAMPLE 3.2.** *Let  $T_0 := \{X^k, k \in \mathbb{N}^n\}$  and for  $1 \leq j \leq n$  let  $T_j$  be the monoid generated by  $\{X_1^{-1} \dots X_n^{-1}\} \cup \{X_1, \dots, \hat{X}_j, \dots, X_n\}$  where the hat symbol indicates that the corresponding element is omitted. In other words,  $T_j$  contains all monomials for which the exponent of  $X_j$  is non-positive and smaller than any other exponent. Then  $(T_j)_{0 \leq j \leq n}$  is a conic decomposition of  $T$  containing  $n + 1$  cones.*

**EXAMPLE 3.3.** *Let  $D_n$  be the set of all maps from  $\{1, \dots, n\}$  to  $\{-1, 1\}$ . For  $d$  in  $D_n$ , let  $T_d$  be the monoid generated by the set  $\{X_1^{d(1)}, \dots, X_n^{d(n)}\}$ . Then  $(T_d)_{d \in D_n}$  is a conic decomposition of  $T$  containing  $2^n$  cones.*

**DEFINITION 3.4** ([9, DEFINITION 2.2]). *Let  $(T_i)_{i \in I}$  be a conic decomposition of  $T$ . A generalized monomial order (or g.m.o) on  $T$  for the decomposition  $(T_i)_{i \in I}$  is a total order  $<$  on  $T$  such that*

- (1)  $\forall t \in T, 1 \leq t$
- (2)  $\forall r \in T, \forall i \in I, (s, t \in T_i \text{ and } r < s) \implies rt < st$

**REMARK 3.5.** *For each  $i \in I$ , the restriction of  $<$  to  $T_i$  is a monomial order (take  $r$  in  $T_i$  in 2. of Definition 3.4).*

Given a conic decomposition, Lemma 3.6 provides a method for constructing a g.m.o by employing an auxiliary function,  $\phi : T \rightarrow \mathbb{Q}_{\geq 0}$ , which exhibits favorable behavior in relation to the decomposition.

LEMMA 3.6 ([9, LEMMA 2.1]). Let  $(T_i)_{i \in I}$  be a conic decomposition of  $T$  and  $E$  be either  $\{1\}$  or one of the  $T_i$ . Let  $<_G$  be a total group order on  $T$  (e.g. the lexicographical order). Let  $\phi : T \rightarrow \mathbb{Q}_{\geq 0}$  be a function fulfilling the following conditions:

- (1)  $\forall t \in T \setminus E, \phi(t) > 0$
- (2)  $\forall s, t \in T, \phi(st) \leq \phi(s) + \phi(t)$
- (3)  $\forall i \in I, \phi|_{T_i}$  is a monoid homomorphism.

Then the order  $<$  defined by

$$r < s \iff \phi(r) < \phi(s) \text{ or } (\phi(r) = \phi(s) \text{ and } r <_G s)$$

is a g.m.o on  $T$  for the decomposition  $(T_i)_{i \in I}$ .

The notions of leading monomial, leading coefficient and leading term are defined for a Laurent polynomial as in the polynomial case.

DEFINITION 3.7. Fix a g.m.o on  $T$  and let  $f \in K[\mathbf{X}^{\pm 1}]$ . The leading monomial  $\text{lm}(f)$  of  $f$  is defined as  $\max(\mathbf{X}^j, j \in \text{supp}(f))$ . The leading coefficient  $\text{lc}(f)$  of  $f$  is the coefficient of  $\text{lm}(f)$  in  $f$ . The leading term  $\text{lt}(f)$  of  $f$  is the product  $\text{lc}(f)\text{lm}(f)$ .

Examples 3.8 to 3.10 illustrate Lemma 3.6 and Definition 3.7. In each case, we take for group order on  $T$  the lexicographical order, and show in the case  $n = 2$  how the monomials of  $f = 2xy^{-2} + x^{-2}y^{-2} + 3x^{-1}y^{-2} + y^2 \in K[x^{\pm 1}, y^{\pm 1}]$  are ordered.

EXAMPLE 3.8. Take the standard decomposition  $(T_0, T_1, \dots, T_n)$  of Example 3.2. Put  $E = \{1\}$  and define  $\phi : T \rightarrow \mathbb{Q}_{\geq 0}$  by  $\phi(i_1, \dots, i_n) = i_1 + \dots + i_n - (n+1) \min(0, i_1, \dots, i_n)$ . We have  $xy^{-2} > x^{-1}y^{-2} > y^2 > x^{-2}y^{-2}$ ,  $\text{lm}(f) = xy^{-2}$ ,  $\text{lc}(f) = 2$  and  $\text{lt}(f) = 2xy^{-2}$ .

EXAMPLE 3.9. Take the standard decomposition  $(T_0, T_1, \dots, T_n)$  of Example 3.2. Put  $E = T_0$  and define  $\phi : T \rightarrow \mathbb{Q}_{\geq 0}$  by  $\phi(i_1, \dots, i_n) = -\min(0, i_1, \dots, i_n)$ . We have  $xy^{-2} > x^{-1}y^{-2} > x^{-2}y^{-2} > y^2$ ,  $\text{lm}(f) = xy^{-2}$ ,  $\text{lc}(f) = 2$  and  $\text{lt}(f) = 2xy^{-2}$ .

EXAMPLE 3.10. Take the conic decomposition  $(T_d)_{d \in D_n}$  of Example 3.3. Put  $E = \{1\}$  and define  $\phi : T \rightarrow \mathbb{Q}_{\geq 0}$  by  $\phi(i_1, \dots, i_n) = |i_1| + \dots + |i_n|$ . We have  $x^{-2}y^{-2} > xy^{-2} > x^{-1}y^{-2} > y^2$ ,  $\text{lm}(f) = x^{-2}y^{-2}$ ,  $\text{lc}(f) = 3$  and  $\text{lt}(f) = 3x^{-2}y^{-2}$ .

In Example 3.11, we illustrate the fact that for  $t \in T$  and  $f \in K[\mathbf{X}^{\pm 1}]$ , the leading monomial of  $tf$  is generally not equal to the product of  $t$  by  $\text{lm}(f)$ .

EXAMPLE 3.11. Take the g.m.o of Example 3.2 in the case  $n = 2$  and  $f = xy + y^{-1} \in K[x^{\pm 1}, y^{\pm 1}]$ . We have  $\text{lm}(yf) = xy^2$  but  $y\text{lm}(f) = 1 \neq xy^2$ .

However, thanks to the compatibility condition (2) of Definition 3.4, the monomial  $t$  of  $f$  such that  $\text{lm}(tf) = t\text{lm}(f)$  relies solely on the cone  $T_i$  in which  $\text{lm}(tf)$  lies, and not on the specific value of  $t$  itself. This is demonstrated in Lemma 3.13 and leads naturally to the definition of "one leading monomial per cone" in Definition 3.14.

For the rest of this section, we fix a g.m.o for a conic decomposition  $(T_i)_{i \in I}$  of  $T$ .

DEFINITION 3.12. For  $i \in I$  and  $f \in K[\mathbf{X}^{\pm 1}]$ , define  $T_i(f) := \{t \in T, \text{lm}(tf) \in T_i\}$ .

LEMMA 3.13 ([9, LEMMA 2.3]). Let  $i \in I, f \in K[\mathbf{X}^{\pm 1}], u, v \in T_i(f)$ . Write  $\text{lm}(uf) = ut_u \in T_i$  and  $\text{lm}(vf) = vt_v \in T_i$  for some monomials  $t_u, t_v$  of  $f$ . Then  $t_u = t_v$ .

DEFINITION 3.14. Let  $i \in I, f \in K[\mathbf{X}^{\pm 1}]$  and  $t \in T_i(f)$ . We define  $\text{lm}_i(f) := \text{lm}(tf)t^{-1}$ . This is well defined by Lemma 3.13 (i.e. it does not depend on a particular  $t \in T_i(f)$ ). We also define  $\text{lc}_i(f)$  as the coefficient of  $\text{lm}_i(f)$  in  $f$ , and  $\text{lt}_i(f) = \text{lc}_i(f)\text{lm}_i(f)$ .

REMARK 3.15. Before giving some examples, we explain how to compute  $\text{lm}_i(f)$ . By Lemma 3.13, we have  $\text{lm}_i(f) = \text{lm}(f)t^{-1}$  whenever  $t \in T_i(f)$ . So we just need to find a  $t \in T_i(f)$ , which can be done as follows. Recall that  $T_i$  is finitely generated and generates  $T$  as a group. This implies that for each monomial  $s$  of  $f$ , we can find  $u_s, v_s \in T_i$  such that  $s = u_s v_s^{-1}$ . Define  $t$  as the product of the monomials  $v_s$  for  $s$  a monomial of  $f$ . Then  $\text{supp}(tf) \subset T_i$ , and  $\text{lm}(tf) \in T_i$ . The later means that  $t \in T_i(f)$  and we are done.

EXAMPLE 3.16 (EXAMPLE 3.8 CONTINUED.). We have  $\text{lm}_0(f) = \text{lm}_1(f) = y^2$ ,  $\text{lm}_2(f) = \text{lm}(f) = xy^{-2}$ ,  $T_0(f) = y^2 T_0$ ,  $T_1(f) = y^2 T_1$  and  $T_2(f) = y T_2$ .

EXAMPLE 3.17 (EXAMPLE 3.9 CONTINUED.). We have  $\text{lm}_0(f) = \text{lm}_2(f) = \text{lm}(f) = xy^{-2}$ ,  $\text{lm}_1(f) = x^{-2}y^{-2}$ ,  $T_0(f) = x^{-2}y^2 T_0$ ,  $T_1(f) = xy^2 T_1$  and  $T_2(f) = x^2 y^2 T_2$ .

EXAMPLE 3.18 (EXAMPLE 3.10 CONTINUED.). We index the 4 cones as in b) of Figure 1. We have  $\text{lm}_0(f) = \text{lm}_1(f) = y^2$ ,  $\text{lm}_2(f) = \text{lm}(f) = x^{-2}y^{-2}$  and  $\text{lm}_3(f) = xy^{-2}$ .

REMARK 3.19. We have  $\text{lm}(f) = \text{lm}_i(f)$  for at least one index  $i \in I$ . For a fixed  $f$ , the  $\text{lm}_i(f)$ 's are not necessarily distincts, and  $\text{lm}_i(f)$  is generally not an element of  $T_i$ .

The following lemma states that a g.m.o shares with monomial orders the crucial property of being a well-order.

LEMMA 3.20 ([9, LEMMA 2.2]). Let  $<$  be a g.m.o on  $T$  (for a conic decomposition  $(T_i)_{i \in I}$  of  $T$ ). Then every strictly descending sequence in  $T$  is finite. In particular, any subset of  $T$  contains a smallest element.

## 4 MULTIVARIATE DIVISION IN $K\{\mathbf{X}; P\}$

In this section, we demonstrate the adaptability of the division algorithm, originally presented in the Tate algebras setting in [1, Proposition 3.1], to the polytopal setting.

The novelty lies in the method employed to eliminate the leading term of an element  $f \in K\{\mathbf{X}; P\}$  by  $g \in K\{\mathbf{X}; P\}$ . We can not solely rely on the leading terms of  $f$  and  $g$  (see Example 4.7), but have to consider also the terms  $\text{lt}_i(g)$  and the sets  $T_i(g)$  for  $i \in I$ .

### 4.1 The monoid of terms $T\{\mathbf{X}; P\}$

DEFINITION 4.1. Let  $\leq_\omega$  be a g.m.o on  $T$ . We define the monoid of terms  $T\{\mathbf{X}; P; \leq_\omega\}$  (or simply  $T\{\mathbf{X}; P\}$ ) as the multiplicative monoid  $\{a\mathbf{X}^u : a \in K^\times, u \in \mathbb{Z}^n\}$ . We define a preorder  $\leq$  on  $T\{\mathbf{X}; P\}$  by

$$a\mathbf{X}^u \leq b\mathbf{X}^v \iff (\text{val}_P(a\mathbf{X}^u) > \text{val}_P(b\mathbf{X}^v)) \text{ or } (\text{val}_P(a\mathbf{X}^u) = \text{val}_P(b\mathbf{X}^v) \text{ and } \mathbf{X}^u \leq_\omega \mathbf{X}^v)$$

REMARK 4.2. The preorder  $\leq$  is not antisymmetric (and so not an order). For terms  $t_1, t_2$ , the fact that  $t_1 \leq t_2$  and  $t_2 \leq t_1$  is equivalent to the existence of  $a \in (K^\circ)^\times$  such that  $t_1 = at_2$ .

REMARK 4.3. The preorder  $\leq$  is compatible with multiplication by elements of  $K^\times$  but is not compatible with multiplication by monomials (because  $\leq_\omega$  itself isn't). It is also not a well-order (there exists infinite strictly decreasing sequences).

However, Lemma 4.4 shows that the preorder  $\leq$  is a topological well-order as in [1, Lemma 2.14].

LEMMA 4.4. Let  $(t_j)_{j \in \mathbb{N}}$  be a strictly decreasing sequence in  $T\{\mathbf{X}; P\}$ . Then  $\lim_{j \rightarrow \infty} \text{val}_P(t_j) = +\infty$ , or equivalently  $\sum_{j \in \mathbb{N}} t_j \in K\{\mathbf{X}; P\}$ .

PROOF. By definition of the preorder  $\leq$ , the sequence  $(\text{val}_P(t_j))_{j \in \mathbb{N}}$  is nondecreasing and takes its values in a discrete subset  $\frac{1}{D}\mathbb{Z}$  of  $\mathbb{R}$  for some integer  $D$  (taking into account the image group of  $\text{val}$  and the product of the denominators of the vertices of  $P$ ). Since  $\leq_\omega$  is a well-order, there can be for a fixed  $v \in \frac{1}{D}\mathbb{Z}$  only a finite number of indices  $j$  for which  $\text{val}_P(t_j) = v$ . Combining these two facts,  $\text{val}_P(t_j)$  must tend to  $+\infty$ .  $\square$

REMARK 4.5. If  $i \neq j \in \mathbb{Z}^n$ , the terms  $a_i \mathbf{X}^i$  and  $b_j \mathbf{X}^j$  are never "equal" (that is  $a_i \mathbf{X}^i \leq b_j \mathbf{X}^j$  and  $b_j \mathbf{X}^j \leq a_i \mathbf{X}^i$ ) for  $\leq$ . Therefore, any nonzero  $f \in K\{\mathbf{X}; P\}$  has a unique leading term for  $\leq$ .

DEFINITION 4.6. Let  $f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha \mathbf{X}^\alpha \in K\{\mathbf{X}; P\}$ . We define  $\text{in}_P(f) := \sum_{\alpha \in \mathbb{Z}^n \text{ s.t. } \text{val}_P(c_\alpha \mathbf{X}^\alpha) = \text{val}_P(f)} c_\alpha \mathbf{X}^\alpha \in K[\mathbf{X}^{\pm 1}]$ . Using the map  $\text{in}_P : K\{\mathbf{X}; P\} \rightarrow K[\mathbf{X}^{\pm 1}]$ , we extend to  $f \in K\{\mathbf{X}; P\}$  the definitions of  $\text{lm}(f)$ ,  $\text{lc}(f)$ ,  $\text{lt}(f)$ ,  $\text{lm}_i(f)$ ,  $\text{lc}_i(f)$ ,  $\text{lt}_i(f)$  and  $T_i(f)$ .

The leading term for the term order on  $T\{\mathbf{X}; P\}$  (as in Remark 4.5) equals  $\text{lt}(f) = \text{lt}(\text{in}_P(f))$ , and same for  $\text{lt}_i(f)$ ,  $T_i(f)$  and so on.

## 4.2 Multivariate division algorithm

In Example 4.7, we illustrate how the classical procedure to cancel the leading term of a polynomial can fail in the Laurent polynomial setting equipped with a g.m.o.

EXAMPLE 4.7. Take the g.m.o of Example 3.8 in the case  $n = 2$  and  $f = x + y$ ,  $g = x^{-1}y + y^{-1} \in K[x^{\pm 1}, y^{\pm 1}]$ . We have  $\text{lt}(f) = x$  and  $\text{lt}(g) = x^{-1}y$ . To cancel out  $\text{lt}(f)$  in  $f$  the classical way, we subtract  $\frac{\text{lt}(f)}{\text{lt}(g)}g$  from  $f$  and obtain  $y - x^2y^{-2}$ . But then we have  $\text{lm}(y - x^2y^{-2}) = x^2y^{-2} > x = \text{lm}(f)$ , that is the leading monomial after cancellation is strictly superior to the leading monomial of  $f$ !

The issue in Example 4.7 arises from assuming that  $\text{lt}(tg) = t\text{lt}(g)$ . For a g.m.o, this equality does not necessarily hold (e.g Example 3.11), and it becomes possible for the leading term of  $tg$  to exceed the leading term of  $f$ . While the leading term of  $f$  is successfully canceled out, a larger term originating from  $tg$  emerges after the cancellation. To properly eliminate the leading term of  $f$  by a multiple of  $g$ , we need instead to identify a term  $t$  such that  $\text{lt}(tg) = \text{lt}(f)$ . According to Lemma 3.13, depending on the specific cone  $T_i$  in which  $\text{lm}(f)$  is contained, this equality can be expressed again as  $\text{lt}(f) = \text{lt}(tg) = t\text{lt}_i(g)$ . This shows that the potential candidates for  $t$  are the terms  $\frac{\text{lt}(f)}{\text{lt}_i(g)}$  for varying indices  $i$  for which  $\text{lm}(f) \in T_i$ . Summarizing the process:

- (1) Find an index  $i$  such that  $\text{lm}(f) \in T_i$  (there exists at least one).

- (2) Check if  $\text{lm}\left(\frac{\text{lm}(f)}{\text{lt}_i(g)}g\right) = \text{lm}(f)$ . If this condition is satisfied, the leading term of  $f$  can be effectively canceled out by subtracting  $\frac{\text{lt}(f)}{\text{lt}_i(g)}g$  from  $f$ , and  $\frac{\text{lm}(f)}{\text{lt}_i(g)} \in T_i(g)$ ; otherwise, it cannot.

REMARK 4.8. The monomial  $\text{lm}(f)$  can be contained in more than one cone, but it suffices to test the condition for cancellation within any one of those cones.

The preceding discussion is applied in Proposition 4.9 and Algorithm 1 to formulate a multivariate division algorithm in the ring  $K\{\mathbf{X}; P\}$ . Same as [1, Algo 1], Algorithm 1 needs a countable amount of steps to terminate, but only a finite amount of steps to reach a given finite precision in  $\text{val}_P$ .

PROPOSITION 4.9. Let  $f \in K\{\mathbf{X}; P\}$  and  $G$  be a finite subset of  $K\{\mathbf{X}; P\}$ . Algorithm 1 produces a family  $(q_g)_{g \in G}$  and  $r$  in  $K\{\mathbf{X}; P\}$  such that:

- (1)  $f = \sum_{g \in G} q_g g + r$
- (2) for all monomial  $t$  in  $r$ ,  $t \notin \bigcup_{i \in I, g \in G} T_i(g) \text{lm}_i(g)$
- (3) for all  $g \in G$  and all monomial  $t$  in  $q_g$ ,  $\text{lt}(tg) \leq \text{lt}(f)$ .

PROOF. We construct by induction sequences  $(f_j)_{j \geq 0}$ ,  $(q_{g,j})_{j \geq 0}$  for  $g \in G$  and  $(r_j)_{j \geq 0}$  such that for all  $j \geq 0$ :

$$f = f_j + \sum_{g \in G} q_{g,j} g + r_j,$$

and  $\text{lt}(f_j)_{j \geq 0}$  is strictly decreasing. We first set  $f_0 = f$ ,  $r_0 = 0$  and  $q_{g,0} = 0$  for all  $g \in G$ . If there exists  $i \in I$  and  $g \in G$  such that

$$\text{lm}\left(\frac{\text{lm}(f_j)}{\text{lt}_i(g)}g\right) = \text{lm}(f_j) \in T_i,$$

we set  $f_{j+1} = f_j - tg$  and  $q_{g,j+1} = q_{g,j} + tg$  where  $t = \frac{\text{lt}(f_j)}{\text{lt}_i(g)}$ , and leave unchanged  $r_j$  and the other  $q_{g,j}$ 's. Otherwise, we set  $f_{j+1} = f_j - \text{lt}(f_j)$  and  $r_{j+1} = r_j + \text{lt}(f_j)$  and leave unchanged the  $q_{g,j}$ 's. By construction, the sequence  $(\text{lt}(f_j))_{j \geq 0}$  is strictly decreasing. By lemma 4.4, we deduce that  $\text{val}_P(r_{j+1} - r_j)$  and the  $\text{val}_P(q_{g,j+1} - q_{g,j})$ 's tend to  $+\infty$ . Thus  $r_j$  and the  $q_{g,j}$ 's converge in  $K\{\mathbf{X}; P\}$ . Their limits satisfy the requirements of the proposition.  $\square$

DEFINITION 4.10. The output  $r$  obtained from Algo 1 with entries  $f$  and  $G = (g_1, \dots, g_m)$  is denoted  $\text{rem}(f, G)$ .

## 5 GRÖBNER BASES FOR POLYTOPAL AFFINOID ALGEBRAS

$\text{Lt} \leq$  be a g.m.o for a conic decomposition  $(T_i)_{i \in I}$ . For an ideal  $J$  in  $K\{\mathbf{X}; P\}$ , define  $\text{lm}(J) := \{\text{lm}(f), f \in J, f \neq 0\}$ . Notice that

$$\text{lm}(J) = \bigcup_{f \in J, i \in I} T_i(f) \text{lm}_i(f). \quad (1)$$

A finite subset  $G$  of  $J$  is then a Gröbner basis when equation (1) still holds while considering in the set union only elements from  $G$  instead of all members of  $J$ .

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**Algorithm 1:** Multivariate division algorithm in  $K\{\mathbf{X}; P\}$ 

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**input** :  $f, g_1, \dots, g_m \in K\{\mathbf{X}; P\}$ **output**:  $q_1, \dots, q_m, r$  satisfying Prop 4.9

```
1  $q_1, \dots, q_m, r \leftarrow 0$ ;  
2 while  $f \neq 0$  do  
3   while  $\exists(i, j) \in I \times \llbracket 1, m \rrbracket$  such that  
4      $\text{lm}\left(\frac{\text{lm}(f)}{\text{lm}_i(g_j)}g_j\right) = \text{lm}(f)$  do  
5        $t \leftarrow \frac{\text{lt}(f)}{\text{lt}_i(g_j)}$ ;  
6        $q_j \leftarrow q_j + t$ ;  
7        $f \leftarrow f - tq_j$ ;  
8      $r \leftarrow r + \text{lt}(f)$   
9    $f \leftarrow f - \text{lt}(f)$ ;  
9 return  $q_1, \dots, q_m, r$ 
```

---

## 5.1 Gröbner bases

DEFINITION 5.1. Let  $J$  be an ideal in  $K\{\mathbf{X}; P\}$  and  $G$  be a finite subset of  $J \setminus \{0\}$ . We say that  $G$  is a Gröbner basis of  $J$  (with respect to the  $g.m.o \leq$  and the conic decomposition  $(T_i)_{i \in I}$ ) when:

$$\text{lm}(J) = \bigcup_{g \in G, i \in I} T_i(g)\text{lm}_i(g).$$

In Proposition 5.3, we prove that every ideal  $J$  within  $K\{\mathbf{X}; P\}$  contains a Gröbner basis. The demonstration relies on the fact that the set  $T_i(f)$  is a finitely generated  $T_i$ -module, as proved in the following lemma:

LEMMA 5.2. For all  $i \in I$ , there is a finite subset  $E_i \subset \text{lm}(J) \cap T_i$  such that  $\text{lm}(J) \cap T_i = T_i \cdot E_i$ . In case  $J = (f)$  is principal, we have  $\text{lm}((f)) \cap T_i = T_i(f)\text{lm}_i(f) = T_i \cdot E_i$ , and so  $T_i(f) = T_i \cdot \left\{ \frac{t}{\text{lm}_i(f)}, t \in E_i \right\}$  is finitely generated over  $T_i$ .

PROOF. By Dickson's lemma, there exists a finite subset  $E_i$  of  $\text{lm}(J) \cap T_i$  which generates the ideal  $\langle \text{lm}(J) \cap T_i \rangle_{K[T_i]}$  of  $K[T_i]$ . Then  $\text{lm}(J) \cap T_i = T_i \cdot E_i$ . If  $(J) = (f)$  is principal, then  $\text{lm}((f)) \cap T_i = T_i(f)\text{lm}_i(f)$  and we can conclude with Def. 3.14.  $\square$

PROPOSITION 5.3. Let  $J$  be an ideal of  $K\{\mathbf{X}; P\}$ . Then  $J$  contains a Gröbner basis.

PROOF. By Lemma 5.2, there are finite subsets  $E_i \subset T_i$  such that  $\text{lm}(J) = \bigcup_{i \in I} T_i \cdot E_i$ . For all  $t \in \bigcup_{i \in I} E_i$  choose an element  $f_t \in J$  such that  $\text{lm}(f_t) = t$ . Thanks to Def 3.14,  $\text{lm}_i(f_t) = \text{lm}(f_t) = t$ . We prove that the finite set  $G = \{f_t \mid t \in \bigcup_{i \in I} E_i\}$  is a Gröbner basis of  $J$ . Let  $u \in \text{lm}(J)$ . Let  $i$  be such that  $u \in T_i$ . Then there is some  $(v, t) \in T_i \times E_i$  such that  $u = vt$ . Let  $c_t = \text{lc}(f_t)$  and  $c_\alpha X^\alpha$  be a term of  $f_t$ . Then  $c_\alpha X^\alpha < c_t t$ . We remark that on the one hand  $\text{val}_P(c_\alpha X^\alpha) > \text{val}_P(c_t t)$  if and only if  $\text{val}_P(c_\alpha v X^\alpha) > \text{val}_P(c_t vt)$ , and on the other hand, if  $\text{val}_P(c_\alpha X^\alpha) = \text{val}_P(c_t t)$ , then by item (2) of Def 3.4 and the fact that  $v$  and  $t$  are in  $T_i$ , we can remark that  $X^\alpha <_\omega t$  implies  $X^\alpha v <_\omega tv$ . In any case,  $c_\alpha X^\alpha v < c_t vt$ . Thus,  $\text{lm}(vf_t) = vt = u \in T_i$ . Consequently,  $v \in T_i(f_t)$  and  $u \in T_i(f_t)\text{lm}_i(f_t)$ . We can then conclude that  $G$  is a GB of  $J$ .  $\square$

PROPOSITION 5.4. Let  $G$  be a Gröbner basis for an ideal  $J$  of  $K\{\mathbf{X}; P\}$ . We have

- (1)  $\forall f \in K\{\mathbf{X}; P\}, f \in J \iff \text{rem}(f, G) = 0$
- (2)  $G$  generates the ideal  $J$

PROOF. For the first part, if  $\text{rem}(f, G) = 0$ , then the multivariate division algorithm gives  $f \in J$ . Reciprocally, if  $f \in J$  and  $G$  is a Gröbner basis of  $J$ , then by (2) of Prop 4.9 the remainder of the division of  $f$  by  $J$  is necessarily 0. For the second part, if  $f \in J$ , then, thanks to the above first part,  $\text{rem}(f, G) = 0$ , and so  $f \in J$ .  $\square$

## 5.2 S-pairs and Buchberger criterion

In the polynomial case, the S-pair of two polynomials  $f$  and  $g$  is formed to eliminate the minimal multiple of their leading terms, relying on the concept of least common multiple in  $N$ .

However, in the Laurent setting, where  $T$  is a group and the notions of lcm and gcd of two monomials in  $T$  are useless, the replacement for the lcm of  $f$  and  $g$  is the finite set introduced in Proposition 5.5. Notice that this set now depends on the cone in which the cancellation occurs.

By the same argument as in Lemma 5.2, we have:

PROPOSITION 5.5. For  $i \in I$  and  $f, g \in K\{\mathbf{X}; P\}$ ,  $\text{lm}_i(f)T_i(f) \cap \text{lm}_i(g)T_i(g) \subset T_i$  is a finitely generated  $T_i$ -module.

DEFINITION 5.6 (S-PAIR). Let  $f, g \in K\{\mathbf{X}; P\}$  and  $i \in I$ . For  $v \in \text{lm}_i(f)T_i(f) \cap \text{lm}_i(g)T_i(g)$ , we define:

$$S(i, f, g, v) := \text{lc}_i(g) \frac{v}{\text{lm}_i(f)} f - \text{lc}_i(f) \frac{v}{\text{lm}_i(g)} g.$$

LEMMA 5.7. With the same notations as in Definition 5.6, we have  $\text{lt}(S(i, f, g, v)) < \text{lc}_i(f)\text{lc}_i(g)v$ .

PROOF. Since  $v \in T_i(f)\text{lm}_i(f) \cap T_i(g)\text{lm}_i(g)$ , there exists  $m_f \in T_i(f)$  and  $m_g \in T_i(g)$  such that  $v = \text{lm}(m_f f) = \text{lm}(m_g g) = m_f \text{lm}_i(f) = m_g \text{lm}_i(g)$ . Then the leading terms of  $\text{lc}_i(g) \frac{v}{\text{lm}_i(f)} f$  and  $\text{lc}_i(f) \frac{v}{\text{lm}_i(g)} g$  both equal  $\text{lc}_i(f)\text{lc}_i(g)v$ . They cancel out leaving  $\text{lt}(S(i, f, g, v)) < \text{lc}_i(f)\text{lc}_i(g)v$ .  $\square$

Since it involves all objects introduced so far, we give a detailed proof of the adaptation of the following classical cancellation lemma (see also e.g. [3, Lemma 5.1]).

LEMMA 5.8. Let  $h_1, \dots, h_m \in K\{\mathbf{X}; P\}$  and  $i \in I$ . For  $1 \leq j \leq m - 1$ , let  $U(i, h_j, h_{j+1})$  be a finite system of generators as in Proposition 5.5. Suppose that there are  $t_1, \dots, t_m \in T\{\mathbf{X}; P\}$ ,  $u \in T_i$  and  $c \in \text{val}(K^\times)$  such that

- for all  $j \in \{1, \dots, m\}$ ,  $\text{lt}(t_j h_j) = c_j u$  with  $\text{val}(c_j) = c$
- $\text{lt}(\sum_{j=1}^m t_j h_j) < c_1 u$ .

Then there are elements  $d_j \in K$ ,  $v_j \in U(i, h_j, h_{j+1})$  for  $1 \leq j \leq m - 1$  and  $t'_m \in K\{\mathbf{X}; P\}$  such that:

- (1)  $\sum_{j=1}^m t_j h_j = \sum_{j=1}^{m-1} d_j \frac{u_j}{v_j} S(i, h_j, h_{j+1}, v_j) + t'_m h_m$ .
- (2)  $\text{val}_P(t'_m h_m) > \text{val}_P(uc_1)$ .
- (3)  $\frac{u_j}{v_j} \in T_i$  for all  $j < m$ .
- (4) For all  $j < m$ ,  $\text{val}(d_j \text{lc}_i(h_j)\text{lc}_i(h_{j+1})) \geq c$ .

PROOF. Write  $p_j = \frac{t_j h_j}{c_j}$ ,  $e_j = \sum_{k=1}^j c_k$  and  $t_j = \gamma_j \tilde{t}_j$  for some  $\gamma_j \in K$  and some monomial  $\tilde{t}_j$ . By hypothesis  $u$  is in  $T_i$  and  $u = \tilde{t}_j \text{lm}_i(h_j) \in T_i(h_j)\text{lm}_i(h_j)$  for all  $j$ . This implies that for all  $j < m$  we have:

$$u \in T_i(h_j)\text{lm}_i(h_j) \cap T_i(h_{j+1})\text{lm}_i(h_{j+1}) = T_i \cdot U(i, h_j, h_{j+1})$$

We deduce that for all  $j < m$ , there exist  $k_j \in T_i$  and  $v_j \in U(i, h_j, h_{j+1})$  such that  $u = k_j v_j$ . Now write

$$\sum_{j=1}^m t_j h_j = e_1(p_1 - p_2) + \cdots + e_{m-1}(p_{m-1} - p_m) + e_m p_m \quad (2)$$

For all  $j < m$ , we have  $\text{lt}(t_j h_j) = c_j u = \gamma_j \text{lc}_i(h_j) \tilde{t}_j \text{lm}_i(h_j)$ , hence  $\frac{t_j}{c_j \tilde{t}_j} = \frac{1}{\text{lc}_i(h_j)}$ . For any  $j < m$ , put  $P_j = p_j - p_{j+1}$ . We can then write:

$$\begin{aligned} P_j &= \frac{u}{v_j} \left( \frac{v_j}{u} p_j - \frac{v_j}{u} p_{j+1} \right) = \frac{u}{v_j} \left( \frac{t_j v_j h_j}{c_j \tilde{t}_j \text{lm}_i(h_j)} - \frac{t_{j+1} v_j h_{j+1}}{c_{j+1} \tilde{t}_{j+1} \text{lm}_i(h_{j+1})} \right) \\ &= \frac{u}{v_j} \left( \frac{1}{\text{lc}_i(h_j) \text{lm}_i(h_j)} h_j - \frac{1}{\text{lc}_i(h_{j+1}) \text{lm}_i(h_{j+1})} h_{j+1} \right) \\ &= \frac{1}{\text{lc}_i(h_j) \text{lc}_i(h_{j+1})} \frac{u}{v_j} \left( \text{lc}_i(h_{j+1}) \frac{v_j}{\text{lm}_i(h_j)} h_j - \text{lc}_i(h_j) \frac{v_j}{\text{lm}_i(h_{j+1})} h_{j+1} \right) \\ &= \frac{1}{\text{lc}_i(h_j) \text{lc}_i(h_{j+1})} \frac{u}{v_j} S(i, h_j, h_{j+1}). \end{aligned}$$

Plugging in the last expression back into equation (2) gives the desired equality with  $d_j = \frac{e_j}{\text{lc}_i(h_j) \text{lc}_i(h_{j+1})}$  and  $t'_m = \frac{e_m}{c_m} t_m$ . It satisfies (1). The hypothesis forces  $\text{val}(e_m) > \text{val}(c_m)$ . Then we have  $\text{val}_P(t'_m h_m) = \text{val}(e_m) + \text{val}_P(u) > \text{val}(c_m) + \text{val}_P(u) = \text{val}_P(c_1 u)$ , which proves (2). In addition,  $\frac{u}{v_j} = k_j \in T_i$ , which proves (3). Finally, using that  $\text{val}(e_j) \geq c$  and  $d_j = \frac{e_j}{\text{lc}_i(h_j) \text{lc}_i(h_{j+1})}$ , one gets (4).  $\square$

To prove the final Buchberger criterion in Proposition 5.10, we need the following lemma:

**LEMMA 5.9.** *If  $f \in K\{X; P\}$  and  $i \in \llbracket 1, m \rrbracket$  are such that  $\text{lt}(f) < u$  for some  $u \in T_i$ , then for any  $v \in T_i$ ,  $\text{lt}(vf) < vu$ .*

**PROOF.** Take  $t$  a term of  $f$ . Then  $t < u$  and  $u, v \in T_i$  so 2. of Definition 3.4 implies  $vt < vu$ . Taking the maximum on the  $vt$ 's, we conclude.  $\square$

**PROPOSITION 5.10 (BUCHBERGER CRITERION).** *Let  $H = (h_1, \dots, h_m)$  be a family in  $K\{X; P\}$  and  $J$  the ideal generated by  $H$ . For each  $i \in I$  and  $h_j \neq h_k \in H$ , let  $U(i, h_j, h_k)$  be a finite system of generators of the  $T_i$ -module  $\text{lm}_i(h_j)T_i(h_j) \cap \text{lm}_i(h_k)T_i(h_k)$ . The following are equivalent:*

- (1)  $H$  is a Gröbner basis of  $J$
- (2) For all  $i \in I$ ,  $h_j \neq h_k, v \in U(i, h_j, h_k)$ ,  $\text{rem}(S(i, h_j, h_k, v), H) = 0$ .

**PROOF.** The fact that (1) implies (2) is immediate from Prop 4.9.

We now prove that (2) implies (1). By contradiction, assume that (2) is true and that  $H$  is not a Gröbner basis of  $J$ . Then there exists  $f \in J$  such that  $\text{lm}(f) \notin \bigcup_{i \in I, 1 \leq j \leq m} T_i(h_j) \text{lm}_i(h_j)$ . Since,  $f \in J = (h_1, \dots, h_m)$ , we can write  $f = \sum_{j=1}^m q_j h_j$  for some  $q_j$  in  $K\{X; P\}$ . Write  $\Delta(j)$  to be the set of terms of  $q_j$ . We can rewrite  $f$  as  $\sum_{j=1}^m \sum_{\alpha \in \Delta(j)} t_{j,\alpha} h_j$ . For such a writing of  $f$ , define  $u = \max\{\text{lt}(t_{j,\alpha} h_j), 1 \leq j \leq m, \alpha \in \Delta(j)\}$  and write the term  $u$  as  $u = c \tilde{u}$  for some  $c \in K$  and some monomial  $\tilde{u}$ . We have  $\text{lt}(f) < u$  because  $\text{lm}(f) \notin \bigcup_{i,j} T_i(h_j) \text{lm}_i(h_j)$  and a cancellation has to appear.

Thus,  $\text{val}_P(u)$  is upper-bounded. Since  $\text{val}$  is discrete, there is a maximal  $\text{val}_P(u)$  among all possible expressions of  $f = \sum_{j=1}^m q_j h_j$ .

Among the expressions reaching this valuation, Lemma 3.20 ensure there is one such that  $u$  is minimal. Let  $i$  be such that  $u \in T_i$ . Define  $Z = \{(j, \alpha) \in \llbracket 1, m \rrbracket \times \Delta(j), \text{s.t. } \text{lt}(t_{j,\alpha} h_j) = \gamma u, \gamma \in K, \text{val}(\gamma) = 0\}$  and  $Z' = \{(j, \alpha) \in \llbracket 1, m \rrbracket \times \Delta(j), \text{s.t. } \text{lt}(t_{j,\alpha} h_j) < u\}$ . We can then write:

$$f = \sum_{(j,\alpha) \in Z} t_{j,\alpha} h_j + \sum_{(j,\alpha) \in Z'} t_{j,\alpha} h_j \quad (3)$$

Let  $g := \sum_{(j,\alpha) \in Z} t_{j,\alpha} h_j$ . We have  $\text{lt}(g) = \text{lt}(f) - \text{lt}(\sum_{(j,\alpha) \in Z'} t_{j,\alpha} h_j) < u$  and  $\text{lt}(t_{j,\alpha} h_j) = c_{j,\alpha} \tilde{u}$  for all  $(j, \alpha) \in Z$ , where the  $c_{j,\alpha}$  all have the same valuation. So  $g$  satisfies the conditions of Lemma 5.8 and we can write

$$g = \sum_{j=1}^{m-1} d_j \frac{\tilde{u}}{v_j} S(i, h_j, h_{j+1}, v_j) + t'_m h_m \quad (4)$$

for some  $d_j \in K$ ,  $v_j \in U(i, h_j, h_{j+1})$ ,  $\text{val}(d_j \text{lc}_i(h_j) \text{lc}_i(h_{j+1})) \geq \text{val}(c)$  and  $\tilde{u}/v_j \in T_i$  for  $j < m$ , and with  $\text{lt}(t'_m h_m) < u$ . Now we use the hypothesis that all the S-pairs of elements of  $H$  reduce to zero. For each  $j < m$  we can write

$$S(i, h_j, h_{j+1}, v_j) = \sum_{l=1}^m q_l^{(j)} h_l, \quad (5)$$

for some  $q_l^{(j)}$ 's in  $K\{X; P\}$  satisfying

$$\begin{aligned} \text{lt}(q_l^{(j)} h_l) &\leq \text{lt}(S(i, h_j, h_{j+1}, v_j)), \\ &< \text{lc}_i(h_j) \text{lc}_i(h_{j+1}) v_j, \end{aligned}$$

where the last inequality comes from Lemma 5.7. Since  $v_j \in T_i$  and  $\tilde{u}/v_j \in T_i$ , we can apply Lemma 5.9:

$$\begin{aligned} \text{lt}\left(\frac{\tilde{u}}{v_j} q_l^{(j)} h_l\right) &< \text{lc}_i(h_j) \text{lc}_i(h_{j+1}) v_j \frac{\tilde{u}}{v_j}, \\ &= \text{lc}_i(h_j) \text{lc}_i(h_{j+1}) \tilde{u}. \end{aligned}$$

Finally, using that  $\text{val}(d_j \text{lc}_i(h_j) \text{lc}_i(h_{j+1})) \geq \text{val}(c)$ , we deduce that for all  $l \in \llbracket 1, m \rrbracket$  and  $j \in \llbracket 1, m-1 \rrbracket$ :

$$\text{lt}\left(d_j \frac{\tilde{u}}{v_j} q_l^{(j)} h_l\right) < u.$$

Inserting the expressions of  $d_j \frac{\tilde{u}}{v_j} S(i, h_j, h_{j+1}, v_j)$  as  $\sum_{l=1}^m d_j \frac{\tilde{u}}{v_j} q_l^{(j)} h_l$  in Equations (4) and then (3), we get an expression of  $f$  in terms of the  $h_j$ 's with strictly smaller  $u$ , contradicting its minimality.  $\square$

### 5.3 Buchberger's algorithm

**PROPOSITION 5.11.** *Algorithm 2 on page 7 is correct and terminates, in the sense that it calls the multivariate division a finite number of times.*

**PROOF.** Correctness of the output is clear thanks to the Buchberger criterion of Proposition 5.10. For the termination, we prove that the addition of a new  $r$  to  $H$  (on Line 10) can only happen a finite amount of times. Indeed, let us assume that there is some input  $J$  such that there is an infinite amount of non-zero  $r$  happening on Line 8. Let  $i \in I$  be an index such that there is infinite amount of  $\text{lt}(r)$  in  $T_i$  and let  $H_j$  be an indexation of all the states of the set  $H$  throughout Algo. 2. Using the second property of multivariate division in Prop. 4.9, we can extract from the non-decreasing

sequence  $(\langle \text{lm}_i(g)T_i(g), g \in H_j \rangle_{K[T_i]})_{j \in \mathbb{N}}$  of  $K[T_i]$ -ideals a strictly increasing one. This is not possible since  $K[T_i]$  is noetherian.  $\square$

We refer to the Annex for a SAGEMATH demo and explicit examples, mostly over  $K[X^{\pm 1}]$ .

---

**Algorithm 2:** Buchberger algorithm in  $K\{X; P\}$

---

```

input :  $J = (h_1, \dots, h_m)$  an ideal of  $K\{X; P\}$ 
output : a Gröbner basis of  $J$ 
1  $H \leftarrow \{h_1, \dots, h_m\}; B \leftarrow \{(h_i, h_j), 1 \leq i < j \leq m\}$ 
2 while  $B \neq \emptyset$  do
3    $(f, g) \leftarrow$  element of  $B; B \leftarrow B \setminus \{(f, g)\};$ 
4   for  $i \in I$  do
5      $U(i, f, g) \leftarrow$  finite set of generators of
        $\text{lm}_i(f)T_i(f) \cap \text{lm}_i(g)T_i(g);$ 
6     for  $v \in U(i, f, g)$  do
7        $\_ , r \leftarrow$  division( $S(i, f, g, v), H$ ); // Algo 1
8       if  $r \neq 0$  then
9          $B \leftarrow B \cup \{(h, r), h \in H\};$ 
10         $H \leftarrow H \cup \{r\}$ 
11 return  $H$ 

```

---

## 6 IMPLEMENTATION

### 6.1 Computation of the $T_i(f)$ 's

To make our Buchberger algorithm fully explicit, we need a method to compute the finite set of generators  $U(i, f, g)$  in line 5 of Algorithm 2. This problem was not addressed when g.m.o.'s for Laurent polynomials were introduced in [9]. Our idea is to compute the  $T_i(f)$ 's, for which we provide a general formula in Theorem 6.2.

The following definitions are motivated by the fact that for particular g.m.o.'s (such as those of Examples 3.8 to 3.10), the sets  $A_i(f)$ 's,  $\Delta_{i,j}$ 's and  $U_i(f)$ 's can be obtained by classical computations in polyhedral geometry.

DEFINITION 6.1. *We define:*

- (1)  $A_i(f) := \{t \in T, \text{tlm}_i(f) \in T_i\}$ .
- (2)  $\Delta_{i,j}(f) := \{t \in T, t \in A_i(f) \cap A_j(f), \text{tlm}_i(f) > \text{tlm}_j(f)\}$ .
- (3)  $U_i(f) := A_i(f) \cap \bigcap_{j \in I, \text{lm}_i(f) \neq \text{lm}_j(f)} (A_j(f)^c \cup \Delta_{i,j}(f))$ .

THEOREM 6.2. *For any  $i \in I$ ,*

$$T_i(f) = U_i(f) \quad (6)$$

PROOF. We first prove that  $T_i(f) \subset U_i(f)$ . Let  $t \in T_i(f)$ . Then  $\text{lm}(tf) \in T_i$  so  $\text{tlm}_i(f) = \text{lm}(tf) \in T_i$ , hence  $t \in A_i(f)$ . Let  $j \in I$  be such that  $\text{lm}_i(f) \neq \text{lm}_j(f)$ . We can write that  $\text{tlm}_i(f) = \text{lm}(tf) \geq \text{tlm}_j(f)$  and since  $\text{lm}_i(f) \neq \text{lm}_j(f)$ , this inequality becomes strict:  $\text{tlm}_i(f) > \text{tlm}_j(f)$ . Depending on whether  $\text{tlm}_j(f) \in T_j$  or not,  $t$  is then in  $A_j^c$  or  $\Delta_{i,j}$ . In conclusion,  $t \in U_i(f)$  and  $T_i(f) \subset U_i(f)$ .

We prove the converse inclusion. Let

$$t \in U_i(f) = A_i \cap \bigcap_{j \in I, \text{lm}_i(f) \neq \text{lm}_j(f)} (A_j^c \cup \Delta_{i,j}).$$

From  $t \in A_i$ , we get that  $\text{tlm}_i(f) \in T_i$ . Let  $j \in I$  be such that  $\text{lm}(tf) \in T_j$ . Then  $\text{lm}(tf) = \text{tlm}_j(f) \in T_j$  and  $t \in A_j(f)$ . If  $j$  is such that  $\text{lm}_i(f) = \text{lm}_j(f)$  then  $\text{tlm}_i(f) = \text{tlm}_j(f)$ , hence  $\text{lm}(tf) = \text{tlm}_i(f)$  and  $\text{lm}(tf) \in T_i$ . Otherwise, our assumptions provide that  $t \in U_i$  but not in  $A_j(f)$ , hence  $t \in \Delta_{i,j}(f)$ . Consequently,  $\text{tlm}_i(f) > \text{tlm}_j(f) = \text{lm}(tf)$  which is in contradiction with the definition of  $\text{lm}(tf)$ . In conclusion,  $\text{lm}(tf) = \text{tlm}_i(f) \in T_i$ , hence  $t \in T_i(f)$  and  $T_i(f) = U_i(f)$ .  $\square$

If the  $T_i$ 's are defined as  $C_i \cap \mathbb{Z}^n$  for  $C_i$  a rational polyhedral cone, we can compute the  $A_i$ 's and  $A_i^c$ 's by solving systems of linear inequalities. Moreover, if the g.m.o. is defined by a function  $\phi : T \rightarrow \mathbb{Q}_{\geq 0}$  as in Lemma 3.6, the  $\Delta_{i,j}(f)$ 's can be computed by computing first the  $t$ 's such that  $\phi(\text{tlm}_i(f)) > \phi(\text{tlm}_j(f))$ . It obviously depends on  $\phi$  but if the restriction of  $\phi$  to each  $T_i$  is linear, the desired set is again obtained by solving systems of linear inequalities. For the case  $\phi(\text{tlm}_i(f)) = \phi(\text{tlm}_j(f))$ , since our tie-break order  $<_G$  is a group order, then  $\text{tlm}_i(f) > \text{tlm}_j(f)$  if and only if  $\text{lm}_i(f) > \text{lm}_j(f)$  which does not depend on  $t$ . In total, one can compute generators of all the  $T_i(f)$ 's relying only on solving systems of linear inequalities from the following:

- (1) Computing the  $\text{lm}_i$ 's,
- (2) Computing the  $A_i(f)$ 's and  $A_i(f)^c$ 's,
- (3) Computing the  $\Delta_{i,j}(f)$ 's (for  $i, j$  such that  $\text{lm}_i(f) \neq \text{lm}_j(f)$ ),
- (4) Use formula (6) to obtain the  $T_i(f)$ 's

In the next part, we will make the computation of the  $T_i(f)$ 's completely explicit for g.m.o.'s whose underlying conic decomposition is the decomposition of Example 3.2.

### 6.2 A particular conic decomposition

Taking a closer look at Algorithm 2, we can identify two areas for minimizing the number of S-pairs to be reduced:

- (1) Within the for loop at line 4, by minimizing the value of  $|I|$  (the number of cones).
- (2) Within the for loop at line 6, by specifying that the set of generators  $U(i, f, g)$  should contain only one element.

The first point is addressed comprehensively in the following proposition, while Theorem 6.5 offers a partial solution to the second point.

PROPOSITION 6.3. *A conic decomposition of  $T \cong \mathbb{Z}^n$  contains at least  $n + 1$  cones.*

PROOF. Assume, for the sake of contradiction, the existence of a conic decomposition containing strictly less than  $n + 1$  cones. Without loss of generality, we may assume that it contains exactly  $n$  cones  $T_1, \dots, T_n$ .

By (1) of Definition 3.1, each  $T_i$  is contained in a cone  $C_i := a_{i,1}\mathbb{R}_{\geq 0} \oplus \dots \oplus a_{i,n}\mathbb{R}_{\geq 0}$  for  $(a_{i,1}, \dots, a_{i,n}) \in \mathbb{Z}^n$  a basis of  $\mathbb{R}^n$ , and by (2) of Definition 3.1,  $\cup T_i = \mathbb{Z}^n$ . Thus  $\cup T_i = \mathbb{Z}^n$  is contained in the closed subset  $C := \cup C_i$  of  $\mathbb{R}^n$ . We show that  $\mathbb{Z}^n \setminus C \neq \emptyset$ , resulting in a contradiction.

We first show that  $\mathbb{R}^n \setminus C \neq \emptyset$ . For each  $i$  there exists  $v_i \in \mathbb{R}^n \setminus \{0\}$  such that  $C_i \setminus \{0\}$  is contained in the linear open half-space  $L(v_i) := \{u \in \mathbb{R}^n \mid (u, v_i) < 0\}$ , and there exists an open neighborhood  $E_i$  of  $v_i$  such that the inclusion  $C_i \subset L(v_i)$  is still true for



$w_i \in E_i$ . Thus we can choose the  $v_i$ 's such that  $(v_1, \dots, v_n)$  is a basis of  $\mathbb{R}^n$ . By Gram-Schmidt process, there exists an orthonormal basis  $b_1, \dots, b_n$  such that the matrice of  $(v_1, \dots, v_n)$  in that basis is upper triangular with strictly positif coefficients on the diagonal. By construction, the vector  $b_n$  satisfies  $\langle b_n, v_i \rangle \geq 0$  for each  $i$ . Thus  $b_n \in \mathbb{R}^n \setminus \cup_i L(v_i) \subset \mathbb{R}^n \setminus \cup_i C_i = \mathbb{R}^n \setminus C$ , and so  $\mathbb{R}^n \setminus C \neq \emptyset$ .

Since  $\mathbb{R}^n \setminus C$  is a non-empty open set, it contains a basis  $(c_1, \dots, c_n)$  of  $\mathbb{R}^n$ , and thus contains the cone  $c_1 \mathbb{R}_{\geq 0} \oplus \dots \oplus c_n \mathbb{R}_{\geq 0}$  which itself contains an infinity of elements of  $\mathbb{Z}^n$ .  $\square$

LEMMA 6.4. *Let  $\leq$  be a g.m.o such that the underlying conic decomposition  $(T_i)_{i \in I}$  satisfies:*

$$\forall i, j \in I, \text{gr}\langle T_i \cap T_j \rangle \cap T_i = T_i \cap T_j.$$

For all  $i, j \in I, f \in K[\mathbf{X}^{\pm 1}]$ , we have:

$$(s \in T_i \cap T_j, t \in T_i(f), st \in T_i(f) \cap T_j(f)) \implies t \in T_j(f).$$

PROOF. Writing  $l := \text{lm}_i(f) = \text{lm}_j(f)$ , we have  $\text{lm}(tf) = tl \in T_i$  and  $\text{lm}(stf) = stl \in T_i \cap T_j$  because  $t \in T_i(f)$  and  $st \in T_i(f) \cap T_j(f)$ . We have to show that  $tl \in T_j$ . Now  $tl = s^{-1}stl \in \text{gr}\langle T_i \cap T_j \rangle \cap T_i = T_i \cap T_j$  by the hypothesis.  $\square$

The conic decompositions of Examples 3.2 and 3.3 satisfy the hypothesis of Lemma 6.4

THEOREM 6.5. *Let  $\leq$  be a g.m.o such that the underlying conic decomposition is  $(T_0, T_1, \dots, T_n)$  as defined in Example 3.2. For all  $f \in K[\mathbf{X}^{\pm 1}]$  and  $0 \leq i \leq n$ ,  $T_i(f)$  is a monogenous  $T_i$ -module.*

PROOF. Assume, for the sake of contradiction, that there exists an index  $i$  and  $f \in K[\mathbf{X}^{\pm 1}]$  such that the set of minimal generators of  $T_i(f)$  contains at least two elements  $a \neq b \in \mathbb{Z}^n$ . We will focus on the case where  $i = 0$ ; the proof for  $i \neq 0$  is essentially the same. We have  $a + T_0 = \bigcap_j \{x \in \mathbb{Z}^n \mid x_j \geq a_j\}$  and  $b + T_0 = \bigcap_j \{x \in \mathbb{Z}^n \mid x_j \geq b_j\}$ . Let  $p \in \mathbb{Z}^n$  be the vector such that  $p_j = \min(a_j, b_j)$  for  $1 \leq j \leq n$ , which is necessarily different from  $a$  and  $b$ . By definition of  $p$ , we have  $a + T_0 \cup b + T_0 \subset p + T_0$ . Therefore,  $p \notin T_0(f)$ ; otherwise,  $\{a, b\}$  wouldn't be contained in a minimal set of generators of  $T_0(f)$ . Since the  $T_i(f)$ 's cover  $\mathbb{Z}^n$ , there exists  $k \neq 0$  such that  $p \in T_k(f)$ . We have  $T_0 \cap T_k = \{x_k = 0\} \cap \left(\bigcap_{j \neq k} \{x_j \geq 0\}\right)$ . Consequently, the set  $p + T_0 \cap T_k$  contains either  $a$  or  $b$ , depending on whether  $\min(a_k, b_k) = a_k$  or  $b_k$ . Without loss of generality, assume it contains  $a$ . From  $a \in p + T_0 \cap T_k$ , we deduce that  $a - p \in T_0 \cap T_k$ . Also, from  $a \in p + T_0 \cap T_k \subset p + T_k$  and  $p \in T_k(f)$ , we get that  $a \in T_k(f)$  and thus  $a \in T_k(f) \cap T_0(f)$ . Consequently we can write  $(a - p) + p = a$  with  $(a - p) \in T_k \cap T_0$ ,  $p \in T_k(f)$  and  $a \in T_k(f) \cap T_0(f)$ . According to Lemma 6.4, this implies  $p \in T_0(f)$ , contradicting  $p \notin T_0(f)$ .  $\square$

Altogether, Proposition 6.3 and Theorem 6.5 assert that, for a g.m.o defined over the decomposition of Example 3.2, the number of S-pairs to be reduced in Algorithm 2 is minimized to the greatest extent possible. For g.m.o's of this nature, we can determine the generator  $g_i$  of  $T_i(f)$  using a straightforward descent algorithm. Recall that  $T_i$  is the monoid generated by  $H_i = \{e_1, \dots, \hat{e}_i, \dots, e_n\} \cup \{-e_1 + \dots + e_n\}$ , where  $(e_1, \dots, e_n)$  denote the canonical basis of  $\mathbb{R}^n$ . We first find a  $t \in T_i(f)$  so that  $T_i(f) = g_i + T_i \subset t + T_i$ . Then, for each generator  $h \in H_i$  of  $T_i$ , we compute  $t = t - h$  until

$t + T_i \not\subset T_i(f)$ , for then we set  $t = t + h$ . At the end,  $t = g_i$ . The procedure is detailed in Algorithm 3.

REMARK 6.6. *For a g.m.o as in Theorem 6.5, it can easily be shown that if  $g_j, g_k$  are the generators of  $T_j(f), T_k(f)$  respectively, then  $\|g_j - g_k\|_\infty \leq 1$  (with  $\|\cdot\|_\infty$  the maximum norm on  $\mathbb{R}^n$ ). To speed up the computation of the  $T_i(f)$ 's, we can first determine  $g_0$  using Algorithm 3. Then the generators of the  $T_i(f)$ 's for  $i \in \llbracket 1, n \rrbracket$  are contained within the (small) set  $\{y \in \mathbb{Z}^n, \|y - g_0\|_\infty \leq 1\}$ .*

REMARK 6.7. *If in addition the g.m.o is defined by a function as outlined in Lemma 3.6, and this function restricts to an integral linear function on each cone (as illustrated in Examples 3.8 and 3.9), then the generator of  $T_i(f)$  can be expressed directly as the intersection of  $n$  affine hyperplanes defined by integral equalities.*

We emphasize that the time spent computing the generators of the  $T_i(f)$ 's is negligible compared to the time spent reducing S-pairs in Algorithm 2.

---

### Algorithm 3: Generator of $T_i(f)$

---

**input** :  $0 \leq i \leq n, f \in K\{\mathbf{X}; P\}, H_i = \{h_1, \dots, h_n\}$   
generators of  $T_i$ , and the g.m.o. of Example 3.2  
**output** :  $g \in T$  such that  $T_i(f) = g + T_i$

- 1  $t \leftarrow$  a monomial such that all monomials of  $t \times \text{in}_P(f)$  are in  $T_i$ ; // Use Remark 3.15
- 2 **for**  $h \in H_i$  **do**
- 3     **while**  $t \in T_i(f)$  **do**
- 4          $t \leftarrow t - h$ ;
- 5          $t \leftarrow t + h$ ;
- 6 **return**  $t$

---

## 7 FUTURE WORKS

### 7.1 Precision and integral bases

We leave to future works the study of the computation of GB in  $K\{\mathbf{X}; P\}^\circ$ , the valuation ring of  $K\{\mathbf{X}; P\}$ , as in [1, §2.3, §2.4]. We believe that if one makes the cancellation of the coefficients in  $v$  in Def. 5.6 happens with  $\text{lcm}(\text{lc}_i(f), \text{lc}_i(g)) \in K^\circ$  instead of  $\text{lc}_i(f) \times \text{lc}_i(g)$ , the good precision behavior of [1, Thm 3.8] will be preserved. We will investigate the reduction to the residue ring  $K^\circ / \pi K^\circ[\mathbf{X}^{\pm 1}]$  of  $K\{\mathbf{X}; P\}^\circ$  and its compatibility with GB as in [1, Prop 2.28].

### 7.2 Applications of Gröbner cases

We intend to investigate natural applications of Gröbner bases, e.g. ideal operations and dimension.

### 7.3 UAGB and tropical analytic geometry

Universal analytic Gröbner bases (UAGB) for polynomial ideals have been defined in [3] as a finite set of polynomial elements of the ideal such that it is a Gröbner basis in any completion of the ideal in Tate algebras. In [14], effective algorithms have been presented to compute them, along with an application to effective tropical (analytic) geometry. This article has also raised questions

and conjectures on their existence and computation in a context of polyhedral affinoid algebras. We intent to build on the presented theory of Gröbner bases over polytopal affinoid algebras to complete the resolution of these questions and conjectures. Furthermore, in [10, Remark 8.8], the author has asked the question of the existence of tropical bases for ideals in polyhedral affinoid algebras, as it would be a cornerstone for tropical analytic geometry. Building on our results and the computation of UAGB, we intent to investigate the existence of finite tropical bases in this context, and elaborate on this to advance toward effective computation of tropical analytic varieties.

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## ANNEX: SAGEMATH IMPLEMENTATION

We have implemented in SAGEMATH all the algorithms presented in this paper for the Laurent polynomial case: Algorithms 1 and 2 can be applied over  $K[X^{\pm 1}]$  as in [9], the main difference being the way leading terms are defined. As far as we know, this is the first ever implementation of the ideas of [9]. It has been made possible thanks to Section 6. The integration of our implementation into the SAGEMATH sources is currently underway. Discussions, comments and progress can be tracked on the official GitHub repository at:

<https://github.com/sagemath/sage/pull/37241>

The case of polytopal affinoid algebras, which is predominantly built upon the polynomial case, is currently a work in progress that can be tracked at

[https://github.com/vilanele/sage/tree/polytopal\\_algebras](https://github.com/vilanele/sage/tree/polytopal_algebras)

We plan to integrate it as well into the SAGEMATH sources as soon as it is ready.

**Short demo.** A new class constructor named `GeneralizedOrder` is introduced. The supported `g.m.o`’s are those for which the underlying conic decomposition is the decomposition of Example 3.2 and for which the order is defined by a group order and a score function as in Lemma 3.6. The group order and score function can be specified using the keywords `group_order` and `score_function` in the constructor. Defaults are the lexicographical order on  $\mathbb{Z}^n$  and the `min` function of Example 3.9.

```
In: from sage.rings.polynomial.generalized_order import
    GeneralizedOrder
In: GeneralizedOrder(3)
Out: Generalized order in 3 variables using (lex, min)
```

Another example, using this time the score function `degmin` of Example 3.8:

```
In: G = GeneralizedOrder(2, score_function='degmin'); G
Out: Generalized order in 2 variables using (lex, degmin)
```

Now we can compare tuples:

```
In: G.greatest_tuple((-2,3), (1,2))
Out: (-2,3)
In: G.greatest_tuple_for_cone(2, (1,3), (-1,2), (-4,-3))
Out: (-4,-3)
```

The `LaurentPolynomialRing` constructor has been updated to accept instances of the new `GeneralizedOrder` class for the keyword `order`. Elements have new methods:

```
In: L.<x,y> = LaurentPolynomialRing(QQ, order=G)
In: f = 2*x^2*y^1 + x^-3*y - 3*y^-5
In: f.leading_monomial() // lm(f)
Out: y^-5
In: f.leadin_monomial_for_cone(1) // lm_1(f)
Out: x^-3*y
In: f.generator_for_cone(2) // T_2(f)
Out: x*y^2
```

We can reduce an element using multivariate division (Algorithm 1):

```
In: L = [x^-2*y^-1 + x*y, x^-2*y + x^2*y^-1]
In: f.generalized_reduction(L)
Out: (-y^3 + 2x^2*y^-1 - 3x^-1*y^-1, [x^-1*y^2 + 3x^-2*y^-2, -3x^-2*y^-4])
```

Lastly, within the `LaurentPolynomialIdeal` class, the method `groebner_basis` has been modified so that it uses Algorithm 2 when a generalized order is specified:

```
In: G = GeneralizedOrder(3, score_function='degmin')
In: L.<x,y,z> = LaurentPolynomialRing(QQ, order=G)
In: I = L.ideal([x^-3*y^-4 + x*y*z, x^3*y^-2 + y^-1*z])
In: I.groebner_basis()
Out: (x^3*y^-4 + x*y*z,
      x^3*y^-2 + y^-1*z,
      -y^-4 + x^-1*y^-2*z^-1)
```

Another example of Gröbner basis computation for an ideal in the ring  $\mathbb{Q}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ :

```
In: G = GeneralizedOrder(3, score_function='min')
In: L.<x,y,z> = LaurentPolynomialRing(QQ, order=G)
In: g1 = 1/2*x^-1*y + 3*y^-4*z^2 + y
In: g2 = 2*x^2*y^3*z^-1 - 1/3*x^-1*y^3*z^-6
In: I = L.ideal([g1, g2])
In: I.groebner_basis()
Out: (y + 1/2*x^-1*y + 3*y^-4*z^2,
      2*x^2*y^3*z^-1 - 1/3*x^-1*y^3*z^-6,
      1/4*y^5*z^5 - 3*x^2*z^7 + 3/2*x*z^7 + 1/3*y^5 + z^2,
      1/4*y^10*z^5 - 3/4*y^5*z^7 + 1/3*y^10 - 2/3*x*z^9 + 2*y^5*z^2 + 3*z^4,
      1/4*y^15*z^5 + 1/3*y^15 + 3*y^10*z^2 + 9*y^5*z^4 + 9*z^6,
      6*x^2*y^4*z^4 + 3*x*y^4*z^4 + 3*x^-1*y^-1*z)
```

Same ideal as above, but using the score function `degmin`:

```
In: G = GeneralizedOrder(3, score_function='degmin')
In: L.<x,y,z> = LaurentPolynomialRing(QQ, order=G)
In: g1 = 1/2*x^-1*y + 3*y^-4*z^2 + y
In: g2 = 2*x^2*y^3*z^-1 - 1/3*x^-1*y^3*z^-6
In: I = L.ideal([g1, g2])
In: I.groebner_basis()
Out: (y + 1/2*x^-1*y + 3*y^-4*z^2,
      2*x^2*y^3*z^-1 - 1/3*x^-1*y^3*z^-6,
      y^5*z^3 + 1/3*x^-2*y^5*z^-2 + x^-2,
      -1/16*y^5*z^6 - 1/12*y^5*z - 1/4*z^3 + 1/8*x^-1*z^3 - 1/16*x^-2*z^3,
      -1/6*x*y^3*z^-1 + 1/24*x^-1*y^3*z^-1 - 1/12*x^-2*y^-2*z^-4 + 1/24*x^-3*y^-2*z^-4,
      -1/36*y^3*z^-1 - 1/72*x^-1*y^3*z^-1 - 1/72*x^-3*y^-2*z^-4)
```

An example in the ring  $\mathbb{F}_9[x^{\pm 1}, y^{\pm 1}]$ :

```
In: G = GeneralizedOrder(2, score_function='degmin')
In: F = GF(9)
In: L.<x,y> = LaurentPolynomialRing(F, order=G)
In: g1 = x^2*y + y^-6
In: g2 = x^3*y^-2 + x^-6*y
In: g3 = x^-2*y + x^-1*y^-2
In: I = L.ideal([g1, g2, g3])
In: I.groebner_basis()
Out: (x^2*y + y^-6,
      x^3*y^-2 + x^-6*y,
      x^-2*y + x^-1*y^-2,
      -x*y + x^-2*y^-3,
      x^2*y + x^-2,
      y^-1 + x^-1,
      -y^2 + x^-1,
      x^-1*y^-1 + x^-2*y^-2)
```