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Higher spin gravity

(Encyclopedia of Mathematical Physics)

Xavier Bekaert

Institut Denis Poisson, Unité Mixte de Recherche 7013 du CNRS
Université de Tours, Université d'Orléans
Parc de Grandmont, 37200 Tours, France
`xavier.bekaert@univ-tours.fr`

Abstract

Key features of higher-spin extensions of ordinary gravity are briefly reviewed. The introduction of consistent interactions between massless fields with spin higher than two is a notoriously difficult challenge of modern theoretical physics, which is well-motivated by string theory and holographic duality. Particular emphasis is put on the fact that the kinematical foundations of the subject are firmly grounded on rigorous mathematical results. Moreover, the introduction of higher-spin interactions admits several formulations that make the former a well-posed problem in mathematical physics. A quick tour is offered through the main results and open issues in this area of research.

Invited contribution to the section “Classical and Quantum Gravity”
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Keywords: AdS/CFT Correspondence, Conformal Field Theory, Higher Spin Gravity, Higher Spin Symmetries, Holographic Duality, Relativistic Wave Equations, String Theory

Key points:

- Definition of higher-spin gravity
- Exotic regime of string theory and holographic duality
- Mathematical foundations grounded in representation theory and advanced topics in contemporary mathematical physics
- Perturbative introduction of interaction as a well-posed problem in mathematical physics
- Some key results

Introduction

Higher-spin gravity (also called higher-spin gauge theory) can be defined as any interacting relativistic field theory satisfying the following conditions, respectively on its vacuum solution, symmetries and spectrum:¹

- (1) It admits a maximally-symmetric (i.e. Minkowski, de Sitter or anti de Sitter) spacetime as vacuum solution (i.e. when all fields are turned off, except the metric).
- (2) Its Lie algebra of rigid symmetries contains:
 - (2.a) the symmetry (isometry or conformal) algebra of the vacuum solution in (1),
 - (2.b) some extra generators that transform, under the previous algebra (2.a), in a representation that is beyond the trivial, adjoint and fundamental (i.e. the vector or spinor) representations, and that is labelled by a Young diagram with at least two columns.
- (3) Its spectrum contains a tower of massless fields with at least:
 - (3.a) one massless field of helicity two (the “graviton”),
 - (3.b) one massless field of helicity greater than two (“higher spin”).

The above definition and terminology applies in any dimension (though the notion of “helicity” requires some adaptation) and allows supersymmetry to be easily included (one simply adds above the adjective “super” before “algebra” and “symmetry”).

The introduction of consistent interactions between massless higher-spin fields can be formulated as a well-posed problem in mathematical physics, offering a difficult but tantalising challenge.² In fact, it is notoriously difficult to introduce consistent (i.e. preserving the number of physical degrees of freedom) interactions between massless higher-spin fields (especially in dimensions strictly greater than 3). There is a long list (whose size monotonously increases over time) of no-go theorems preventing a naive construction of such interactions. See e.g. (Bekaert, Boulanger, Sundell, 2012; Ponomarev, 2023) for some extensive discussions of the

¹The rationale behind this broad definition is as follows. *Higher-spin symmetries* are characterised by the condition (2.b). The conditions (1), (2.a) and (2.b) have a hierarchical structure: the next condition requires the previous one in order to be even formulated. Similarly, the global symmetries mentioned in (2.a)-(2.b) correspond, respectively, to the local symmetries in (3.a)-(3.b) of the theory preserving the vacuum solution (remember that the usual covariant description of massless fields is as gauge fields). The condition (3.a) is a minimal requirement for talking about a “gravity” theory. Analogously, a theory satisfying the condition (3.b) is generically called a *higher-spin theory*, even if the condition (3.a) and/or the adjectives “massless” are relaxed everywhere in (3). For instance, from this point of view string theory can be seen as higher-spin theory where all fields of spin greater than two are massive.

²Strictly speaking, several (mathematically inequivalent) formulations of this physical problem exist, depending on the approach (e.g. variational, light-cone, formal PDE).

main no-go theorems and some ways out. Each time obstacles were overcome, various modern mathematical notions played a crucial role in the construction. Several objects of importance in contemporary mathematical physics have already appeared in higher-spin gravity: they originate from conformal geometry (Cartan connections, Fefferman-Graham ambient metric, tractor bundles), formal geometry of Partial Differential Equations (e.g. infinite jet bundles and their Cartan distributions), differential graded geometry (such as Q-manifolds, Batalin-Vilkovisky and Becchi-Rouet-Stora-Tyutin formalisms, Alexandrov-Kontsevich-Schwarz-Zaboronky sigma models) and deformation quantisation (for instance star products, Fedosov-like connections, formality theorems, strong homotopy Lie algebras, etc).

Two obstacles met by the construction of a unitary higher-spin gravity in dimension four (or more) are the following generic features:

- The spectrum must necessarily contain an infinite tower of fields of unbounded spin. Indirectly, this implies some tension with locality (at least in its strictest sense).
- In the formulations in terms of tensor gauge fields, there is a tension between higher-spin gauge symmetries and minimal coupling to gravity, especially around flat spacetime and assuming parity symmetry.

If one either considers lower spacetime dimensions (two or three), or go to Euclidean signature or relax unitarity, then there exists a variety of manageable examples of higher-spin gravity theories (e.g. Jackiw-Teitelboim, Chern-Simons, chiral, conformal). Higher-spin gravity theories are usually named after the corresponding gravity theory (whose Lie algebra is extended by higher-spin generators), e.g. “conformal higher-spin gravity” is understood as the higher-spin extension of Weyl’s conformal gravity.

In what follows, we give a quick tour through the main results and open issues in this area of research. The discussion is divided into four parts. In the first one, the early history of the subject is summarised from a mathematical physics perspective. In the second part, the problem of perturbatively introducing interactions between massless higher-spin fields is introduced, together with some of its difficulties. A list of some yes-go examples is also provided. In the third part, the modern perspectives brought on the subject by string theory and holographic duality are presented. In the fourth part, some strategies and mathematical tools used for attacking the interaction problem are sketched. We end up with a brief conclusion.

Many pedagogical reviews of various levels are available by now, from advanced ones to introductory ones. A non-exhaustive list is proposed at the end for further readings, together with few seminal papers cited in the body of the text.

Early history

The early theoretical works on higher-spins can be structured around 4 main questions, formulated here as mathematical classification research programs (ordered in logical progression and suitably generalised to suit modern considerations):

- **Wigner's programme (1939):** Classify all unitary irreducible representations of the isometry groups of maximally-symmetric spacetimes.
- **Bargmann-Wigner's programme (1948):** Associate a covariant linear differential equation to each unitary irreducible representation of the isometry groups of maximally-symmetric spacetimes, such that the space of inequivalent solutions carries the corresponding representation.
- **Fierz-Pauli's programme (1939):** Associate a quadratic covariant local Lagrangian to each unitary irreducible representation of the isometry groups of maximally-symmetric spacetimes, such that the space of inequivalent solutions to Euler-Lagrange equations carries the corresponding representation.
- **Fronsdal's programme (1978):** List all consistent, covariant, local, perturbative interactions deforming a positive sum (finite or not) of quadratic covariant local Lagrangians associated with unitary irreducible representations of the isometry group of a maximally symmetric spacetime.

Obviously, these programmes were originally restricted to four-dimensional Minkowski spacetime. The first two programmes involved Wigner. They were motivated by his modern and extremely profound view on the classification of elementary particles: combining the axioms of quantum mechanics and the principles of special relativity necessarily leads to the fact that the space of states (rays) of a free quantum relativistic particle is a unitary module of the spacetime isometry group. Consequently, there is a one-to-one correspondence between wave equations describing a free quantum relativistic particle and unitary representations of the spacetime isometry group. The Wigner programme was one of the seminal motivations behind the development of the mathematical theory of unitary representations, through the subsequent works of Bargmann, Gel'fand, Harish-Chandra, ... The second programme (i.e. Bargmann-Wigner's) consists in going from the abstract representation (classified by mathematicians) to a more concrete realisation (as solution space of a differential equation). This step is not trivial because it is not algorithmic: writing suitable relativistic equations is some sort of art. In fact, these classification programmes allowed to put some order in the plethora of relativistic wave equations available at that time: d'Alembert, Dirac, Fierz, Klein-Gordon, Majorana, Maxwell, Proca, Weyl, ...

At the beginning of the sixties, the proliferation of hadrons with “high” ($\geq 3/2$) spin was one of the main mystery of the strong nuclear interaction. Experimental plots suggested the existence of an infinite tower of hadrons, with unbounded spin. The attempts to model scattering cross-sections of higher-spin hadrons required the knowledge of the propagators for fields of arbitrary spin. This provided a new motivation for the Fierz-Pauli programme, i.e. the inverse problem of variational calculus with respect to the equations obtained in the Bargmann-Wigner programme. The Fierz-Pauli programme in four-dimensional Minkowski spacetime was completed in the late seventies, by Singh and Hagen for massive fields, and by Fang and Fronsdal for massless fields. These works (and their subsequent extensions) set the mathematical foundations of the subject, as far as the free theory is concerned.

The perturbative introduction of consistent interactions for massless spin-two fields and the corresponding reconstruction of Einstein’s gravity is sometimes called Gupta’s programme. The completion of Fierz-Pauli’s programme and the generalisation of Gupta’s programme for all “low” spins ($s \leq 2$) via supergravity lead to further generalise the latter for arbitrary spins (Fronsdal, 1978). It is important to point out that, from the point of view of mathematical physics, the Fronsdal program can be formulated as a well-posed problem which must therefore have an unambiguous answer. Even if it admits no solution in dimension four corresponding to a higher-spin gravity (as defined above), at least this would explain the experimental fact that particles of zero mass and high spin have never been observed. As explained later, a wealth of results on higher-spin interactions, supplemented with arguments coming from string theory, support the existence of an interacting solution, possibly with exotic locality properties.

Perturbative approach

As was explained above, the covariant gauge theories describing arbitrary free massless fields on constant-curvature spacetimes of dimension $D \geq 2$ are firmly established by means of the unitary representation theory of their isometry groups. For instance, the massless unitary irreducible representations of the Poincaré group $ISO(D-1, 1) = SO(D-1, 1) \ltimes \mathbb{R}^{D-1,1}$ are induced from unitary irreducible representations of the stabiliser subgroup $ISO(D-2) = SO(D-2) \ltimes \mathbb{R}^{D-2}$. The latter representations are finite-dimensional if and only if the Abelian normal subgroup \mathbb{R}^{D-2} acts trivially. The corresponding massless representations of the Poincaré group $ISO(D-1, 1)$ are usually called “helicity” (or “finite spin”) representations. It is standard to restrict one’s attention to the helicity representations induced from the irreducible representations of $SO(D-2)$ spanned by completely-symmetric traceless tensors of rank $s \in \mathbb{N}$. In order to have Lorentz invariance manifest, as well as second order local field equations with minimal field content, the theory is usually expressed (Fronsdal, 1978) in terms of completely-symmetric

and double-traceless tensor fields $h_{\mu_1 \dots \mu_s}$ ($\mu = 0, 1, \dots, D$) of rank s , in a way analogous to the metric formulation of gravity ($s = 2$). In Minkowski space-time $\mathbb{R}^{D-1,1}$ with flat metric $\eta_{\mu\nu} = \text{diag}(-1, +1, \dots, +1)$, the Fronsdal action for a free massless spin- s field is

$$\begin{aligned} S^{(0)} [h_{\mu_1 \dots \mu_s}] &= \frac{1}{2} \int d^D x \left(\partial_\nu h_{\mu_1 \dots \mu_s} \partial^\nu h^{\mu_1 \dots \mu_s} \right. \\ &\quad - \frac{s(s-1)}{2} \partial_\nu h^\lambda{}_{\lambda\mu_3 \dots \mu_s} \partial^\nu h^\rho{}_{\rho\mu_3 \dots \mu_s} + s(s-1) \partial_\nu h^\lambda{}_{\lambda\mu_3 \dots \mu_s} \partial_\rho h^{\nu\rho\mu_3 \dots \mu_s} \\ &\quad \left. - s \partial_\nu - h^\nu{}_{\mu_2 \dots \mu_s} \partial_\rho h^{\rho\mu_2 \dots \mu_s} - \frac{s(s-1)(s-2)}{4} \partial_\nu h^{\nu\rho}{}_{\rho\mu_2 \dots \mu_s} \partial_\lambda h^\sigma{}_{\lambda\sigma\mu_2 \dots \mu_s} \right), \quad [1] \end{aligned}$$

where the tensor field is double-traceless ($\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}h_{\mu_1 \dots \mu_s} = 0$). This action is invariant under the gauge transformations

$$\delta^{(0)}_\xi h_{\mu_1 \dots \mu_s} = \partial_{\mu_1} \xi_{\mu_2 \dots \mu_s} + \text{cyclic}, \quad [2]$$

where the gauge parameter $\xi_{\mu_1 \dots \mu_{s-1}}$ is a completely-symmetric and traceless ($\eta^{\mu_1\mu_2}\xi_{\mu_1 \dots \mu_{s-1}} = 0$) tensor field of rank $s-1$ and “cyclic” stands for the sum of terms necessary to ensure the symmetry of the right-hand side under permutations of the indices. The Lie algebra of field-independent gauge transformations such as [2] is of course Abelian. For spin $s = 1$, the quadratic action [1] is the Maxwell action of a vector gauge field, for which [2] are the usual $U(1)$ gauge transformations. For spin $s = 2$, the action [1] is the Pauli-Fierz action that is obtained from the Einstein-Hilbert action $S_{EH}[g]$ via the substitution $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and keeping only the quadratic term in the expansion. Moreover, the gauge transformations [2] correspond to linearised diffeomorphisms.

Non-Abelian gauge theories for “low spins” ($s \leq 2$) are well known and essentially correspond to Yang-Mills ($s = 1$) and Einstein ($s = 2$) theories for which the underlying geometries (principal bundles and pseudo-Riemannian manifolds) were familiar to mathematicians before the construction of the physical theory. In contrast, the situation is rather different for “higher spins” ($s > 2$) for which the underlying geometry remains obscure. Due to this lack of information, it is natural to look for inspiration in the perturbative reconstruction of Einstein gravity as the non-Abelian gauge theory of a spin-two particle propagating on a constant-curvature spacetime.

The condition (1) in the definition of higher-spin gravity allows one to expand around the vacuum solution. Furthermore, it follows from conditions (2)-(3) that the Lie algebra of gauge symmetries must be non-Abelian already at first order in the deformation parameter. Two extra assumptions are most often added in this perturbative setting:

(4) **perturbative unitarity**: absence of “ghosts” (i.e. kinetic terms with the wrong sign in

the Lagrangian or negative norm states in the Hilbert space of physical states).³

- (5) **perturbative locality**: each term in the expansion of equations of motion (or Lagrangian) in powers of field fluctuations (around the vacuum solution) is a local function for any fixed set of fields in the vertex (e.g. for a prescribed collection of individual helicities).

Note that the assumption (4) is sometimes dropped, e.g. in conformal higher-spin gravity (consistently with the fact that Weyl’s gravity also has ghost terms in its Lagrangian). Nevertheless, unitarity is a very reasonable assumption for any candidate theory of fundamental interactions. The assumption (5) allows for a mild form of non-locality, controlled by one (or more) parameter(s). Locality seems a pretty natural requirement in the realm of field theory, but it becomes questionable once an infinite collection of fields is present in the spectrum (as in higher-spin gravity or string theory). Traditionally, locality is required because it allows (with Lorentz invariance) to guarantee the causality of the quantum field theory (QFT). Nevertheless, a mild form of non-locality is allowed: an equation of motion (or a Lagrangian) is perturbatively local if each of the coefficients in its formal power series expansion (in the deformation parameters) is a local function (i.e. a polynomial function in the derivatives of the fields). Concretely, this means that the complete series can contain an infinite number of derivatives but, order by order, each partial sum can contain only a finite number of derivatives. This type of controlled non-locality seems physically acceptable, at least at the perturbative level. Indeed, most of the effective theories belong to this class (where the non-locality is controlled by the cut-off length) as well as string field theory (where the non-locality is controlled by the string length ℓ_s). From the mathematical point of view, the assumption of perturbative locality makes it possible to have a tractable well-posed problem. However, as will be discussed below, the extra physical condition (5) might need to be refined because it seems too strong for allowing solutions to the higher-spin interaction problem at quartic level (and higher), at least in flat spacetime of dimension $D \geq 4$. In any case, it appears that the notion of locality should be re-examined carefully in the context of higher-spin gravity and replaced with a weaker notion, whose precise definition remains to be unravelled. This issue has been debated and is under intense investigations, see e.g. the lecture notes (Ponomarev, 2023) for some discussion and references.

Putting together the assumptions (1)-(5), in the metric-like formulation set by the free theory from Eqs [1]-[2], allows to formulate the Fronsdal programme as a well-posed deformation problem (that can be addressed via a perturbative analysis, often referred to as the “Noether method”, with a slight abuse of terminology):

³Concretely, this implies that the total quadratic action is a positive sum of Fronsdal actions (i.e. there is no relative negative sign). This seems like a simple criterion to satisfy but it is often the one that is violated in bold attempts at solving Fronsdal’s programme (without making use of conventional free formulations).

Fronsdal programme (tower of massless fields): List all Poincaré-invariant local deformations

$$S[h] = S^{(0)}[h] + \varepsilon S^{(1)}[h] + \varepsilon^2 S^{(2)}[h] + \mathcal{O}(\varepsilon^3) \quad [3]$$

of a positive sum, including $s = 2$ and with at least one $s > 2$,

$$S^{(0)}[h] = \sum_s S^{(0)}[h_{\mu_1 \dots \mu_s}] \quad [4]$$

of quadratic actions such that the deformed local gauge symmetries

$$\delta_\xi h = \delta_\xi^{(0)} h + \varepsilon \delta_\xi^{(1)} h + \varepsilon^2 \delta_\xi^{(2)} h + \mathcal{O}(\varepsilon^3) \quad [5]$$

are already non-Abelian at first order in the deformation parameter(s) ε and do not arise from local redefinitions

$$h \rightarrow h + \varepsilon \phi(h) + \mathcal{O}(\varepsilon^2), \quad \xi \rightarrow \xi + \varepsilon \zeta(h, \xi) + \mathcal{O}(\varepsilon^2) \quad [6]$$

of the gauge fields and parameters.

The assumption that the deformations are formal power series in a deformation parameter ε enables one to investigate the problem order by order. The crucial observations of any perturbative analysis is that (i) the first order deformations $S^{(1)}[h]$ of the action are constrained by the gauge symmetries $\delta_\xi^{(0)} h$ of the undeformed action $S^{(0)}[h]$ and (ii) the first order deformations of the gauge symmetries $\delta_\xi^{(1)} h$ are constrained by the rigid symmetries of the undeformed action $S^{(0)}[h]$. The so-called Noether method scrutinises the gauge symmetry of the action, $\delta_\xi S = 0$. At zeroth order, the latter equation is satisfied by hypothesis: $\delta_\xi^{(0)} S^{(0)} = 0$. At first order, it reads

$$\delta_\xi^{(0)} S^{(1)} + \delta_\xi^{(1)} S^{(0)} = 0. \quad [7]$$

This equation may be used to constrain the possible deformations by reinterpreting them as familiar objects of the undeformed gauge theory. By definition, an *observable* of a gauge theory is a functional which is gauge-invariant on-shell, while a *reducibility parameter* of a gauge theory is a gauge parameter such that the corresponding gauge variation vanishes off-shell. For instance, a reducibility parameter of Fronsdal transformations [2] is a traceless tensor field $\bar{\xi}_{\mu_1 \dots \mu_{s-1}}$ such that

$$\delta_{\bar{\xi}}^{(0)} h = \partial_{\mu_1} \bar{\xi}_{\mu_2 \dots \mu_s} + \text{cyclic} = 0, \quad [8]$$

that is to say it is a Killing tensor field on Minkowski spacetime. The following lemma provides the basic constraints on the deformations at first order.

Lemma (First-order deformations in terms of the undeformed theory):

- (i) First-order deformations of the action are observables of the undeformed theory, i.e. $\delta_{\xi}^{(0)} S^{(1)}$ vanishes when $\delta S^{(0)}/\delta h$ vanishes.
- (ii) First-order deformations of the gauge symmetries, $\delta_{\xi}^{(1)} h$, evaluated at reducibility parameters $\bar{\xi}$ of the undeformed gauge theory (i.e. $\delta_{\bar{\xi}}^{(0)} h = 0$), span a Lie algebra of global symmetries $\delta_{\bar{\xi}}^{(1)} h$ of the undeformed theory (i.e. $\delta_{\bar{\xi}}^{(1)} S^{(0)} = 0$).

Statements (i)-(ii) easily follow from the previous definitions applied to Eq. (7).

The reformulation of the Fronsdal programme in terms of the local Becchi-Rouet-Stora-Tyutin (BRST) cohomology allowed the complete classification of non-Abelian deformations in various particular cases. Other techniques have also been applied to (i) in order to classify all first-order local deformations of Fronsdal's free theory. These results can be combined with three stringent constraints on the global symmetries following from (ii). The first constraint is that there should be a non-Abelian Lie algebra whose vector space is spanned by the traceless Killing tensors. Second, it is necessary that this Lie algebra has a faithful representation on the space spanned by off-shell gauge fields modulo pure gauge degrees of freedom. Third, the action functional of the free theory must be invariant under these linear transformations. Combining previous classifications results of consistent vertices together with these stringent conditions, one can show⁴ that the Fronsdal programme, as formulated above, does not admit solutions in $D \geq 4$ flat spacetime, under the above list of assumptions (1)-(5). To qualify this no-go theorem, several comments are in order.

Firstly, if one drops one of the implicit assumptions (spacetime dimension $D > 3$, parity invariance or unitarity), then there exists at least 4 examples of fully interacting higher-spin gravity theories (without matter) extending the following spin-two gravity theories:

1. Jackiw-Teitelboim gravity ($D = 2$),
2. Chern-Simons gravity ($D = 3$),
3. Self-dual gravity ($D = 4$),
4. Conformal gravity (even D or Chern-Simons for $D = 3$).

The corresponding higher-spin gravity theories share the following features: they

- are consistent with minimal coupling to gravity, even around flat background,

⁴See e.g. (Ponomarev, 2023) where the chain of arguments is spelled out. See also (Bekaert, Boulanger, Sundell, 2012) for a different proof leading to the same no-go theorem. Similar conclusions are reached in the light-cone formulation. So this obstruction seems quite robust, in agreement with S-matrix no-go theorems.

- do not require the unfolding procedure (instrumental in Vasiliev’s equations),
- can be truncated consistently to the usual low-spin ($s \leq 2$) sector,
- admit a perturbatively local action principle.

However, they are either topological, or in Euclidean signature, or not unitary.

Secondly, one may consider the Fronsdal programme for a tower of massless fields around (anti) de Sitter spacetime instead. In this way, the tension between higher-spin gauge symmetries and minimal coupling to gravity is evaded (in a subtle manner, sometimes called “quasi-minimal coupling”) which allows to bypass a crucial step in the proof of the no-go theorem. A formally consistent system of nonlinear equations whose linearisation is equivalent to the equations of the free theory (e.g. Fronsdal equations) for a tower of massless fields, has been proposed in (Vasiliev, 1990, 2003) as a complete solution to the interaction problem, at the level of equations of motion. See e.g. (Bekaert, Cnockaert, Iazeolla, Vasiliev, 2005; Didenko & Skvortsov, 2014; Vasiliev, 2015; Giombi, 2016) for reviews. A fundamental ingredient of Vasiliev equations is the “unfolded” formulation, which is somewhat exotic since each individual particle is described by an infinity of fields (of which almost all of them are auxiliary or purely gauge). This exotic feature makes the issue of their perturbative (non)locality a technically challenging problem which is still under detailed investigation.

Thirdly, the spectrum of fields in the Fronsdal programme can be modified to contain massive fields instead, in which case consistent interactions are known to be possible, even for higher-spin fields around flat spacetime. This arises for instance in string theory. The latter provided several modern motivations for higher-spin gravity, to which we now turn.

Modern developments and motivations

The main physical motivation for studying higher-spin gravity theories is that they could provide toy models of quantum gravity. The line of reasoning is as follows. The global symmetry algebras underlying higher-spin gravity theories with unbounded spin are infinite-dimensional Lie algebras realised as differential operators of arbitrarily high order. The corresponding higher-spin gauge symmetries are so huge and stringent that higher-spin gravity is expected to be uniquely fixed by the higher-spin symmetry algebra. If so, then no counter-terms are allowed in the action, which suggests that the quantum theory must be ultraviolet (UV) finite. This expectation also fits nicely with known features of string theory.

It was the theoretical study of hadronic physics that gave birth to string theory, the spectrum of which is made of an infinite pyramid of particles with unbounded spin. All particles in exotic representations (higher spin, mixed symmetries, etc) have a mass above (or of the order

of) Planck mass ($\approx 10^{19}$ proton mass). This infinite pyramid of extremely massive higher-spin particles is responsible for the very good ultraviolet behaviour (UV softness and finiteness) of string theory. From the point of view of Fronsdal's programme, string field theory is a highly nontrivial example of consistent interacting theory of *massive* higher-spin fields. Conversely, a better understanding of higher-spin interactions could shed new light on string theory, by providing an underlying symmetry principle responsible for its extremely soft high-energy behaviour. In fact, an old conjecture (by Fradkin, Gross and others) is that string theory would arise from a higher-spin gravity (understood in a very broad sense, c.f. the definition proposed above) via some gigantic spontaneous symmetric breaking (where all but a finite number of fields, with spin not higher than two, become massive). This qualitative scenario was supported and made somewhat more concrete in the setting of the Maldacena conjecture, as will be explained later.

String theory agrees with the folk theorem that jumping beyond the spin two barrier requires including (I) spin two itself (therefore gravity) and also (II) an infinite tower of arbitrarily high spins. Although an infinite tower of exotic particles may seem like a high price to pay for the consistency of the theory, the point is that, in return, this infinity of additional fields beyond the barrier (gravity) solves the problem of the barrier itself (its perturbative non-renormalizability). In a sense, adding the infinite pyramid of high-spin massive fields can be interpreted as a procedure for regularising the ultraviolet divergences of the graviton (and all other particles at the same time) while preserving covariance, unitarity, etc. From the perspective of QFT, the mechanism of ultraviolet regularisation at work in string theory is the following: although the interactions including a finite number of particles of spin 2 (or more) are non-renormalizable (by power-counting), the sum of the contributions of an infinite collection of particles with unbounded spin, can be finite. The origin of this intriguing mechanism consists in turning two vices, peculiar to high spins, into two virtues. Firstly, the ultraviolet divergences worsen with the spin of the fields (like the number of derivatives in the interaction vertices) but the corresponding power series for the total amplitude can converge in the ultraviolet limit, even if each of the partial sums diverges. Secondly, massless higher-spin particles appear not to admit non-trivial scattering matrix in flat spacetime, hence one may expect that scattering amplitudes including exchanges of massive higher-spin particles should go to zero in the ultraviolet limit. More precisely, the tree-level scattering amplitudes can effectively become very soft (i.e. quickly tend to zero) in the high-energy limit. Loop diagrams are constructed by gluing legs of loop diagrams trees, thus a good ultraviolet behaviour of the tree diagrams allows for the perturbative finiteness of loop diagrams.

At the beginning of this century, a major conceptual change of perspective on higher-spin gravity and its potential applications in physics was brought by holographic duality (aka the "AdS/CFT correspondence"). In its strong form, a *holographic duality* is an equivalence between

a theory of quantum gravity in the bulk of (an asymptotically) anti de Sitter (AdS) spacetime M and a conformal field theory (CFT) on the conformal boundary ∂M . The semi-classical limit (tree level approximation) in the bulk gravitational theory corresponds to the limit of large number of fields on the boundary. The seminal example is the Maldacena conjecture postulating a holographic duality between maximal super-Yang-Mills (SYM) theory in four dimensions and superstring theory around $AdS_5 \times S^5$. One often considers in the bulk the semi-classical limit of small string coupling $g_s \ll 1$ (which is equivalent to a large string length with respect to the Planck length, $\ell_s \gg \ell_P$) that corresponds on the boundary to a large number of colors with respect to 't Hooft coupling, $N \gg \lambda$. In addition, usually the bulk spacetime is assumed weakly curved, this is the supergravity approximation of superstring theory. The latter regime (equivalent to a large curvature radius with respect to the string length, $\ell_{AdS} \gg \ell_s$) corresponds to a strongly coupled CFT on the boundary, in the sense of a large 't Hooft coupling $\lambda \gg 1$. But the opposite regime (small curvature radius/'t Hooft coupling: $\ell_s \gg \ell_{AdS} \Leftrightarrow \lambda \ll 1$) is also of interest, as argued early by Sezgin, Sundell, Sundborg, Witten, ... In this exotic regime one should, in principle, be able to reconstruct perturbatively the gravitational theory in the strongly curved bulk from the weakly coupled SYM theory. In particular, the bulk theory dual to free SYM theory must be a higher-spin gravity with unbroken higher-spin symmetries. This corollary of the strong form of Maldacena conjecture has been extensively generalised to other types of higher-spin holographic dualities (not involving strings nor supersymmetry).

The basic ideas behind higher-spin holographic duality are the following generic observations. Free (or integrable) CFTs have an infinite number of higher-order conformal symmetries. By Noether theorem, their spectrum of primary operators contains an infinite tower of traceless conserved currents with unbounded spin, including the energy-momentum tensor (since it is a CFT). According to the AdS/CFT dictionary, a conformal current at the boundary couples to the asymptotic value of a gauge field (of the same rank) inside. Therefore the spectrum of the bulk theory contains an infinite tower of gauge fields with unbounded spin, including spin two. This leads to the following conclusion:

Higher-spin holographic duality: Free (or integrable) CFTs should be dual to higher-spin gravity theories with unbroken higher-spin symmetry.

Some remarks are in order. First, one should stress the important point that, although the boundary dual is free, the n -point correlators are non-vanishing (they are computed by Wick contractions of $2n$ elementary dynamical fields in the boundary theory), thus the n -point interaction vertices of the bulk higher-spin gravity must be non-vanishing. In this sense, the above duality actually predicts non-vanishing interactions of bulk higher-spin gauge fields.⁵ This

⁵Nevertheless, these interactions are sometimes argued to be nevertheless “trivial”, in the sense that they reconstruct the correlators of a free theory.

is in contrast with S -matrix no-go theorems (Coleman-Mandula theorem, generalised Weinberg-Witten theorem, Weinberg low-energy theorems, etc) which predict the absence of scattering in flat spacetime. Second, the above duality supports the expectation that higher-spin gravity theories could provide toy model examples for quantum gravity (in the sense that they are UV-finite QFTs including interacting gravitons) because free CFTs have no $1/N$ corrections therefore bulk higher-spin theories should be “tree-level exact” (i.e. receive no quantum correction).

A natural question arises: what about interacting CFT? After the perturbative introduction of interactions on the boundary, the conformal currents are no longer conserved in general. This should correspond to a breaking of the gauge symmetries in the interior. The ultraviolet cut-off of a QFT with asymptotic freedom corresponds holographically to the infrared cutoff of classical high-spin gravity around AdS. In the case of the Maldacena conjecture, this suggests that the perturbative regime of the SYM theory would contain (in a form very hard to decipher) all necessary information about the tensionless limit of superstring theory (around $AdS_5 \times S^5$) as well as on the mysterious Brout-Englert-Higgs-like mechanism responsible for the higher-spin symmetry breaking. Another interesting possibility is when the interacting deformation of the free CFT on the boundary remains integrable, at least in the large- N limit. One of the most inspiring (though simple) example of such holographic duality concerns a collection of massless scalar fields on the boundary (c.f. the seminal observations by Klebanov, Polyakov, Sezgin and Sundell). The idea is to consider “double-trace” deformations of large- N vector models and look for a semi-classical bulk description of the singlet sector. Applying general observations on double-trace deformations of CFTs to the $O(N)$ vector model, one finds that, in the large- N limit, the free energies of the free and the critical vector model are related by a Legendre transformation (with respect to the source for the charge density, and modulo proper rescalings). The two fixed points have the same infinite set of conserved currents and symmetries, most of which are (softly) broken by $1/N$ corrections in the interacting theory. Both fixed points (Gaussian and Wilson-Fisher) should correspond to different choices of boundary conditions for the *same* bulk theory (holographic degeneracy). This chain of observations leads to the following example of higher-spin holographic duality, which will be our main focus from now on.

Higher-spin/Vector-model duality: The higher-spin gravity around AdS_{d+1} whose spectrum is a tower of totally-symmetric tensor gauge fields of all integer spins ($s = 0, 1, 2, 3, \dots$) is dual to the singlet sector of the $U(N)$ vector model in d dimension. In particular, for unbroken higher-spin symmetries the free/critical vector model (i.e. at the Gaussian/Wilson-Fisher fixed point) is dual to higher-spin gravity with Dirichlet/Neumann boundary conditions.

The singlet primary operators of the vector model are bilinear in the fundamental scalar fields (taking values in the vector representation of some internal compact symmetry group). They form an infinite tower of composite conserved currents, dual to bulk gauge fields. This conjecture

(type A model) can be extended to the vector model based on spinor fields (type B model). In three dimensions, the scalar and spinor are naturally coupled to a Chern-Simons gauge field, thereby justifying the focus on gauge-invariant singlets. These “Chern-Simons-matter” theories have attracted considerable attention in QFT literature.

Higher-spin/vector-model dualities are among the simplest possible examples of holographic dualities, not only because the hologram is most simple (the bilinear sector of a free CFT) but also because both sides of the correspondence are in principle under computational control in the same regime (such dualities are sometimes called “weak/weak” dualities to emphasise this fact). Accordingly, they might be explicitly provable, even with mathematical level of rigour, which could be a tantalising goal for mathematical physicists interested in holographic duality. In this sense, one may speculate that the explicit realisation of Fronsdal programme in AdS could be key to proving the conjectured holographic duality. Conversely, the AdS/CFT dictionary suggests a perturbative algorithm for the holographic reconstruction of the higher-spin gravity theory in the interior from the boundary CFT. In this sense, the holographic reconstruction can be interpreted as another angle of attack for the Fronsdal program around AdS, on the same footing than the Noether method. Taking this view to the extreme, one could even be tempted to interpret the CFT as providing a definition (albeit indirect) of the higher-spin gravity inside. However, it is bulk locality that makes the AdS/CFT correspondence so highly non-trivial (when it applies). Precisely, the problem with a holographic definition of higher-spin gravity is that such a formal definition does not guarantee the existence of a perturbatively local solution to the Fronsdal programme in AdS. The point is that holographic reconstruction is non-local in essence (it involves boundary to bulk propagators, etc).

To sum up, string theory and holographic duality keep bringing many insights on higher-spin gravity and point towards explicit solutions to the higher-spin interaction problem, though many open questions remain.

Mathematics of higher-spin gravity: some basic ingredients and results

Higher-spin symmetry is so huge that the dynamics of the theory is essentially fixed by its kinematics. In fact, the kinematical ingredients (i.e. spectrum and symmetry) essentially fix the theory uniquely (i.e. interactions in the bulk, correlators on the boundary, etc). For this reason, before turning to a sketch of the main strategies for building interactions two seminal results (respectively on the symmetry algebra and on the spectrum of fields) are reviewed. They are best described via the representation theory of the Lie algebra $\mathfrak{so}(d, 2)$, the AdS_{d+1}/CFT_d

isometry/conformal algebra.

In group-theoretical language, the one-to-one relation between bulk and boundary fields is an intertwiner of equivalent representations realised either as an on-shell field on AdS_{d+1} or as a conformal primary field on compactified Minkowski spacetime $S^1 \times S^{d-1}$. The jet space of a conformal primary field (together with all their descendants) is dual to a generalized Verma module

$$\mathcal{V}(\Delta; \vec{s}) = \mathcal{U}(\mathfrak{so}(d, 2)) \otimes_{\mathcal{U}(\mathbb{R}\text{iso}(d,1))} V(\Delta; \vec{s}) \quad [9]$$

defined⁶ as the lowest-weight $\mathfrak{so}(d, 2)$ -module induced from the irreducible finite-dimensional module $V(\Delta; \vec{s})$ of the maximal compact subalgebra $\mathfrak{so}(2) \oplus \mathfrak{so}(d)$ with weight $[\Delta; \vec{s}]$ where:

- $\Delta \equiv E_0$ is the scaling dimension Δ of the CFT_d primary field or, equivalently, the lowest energy E_0 of the AdS_{d+1} particle.
- $\vec{s} = (s_1, \dots, s_r)$ is the “spin” or, better, the weight with respect to the rotation subalgebra $\mathfrak{so}(d)$ of rank r . Since our focus is on totally-symmetric traceless tensors, for which $\vec{s} = (s, 0, \dots, 0)$, this weight will consist in a single number (so the notation will be abbreviated) in the concrete examples.

The irreducible $\mathfrak{so}(d, 2)$ -module obtained as a quotient of the generalised Verma module $\mathcal{V}(\Delta, \vec{s})$ by its maximal submodule is usually denoted $\mathcal{D}(\Delta, \vec{s})$ in the physics literature.

The main players in the higher-spin holographic duality are the unitary irreducible representations of $\mathfrak{so}(d, 2)$ that saturate the unitarity bound, such as

$$\mathcal{D}(d + s - 2, s) \cong \mathcal{V}(d + s - 2, s) / \mathcal{V}(d + s - 1, s - 1) \quad [10]$$

for $s \geq 1$. On the boundary, this $\mathfrak{so}(d, 2)$ -module corresponds to a spin- s conserved conformal current. In the bulk, it corresponds to a spin- s gauge field with Dirichlet boundary condition. The unitary irreducible representation of $\mathfrak{so}(d, 2)$ that saturates the unitarity bound for $s = 0$ is the *minimal representation* (aka the *scalar singleton* in the higher-spin literature)

$$\mathcal{D}\left(\frac{d-2}{2}, 0\right) \cong \mathcal{V}\left(\frac{d+2}{2}, 0\right) / \mathcal{V}\left(\frac{d+2}{2}, 0\right). \quad [11]$$

On the boundary, it corresponds to an on-shell conformal scalar field ϕ satisfying the wave equation $\square\phi = 0$. It corresponds to the leading boundary data of an on-shell bulk scalar field Φ

⁶Consider a parabolic subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ of a finite-dimensional semisimple Lie algebra \mathfrak{g} . The \mathfrak{g} -module induced from a finite-dimensional \mathfrak{h} -module V is a filtered \mathfrak{g} -module $\mathcal{V} = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} V$ called the *generalized Verma module* induced from the finite-dimensional module V .

satisfying the Klein-Gordon equation

$$\left(g_{AdS}^{\mu\nu} \nabla_\mu \nabla_\nu + \frac{\left(\frac{d}{2}\right)^2 - 1}{\ell_{AdS}^2} \right) \Phi = 0. \quad [12]$$

Higher-spin algebras admit various equivalent definitions (singleton symmetry algebra, oscillator realisation, etc). Among them, one may pick the abstract definition of higher-spin algebras, i.e. as quotients of the universal enveloping algebra $\mathcal{U}(\mathfrak{so}(d, 2))$ of the conformal algebra. More precisely, consider the quotient of the latter by its Joseph ideal, i.e. the annihilator of the scalar singleton, $\text{Ann } \mathcal{D}(\frac{d-2}{2}, 0) \subset \mathcal{U}(\mathfrak{so}(d, 2))$. A concrete realisation of this higher-spin algebra is as the maximal algebra of symmetries of the scalar singleton, as made precise by the following theorem.

Theorem 1: [Nikitin (1991), Shapovalov-Shirokov (1992), Eastwood (2002)] The associative algebra of higher symmetries of the wave equation $\square\phi = 0$ on $\mathbb{R}^{d-1,1}$ (i.e. differential operators \hat{A} on $\mathbb{R}^{d-1,1}$ such that $\square \circ \hat{A} = \hat{B} \circ \square$ and modulo trivial generators $\hat{A} = \hat{C} \circ \square$, for some differential operators \hat{B}, \hat{C} on $\mathbb{R}^{d-1,1}$) is isomorphic to the quotient algebra $\mathcal{U}(\mathfrak{so}(d, 2)) / \text{Ann } \mathcal{D}(\frac{d-2}{2}, 0)$. Furthermore, as a vector space it is isomorphic to the space of conformal Killing tensors fields on $\mathbb{R}^{d-1,1}$.

Vasiliev's equations of bosonic higher-spin gravity around AdS_{d+1} (Vasiliev, 2003) are based on a suitable real form $\mathfrak{hs}(\mathfrak{so}(d, 2))$ of the above algebra, endowed with the commutator as Lie bracket. Furthermore, the reducibility parameters for a totally-symmetric tensor gauge field on AdS_{d+1} are Killing tensors on AdS_{d+1} . The latter are in one-to-one correspondence with the conformal Killing tensor fields of the same rank defined on the boundary. Consequently, Theorem 1 ensures the match of symmetries in the higher-spin/vector-model duality. The match of spectra is settled by another theorem, reviewed below.

By construction, the scalar singleton [11] is an irreducible representation of the higher-spin algebra $\mathfrak{hs}(\mathfrak{so}(d, 2))$ with the remarkable property that it remains irreducible under restriction to the conformal algebra $\mathfrak{so}(d, 2)$. On the contrary, its tensor product $\mathcal{D}(d + s - 2, s)^{\otimes 2}$ spans a reducible $\mathfrak{so}(d, 2)$ -module whose decomposition in irreducible $\mathfrak{so}(d, 2)$ -modules is provided by the following theorem.

Theorem 2: [$d = 3$: (Flato & Fronsdal, 1978), $d > 3$: (Vasiliev, 2004)] The tensor product of two scalar singletons decomposes as the following sum

$$\mathcal{D}\left(\frac{d}{2} - 1, 0\right) \otimes \mathcal{D}\left(\frac{d}{2} - 1, 0\right) = \bigoplus_{s=0}^{\infty} \mathcal{D}(d + s - 2, s) \quad [13]$$

of the unitary irreducible $\mathfrak{so}(d, 2)$ -modules describing the conformal currents of all ranks $s \in \mathbb{N}$.

The corresponding decomposition gives the spectrum on both sides of the higher-spin/vector-model duality: the collection of bilinear conformal primary fields of the free boundary theory (e.g. the conserved currents) and the collection of on-shell elementary fields in the interacting bulk theory (e.g. the gauge fields). The $\mathfrak{so}(d, 2)$ -modules [10] on the right-hand side of Eq. [13] can be interpreted either as the Noether currents associated to the symmetry algebra of the free singleton on the boundary, or as on-shell massless fields in the bulk. The left-hand side of Eq. [13] can be interpreted as the bilinears in the scalar singleton while the right-hand side reproduces the spectrum of the higher-spin gravity in the higher-spin/vector-model duality. Hence, Theorem 2 ensures the match of spectra.

Furthermore, a corollary of Theorem 2 is that the sum of all vacuum bubble diagrams of the higher-spin gravity vanishes in the zeta-regularisation (this can be shown via the corresponding $\mathfrak{so}(d, 2)$ -characters). More generally, the vanishing of vacuum one-loop quantities (partition function, Casimir energy, etc) is by-now well-established, see e.g. Section 11 in (Giombi, 2016). Finally, the higher-spin symmetries are known to impose that the CFT must have correlators equivalent to the ones of the free vector model (c.f. the Maldacena-Zhiboedov theorem and its avatars). In this sense, the boundary correlators that would be obtained via the Gubser-Klebanov-Polyakov-Witten prescription from the on-shell value of a bulk action for higher-spin gravity should necessarily be equivalent to the ones of the free vector model. Although the last argument does not yet yield a mathematically complete and rigorous proof, these results makes a very strong case in favour of the conjectured duality. Still, a rigorous proof of the duality cannot make the economy of facing the Fronsdal programme in AdS spacetime in one way or another. Therefore, let us now briefly mention some results on the latter programme.

There are two celebrated manifestly-covariant tensorial⁷ formulations of higher-spin gauge fields: the metric-like and the frame-like formulations. Accordingly, there are two main paths for introducing interactions:

- **Metric-like formulation:** Higher-spin gauge fields are described as completely-symmetric tensor fields $h_{\mu_1 \dots \mu_s} \in \Gamma(\odot^s T^* M)$ of higher rank $s > 2$ on the spacetime manifold M . The Fronsdal formulation reviewed above is a particular example of metric-like formulation of the free theory.

The main results on higher-spin interactions that have been obtained in this formulation are perturbative:

- the systematic classification of vertices, which are consistent with gauge symmetries and modulo field redefinitions.

⁷In four dimensions, there are also formulations (such as the light-cone and twistor ones) which are not necessarily tensorial. They allow to bypass some of the obstructions met by the Fronsdal programme. Due to lack of space, they are not reviewed here.

- the (re)construction of vertices (and/or study of their physical properties, such as locality) either as scattering amplitudes in flat or AdS spacetimes, or as effective action vertices (in the case of conformal higher-spin gravity).

- **Frame-like formulation:** Higher-spin gauge fields are described in terms of generalised coframes, i.e. tensor-valued differential one-forms $e^{a_1 \dots a_{s-1}} = dx^\mu e_\mu^{a_1 \dots a_{s-1}} \in \Omega^1(M) \otimes \odot^{s-1} \mathbb{R}^D$ on the D -dimensional spacetime manifold M (where Greek indices are holonomic while Latin indices are fibre indices in a given tangent frame), together with a collection of generalised spin-connections $\omega^{a_1 \dots a_{s-1} | b_1 \dots b_t} = dx^\mu \omega_\mu^{a_1 \dots a_{s-1} | b_1 \dots b_t}$ ($t = 1, 2, \dots, s-1$). The collection of one-forms (frame-like and generalised spin-connection) must be in one-to-one correspondence with the generators of a higher-spin algebra.

In this formulation, one can construct fully interacting theories by looking for inspiration in three seminal contributions by Élie Cartan:

- *Cartan’s calculus* (Differential forms and exterior calculus): The action, equations of motion and gauge symmetries of the free theory are rewritten in the coordinate-free language of differential forms in terms of the above collection of one-forms.
- *Cartan’s view of gravity* (Cartan connection on a principal bundle): The idea is to replace, in the Cartan connection of gravity, the isometry algebra by its higher-spin extension. In this way, one builds a Cartan-like connection one-form (as well as its curvature two-form and gauge parameter zero-form) taking values in the higher-spin algebra. The flat Cartan connection provides a coordinate-free description of the vacuum solution. The linearisation of the Cartan-like formulation must reproduce the free theory.
- *Cartan’s method of prolongation* (Geometry of PDEs): One reformulates the equations of motion as an exterior differential system or its modern avatars (such as L_∞ -algebras, Q -manifolds, etc). This procedure is known as “unfolding” in higher-spin literature. This reformulation lifts the deformation problem to the infinite-dimensional target space, which allows to apply modern techniques from the deformation theory of associative algebras to the higher-spin interaction problem.

One can conclude this brief survey (for more recent applications and developments, see e.g. the white paper (Bekaert et al, 2022)) with a list of the two major open problems in higher-spin gravity:

- **Problem 1 (locality vs formality):** Clarify the status of perturbative locality (or find a mathematically precise substitute) in unitary higher-spin gravity in dimension four.

- **Problem 2 (higher-spin symmetry breaking):** Design a suitable mechanism for spontaneously breaking higher-spin gauge symmetries to the usual low-spin ones (such as diffeomorphisms). At interacting level, this issue essentially is uncharted territory. It is not even clear how to address it. Nevertheless, this issue remains cardinal in order to relate higher-spin gravity to low-spin physics, e.g.
 - define a low-energy limit and make contact with standard gravity, tensile strings, ...
 - obtain a nontrivial scattering matrix and make contact with non-integrable interacting CFTs, string amplitudes, ...

A reasonable expectation is that Problems 1 and 2 should be intimately related.

Conclusion

Higher-spin gauge fields have a long history and provided a wealth of well-posed mathematical problems, notably the introduction of consistent interactions among them. Understanding better the properties of higher-spin interactions remains a major challenge in theoretical physics, motivated by general considerations in quantum field theory, string theory and holographic duality. The deep relation between higher-spin gravity and important topics of contemporary mathematics might also keep some surprises in store.

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