



**HAL**  
open science

# Hybrid maximum principle for regional optimal control problems with non-smooth interfaces

Térence Bayen, Anas Bouali, Florent Nacry

► **To cite this version:**

Térence Bayen, Anas Bouali, Florent Nacry. Hybrid maximum principle for regional optimal control problems with non-smooth interfaces. 2024. hal-04508454

**HAL Id: hal-04508454**

**<https://hal.science/hal-04508454v1>**

Preprint submitted on 18 Mar 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Hybrid maximum principle for regional optimal control problems with non-smooth interfaces

Térence Bayen\*      Anas Bouali†      Florent Nacry‡

March 18, 2024

## Abstract

In this paper, we consider a general Mayer optimal control problem whose dynamics is defined regionally, and, additionally, we suppose that the interface between two regions is nonsmooth in the sense that it is described by a locally Lipschitz continuous function. Our objective is to derive a hybrid maximum principle in this setting. Doing so, we consider a sequence of mollifiers which allows us to approximate uniformly the interface between two regions by a sequence of smooth functions. This makes possible to apply the hybrid maximum principle on a sequence of approximated optimal control problems involving the smooth interface in place of the nonsmooth one. By passing to the limit as the approximation parameter tends to zero, we obtain the desired necessary optimality conditions in the form of a nonsmooth hybrid maximum principle.

## 1 Introduction

The Pontryagin Maximum Principle (PMP in short) developed in the late 1950's in [24] was a major breakthrough in Mathematics particularly in terms of its many applications. It provides first order necessary optimality conditions for optimal control problems governed by ordinary differential equations extending that way the theory of calculus of variations. It has now been developed in other frameworks such as for problems governed by partial differential equations, ordinary differential equations with delays, and hybrid control systems, which is the main subject of this paper.

By hybrid control system, we mean a dynamical control system that has the particularity to be discontinuous with respect to the state, *i.e.*, the dynamics may change over the time frame. This discontinuity may have several origins. It can be controlled by an automaton like for switched systems (see, *e.g.*, [23]). In that case, switchings of the dynamics occur at so-called *switching times* that can be controlled. Another important framework where discontinuities of the dynamics arise is when the dynamical system is defined over a partition of the state space into a countable number of open regions (see, *e.g.*, [11]). The theory of hybrid systems is very broad and we restrict our attention in this paper only to those hybrid control systems that enter into the previous setting. In this context, the dynamics is thus discontinuous w.r.t. the state each time the trajectory crosses an interface between two regions. In order to characterize optimal solutions to optimal control problems governed by a hybrid control system, the PMP was extended to the hybrid setting leading to the co-called *Hybrid Maximum Principle* (HMP in short), see, *e.g.*, [15, 20, 28]. In the case where the hybrid control system is defined over a partition of the state space (as what is considered in this paper), we will refer to *regional optimal control problems*. For the derivation of the corresponding HMP in this context, we can cite for instance [3, 4, 5, 19].

As a consequence of discontinuities (w.r.t. the state) arising in hybrid control systems, the covector in the HMP is no longer absolutely continuous (as it is in the PMP), but only piecewise absolutely continuous. More precisely, at each crossing time (between two regions), the covector has a discontinuity that is colinear to the outward unit vector to the interface at the crossing point. This condition requires the boundary of the

---

\*Avignon Université, Laboratoire de Mathématiques d'Avignon (EA 2151) F-84018. [terence.bayen@univ-avignon.fr](mailto:terence.bayen@univ-avignon.fr)

†Avignon Université, Laboratoire de Mathématiques d'Avignon (EA 2151) F-84018. [anas.bouali@univ-avignon.fr](mailto:anas.bouali@univ-avignon.fr)

‡Laboratoire de Mathématiques et Physique, Université de Perpignan Via Domitia, Perpignan, France, [florent.nacry@univ-perp.fr](mailto:florent.nacry@univ-perp.fr)

interface to be locally the graph of a  $C^1$  function (as it is usually done in this setting, see, *e.g.*, [3, 4, 5, 19]) in order to properly define the discontinuity condition of the covector. The objective of this paper is to address the case of an interface between two regions that is nonsmooth in the sense that it coincides with the graph of a locally Lipschitz continuous function (see, *e.g.*, [10, 17, 18] in which related questions were studied for state constraints optimal control problems). Our aim here is to provide the corresponding necessary optimality conditions in this case. Regions with nonsmooth interfaces arise typically when a partition of the state space consists of polyhedra or intersections of half spaces. In that case, a trajectory may leave a region at some corner point, so that the HMP as developed for instance in [5] no longer applies.

To handle a nonsmooth interface between two regions, our methodology relies on a regularization technique of the interface. We consider a sequence of  $C^1$  functions converging uniformly to the Lipschitz continuous function describing a nonsmooth interface. This sequence is defined thanks to mollifiers following the approach in [27]. Besides the uniform convergence, it is shown in [27] that subgradients of the original function can be recovered as the convex envelope of the limit gradients of the approximated function (it is also called *gradient consistency*, see [29]). This property is of utmost importance in our approach since it will allow us to extend the discontinuity condition at some point of non-differentiability of the interface. Thanks to this sequence that approaches the nonsmooth interface, we can associate a sequence of approximated optimal control problems. We know that in this case, the HMP as in [5] can be applied on the approximated optimal control problems. By letting the approximation parameter tends to zero, we obtain our main result (Theorem 4.1) which provides first-order necessary optimality conditions on an optimal control. It is obtained using the gradient consistency property of the sequence of regularizing functions that provides the discontinuity condition on the covector.

For proving Theorem 4.1, we use a strong transverse hypothesis on trajectories. Roughly speaking, this hypothesis means that locally, every admissible trajectory crosses the interface transversally. We refer to [7] for a detailed presentation of varied transverse hypotheses in the present hybrid setting. This assumption is needed in particular to ensure that the structure of an optimal trajectory remains unchanged whenever the nonsmooth interface is replaced by a smooth one (which is a non-evident issue as far as we know). There is indeed in general no reason that the structure of an optimal trajectory remains the same whenever considering (small) perturbations of the interface, but this property is kept under a strong transverse hypothesis as we introduce in this paper.

In order to simplify the layout, we restrict our attention in this paper to the case of one nonsmooth interface delimiting two open regions, but, we would like to emphasize that several possible extensions are possible and that some of them (in particular when handling a partition involving a countable number of regions) are presented at the end of the paper. The paper is organized as follows. In Section 2, we introduce some notation and the hybrid optimal control problem. In Section 3, we introduce a sequence of smooth functions that approaches the nonsmooth interface and we also define a sequence of optimal control problems governed by a perturbed differential inclusion (defined thanks to the sequence of approximating functions). We start by proving a stability result concerning solutions to these differential inclusions based on graphical convergence of sets (without the use of a transverse hypothesis). Next, we prove that under a strong transverse hypothesis, optimal solutions to the approximated optimal control problems converge (up to a sub-sequence) to an optimal solution to the original optimal control problem. Finally, in Section 4, we derive the nonsmooth HMP following [8] to address convergence of covectors (coming from the application of the HMP to the sequence of approximated optimal control problems). The paper ends up with a list of remarks on the employed methodology and perspectives.

## 2 Statement of the hybrid optimal control problem

We start by introducing some notation. The letter  $\mathbb{N}$  stands for the set of natural integers starting from 0, that is,  $\mathbb{N} = \{0, 1, 2, \dots\}$ . Let  $m, n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ . For any extended-real number  $q \in [1, \infty]$  and any real interval  $I \subset \mathbb{R}$ , we denote by  $L^q(I, \mathbb{R}^n)$  the usual Lebesgue space of  $q$ -integrable functions defined on  $I$  with values in  $\mathbb{R}^n$ , endowed with its usual norm  $\|\cdot\|_{L^q}$ . A sequence  $(x_k)$  of absolutely continuous functions from  $[0, T]$  into  $\mathbb{R}^n$  is said to strongly-weakly converge (over the interval  $[0, T]$ ) to an absolutely continuous function  $x^* : [0, T] \rightarrow \mathbb{R}^n$  whenever  $(x_k)$  converges to  $x^*$  in  $L^\infty(I, \mathbb{R}^n)$  and  $(\dot{x}_k)$  weakly converges in  $L^2(I, \mathbb{R}^n)$  to  $\dot{x}^*$ . It will be convenient to denote (as usual)  $\dot{x}_k \rightharpoonup \dot{x}^*$  the latter weak convergence. For convenience, we

shall also note indifferently  $x$  or  $x(\cdot)$  a function  $x : [0, T] \rightarrow \mathbb{R}^n$ .

Next,  $B(x_0, r)$  (resp.  $B[x_0, r]$ ) stands for the open (resp. closed) ball centered at  $x_0 \in \mathbb{R}^n$  with radius  $r > 0$ . The closed unit ball  $B[0, 1]$  of  $\mathbb{R}^n$  is denoted  $\mathbb{B}_{\mathbb{R}^n}$ . We also write  $\cdot$  (resp.  $|\cdot|$ ) for the standard inner product (resp. Euclidean norm) of  $\mathbb{R}^n$ . For any subset  $X \subset \mathbb{R}^n$ , we denote by  $\partial X$  the boundary of  $X$  defined by  $\partial X := \overline{X} \setminus \text{int}(X)$ , where  $\overline{X}$  and  $\text{int}(X)$  stand respectively for the closure and the interior of  $X$ . Given a function  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ , we define (as usual) its support as the closure of the set  $\{\rho \neq 0\}$ . The convex hull of the set  $X$  is denoted by  $\text{co}(X)$ .

The *Clarke subdifferential* of a locally-Lipschitz continuous function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is given by

$$\partial_C f(x) = \text{co} \left\{ \lim_{k \rightarrow +\infty} \nabla f(x_k) : \Delta_f \ni x_k \rightarrow x \right\},$$

where  $\Delta_f$  denotes the set of points  $x \in \mathbb{R}^n$  where the (Fréchet) gradient  $\nabla f(x)$  exists. The *Clarke normal cone* of a set  $S \subset \mathbb{R}^n$  at  $x \in S$  is defined as the polar of the *Clarke tangent cone*  $T^C(S; x)$ , that is,

$$N^C(S; x) := \left\{ \zeta \in \mathcal{H} : \langle \zeta, h \rangle \leq 0, \forall h \in T^C(S; x) \right\},$$

where

$$T^C(S; x) := \{h \in X : \forall S \ni x_n \rightarrow x, \forall t_n \downarrow 0, \exists h_n \rightarrow h, \forall n \in \mathbb{N}, x_n + t_n h_n \in S\}.$$

The *limit inferior* (resp., *limit superior*) of a sequence  $(S_k)$  of sets in  $\mathbb{R}^n$  is defined as

$$\text{Lim inf}_{k \rightarrow +\infty} S_k := \{x \in \mathbb{R}^n : \forall \varepsilon > 0, \exists K \in \mathbb{N}, \forall k \geq K, S_k \cap B(x, \varepsilon) \neq \emptyset\}.$$

(resp.,

$$\text{Lim sup}_{k \rightarrow +\infty} S_k := \{x \in \mathbb{R}^n : \forall \varepsilon > 0, \forall K \in \mathbb{N}, \exists k \geq K, S_k \cap B(x, \varepsilon) \neq \emptyset\}.$$

Whenever both sets coincides, one says that the sequence  $(S_k)$  *Painlevé-Peano-Kuratowski converges*. It is well-known (and not difficult to check) that  $x \in \text{Lim sup}_{k \rightarrow +\infty} S_k$  if and only if there exist a a sequence  $(x_k)$  converging to  $x$  and an increasing mapping  $s : \mathbb{N} \rightarrow \mathbb{N}$  such that  $x_k \in S_{s(k)}$  for all  $k \in \mathbb{N}$ . Such a sequential characterization of the upper-limit easily ensures ([27, Theorem 4.10]) that any (not necessarily closed) set  $S \supset \text{Lim sup}_{k \rightarrow +\infty} S_k$  satisfies for all real  $\rho > 0$  and all real  $\varepsilon > 0$ , the existence of  $n_0 \in \mathbb{N}$  such that

$$S_n \cap \rho \mathbb{B}_{\mathbb{R}^n} \subset S + \varepsilon \mathbb{B}_{\mathbb{R}^n} \quad \text{for all } n \geq n_0. \quad (2.1)$$

Let  $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  be a multimapping. A sequence of multimappings  $(G_k)$  from  $\mathbb{R}^n$  into itself *graphically converges* to a multimapping  $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  whenever the sequence of graphs  $(\text{gph } G_k)$  Peano-Painlevé-Kuratowski converges to  $\text{gph } G$ . One says that  $G$  is *outer semicontinuous* at  $x \in \mathbb{R}^n$  if for every sequence  $x_k \rightarrow x$  and every sequence  $y_k \rightarrow y$  such that  $y_k \in F(x_k)$  for all  $k \in \mathbb{N}$ , then one has  $y \in F(x)$ . Obviously, the multimapping  $G$  is outersemicontinuous at each point of  $\mathbb{R}^n$  if and only if its graph  $\text{gph } G := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n ; y \in G(x)\}$  is closed in  $\mathbb{R}^n \times \mathbb{R}^n$ . One says that the multimapping  $M$  is *upper semicontinuous* at  $x$  whenever for each open set  $V$  of  $\mathbb{R}^n$  with  $G(x) \subset V$  there is a real  $\varepsilon > 0$  such that

$$G(B(x, \varepsilon)) \subset V.$$

Recall that (see, e.g., [30, Proposition 1.39]) the outer semicontinuity of  $G$  at  $\bar{x} \in \mathbb{R}^n$  is equivalent to its upper semicontinuity at  $\bar{x}$  provided that there are  $V$  a neighborhood of  $\bar{x}$  in  $\mathbb{R}^n$  and a compact  $K \subset \mathbb{R}^n$  such that  $M$  is closed-valued on  $V$  and  $M(V) \subset K$ .

In the sequel, we consider two functions  $f_1, f_2 : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  of class  $C^1$  (representing the dynamics) and a nonempty compact subset  $U$  of  $\mathbb{R}^m$ . We suppose that for  $i = 1, 2$  and that for every  $x \in \mathbb{R}^n$ , the velocity set

$$\widehat{F}_i(x) := \{f_i(x, u) ; u \in U\}$$

is a nonempty (compact) and convex subset in  $\mathbb{R}^n$ . For technical reasons in the proof of the HMP (based on an approximating sequence of the nominal control), we also suppose that for  $i = 1, 2$  and for every  $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$  the set

$$\widehat{G}_i(x, p) := \{D_x f_i(x, u)^\top p ; u \in U\}$$

is a nonempty compact convex subset in  $\mathbb{R}^n$  where as usual  $D_x f_i(x, u)$  stands for the Jacobian matrix of  $f_i(\cdot, u)$  at  $x$  and where  $^\top$  denotes the transpose operator. Note that the preceding convexity hypotheses are automatically verified by affine-control systems w.r.t. the control which already represent a large class of control systems in application models. Additionally, we suppose (in order to prevent blow-up of trajectories) that the function  $f_i$  has a linear growth, *i.e.*, there is some  $c \geq 0$  such that

$$|f_i(x, u)| \leq c(|x| + 1) \quad \text{for all } (x, u) \in \mathbb{R}^n \times U. \quad (2.2)$$

Let also  $\Gamma$  be a subset of  $\mathbb{R}^n$  for which there is a locally Lipschitz continuous function  $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\Gamma$  is the zero level-set of  $\psi$ , *i.e.*,

$$\Gamma := \{x \in \mathbb{R}^n ; \psi(x) = 0\}.$$

In what follows, we set  $X_1 := \{x \in \mathbb{R}^n ; \psi(x) < 0\}$  and  $X_2 := \{x \in \mathbb{R}^n ; \psi(x) > 0\}$ . We also consider a function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  of class  $C^1$  which represents in what follows the terminal pay-off and the admissible control set is the set  $\mathcal{U}$  that contains all measurable functions  $u : [0, T] \rightarrow \mathbb{R}^n$  such that  $u(t) \in U$  for a.e.  $t \in [0, T]$ . Throughout this paper, we study the Mayer optimal control problem

$$\inf_{u \in \mathcal{U}} g(x(T)), \quad (2.3)$$

where  $T > 0$  is given,  $x$  is a solution to the Cauchy problem

$$\begin{cases} \dot{x}(t) &= f(x(t), u(t)) \quad \text{a.e. } t \in [0, T], \\ x(0) &= x_0, \end{cases} \quad (2.4)$$

with  $x_0 \in X_1$  and where the hybrid dynamics  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  satisfies for every  $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$

$$f(x, u) := \begin{cases} f_1(x, u) & \text{if } \psi(x) > 0, \\ f_2(x, u) & \text{if } \psi(x) < 0. \end{cases} \quad (2.5)$$

Note that  $f$  is not defined on  $\Gamma \times \mathbb{R}^m$  which has no importance when no sliding mode occurs (see [23]), *i.e.*, typically, under a transverse hypothesis on trajectories. At this step, we do not consider such a hypothesis, that is why, it is convenient to replace the preceding control system by the differential inclusion

$$\begin{cases} \dot{x}(t) &\in F(x(t)) \quad \text{a.e. } t \in [0, T], \\ x(0) &= x_0, \end{cases} \quad (2.6)$$

where  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is the Filippov multimapping associated with  $f$  which is defined for all  $x \in \mathbb{R}^n$  by

$$F(x) := \begin{cases} \hat{F}_1(x) & \text{if } \psi(x) > 0, \\ \hat{F}_2(x) & \text{if } \psi(x) < 0, \\ \hat{F}_{1,2}(x) & \text{if } \psi(x) = 0, \end{cases} \quad (2.7)$$

where for all  $x \in \mathbb{R}^n$ , the set  $\hat{F}_{1,2}(x)$  is the closed convex hull of  $\hat{F}_1(x)$  and  $\hat{F}_2(x)$ . For more details on the Filippov multimapping, we refer to [16]. It is an exercise (having in mind that the multimappings  $\hat{F}_1$  and  $\hat{F}_2$  are convex-valued) to check that

$$\hat{F}_{1,2}(x) := \{\theta f_1(x, u) + (1 - \theta)f_2(x, v) ; \theta \in [0, 1] ; u, v \in U\}. \quad (2.8)$$

It is readily seen that the  $F$  is with nonempty convex compact values. It is also not difficult to check that  $F$  is outer semicontinuous, hence it is upper semicontinuous. This and the fact that the mapping  $F$  is with linear growth allows us to get the existence of an absolutely continuous function  $x : [0, T] \rightarrow \mathbb{R}^n$  satisfying the Cauchy problem (2.6). We refer for instance to [1, Chapter 2-Theorem 3]) for more details on this point. In the sequel, we denote by  $\mathcal{S}_T(x_0)$  all solutions to (2.6). We can now state an existence result related to (2.3).

**Proposition 2.1.** *There exists a solution to the optimal control problem*

$$\inf_{x(\cdot) \in \mathcal{S}_T(x_0)} g(x(T)). \quad (2.9)$$

*Proof.* The existence of a solution to (2.9) follows from the continuity of  $g$  and [2, Theorem 19.2.3] (see also [14, Theorem 1.11]) taking a minimizing sequence.  $\square$

Our objective is to provide necessary optimality condition for (2.3). Since the function  $\psi$  is only Lipschitz-continuous w.r.t. the state, we cannot directly apply the HMP as in [5] or in [19] (because if  $x \in \Gamma$ ,  $\partial_C \psi(x)$  may not be a singleton). To derive a nonsmooth HMP (meaning that regions are with nonsmooth boundaries), we will proceed in two steps. First, we introduce a regularization of  $\Gamma$  and we prove stability results of optimal solutions to the auxiliary optimal control problems (Section 3). Next, we apply the HMP on this sequence of problems for which the boundary is smooth, and necessary optimality conditions for (2.3) will be obtained by passing to the limit as the regularization parameter tends to zero (Section 4).

### 3 Regularization of the interface and convergence analysis

#### 3.1 Approximated optimal control problems

In this section, we introduce a sequence of optimal control problems approaching (2.3) that are based on the regularization of the function  $\psi$  describing the interface between  $X_1$  and  $X_2$ . Doing so, let us start by recalling a classical result concerning the convergence of the gradients of a sequence of smooth functions (defined by convolution) approximating uniformly a locally Lipschitz-continuous function (see [27]). Let  $\rho_k : \mathbb{R}^n \rightarrow \mathbb{R}_+$  be a sequence of bounded and continuous mollifiers<sup>1</sup> satisfying  $\int_{\mathbb{R}^n} \rho_k(y) dy = 1$  and  $\text{supp}(\rho_k) \subset B[0, \varepsilon_k]$  for any  $k$ , where  $\varepsilon_k \downarrow 0$  as  $k \rightarrow +\infty$ . If  $\psi_k(x) := \int_{\mathbb{R}^n} \psi(x-y)\rho_k(y) dy$  denotes the convolution of  $\psi$  with  $(\rho_k)_k$ , the following properties are satisfied (see [27, Theorem 9.67] or [29]):

- the sequence  $(\psi_k)$  uniformly converges to  $\psi$  over every compact subset of  $\mathbb{R}^n$  ;
- the gradient consistency (allowing to obtain subgradients of  $\psi$  via mollifiers) holds true:

$$\partial_C \psi(x) = \text{co} \left\{ \lim_{k \rightarrow +\infty} \nabla \psi_k(x_k) ; x_k \rightarrow x \right\}. \quad (3.1)$$

In order to approach (2.3) by a sequence of optimal control problems involving a smooth boundary, we proceed as follows. For  $k \in \mathbb{N}$ , define a smooth interface

$$\Gamma_k := \{x \in \mathbb{R}^n ; \psi_k(x) = 0\},$$

as well as the set-valued map  $F_k : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$

$$F_k(x) := \begin{cases} \hat{F}_1(x) & \text{if } \psi_k(x) > 0, \\ \hat{F}_2(x) & \text{if } \psi_k(x) < 0, \\ \hat{F}_{1,2}(x) & \text{if } \psi_k(x) = 0. \end{cases} \quad (3.2)$$

Similarly as for  $F$ , the set-valued map  $F_k$  is upper semi-continuous for every  $k \in \mathbb{N}$ . In addition, for every  $k \in \mathbb{N}$  and for every  $x \in \mathbb{R}^n$ ,  $F_k(x)$  is a nonempty compact convex subset of  $\mathbb{R}^n$ . Since  $F_k$  is with linear growth (because of (2.2)), for every  $k \in \mathbb{N}$ , the Cauchy problem

$$\begin{cases} \dot{x}(t) \in F_k(x(t)) & \text{a.e. } t \in [0, T], \\ x(0) = x_0, \end{cases} \quad (3.3)$$

has an absolutely continuous solution over  $[0, T]$  denoted by  $x_k$ . Hence, using a similar argumentation as in the proof of Proposition 2.1, for every  $k \in \mathbb{N}$ , the optimal control problem

$$\inf_{x(\cdot) \in \mathcal{S}_T^k(x_0)} g(x_k(T)), \quad (3.4)$$

has a solution  $x_k^*$  where for any  $k \in \mathbb{N}$ , the set  $\mathcal{S}_T^k(x_0)$  denotes all solutions to (3.3).

---

<sup>1</sup>e.g.,  $\rho_k(y) := \frac{1}{\varepsilon_k^n} \rho(\frac{y}{\varepsilon_k})$  where  $\rho(y) := C \exp(-\frac{1}{1-|y|^2})$  for  $|y| < 1$  and  $\rho(y) := 0$  for  $|y| \geq 1$  with  $C > 0$  such that  $\int_{\mathbb{R}^n} \rho = 1$ .

### 3.2 A stability result for hybrid differential inclusions

In this section, we address the following question : given a sequence of solutions to (3.3), does there exist a solution  $x$  to (2.6) such that, up to a sub-sequence,  $(x_k)$  strongly-weakly converges to  $x$ ? Such a problem is developed in the literature through different approaches and under various assumptions. For the case of one Cauchy problem in  $\mathbb{R}^n$ , say  $F_k \equiv F$ , we refer the reader to the result [14, Chapter 4-Theorem 1.11] (which is based on Arzelà-Ascoli theorem). In [26], the strong-weak convergence is studied in the context of a general Banach space for a sequence of Cauchy problems under an appropriate Kuratowski-Mosco convergence (namely,  $F_k(x_k) \rightarrow F(x)$  for every sequence  $(x_k)$  which weakly converges to  $x$ ). As far as we know, the hypotheses required to apply [26] are not completely fulfilled by  $(F_k)$  and  $F$  as defined in (2.7) and (2.7). Another possible way is to follow the approach in [21] based on graphical convergence of sets.

**Definition 3.1.** *Let  $(G_k)$  be a sequence of multimappings from  $\mathbb{R}^n$  into itself. The (graphical) upper limit of the sequence  $(G_k)$  is the multimapping  $G^\# : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  defined by*

$$\text{Graph}(G^\#) := \limsup_{k \rightarrow +\infty} \text{Graph}(G_k),$$

According to the above sequential characterization of the upper-limit, it is readily seen that  $(x, y) \in \text{Graph}(G^\#)$  if and only if there are two sequences  $(x_k)$  and  $(y_k)$  satisfying  $x_k \rightarrow x$ ,  $y_k \rightarrow y$  and an increasing function  $s : \mathbb{N} \rightarrow \mathbb{N}$  such that  $(x_k, y_k) \in \text{gph} F_{s(k)}$  (or equivalently,  $y_k \in F_{s(k)}(x_k)$ ) for all  $k \in \mathbb{N}$ .

**Proposition 3.1.** *Given a sequence  $(x_k)$  such that  $x_k \in \mathcal{S}_T^k(x_0)$  for every  $k \in \mathbb{N}$ , there is  $x \in \mathcal{S}_T(x_0)$  such that, up to a sub-sequence,  $(x_k)$  strongly-weakly converges to  $x$ .*

*Proof.* Let  $(x_k)$  be a sequence of functions from  $[0, T]$  into  $\mathbb{R}^n$  satisfying  $x_k \in \mathcal{S}_T^k(x_0)$  for every  $k \in \mathbb{N}$ . Thanks to (2.2), there exists  $c' \geq 0$  such that for all  $k \in \mathbb{N}$ , one has

$$|\dot{x}_k(t)| \leq c' \quad \text{a.e. } t \in [0, T].$$

This implies that  $(x_k)$  is uniformly bounded in  $L^\infty([0, T], \mathbb{R}^n)$ . Using the Arzelà-Ascoli theorem, we can find an absolutely continuous function  $x : [0, T] \rightarrow \mathbb{R}^n$  such that  $(x_k)$  uniformly converges to  $x$  and  $(\dot{x}_k)$  weakly converges to  $\dot{x}$  in  $L^2([0, T], \mathbb{R}^n)$ . Pick any real  $\rho > 0$  such that for all  $k \in \mathbb{N}$  and for almost every  $t \in [0, T]$  one has

$$(x_k(t), \dot{x}_k(t)) \in \text{gph } F_k \cap \rho \mathbb{B}_{\mathbb{R}^{2n}}.$$

Given any real  $\varepsilon > 0$  we easily derive from the inclusion  $\limsup_{k \rightarrow +\infty} \text{gph } F_k \subset \text{gph } F$  (see (2.1)) that

$$\text{gph } F_k \cap \rho \mathbb{B}_{\mathbb{R}^{2n}} \subset \text{gph } F + \varepsilon \mathbb{B}_{\mathbb{R}^{2n}}.$$

It remains to apply [1, Theorem 1, Chapter 1] to obtain the inclusion  $x \in \mathcal{S}_T(x_0)$ . □

### 3.3 Convergence of optimal solutions to (3.4)

In this subsection, we provide sufficient conditions ensuring the convergence (up to a subsequence) of the sequence  $(x_k^*)$  to an optimal solution to (2.9). Doing so, we need to prove that every admissible solution  $x$  of (2.6) can be approached by a sequence  $(x_k)$  satisfying (3.3) (this point is proved in Proposition 3.2) For proving this point, we introduce the following transverse hypothesis (which is also standard in the hybrid setting [6, 7, 19]). Remind that since  $x_0$  and  $T > 0$  are fixed and since the dynamics  $f_i$ ,  $i = 1, 2$  is with linear growth, there is  $R \geq 0$  such that for all  $x(\cdot) \in \mathcal{S}_T(x_0)$ , for every  $t \in [0, T]$ , one has  $x(t) \in B[0, R]$ .

**Assumption 3.1.** *There is  $\kappa > 0$  such that for every  $x \in \Gamma \cap B[0, R]$ , for every  $v \in \partial_C \psi(x)$ , for every  $u \in U$ , and for  $i = 1, 2$ :*

$$v \cdot f_i(x, u) \geq \kappa. \tag{3.5}$$

Following [7], this assumption is called *strong transverse hypothesis* (see [7] for a discussion on various transverse hypotheses in the hybrid setting). Note also that under such a hypothesis, then the differential inclusion in (2.6) can be replaced by the control system in (2.4) since (3.5) excludes sliding modes. In particular, (2.3) and (2.9) are equivalent. In the sequel, we suppose that this hypothesis is fulfilled.

**Definition 3.2.** Given a solution  $x$  to the Cauchy problem (2.6), one says that a real  $\tau \in (0, T)$  is a crossing time from  $X_1$  to  $X_2$  whenever there exists a real  $\alpha > 0$  (small enough) such that

$$(\psi(x(t)) - \psi(x(\tau)))(t - \tau) > 0 \quad \text{for all } t \in [\tau - \alpha, \tau + \alpha] \setminus \{\tau\}.$$

A similar definition with  $\Gamma_k$  and (3.3) in place of  $\Gamma$  and (2.6) is omitted. Obviously, if  $\tau$  is a crossing time of  $x(\cdot) \in \mathcal{S}_T(x_0)$ , then, one must have  $\psi(x(\tau)) = 0$ .

**Lemma 3.1.** Every admissible trajectory of (2.3) has at most one crossing time on  $\Gamma$  from  $X_1$  to  $X_2$ .

*Proof.* First, every crossing time of an admissible solution  $x \in \mathcal{S}_T(x_0)$  is isolated. Indeed, let  $\tau \in (0, T)$  be a crossing time of  $x(\cdot)$ . We then have  $\psi(x(\tau)) = 0$ . In addition, by composition,  $t \mapsto \psi(x(t))$  is absolutely continuous, so for a.e.  $t$ , one has  $\frac{d}{dt}\psi(x(t)) = \nabla\psi(x(t)) \cdot \dot{x}(t) = \nabla\psi(x(t)) \cdot f_i(x(t), u(t))$  where  $i = 1, 2$ . Now, from (3.5), we get the existence of  $\eta > 0$  such that for a.e.  $t \in [\tau, \tau + \eta]$ ,

$$\nabla\psi(x(t)) \cdot f_2(x(t), u(t)) \geq \frac{\kappa}{2}.$$

Hence,  $t \mapsto \psi(x(t))$  is increasing in a right neighborhood of  $\tau$ , whence the result. Suppose now that the mapping  $x$  has a crossing time and set

$$\tau := \min\{t \in [0, T] ; \psi(x(t)) = 0\}.$$

Suppose now that there is  $\tau' \in (\tau, T)$  such that  $\psi(x(\tau')) = 0$  and  $x(t) \in X_2$  for  $t \in (\tau, \tau')$  so that  $\psi(x(t)) > 0$  for every  $t \in (\tau, \tau')$ . Take  $r > 0$  small enough. We have (keeping in mind that  $\psi \circ x$  is absolutely continuous on  $[0, T]$ )

$$\psi(x(\tau')) - \psi_k(x(\tau' - r)) = \int_{\tau' - r}^{\tau'} \nabla\psi(x(t)) \cdot \dot{x}(t) dt = \int_{\tau' - r}^{\tau'} \nabla\psi(x(t)) \cdot f_2(x(t), u(t)) dt.$$

On the one hand, for  $r$  small enough, the integral above is bounded below by a positive constant (using the transversality condition and a continuity argumentation). On the other hand,

$$\psi(x(\tau')) - \psi(x(\tau' - r)) = -\psi(x(\tau' - r)) < 0,$$

for every  $r > 0$  small enough. This is thus a contradiction which ends the proof.  $\square$

The preceding property can be transferred to admissible trajectories of (3.4) for  $k$  large enough as we show in the next lemma.

**Lemma 3.2.** For  $k$  large enough, every admissible solution of (3.3) has zero or one crossing time on  $\Gamma_k$  from  $X_1$  to  $X_2$ .

*Proof.* Suppose that for every  $k \in \mathbb{N}$ , there are  $n_k \in \mathbb{N}$ ,  $x_{n_k} \in \Gamma_{n_k} \cap B[0, R]$ ,  $u_{n_k} \in U$  such that

$$\nabla\psi_{n_k}(x_{n_k}) \cdot f_i(x_{n_k}, u_{n_k}) < \frac{\kappa}{2},$$

where  $i = 1, 2$  is fixed. Since  $U$  is compact, we may assume that  $(u_{n_k})$  converges to some  $u \in U$  and that  $(x_{n_k})$  also converges to some vector  $\tilde{x} \in \Gamma$ . By using the gradient consistency property, we deduce that (up to a sub-sequence), there exists  $v \in \partial_C\psi(\tilde{x})$  such that  $v = \lim_{k \rightarrow +\infty} \nabla\psi_{n_k}(x_{n_k})$ . It follows that  $v \cdot f_i(\tilde{x}, u) \leq \frac{\kappa}{2}$  which contradicts assumption 3.5. Hence, we deduce that there is  $k_0 \in \mathbb{N}$  such that for every  $k \geq k_0$ , one has:

$$\forall x \in \Gamma_k \cap B[0, R], \forall u \in U, \nabla\psi_k(x) \cdot f_i(x, u) \geq \frac{\kappa}{2}, \quad (3.6)$$

for  $i = 1, 2$ . Let  $k \geq k_0$  and  $x \in \mathcal{S}_T^k(x_0)$ . Suppose that  $x$  has at least two crossing times  $\tau < \tau'$  (isolated because of (3.6)) such that  $\psi_k(x(\tau)) = \psi_k(x(\tau')) = 0$  so that  $x(t) \in X_2$  for  $t \in (\tau, \tau')$ . We thus have  $\psi(x(t)) > 0$  for every  $t \in (\tau, \tau')$ . By taking  $r$  small enough, we find that

$$\forall k \geq k_0, \psi_k(x(\tau')) - \psi_k(x(\tau' - r)) = \int_{\tau' - r}^{\tau'} \nabla\psi_k(x(t)) \cdot f_2(x(t), u(t)) dt \geq \frac{r\kappa}{4} > 0.$$

But, in the preceding equality, one has  $\psi_k(x(\tau')) - \psi_k(x(\tau' - r)) = -\psi_k(x(\tau' - r)) < 0$  which is a contradiction. This ends the proof.  $\square$



**Proposition 3.2.** *Up to a sub-sequence,  $(x_k^*)$  strongly-weakly converges to an optimal solution  $x^*$  of (2.3).*

*Proof.* We know from Proposition 3.1 that there is an admissible solution  $x^*$  to (2.3) such that, up to a sub-sequence,  $(x_k^*)$  strongly-weakly converges to  $x^*$ . Now, let us given an admissible solution  $x$  of (2.3) associated with some control  $u \in \mathcal{U}$ . We define  $x_k : [0, T] \rightarrow \mathbb{R}^n$  as the unique solution to the system

$$\begin{cases} \dot{x}_k(t) = f_1(x_k(t), u(t)) & \text{if } \psi_k(x_k(t)) < 0, \\ \dot{x}_k(t) = f_2(x_k(t), u(t)) & \text{if } \psi_k(x_k(t)) > 0, \end{cases} \quad (3.7)$$

such that  $x_k(0) = x_0$ . Let us prove that  $(x_k)$  strongly-weakly converges to  $x$  over  $[0, T]$ . If  $x$  has no crossing time, the result is obvious. Suppose then that  $x$  has a (single) crossing time  $\tau \in (0, T)$ . It follows from Lemma 3.2 that for  $k$  large enough,  $x_k$  also has a unique crossing time over  $(0, T)$ . Up to a sub-sequence, we may assume that  $(x_k)$  strongly-weakly converges to some absolutely continuous function  $\hat{x}$  over  $[0, T]$  where  $\hat{x}$  satisfies (2.6) (Proposition 3.1). By using (3.7), we easily get that  $\hat{x}(t) = x(t)$  for every  $t \in [0, \tau - \alpha]$  for every  $\alpha > 0$  small enough. It follows that  $\hat{x} = x$  over  $[0, \tau]$  (using the continuity of  $x$  and  $\hat{x}$  at  $t = \tau^-$ ). Next,  $\hat{x}$  and  $x$  satisfy the same ordinary differential equation over  $[\tau, T]$ . Since  $\hat{x}(\tau) = x(\tau)$ , we get that  $\hat{x} = x$  over  $[\tau, T]$ . This proves that  $(x_k)$  strongly-weakly converges to  $x$  over  $[0, T]$ .

To conclude the proof, take any admissible trajectory  $x$  of (2.3) and construct an admissible sequence  $(x_k)$  for (3.4) by (3.7). Since we have

$$g(x_k^*(T)) \leq g(x_k(T))$$

for every  $k \in \mathbb{N}$ , we deduce that  $g(x^*(T)) \leq g(x(T))$  by letting  $k \rightarrow +\infty$  and using the continuity of  $g$ . Since  $x$  is any admissible trajectory of (2.3), the result follows.  $\square$

## 4 Nonsmooth hybrid maximum principle

Our goal now is to exploit optimality conditions on (3.4) to deduce optimality conditions on Problem (2.3). Let us then consider  $x^* \in \mathcal{S}_T(x_0)$  an optimal trajectory of (2.3). We suppose that  $x^*$  has a single crossing time  $\tau^* \in (0, T)$  from  $X_1$  to  $X_2$  (remind Lemma 3.1). By using the results of the previous section, for every  $k \in \mathbb{N}$ , there exists a solution  $x_k^*$  to (3.4) such that (up to a sub-sequence),  $(x_k^*)$  strongly-weakly converges to  $x^*$ . So, for every  $k \in \mathbb{N}$ , there is an admissible control  $u_k^* \in \mathcal{U}$  associated with  $x_k^*$ . Our next step is to apply the HMP from [5] on  $x_k^*$  (for  $k$  large enough according to Lemma 3.2) which is possible since  $\Gamma_k$  is smooth.

### 4.1 Application of the HMP on (3.4) and consequences

Since  $x^*$  has a single crossing time, for every  $k$  large enough,  $x_k^*$  also has a single crossing time  $\tau_k$  and by uniform convergence of  $(x_k^*)$ , we get that  $\tau_k \rightarrow \tau^*$  as  $k \rightarrow +\infty$ . We can now straightforwardly apply the HMP from [5] on (3.4) since it does not involve mixed initial-terminal constraints. For every  $k$  large enough, there is a piecewise absolutely continuous function  $p_k : [0, T] \rightarrow \mathbb{R}^n$  (a covector) such that the following properties are fulfilled:

(i) Adjoint equation :

$$\dot{p}_k(t) = -D_x f(x_k^*(t), u_k^*(t))^\top p_k(t) \quad \text{a.e. } t \in [0, T], \quad (4.1)$$

(ii) Terminal condition :

$$p_k(T) = -\nabla g(x_k^*(T)), \quad (4.2)$$

(iii) Hamiltonian maximization condition :

$$u_k^*(t) \in \operatorname{argmax}_{\omega \in \mathcal{U}} p_k(t) \cdot f(x_k^*(t), \omega) \quad \text{a.e. } t \in [0, T], \quad (4.3)$$

(iv) discontinuity condition of the covector : there is  $\nu_k \in \mathbb{R}$  such that

$$p_k^+(\tau_k) - p_k^-(\tau_k) = \nu_k \nabla \psi_k(x_k^*(\tau_k)). \quad (4.4)$$

Observe that the adjoint equation (4.1) can be rewritten as

$$\begin{cases} \dot{p}_k(t) = -D_x f_1(x_k^*(t), u_k^*(t))^\top p_k(t) & \text{for a.e. } t \in (0, \tau_k), \\ \dot{p}_k(t) = -D_x f_2(x_k^*(t), u_k^*(t))^\top p_k(t) & \text{for a.e. } t \in (\tau_k, T). \end{cases} \quad (4.5)$$

Finally, the Hamiltonian function  $t \mapsto p_k(t) \cdot f(x_k^*(t), u_k^*(t))$  is constant almost everywhere from [5] so that there is  $c_k \in \mathbb{R}$  such that

$$p_k(t) \cdot f_1(x_k^*(t), u_k^*(t)) = p_k(t') \cdot f_2(x_k^*(t'), u_k^*(t')) = c_k, \quad (4.6)$$

for a.e.  $(t, t') \in (0, \tau_k) \times (\tau_k, T)$ . The main issue now is to prove the uniform boundedness of  $(p_k)$  in order to prove the existence of a covector associated with  $x^*$  that satisfies the adjoint equation together with a discontinuity condition at the crossing time.

**Proposition 4.1.** *The sequence  $(\nu_k)$  is bounded.*

*Proof.* By contradiction, suppose that  $|\nu_k| \rightarrow +\infty$ . Observe first that the sequences  $(g(x_k(T)))$  is bounded. It follows using (4.1) backward in time together with (4.2) that  $(p_k(\tau_k^+))$  is also bounded. We also deduce that there is a constant  $M \geq 0$  such that

$$\forall k \in \mathbb{N}, \forall t \in [0, \tau_k], \quad |p_k(t)| \leq M. \quad (4.7)$$

Now, from the uniform convergence of  $(x_k^*)$  to  $x^*$  over  $[0, T]$ , we get that  $\psi(x_k^*(\tau_k)) \rightarrow \psi(x^*(\tau^*))$  as  $k \rightarrow +\infty$ . Using the gradient consistency property (3.1), we may assume that there exists  $v \in \partial_C \psi(x^*(\tau^*))$  such that

$$\nabla \psi_k(x_k^*(\tau_k)) \rightarrow v,$$

as  $k \rightarrow +\infty$  (extracting a sub-sequence if necessary). Let us now prove that the sequence  $(p_k(\tau_k^-))$  is unbounded. By contradiction, if  $(p_k(\tau_k^-))$  is bounded, then, dividing (4.4) by  $\nu_k$  and letting  $k \rightarrow +\infty$  yields that  $v = 0$ . Since  $x^*(\tau^*) \in \Gamma$ , this contradicts (3.5), hence,  $(p_k(\tau_k^-))$  is unbounded so that we may assume that  $|p_k(\tau_k^-)| \rightarrow +\infty$  as  $k \rightarrow +\infty$  (extracting a sub-sequence if necessary). For every  $k \in \mathbb{N}$ , the sequence  $(p_k(\tau_k^-)/|p_k(\tau_k^-)|)$  belongs to the unit sphere of  $\mathbb{R}^n$  so that (extracting a sub-sequence if necessary), we may assume that there is  $\xi \in \mathbb{R}^n$  such that  $|\xi| = 1$  and  $p_k(\tau_k^-)/|p_k(\tau_k^-)| \rightarrow \xi$  as  $k \rightarrow +\infty$ . Dividing now (4.4) by  $|p_k(\tau_k^-)|$  yields

$$\frac{p_k(\tau_k^+)}{|p_k(\tau_k^-)|} - \frac{p_k(\tau_k^-)}{|p_k(\tau_k^-)|} = \frac{\nu_k}{|p_k(\tau_k^-)|} \nabla \psi_k(x_k^*(\tau_k))$$

Since the left member of the preceding equality goes to  $-\xi$  and  $\nabla \psi_k(x_k^*(\tau_k)) \rightarrow v$  as  $k \rightarrow +\infty$ , we obtain that the sequence  $(\nu_k/|p_k(\tau_k^-)|)$  is bounded. Hence, extracting a sub-sequence if necessary, we may suppose that there is  $r \in \mathbb{R}$  such that  $(\nu_k/|p_k(\tau_k^-)|) \rightarrow r$  as  $k \rightarrow +\infty$ . It follows that

$$-\xi = rv. \quad (4.8)$$

On the other hand, for all  $k \in \mathbb{N}$ ,  $t \mapsto u_k^*(t)$  is a Lebesgue measurable function such that  $u_k^*(t) \in U$  for a.e.  $t \in [0, T]$ . Since  $U$  is bounded,  $u_k^* \in L^1([0, 1], \mathbb{R}^m)$ , thus, almost every point of  $[0, T]$  is a Lebesgue point of  $u_k^*$ . It follows that for every  $k \in \mathbb{N}$ , there exists  $t_k \in (0, \tau_k)$  such that  $t_k$  is a Lebesgue point of  $u_k^*$ ,  $u_k(t_k) \in U$ , and moreover, we may suppose that  $|t_k - \tau_k| \rightarrow 0$  as  $k \rightarrow +\infty$ . Now, observe that one has:

$$\begin{aligned} \frac{c_k}{|p_k(\tau_k^-)|} &= \frac{p_k(t_k)}{|p_k(\tau_k^-)|} \cdot f_1(x_k^*(t_k), u_k^*(t_k)) \\ &= \frac{(p_k(t_k) - p_k(\tau_k^-))}{|p_k(\tau_k^-)|} \cdot f_1(x_k^*(t_k), u_k^*(t_k)) + \frac{p_k(\tau_k^-)}{|p_k(\tau_k^-)|} \cdot f_1(x_k^*(t_k), u_k^*(t_k)). \end{aligned} \quad (4.9)$$

Next,  $c_k = p_k(t) \cdot f_2(x_k(t), u_k^*(t))$  for a.e.  $t \in (\tau_k, T)$ . Hence, taking for every  $k \in \mathbb{N}$  a Lebesgue point of  $u_k^*$  in  $(\tau_k, T)$  such that  $u_k^*(t) \in U$  and using (4.7), we deduce that the sequence  $(c_k)$  is bounded. Additionally, by using (4.1) backward in time and Gronwall's Lemma, it follows that

$$\forall t \in [0, \tau_k], \quad |p_k(t)| \leq c_1 |p_k(\tau_k^-)|,$$

where  $c_1 \geq 0$  is a constant. We deduce from (4.1) that  $|\dot{p}_k(t)| \leq c'_1 |p_k(\tau_k^-)|$  for every  $t \in [0, \tau_k]$  where  $c'_1 \geq 0$  is a constant. Finally, we get that

$$|p_k(t_k) - p_k(\tau_k^-)| \leq c'_1 |p_k(\tau_k^-)| |t_k - \tau_k|.$$

Combining the preceding inequality with (4.9), we obtain that

$$\frac{p_k(\tau_k^-)}{|p_k(\tau_k^-)|} \cdot f_1(x_k^*(t_k), u_k^*(t_k)) = o(1),$$

where  $o(1) \rightarrow 0$  as  $k \rightarrow +\infty$ . Extracting a sub-sequence if necessary (remind that  $U$  is compact), we may suppose that there is  $\omega \in U$  such that  $u_k^*(t_k) \rightarrow \omega$  as  $k \rightarrow +\infty$ . Hence, we get that

$$\xi \cdot f_1(x^*(\tau^*), \omega) = 0. \quad (4.10)$$

Combining (4.8) and (4.10), we find that  $v \cdot f_1(x^*(\tau^*), \omega) = 0$ . This equality is a contradiction with the transverse hypothesis (3.5) since  $v \in \partial_C \psi(x^*(\tau^*))$ . This ends the proof.  $\square$

**Remark 4.1.** *This proof mainly relies on the (strong) transverse hypothesis (3.5) made on the dynamics at the interface  $\Gamma$ . The main difficulty is that we cannot handle  $f_i(x_k^*(\tau_k), u_k^*(\tau_k))$  for  $i = 1, 2$  since  $u_k^*$  is defined almost everywhere. Our methodology consists in considering an optimal solution to (3.4), but we have a priori no information on the regularity of the control  $u_k^*$  which is a drawback of this approach. But, note that if  $u_k^*$  is right and left continuous at  $t = \tau_k$ , then, following [5], one can show that*

$$\nu_k = \frac{p_k^+(\tau_k) \cdot (f(x_k^*(\tau_k), u_k^*(\tau_k^-)) - f(x_k^*(\tau_k), u_k^*(\tau_k^+)))}{\nabla \psi_k(x_k(\tau_k)) \cdot f(x_k^*(\tau_k), u_k(\tau_k^-))}.$$

Combining this expression together with the transverse hypothesis, we would more easily deduce that  $(\nu_k)$  is bounded. Unfortunately, our approach does not allow us a priori to use this expression since we do not know in advance if  $u_k^*$  is defined at  $t = \tau_k^\pm$ .

We can now prove the following convergence result on the sequence  $(p_k)$ .

**Proposition 4.2.** *There exists a function  $p : [0, T]$  such that its restriction to  $[0, \tau]$ , resp. to  $(\tau, T]$  can be extended to an absolutely continuous function over  $[0, \tau]$ , resp. over  $[\tau, T]$ . Additionally, up to a sub-sequence,  $(p_k)$  strongly-weakly converges to  $p$  over every set  $[0, \tau - \alpha] \cup [\tau + \alpha, T]$  (for  $\alpha$  small enough). Finally,  $p$  satisfies*

$$\begin{cases} \dot{p}(t) = -D_x f_1(x^*(t), u^*(t))^\top p(t) & \text{for a.e. } t \in [0, \tau^*], \\ \dot{p}(t) = -D_x f_2(x^*(t), u^*(t))^\top p(t) & \text{for a.e. } t \in [\tau^*, T]. \end{cases} \quad (4.11)$$

*Proof.* The proof of this result is done in four steps:

**Step 1. Definition of auxiliary trajectories and covectors.** First, let us denote by  $\tilde{u}^*$  the optimal control corresponding to the optimal trajectory  $x^*$  of Problem (2.3). Then, given  $\eta > 0$ , let us define an auxiliary trajectory  $x_1^*$  as the unique solution to the following Cauchy problem:

$$\begin{cases} \dot{x}_1^*(t) = f_1(x_1^*(t), \tilde{u}^*(t)), & \text{for a.e. } t \in [0, \tau^* + \eta], \\ x_1^*(0) = x_0, \end{cases}$$

As well, we can define an auxiliary trajectory  $x_2^*$  as the unique solution to the following Cauchy problem:

$$\begin{cases} \dot{x}_2^*(t) = f_2(x_2^*(t), \tilde{u}^*(t)), & \text{for a.e. } t \in [\tau^* - \eta, T], \\ x_2^*(T) = x^*(T). \end{cases}$$

Note that the existence of  $\eta > 0$  (such that  $\tau^* - \eta \geq 0$  and  $\tau^* + \eta \leq T$ ) is guaranteed by Cauchy-Lipschitz's Theorem. Furthermore, we can define  $p_2$  as the unique solution to the following Cauchy problem:

$$\begin{cases} \dot{p}_2(t) = -D_x f_2(x_2^*(t), \tilde{u}^*(t))^\top p_2(t), & \text{for a.e. } t \in [\tau^* - \eta, T], \\ p_2(T) = -\nabla g(x^*(T)), \end{cases}$$

and, similarly  $p_1$  is defined as the unique solution to the Cauchy problem:

$$\begin{cases} \dot{p}_1(t) = -D_x f_1(x_1^*(t), \tilde{u}^*(t))^\top p_1(t), & \text{for a.e. } t \in [0, \tau^* + \eta], \\ p_1(\tau^*) = p_2(\tau^*) - \nu v, \end{cases}$$

where  $\nu \in \mathbb{R}$  and  $v \in \partial_C \psi(x^*(T))$  are limits (up to a subsequence) of the sequences  $(\nu_k)$  and  $(\nabla \psi_k(x_k^*(\tau_k)))$  respectively. Note that the restriction of  $x_1^*$  and  $p_1$ , resp.  $x_2^*$  and  $p_2$ , coincides with  $x^*$  and  $p$  over  $[0, \tau^*]$ , resp. over  $[\tau^*, T]$ . Now, let us define  $x_{k,1}^*$  as the unique solution to the following Cauchy problem:

$$\begin{cases} \dot{x}_{k,1}^*(t) &= f_1(x_{k,1}^*(t), u_k^*(t)), \quad \text{for a.e. } t \in [0, \tau^* + \eta], \\ x_{k,1}^*(0) &= x_0, \end{cases}$$

and similarly,  $x_{k,2}^*$  is defined as the solution to the following Cauchy problem:

$$\begin{cases} \dot{x}_{k,2}^*(t) &= f_2(x_{k,2}^*(t), u_k^*(t)), \quad \text{for a.e. } t \in [\tau^* - \eta, T], \\ x_{k,2}^*(T) &= x_k^*(T). \end{cases}$$

Furthermore, we define  $p_{k,2}$  as the unique solution to the following Cauchy problem:

$$\begin{cases} \dot{p}_{k,2}(t) &= -D_x f_2(x_{k,2}^*(t), u_{k,2}^*(t))^\top p_{k,2}(t), \quad \text{for a.e. } t \in [\tau^* - \eta, T], \\ p_{k,2}(T) &= -\nabla g(x_{k,2}^*(T)), \end{cases}$$

and  $p_{k,1}$  as the unique solution to the following Cauchy problem:

$$\begin{cases} \dot{p}_{k,1}(t) &= -D_x f_1(x_{k,1}^*(t), u_{k,1}^*(t))^\top p_{k,1}(t), \quad \text{for a.e. } t \in [0, \tau^* + \eta], \\ p_{k,1}(\tau_k) &= p_{k,2}(\tau_k) - \nu_k \nabla \psi_k(x_{k,2}(\tau_k)). \end{cases}$$

By using a standard argumentation (based on the boundedness of  $U$ , the boundedness of the sequence  $(x_k)$  for the  $L^\infty$ -norm, and the continuity of  $f_i$  and  $D_x f_i$ ), we deduce that there exists  $k_0 \in \mathbb{N}^*$  such that  $x_{k,1}^*$  and  $p_{k,1}$ , resp.  $x_{k,2}^*$  and  $p_{k,2}$ , are well defined over  $[0, \tau^* + \eta]$ , resp. over  $[\tau^* - \eta, T]$ , for all  $k \geq k_0$ .

**Step 2. Convergence over  $[\tau^* - \eta, T]$ .** One can see that the pair  $z_{k,2} := (x_{k,2}^*, p_{k,2})$  satisfies the following differential inclusion

$$\dot{z}_{k,2}(t) \in H_2(z_{k,2}(t)) \quad \text{a.e. } t \in [\tau^* - \eta, T],$$

where  $H_2 : B(0, R) \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n \times \mathbb{R}^n$  is defined as  $H_2(z) := \widehat{F}_2(\tilde{x}) \times \widehat{G}_2(z)$  for every  $z := (\tilde{x}, p) \in \mathbb{R}^n \times \mathbb{R}^n$ . From the hypotheses on the velocity sets, we check that for each  $z \in B(0, R) \times \mathbb{R}^n$ ,  $H_2(z)$  is with nonempty compact and convex values. In addition,  $H_2$  is also with linear growth (using the linearity of the adjoint equation, the fact that  $x \in B(0, R)$  and the compactness of  $U$ ). Moreover, from the compactness of  $U$  and the uniform continuity of the mapping  $(x, u, p) \mapsto (f_2(x, u), -D_x f_2(x, u)^\top p)$  over compact sets one can deduce that  $H_2$  upper semi-continuous at each  $(x, p) \in B(0, R) \times \mathbb{R}^n$ . Hence, we are in a position to apply [14, Theorem 1.11]. It follows that there is an absolutely continuous function  $z_2 = (x_2^*, p_2) : [\tau^* - \eta, T] \rightarrow \mathbb{R}^n \times \mathbb{R}^n$  satisfying

$$\dot{z}_2(t) \in H_2(z_2(t)) \quad \text{a.e. } t \in [\tau^* - \eta, T],$$

such that  $(z_{k,2})$  strongly-weakly converges to  $z_2$  over  $[\tau^* - \eta, T]$ . Moreover, one also gets  $x_2(T) = x^*(T)$  and  $p_2(T) = -\nabla g(x^*(T))$ . From the uniform convergence of  $x_{k,2}^*$  towards  $x_2^*$  and the definition of auxiliary trajectories we get  $x_2^* = x^*$  over  $[\tau^*, T]$ . Furthermore, from the Filippov selection Lemma, there exists an admissible control  $u_2^*$  such that

$$\dot{x}_2^*(t) = f_2(x_2^*(t), u_2^*(t)), \quad \dot{p}_2(t) = -D_x f_2(x_2^*(t), u_2^*(t))^\top p_2(t),$$

for almost every  $t \in [\tau^* - \eta, T]$ .

**Step 3. Convergence over  $[0, \tau^* + \eta]$ .** Similarly, one can see that the pair  $z_{k,1} := (x_{k,1}^*, p_{k,1})$  satisfies the following differential inclusion

$$\dot{z}_{k,1}(t) \in H_1(z_{k,1}(t)) \quad \text{a.e. } t \in [0, \tau^* + \eta],$$

where  $H_1 : B(0, R) \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n \times \mathbb{R}^n$  is defined as  $H_1(z) := \widehat{F}_1(\tilde{x}) \times \widehat{G}_1(z)$  for every  $z := (\tilde{x}, p) \in \mathbb{R}^n \times \mathbb{R}^n$ . Using similar arguments as in Step 2, it follows that there is an absolutely continuous function  $z_1 = (x_1^*, p_1) : [0, \tau^* + \eta] \rightarrow \mathbb{R}^n \times \mathbb{R}^n$  satisfying

$$\dot{z}_1(t) \in H_1(z_1(t)) \quad \text{a.e. } t \in [0, \tau^* + \eta],$$

such that  $(z_{k,1})$  strongly-weakly converges to  $z_1$  over  $[0, \tau^* + \eta]$ . Moreover, one also gets  $x_1(0) = x^*(0) = x_0$  and  $p_1(\tau^*) = p_2(\tau^*) - \nu v$ . From the uniform convergence of  $x_{k,1}^*$  towards  $x_1^*$  and the definition of auxiliary trajectories we get that  $x_1^* = x^*$  over  $[0, \tau^*]$ . Furthermore, from the Filippov selection lemma, we get the existence of an admissible control  $u_1$  such that

$$\dot{x}_1^*(t) = f_1(x_1^*(t), u_1^*(t)), \quad \dot{p}_1(t) = -D_x f_1(x_1^*(t), u_1^*(t))^\top p_1(t),$$

for almost every  $t \in [0, \tau^* + \eta]$ .

**Step 4. Definition of the covector  $p$ .** Finally, we define  $u^*$  and  $p$  as follows

$$u^*(t) := \begin{cases} u_1^*(t) & \text{a.e. } t \in [0, \tau^*], \\ u_2^*(t) & \text{a.e. } t \in [\tau^*, T]. \end{cases}$$

and

$$p(t) := \begin{cases} p_1(t) & \text{for all } t \in [0, \tau^*), \\ p_2(t) & \text{for all } t \in (\tau^*, T]. \end{cases}$$

One can note that  $u^*$  is a corresponding control to the optimal trajectory  $x^*$ . Moreover, we get that the covector  $p$  (that is piecewise absolutely continuous) satisfies (4.11) which completes the proof.  $\square$

## 4.2 Derivation of a nonsmooth HMP

In this section, we provide necessary optimality conditions for an optimal solution to (2.3) in terms of a nonsmooth hybrid maximum principle. Our main result is as follows.

**Theorem 4.1.** *Suppose that the hypotheses in Section 2 and Assumption 3.1 are fulfilled. Let  $x^* \in \mathcal{S}_T(x_0)$  be an optimal solution to (2.3) and  $u^* \in \mathcal{U}$  the associated optimal control. Then,  $x^*$  has zero or one crossing time and there is a piecewise absolutely continuous function  $p : [0, T] \rightarrow \mathbb{R}^n$  satisfying:*

(i) *the adjoint equation*

$$\dot{p}(t) = -D_x f(x^*(t), u^*(t))^\top p(t) \quad \text{a.e. } t \in [0, T],$$

(ii) *the terminal condition*

$$p(T) = -\nabla g(x^*(T)),$$

(iii) *the Hamiltonian maximization condition*

$$u^*(t) \in \operatorname{argmax}_{\omega \in \mathcal{U}} p(t) \cdot f(x^*(t), \omega) \quad \text{a.e. } t \in [0, T],$$

(iv) *the discontinuity condition*

$$p^+(\tau) - p^-(\tau) = \nu v, \tag{4.12}$$

where  $\nu \in \mathbb{R}$  and  $v \in \partial_C \psi(x^*(\tau^*))$  if  $x^*$  has a crossing time  $\tau^* \in (0, T)$  (otherwise  $p$  has no discontinuity),

(v) *the Hamiltonian constancy : there is  $\tilde{c} \in \mathbb{R}$  such that for a.e.  $(t, t') \in (0, \tau^*) \times (\tau^*, T)$*

$$p(t) \cdot f_1(x^*(t), u^*(t)) = p(t') \cdot f_2(x^*(t'), u^*(t')) = \tilde{c}.$$

*Proof.* We prove each item step by step.

(i) Adjoint equation : the existence of a piecewise absolutely continuous function  $p : [0, T] \rightarrow \mathbb{R}^n$  satisfying the adjoint equation follows from Proposition 4.2.

(ii) Terminal condition : since for all  $k \in \mathbb{N}$ ,  $p_k(T) = -\nabla g(x_k(T))$ , the result follows from Proposition 4.2 letting  $k \rightarrow +\infty$ .

(iii) Hamiltonian maximization condition : from (4.3), we get

$$p_k(t) \cdot f(x_k^*(t), \omega) \leq p_k(t) \cdot f(x_k^*(t), u_k^*(t)) \quad \text{a.e. } t \in [0, T].$$

Fix  $t_0 \in (0, T) \setminus \{\tau^*\}$  and  $\varepsilon > 0$ . It follows that

$$\frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} p_k(t) \cdot f(x_k^*(t), \omega) dt \leq \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} p_k(t) \cdot \dot{x}_k^*(t) dt.$$

By using the uniform convergence of  $(x_k)$  to  $x^*$  over  $[0, T]$ , the uniform convergence of  $(p_k)$  to  $p$  over every compact subset of  $[0, \tau^*) \cup (\tau^*, T]$ , and the weak convergence of  $\dot{x}_k^*$  to  $\dot{x}^*$ , we find that

$$\frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} p(t) \cdot f(x^*(t), \omega) dt \leq \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} p(t) \cdot \dot{x}^*(t) dt.$$

Letting  $\varepsilon \downarrow 0$ , we obtain the Hamiltonian maximization condition (ii) since almost every point of  $[0, T]$  is a Lebesgue point of  $u^*$ .

(iv) Discontinuity condition. Suppose that  $x^*$  has a crossing point at some time  $\tau^*$ . From Lemma 3.1, we know that  $\tau^*$  is unique. Now, for every  $k \in \mathbb{N}$  large enough, one has  $p_k(\tau_k^*) - p_k(\tau_k^-) = \nu_k \nabla \psi_k(x_k(\tau_k))$ . Remind that the sequence  $(\nu_k)$  is bounded, so, extracting a sub-sequence if necessary, we may assume that  $\nu_k \rightarrow \nu$  as  $k \rightarrow +\infty$  where  $\nu \in \mathbb{R}$ . From the gradient consistency, there is  $v \in \partial_C \psi(x^*(\tau^*))$  such that (up to a sub-sequence),  $\psi_k(x_k(\tau_k)) \rightarrow +\infty$  as  $k \rightarrow +\infty$ . Hence, by passing to the limit as  $k \rightarrow +\infty$ , we find that  $p^+(\tau^*) - p^-(\tau^*) = \nu v$ , whence the result.

(v) Since  $(c_k)$  is bounded, we may assume that there is  $\tilde{c} \in \mathbb{R}$  such that, up to a subsequence, one has  $c_k \rightarrow \tilde{c}$  as  $k \rightarrow +\infty$ . Next, we proceed as for proving (iii). Let  $t_0 \in (0, \tau) \setminus \{\tau^*\}$  be a Lebesgue point of  $u^*$  such that  $u^*(t_0) \in U$  (we know that a.e. point of  $(0, T)$  satisfies this property since  $u^*$  is a measurable (bounded) function such that  $u^*(t) \in U$  for a.e.  $t \in [0, T]$ ). We get for  $\varepsilon > 0$  small enough

$$\frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} p_k(t) \cdot \dot{x}_k^*(t) dt = \tilde{c}_k.$$

As for proving (iii), we let  $k \rightarrow +\infty$  (using weak convergence of  $(x_k^*)$ ) and then  $\varepsilon \downarrow 0$ , and we obtain the result over  $(0, \tau^*)$ . Next, the same reasoning is done over  $(\tau^*, T)$  which ends the proof.  $\square$

**Remark 4.2.** Whenever  $\nu \geq 0$ , (4.12) becomes

$$p^+(\tau^*) - p^-(\tau^*) \in N^C(\overline{X}_1; x^*(\tau^*)).$$

Indeed, in that case, one has  $N^C(\overline{X}_1; x^*(\tau^*)) = \mathbb{R}_+ \partial_C \psi(x^*(\tau^*))$  since  $0 \notin \partial_C \psi(x^*(\tau^*))$  because of the transversality condition (3.5).

We end-up this section by giving a simple example highlighting the application of the nonsmooth HMP. Consider the Lipschitz-continuous function  $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by

$$\psi(x_1, x_2) := \max(x_1, x_1 + x_2 - 1),$$

so that  $\Gamma = \{(x_1, x_2) \in \mathbb{R}^2 ; \psi(x_1, x_2) = 0\}$ . It is easily seen that at the point  $(0, 1)$  the sub-differential of  $\psi$  is the convex combination of the gradients of the maps  $(x_1, x_2) \mapsto x_1$  and  $(x_1, x_2) \mapsto x_1 + x_2 - 1$ , hence

$$\partial_C \psi(0, 1) = \{(1, \theta) ; \theta \in [0, 1]\}.$$

Now, we consider the hybrid control system

$$\begin{cases} \dot{x}_1 = \frac{\sqrt{3}}{2}(1-u) & \text{if } \psi(x_1, x_2) < 0 \\ \dot{x}_2 = \frac{1}{2}(1-u) & \end{cases} \quad \text{and} \quad \begin{cases} \dot{x}_1 = x_2 & \text{if } \psi(x_1, x_2) > 0, \\ \dot{x}_2 = \frac{u}{2} & \end{cases}$$

together with the initial condition  $x(0) = (-\sqrt{3}, 0)$  and the optimal control problem

$$-x_1(2) \rightarrow \min$$

where the minimum is taken w.r.t. measurable control functions  $u : [0, 2] \rightarrow [-1, \frac{1}{2}]$ . We see that every admissible trajectory crosses the interface at the point  $(0, 1)$ . Hence, the transverse condition needs only to be verified at the point  $(x_1, x_2) = (0, 1)$ . For all  $\theta \in [0, 1]$  and all  $u \in [-1, \frac{1}{2}]$ , one has

$$\left(\frac{\sqrt{3}(1-u)}{2}, \frac{1-u}{2}\right) \cdot (1, \theta) = \frac{(1-u)}{2}(\sqrt{3} + \theta) \geq \frac{\sqrt{3}}{4}; \quad \left(1, \frac{u}{2}\right) \cdot (1, \theta) = 1 + \frac{u\theta}{2} \geq \frac{1}{2},$$

so that the transverse condition is verified. Now, since the objective is equivalent to maximize  $x_1(2)$ , an optimal control will necessarily be such that  $u(t) = -1$  as long  $x(t) \in X_1$  (where  $X_1 = \{x \in \mathbb{R}^2; \psi(x) < 0\}$ ). We thus get that an optimal trajectory has a single crossing time at  $\tau = +1$ . If we write the HMP, we introduce the (conserved) Hamiltonian

$$H = \frac{\sqrt{3}}{2}p_1(1-u) + \frac{1}{2}p_2(1-u) = q_1x_2 + q_2\frac{u}{2},$$

where  $(p_1, p_2)$  is the covector over  $[0, 1]$  such that  $\dot{p}_1 = \dot{p}_2 = 0$  over  $[0, 1]$  (hence,  $(p_1, p_2)$  is constant) and where  $(q_1, q_2)$  is the covector over  $[1, 2]$  such that  $\dot{q}_1 = 0$  and  $\dot{q}_2 = -1$ . Additionally, the terminal condition gives us  $q_1(2) = 1$  and  $q_2(2) = 0$  so that  $q_1(t) = +1$  and  $q_2(t) = 2 - t$  for  $t \in [1, 2]$  and consequently  $u(t) = \frac{1}{2}$  a.e. over  $[1, 2]$  (using the maximization condition). If we write the jump condition and the conservation of the Hamiltonian, we get the three equalities

$$\begin{array}{rcl} q_1 - p_1 & = & \nu \\ q_2(\tau) - p_2 & = & \nu\theta \\ \sqrt{3}p_1 + p_2 & = & 2 \end{array} \Rightarrow \begin{array}{rcl} 1 - p_1 & = & \nu \\ 1 - p_2 & = & \nu\theta \\ \sqrt{3}p_1 + p_2 & = & 2 \end{array}$$

where unknown parameters are  $p_1, p_2, \nu \in \mathbb{R}$  and  $\theta \in [0, 1]$ . This gives the relations  $p_2 = 2 - \sqrt{3}p_1$  and  $p_1 = 1 - \nu$  so that  $p_2 = 2 - \sqrt{3}(1 - \nu) = 1 - \nu\theta$ . Finally, we get  $(\sqrt{3} + \theta)\nu = \sqrt{3} - 1$ . We thus deduce from the preceding equality that  $\nu$  cannot be null. Hence,  $\nu$  is necessarily such that  $\nu \neq 0$  so that the covector is necessarily discontinuous at the crossing time (it has a jump). In addition, to ensure that  $u = -1$  over  $[0, 1]$ , one must have  $\frac{\sqrt{3}}{2}p_1 + \frac{1}{2}p_2 > 0$  from the maximization condition. We see that it is possible to choose adequately the pair  $(\nu, \theta) \in \mathbb{R}^* \times [0, 1]$  to fulfill this constraint. For instance, taking  $\theta = 0$  gives us  $\nu = 1 - 1/\sqrt{3}$ ,  $p_1 = 1/\sqrt{3}$  and  $p_2 = 1$  so that  $(\sqrt{3}/2)p_1 + p_2/2 > 0$  as wanted.

In conclusion, this example highlights the fact that (depending on the problem) the covector must have a jump at the crossing time but the direction of the jump in the subdifferential of  $\psi$  may be non-unique.

## 5 Discussions and conclusion

In this paper, we have given a so-called nonsmooth HMP for a Mayer optimal control problem governed by a hybrid control system defined regionally. Additionally, we supposed the partition of  $\mathbb{R}^n$  to have two regions delimited by a nonsmooth interface. This framework has been made simple in order to avoid further technical issues and also to highlight the possibility of using the "classical" HMP (valid for regions with smooth boundaries). The nonsmooth HMP was obtained thanks to the gradient consistency property arising from the regularization of a locally Lipschitz continuous function via mollifiers. The resulting discontinuity condition naturally generalizes the one encountered in the hybrid setting when interfaces between regions are smooth, see, *e.g.*, [5, 19]. It is also in line with the one in [13, Theorem 22.20] which (as far as we know) is valid only for temporally hybrid optimal control problems (in the sense that no partition of the state space is considered and the change of dynamics occurs at free instants).

We are confident that our approach could be re-employed in a more general setting such as when the partition of the state space has a countable number of regions, *i.e.*,  $\mathbb{R}^n = \cup_{i \in \mathcal{I}} \bar{X}_i$ , when the system involves mixed initial terminal constraints  $\phi(x(0), x(T)) \in C$ , a Bolza cost and a free terminal time. Note that when handling mixed initial terminal constraints, controllability hypotheses should probably be added in order to guarantee that one can approach any admissible trajectory to the original optimal control problem by an admissible trajectory solution to the approximated optimal control problems. This point is crucial in order to make this approach possible (note that here, no terminal constraints were considered, that is why

we did not need such hypotheses in Proposition 3.2). Next, the two main issues will be first to approach any admissible trajectory solution to the original optimal control problem and then to give bounds on the sequence  $(\nu_k)$  arising in the discontinuity condition of the covector  $p_k$ . This will be made possible thanks to a strong transverse hypothesis similar to our but at every interface between two regions.

Note that our methodology differs from other approaches like the augmentation technique as in [5] or the sensitivity analysis as it is done in [4, 19] (which follows the standard techniques based on the variation vector for proving the PMP, see [24]). At first glance, our approach seems rather straightforward, however, when introducing an approximated optimal control problem, we have in general no information on the regularity of the approximated optimal control (such as right and left continuity at a crossing time). This is a difficulty that is inherent to this methodology. As far as we know, we can point out a similar difficulty if Ekeland's variational principle is applied on some (regional) hybrid optimal control problem involving mixed initial terminal constraints. Indeed, in that case, no information on the regularity of an approximated control (lie right and left continuity at a crossing time) follows from the application of this principle. Overcoming such difficulties seems an interesting (but difficult) question.

Finally, another possible way to derive a nonsmooth HMP in our setting could be to follow [5] and to use a nonsmooth PMP (in place of the usual one) on some augmented optimal control problem involving nonsmooth data (since the interface is nonsmooth). This could also be an interesting perspective for future works on the subject. As well, it could be interesting to pursue the analysis made in this work, but under a weak transverse hypothesis at the interface (see [7]).

## Acknowledgements

This work has benefited from several fruitful exchanges with colleagues. We thank Abderrahim Jourani for suggesting to us the employed methodology to derive necessary optimality conditions. We also would like to thank Edouard Pauwels for fruitful exchanges on the chain rule formula. We are also grateful to Alberto Seeger for helpful discussions on the gradient consistency property.

## References

- [1] J.-P. Aubin, A. Cellina, *Differential Inclusions: Set-Valued Maps and Viability Theory*, Springer-Verlag Berlin Heidelberg 1984.
- [2] J.-P. Aubin, A.M. Bayen, P. Saint-Pierre, *Viability Theory, New Directions*, Second Editions, Springer, 2011.
- [3] G. Barles, A. Briani, E. Trélat, *Value function for regional control problems via dynamic programming and Pontryagin maximum principle*, Math. Control Relat. Fields, vol. 8, 3-4, pp. 509–533, 2018.
- [4] T. Bayen, A. Bouali, L. Bourdin, *Hybrid maximum principle with regionally switching parameter*, submitted, 2023, [hal-03638701](#).
- [5] T. Bayen, A. Bouali, L. Bourdin, *The hybrid maximum principle for optimal control problems with spatially heterogeneous dynamics is a consequence of a Pontryagin maximum principle for  $L^1_{\square}$  local solutions*, submitted, 2023, [hal-03985420v1](#).
- [6] T. Bayen, A. Bouali, L. Bourdin, *Minimum time problem for the double integrator with a loss control region*, submitted, 2023, [hal-03928967v2](#).
- [7] T. Bayen, A. Bouali, L. Bourdin, O. Cots, *On the reduction of a spatially hybrid optimal control problem into a temporally hybrid optimal control problem*, to appear in American Institute of Mathematical Sciences (tribute to I. Kupka), 2024, [hal-04319045v1](#).
- [8] T. Bayen, K. Boumaza, A. Rapaport, *Necessary optimality condition for the minimal time crisis relaxing transverse condition via regularization*, ESAIM Control Optim. Calc. Var., vol. 27, pp. 105, 2021.



- [9] T. Bayen, A. Rapaport, *About Moreau-Yosida regularization of the minimal time crisis problem*, J. Convex Anal., vol. 2, 1, pp. 63–290, 2016.
- [10] P. Bettiol, H. Frankowska, *Normality of the maximum principle for nonconvex constrained Bolza problems*, J. Diff. Eqs., vol. 243, 2, pp. 256–269, 2007.
- [11] A. Bressan, Y. Hong, *Optimal control problems on stratified domains*, Netw. Heterog. Media, vol. 2, 2, pp. 313–331, 2007.
- [12] F.H. Clarke, *Optimization and nonsmooth analysis*, Society for Industrial and Applied Mathematics, 1990.
- [13] F.H. Clarke, *Functional analysis, calculus of variations and optimal control*, London, Springer, 2013.
- [14] F.H. Clarke, Y.S. Ledyaev, R.J. Stern, P.R. Wolenski, *Nonsmooth analysis and control theory*, vol. 178, Springer Science & Business Media, 2008.
- [15] F.H. Clarke, R.B. Vinter, *Optimal multiprocesses*, SIAM J. Control Optim., vol.27, pp. 1072–1091, 1989.
- [16] A.F. Filippov, *Classical solutions of differential equations with multi-valued right-hand side*, SIAM J. Control, vol. 5, 4, pp. 609–621, 1967.
- [17] H. Frankowska, *Regularity of minimizers and of adjoint states in optimal control under state constraints*, J. of Convex Anal., vol. 13, 2, pp. 299–328, 2006.
- [18] H. Frankowska, F. Rampazzo, *Filippov’s and Filippov-Wazewski’s theorems on closed domains*, J. Diff. Eqs., vol. 161, pp. 449–478, 2000.
- [19] T. Haberkorn, E. Trélat, *Convergence results for smooth regularizations of hybrid nonlinear optimal control problems*, SIAM J. Control Optim., vol. 49, 4, pp. 1498–1522, 2011.
- [20] M. Garavello, B. Piccoli, *Hybrid necessary principle*, SIAM J. Control Optim., vol. 43, pp. 1867–1887, 2005.
- [21] R. Goebel, A.R. Teel, *Solutions to hybrid inclusions via set and graphical convergence with stability theory applications*, Automatica, vol. 42, 4, pp. 573–587, 2006.
- [22] I.A. Kupka, *The ubiquity of Fuller’s phenomenon*, In : Nonlinear controllability and optimal control, Dekker, New York, vol. 133, pp. 313–350, 1990.
- [23] D. Liberzon, *Switching in systems and control*, Boston, Birkhäuser, 2003.
- [24] L.S. Pontryagin, V.G. Boltyanskiy, R.V. Gamkrelidze, E.F. Mishchenko, *Mathematical theory of optimal processes*, The Macmillan Company, 1964.
- [25] A. Pakniyat, P.E. Caines, *On the hybrid minimum principle: the Hamiltonian and adjoint boundary conditions*, IEEE Trans. Automat. Contr., vol. 66, pp. 1246–1253, 2020.
- [26] N.S. Papageorgiou, *A stability result for differential inclusions in Banach spaces*, J. Math. Anal. Appl., vol. 118, 1, pp. 232–246, 1986.
- [27] R.T. Rockafellar, R.J.-B. Wets, *Variational Analysis*, Springer-Verlag, 2009.
- [28] H.J. Sussmann, *A maximum principle for hybrid optimal control problems*, Proc. 38th IEEE Conf. Decis. Control, pp. 425–430, 1999.
- [29] C. Xiaojun, *Smoothing methods for nonsmooth, nonconvex minimization*, Math. Program., vol. 134, pp. 71–99, 2012.
- [30] L. Thibault, *Unilateral variational analysis in Banach spaces*, World Scientific, 2023.