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CONTROLLING THE RATES OF A CHAIN OF HARMONIC OSCILLATORS WITH A POINT LANGEVIN THERMOSTAT

AMIRALI HANNANI, MINH NHAT PHUNG, MINH-BINH TRAN, AND EMMANUEL TRÉLAT

Abstract. We consider the control problem for an infinite chain of coupled harmonic oscillators with a Langevin thermostat at the origin. We study the effect of two types of open-loop boundary controls, impulsive control and linear memory-feedback control, in the high frequency limit. We investigate their action on the reflection-transmission coefficients for the wave energy for the scattering of the thermostat. Our study shows that impulsive boundary controls have no impact on the rates and are thus not appropriate to act on the system, despite their physical meaning and relevance. In contrast, the second kind of control that we propose, which is less standard and uses the past of the state solution of the system, is adequate and relevant. We prove that any triple of rates satisfying appropriate assumptions is asymptotically reachable thanks to linear memory-feedback controls that we design explicitly.

Contents

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1. Introduction

Heat reservoirs at temperature T are commonly modeled by the Langevin stochastic dynamics. When the bulk evolution is governed by a discrete wave equation, a small parameter $\varepsilon > 0$ is introduced to dictate the ratio between microscopic and macroscopic space-time units. In general, the noise is chosen so that by the stochastic mechanism, there is only a finite amount of momentum exchanged in a finite interval of time. As thus, each particle undergoes only a finite number of collisions in any finite interval of time. It is common to let ε tend to 0, which is often referred to as taking the kinetic limit for the system (for instance, see [10], [11], [13]). When a chain has no microscopic boundary, the energy density evolution is often described by a linear kinetic equation.

A useful tool to localize in space the energy per frequency mode is the Wigner distribution, introduced in [15]. In the absence of the thermostat, by adding a small conservative noise exchanging velocities, the authors of [2] prove that, in the kinetic limit $\varepsilon \to 0$, the Wigner distribution converges to the solution of the kinetic transport equation

$$\partial_t W(t,x,k) + v_g(k)\partial_x W(t,x,k) = 2\gamma_0 \int_{\mathbb{T}} \mathcal{R}(k,k') \big(W(t,x,k') - W(t,x,k)\big)\, dk, \tag{1}$$

for all $(t,x,k) \in [0,+\infty) \times \mathbb{T} \times \mathbb{R}$. The explicit scattering kernel $\mathcal{R}(k,k') \geq 0$ is given by

$$\mathcal{R}(k) := \int_{\mathbb{T}} dk' \mathcal{R}(k,k') \sim |k|^2 \quad \text{for } |k| \ll 1.$$

We also refer to [3] for a related situation. Here and in the sequel $\mathbb{T}$ is the unit torus, identify with the interval $[-1/2, 1/2]$ with periodic endpoints. The parameter $\gamma_0 > 0$ is the scattering rate for the microscopic chain. The group velocity is defined by $v_g(k) = \omega(k)/2\pi$, $\omega(k)$ is the dispersion relation of the chain.

When a heat bath at temperature T is applied to one particle, which is labeled 0, with a coupling strength $\gamma_1 > 0$, the action of the heat bath is not affected by the scale of the small parameter ε. As a consequence, when a thermostat is included in the system, its presence can be regarded as a singular perturbation of the dynamics of the system. Mathematically speaking, when $\varepsilon \to 0$, in [6, 9], it has been proved that the thermostat enforces that phonons of wave number k are generated with rate $r_a(k)T$, incoming k-phonons can be transmitted with probability $r_t(k)$ and reflected with probability $r_r(k)$, which means that one needs to introduce the boundary conditions at $y = 0$ on (1):

$$\begin{aligned} W(t,0^+,k) &= r_r(k)W(t,0^+,-k) + r_t(k)W(t,0^-,k) + r_a(k)T, \quad \text{for } 0 < k \leq 1/2, \\ W(t,0^-,k) &= r_r(k)W(t,0^-,-k) + r_t(k)W(t,0^+,k) + r_a(k)T, \quad \text{for } -1/2 < k < 0. \end{aligned} \tag{2}$$

Those quantities are properly normalized according to $r_a(k)+r_t(k)+r_r(k) = 1$, so that $W(t,y,k) = T$ is a thermal equilibrium.

In the recent years, there have been significant progresses on the control theory for kinetic models [1, 4, 12]. The goal of our work is to initiate the study of the Wigner distributions for stochastic discrete wave equations under the point of view of control theory. To be more precise, in the setting of the stochastic discrete wave equations considered in [6, 9], the three important parameters $r_a(k), r_t(k), r_r(k)$ are respectively the probabilities for absorption, transmission and reflection, and the rate of creation of a phonon of mode k. Within the control theory viewpoint, we investigate the following question:

Question A - Controllability at the kinetic limit: *if we add a control function, having a relevant physical meaning, to the wave system/equation, can we control the above three important rates: transmission, reflection/ absorption, and creation of a phonon of mode k in* (2)*?*

To address Question A, we consider two possible types of open-loop boundary controls.

The first boundary control is inpulsive (see Section 3) and consists of adding the term $F(t)\delta_{0,n}$ to the system. At the kinetic limit such a control has an impact only at the boundary, causing a shoot-up explained by the friction ν at the boundary. Like the thermostat, the control force $F(t)$ can be seen as a wave at all frequencies but only the frequency of the oscillators are kept; all other frequencies are damped by oscillations at the macroscopic limit. Physically, this type of control corresponds to adding a force $F(t)$ at the boundary. Similar physical phenomena (subjecting a chain of oscillators to a point force) have been considered recently in [7]. The difference with our work lies in the nature of the force: L^1 in time in our case vs. periodic in time in theirs. Another difference is the scaling regime that we consider for the control of the kinetic limit, in contrast to their setup that corresponds to the diffusive scaling. Our control is impulsive in the sense that it creates a new term in the kinetic limit equation (see (18) further), namely, a Dirac delta measure along the characteristic $\delta(x - v_g(k)t)$. This means that the control is too strong and the whole dynamics of the chain follows the control and not its own dynamics. Even when we take a smooth function F, an extra Dirac delta measure along the characteristic $\delta(x - v_g(k)t)$ pops up in the final equation, as proved in (18). It follows that the probabilities for absorption, transmission and reflection, and the rate of creation of a phonon of mode k are not controllable using this control (see Remark 2).

The second type of open-loop boundary control is a linear memory-feedback control (see Section 4) on the stochastic process of the wave consisting of adding the convolution term $F \star \alpha_0\delta_{0,n}$. In control theory, the time-convolution integral for a continuous-time system calculates the output of a system to a given input using the response of the system. Therefore it is natural to utilize time-convolution as a tool to control the system at any time t given that the anterior states of the system are known. In this respect, the control uses feedback from previous times to control outputs at the current time and is often called memory-feedback. Since the control in this case follows closely the dynamics of the system, the "impulsive" behavior in which a delta function pops up in the final result can be avoided. Noticing that the three quantities r_a, r_t, r_r depend on the friction ν, a control $F \star \alpha_0\delta_{0,n}$ can be imposed on this parameter ν, which sticks to the stochastic process α_0. Thanks to this feedback-type boundary control, eventually the effect of the control on the kinetic limit is much better: the three rates (r_a, r_t, r_r) can be changed at the kinetic limit. We will prove that for a given class of function rates (r_a, r_t, r_r), enjoying appropriate conditions, it is possible to find a control F asymptotically steering the rates to desired target functions (see Corollary 6).

Another type of control, which seems to be technically harder is the so-called "internal control", for which, the control acts on several points of the chain, will be studied in future work. Let us mention that instead of a Langevin thermostat, one may also consider a Poisson scattering mechanism at the boundary. Such mechanisms are studied in [8]. Controlling the rate of such problems seems doable via our methods.

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2. Setting and Notations

Following [6, 9], we consider the evolution of an infinite particle system governed by the Hamiltonian

$$\mathcal{H}(\alpha, \beta) := \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_n^2 + \frac{1}{2}\sum_{n,n'\in\mathbb{Z}} \sigma_{n-n'}\beta_n\beta_{n'} = \frac{1}{2}\sum_{n\in\mathbb{Z}} \|\alpha\|^2 + \langle\beta, \sigma \star \beta\rangle,$$

where $n \in \mathbb{Z}$, (α_n, β_n) is the position and momentum of the particle n, and $(\alpha, \beta) = \{(\alpha_n, \beta_n),\, n \in \mathbb{Z}\}$. The assumption on σ will be specified later. The Hamiltonian dynamics with stochastic source without control reads

$$\begin{aligned}\dot\beta_n(t) &= \alpha_n(t),\\ d\alpha_n(t) &= -(\sigma \star \beta(t))_n dt + \big(-\nu\alpha_0(t)dt + \sqrt{2\nu T}dR(t)\big)\delta_{0,n}, \quad n \in \mathbb{Z},\end{aligned}$$

where $\{R(t),\, t \geq 0\}$ is a Wiener process on a probability space with proper filtration $(\Omega, \mathfrak{F}_t, \mathbb{E})$ and an initial probability measure μ_ε on $\ell^2(\mathbb{Z})$. In this setting, we couple the particle whose label is 0 to a Langevin thermostat at temperature T and we assume that the friction is $\nu > 0$. The convolution of two functions on $\mathbb{Z}$ is

$$(f \star g)_n = \sum_{n' \in \mathbb{Z}} f_{n-n'} g_{n'}. \tag{3}$$

The wave function is given by

$$\psi_n(t) := (\tilde\omega \star \beta(t))_n + i\alpha_n(t) \tag{4}$$

in which $\{\tilde\omega_n,\, n \in \mathbb{Z}\}$ is $\omega(k) := \sqrt{\hat\sigma(k)}$. The Fourier transform of the wave function is

$$\hat\psi(t,k) := \omega(k)\hat\beta(t,k) + i\hat\alpha(t,k) \qquad \forall k \in \mathbb{T}.$$

The Fourier transform of $f_n \in \ell^2(\mathbb{Z})$ and the inverse Fourier transform of $\hat f \in L^2(\mathbb{T})$ are

$$\hat f(k) = \sum_{x \in \mathbb{Z}} f_n \exp\{-2\pi i n k\},\ f_n = \int_{\mathbb{T}} \hat f(k) \exp\{2\pi i n k\} dk, \quad n \in \mathbb{Z},\ k \in \mathbb{T}.$$

We also have

$$\hat\alpha\,(t,k) = \frac{1}{2i}[\hat\psi(t,k) - \hat\psi^*(t,-k)],\ \alpha_0(t) = \int_{\mathbb{T}} \operatorname{Im}\hat\psi(t,k) dk.$$

The initial condition can be written as $\hat\psi(k) = \hat\psi(0,k)$.

Following [9], the energy and correlation are assumed to satisfy

$$\begin{aligned}\langle\hat\psi(k), \hat\psi(h)\rangle_{\mu_\varepsilon} &= 0, \quad k, h \in \mathbb{T},\\ \sup_{\varepsilon\in(0,1]} \sum_{n\in\mathbb{Z}} \varepsilon\langle|\psi_n|^2\rangle_{\mu_\varepsilon} &= \sup_{\varepsilon\in(0,1]} \varepsilon\langle\|\hat\psi\|^2_{L^2(\mathbb{T})}\rangle_{\mu_\varepsilon} < \infty,\end{aligned} \tag{5}$$

where the expectation for μ_ε is denoted $\langle\cdot\rangle_{\mu_\varepsilon}$. The expectation for the two processes, the Wiener process and the initial measure μ_ε, is denoted by $\mathbb{E}_\varepsilon$.

For a function $O(x,k)$, we denote by $\tilde O : \mathbb{R}\times\mathbb{Z} \to \mathbb{C}$, $\hat O : \mathbb{R}\times\mathbb{T} \to \mathbb{C}$ the Fourier transforms of O in the k and x variables, respectively, and the Laplace transform for the time variable is

$$\mathcal{L}(f)(Z) = \int_0^\infty e^{-Zt} f(t) dt$$

We have some specific notations for the Laplace transform of few functions in this work and we define them when we need. Also, for now, if we don't say anything further, we assume the domain for convergence of the Laplace transform is $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$. We keep the notations $\operatorname{Re}(z), \operatorname{Im}(z), z^*$ for the real part, imaginary part and complex conjugate of z.

The Laplace transform of the Wiener process is denoted by

$$\tilde R = \mathcal{L}(R).$$

One can observe that $\tilde{R}$ is a Gaussian process, determined by its covariance which is given by:

$$\mathbb{E}(\tilde{R}(Z_1)\tilde{R}(Z_2)) = \frac{1}{Z_1+Z_2}, \qquad \mathrm{Re}Z_1, \mathrm{Re}Z_2 > 0.$$

Recall the convolution notation in (3). We also use the notation $\star$ for convolutions with respect to the time variable t, as

$$f \star g(t) = \int_0^t f(t-s)g(s)ds.$$

We use this for convenience. To avoid confusion, we have a convention that we only use this notation when dealing with functions $F, \Theta, \Psi, C_\omega$. We define those functions later.

The ε time scaling of a function f is defined by

$$f^{(\varepsilon)}(t) = f(t/\varepsilon). \tag{6}$$

For the estimates, we use the following symbols: for $f, g : D \to \mathbb{R}$, we write

$f \lesssim g$ if there exists $C > 0 : f(x) \leq Cg(x), x \in D$.
We write $f \approx g$ if $f \lesssim g$ and $g \lesssim f$.

Our estimation usually involves $\varepsilon \to 0^+$, hence when we use those symbols we mean that D is a small positive neighborhood of 0 for the variable ε and C does not depend on ε.

We now state a few basic assumptions.

2.1. **Assumptions on the initial wave.** The Wigner distribution is defined by

$$\langle O, W^{(\varepsilon)}(t)\rangle_{\mathscr{N}} = \frac{\varepsilon}{2}\sum_{n,n'\in\mathbb{Z}} \mathbb{E}_\varepsilon\left[\hat{\psi}_n^{(\varepsilon)}(t)\hat{\psi}_{n'}^{(\varepsilon)*}(t)\right]\tilde{O}^*(\varepsilon\frac{n+n'}{2}, n-n'), \tag{7}$$

for every test function $O \in \mathscr{S}(\mathbb{R}\times\mathbb{T})$ in the Schwartz space. Recalling that G has two variables, $\hat{O}$ is the Fourier transform with respect to the first variable, and $\tilde{O}$ is the Fourier transform with respect to the second variable, defined by

$$\hat{O}(\xi, k) = \int_{\mathbb{R}} e^{-2\pi i\xi x}O(x,k)dx,$$
$$\tilde{O}(x, n) = \int_{\mathbb{T}} e^{-2\pi ink}O(x,k)dk.$$

The Wigner distribution is also defined by its Fourier transform, as

$$\hat{W}_\varepsilon(t,\xi,k) = \frac{\varepsilon}{2}\mathbb{E}_\varepsilon\left[\hat{\psi}^{(\varepsilon)*}(t, k-\varepsilon\xi/2)\hat{\psi}^{(\varepsilon)}(t,k+\varepsilon\xi/2)\right], \tag{8}$$
$$\langle O, W^{(\varepsilon)}(t)\rangle_{\mathscr{N}} = \int_{\mathbb{T}\times\mathbb{R}} \hat{W}_\varepsilon(t,\xi,k)\hat{O}^*(\xi,k)d\xi dk.$$

For the Wigner distribution's Laplace transform, we use the notation

$$\mathscr{W}_\varepsilon(Z,\xi,k) = \mathcal{L}(\hat{W}_\varepsilon(\cdot,\xi,k))(Z).$$

For most parts, we will find the limit of the Laplace transform $\mathscr{W}_\varepsilon$ instead of directly find the limit of $\hat{W}_\varepsilon$. When we want to mention about the initial condition of the Wigner distribution, we omit the time variable and we write

$$\hat{W}_\varepsilon(\xi,k) = \hat{W}_\varepsilon(0,\xi,k).$$

We define the space $\mathscr{N}$ as the completion of $\mathscr{S}(\mathbb{R}\times\mathbb{T})$ for the norm

$$\|O\|_{\mathscr{N}} = \int_{\mathbb{R}} \sup_{k\in\mathbb{T}} |\hat{O}(\xi,k)| d\xi$$

and we consider it dual space $\mathscr{N}'$.

The energy of the wave grows at most linearly in time, hence

$$\sup_{t\in[0,\tau]} \|W^{(\varepsilon)}(t)\|_{\mathscr{N}'} < \infty,$$

for every $\tau > 0$. Therefore, the family of all $W^{(\varepsilon)}$ is sequentially weak-star compact in $(L^1([0,\tau];\mathscr{N}))^\star$. Considering subsequences is necessary, we can then assume that

$$\hat{W}_\varepsilon(\xi,k) \text{ converges weakly in } \mathscr{N}' \text{ to } \hat{W}_0 \in L^1(\mathbb{R}\times\mathbb{T}).$$

For the estimates later on, an additional assumption on the initial state is needed, namely, that

$$|\hat{W}_\varepsilon(\xi,k)| \leq \frac{C_1}{\langle\xi\rangle_{\mathbb{R}}^{3+\kappa}}, \tag{9}$$

for $(\xi,k)\in\mathbb{T}_{2/\varepsilon}\times\mathbb{T}, \varepsilon\in(0,1]$ ($\mathbb{T}_{2/\varepsilon}$ is the periodic torus $[-\varepsilon^{-1},\varepsilon^{-1}]$). Here, C_1,κ are positive constants and the Japanese bracket is defined by

$$\langle x\rangle_{\mathbb{R}} = \sqrt{1+x^2}.$$

Note that, for any $\kappa>0$, $\int_{\mathbb{R}}\langle x\rangle_{\mathbb{R}}^{-1-\kappa}dx$ converges.

2.2. **Assumptions on the dispersion relation.** The coupling between two points $n,n'\in\mathbb{Z}$ is denoted by

$$\sigma_{n-n'}\in\mathbb{C}.$$

We assume that there exist $C_2,C_3>0$ such that

$$|\sigma_n| \leq C_2 e^{-|n|/C_3}, \qquad \forall n\in\mathbb{Z}.$$

This assumption implies that the Fourier transform $\hat{\sigma}$ is smooth. We also assume that

$$\begin{aligned}
&\hat{\sigma} \text{ is even,}\\
&\hat{\sigma}(k) > 0, \quad k\neq 0,\\
&\hat{\sigma}''(0) > 0, \quad \text{if } \hat{\sigma}(0) = 0.
\end{aligned}$$

The dispersion relation is defined using this coupling

$$\omega(k) = \sqrt{\hat{\sigma}(k)}.$$

By definition, the function $k\mapsto\omega(k)$ is non-negative, even, and is smooth on $\mathbb{T}\setminus\{0\}$. In our work, we assume that ω is a smooth and positive function.

The torus $\mathbb{T}$ is divided into the negative branch $[-1/2,0]$ and positive branch $[0,1/2]$. Since ω is an even function, it makes sense to work on the positive branch, then pass the result to the negative branch. We add the assumption that ω is decreasing on the negative branch, or equivalently, increasing on the positive branch. The minimum and maximum values of the dispersion relation are

$$\begin{aligned}
\omega_{\min} &= \omega(0),\\
\omega_{\max} &= \omega(-1/2) = \omega(1/2).
\end{aligned}$$

We also define two inverse functions $\varphi_+ : [\omega_{\min}, \omega_{\max}] \to [0, 1/2]$ and $\varphi_- : [\omega_{\min}, \omega_{\max}] \to [-1/2, 0]$. Since ω increases on the positive branch and is continuous, given any $\overline{\omega} \in [\omega_{\min}, \omega_{\max}]$ there exists a unique $k \in [0, 1/2]$ such that $\omega(k) = \overline{\omega}$. The value of $\varphi(\overline{\omega})$ is set as k. We define φ_+ by

$$\varphi_+(\omega(k)) = |k|$$

and φ_- by

$$\varphi_-(\omega(k)) = -|k|.$$

We assume that the derivatives of the inverses have the following properties

$$\varphi'_\pm(\overline{\omega}) = \pm(\overline{\omega} - \omega_{\min})^{-1/2}\varphi_1(\overline{\omega}), \quad \overline{\omega} - \omega_{\min} \ll 1, \tag{10}$$

$$\varphi'_\pm(\overline{\omega}) = \pm(\omega_{\max} - \overline{\omega})^{-1/2}\varphi_2(\overline{\omega}), \quad \omega_{\max} - \overline{\omega} \ll 1, \tag{11}$$

where φ_1 and φ_2 are smooth positive functions. For example (see [2]):

$$\omega(k) = \sqrt{\omega_0^2 + \gamma(1 - \cos(2\pi k))},$$

where $\omega_0, \gamma > 0$. For $\overline{\omega} \in [\omega_0, \sqrt{\omega_0^2 + 2\gamma}]$, we have

$$\varphi_\pm(\overline{\omega}) = \pm\frac{1}{\pi}\arcsin\sqrt{\frac{\overline{\omega}^2 - \omega_0^2}{2\gamma}}$$

and

$$\varphi'_\pm(\overline{\omega}) = \pm\frac{\overline{\omega}}{\pi\sqrt{2\gamma + \omega_0^2 - \overline{\omega}^2}\sqrt{\overline{\omega}^2 - \omega_0^2}}.$$

When $\overline{\omega} - \omega_0 \ll 1$, we define

$$\varphi_1(\overline{\omega}) = \frac{\overline{\omega}}{\pi\sqrt{2\gamma + \omega_0^2 - \overline{\omega}^2}\sqrt{\overline{\omega} + \omega_0}},$$

and when $\sqrt{\omega_0^2 + 2\gamma} - \overline{\omega} \ll 1$, we define

$$\varphi_2(\overline{\omega}) = \frac{\overline{\omega}}{\pi\sqrt{\sqrt{\omega_0^2 + 2\gamma} + \overline{\omega}}\sqrt{\overline{\omega}^2 - \omega_0^2}}.$$

Finally, we introduce some notations related to the dispersion relation:

$$\begin{aligned} D_\varepsilon(k,\xi) &= \varepsilon^{-1}\left[\omega(k + \varepsilon\xi/2) - \omega(k - \varepsilon\xi/2)\right] \qquad \text{for } k \in \mathbb{T}, \xi \in \mathbb{T}_{2/\varepsilon}, \\ D^+_\varepsilon(k,\xi) &= \varepsilon^{-1}\left[\omega(k + \varepsilon\xi) - \omega(k)\right] \qquad \text{for } k \in \mathbb{T}, \xi \in \mathbb{T}_{1/\varepsilon}, \\ D^-_\varepsilon(k,\xi) &= \varepsilon^{-1}\left[\omega(k) - \omega(k - \varepsilon\xi)\right] \qquad \text{for } k \in \mathbb{T}, \xi \in \mathbb{T}_{1/\varepsilon}. \end{aligned} \tag{12}$$

When $\varepsilon \to 0$, $D_\varepsilon(k,\xi), D^+_\varepsilon(k,\xi), D^-_\varepsilon(k,\xi)$ all converge to $\omega'(k)\xi$.

3. Impulsive control

3.1. **Setting and Main Theorem.** We further assume that $\langle\hat{\psi}(k)\rangle_{\mu_\varepsilon} = 0$ for any k so that scattering terms containing this expectation in their product vanish. Actually, it would be enough to assume that

$$\begin{aligned} &P(\varepsilon,k) = \langle\hat{\psi}(k)\rangle_{\mu_\varepsilon} \text{ and} \\ &\varepsilon^{1/2-d}\left|P(\varepsilon, k - \varepsilon\xi/2)\right| \text{ is bounded for each fixed } k, \xi, \end{aligned} \tag{13}$$

In (13), d is such that P is small enough so that when the control acts on the wave, the new term involving P will disappear when taking the limit. To be more precise, d is such that $1/2 - d$ is small.

In this part, our control term is associated with a real-valued function $F(t)$ and we assume that

$$\varepsilon^{1/2}\mathcal{L}(F)(\varepsilon Z + i\omega(k - \varepsilon\xi/2)) \text{ converges to } \mathcal{F}(k) \text{ for fixed } \xi, \text{ for some continuous function } \mathcal{F}. \tag{14}$$

This assumption comes from a basic relation between Laplace and Fourier transforms. For $F \in L^1([0,+\infty))$ we have

$$\lim_{Z\to 0} \mathcal{L}(F)(Z + i\omega) = \lim_{Z\to 0} \int_0^{+\infty} e^{-Zt - i\omega t} F(t)dt = \int_0^{+\infty} e^{-i\omega t} F(t)dt = \hat{F}(\omega/2\pi). \tag{15}$$

We use the notation $\hat{F}$ for the Fourier transform in the time variable.

To ensure that the energy will grow at most linearly in time, we make another assumption on F, namely, that there exists $C_4 > 0$ such that

$$\sup_{\varepsilon\in(0,1]} \int_0^{t/\varepsilon} \varepsilon F^2(s)ds \leq C_4 t, \forall t > 0.$$

We now state our main theorem for inpulsive control. We consider the system of equations

$$\dot{\beta}_n(t) = \alpha_n(t), \tag{16}$$

$$d\alpha_n(t) = -(\sigma \star \beta(t))_n dt + (-\nu\alpha_0 dt + F(t)dt + \sqrt{2\nu T}dR(t))\delta_{0,n}. \tag{17}$$

Theorem 1. *Consider the wave system governed by* (16) *and* (17) *along with all assumptions stated in sections 2 and 3. For any $\tau > 0$ and $O \in L^1([0,\tau];\mathscr{N})$ we have*

$$\lim_{\varepsilon\to 0} \int_0^{\tau} \langle O(t), W_\varepsilon(t)\rangle_{\mathscr{N}} dt = \int_0^{\tau} \int_{\mathbb{R}\times\mathbb{T}} O^*(t,x,k)W(t,x,k)dxdkdt, \tag{18}$$

where

$$\begin{aligned} W(t,x,k) &= W_0(x - v_g(k)t, k)1_{[0,v_g(k)t]^c}(x) + r_a(k)T1_{[0,v_g(k)t]}(x) \\ &+ r_t(k)W_0(x - v_g(k)t, k)1_{[0,v_g(k)t]}(x) + r_r(k)W_0(-x + v_g(k)t, -k)1_{[0,v_g(k)t]}(x) \\ &+ \frac{|v_g(k)|r_a(k)|\mathcal{F}(k)|^2}{\nu}\delta(x - v_g(k)t). \end{aligned} \tag{19}$$

Remark 2. In view of Question A, posed in our introduction, we observe that the three rates r_a, r_t, r_r are not modified under the influence of the impulsive control $F(t)\delta_{0,n}$. The control appears in the term $\frac{|v_g(k)|r_a(k)|\mathcal{F}(k)|^2}{\nu}\delta(x - \bar{\omega}'(k)t)$ in (19)(recall that $\mathcal{F}$ is given by (14): this is where the control $F(t)\delta_{0,n}$ appears in (19)). With respect to the main result of [9], the latter term is the new additional one.

Therefore, despite its physical meaning and relevance, such an impulsive control is not appropriate to act on the thermostat system. In the next section, we will propose another type of control called memory-feedback because it uses the information provided by the past of the state solution. This alternative kind of control enables us to act on the triple rates.

We recall some definitions. The group velocity is denoted

$$v_g(k) = \frac{\omega'(k)}{2\pi}.$$

We then define

$$r_a(k) = \frac{\nu|\theta(k)|^2}{|v_g(k)|},$$

$$r_t(k) = 1 - \mathrm{Re}(\theta(k))\frac{\nu}{|v_g(k)|} + \frac{\nu r_a(k)}{4|v_g(k)|},$$
$$r_r(k) = \frac{\nu r_a(k)}{4|v_g(k)|},$$

where θ is defined further in (31). Recall that the terms r_a, r_t and r_r represent the absorption rate, transmission rate and reflection rate. Section [9] shows the properties of θ yielding some other properties of r_a, r_t, r_r.

In the proof of Theorem 1, to analyze the kinetic limit, the Wigner distribution is splitted into 13 terms that we categorize into 4 types.

(1) Terms do not involving the control: one of the thermal term (38), the ballistic term (52), the first and the second scattering terms (58), (59). These terms are estimated using the results of [9].
(2) Terms with a single Wiener process: (36) and (37). Since the expectation of the Wiener process is zero, thus these terms are zero.
(3) Terms involving one occurence of F in the product: those terms are (54), the third, the fourth, the fifth and the sixth scattering terms, (60), (61), (62), (63).
(4) Terms involving two occurences of F in the product: (35) and (64).

We will see that only the first and the fourth types have non-zero kinetic limits. All others vanish at the limit.

3.2. **Preliminary computations.** We write $d\psi$ using the system of equations (see [5, 14] for a related nonlinear problem)

$$\begin{aligned} d\hat{\psi}(t,k) &= d(\omega(k)\hat{\beta}(t,k)) + id\hat{\alpha}(t,k) \\ &= \omega(k)\hat{\alpha}(t,k)dt + i(-\omega^2(k)\hat{\beta}dt - \nu\alpha_0(t)dt + F(t)dt + \sqrt{2\nu T}dR(t)) \\ &= -i\omega(k)\hat{\psi}(t,k)dt - i\nu\alpha_0(t)dt + iF(t)dt + \sqrt{2\nu T}idR(t). \end{aligned} \tag{20}$$

Solving the linear ODE (20) gives

$$\begin{aligned} \hat{\psi}(t,k) = e^{-i\omega(k)t}\hat{\psi}(k) - i\nu\int_0^t e^{-i\omega(k)(t-s)}\alpha_0(s)ds + i\int_0^t e^{-i\omega(k)(t-s)}F(s)ds \\ + i\sqrt{2\nu T}\int_0^t e^{-i\omega(k)(t-s)}dR(s). \end{aligned} \tag{21}$$

Noting that by definition of inverse Fourier transform,

$$\alpha_0(t) = \int_{\mathbb{T}} \mathrm{Im}(\hat{\psi}(t,k))dk, \tag{22}$$

and that the real part of $e^{-i\omega(k)(t-s)}$ is $\cos(\omega(t-s))$, we infer from (21) and (22) that

$$\begin{aligned} \alpha_0(t) &= \Psi(t) - \nu\int_{\mathbb{T}}\int_0^t \cos\left(\omega(k)(t-s)\right)\alpha_0(s)dsdk + \int_{\mathbb{T}}\int_0^t \cos\left(\omega(k)(t-s)\right)F(s)dsdk \\ &\quad + \sqrt{2\nu T}\int_{\mathbb{T}}\int_0^t \cos\left(\omega(k)(t-s)\right)dR(s)dk, \\ &= \Psi(t) - \nu C_\omega \star \alpha_0(t) + C_\omega \star F(t) + \sqrt{2\nu T}C_\omega \star dR(t), \end{aligned} \tag{23}$$

where

$$\Psi(t) = \int_{\mathbb{T}} \mathrm{Im}\left(\hat{\psi}(k)e^{-i\omega(k)t}\right)dk, \tag{24}$$

and

$$C_\omega(t) = \int_{\mathbb{T}} \cos(\omega(k)t)dk. \tag{25}$$

We can compute the Laplace transform

$$\mathcal{L}(C_\omega)(Z) = \int_{\mathbb{T}} \frac{Z}{Z^2 + \omega^2(k)}dk. \tag{26}$$

Recall that $\mathrm{Re}(Z) > 0$ so $\mathrm{Re}(\mathcal{L}(C_\omega)(Z)) > 0$.

Using the Laplace transform on (23) and the fact that

$$\tilde{\Theta}(Z) = \frac{1}{1 + \nu\mathcal{L}(C_\omega)(Z)} = \sum_{j=0}^{\infty}(-\nu\mathcal{L}(C_\omega)(Z))^j, \tag{27}$$

we get

$$\begin{aligned}\tilde{\alpha}_0(Z) = \mathcal{L}(\alpha_0)(Z) &= \tilde{\Theta}\mathcal{L}(\Psi)(Z) + \tilde{\Theta}\mathcal{L}(C_\omega)(Z)\mathcal{L}(F)(Z) + \sqrt{2\nu T}\tilde{\Theta}(Z)\mathcal{L}(C_\omega)(Z)\tilde{R}(Z)\\ &= \tilde{\Theta}\mathcal{L}(\Psi) + \frac{1}{\nu}(1 - \tilde{\Theta})\mathcal{L}(F) + \sqrt{2\nu T}\tilde{\Theta}\mathcal{L}(C_\omega)\tilde{R}(Z).\end{aligned} \tag{28}$$

We will write Θ as the inverse Laplace transform of $\tilde{\Theta}$ and it can be understood using the infinite sum of convolution terms

$$\Theta(dt) = \delta_{0,t}(dt) + \mathcal{L}^{-1}(\tilde{\Theta} - 1) = \delta_{0,t}(dt) + \sum_{n=1}^{\infty}(-\nu)^n(\star^n C_\omega)dt,$$

where $(\star^n C_\omega)$ is the n-time convolution of C_ω with itself.

Applying the Laplace transform to (21), and using (28) we obtain

$$\begin{aligned}\mathcal{L}(\hat{\psi}(\cdot,k))(Z) &= \frac{\hat{\psi}(k) - i\nu\tilde{\alpha}_0(Z) + i\mathcal{L}(F)(Z) + i\sqrt{2\nu T}\tilde{R}(Z)}{Z + i\omega(k)}\\ &= \frac{\hat{\psi}(k) - i\nu\mathcal{L}(\Psi)(Z) + i\tilde{\Theta}(Z)\mathcal{L}(F)(Z) + i\sqrt{2\nu T}\tilde{\Theta}(Z)\tilde{R}(Z)}{Z + i\omega(k)}.\end{aligned}$$

Therefore, by inverse Laplace transform,

$$\begin{aligned}\hat{\psi}(t,k) = e^{-i\omega(k)t}\hat{\psi}(k) - i\nu\int_0^t \Theta_\omega(t-s,k)\Psi(s)ds + i\int_0^t \Theta_\omega(t-s,k)F(s)ds\\ + i\sqrt{2\nu T}\int_0^t \Theta_\omega(t-s,k)dR(s),\end{aligned} \tag{29}$$

where

$$\Theta_\omega(t,k) = \int_0^t e^{-i\omega(k)(t-s)}\Theta(ds).$$

Denoting by

$$\Theta^\omega(t,k) = \int_0^t e^{i\omega(k)s}g(ds) = e^{i\omega(k)t}\Theta_\omega(t,k), \tag{30}$$

we have the important identity

$$\mathcal{L}((\Theta^\omega)^{(\varepsilon)}(\cdot,k))(Z) = \frac{\tilde{\Theta}(\varepsilon Z - i\omega(k))}{Z}.$$

It can be proved that $\tilde{\Theta}(\varepsilon Z - i\omega(k))$ is bounded by 1. Hence, using (25), (26), (27), and Fatou's Lemma, $\tilde{\Theta}(\varepsilon\lambda - i\omega(k))$ converges (a.e. and in any L^p) when $Z \to 0$. We also write

$$\lim_{\varepsilon\to 0^+} \tilde{\Theta}(\varepsilon Z - i\omega(k)) = \theta(k). \tag{31}$$

Note that we can truncate the wave so that it vanishes around k where $\omega'(k) = 0$. We have

$$\hat{\psi}^1(0,k) = \hat{\psi}(k)\chi_\varrho(k), \tag{32}$$

$$d\hat{\psi}^1(t,k) = \left\{-i\omega(k)\hat{\psi}^1(t,k) - \frac{\nu}{2i}\int_{\mathbb{T}}[\hat{\psi}^1(t,k') - \hat{\psi}^{1*}(t,k')]dk' + F(t)\right\}dt + i\sqrt{2\nu T}dR(t). \tag{33}$$

$$\hat{\psi}^2(0,k) = \hat{\psi}(k)(1-\chi_\varrho(k)),$$

$$d\hat{\psi}^2(t,k) = \left\{-i\omega(k)\hat{\psi}^1(t,k) - \frac{\nu}{2i}\int_{\mathbb{T}}[\hat{\psi}^2(t,k') - \hat{\psi}^{2*}(t,k')]dk'\right\}dt.$$

In here, $\chi_\varrho(k)$ is a smooth function with values in $[0,1]$ such that it is equal to 0 on $[\mathbb{T}](\varrho)$ and to 1 on $\mathbb{T}\setminus[\mathbb{T}](\varrho)$ with

$$\begin{aligned}[\mathbb{T}](\varrho) &= \{k : \operatorname{dist}(k, [\mathbb{T}](0)) < \varrho\}, \\ [\mathbb{T}](0) &= \{k : \omega'(k) = 0\}.\end{aligned} \tag{34}$$

We will choose ϱ sufficiently small so that the part near the place where ω' vanishes, will also be sufficiently small. Note that, since ω is smooth, we have $\omega'(k) = 0$ at $-\frac{1}{2}, 0$ and $\frac{1}{2}$.

We now split the wave into several terms to determine the kinetic limit.

3.3. **Thermal terms.** The thermal part of the wave is considered independently from the initial state of the wave system. Therefore, for the thermal part, we put $\hat{\psi}(k) \equiv 0$. With that assumption, (29) becomes

$$\hat{\psi}(t,k) = i\int_0^t \Theta_\omega(t-s,k)F(s)ds + i\sqrt{2\nu T}\int_0^t \Theta_\omega(t-s,k)dR(s).$$

The Wigner distribution's definition (8) gives

$$\hat{W}_\varepsilon(t,\xi,k) = \frac{\varepsilon}{2}\mathbb{E}_\varepsilon\left[\int_0^{t/\varepsilon}\Theta_\omega(t/\varepsilon - s, k+\varepsilon\xi/2)F(s)ds\int_0^{t/\varepsilon}\Theta^*_\omega(t/\varepsilon - s, k-\varepsilon\xi/2)F(s)ds\right. \tag{35}$$

$$+\sqrt{2\nu T}\int_0^{t/\varepsilon}\Theta_\omega(t/\varepsilon - s, k+\varepsilon\xi/2)F(s)ds\int_0^{t/\varepsilon}\Theta^*_\omega(t/\varepsilon - s, k-\varepsilon\xi/2)dR(s) \tag{36}$$

$$+\sqrt{2\nu T}\int_0^{t/\varepsilon}\Theta^*_\omega(t/\varepsilon - s, k-\varepsilon\xi/2)F(s)ds\int_0^{t/\varepsilon}\Theta_\omega(t/\varepsilon - s, k+\varepsilon\xi/2)dR(s) \tag{37}$$

$$\left.+\,2\nu T\int_0^{t/\varepsilon}\Theta_\omega(t/\varepsilon - s, k+\varepsilon\xi/2)dR(s)\int_0^{t/\varepsilon}\Theta^*_\omega(t/\varepsilon - s, k-\varepsilon\xi/2)dR(s)\right]. \tag{38}$$

3.3.1. *First thermal term.* Since the term (35) is independent from any random process, its Laplace transform is

$$\frac{\varepsilon}{2}\mathcal{L}\left((\Theta_\omega \star F)^{(\varepsilon)}(\cdot, k+\varepsilon\xi/2)(\Theta^*_\omega \star F)^{(\varepsilon)}(\cdot, k-\varepsilon\xi/2)\right). \tag{39}$$

We rewrite the convolution $\Theta_\omega \star F$ using (30) as

$$\Theta_\omega \star F(t,k) = \int_0^t \Theta_\omega(t-s,k)F(s)ds = \int_0^t \Theta^\omega(t-s,k)e^{i\omega(k)(s-t)}F(s)ds$$

$$= \int_0^t \Theta^\omega(t-s,k)e^{-i\omega(k)t}F_k^*(s)ds$$
$$= e^{-i\omega(k)t}\Theta^\omega \star F_k^*(t), \tag{40}$$

where

$$F_k(t) = F(t)e^{-i\omega(k)t}. \tag{41}$$

We also rewrite the conjugation,

$$\Theta_\omega^* \star F(t,k) = e^{i\omega(k)t}(\Theta^\omega)^* \star F_k(t). \tag{42}$$

Using (12), (40) and (42), the product of the two convolutions is

$$\begin{aligned}&(\Theta_\omega \star F(t,k+\varepsilon\xi/2))(\Theta_\omega^* \star F(t,k-\varepsilon\xi/2))\\ &\quad = e^{-i\varepsilon D_\varepsilon(k,\xi)t}(\Theta^\omega \star F^*_{k+\varepsilon\xi/2})(t,k+\varepsilon\xi/2)((\Theta^\omega)^* \star F_{k-\varepsilon\xi/2})(t,k-\varepsilon\xi/2).\end{aligned} \tag{43}$$

Now, we use (43), (39), and the formula of the Laplace transform of a product to get

$$\begin{aligned}&\frac{\varepsilon}{4i\pi}\lim_{L\to\infty}\int_{c-iL}^{c+iL}\mathcal{L}((\Theta^\omega \star F^*_{k+\varepsilon\xi/2})^{(\varepsilon)})(\zeta)\mathcal{L}(((\Theta^\omega)^* \star F_{k-\varepsilon\xi/2})^{(\varepsilon)})(Z+iD_\varepsilon(k,\xi)-\zeta)d\zeta\\ &\quad = \frac{\varepsilon}{4i\pi}\lim_{L\to\infty}\int_{c-iL}^{c+iL}\mathcal{L}((\Theta^\omega)^{(\varepsilon)})(\zeta,k+\varepsilon\xi/2)\mathcal{L}(F^*_{k+\varepsilon\xi/2})(\varepsilon\zeta)\\ &\qquad \mathcal{L}((\Theta^\omega)^{(\varepsilon)*})(Z+iD_\varepsilon(k,\xi)-\zeta,k-\varepsilon\xi/2)\mathcal{L}(F_{k-\varepsilon\xi/2})(\varepsilon(Z+iD_\varepsilon(k,\xi)-\zeta))d\zeta\\ &\quad = \frac{\varepsilon}{4i\pi}\lim_{L\to\infty}\int_{c-iL}^{c+iL}\frac{\tilde\Theta(\varepsilon\zeta-i\omega(k+\varepsilon\xi/2))}{\zeta}\mathcal{L}(F)(\varepsilon\zeta-i\omega(k+\varepsilon\xi/2))\\ &\qquad \frac{\tilde\Theta(\varepsilon(Z+iD_\varepsilon(k,\xi)-\zeta)+i\omega(k-\varepsilon\xi/2))}{Z+iD_\varepsilon(k,\xi)-\zeta}\mathcal{L}(F)(\varepsilon(Z+iD_\varepsilon(k,\xi)-\zeta)+i\omega(k-\varepsilon\xi/2))d\zeta,\end{aligned}$$

where we follow the notation of (6). From (31) and (14), taking the limit $\varepsilon \to 0$ we obtain

$$\frac{1}{4i\pi}\lim_{L\to\infty}\int_{c-iL}^{c+iL}\frac{|\theta(k)|^2|\mathcal{F}(k)|^2}{\zeta(Z+i\omega'(k)\xi-\zeta)}d\zeta. \tag{44}$$

The next few steps are just basic calculus. We have

$$\frac{1}{\zeta(Z+i\omega'(k)\xi-\zeta)} = \frac{1}{Z+i\omega'(k)\xi}\left(\frac{1}{\zeta}+\frac{1}{Z+i\omega'(k)\xi-\zeta}\right).$$

We also have the identity

$$\begin{aligned}\lim_{L\to\infty}\int_{c-iL}^{c+iL}\frac{1}{\zeta}d\zeta &= i\lim_{L\to\infty}\left(\int_{-L}^{L}\frac{c}{c^2+x^2}dx - i\int_{-L}^{L}\frac{x}{c^2+x^2}dx\right)\\ &= i\pi.\end{aligned} \tag{45}$$

Hence, (44) equals

$$\frac{|\theta(k)|^2|\mathcal{F}(k)|^2}{2(Z+i\omega'(k)\xi)} = \frac{|v_g(k)|r_a(k)|\mathcal{F}(k)|^2}{2\nu(Z+i\omega'(k)\xi)}.$$

3.3.2. *Second and third thermal terms.* Consider the terms (36),(37). They have one $dR(s)$ each, and because they follow a Gaussian distribution, those terms vanish.

3.3.3. *Last thermal term.* Consider the term (38).

Because of the correlation of Wiener process, the term (38) is the same as the thermal term in [9, chapter 4]. Therefore, it contributes to the limit of $\mathscr{W}_\varepsilon$ as

$$\frac{\nu T|\theta(k)|^2}{Z(Z+i\omega'(k)\xi)}.$$

The limit can be rewritten as

$$\frac{T|v_g(k)|r_a(k)}{Z(Z+i\omega'(k)\xi)}.$$

3.4. **Scattering terms.** From (8), we have

$$\partial_t\hat{W}_\varepsilon(t,\xi,k)=\frac{\varepsilon}{2}\mathbb{E}_\varepsilon\left[\left(\partial_t\hat{\psi}^{(\varepsilon)}(t,k+\varepsilon\xi/2)\right)\hat{\psi}^{(\varepsilon)*}(t,k-\varepsilon\xi/2)+\left(\partial_t\hat{\psi}^{(\varepsilon)*}(t,k-\varepsilon\xi/2)\right)\hat{\psi}^{(\varepsilon)}(t,k+\varepsilon\xi/2)\right]. \tag{46}$$

Then, we compute each term of (46) by replacing $T=0$ in (20):

$$\begin{aligned}&\left(\partial_t\hat{\psi}^{(\varepsilon)}(t,k+\varepsilon\xi/2)\right)\hat{\psi}^{(\varepsilon)*}(t,k-\varepsilon\xi/2) &(47)\\ &\qquad=\frac{1}{\varepsilon}\left[-i\omega(k+\varepsilon\xi/2)\hat{\psi}^{(\varepsilon)}(t,k+\varepsilon\xi/2)-i\nu\alpha_0^{(\varepsilon)}(t)+iF^{(\varepsilon)}(t)\right]\hat{\psi}^{(\varepsilon)*}(t,k-\varepsilon\xi/2),\\ &\left(\partial_t\hat{\psi}^{(\varepsilon)*}(t,k-\varepsilon\xi/2)\right)\hat{\psi}^{(\varepsilon)}(t,k+\varepsilon\xi/2) &(48)\\ &\qquad=\frac{1}{\varepsilon}\left[i\omega(k-\varepsilon\xi/2)\hat{\psi}^{(\varepsilon)*}(t,k-\varepsilon\xi/2)+i\nu\alpha_0^{(\varepsilon)}(t)-iF^{(\varepsilon)}(t)\right]\hat{\psi}^{(\varepsilon)}(t,k+\varepsilon\xi/2).\end{aligned}$$

We infer from (47) and (48) that

$$\begin{aligned}&\partial_t\hat{W}_\varepsilon(t,\xi,k)=-iD_\varepsilon(k,\xi)\hat{W}_\varepsilon(t,\xi,k) &(49)\\ &-\frac{\nu}{2}\mathbb{E}_\varepsilon\left[i\alpha_0^{(\varepsilon)}(t)\hat{\psi}^{(\varepsilon)*}(t,k-\varepsilon\xi/2)-i\alpha_0^{(\varepsilon)}(t)\hat{\psi}^{(\varepsilon)}(t,k+\varepsilon\xi/2)\right] &(50)\\ &\frac{i}{2}\mathbb{E}_\varepsilon\left[F^{(\varepsilon)}(t)\hat{\psi}^{(\varepsilon)*}(t,k-\varepsilon\xi/2)-F^{(\varepsilon)}(t)\hat{\psi}^{(\varepsilon)}(t,k+\varepsilon\xi/2)\right]. &(51)\end{aligned}$$

Using the Laplace transform on (49), (50) and (51), we obtain

$$\begin{aligned}&(Z+iD_\varepsilon(k,\xi))\mathscr{W}_\varepsilon(Z,\xi,k)=\hat{W}_\varepsilon(\xi,k) &(52)\\ &-\frac{\nu}{2}\left[S_\varepsilon(Z,k-\varepsilon\xi/2)+S_\varepsilon^*(Z,k+\varepsilon\xi/2)\right] &(53)\\ &-\mathcal{L}\left(\frac{iF^{(\varepsilon)}(t)}{2}(\mathbb{E}_\varepsilon[\hat{\psi}^{(\varepsilon)}(t,k+\varepsilon\xi/2)]-\mathbb{E}_\varepsilon[\hat{\psi}^{(\varepsilon)*}(t,k-\varepsilon\xi/2)])\right)(Z), &(54)\end{aligned}$$

where

$$\begin{aligned}S_\varepsilon(Z,k)&=i\mathcal{L}\left(\mathbb{E}_\varepsilon\alpha_0^{(\varepsilon)}(t)\hat{\psi}^{(\varepsilon)*}(t,k)\right), &(55)\\ S_\varepsilon^*(Z,k)&=-i\mathcal{L}\left(\mathbb{E}_\varepsilon\alpha_0^{(\varepsilon)*}(t)\hat{\psi}^{(\varepsilon)}(t,k)\right).\end{aligned}$$

To handle the term (53) we split (55) into seven terms. First, rewrite (29) and (28) in convolution form with $T=0$:

$$\begin{aligned}\hat{\psi}(t,k)&=e^{-i\omega(k)t}\hat{\psi}(k)-i\nu\Theta_\omega\star\Psi(t)+i\Theta_\omega\star F(t), &(56)\\ \alpha_0(t)&=\Theta\star\Psi(t)+\frac{1}{\nu}F(t)-\frac{1}{\nu}\Theta\star F(t). &(57)\end{aligned}$$

Then S_ε is the sum of the Laplace transforms of $I_\epsilon^{(\varepsilon)}, II_\epsilon^{(\varepsilon)}, III_\epsilon^{(\varepsilon)}, IV_\epsilon^{(\varepsilon)}, V_\epsilon^{(\varepsilon)}, VI_\epsilon^{(\varepsilon)}, VII_\epsilon^{(\varepsilon)}$:

(1) I_ε is obtained as the product of the first term of (57) and the first term of (56):

$$I_\varepsilon(t,k) = i\mathbb{E}_\varepsilon\left[g \star \Psi(t) e^{i\omega(k)t}\hat{\psi}^*(k)\right] = ie^{i\omega(k)t}\int_0^t \langle \Psi(t-s)\hat{\psi}^*(k)\rangle_{\mu_\varepsilon}\Theta(ds). \tag{58}$$

(2) II_ε is obtained as the product of the first term of (57) and the second term of (56):

$$II_\varepsilon(t,k) = i\mathbb{E}_\varepsilon\left[\Theta \star \Psi(t)\left(i\nu\Theta_\omega \star \Psi(t)\right)\right] = -\nu\int_0^t\int_0^t \Theta_\omega^*(t-s,k)\langle\Psi(s)\Psi(t-s')\rangle_{\mu_\varepsilon} ds\Theta(ds'). \tag{59}$$

(3) III_ε is obtained as the product of the first term of (57) and the third term of (56):

$$III_\varepsilon(t,k) = i\mathbb{E}_\varepsilon\left[\left(\Theta \star \Psi(t)\right)\left(-i\Theta_\omega^* \star F(t)\right)\right] = \int_0^t \Theta_\omega^*(s,k)F(t-s)ds\langle\int_0^t \Psi(t-s')\Theta(ds')\rangle_{\mu_\varepsilon}. \tag{60}$$

(4) IV_ε is obtained as the product of the second term of (57) and $\hat{\psi}^*$:

$$IV_\varepsilon(t,k) = i\mathbb{E}_\varepsilon\left[\left(\frac{1}{\nu}F(t)\right)\hat{\psi}^*(t,k)\right] = \frac{i}{\nu}F(t)\langle\hat{\psi}^*(t,k)\rangle_{\mu_\varepsilon}. \tag{61}$$

(5) V_ε is obtained as the product of the third term of (57) and the first term of (56):

$$V_\varepsilon(t,k) = i\mathbb{E}_\varepsilon\left[\left(-\frac{1}{\nu}\Theta \star F(t)\right)\left(e^{i\omega(k)t}\hat{\psi}^*(k)\right)\right] = \frac{-i}{\nu}\int_0^t F(t-s)\Theta(ds)e^{i\omega(k)t}\langle\hat{\psi}^*(k)\rangle_{\mu_\varepsilon}. \tag{62}$$

(6) VI_ε is obtained as the product of the third term of (57) and the second term of (56):

$$VI_\varepsilon(t,k) = i\mathbb{E}_\varepsilon\left[\left(-\frac{1}{\nu}\Theta \star F(t)\right)\left(i\nu\Theta_\omega^* \star \Psi(t)\right)\right] = \int_0^t F(t-s)\Theta(ds)\langle\int_0^t \Theta_\omega^*(s',k)\Psi(t-s')ds'\rangle_{\mu_\varepsilon}. \tag{63}$$

(7) VII_ε is obtained as the product of the third term of (57) and the third term of (56):

$$VII_\varepsilon(t,k) = i\mathbb{E}_\varepsilon\left[\left(-\frac{1}{\nu}\Theta \star F(t)\right)\left(-i\Theta_\omega^* \star F(t)\right)\right] = -\frac{1}{\nu}\int_0^t F(t-s)\Theta(ds)\langle\int_0^t \Theta_\omega^*(s',k)F(t-s')ds'\rangle_{\mu_\varepsilon}. \tag{64}$$

We next show how to treat the scattering terms.

The ballistic term is the term generated from $\hat{W}_\varepsilon(\eta,k)$ in (52). The limit is

$$\int_{\mathbb{R}\times\mathbb{T}} \hat{O}(\xi,k)\frac{\hat{W}_\varepsilon(\xi,k)}{Z+iD_\varepsilon(k,\xi)}d\xi dk \to \int_{\mathbb{R}\times\mathbb{T}} \hat{O}(\xi,k)\frac{\hat{W}_0(\xi,k)}{Z+i\omega'(k)\xi}d\xi dk, \text{ as } \varepsilon\to 0, \tag{65}$$

where O is the test function used in (7).

3.4.1. *The first scattering term.* The first scattering term is obtained in (53) using I_ε, defined by (58). The result is straightforward from [9]. A more general result is obtained in Section 4.4.2. In this case, it is

$$-\nu\int_{\mathbb{R}\times\mathbb{T}} \mathrm{Re}(\theta(k))\frac{\hat{W}_0(\xi',k)}{Z+i\omega'(k)\xi'}\int_{\mathbb{R}}\frac{\hat{O}^*(\xi,k)}{Z+i\omega'(k)\xi}d\xi d\xi' dk,$$

under the same limit with (65).

3.4.2. *The second scattering term.* The second scattering term is obtained in (53) using II_ε, defined by (59). Like the first scattering term, we use the computations in Section 4.4.3

$$\frac{\nu}{4}\int_{\mathbb{R}\times\mathbb{T}}\frac{r_a(k)\hat{W}_0(\xi',k)}{Z+i\omega'(k)\xi'}\int_{\mathbb{R}}\frac{\hat{O}^*(\xi,k)}{Z+i\omega'(k)\xi}d\xi d\xi' dk$$
$$+\frac{\nu}{4}\int_{\mathbb{R}\times\mathbb{T}}\frac{r_a(k)\hat{W}_0(\xi',-k)}{Z-i\omega'(k)\xi'}\int_{\mathbb{R}}\frac{\hat{O}^*(\xi,k)}{Z+i\omega'(k)\xi}d\xi d\xi' dk,$$

under the same limit with (65).

The sum of the first and second scattering terms is

$$\int_{\mathbb{R}\times\mathbb{T}}(r_t(k)-1)\int_{\mathbb{R}}\frac{\hat{W}_0(\xi',k)}{Z+i\omega'(k)\xi'}\frac{\hat{O}^*(\xi,k)|v_g(k)|}{Z+i\omega'(k)\xi}d\xi' d\xi dk$$
$$+\int_{\mathbb{R}\times\mathbb{T}}(r_r(k))\int_{\mathbb{R}}\frac{\hat{W}_0(\xi',-k)}{Z-i\omega'(k)\xi'}\frac{\hat{O}^*(\xi,k)|v_g(k)|}{Z+i\omega'(k)\xi}d\xi' d\xi dk,$$

under the same limit with (65).

3.4.3. *The fourth scattering terms.* The fourth scattering terms are obtained in (53) using IV_ε, defined by (61). We see that $-\frac{\nu}{2}(\mathcal{L}(IV_\varepsilon^{(\varepsilon)}(\cdot,k-\varepsilon\xi/2))+\mathcal{L}(IV_\varepsilon^{(\varepsilon)*}(\cdot,k+\varepsilon\xi/2))$ is the same as the term (54) but with a different sign. Therefore, the fourth scattering term and term (54) cancel each other. The limit is 0.

3.4.4. *The fifth scattering terms.* The fifth scattering terms are obtained in (53) using V_ε, defined by (62). We have

$$\begin{aligned}\mathcal{L}(V_\varepsilon^{(\varepsilon)})(Z,k-\varepsilon\xi/2)&=\varepsilon\mathcal{L}(V_\varepsilon)(\varepsilon Z,k-\varepsilon\xi/2)\\&=\frac{-i\varepsilon}{\nu}\mathcal{L}(\Theta\star F)(\varepsilon Z-i\omega(k-\varepsilon\xi/2))P(\varepsilon,k-\varepsilon\xi/2)\\&=\frac{-i\varepsilon}{\nu}\tilde{\Theta}(\varepsilon Z-i\omega(k-\varepsilon\xi/2))\mathcal{L}(F)(\varepsilon Z-i\omega(k-\varepsilon\xi/2))P(\varepsilon,k-\varepsilon\xi/2).\end{aligned}$$

By (13),(14) and (31), the limit is 0.

3.4.5. *The third and the sixth scattering terms.* From (53), (60), (63), we compute

$$\begin{aligned}&-\frac{\nu}{2}\int_{\mathbb{R}\times\mathbb{T}}\mathcal{L}\left(III_\varepsilon^{(\varepsilon)}(\cdot,k-\varepsilon\xi/2)+III_\varepsilon^{(\varepsilon)*}(\cdot,k+\varepsilon\xi/2)\right)(Z)\frac{\hat{O}^*(\xi,k)}{Z+iD_\varepsilon(k,\xi)}d\xi dk\\&-\frac{\nu}{2}\int_{\mathbb{R}\times\mathbb{T}}\mathcal{L}\left(VI_\varepsilon^{(\varepsilon)}(\cdot,k-\varepsilon\xi/2)+VI_\varepsilon^{(\varepsilon)*}(\cdot,k+\varepsilon\xi/2)\right)(Z)\frac{\hat{O}^*(\xi,k)}{Z+iD_\varepsilon(k,\xi)}d\xi dk\\&=-\frac{\nu}{2}\left[\int_{\mathbb{R}\times\mathbb{T}}\mathcal{L}\left(III_\varepsilon^{(\varepsilon)}(\cdot,k)\right)(Z)\frac{\hat{O}^*(\xi,k+\varepsilon\xi/2)}{Z+iD_\varepsilon^+(k,\xi)}d\xi dk+\int_{\mathbb{R}\times\mathbb{T}}\mathcal{L}\left(III_\varepsilon^{(\varepsilon)*}(\cdot,k)\right)(Z)\frac{\hat{O}^*(\xi,k-\varepsilon\xi/2)}{Z+iD_\varepsilon^-(k,\xi)}d\xi dk\right]\\&\quad-\frac{\nu}{2}\left[\int_{\mathbb{R}\times\mathbb{T}}\mathcal{L}\left(VI_\varepsilon^{(\varepsilon)}(\cdot,k)\right)(Z)\frac{\hat{O}^*(\xi,k+\varepsilon\xi/2)}{Z+iD_\varepsilon^+(k,\xi)}d\xi dk+\int_{\mathbb{R}\times\mathbb{T}}\mathcal{L}\left(VI_\varepsilon^{(\varepsilon)*}(\cdot,k)\right)(Z)\frac{\hat{O}^*(\xi,k-\varepsilon\xi/2)}{Z+iD_\varepsilon^-(k,\xi)}d\xi dk\right]\\&=-\frac{\nu}{2}\left[\int_{\mathbb{R}\times\mathbb{T}}S_\varepsilon^{III}(Z,k)\frac{\hat{O}^*(\xi,k+\varepsilon\xi/2)}{Z+iD_\varepsilon^+(k,\xi)}d\xi dk+\int_{\mathbb{R}\times\mathbb{T}}S_\varepsilon^{III*}(Z,k)\frac{\hat{O}^*(\xi,k-\varepsilon\xi/2)}{Z+iD_\varepsilon^-(k,\xi)}d\xi dk\right]\\&\quad-\frac{\nu}{2}\left[\int_{\mathbb{R}\times\mathbb{T}}S_\varepsilon^{VI}(Z,k)\frac{\hat{O}^*(\xi,k+\varepsilon\xi/2)}{Z+iD_\varepsilon^+(k,\xi)}d\xi dk+\int_{\mathbb{R}\times\mathbb{T}}S_\varepsilon^{VI*}(Z,k)\frac{\hat{O}^*(\xi,k-\varepsilon\xi/2)}{Z+iD_\varepsilon^-S(k,\xi)}d\xi dk\right].\end{aligned}$$

Here, we use the notations

$$S_\varepsilon^{III}(Z,k) = \int_0^\infty \varepsilon e^{-Z\varepsilon t} III_\varepsilon(t,k)dt, \tag{66}$$

$$S_\varepsilon^{III*}(Z,k) = \int_0^\infty \varepsilon e^{-Z\varepsilon t} III_\varepsilon^*(t,k)dt, \tag{67}$$

$$S_\varepsilon^{VI}(Z,k) = \int_0^\infty \varepsilon e^{-Z\varepsilon t} VI_\varepsilon(t,k)dt, \tag{68}$$

$$S_\varepsilon^{VI*}(Z,k) = \int_0^\infty \varepsilon e^{-Z\varepsilon t} VI_\varepsilon^*(t,k)dt. \tag{69}$$

Let us compute (66) + (68). The sum (67) + (69) is similar.

To this end, we have

$$\begin{aligned} S_\varepsilon^{III}(Z,k) &= \varepsilon \int_0^\infty e^{-Z\varepsilon t} III_\varepsilon(t,k)dt \\ &= \varepsilon \int_0^\infty e^{-Z\varepsilon t} \int_0^t e^{i\omega(k)(t-s)} \Theta \star F(s) \langle \Theta \star \Psi(t) \rangle_{\mu_\varepsilon} ds dt. \end{aligned}$$

Hence

$$\begin{aligned} \operatorname{Re} S_\varepsilon^{III}(Z,k) = \varepsilon \int_0^\infty e^{-Z\varepsilon t} \int_0^t \cos(\omega(k)s)(\Theta \star F)(s) \cos(\omega(k)t) \langle \Theta \star \Psi(t) \rangle_{\mu_\varepsilon} ds dt \\ + \varepsilon \int_0^\infty e^{-Z\varepsilon t} \int_0^t \sin(\omega(k)s)(\Theta \star F)(s) \sin(\omega(k)t) \langle \Theta \star \Psi(t) \rangle_{\mu_\varepsilon} ds dt. \end{aligned}$$

Proceeding similarly with VI_ε we get

$$\begin{aligned} \operatorname{Re} S_\varepsilon^{VI}(Z,k) = \varepsilon \int_0^\infty e^{-Z\varepsilon t} \cos(\omega(k)t)(\Theta \star F)(t) \langle \int_0^t ds \cos(\omega(k)s) \Theta \star \Psi(s) ds \rangle_{\mu_\varepsilon} dt \\ + \varepsilon \int_0^\infty e^{-Z\varepsilon t} \sin(\omega(k)t)(\Theta \star F)(t) \langle \int_0^t ds \sin(\omega(k)s) \Theta \star \Psi(s) ds \rangle_{\mu_\varepsilon} dt. \end{aligned}$$

Therefore, using integrations by parts, we have

$$\begin{aligned} &\operatorname{Re} S_\varepsilon^{III}(Z,k) + \operatorname{Re} S_\varepsilon^{VI}(Z,k) \\ &= \varepsilon^2 Z \int_0^\infty e^{-Z\varepsilon t} \left(\int_0^t \cos(\omega(k)s)(\Theta \star F)(s) ds \right) \langle \int_0^t \cos(\omega(k)s)(\Theta \star \Psi)(s) ds \rangle_{\mu_\varepsilon} dt \\ &+ \varepsilon^2 Z \int_0^\infty e^{-Z\varepsilon t} \left(\int_0^t \sin(\omega(k)s)(\Theta \star F)(s) ds \right) \langle \int_0^t \sin(\omega(k)s)(\Theta \star \Psi)(s) ds \rangle_{\mu_\varepsilon} dt. \end{aligned} \tag{70}$$

Using the identity:

$$\Psi(t) = \frac{1}{2i} \left(\int_{\mathbb{T}} e^{-i\omega(k)t} \hat{\psi}(0,k) - \int_{\mathbb{T}} e^{i\omega(k)t} \hat{\psi}^*(0,k) \right),$$

we expand the term (70) as

$$\begin{aligned} &\frac{\varepsilon^2 Z}{2i} \int_0^\infty e^{-Z\varepsilon t} \int_0^t \int_0^t \cos(\omega(k)s) \cos(\omega(k)s') \\ &\quad \times \int_0^s \int_0^{s'} F(s-u) \left\langle \int_{\mathbb{T}} e^{-i\omega(h)(s'-u')} \hat{\psi}(0,h) dh \right\rangle_{\mu_\varepsilon} \Theta(du)\Theta(du') ds ds' dt \end{aligned} \tag{71}$$

$$- \frac{\varepsilon^2 Z}{2i} \int_0^\infty e^{-Z\varepsilon t} \int_0^t \int_0^t \cos(\omega(k)s)\cos(\omega(k)s') \times \int_0^s \int_0^{s'} F(s-u) \left\langle \int_{\mathbb{T}} e^{i\omega(h)(s'-u')} \hat{\psi}^*(0,h)dh \right\rangle_{\mu_\varepsilon} \Theta(du)\Theta(du')dsds'dt. \tag{72}$$

Let us estimate (71), the term (72) being estimated similarly. After that, we subtract (71) − (72) to get (70). Using the identity

$$\delta_{t,t'} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ia(t'-t)} da,$$

and changing $e^{-\lambda\varepsilon t}$ into $e^{-\lambda\varepsilon(t+t')/2}$, the term (71) becomes

$$\begin{aligned}
&\frac{\varepsilon^2 Z}{2i} \int_0^\infty e^{-Z\varepsilon(t+t')/2} \int_0^t \int_0^{t'} \cos(\omega(k)s)\cos(\omega(k)s') \\
&\qquad \int_0^s \int_0^{s'} F(s-u) \left\langle \int_{\mathbb{T}} e^{-i\omega(h)(s'-u')} \hat{\psi}(0,h)dh \right\rangle_{\mu_\varepsilon} \Theta(du)\Theta(du')dsds'\delta_{t,t'}dtdt' \\
&= \frac{\varepsilon^2 Z}{4i\pi} \int_{\mathbb{R}} \int_0^\infty \int_0^\infty e^{ia(t'-t)} e^{-Z\varepsilon(t+t')/2} \int_0^t \int_0^{t'} \cos(\omega(k)s)\cos(\omega(k)s') \\
&\qquad \int_0^s \int_0^{s'} F(s-u) \left\langle \int_{\mathbb{T}} e^{-i\omega(h)(s'-u')} \hat{\psi}(0,h)dh \right\rangle_{\mu_\varepsilon} \Theta(du)\Theta(du')dsds'dtdt'da.
\end{aligned}$$

Rewriting the domain of integration,

$$(u,s,t) \in [0,s]\times[0,t]\times[0,\infty] \to (t,s,u) \in [s,\infty]\times[u,\infty]\times[0,\infty],$$

this term is equal to

$$\frac{\varepsilon Z}{4i\pi} \int_{\mathbb{R}} \int_{\mathbb{T}} \varepsilon P(\varepsilon,h) A(h,k,Z) A_F(k,Z) dhda,$$

where

$$A(h,k,Z) = \frac{1}{2(Z\varepsilon/2 - ia)} \left(\frac{\tilde{\Theta}(Z\varepsilon/2 - ia - i\omega(k))}{Z\varepsilon/2 - ia - i(\omega(k)-\omega(h))} + \frac{\tilde{\Theta}(Z\varepsilon/2 - ia + i\omega(k))}{Z\varepsilon/2 - ia + i(\omega(k)+\omega(h))} \right),$$

and

$$\begin{aligned}
A_F(k,Z) &= \int_0^\infty \int_u^\infty \cos(\omega(k)s) F(s-u) \int_s^\infty e^{-(Z\varepsilon/2+ia)t} dtds\Theta(du) \\
&= \frac{1}{Z\varepsilon/2 + ia} \int_0^\infty \int_u^\infty \cos(\omega(k)s) F(s-u) e^{-(Z\varepsilon/2+ia)s} ds\Theta(du) \\
&= \frac{1}{2(Z\varepsilon/2 + ia)} \int_0^\infty \Big(e^{-(Z\varepsilon/2+ia-i\omega(k))u} \mathcal{L}(F)(Z\varepsilon/2 + ia - i\omega(k))\Theta(du) \\
&\qquad + e^{-(Z\varepsilon/2+ia+i\omega(k))u} \mathcal{L}(F)(Z\varepsilon/2 + ia + i\omega(k)) \Big) \\
&= \frac{1}{2(Z\varepsilon/2 + ia)} \Big(\tilde{\Theta}(Z\varepsilon/2 + ia - i\omega(k)) \mathcal{L}(F)(Z\varepsilon/2 + ia - i\omega(k)) \\
&\qquad + \tilde{\Theta}(Z\varepsilon/2 + ia + i\omega(k)) \mathcal{L}(F)(Z\varepsilon/2 + ia + i\omega(k)) \Big).
\end{aligned}$$

Making the change variable $a \to \varepsilon a$, (71) is equal to:

$$\begin{aligned}
&\frac{Z}{16i\pi}\int_{\mathbb{R}}\int_{\mathbb{T}} P(\varepsilon,h)\frac{1}{(\lambda/2)^2+a^2}\\
&\times\Big\{\tilde{\Theta}(\varepsilon(Z/2+ia)-i\omega(k))\mathcal{L}(F)(\varepsilon(Z/2+ia)-i\omega(k))\\
&+\tilde{\Theta}(\varepsilon(Z/2+ia)+i\omega(k))\mathcal{L}(F)(\varepsilon(Z/2+ia)+i\omega(k))\Big\}\frac{\tilde{\Theta}(\varepsilon(Z/2-ia)-i\omega(k))}{Z/2-ia-i\varepsilon^{-1}(\omega(k)-\omega(h))}dhda \quad (73)\\
&+\frac{Z}{16i\pi}\int_{\mathbb{R}}\int_{\mathbb{T}} P(\varepsilon,h)\frac{1}{(Z/2)^2+a^2}\\
&\times\Big\{\tilde{\Theta}(\varepsilon(Z/2+ia)-i\omega(k))\mathcal{L}(F)(\varepsilon(Z/2+ia)-i\omega(k))\\
&+\tilde{\Theta}(\varepsilon(Z/2+ia)+i\omega(k))\mathcal{L}(F)(\varepsilon(Z/2+ia)+i\omega(k))\Big\}\frac{\varepsilon\tilde{\Theta}(\varepsilon(Z/2-ia)+i\omega(k))}{\varepsilon(Z/2-ia)+i(\omega(k)+\omega(h))}dhda. \quad (74)
\end{aligned}$$

We expect that most of the contribution at the limit comes from (73). We will prove that the limit of (74) is zero. From (13), (14), (31), using the bound of test function $\hat{O}^*$, and the bound $|Z+iD_\varepsilon^+(k,\eta)|\geq \mathrm{Re}(Z)$, it suffices to estimate

$$\left|\frac{\varepsilon^d}{\varepsilon(Z/2-ia)+i\omega(k)+i\omega(h)}\right|\lesssim \varepsilon^d. \quad (75)$$

For ε sufficient small, $|\varepsilon(Z/2-ia)+i\omega(k)+i\omega(h)|\gtrsim 2\omega_{min}>0$, and (75) follows.

Next, we handle the term (73). If $|\omega(k)-\omega(h)|\gtrsim\varepsilon^{d_1}$ for some $d_1\in[0,d)$ then we do the same as (75) to show that the integral on this domain is small. We consider the domain $|\omega(k)-\omega(h)|\leq \varrho_1\varepsilon^{d_1}$ for some ϱ_1, d_1 to be specified later.

Recall that we consider the truncated wave (33) with some small number ϱ to be chosen later. In the domain, $k\in\left[-\frac{1}{2},-\frac{1}{2}+\frac{\varrho}{2}\right]\cup\left[-\frac{\varrho}{2},\frac{\varrho}{2}\right]\cup\left[\frac{1}{2}-\frac{\varrho}{2},\frac{1}{2}\right]$, we choose $\varrho_1\ll\frac{\varrho^2}{4}, d_1=0$. Then, $h\in[\mathbb{T}](\varrho)$ according to (10) and (11). As a result, $P(\varepsilon,h)=0$ on this domain.

We will focus on: $k\in\left[-\frac{1}{2}+\frac{\varrho}{2},-\frac{\varrho}{2}\right]\cup\left[\frac{\varrho}{2},\frac{1}{2}-\frac{\varrho}{2}\right]$ and h such that $|\omega(k)-\omega(h)|\leq\varrho_1\varepsilon^{d_1}$ for some small ϱ_1. Because $|\omega(k)-\omega(h)|=|\omega'(k')||k-h|\gtrsim|k-h|$, we consider $h\in[-\varrho_1\varepsilon^{d_1},\varrho_1\varepsilon^{d_1}]$ for some small ϱ_1.

We perform the change of variables $h\to k-\varepsilon\xi'$ where $\xi'\in[-\varrho_1\varepsilon^{d_1-1},\varrho_1\varepsilon^{d_1-1}]$. The term (73) becomes

$$\begin{aligned}
&\frac{-Z\varepsilon}{16i\pi}\int_{\mathbb{R}}\int_{-\varrho_1\varepsilon^{d_1-1}}^{\varrho_1\varepsilon^{d_1-1}} P(\varepsilon,k-\varepsilon\xi')\frac{1}{(Z/2)^2+a^2}\\
&\times\Big\{\tilde{\Theta}(\varepsilon(Z/2+ia)-i\omega(k))\mathcal{L}(F)(\varepsilon(Z/2+ia)-i\omega(k))\\
&+\tilde{\Theta}(\varepsilon(Z/2+ia)+i\omega(k))\mathcal{L}(F)(\varepsilon(Z/2+ia)+i\omega(k))\Big\}\frac{\tilde{\Theta}(\varepsilon(Z/2-ia)-i\omega(k))}{Z/2-ia-i\varepsilon^{-1}(\omega(k)-\omega(k-\varepsilon\xi'))}d\xi' da.
\end{aligned} \quad (76)$$

We approximate (76) using the derivative of ω, that is

$$\begin{aligned}
&\frac{-Z\varepsilon}{16i\pi}\int_{\mathbb{R}}\int_{-\varrho_1\varepsilon^{d_1-1}}^{\varrho_1\varepsilon^{d_1-1}} P(\varepsilon,k-\varepsilon\xi')\frac{1}{(Z/2)^2+a^2}\\
&\times\Big\{\tilde{\Theta}(\varepsilon(Z/2+ia)-i\omega(k))\mathcal{L}(F)(\varepsilon(Z/2+ia)-i\omega(k))
\end{aligned}$$

$$+\tilde{\Theta}(\varepsilon(Z/2+ia)+i\omega(k))\mathcal{L}(F)(\varepsilon(Z/2+ia)+i\omega(k))\Big\}\frac{\tilde{\Theta}(\varepsilon(Z/2-ia)-i\omega(k))}{Z/2-ia-i\omega'(k)\xi'}d\xi' da. \quad (77)$$

We have

$$\begin{aligned}&\left|\frac{\varepsilon}{Z/2-ia-i\varepsilon^{-1}(\omega(k)-\omega(k-\varepsilon\xi'))}-\frac{\varepsilon}{Z/2-ia-\omega'(k)\xi'}\right|\\&=\left|\frac{\omega(k)-\omega(k-\varepsilon\xi')+\omega'(k)\varepsilon\xi'}{(Z/2-ia-i\varepsilon^{-1}(\omega(k)-\omega(k-\varepsilon\xi')))(Z/2-ia-i\omega'(k)\xi')}\right|,\end{aligned}$$

and $|\omega(k)-\omega(k-\varepsilon\xi')+\omega'(k)\varepsilon\xi'|\approx|\omega''(k')(\varepsilon\xi')^2|\lesssim\varrho_1^2\varepsilon^{2d_1}$. Besides, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\left|\frac{1}{(Z/2-ia-i\varepsilon^{-1}(\omega(k)-\omega(k-\varepsilon\xi')))(Z/2-ia-i\omega'(k)\xi')}\right|\\&\lesssim\frac{1}{|Z/2-ia-i\varepsilon^{-1}(\omega(k)-\omega(k-\varepsilon\xi'))|^2}+\frac{1}{|Z/2-ia-i\omega'(k)\xi'|^2}\\&\approx\frac{1}{1+|a+\varepsilon^{-1}(\omega(k)-\omega(k-\varepsilon\xi'))|^2}+\frac{1}{1+|a+\omega'(k)\xi'|^2}.\end{aligned}$$

Next, we apply the estimate

$$\int_{\mathbb{R}}\frac{da}{(1+a^2)(1+(a+b)^2)}\lesssim\frac{1}{1+b^2},$$

the difference (76) − (77) is estimated by

$$\frac{\varrho_1^2\varepsilon^{2d_1-1+d}}{\inf_{k\in[\varrho/2-\varrho_1,1/2-\varrho/2+\varrho_1]}|\omega'(k)|}\int_{\mathbb{R}}\frac{dx}{1+x^2}\lesssim\varrho_1^2\varepsilon^{d+2d_1-1}.$$

In the latter estimation, we recall that $1/2-d$ is small, so $d>1/3$, and we choose $d_1=1/3$. As a result, ε^{d+2d_1-1} tends to 0 as $\varepsilon\to0$.

We now estimate (77) for some small ϱ_1. Using (13), (14) and (31), we only need to estimate

$$\begin{aligned}\varepsilon^d\int_{-\varrho_1/\varepsilon^{d_1}}^{\varrho_1/\varepsilon^{d_1}}\frac{d\xi'}{|Z/2-ia-i\omega'(k)\xi'|}&=\varepsilon^d\int_{-\varrho_1\varepsilon^{d_1-1}}^{\varrho_1\varepsilon^{d_1-1}}\frac{d\xi'}{\sqrt{\mathrm{Re}(Z/2)^2+(\omega'(k)\xi'+a-\mathrm{Im}(Z/2))^2}}\\&\lesssim\frac{\varepsilon^d}{\omega'(k)}\operatorname{arcsinh}(x)\Big|_{-\omega'(k)\varrho_1\varepsilon^{d_1-1}+a-\mathrm{Im}(Z/2)}^{\omega'(k)\varrho_1\varepsilon^{d_1-1}+a-\mathrm{Im}(Z/2)}\approx\varepsilon^d\log(\varepsilon^{d_1-1}). \quad (78)\end{aligned}$$

The limit of term (78) is 0. This implies that the term (73) also has limit 0.

3.4.6. *The seventh scattering terms.* The seventh scattering terms are obtained in (53) using VII_ε, defined by (64). We have a transformation for $VII_\varepsilon(t,k)$ using (30) and (41)

$$\begin{aligned}VII_\varepsilon(t,k)&=\frac{-1}{\nu}\int_0^t F(t-s')\Theta(ds')\int_0^t\Theta_\omega^*(t-s,k)F(s)ds\\&=\frac{-1}{\nu}\int_0^t e^{i\omega(k)s'}e^{i\omega(k)(t-s')}F(t-s')\Theta(ds')\int_0^t(\Theta^\omega)^*(t-s,k)e^{-i\omega(k)s}F(s)ds\\&=\frac{-1}{\nu}((e^{i\omega(k)\cdot}\Theta)\star F_k^*(t))((\Theta^\omega)^*\star F_k(t)).\end{aligned}$$

Next, we compute $\mathcal{L}(VII_\varepsilon^{(\varepsilon)})(Z,k-\varepsilon\xi/2)$ using the formula for the Laplace transform of a product

$\mathcal{L}(VII_\varepsilon^{(\varepsilon)})(Z,k-\varepsilon\xi/2)$

$$
\begin{aligned}
&= \frac{-1}{2\nu i\pi} \lim_{L\to\infty} \int_{c-iL}^{c+iL} \mathcal{L}((e^{i\omega(\cdot)\cdot}\Theta) \star F_\cdot^{*(\varepsilon)})(\sigma, k-\varepsilon\xi/2)\mathcal{L}((\Theta^\omega)^* \star F_\cdot^{(\varepsilon)})(Z-\zeta, k-\varepsilon\xi/2)d\zeta \\
&= \frac{-\varepsilon^2}{2\nu i\pi} \lim_{L\to\infty} \int_{c-iL}^{c+iL} \tilde{\Theta}(\varepsilon\zeta - i\omega(k-\varepsilon\xi/2))\mathcal{L}(F)(\varepsilon\zeta - i\omega(k-\varepsilon\xi/2)) \\
&\quad \mathcal{L}((\Theta^\omega)^*)(\varepsilon(Z-\zeta), k-\varepsilon\xi/2)\mathcal{L}(F)(\varepsilon(Z-\zeta) + i\omega(k-\varepsilon\xi/2))d\zeta \\
&= \frac{-\varepsilon}{2\nu i\pi} \lim_{L\to\infty} \int_{c-iL}^{c+iL} \tilde{\Theta}(\varepsilon\zeta - i\omega(k-\varepsilon\xi/2))\mathcal{L}(F)(\varepsilon\zeta - i\omega(k-\varepsilon\xi/2)) \\
&\quad \mathcal{L}((\Theta^\omega)^{(\varepsilon)*})(Z-\zeta, k-\varepsilon\xi/2)\mathcal{L}(F)(\varepsilon(Z-\zeta) + i\omega(k-\varepsilon\xi/2))d\zeta \\
&= \frac{-\varepsilon}{2\nu i\pi} \lim_{L\to\infty} \int_{c-iL}^{c+iL} \tilde{\Theta}(\varepsilon\zeta - i\omega(k-\varepsilon\xi/2))\mathcal{L}(F)(\varepsilon\zeta - i\omega(k-\varepsilon\xi/2)) \\
&\quad \frac{\tilde{\Theta}(\varepsilon(Z-\zeta) + i\omega(k-\varepsilon\xi/2))}{Z-\zeta}\mathcal{L}(F)(\varepsilon(Z-\zeta) + i\omega(k-\varepsilon\xi/2))d\zeta. \qquad (79)
\end{aligned}
$$

At the limit $\varepsilon \to 0$, the term (79) gives

$$\frac{-1}{2\nu i\pi} \lim_{L\to\infty} \int_{c-iL}^{c+iL} \frac{|\theta(k)|^2|\mathcal{F}(k)|^2}{Z-\zeta} d\zeta.$$

Now, the integral is estimated using (45). We get

$$\frac{-|\theta(k)|^2|\mathcal{F}(k)|^2}{2\nu}. \qquad (80)$$

Expanding (80) in (53), the result is

$$\frac{|\theta(k)|^2|\mathcal{F}(k)|^2}{2(Z + i\omega'(k)\xi)}.$$

3.5. **Proof of Theorem 1.** We use $\hat{\psi}^1$ in (32) and use the bound on the initial condition (5), and the growth of energy of $\hat{\psi}^1$ is at most linear, which is similar to $\hat{\psi}$; meanwhile, the energy of $\hat{\psi}^2$ is decreasing

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0^+} \sup_{\xi\in\mathbb{T}_{2/\varepsilon}} \int_{\mathbb{T}} |\mathscr{W}_\varepsilon(Z,\xi,k) - \mathscr{W}_\varepsilon^1(Z,\xi,k)|dk \\
&= \limsup_{\varepsilon\to 0^+} \sup_{\xi\in\mathbb{T}_{2/\varepsilon}} \frac{\varepsilon}{2}\mathbb{E}_\varepsilon \left[\int_{\mathbb{T}} \left| \int_0^\infty e^{-Zt} \left(\hat{\psi}^{(\varepsilon)}(t, k+\frac{\varepsilon\xi}{2})\hat{\psi}^{(\varepsilon)*}(t, k-\frac{\varepsilon\xi}{2}) \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. -\hat{\psi}^{1(\varepsilon)}(t, k+\frac{\varepsilon\xi}{2})\hat{\psi}^{1(\varepsilon)*}(t, k-\frac{\varepsilon\xi}{2}) \right) dt \right| dk \right] \\
&\le \limsup_{\varepsilon\to 0^+} \sup_{\xi\in\mathbb{T}_{2/\varepsilon}} \frac{\varepsilon}{2}\mathbb{E}_\varepsilon \left[\int_0^\infty e^{-\operatorname{Re} Zt} \int_{\mathbb{T}} \left| \left(\hat{\psi}^{(\varepsilon)}(t, k+\frac{\varepsilon\xi}{2})\hat{\psi}^{(\varepsilon)*}(t, k-\frac{\varepsilon\xi}{2}) \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. -\hat{\psi}^{1(\varepsilon)}(t, k+\frac{\varepsilon\xi}{2})\hat{\psi}^{1(\varepsilon)*}(t, k-\frac{\varepsilon\xi}{2}) \right) dt \right| dk \right] \\
&\le \limsup_{\varepsilon\to 0^+} \sup_{\xi\in\mathbb{T}_{2/\varepsilon}} \frac{\varepsilon}{2}\mathbb{E}_\varepsilon \left[\int_0^\infty e^{-\operatorname{Re} Zt} \int_{\mathbb{T}} \left| \hat{\psi}^{2(\varepsilon)}(t, k+\frac{\varepsilon\xi}{2})\hat{\psi}^{2(\varepsilon)*}(t, k-\frac{\varepsilon\xi}{2})dt \right| dk \right] \\
&+ \limsup_{\varepsilon\to 0^+} \sup_{\xi\in\mathbb{T}_{2/\varepsilon}} \varepsilon\mathbb{E}_\varepsilon \left[\int_0^\infty e^{-\operatorname{Re} Zt} \int_{\mathbb{T}} \left| \hat{\psi}^{2(\varepsilon)}(t, k+\frac{\varepsilon\xi}{2})\hat{\psi}^{1(\varepsilon)*}(t, k-\frac{\varepsilon\xi}{2})dt \right| dk \right]
\end{aligned}
$$

$$\lesssim \int_0^\infty e^{-\operatorname{Re} Zt} \limsup_{\varepsilon\to 0^+} \varepsilon \mathbb{E}_\varepsilon \left\|\hat\psi^2(t/\varepsilon)\right\|_{L^2(\mathbb{T})} \left(\left\|\hat\psi^2(t/\varepsilon)\right\|_{L^2(\mathbb{T})} + \left\|\hat\psi^1(t/\varepsilon)\right\|_{L^2(\mathbb{T})}\right) dt$$
$$\lesssim \rho(\varrho)\left(\rho(\varrho) + \int_0^t e^{-\operatorname{Re} Zt} t dt\right) \ll 1.$$

In this estimate, $\rho(\varrho)^2$ is a bound for $\sup_\varepsilon \varepsilon\mathbb{E}_\varepsilon\|\hat\psi^2\|_{L^2}^2$. We see that the limit $\rho(\varrho)$ is 0 as ϱ tends to 0 by the dominated convergence theorem. If we denote $\mathscr{W}^1$ to be the limit of $\mathscr{W}_\varepsilon^1$ then for $\hat O^* \in \mathscr{S}(\mathbb{R}\times\mathbb{T})$ we have

$$\begin{aligned}
&\limsup_{\varepsilon\to 0^+} \left|\int_{\mathbb{R}\times\mathbb{T}} \hat O^*(\xi,k)\mathscr{W}^1(Z,\xi,k)d\xi dk - \int_{\mathbb{R}\times\mathbb{T}} \hat O^*(\xi,k)\mathscr{W}_\varepsilon(Z,\xi,k)d\xi dk\right| \\
&\quad\le \left|\int_{\mathbb{R}\times\mathbb{T}} \hat O^*(\xi,k)\mathscr{W}_\varepsilon^1(Z,\xi,k)d\xi dk - \int_{\mathbb{R}\times\mathbb{T}} \hat O^*(\xi,k)\mathscr{W}^1(Z,\xi,k)d\xi dk\right| \\
&\qquad + \left|\int_{\mathbb{R}\times\mathbb{T}} \hat O^*(\xi,k)\mathscr{W}_\varepsilon^1(Z,\xi,k)d\xi dk - \int_{\mathbb{R}\times\mathbb{T}} \hat O^*(\xi,k)\mathscr{W}_\varepsilon(Z,\xi,k)d\xi dk\right| \\
&\quad\lesssim \rho(\varrho).
\end{aligned}$$

With the assumption that $\hat\psi$ vanishes on $[\mathbb{T}](\varrho)$, we combine all mentioned 13 terms and get

$$\begin{aligned}
\mathscr{W}(Z,\xi,k) = {}& \frac{T|v_g(k)|r_a(k)}{Z(Z+i\omega'(k))\xi} + \frac{\hat W_0(\xi,k)}{Z+i\omega'(k)\xi} + \frac{|v_g(k)|(r_t(k)-1)}{Z+i\omega'(k)\xi}\int_{\mathbb{R}} \frac{\hat W_0(\xi',k)d\xi'}{Z+i\omega'(k)\xi'} \\
&+ \frac{|v_g(k)|r_r(k)}{Z+i\omega'(k)\xi}\int_{\mathbb{R}} \frac{\hat W_0(\xi',-k)d\xi'}{Z-i\omega'(k)\xi'} + \frac{|v_g(k)|r_a(k)|\mathcal{F}(k)|^2}{\nu(Z+i\omega'(k)\xi)}.
\end{aligned}$$

This result remain true without the above assumption since it does not depend on ϱ.

To get W in the main theorem, we take the inverse Laplace and Fourier transform of each term. Let us make precise the technical computations:

Term with T:

$$\begin{aligned}
\int_{\mathbb{R}} \hat O^*(\xi,k)\mathcal{L}^{-1}\left(\frac{1}{Z(Z+i\omega'(k)\xi)}\right)(t)d\xi &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1-e^{-i2\pi v_g(k)t\xi}}{i2\pi v_g(k)\eta} e^{i2\pi x\xi} O^*(x,k)dxd\xi \\
&= \int_{\mathbb{R}} O^*(x,k)\left(1_{[0,\infty)}(x) - 1_{[v_g(k)t,\infty)}(x)\right)dx = \int_{\mathbb{R}} O^*(x)1_{[0,v_g(k)t]}(x)dx. \qquad (81)
\end{aligned}$$

Ballistic term:

$$\begin{aligned}
\int_{\mathbb{R}} \hat O^*(\xi,k)\mathcal{L}^{-1}\left(\frac{\hat W_0(\xi,k)}{Z+i\omega'(k)\xi}\right)(t)d\xi &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i2\pi v_g(k)t\xi} e^{i2\pi x\xi} O^*(x,k)\hat W_0(\xi,k)dxd\xi \\
&= \int_{\mathbb{R}} O^*(x,k)W_0(x-v_g(k)t,k)dx. \qquad (82)
\end{aligned}$$

Transmitted term:

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{L}^{-1}\left(\frac{\hat W_0(\xi',k)\hat O^*(\xi,k)|v_g(k)|}{(Z+i\omega'(k)\xi')(Z+i\omega'(k)\xi)}\right)(t)d\xi'd\xi \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i2\pi x\xi} O^*(x,k)|v_g(k)| \int_{\mathbb{R}} \left(e^{-i2\pi v_g(k)\xi\cdot} \star (e^{-i2\pi v_g(k)\xi'}\hat W_0(\xi',k))(t)\right)d\xi'd\xi dx \\
&= \int_{\mathbb{R}}\int_0^t |v_g(k)|O^*(x,k)W_0(-v_g(k)(t-s),k)\delta(x-v_g(k)s)dsdx
\end{aligned}$$

$$= \int_{\mathbb{R}} O^*(x,k) W_0(x - v_g(k)t, k) 1_{[0, v_g(k)t]}(x) dx. \tag{83}$$

Reflecting term:

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{L}^{-1}\left(\frac{\hat{W}_0(\xi', -k)\hat{O}^*(\xi, k)|v_g(k)|}{(Z - i\omega'(k)\xi')(Z + i\omega'(k)\xi)} \right)(t) d\xi' d\xi \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i2\pi x\xi} O^*(x,k)|v_g(k)| \int_{\mathbb{R}} \left(e^{-i2\pi v_g(k)\xi \cdot} \star (e^{i2\pi v_g(k)\xi'} \hat{W}_0(\xi', -k))(t) \right) d\xi' d\xi dx \\
&= \int_{\mathbb{R}} \int_0^t |v_g(k)| O^*(x,k) W_0(v_g(k)(t-s), -k) \delta(x - v_g(k)t) ds dx \\
&= \int_{\mathbb{R}} O^*(x,k) W_0(v_g(k)t - x, -k) 1_{[0, v_g(k)t]}(x) dx.
\end{aligned} \tag{84}$$

The new boundary control term:

$$\begin{aligned}
\int_{\mathbb{R}} \hat{O}^*(\xi, k) \mathcal{L}^{-1}\left(\frac{1}{Z + i\omega'(k)\xi} \right)(t) d\xi &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i2\pi(x - v_g(k)t)\xi} O^*(x,k) d\xi dx \\
&= \int_{\mathbb{R}} O^*(x,k) \delta(x - v_g(k)t) dx.
\end{aligned}$$

We have proved that

$$\lim_{\varepsilon} \langle O, W_\varepsilon(t) \rangle_{\mathscr{N}} = \int_{\mathbb{R}\times\mathbb{T}} O^*(x,k) W(t,x,k) dk dx$$

for any t and any $O \in \mathscr{S}(\mathbb{R} \times \mathbb{T})$. The theorem follows.

4. Linear memory-feedback control

4.1. **Setting and Main Theorem.** In this section, we add a convolution control term $F \star \alpha_0(t)\delta_{0,n}$. To simplify the problem we actually add $(F + \nu\delta_0(dt)) \star \alpha_0(t)\delta_{0,n}$. Note that $\nu\delta_0(dt) \star \alpha_0(t) = \nu\alpha_0$ cancels $-\nu\alpha_0$ in the original equation. Hence, we will consider the system of equations:

$$\dot{\beta}_n(t) = \alpha_n(t), \tag{85}$$

$$d\alpha_n(t) = -(\sigma \star \beta(t))_n dt + (F \star \alpha_0(t) dt + \sqrt{2\nu T} dR(t))\delta_{0,n}. \tag{86}$$

We assume that $F \in C^\infty((0, +\infty)) \cap L^m((0, +\infty)), m \in [1, \infty)$. Recall that we have (15) and the Fourier transform $\hat{F}$ is a bounded and continuous function.

We assume that

$$\mathrm{Re}\left(\hat{F}\left(\frac{\omega(k)}{2\pi} \right) \right) < 0, \forall k \in \mathbb{T}. \tag{87}$$

This inequality allows us to keep the control over the energy of the wave in the domain $[\mathbb{T}](\varrho)$ defined in (34).

We redefine the three new rates and use them for our main theorem

$$r_a^F(k) = -\mathrm{Re}(\hat{F}(\omega(k)/2\pi)) \frac{|\theta_F(k)|^2}{|v_g(k)|}, \tag{88}$$

$$r_t^F(k) = 1 + \frac{\mathrm{Re}(\hat{F}^*(\omega(k)/2\pi)\theta_F(k))}{|v_g(k)|} + \frac{|\hat{F}(\omega(k)/2\pi)|^2 |\theta_F(k)|^2}{4|v_g(k)|^2}, \tag{89}$$

$$r_r^F(k) = \frac{|\hat{F}(\omega(k)/2\pi)|^2 |\theta_F(k)|^2}{4|v_g(k)|^2}. \tag{90}$$

Theorem 3. *Consider the system of equations* (85) *and* (86) *with the initial conditions stated in sections 2 and 4. Then, for any* $\tau > 0$ *and* $G \in L^1([0,\tau]; \mathscr{A})$ *we have*

$$\int_0^\tau \langle G(t), W_\varepsilon(t)\rangle \, dt = \int_0^\tau \int_{\mathbb{R}\times\mathbb{T}} G^*(t,x,k) W(t,x,k) dx dk dt,$$

where

$$\begin{aligned} W(t,x,k) = {} & W_0(x - v_g(k)t, k) 1_{[0,v_g(k)t]^c}(x) - \frac{\nu T r_a^F(k)}{\mathrm{Re}(\hat{F}(\omega(k)/2\pi))} 1_{[0,v_g(k)t]}(x) \\ & + r_t^F(k) W_0(x - v_g(k)t, k) 1_{[0,v_g(k)t]}(x) + r_r^F(k) W_0(-x + v_g(k)t, -k) 1_{[0,v_g(k)t]}(x). \end{aligned}$$

Remark 4. On the well-posedness of (85)-(86), we recall (24) in which the solution $\hat{\psi}$ is uniquely determined by (96) for the full interval $[0,\infty)$ of time.

Remark 5. In view of addressing Question A, posed in the introduction, we observe that the three rates (r_a^F, r_t^F, r_r^F) can now be controlled under the influence of the linear memory-feedback control. An "inverse problem" question is: given a triple of functions (r_a, r_t, r_r) as target rates, can we determine if (r_a, r_t, r_r) can be asymptotically reached following (88)-(89)-(90), i.e. can we find F that satisfies there exists a family F_N such that F_N is equal to F in finite time and $r_a^{F_N}, r_t^{F_N}, r_r^{F_N}$ respectively converge almost everywhere to r_a, r_t, r_r on $\mathbb{T}$. If there exists such a control function F, we say that a triple of functions (r_a, r_t, r_r) is "asymptotically reachable". We state some assumptions on the target rates to make a sufficient condition for "asymptotically reachable":

(H1) The three functions are even on $\mathbb{T}$;
(H2) The three functions have their sum equal to 1;
(H3) $r_t(k), r_r(k) > 0$ and there exists $c_1 > 0$ such that $r_a(k) > c_1$ for $k \in (0, 1/2)$;
(H4) The three functions are continuous on $(0, 1/2)$;
(H5) $\sqrt{r_t(k)} + \sqrt{r_r(k)} \geq 1$ for all $k \in \mathbb{T}$;
(H6) $\int_0^\infty \mathrm{Re}(\bar{F}(u)) \cos(ut) du = \int_0^\infty \mathrm{Im}(\bar{F}(u)) \sin(ut) du$ for all $t > 0$.

In (H6), $\bar{F}$ is defined by

- $RE(k) = |v_g(k)|(r_t(k) - r_r(k) - 1)$ for $k \in (0, 1/2)$;
- $IM(k) = \sqrt{4|v_g(k)|^2 r_r(k) - RE(k)^2}$ for $k \in (0, 1/2)$;
- $FT(k) = RE(k) + iIM(k)$ for $k \in (0, 1/2)$;
- $TH(k) = 1 + FT(k) \lim_{Z\to 0} \mathcal{L}(C_\omega)(Z - i\omega(k))$ for $k \in (0, 1/2)$;
- $\bar{F}(u) = FT(\varphi_+(u))/TH(\varphi_+(u))$ for $u \in (\omega_{\min}, \omega_{\max})$.

Corollary 6. *If a triple* (r_a, r_t, r_r) *satisfies (H1) - (H6) then it is asymptotically reachable.*

4.2. **Preliminary computations.** Recalling (4), we write (85), (86) as:

$$d\hat{\psi}(t,k) = -i\omega(k)\hat{\psi}(t,k)dt + iF \star \alpha_0(t)dt + i\sqrt{2\nu T} dR(t). \tag{91}$$

Solving (91), we obtain

$$\hat{\psi}(t,k) = e^{-i\omega(k)t}\hat{\psi}(k) + ie^{-i\omega(k)\cdot} \star F \star \alpha_0(t) + i\sqrt{2\nu T} \int_0^t e^{-i\omega(k)(t-s)} dR(s). \tag{92}$$

Hence,

$$\alpha_0(t) = \Psi(t) + C_\omega \star F \star \alpha_0(t) + \sqrt{2\nu T} C_\omega \star dR(t), \tag{93}$$

where we have used the notation of (25). To solve (93), we define, formally,

$$\tilde{\Theta}_F(Z) = \frac{1}{1 - \mathcal{L}(F)(Z)\mathcal{L}(C_\omega)(Z)} \quad \text{for } Z \in \mathbb{C}_+.$$

We are going to see that $\tilde{\Theta}_F$ is well defined and is bounded, similarly to $\tilde{\Theta}$. First, we infer from (15) that

$$\lim_{Z\to 0} \tilde{\Theta}_F(Z - i\omega(k)) = \frac{1}{1 - \hat{F}^*(\omega(k)/2\pi)\lim_{Z\to 0}\mathcal{L}(C_\omega)(Z - i\omega(k))}. \tag{94}$$

Note that, $\lim_{Z\to 0}\mathcal{L}(C_\omega)(Z - i\omega(k)) = \frac{1}{\nu}\left(\frac{1}{\theta(k)} - 1\right)$ if $\theta(k) \neq 0$. By (87), the limit $\lim_{Z\to 0}\tilde{\Theta}_F(Z - i\omega(k))$ exists and is denoted by $\theta_F(k)$. When $\theta(k) = 0$, we have $|\mathcal{L}(C_\omega)(Z - i\omega(k))| \to \infty$, and we also get $\theta_F(k) = 0$. Since $\theta(k)$ is defined almost everywhere, so is $\theta_F(k)$.

The boundedness of $\tilde{\Theta}_F(Z - i\omega(k))$ comes from (87) and it is satisfied almost everywhere. This can be proved as follows. By (87), we get $\left|1 - \frac{\hat{F}(\omega(k)/(2\pi))}{\nu}\left(\frac{1}{\theta(k)} - 1\right)\right| \geq C > 0$, for a constant C. Indeed, the function $\mathrm{Re}(\hat{F})$ is bounded continuous and $\mathrm{Re}(\hat{F}(\omega(k)/(2\pi))) < 0$ so there exist $C', C'' > 0$ such that $C' \geq |\hat{F}(\omega(k)/(2\pi))| \geq |\mathrm{Re}(\hat{F}(\omega(k)/(2\pi)))| \geq C''$. Thus

$$\begin{aligned}
\left|1 - \frac{\hat{F}(\omega(k)/(2\pi))}{\nu}\left(\frac{1}{\theta(k)} - 1\right)\right| &= \frac{|\hat{F}(\omega(k)/(2\pi))|}{\nu}\left|\frac{\nu}{\hat{F}(\omega(k)/(2\pi))} - \left(\frac{1}{\theta(k)} - 1\right)\right| \\
&\geq \frac{C''}{\nu}\left|\mathrm{Re}\left(\frac{\nu}{\hat{F}(\omega(k)/(2\pi))}\right) - \mathrm{Re}\left(\frac{1}{\theta(k)} - 1\right)\right| \\
&\geq \frac{C''}{\nu}\frac{\nu(-\mathrm{Re}(\hat{F}))}{|\hat{F}|^2} \geq \left(\frac{C''}{C'}\right)^2 > 0.
\end{aligned}$$

Using (94), let us prove that the sum of the three rates is 1

$$\begin{aligned}
\hat{F}^*(\omega(k)/2\pi)\theta_F(k) &= \frac{\hat{F}^*(\omega(k)/2\pi)}{1 - \hat{F}^*(\omega(k)/2\pi)\lim_{\varepsilon\to 0}\mathcal{L}(C_\omega)(\varepsilon - i\omega(k))} \\
&= |\hat{F}(\omega(k)/2\pi)|^2 \frac{1}{\hat{F}(\omega(k)/2\pi) - |\hat{F}(\omega(k)/2\pi)|^2\lim_{\varepsilon\to 0}\mathcal{L}(C_\omega)(\varepsilon - i\omega(k))}.
\end{aligned}$$

From [9, Section 10], we have $\mathrm{Re}(\lim_{\varepsilon\to 0}\mathcal{L}(C_\omega)(\varepsilon - i\omega(k))) = \frac{\pi}{|\omega'(k)|}$. Thus, using $\mathrm{Re}\, z = |z|^2\,\mathrm{Re}\,\frac{1}{z}$ we get

$$\mathrm{Re}(\hat{F}^*(\omega(k)/2\pi)\theta_F(k)) = \left(\mathrm{Re}(\hat{F}(\omega(k)/2\pi)) - |\hat{F}(\omega(k)/2\pi)|^2\frac{\pi}{|\omega'(k)|}\right)|\theta_F(k)|^2.$$

As a consequence,

$$r_a^F(k) + r_t^F(k) + r_r^F(k) = 1. \tag{95}$$

We write (92) in the form

$$\hat{\psi}(t,k) = e^{-i\omega(k)t}\hat{\psi}(k) + i\Theta_\omega^F \star F \star \Psi(t) + i\sqrt{2\nu T}\Theta_\omega^F \star R(t), \tag{96}$$

where

$$\Theta_\omega^F(t,k) = \int_0^t e^{-i\omega(k)(t-s)}\Theta_F(ds).$$

We will also consider

$$\Theta_F^\omega(t,k) = \int_0^t e^{i\omega(k)s}\Theta_F(ds) = e^{i\omega(k)t}\Theta_\omega^F(t,k),$$
$$\mathcal{L}((\Theta_F^\omega)^{(\varepsilon)})(Z,k) = \frac{\tilde{\Theta}_F(\varepsilon Z - i\omega(k))}{Z}.$$

Using $[\mathbb{T}](\varrho)$ defined by (34), we define two waves similar to (32):

$$\hat{\psi}^1(0,k) = \hat{\psi}(k)\chi_\varrho(k),$$
$$d\hat{\psi}^1(t,k) = \left\{-i\omega(k)\hat{\psi}^1(t,k) + \frac{1}{2i}\int_{\mathbb{T}}\int_0^t F(s)[\hat{\psi}^1(t-s,k') - \hat{\psi}^{1*}(t-s,k')]dsdk'\right\}dt + i\sqrt{2\nu T}dR(t),$$
$$\hat{\psi}^2(0,k) = \hat{\psi}(k)(1-\chi_\varrho(k)),$$
$$d\hat{\psi}^2(t,k) = \left\{-i\omega(k)\hat{\psi}^1(t,k) + \frac{1}{2i}\int_{\mathbb{T}}\int_0^t F(s)[\hat{\psi}^2(t-s,k') - \hat{\psi}^{2*}(t-s,k')]dsdk'\right\}dt.$$

4.3. **Thermal term.** For this term we consider the initial wave to be 0 due to the independence between the initial wave and the thermostat. In that case, we compute

$$\mathscr{W}_\varepsilon(Z,\xi,k) = \frac{\nu T}{Z}\int_0^\infty e^{-(Z+iD_\varepsilon(k,\xi))s}\Theta_F^\omega\left(\varepsilon^{-1}s, k+\varepsilon\xi/2\right)\left(\Theta_F^\omega\right)^*\left(\varepsilon^{-1}s, k-\varepsilon\xi/2\right)ds$$
$$= \frac{\nu T}{Z}\frac{1}{2\pi i}\lim_{L\to\infty}\int_{c-iL}^{c+iL}\frac{\tilde{\Theta}_F(\varepsilon\zeta - i\omega(k+\varepsilon\xi/2))\tilde{\Theta}_F^*(\varepsilon(Z+iD_\varepsilon(k,\xi)-\zeta) - i\omega(k-\varepsilon\xi/2))}{\zeta(Z+iD_\varepsilon(k,\xi)-\zeta)}d\zeta.$$

Taking the limit $\varepsilon\to 0$ and using (45), we obtain

$$\frac{\nu T|\theta_F(k)|^2}{Z(Z+i\omega'(k)\xi)} = \frac{\nu T|v_g(k)|v_a^F(k)}{\mathrm{Re}(\hat{F}(\omega(k)/(2\pi)))Z(Z+i\omega'(k)\xi)}.$$

4.4. **Scattering terms.** For scattering terms, we consider $T=0$ like in the previous part. The derivative of the Wigner transform is given by

$$\partial_t\hat{W}_\varepsilon(t,\xi,k) = -iD_\varepsilon(k,\xi)\hat{W}_\varepsilon(t,\xi,k) + \mathbb{E}_\varepsilon\left[i(F\star\Psi)^{(\varepsilon)}(t)\hat{\psi}^{(\varepsilon)*}(t,k-\varepsilon\xi/2) - i(F\star\Psi)^{(\varepsilon)}(t)\hat{\psi}^{(\varepsilon)}(t,k+\varepsilon\xi/2)\right].$$

4.4.1. *The ballistic term.* The ballistic term is unchanged with respect to (65). The term only involves the initial condition of the wave, neither the control nor the thermostat affect this term.

4.4.2. *The first scattering term.* The first scattering term is given by

$$I_\varepsilon(t,k,F) = ie^{i\omega(k)t}\int_0^t\langle F\star\Psi(t-s)\hat{\psi}^*(k)\rangle_{\mu_\varepsilon}\Theta_F(ds)$$
$$= \frac{1}{2}\int_0^t\int_{\mathbb{T}}\int_0^{t-s}F(s')\langle\hat{\psi}^*(k)\hat{\psi}(h)\rangle_{\mu_\varepsilon}e^{i(\omega(k)-\omega(h))t+i\omega(h)(s+s')}ds'dh\Theta(ds).$$

We have

$$S_\varepsilon^I(Z,k,F) = \mathcal{L}(I_\varepsilon^{(\varepsilon)})(Z,k,F)$$
$$= \frac{\varepsilon}{2}\int_0^\infty e^{-\varepsilon Zt}\int_0^t\int_{\mathbb{T}}\int_0^{t-s}F(s')\langle\hat{\psi}^*(k)\hat{\psi}(h)\rangle_{\mu_\varepsilon}e^{i(\omega(k)-\omega(h))t+i\omega(h)(s+s')}ds'dh\Theta(ds)dt$$

$$= \frac{\varepsilon}{2}\int_{\mathbb{T}} \langle \hat{\psi}^*(k)\hat{\psi}(h)\rangle_{\mu_\varepsilon} \int_0^\infty F(s') \int_0^\infty \int_{s+s'}^\infty e^{-\varepsilon Z t} e^{i(\omega(k)-\omega(h))t + i\omega(h)(s+s')} dt\Theta(ds)ds'dh$$
$$= \frac{\varepsilon}{2}\int_{\mathbb{T}} \frac{\langle \hat{\psi}^*(k)\hat{\psi}(h)\rangle_{\mu_\varepsilon}}{\lambda\varepsilon + i(\omega(h)-\omega(k))} \int_0^\infty F(s') \int_0^\infty e^{-\varepsilon Z(s+s')} e^{i\omega(k)(s+s')}\Theta(ds)ds'dh$$
$$= \frac{\varepsilon}{2}\int_{\mathbb{T}} \frac{\langle \hat{\psi}^*(k)\hat{\psi}(h)\rangle_{\mu_\varepsilon}}{\lambda\varepsilon + i(\omega(h)-\omega(k))} \mathcal{L}(F)(\varepsilon Z - i\omega(k))\tilde{\Theta}_F(\varepsilon Z - i\omega(k))dh.$$

Changing the variables from k to $k' - \varepsilon\xi'/2$ and h to $k' + \varepsilon\xi'/2$, we call the new domain $T_\varepsilon \subset \mathbb{T}_{2/\varepsilon} \times \mathbb{T}$. We get

$$\int_{\mathbb{R}\times\mathbb{T}} \frac{\hat{O}^*(\xi, k+\varepsilon\xi/2)S^1_\varepsilon(Z,k,F)}{Z + iD^+_\varepsilon(k,\xi)} d\xi dk$$
$$= \int_{\mathbb{R}}\int_{T_\varepsilon} \frac{\hat{W}_\varepsilon(\xi',k')\mathcal{L}(F)(\varepsilon Z - i\omega(k'-\varepsilon\xi/2))\tilde{\Theta}_F(\varepsilon Z - i\omega(k'-\varepsilon\xi'/2))\hat{O}^*(\xi, k'+\varepsilon\xi/2-\varepsilon\xi'/2)}{(\varepsilon Z + i(\omega(k'+\varepsilon\xi'/2)-\omega(k'-\varepsilon\xi'/2)))(Z + iD^+_\varepsilon(k'-\varepsilon\xi'/2,\xi))} d\xi' d\xi dk$$
$$\xrightarrow{\varepsilon\to 0} \int_{\mathbb{R}^2\times\mathbb{T}} \frac{\hat{W}_0(\xi',k')\hat{F}^*(\omega(k')/(2\pi))\theta_F(k)\hat{O}^*(\xi,k')}{(Z+i\omega'(k')\xi)(Z+i\omega'(k')\xi')} d\xi' d\xi dk.$$

Similarly, we have

$$\int_{\mathbb{R}\times\mathbb{T}} \frac{\hat{O}^*(\xi, k-\varepsilon\xi/2)S^{1*}_\varepsilon(Z,k,F)}{Z + iD^-_\varepsilon(k,\xi)} d\xi dk \xrightarrow{\varepsilon\to 0} \int_{\mathbb{R}^2\times\mathbb{T}} \frac{\hat{W}^*_0(\xi',k')\hat{F}(\omega(k')/(2\pi))\theta^*_F(k)\hat{O}^*(\xi,k')}{(Z+i\omega'(k')\xi)(Z-i\omega'(k')\xi')} d\xi' d\xi dk.$$

Taking the sum and using the fact that $\hat{W}^*_0(\eta,k) = \hat{W}_0(-\eta,k)$, we get the limit of the first scattering term

$$\int_{\mathbb{R}^2\times\mathbb{T}} \mathrm{Re}\left(\hat{F}^*(\omega(k)/(2\pi))\theta_F(k)\right) \frac{\hat{W}_0(\xi',k)O^*(\xi,k)}{(Z+i\omega'(k)\xi')(Z+i\omega'(k)\xi)} d\xi' d\xi dk.$$

4.4.3. *The second scattering term.* The second scattering term is given by

$$II_\varepsilon(t,k,F) = \int_0^t \int_0^t \Theta^F_\omega(t-s,k)\langle F\star\Psi(s)F\star\Psi(t-s')\rangle_{\mu_\varepsilon} ds\Theta(ds').$$

We compute

$$\mathrm{Re}\, S^{II}_\varepsilon(Z,k,F) = \mathrm{Re}\,\mathcal{L}(II^{(\varepsilon)}_\varepsilon)(Z,k,F)$$
$$= \varepsilon^2 Z \int_0^t e^{-\varepsilon Z t} \left\langle \left(\int_0^t \cos(\omega(k)s)(\Theta_F\star F\star\Psi)(s)ds\right)\left(\int_0^t \cos(\omega(k)s)(\Theta_F\star F\star\Psi)(s)ds\right)\right\rangle_{\mu_\varepsilon} dt \tag{97}$$
$$+ \varepsilon^2 Z \int_0^t e^{-\varepsilon Z t} \left\langle \left(\int_0^t \sin(\omega(k)s)(\Theta_F\star F\star\Psi)(s)ds\right)\left(\int_0^t \sin(\omega(k)s)(g_F\star F\star\Psi)(s)ds\right)\right\rangle_{\mu_\varepsilon} dt. \tag{98}$$

Let us handle the term (97). The term (98) is handled similarly.

We write

$$\frac{\varepsilon^2 Z}{2}\int_0^\infty e^{-\varepsilon Z t}\int_0^t\int_0^t \cos(\omega(k)s)\cos(\omega(k)s')\int_0^s\int_0^{s'}$$
$$\times \int_{\mathbb{T}^2} e^{-i\omega(h)(s-u)}e^{i\omega(h')(s'-u')}\left\langle \hat{\psi}(h)\hat{\psi}^*(h')\right\rangle_{\mu_\varepsilon} dh'dh\Theta_F\star F(du')\Theta_F\star F(du)ds'dsdt$$

$$= \frac{\varepsilon Z}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{T}^2}\varepsilon\left\langle\hat\psi(h)\hat\psi^*(h')\right\rangle_{\mu_\varepsilon}\int_0^\infty\int_0^\infty e^{ia(t-t')}e^{-\varepsilon Z(t+t')/2}\int_0^t\int_0^t\cos(\omega(k)s)\cos(\omega(k)s')$$
$$\times\int_0^s\int_0^{s'}e^{-i\omega(h)(s-u)}e^{i\omega(h')(s'-u')}\Theta_F\star F(du')\Theta_F\star F(du)ds'dsdt'dtdh'dhda$$
$$= \frac{\varepsilon Z}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{T}^2}\varepsilon\left\langle\hat\psi(h)\hat\psi^*(h')\right\rangle_{\mu_\varepsilon}A(h,k,\lambda,F)A^*(h',k,\lambda,F)dh'dhda, \tag{99}$$

in which

$$A(h,k,Z,F) = \int_0^\infty\cos(\omega(k)s)\int_0^s e^{-i\omega(h)(s-u)}\int_s^\infty e^{(-\varepsilon Z/2+ia)t}dt\Theta_F\star F(du)ds$$
$$= \frac{1}{2(\varepsilon Z/2-ia)}\frac{\mathcal{L}(F)(\varepsilon Z/2-ia-i\omega(k))\tilde\Theta_F(\varepsilon Z/2-ia-i\omega(k))}{\varepsilon Z/2-ia-i(\omega(k)-\omega(h))} \tag{100}$$
$$+\frac{1}{2(\varepsilon Z/2-ia)}\frac{\mathcal{L}(F)(\varepsilon Z/2-ia+i\omega(k))\tilde\Theta_F(\varepsilon Z/2-ia+i\omega(k))}{\varepsilon Z/2-ia+i(\omega(k)+\omega(h))}, \tag{101}$$

and A^* is defined similarly, that is

$$A^*(h',k,Z,F) = \frac{1}{2(\varepsilon Z/2+ia)}\frac{\mathcal{L}(F)(\varepsilon Z/2+ia+i\omega(k))\tilde\Theta_F(\varepsilon Z/2+ia+i\omega(k))}{\varepsilon Z/2+ia+i(\omega(k)-\omega(h'))} \tag{102}$$
$$+\frac{1}{2(\varepsilon Z/2+ia)}\frac{\mathcal{L}(F)(\varepsilon Z/2+ia-i\omega(k))\tilde\Theta_F(\varepsilon Z/2+ia-i\omega(k))}{\varepsilon Z/2+ia-i(\omega(k)+\omega(h'))}. \tag{103}$$

We will see that among the four terms in the expansion of (99), using (100), (101), (102) and (103) only the term generated by (100) and (102) contributes to the high frequency limit. To derive the limit, we need some lemmas. Those lemmas are a generalized version of those in [9].

Lemma 7. *The following limit holds true*

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_{\mathbb{R}}\frac{1}{(Z/2)^2+a^2}\int_{\mathbb{T}^3}\left|\left\langle\hat\psi(h)\hat\psi^*(h')\right\rangle_{\mu_\varepsilon}\right|$$
$$\times\left|\frac{\mathcal{L}(F)(\varepsilon Z/2-i\varepsilon a-i\omega(k))\tilde\Theta_F(\varepsilon Z/2-i\varepsilon a-i\omega(k))}{Z/2-ia-i\varepsilon^{-1}(\omega(k)-\omega(h))}\right|$$
$$\times\left|\frac{\mathcal{L}(F)(\varepsilon Z/2+i\varepsilon a-i\omega(k))\tilde\Theta_F(\varepsilon Z/2+i\varepsilon a-i\omega(k))}{Z/2+ia-i\varepsilon^{-1}(\omega(k)+\omega(h'))}\right|dh'dhdkda=0$$

To use this lemma, we change a into εa in (99). It implies that the term

$$\frac{Z}{16\pi\varepsilon}\int_{\mathbb{R}}\frac{1}{(Z/2)^2+a^2}\int_{\mathbb{T}^2}\varepsilon\left\langle\hat\psi(h)\hat\psi^*(h')\right\rangle_{\mu_\varepsilon}$$
$$\times\frac{\mathcal{L}(F)(\varepsilon Z/2-i\varepsilon a-i\omega(k))\tilde\Theta_F(\varepsilon Z/2-i\varepsilon a-i\omega(k))}{Z/2-ia-i\varepsilon^{-1}(\omega(k)-\omega(h))}$$
$$\times\frac{\mathcal{L}(F)(\varepsilon Z/2+i\varepsilon a+i\omega(k))\tilde\Theta_F(\varepsilon Z/2+i\varepsilon a+i\omega(k))}{Z/2+ia+i\varepsilon^{-1}(\omega(k)-\omega(h'))}dh'dhda$$

is the only term contributing in the limit. By changing the variables from h into $k'+\varepsilon\xi'/2$ and h' into $k'-\varepsilon\xi'/2$, we need to consider

$$\frac{Z}{16\pi}\int_{\mathbb{R}\times\mathbb{T}_{2/\varepsilon}}\frac{1}{(Z/2)^2+a^2}\int_{\mathbb{T}\times T^2_\varepsilon}\frac{\hat W_\varepsilon(\xi',k')}{Z/2-ia-i\varepsilon^{-1}(\omega(k)-\omega(k'+\varepsilon\xi'/2))}$$

$$\times \frac{\left|\mathcal{L}(F)(\varepsilon Z/2 - i\varepsilon a - i\omega(k))\tilde{\Theta}_F(\varepsilon Z/2 - i\varepsilon a - i\omega(k))\right|^2}{Z/2 + ia + i\varepsilon^{-1}(\omega(k) - \omega(k' - \varepsilon\xi'/2))} \times \frac{\hat{O}^*(\xi, k + \varepsilon\xi/2)}{Z + iD_\varepsilon^+(k,\xi)} dk' d\xi' dk d\xi da.$$

In the integral, we have the set $T_\varepsilon^2 = \left\{(\xi', k') : |\xi'| \leq \frac{\varrho}{2^{100}\varepsilon}, |k'| \leq \frac{1-\varepsilon|\xi'|}{2}\right\} \subset \mathbb{T}_{2/\varepsilon} \times \mathbb{T}$, ϱ is in (34). The set is defined this way due to the assumption (9). If $|\xi'| > \frac{\varrho}{2^{100}\varepsilon}$ then $|\hat{W}_\varepsilon| \lesssim \varepsilon^{3/2}$; it makes the whole term tend to 0.

We define

$$\mathcal{I}_\pm(Z,\varepsilon,F) = \frac{Z}{32\pi}\int_{\mathbb{R}\times\mathbb{T}_{2/\varepsilon}} \frac{1}{(Z/2)^2 + a^2} \int_{T^3_{\varepsilon,\pm}} \frac{\hat{W}_\varepsilon(\xi', \pm k + \varepsilon\xi''/2)}{Z/2 - ia - i\varepsilon^{-1}(\omega(k) - \omega(k \pm \varepsilon(\xi' + \xi'')/2))}$$
$$\times \frac{\left|\mathcal{L}(F)(\varepsilon Z/2 - i\varepsilon a - i\omega(k))\tilde{\Theta}_F(\varepsilon Z/2 - i\varepsilon a - i\omega(k))\right|^2}{Z/2 + ia + i\varepsilon^{-1}(\omega(k) - \omega(k \mp \varepsilon(\xi' - \xi'')/2))} \times \frac{\hat{O}^*(\xi, k + \varepsilon\xi/2)}{Z + iD_\varepsilon^+(k,\xi)} d\xi'' d\xi' dk d\xi da.$$

In this integral,

$$T^3_{\varepsilon,\pm} = \left\{(k, \xi', \xi'') : k \in \mathbb{T}, |\xi'| \leq \frac{\varrho}{2^{50}\varepsilon}, |k \pm \varepsilon\xi''/2| \leq \frac{1 - \varepsilon|\xi'|}{2}, \operatorname{sign} k = \operatorname{sign}(k \pm \varepsilon(\xi' + \xi'')/2)\right\}.$$

The set is defined thanks to the change of variables k' into $\pm k + \varepsilon\xi''/2$.

We also define the following integrals:

$$\mathcal{I}^{(1)}_\pm(Z,\varepsilon,F) = \frac{Z}{32\pi}\int_{\mathbb{R}\times\mathbb{T}_{2/\varepsilon}} \frac{1}{(Z/2)^2 + a^2} \int_{T^3_{\varepsilon,\pm}} \frac{\hat{W}_\varepsilon(\xi', \pm k + \varepsilon\xi''/2)}{Z/2 - ia \pm i\omega'(k)(\xi' + \xi'')/2}$$
$$\times \frac{\left|\mathcal{L}(F)(\varepsilon Z/2 - i\varepsilon a - i\omega(k))\tilde{\Theta}_F(Z\varepsilon/2 - i\varepsilon a - i\omega(k))\right|^2}{Z/2 + ia + i\varepsilon^{-1}(\omega(k) - \omega(k \mp \varepsilon(\xi' - \xi'')/2))} \times \frac{\hat{O}^*(\xi, k + \varepsilon\xi/2)}{Z + iD_\varepsilon^+(k,\xi)} d\xi'' d\xi' dk d\xi da;$$

$$\mathcal{I}^{(2)}_i(Z,\varepsilon,F) = \frac{Z}{32\pi}\int_{\mathbb{R}\times\mathbb{T}_{2/\varepsilon}} \frac{1}{(Z/2)^2 + a^2} \int_{T^3_{\varepsilon,\pm}} \frac{\hat{W}_\varepsilon(\xi', \pm k + \varepsilon\xi''/2)}{Z/2 - ia \pm i\omega'(k)(\xi' + \xi'')/2}$$
$$\times \frac{\left|\mathcal{L}(F)(\varepsilon Z/2 - i\varepsilon a - i\omega(k))\tilde{\Theta}_F(\varepsilon Z/2 - i\varepsilon a - i\omega(k))\right|^2}{Z/2 + ia \pm i\omega'(k)(\xi' - \xi'')/2} \times \frac{\hat{O}^*(\xi, k + \varepsilon\xi/2)}{Z + iD_\varepsilon^+(k,\xi)} d\xi'' d\xi' dk d\xi da;$$

$$\mathcal{I}^{(3)}_\pm(Z,\varepsilon,F) = \frac{Z}{32\pi}\int_{\mathbb{R}\times\mathbb{T}_{2/\varepsilon}} \frac{1}{(Z/2)^2 + a^2} \int_{T^3_{\varepsilon,\pm}} \frac{\hat{W}_\varepsilon(\xi', \pm k + \varepsilon\xi''/2)}{Z/2 - ia \pm i\omega'(k)(\xi' + \xi'')/2}$$
$$\times \frac{\left|\hat{F}(\omega(k)/(2\pi))\theta_F(k)\right|^2}{Z/2 + ia \pm i\omega'(k)(\xi' - \xi'')/2} \times \frac{\hat{O}^*(\xi, k + \varepsilon\xi/2)}{Z + iD_\varepsilon^+(k,\xi)} d\xi'' d\xi' dk d\xi da;$$

$$\mathcal{I}^{(4)}_\pm(Z,F) = \frac{Z}{32\pi}\int_{\mathbb{R}^2} \frac{1}{(Z/2)^2 + a^2} \int_{\mathbb{T}\times\mathbb{R}^2} \frac{\hat{W}_0(\xi', \pm k)}{Z/2 - ia \pm i\omega'(k)(\xi' + \xi'')/2}$$
$$\times \frac{\left|\hat{F}(\omega(k)/(2\pi))\theta_F(k)\right|^2}{Z/2 + ia \pm i\omega'(k)(\xi' - \xi'')/2} \times \frac{\hat{O}^*(\xi, k)}{Z + i\omega'(k)\xi} d\xi'' d\xi' dk d\xi da.$$

Lemma 8. *We have the following approximations:*

$$\lim_{\varepsilon\to 0}\left|\mathcal{I}_{\pm}(Z,\varepsilon,F)-\mathcal{I}_{\pm}^{(1)}(Z,\varepsilon,F)\right|=0;$$

$$\lim_{\varepsilon\to 0}\left|\mathcal{I}_{\pm}^{(1)}(Z,\varepsilon,F)-\mathcal{I}_{\pm}^{(2)}(Z,\varepsilon,F)\right|=0;$$

$$\lim_{\varepsilon\to 0}\left|\mathcal{I}_{\pm}^{(2)}(Z,\varepsilon,F)-\mathcal{I}_{\pm}^{(3)}(Z,\varepsilon,F)\right|=0;$$

$$\lim_{\varepsilon\to 0}\left|\mathcal{I}_{\pm}^{(3)}(Z,\varepsilon,F)-\mathcal{I}_{\pm}^{(4)}(Z,F)\right|=0.$$

As a consequence,

$$\lim_{\varepsilon\to 0}\mathcal{I}_{\pm}(Z,\varepsilon,F)=\mathcal{I}_{\pm}^{(4)}(Z,F).$$

Lemma 9.

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}\times\mathbb{T}}\operatorname{Im}S_{\varepsilon}^{II}(Z,k,F)\left(\frac{\hat{O}^*(\xi,k+\varepsilon\xi/2)}{Z+iD_{\varepsilon}^{+}(k,\xi)}-\frac{\hat{O}^*(\xi,k-\varepsilon\xi/2)}{Z+iD_{\varepsilon}^{-}(k,\xi)}\right)d\xi dk=0.$$

We apply to $\mathcal{I}_i^{(4)}$ the following identity

$$\int_{\mathbb{R}}\frac{dz}{(z-z_+)(z-z_-)}=\frac{2\pi i}{z_+-z_-}\ \text{when}\ \operatorname{Im}(z_+)>0>\operatorname{Im}(z_-).$$

The variable z is $a\mp\omega'(k)\xi''$ in our case, $d\xi''$ is integrated out first. Then, we also integrate with respect to a:

$$\begin{aligned}\mathcal{I}_{\pm}^{(4)}(\lambda,F)&=\frac{Z}{16}\int_{\mathbb{R}^2}\frac{1}{(Z/2)^2+a^2}\int_{\mathbb{T}\times\mathbb{R}}\frac{|\hat{F}(\omega(k)/(2\pi))|^2|\theta_F(k)|^2\hat{W}_0(\xi',\pm k)}{|\omega'(k)|(Z\pm i\omega(k)\xi')}\times\frac{\hat{O}(\xi,k)}{Z+i\omega'(k)\xi}d\xi' dk d\xi da\\&=\int_{\mathbb{T}\times\mathbb{R}^2}\frac{|\hat{F}(\omega(k)/(2\pi))|^2|\theta_F(k)|^2\hat{W}_0(\xi',\pm k)}{16|v_g(k)|(Z\pm i\omega(k)\eta')}\times\frac{\hat{O}(\xi,k)}{Z+i\omega'(k)\xi}d\xi' dk d\xi.\end{aligned}$$

By repeating these steps, the term (98) will produce the same result as (97). That concludes the computation for $S_{\varepsilon}^{II}(Z,k,F)$. We also have the same result for $\frac{\hat{O}^*(k-\varepsilon\xi/2)}{Z+iD_{\varepsilon}^{-}(k,\xi)}$, so the result get doubled. The second scattering term limit is

$$\begin{aligned}&\int_{\mathbb{T}\times\mathbb{R}^2}\frac{|\hat{F}(\omega(k)/(2\pi))|^2|\theta_F(k)|^2\hat{W}_0(\xi',k)}{4|v_g(k)|(Z+i\omega(k)\xi')}\times\frac{\hat{O}(\xi,k)}{Z+i\omega'(k)\xi}d\xi' d\xi dk\\&+\int_{\mathbb{T}\times\mathbb{R}^2}\frac{|\hat{F}(\omega(k)/(2\pi))|^2|\theta_F(k)|^2\hat{W}_0(\xi',-k)}{4|v_g(k)|(Z-i\omega(k)\xi')}\times\frac{\hat{O}(\xi,k)}{Z+i\omega'(k)\xi}d\xi' d\xi dk.\end{aligned}$$

4.5. **Proof of Theorem 3.** The technical detail for this part is similar to that of the impulsive control. We see that (81), (82), (83), and (84) do not depend on the rates. Thus, we do not repeat the same arguments to derive the theorem although the rates are different.

4.6. **Proof of Corollary 5.** In this section, we show how to design controls that asymptotically steer the system to some desired rates (r_a, r_t, r_r) in the kinetic limit. First, we state some basic properties that the rates must satisfy and the implications in finding a control.

(1) By (88), (89), and (90), the target rates must be defined when $\omega'(k) \neq 0$. Hence the domain of definition is $(-1/2, 0) \cup (0, 1/2)$. Since $\omega(\cdot)$ is an even function, the three rates are even functions. We reduce the domain of definition to the positive branch $(0, 1/2)$.
(2) By (95), the sum of the three rates is 1 and hence we only need to know two among the three rates (r_a, r_t, r_r). We focus on r_t and r_r.
(3) All the rates are positive functions. As a result, we have $r_t^F + r_r^F < 1$. To simplify we assume that there exists $c_1 > 0$ such that
$$r_t(k) + r_r(k) \leq 1 - c_1 \qquad \forall k \in \mathbb{T}.$$
(4) For $k \in (0, 1/2)$, from [9, Section 10], we have
$$\lim_{Z\to 0} \mathcal{L}(C_\omega)(Z - i\omega(k)) = \frac{\pi}{|\omega'(k)|} + i\left(\int_0^{1/2} \frac{dh}{\omega(k)+\omega(h)} + \log\frac{\omega(k)-\omega_{min}}{\omega_{max}-\omega(k)} + \frac{C_{\omega,0}(\omega(k))}{|\omega'(k)|}\right),$$
where $C_{\omega,0}$ is a continuous, bounded and real-valued function. Combining this with the continuity of $\hat{F}$, it follows that the rates are continuous on (0,1/2).
(5) By the triangular inequality, we have
$$\sqrt{r_t^F(k)} + \sqrt{r_r^F(k)} = \left|1 + \frac{\hat{F}^*(\omega(k)/(2\pi))\theta_F(k)}{2|v_g(k)|}\right| + \left|\frac{\hat{F}^*(\omega(k)/(2\pi))\theta_F(k)}{2|v_g(k)|}\right| \geq 1. \tag{104}$$

To sum up, we consider two continuous functions r_t, r_r on $(0, 1/2)$ such that $r_t(k) + r_r(k) \leq 1 - c_1$ and $\sqrt{r_t(k)} + \sqrt{r_r(k)} \geq 1$. These are assumptions (H1) - (H5).

We define
$$RE(k) = |v_g(k)|\,(r_t(k) - r_r(k) - 1) \qquad \forall k \in (0, 1/2).$$
It follows from (89) and (90) that $RE(k) = \mathrm{Re}(\hat{F}(\omega(k)/(2\pi))\theta_F(k))$.

We also define
$$IM(k) = \sqrt{4|v_g(k)|^2 r_r(k) - RE(k)^2} \qquad \forall k \in (0, 1/2),$$
which is well-defined because, using (104),
$$\begin{aligned} 4|v_g(k)|^2 r_r(k) - RE(k)^2 = |v_g(k)|^2(\sqrt{r_t(k)} + \sqrt{r_r(k)} - 1)(\sqrt{r_t(k)} - \sqrt{r_r(k)} + 1) \\ \times (-\sqrt{r_t(k)} + \sqrt{r_r(k)} + 1)(\sqrt{r_t(k)} + \sqrt{r_r(k)} + 1) \geq 0. \end{aligned}$$

Finally, we define a complex-valued function $FT(k)$, which corresponds to $\hat{F}(\omega(k)/(2\pi))\theta_F(k)$ by
$$FT(k) = RE(k) + iIM(k) \qquad \forall k \in (0, 1/2).$$

By (94), the function corresponding to $\theta_F(k)$ is defined by
$$TH(k) = 1 + FT(k) \lim_{Z\to 0} \mathcal{L}(C_\omega)(Z - i\omega(k)).$$

We claim that $|TH(k)| \geq c_1/4$ for any $k \in (0, 1/2)$. To prove this claim, we consider two cases: either $\mathrm{Im}(\lim_{\lambda\to 0} \mathcal{L}(C_\omega)(\lambda - i\omega(k))) \geq 0$ or $\mathrm{Im}(\lim_{\lambda\to 0} \mathcal{L}(C_\omega)(\lambda - i\omega(k))) < 0$. In the first case,
$$\mathrm{Re}(TH(k)) = 1 + RE(k)\,\mathrm{Re}(\lim_{Z\to 0} \mathcal{L}(C_\omega)(Z - i\omega(k))) + IM(k)\,\mathrm{Im}(\lim_{Z\to 0} \mathcal{L}(C_\omega)(Z - i\omega(k)))$$

$$\geq 1 - \frac{1 + r_r(k) - r_t(k)}{2} = \frac{1 + r_t(k) - r_r(k)}{2} \geq r_t(k) + c_1/2.$$

In the second case, we have either

$$|IM(k)\,\mathrm{Im}(\lim_{Z\to 0}\mathcal{L}(C_\omega)(Z - i\omega(k)))| \leq \frac{1 + r_t(k) - r_r(k)}{4}$$
$$\Rightarrow \mathrm{Re}(TH(k)) \geq \frac{1 + r_t(k) - r_r(k)}{4} \geq \frac{r_t(k)}{2} + \frac{c_1}{4},$$

or

$$|IM(k)\,\mathrm{Im}(\lim_{Z\to 0}\mathcal{L}(C_\omega)(Z - i\omega(k)))| > \frac{1 + r_t(k) - r_r(k)}{4},$$

and then

$$\mathrm{Im}(TH(k)) = RE(k)\,\mathrm{Im}(\lim_{Z\to 0}\mathcal{L}(C_\omega)(Z - i\omega(k))) + IM(k)\,\mathrm{Re}(\lim_{Z\to 0}\mathcal{L}(C_\omega)(Z - i\omega(k)))$$
$$> \sqrt{(1 + r_r(k) - r_t(k))(1 + r_t(k) - r_r(k))}/\sqrt{2} \geq \frac{\sqrt{c_1 + 2r_t(k)}\sqrt{c_1 + 2r_r}(k)}{\sqrt{2}}.$$

The claim is proved. In particular, $TH(k) \neq 0$ for every $k \in (0, 1/2)$, so we can define $\bar{F}(u) = FT(\varphi_+(u))/TH(\varphi_+(u)), u \in (\omega_{\min}, \omega_{\max})$, which corresponds to $\hat{F}^*(\omega(k)/2\pi)$.

The next step consists of computing the inverse Fourier of $\bar{F}$. Recall that the control function F is real-valued and is only defined on $[0, +\infty)$ to facilitate the Laplace transform. It is more convenient to use the half-line Fourier transform defined by

$$f^c(s) = \int_0^\infty f(t)\cos(st)dt,$$
$$f^s(s) = \int_0^\infty f(t)\sin(st)dt,$$

for all $f \in L^1([0, +\infty)) \cap L^2([0, +\infty))$. We obtain $f^c, f^s \in L^2([0, +\infty))$. It is also possible to define the inverses in the case the transformed functions are also in L^1 by

$$f(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(t)\cos(st)dt\cos(sx)ds,$$
$$f(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(t)\sin(st)dt\sin(sx)ds.$$

By (15), if there is a control $F \in L^1$, we have

$$\lim_{Z\to 0}\mathcal{L}(F)(Z + i\omega) = F^c(\omega) - iF^s(\omega).$$

We see that $\mathrm{Re}(\bar{F})$ and $\mathrm{Im}(\bar{F})$ correspond to F^c and F^s, respectively. Thus, the existence of F depends on whether we have

$$\int_0^\infty \mathrm{Re}(\bar{F}(u))\cos(ut)du = \int_0^\infty \mathrm{Im}(\bar{F}(u))\sin(ut)du,$$

for all $t > 0$. This happens because if F exists then both sides equal $\frac{\pi}{2}F(t)$. Therefore, on the target rates, we expect them to satisfy assumptions (H1) - (H6).

For a triple of rates that satisfy assumptions (H1) - (H6), let us design an explicit control function F by

$$F(t) = \frac{2}{\pi}\int_0^\infty \mathrm{Re}(\bar{F}(u))\cos(ut)du.$$

It is easy to see that $\bar{F}$ is continuous. Therefore,

$$\frac{d^n}{dt^n}F(t) = \frac{2}{\pi}\int_0^\infty \mathrm{Re}(\bar{F}(u))\frac{d^n}{dt^n}\cos(ut)du$$

are continuous for all n. This means that F must belong to $C^\infty(0,+\infty)$. We also note that $|\bar{F}| \leq \frac{8|v_g(k)|r_r(k)}{c_1}$ and $\bar{F}$ is supported on $[\omega_{\min}, \omega_{\max}]$, in particular $\bar{F} \in L^2(0,+\infty)$, and thus also $F \in L^2(0,+\infty)$.

We define a family of function $F_N \in C_c^\infty \subset C^\infty \cap L^m, m \geq 1$. For each $N \in \mathbb{N}$, F_N is the product of F with a smooth function with values in $[0,1]$ such that it is equal to 1 on $[0,N]$ and to 0 on $[N+1,+\infty)$. It is clear that F_N converges to F in L^2. Thus, F_N^c, F_N^s converge to $F^c = \mathrm{Re}(\bar{F}), F^s = \mathrm{Im}(\bar{F})$ in L^2, respectively. Take a subsequence if necessary, we obtain a family F_N such that $\hat{F}_N^*(\cdot/2\pi)$ converges almost everywhere to $\bar{F}$. We have

$$FT(k) = \frac{\bar{F}(\omega(k))}{1 - \bar{F}(\omega(k))\lim_{Z\to 0}\mathcal{L}(C_\omega)(Z - i\omega(k))}$$

and

$$\hat{F}_N^*(\omega(k)/2\pi)\theta_F(k) = \frac{\hat{F}_N^*(\omega(k)/2\pi)}{1 - \hat{F}_N^*(\omega(k)/2\pi)\lim_{Z\to 0}\mathcal{L}(C_\omega)(Z - i\omega(k))}.$$

Hence, $\hat{F}_N^*(\omega(k)/2\pi)\theta_F(k)$ converges almost everywhere to $FT(k)$. We also have

$$r_r(k) = \frac{RE(k)^2 + IM(k)^2}{4|v_g(k)|^2} = \frac{|FT(k)|^2}{4|v_g(k)|^2},$$
$$r_t(k) = 1 + \frac{RE(k)}{|v_g(k)|} + r_r(k) = 1 + \frac{\mathrm{Re}(FT(k))}{|v_g(k)|} + \frac{|FT(k)|^2}{4|v_g(k)|^2}.$$

By (89) and (90), we get that $r_t^{F_N}, r_r^{F_N}$ converge almost everywhere to r_t, r_r on $(0,1/2)$. By (H1) and (H2), the result is extended to $r_a^{F_N}, r_t^{F_N}, r_r^{F_N}$ converge almost everywhere to r_a, r_t, r_r on $\mathbb{T}$.

References

[1] Giacomo Albi, Michael Herty, and Lorenzo Pareschi. Kinetic description of optimal control problems and applications to opinion consensus. *Communications in Mathematical Sciences*, 13:1407–1429, 2015.

[2] G. Basile, S. Olla, and H. Spohn. Energy transport in stochastically perturbed lattice dynamics. *Archive for Rational Mechanics and Analysis*, 195(1):171–203, 2010.

[3] L. Borcea and J. Garnier. Derivation of a one-way radiative transfer equation in random media. *Physical Review E*, 93(2):022115, 2016.

[4] J. A. Carrillo, D. Kalise, F. Rossi, and E. Trélat. Controlling swarms toward flocks and mills. *SIAM Journal on Control and Optimization*, 60(3):1863–1891, 2022.

[5] A. Hannani, M. Rosenzweig, G. Staffilani, and M.-B. Tran. On the wave turbulence theory for a stochastic KdV type equation–generalization for the inhomogeneous kinetic limit. *arXiv preprint arXiv:2210.17445*, 2022.

[6] T. Komorowski and S. Olla. Kinetic limit for a chain of harmonic oscillators with a point langevin thermostat. *Journal of Functional Analysis*, 279(12):108764, 2020.

[7] Tomasz Komorowski, Joel L Lebowitz, and Stefano Olla. Heat flow in a periodically forced, thermostatted chain. *Communications in Mathematical Physics*, pages 1–45, 2023.

[8] Tomasz Komorowski and Stefano Olla. Asymptotic scattering by Poissonian thermostats. In *Annales Henri Poincaré*, volume 23, pages 3753–3790. Springer, 2022.

[9] Tomasz Komorowski, Stefano Olla, Lenya Ryzhik, and Herbert Spohn. High frequency limit for a chain of harmonic oscillators with a point Langevin thermostat. *Archive for Rational Mechanics and Analysis*, 237, 07 2020.

[10] Jani Lukkarinen and Herbert Spohn. Kinetic limit for wave propagation in a random medium. *Archive for Rational Mechanics and Analysis*, 183:93–162, 01 2007.

[11] Alexander Mielke. Macroscopic behavior of microscopic oscillations in harmonic lattices via wigner-husimi transforms. *Archive for Rational Mechanics and Analysis*, 181:401–448, 01 2006.
[12] B. Piccoli, F. Rossi, and E. Trélat. Control to flocking of the kinetic Cucker–Smale model. *SIAM Journal on Mathematical Analysis*, 47(6):4685–4719, 2015.
[13] Leonid Ryzhik, George Papanicolaou, and Joseph Keller. Transport equations for elastic and other waves in random media. *Wave Motion*, 24:327–370, 12 1996.
[14] G. Staffilani and M.-B. Tran. On the wave turbulence theory for a stochastic KdV type equation. *arXiv preprint arXiv:2106.09819*, 2021.
[15] E. Wigner. On the quantum correction for thermodynamic equilibrium. *Phys. Rev.*, 40:749–759, Jun 1932.

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