



HAL
open science

Optimal structures of crop irrigation strategies with state constraints

Ruben Chenevat, Bruno Cheviron, Sébastien Roux, Alain Rapaport

► **To cite this version:**

Ruben Chenevat, Bruno Cheviron, Sébastien Roux, Alain Rapaport. Optimal structures of crop irrigation strategies with state constraints. *Journal of Optimization Theory and Applications*, 2026, 208 (18), <10.1007/s10957-025-02854-7>. <hal-04506794v2>

HAL Id: hal-04506794

<https://hal.science/hal-04506794v2>

Submitted on 28 Mar 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Optimal structures of crop irrigation strategies with state constraints

Ruben Chenevat¹, Bruno Cheviron², Sébastien Roux¹ and Alain Rapaport^{1‡}

March 28, 2025

Abstract

We investigate an optimal control problem of crop irrigation with non-autonomous and non-smooth dynamics. Depending on contexts and objectives, several formulations associated to different constraints and criteria can be derived. Our work aims at providing optimal feedback solutions for these problems by deriving and analyzing the optimality necessary conditions. To this end, we assemble the different problems into a common formulation, and we carry out a dedicated way of handling state constraints. We show that all optimal irrigation strategies belong to a family of simple parameterized time-varying feedback controls, independently of the context and objective, and suitable for computational purposes.

Contents

1	Introduction	2
2	Model description and hypotheses	2
2.1	Basis model description	2
2.2	Framework and constraints	4
2.3	Problems formulation	4
2.4	Preliminaries about the trajectory of S	5
2.4.1	First lemmas	5
2.4.2	Most Rapid Approach Path	6
3	Equivalent formulation and preliminary properties	8
3.1	Compensation technique	8
3.2	Properties of optimal solutions	10
4	Optimal synthesis	14
4.1	Application of the Pontryagin Maximum Principle	14
4.1.1	Optimality necessary conditions	14
4.1.2	Properties of adjoints	15
4.2	Structures of optimal solutions	17
4.2.1	Singular arcs	17
4.2.2	Analysis of switching function	17
4.3	Main result	18
5	Conclusion	19

*This work is funded by INRAE-MathNum and Labex Numev (ANR-10-LABX-0020), and supported by the French National Research Agency under the Investments for the Future Program, referred as ANR-16-CONV-0004. The authors are also thankful to the French Ministry for Europe and Foreign Affairs, and the French Ministry for Higher Education and Research for the “PHC PESSOA” programme (project number 52874SG : OITE (Optimizing Irrigation efficiency under Technical and Environmental constraints)).

^{†1}Ruben Chenevat, Sébastien Roux and Alain Rapaport are with MISTEA, Univ. Montpellier, INRAE, Institut Agro, 34060 Montpellier, France {ruben.chenevat, sebastien.roux, alain.rapaport} @inrae.fr

^{‡2}Bruno Cheviron is with G-EAU, Univ Montpellier, AgroParisTech, BRGM, CIRAD, INRAE, Institut Agro, IRD, Montpellier, France bruno.cheviron@inrae.fr

1 Introduction

The development of agriculture plays a significant role in the worldwide management of natural resources. Among all the consequent challenges, the management of water resources is critical especially in areas where the scarce precipitations need to be more and more supported by irrigation. It is possible to save water through different agronomic and technological approaches, such as selecting stress resilient species, upgrading equipment performances, improving irrigation scheduling. Practices based on enhancing irrigation planning [24], coupled with a framework of deficit irrigation [8, 20] show promising potential to increase efficiency.

Numerous studies tackle the agricultural water management by incorporating modeling tools, such as complex numerical models to describe precisely the physical processes of plant growth [1, 4, 13]. They typically present a detailed structure for predefined crop or regional context [17, 30], and contain significant constraints due to biological requirements. Their complexity implies computational limitations, it is thus reasonable to allow a more parsimonious description in order to improve flexibility and analytical capacity. To this end, irrigation planning is seen as a control problem, and using advanced control theory show good potential to improve efficiency in agriculture [7, 21]. Some studies have explored model predictive control [6, 22, 25] targeting adaptive and robust strategies regarding unknown entries like weather or model uncertainties, through numerical simulations. In a more deterministic approach towards analytical strategies, some studies addressed the irrigation problem using optimal control tools [2, 16, 18, 26]. These works explore models based on simple dynamical systems and decision making through only a few variables, and require a limited number of hypotheses on the physical processes. According to their various assumptions and objectives some models consider more or less detailed soil water balance and more or less constrained contexts.

The desired benefit of using analytical models is to aim for strategies depicting optimal irrigation thanks to time-varying feedback controls suitable for the low number of model inputs and state variables that are related to quantities accessible in real world. Several major agronomic objectives match with a formulation of a constrained optimization problem : maximizing the biomass production with a water availability constraint, minimizing the consumption of water to reach a yield target, maximizing a financial cost related to production sales and water expenses, avoiding harmful stress due to critically low level of soil moisture. In this work, we investigate these problems with a unified methodological point of view, and we employ a model that we call Controlled Crop Irrigation (CCI) model, inspired from [2, 14, 19]. This model is prone to analytical optimal control techniques (even though it presents non-smooth features) and is thought to possess enough physical sense to describe the order-1 biophysical processes, especially for greenhouse application. The various objectives discussed previously lead to the formulation of three optimization problems (according to three different couples of criterion and target, all fulfilling a state constraint) that we study in a unified procedure. In order to look for a common structure of the optimal irrigation strategies in terms of feedback, we shall establish and derive the optimality necessary conditions. There exists various ways to deal with state constraints ([11, 23]), but in our study we introduce an original "compensation technique", and we work out an equivalent problem formulation without state constraint on which we can apply a standard unconstrained Pontryagin Maximum Principle [5].

In Section 2, we provide a description of the CCI model and its hypotheses, then we present the different criteria and targets that we will study and we formalize their associated optimal control problems. After explaining the compensation technique, that we use, leading to equivalent problem formulations freed from state constraints, we show in Section 3 some necessary properties of optimal solutions. Then we apply in Section 4 the non-smooth Pontryagin Maximum Principle and we derive the optimality necessary conditions to finally get optimal structures of irrigation strategies.

2 Model description and hypotheses

2.1 Basis model description

We consider the Controlled Crop Irrigation model developed by [14], inspired from [19] and simplified for control theory purposes. A first analysis of these dynamics was made in [2] and we extend the study with targets, criteria and constraints as discussed below. The CCI model relies on a soil moisture balance incorporating plant transpiration, soil

evaporation and water irrigation input. It also tracks biomass production and water consumed through irrigation.

$$\begin{cases} \dot{S}(t) = k_1 \left(-\varphi(t)K_S(S(t)) - (1 - \varphi(t))K_R(S(t)) + k_2 u(t) \right) \\ \dot{B}(t) = \varphi(t)K_S(S(t)) \\ \dot{V}(t) = u(t) \end{cases} \quad (1)$$

The system is supposed to progress over a fixed time period $[0, T]$ corresponding to the crop season from sowing to harvest. The soil moisture is represented by $S \in [0, 1]$, B is the production of biomass and V denotes the water consumption. Irrigation is represented by the control $u(\cdot)$ which is a measurable function defined on $[0, T]$ and taking values in $[0, 1]$. The parameters k_1 and k_2 are constants depending on agronomic and operational characteristics. The value k_1 is the ratio of the daily reference evapotranspiration over the volume of water reservoir, and the value k_2 is the ratio of the maximal irrigation rate over the daily reference evapotranspiration.

Hypothesis 1. *The constants k_1 and k_2 satisfy :*

$$k_1 > 0 \quad , \quad k_2 > 1$$

The function φ models the radiation interception efficiency, related to the growth of plant cover.

Hypothesis 2. *Assume that $\varphi : [0, T] \rightarrow [0, 1]$ is increasing and of class C^1 .*

The functions K_S and K_R respectively describe the variations of soil water content through plant transpiration and soil evaporation.

Hypothesis 3. *The functions K_S and K_R are defined for $S \in [0, 1]$, and are non-decreasing, piecewise affine, continuous. Typically, K_S and K_R are found in the literature ([2, 19]) as :*

$$K_S(S) = \begin{cases} 0 & , S \in [0, S_w) \\ \frac{S - S_w}{S^* - S_w} & , S \in [S_w, S^*) \\ 1 & , S \in [S^*, 1] \end{cases} \quad (2a)$$

$$K_R(S) = \begin{cases} 0 & , S \in [0, S_h) \\ \frac{S - S_h}{1 - S_h} & , S \in [S_h, 1] \end{cases} \quad (2b)$$

where $0 < S_h < S_w < S^* < 1$ represent respectively soil water levels depending on soil and plant characteristics (residual water point, wilting point, point of incipient water stress).

Remark 1. *It is possible to consider more generic shapes than Eqs. (2a) and (2b) (with more non-differentiable points for example), as long as they remain non-decreasing, piecewise affine, continuous, and verify K_S concave over $[S_w, 1]$ and K_R convex over $[0, 1]$. Notice that K_S and K_R are Lipschitz, and denote $L_S > 0$ and $L_R > 0$ their respective Lipschitz constant.*

Hypothesis 4. *The system has the following initial conditions :*

$$S(0) = S_0 \in (S^*, 1] \quad , \quad B(0) = 0 \quad , \quad V(0) = 0$$

Remark 2. *It is reasonable to consider a quite high initial soil moisture, since the crop season begins after the year period with usually wet weather.*

Remark 3. *The condition $k_2 > 1$ ensures local controllability of the S variable, in the sense that it is possible to select controls in order to chose the sign of \dot{S} over $[0, T]$. Moreover, the control keeping soil moisture constant at a fixed value is locally admissible.*

Definition 2.1. Under Hypothesis 1, the following control :

$$u_{sing}(t, \tilde{S}) = \frac{\varphi(t)K_S(\tilde{S}) + (1 - \varphi(t))K_R(\tilde{S})}{k_2} \quad (3)$$

belongs to $[0, 1]$, for any $\tilde{S} \in [0, 1]$.

2.2 Framework and constraints

According to the context of deficit irrigation, we restrict our study to (optimal) trajectories inducing water stress. In other words, avoiding water stress means keeping the term $K_S(S(t))$ equal to 1, i.e. maintaining the soil moisture S above the threshold S^* . Denote by $S_{i_i, S_i, u(\cdot)}(\cdot)$ the trajectory of S that takes the value S_i at the time t_i and is controlled by $u(\cdot)$. Let us introduce $t^* = \sup\{t \in [0, T], S_{0, S_0, 0}(t) > S^*\}$, and define :

$$u^*(t) = \begin{cases} 0 & , t \in [0, t^*) \\ u_{sing}(t, S^*) & , t \in [t^*, T] \end{cases} \quad (4)$$

where u_{sing} is defined in Eq. (3). This control thus induces a limit trajectory where no water stress is encountered. We introduce the corresponding limit values for biomass production and water consumption in this context.

Definition 2.2. Using the control u^* defined above, the associated biomass production and water consumption are :

$$B^* = \int_0^T \varphi(t) dt \quad (5a)$$

$$V^* = \int_0^T u^*(t) dt \quad (5b)$$

In addition to the context of deficit irrigation, we consider a state constraint on the soil moisture, representing the biological need to prevent critically low levels of soil water content. More precisely, the system is subject to :

$$\forall t \in [0, T], S_{tol} \leq S(t) \leq 1 \quad (6)$$

where S_{tol} is called tolerance threshold, and verifies $S_w < S_{tol} < S^*$.

Similarly as above, we introduce $t_{tol} = \sup\{t \in [0, T], S_{0, S_0, 0}(t) > S_{tol}\}$, and we define the control inducing the limit trajectory where soil moisture merely fulfills the tolerance threshold constraint.

$$u_{tol}(t) = \begin{cases} 0 & , t \in [0, t_{tol}) \\ u_{sing}(t, S_{tol}) & , t \in [t_{tol}, T] \end{cases} \quad (7)$$

As previously we introduce the corresponding limit values for biomass production and water consumption.

Definition 2.3. Using the control u_{tol} defined above, the associated biomass production and water consumption are :

$$B_{tol} = \int_0^T \varphi(t) K_S(S_{0, S_0, u_{tol}(\cdot)}(t)) dt \quad (8a)$$

$$V_{tol} = \int_0^T u_{tol}(t) dt \quad (8b)$$

As a consequence, we set our study gathering deficit irrigation and biological constraint : each biomass target Y considered will satisfy $B_{tol} < Y < B^*$, and each water consumption target Q considered must fulfill $V_{tol} < Q < V^*$.

2.3 Problems formulation

The problem of optimizing crop irrigation might be seen in various ways according to classical agronomic objectives. We introduce three optimization problems corresponding to three natural points of view about the operational targets and the criteria to be optimized.

- (P1) : maximizing the biomass production at the end of the season, when using a limited quota of water, i.e. maximizing $B(T)$ under the terminal condition $V(T) \leq Q$ where Q is the initial available water.
- (P2) : minimizing the water consumption to reach a certain target of biomass, i.e. minimizing $V(T)$ under the terminal condition $B(T) \geq Y$ where Y is the objective of biomass production at the end of the season.
- (P3) : maximizing a financial balance between biomass sales and water consumption expenses, i.e. maximizing the value $g(B(T)) - c(V(T))$ where g and c are functions representing the earnings and spendings, with $g(0) = c(0) = 0$, $g' > 0$ and $c' > 0$.

We introduce the associated sets of admissible controls for each problem.

Definition 2.4. For any control $u(\cdot)$, consider its associated solution to Eq. (1) denoted by $(S(\cdot), B(\cdot), V(\cdot))$, and define :

$$\begin{aligned}\mathcal{U}^{P1} &= \{u \in L^1([0, T], [0, 1]); S(\cdot) \text{ verifies (6), } V(T) \leq Q, V_{tol} < Q < V^*\} \\ \mathcal{U}^{P2} &= \{u \in L^1([0, T], [0, 1]); S(\cdot) \text{ verifies (6), } B(T) \geq Y, B_{tol} < Y < B^*\} \\ \mathcal{U}^{P3} &= \{u \in L^1([0, T], [0, 1]); S(\cdot) \text{ verifies (6)}\}\end{aligned}$$

Consequently, the three considered problems are formally written as the following optimal control problems.

Definition 2.5.

$$\begin{aligned}(P1) : & \sup_{u(\cdot) \in \mathcal{U}^{P1}} B(T) \\ (P2) : & \inf_{u(\cdot) \in \mathcal{U}^{P2}} V(T) \\ (P3) : & \sup_{u(\cdot) \in \mathcal{U}^{P3}} g(B(T)) - c(V(T)) \quad , g' > 0, c' > 0\end{aligned}$$

Remark 4. From the theory of optimal control (see [15, 29] for instance), our problems satisfy the usual assumptions that guarantee that the above supremum and infimum are reached.

As mentioned in the introduction, these optimal control problems present several levels of difficulty. Indeed the model dynamics are non-autonomous and non-smooth, trajectories are subject to running state constraints and admissible controls are constrained to additional terminal conditions.

2.4 Preliminaries about the trajectory of S

2.4.1 First lemmas

Lemma 2.1. Consider a control $u \in \mathcal{U}^{Pj}$ and denote $S(\cdot)$ its associated solution of Eq. (1). Then :

$$\forall t \in [0, T], S(t) > S_h$$

Proof Denote $S^0(\cdot)$ the solution of the system Eq. (1) using the constant control $u^0 = 0$. Since $u \geq 0$, we have $\dot{S}(t) \geq \dot{S}^0(t)$ for all $t \in [t_0, T]$, and then $S(t) \geq S^0(t)$ for all $t \in [t_0, T]$.

Moreover the trajectory $S_c(\cdot)$ such that $S_c(t) = S_h$ for all $t \in [0, T]$ is a solution of the differential equation of the state variable S using the constant control u^0 . By Cauchy-Lipschitz Theorem, we know that two trajectories of the same equation with different initial conditions don't coincide at any time. Thus for all $t \in [0, T]$ one has $S^0(t) > S_c(t)$. As a consequence, one gets $S(t) > S_h$ for all $t \in [0, T]$. \square

Lemma 2.2. Let $S_1(\cdot)$ and $S_2(\cdot)$ be two solutions of the system Eq. (1) and $0 \leq t_1 < t_2 \leq T$ such that :

$$S_1(t_1) = S_2(t_1) \quad \text{and} \quad S_1(t) \geq S_2(t) \quad \text{for all } t \in [t_1, t_2]$$

Let u_1, u_2 be their corresponding controls. Then :

$$\forall t \in [t_1, t_2], \quad \int_{t_1}^t u_1(\tau) d\tau \geq \int_{t_1}^t u_2(\tau) d\tau$$

In addition, if the controls u_1 and u_2 belong to \mathcal{U}^{Pj} , then this inequality is strict unless the trajectories of S_1 and S_2 are identical.

Proof Let $t \in [t_1, t_2]$. One has :

$$S_1(t) - S_2(t) = k_1 \int_{t_1}^t h(\tau, S_2(\tau)) - h(\tau, S_1(\tau)) + k_1 k_2 \int_{t_1}^t u_1(\tau) - u_2(\tau) d\tau \quad (9)$$

where $h(t, S) = \varphi(t)K_S(S) + (1 - \varphi(t))K_R(S)S$ is an non-decreasing function with respect to the variable S . Since $S_1 \geq S_2$ over the interval $[t_1, t_2]$, we get from Eq. (9) :

$$\forall t \in [t_1, t_2], 0 \leq S_1(t) - S_2(t) \leq k_1 k_2 \int_{t_1}^t u_1(\tau) - u_2(\tau) d\tau$$

Hence one has :

$$\forall t \in [t_1, t_2], \int_{t_1}^t u_1(\tau) d\tau \geq \int_{t_1}^t u_2(\tau) d\tau \quad (10)$$

Moreover, if $u_1, u_2 \in \mathcal{U}^{Pj}$, we know from Lemma 2.1 that their associated trajectories are above S_h . Since the function h is increasing w.r.t. $S \in [S_h, 1]$, then the inequality Eq. (10) is strict unless $S_1(\cdot)$ and $S_2(\cdot)$ are identical. \square

2.4.2 Most Rapid Approach Path

In this section, we introduce the notion of Most Rapid Approach Path (MRAP, as in [9, 10, 12]), which entail particular trajectories similar to exact turnpike ([27, 28]). They are useful when building a control to be compared to another, as in Propositions 3.1 and 3.3. We illustrate their definition in Fig. 1.

Definition 2.6. Let $0 < \bar{S} < 1$.

For any $(\theta, \sigma) \in [0, T] \times [\bar{S}, 1]$, we define :

$$t_{\bar{S}}^+(\theta, \sigma) = \begin{cases} T & , \text{ if } S_{\theta, \sigma, 0}(t) > \bar{S} \text{ for all } t \in [\theta, T] \\ \inf\{t > \theta, S_{\theta, \sigma, 0}(t) = \bar{S}\} & , \text{ otherwise} \end{cases}$$

For any $(\theta, \sigma) \in (0, T] \times [\bar{S}, 1]$, we define :

$$t_{\bar{S}}^-(\theta, \sigma) = \begin{cases} 0 & , \text{ if } S_{\theta, \sigma, 1}(t) > \bar{S} \text{ for all } t \in [0, \theta] \\ \sup\{t < \theta, S_{\theta, \sigma, 1}(t) = \bar{S}\} & , \text{ otherwise} \end{cases}$$

Definition 2.7 (MRAP from above). Let $0 < \bar{S} < 1$.

For any $(\theta_1, \sigma_1) \in [0, T] \times [\bar{S}, 1]$ and $(\theta_2, \sigma_2) \in (0, T] \times [\bar{S}, 1]$ such that σ_2 is attainable at the time θ_2 from (θ_1, σ_1) with some admissible control $u(\cdot)$, we call "MRAP to the value \bar{S} from above" the control $\tilde{u}_{\bar{S}}^{\bar{S}}(\cdot)$ defined on $[\theta_1, \theta_2]$ by the following :

(i) If $t_{\bar{S}}^+(\bar{S}, \theta_1, \sigma_1) \leq t_{\bar{S}}^-(\bar{S}, \theta_2, \sigma_2)$:

$$\tilde{u}_{\bar{S}}^{\bar{S}}(t) = \begin{cases} 0 & , \text{ if } t \in [\theta_1, t_{\bar{S}}^+(\bar{S}, \theta_1, \sigma_1)) \\ u_{sing}(t, \bar{S}) & , \text{ if } t \in [t_{\bar{S}}^+(\bar{S}, \theta_1, \sigma_1), t_{\bar{S}}^-(\bar{S}, \theta_2, \sigma_2)] \\ 1 & , \text{ if } t \in (t_{\bar{S}}^-(\bar{S}, \theta_2, \sigma_2), \theta_2] \end{cases}$$

(ii) If $t_{\bar{S}}^-(\bar{S}, \theta_2, \sigma_2) < t_{\bar{S}}^+(\bar{S}, \theta_1, \sigma_1)$:

$$\tilde{u}_{\bar{S}}^{\bar{S}}(t) = \begin{cases} 0 & , \text{ if } t \in [\theta_1, \bar{t}_{\bar{S}}(\theta_1, \sigma_1, \theta_2, \sigma_2)) \\ 1 & , \text{ if } t \in [\bar{t}_{\bar{S}}(\theta_1, \sigma_1, \theta_2, \sigma_2), \theta_2] \end{cases}$$

where the time $\bar{t}_{\bar{S}}(\theta_1, \sigma_1, \theta_2, \sigma_2)$ is defined as the unique $\bar{t} \in [\theta_1, \theta_2]$ such that $S_{\theta_1, \sigma_1, 0}(\bar{t}) = S_{\theta_2, \sigma_2, 1}(\bar{t}) > \bar{S}$. Indeed the function $I(t) = S_{\theta_1, \sigma_1, 0}(t) - S_{\theta_2, \sigma_2, 1}(t)$ is decreasing on $[\theta_1, \theta_2]$, and one has $I(\theta_1) \geq 0$ and $I(\theta_2) \leq 0$, hence the existence and uniqueness of $\bar{t}_{\bar{S}}(\theta_1, \sigma_1, \theta_2, \sigma_2)$.

Definition 2.8. Let $0 < \bar{S} < 1$.

For any $(\theta, \sigma) \in [0, T] \times [0, \bar{S}]$, we define :

$$t_{\bar{S}}^+(\bar{S}, \theta, \sigma) = \begin{cases} T & , \text{ if } S_{\theta, \sigma, 1}(t) < \bar{S} \text{ for all } t \in [\theta, T] \\ \inf\{t > \theta, S_{\theta, \sigma, 1}(t) = \bar{S}\} & , \text{ otherwise} \end{cases}$$

For any $(\theta, \sigma) \in (0, T] \times [0, \bar{S}]$, we define :

$$t_{\bar{S}}^-(\bar{S}, \theta, \sigma) = \begin{cases} 0 & , \text{ if } S_{\theta, \sigma, 0}(t) < \bar{S} \text{ for all } t \in [0, \theta] \\ \sup\{t < \theta, S_{\theta, \sigma, 0}(t) = \bar{S}\} & , \text{ otherwise} \end{cases}$$

Definition 2.9 (MRAP from below). Let $0 < \bar{S} < 1$.

For any $(\theta_1, \sigma_1) \in [0, T] \times [0, \bar{S}]$ and $(\theta_2, \sigma_2) \in (0, T] \times [0, \bar{S}]$ such that σ_2 is attainable at the time θ_2 from (θ_1, σ_1) with some admissible control $u(\cdot)$, we call "MRAP to the value \bar{S} from below" the control $\tilde{u}_\lambda^{\bar{S}}(\cdot)$ defined on $[\theta_1, \theta_2]$ by the following :

(i) If $t_\lambda^+(\bar{S}, \theta_1, \sigma_1) \leq t_\lambda^-(\bar{S}, \theta_2, \sigma_2)$:

$$\tilde{u}_\lambda^{\bar{S}}(t) = \begin{cases} 1 & , \text{ if } t \in [\theta_1, t_\lambda^+(\bar{S}, \theta_1, \sigma_1)) \\ u_{sing}(t, \bar{S}) & , \text{ if } t \in [t_\lambda^+(\bar{S}, \theta_1, \sigma_1), t_\lambda^-(\bar{S}, \theta_2, \sigma_2)] \\ 0 & , \text{ if } t \in (t_\lambda^-(\bar{S}, \theta_2, \sigma_2), \theta_2] \end{cases}$$

(ii) If $t_\lambda^-(\bar{S}, \theta_2, \sigma_2) < t_\lambda^+(\bar{S}, \theta_1, \sigma_1)$:

$$\tilde{u}_\lambda^{\bar{S}}(t) = \begin{cases} 1 & , \text{ if } t \in [\theta_1, \bar{t}_\lambda(\theta_1, \sigma_1, \theta_2, \sigma_2)) \\ 0 & , \text{ if } t \in [\bar{t}_\lambda(\theta_1, \sigma_1, \theta_2, \sigma_2), \theta_2] \end{cases}$$

where the time $\bar{t}_\lambda(\theta_1, \sigma_1, \theta_2, \sigma_2)$ is defined as the unique $\bar{t} \in [\theta_1, \theta_2]$ such that $S_{\theta_1, \sigma_1, 1}(\bar{t}) = S_{\theta_2, \sigma_2, 0}(\bar{t}) < \bar{S}$. Indeed the function $I(t) = S_{\theta_1, \sigma_1, 1}(t) - S_{\theta_2, \sigma_2, 0}(t)$ is increasing on $[\theta_1, \theta_2]$, and one has $I(\theta_1) \leq 0$ and $I(\theta_2) \geq 0$, hence the existence and uniqueness of $\bar{t}_\lambda(\theta_1, \sigma_1, \theta_2, \sigma_2)$.

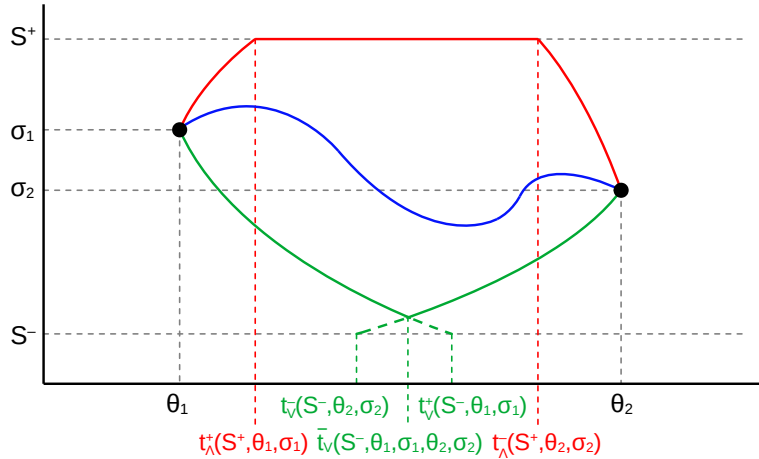


Figure 1: Example of a reference $S(\cdot)$ trajectory (in blue), with its induced MRAP trajectories associated to the MRAP controls "from below" $\tilde{u}_\lambda^{S^+}$ (case (i), in red), and "from above" $\tilde{u}_\lambda^{S^-}$ (case (ii), in green).

With the above definitions and Lemma 2.2, we easily deduce the following property about comparing trajectories associated to MRAP control, and their respective water consumption.

Proposition 2.3. Let $S_h < \underline{S} < \bar{S} \leq 1$.

Let $S(\cdot)$ be a solution to Eq. (11) on $[t_1, t_2]$ (with $0 \leq t_1 < t_2 \leq T$) for an admissible control $u(\cdot)$ such that $\underline{S} \leq S(t) \leq \bar{S}$ for all $t \in [t_1, t_2]$. We denote $S_1 = S(t_1)$ and $S_2 = S(t_2)$. Then, the respective MRAP controls $\tilde{u}_\lambda^{\underline{S}}(\cdot)$ and $\tilde{u}_\lambda^{\bar{S}}(\cdot)$ over $[t_1, t_2]$, and their respective trajectory $\tilde{S}_\lambda^{\underline{S}}(\cdot)$ and $\tilde{S}_\lambda^{\bar{S}}(\cdot)$ with initial conditions $\tilde{S}_\lambda^{\underline{S}}(t_1) = \tilde{S}_\lambda^{\bar{S}}(t_1) = S_1$, satisfy the following properties :

$$\begin{aligned} \tilde{S}_\lambda^{\underline{S}}(t_2) &= \tilde{S}_\lambda^{\bar{S}}(t_2) = S_2 \\ \forall t \in [t_1, t_2], \tilde{S}_\lambda^{\underline{S}}(t) &\leq S(t) \leq \tilde{S}_\lambda^{\bar{S}}(t) \\ \int_{t_1}^{t_2} \tilde{u}_\lambda^{\underline{S}}(t) dt &\leq \int_{t_1}^{t_2} u(t) dt \leq \int_{t_1}^{t_2} \tilde{u}_\lambda^{\bar{S}}(t) dt \end{aligned}$$

Moreover the last inequalities are strict unless $S(\cdot)$ is identical to $\tilde{S}_\lambda^{\underline{S}}(\cdot)$, or identical to $\tilde{S}_\lambda^{\bar{S}}(\cdot)$.

3 Equivalent formulation and preliminary properties

3.1 Compensation technique

We aim at deriving optimality necessary conditions through the use of a Pontryagin Maximum Principle, and we need to overcome the difficulty of analysis resulting from the presence of running state constraints Eq. (6). To this end, we deal with the lower inequality $S_{tol} \leq S(t)$ by introducing in this section a slightly modified formulation of the dynamical system Eq. (1). We replace this running state constraint by a mere terminal condition, and more importantly we follow an idea discussed in [11] by defining the function $f(S) = \max(0, S_{tol} - S)$, that we add in the dynamics of the water consumption as follows, with some $C > 0$.

$$\begin{cases} \dot{S}(t) = k_1 \left(-\varphi(t)K_S(S(t)) - (1 - \varphi(t))K_R(S(t)) + k_2 u(t) \right) \\ \dot{B}(t) = \varphi(t)K_S(S(t)) \\ \dot{W}(t) = u(t) + Cf(S(t)) \end{cases} \quad (11)$$

This system keeps the same initial conditions, adapted to the variable W :

$$S(0) = S_0 \in (S^*, 1] \quad , \quad B(0) = 0 \quad , \quad W(0) = 0$$

Let us define the three new problems corresponding to this formulation, recalling that this formulation now allows a wider range of trajectories and bigger sets of admissible controls.

Definition 3.1. For any control $u(\cdot)$, denote its associated solution to Eq. (11) by $(S(\cdot), B(\cdot), W(\cdot))$, and define :

$$\begin{aligned} \mathcal{U}^{\widetilde{P1}} &= \{u \in L^1([0, T], [0, 1]); S(\cdot) \leq 1, S(T) \geq S_{tol}, W(T) \leq Q, V_{tol} < Q < V^*\} \\ \mathcal{U}^{\widetilde{P2}} &= \{u \in L^1([0, T], [0, 1]); S(\cdot) \leq 1, S(T) \geq S_{tol}, B(T) \geq Y, B_{tol} < Y < B^*\} \\ \mathcal{U}^{\widetilde{P3}} &= \{u \in L^1([0, T], [0, 1]); S(\cdot) \leq 1, S(T) \geq S_{tol}\} \end{aligned}$$

Definition 3.2.

$$\begin{aligned} (\widetilde{P1}) : \quad & \max_{u(\cdot) \in \mathcal{U}^{\widetilde{P1}}} B(T) \\ (\widetilde{P2}) : \quad & \min_{u(\cdot) \in \mathcal{U}^{\widetilde{P2}}} W(T) \\ (\widetilde{P3}) : \quad & \max_{u(\cdot) \in \mathcal{U}^{\widetilde{P3}}} g(B(T)) - c(W(T)) \quad , \quad g' > 0, \quad c' > 0 \end{aligned}$$

Remark 5. We evidently have $\mathcal{U}^{P1} \subset \mathcal{U}^{\widetilde{P1}}$, $\mathcal{U}^{P2} \subset \mathcal{U}^{\widetilde{P2}}$ and $\mathcal{U}^{P3} \subset \mathcal{U}^{\widetilde{P3}}$, thus any admissible control of a problem (Pj) is admissible for (\widetilde{Pj}) .

The additional term $Cf(S(t))$ plays the role of a cost penalization for the problems $(\widetilde{P2})$ and $(\widetilde{P3})$. However for the problem $(\widetilde{P1})$ it may be seen as some room of maneuver for the input $u(\cdot)$ which is restricted by the terminal target. The proposed technique consists in determining a well-suited (large enough) constant $C > 0$ such that a variation of a trajectory of S travelling below S_{tol} over some time interval I , induces a variation of the quantity $Cf(S(t))$ on the interval I that matches with the variation of corresponding input $u(t)$ over I . This way, we will be able to claim that the optimization of the new problems $(\widetilde{P1})$, $(\widetilde{P2})$ and $(\widetilde{P3})$ is equivalent to the optimization of the initial ones $(P1)$, $(P2)$ and $(P3)$. That is the purpose of the following proposition.

Proposition 3.1. Let $C = \frac{L_S + L_R}{k_2} > 0$. Suppose that $u_{\widetilde{Pj}} \in \mathcal{U}^{\widetilde{Pj}}$ is an optimal solution of the problem (\widetilde{Pj}) . Denote $(S_{\widetilde{Pj}}(\cdot), B_{\widetilde{Pj}}(\cdot), W_{\widetilde{Pj}}(\cdot))$ its associated trajectory. Then :

$$\forall t \in [0, T], f(S_{\widetilde{Pj}}(t)) = 0$$

As a consequence, one has $u_{\widetilde{Pj}} \in \mathcal{U}^{Pj}$, and $u_{\widetilde{Pj}}$ is an optimal solution of the initial problem (Pj) .

Proof First note that the function f is non-negative. Consider a feasible control $u(\cdot)$, and denote $S_u(\cdot), B_u(\cdot), W_u(\cdot)$ the corresponding solutions. Suppose that u is optimal for the problem $(\widehat{P}j)$, and suppose that there exists an interval $[t_1, t_2]$ with $0 \leq t_1 < t_2 \leq T$ such that $f(S_u(t)) > 0$ for all $t \in (t_1, t_2)$ and $f(S_u(t_1)) = f(S_u(t_2)) = 0$. In other words, one has $S_u(t_1) = S_u(t_2) = S_{tol}$ and $S_u(t) < S_{tol}$ for all $t \in (t_1, t_2)$. Denote \underline{S} the local minimum of S_u over $[t_1, t_2]$, which is reached at the time $\underline{t} \in (t_1, t_2)$. By continuity there exists $t^- \in (t_1, \underline{t})$ and $t^+ \in (\underline{t}, t_2)$ such that $S_u(t^-) = S_u(t^+) = (S_{tol} + \underline{S})/2$.

We consider the control $v(\cdot)$, and its corresponding solutions $S_v(\cdot), B_v(\cdot), W_v(\cdot)$, defined by :

$$\begin{cases} v(t) = u(t) & , \text{ if } t \in [0, t^-) \\ v(t) = \widetilde{u}_{\wedge}^{S_{tol}}(t) & , \text{ if } t \in [t^-, t^+) \\ v(t) = u(t) & , \text{ if } t \in (t^+, T] \end{cases} \quad (12)$$

where $\widetilde{u}_{\wedge}^{S_{tol}}$ is the MRAP to the value S_{tol} from below, over the time interval $[t^-, t^+]$. The associated trajectory $S_v(\cdot)$ satisfies :

$$\forall t \in [0, T] \setminus (t^-, t^+), S_v(t) = S_u(t) \quad \text{and} \quad \forall t \in (t^-, t^+), S_v(t) \geq S_u(t)$$

with :

$$\int_{t^-}^{t^+} S_v(t) dt > \int_{t^-}^{t^+} S_u(t) dt$$

Moreover, from the dynamics of S , we get :

$$\begin{aligned} 0 &= \int_{t^-}^{t^+} \dot{S}_v(t) - \dot{S}_u(t) dt \\ &= k_1 \int_{t^-}^{t^+} -\varphi(t) (K_S(S_v(t)) - K_S(S_u(t))) \\ &\quad - (1 - \varphi(t)) (K_R(S_v(t)) - K_R(S_u(t))) + k_2(v(t) - u(t)) dt \end{aligned}$$

Using the definition of K_S and K_R and the fact that $\varphi(t) \in (0, 1)$ for all $t \in (t^-, t^+)$, one obtains :

$$\int_{t^-}^{t^+} v(t) - u(t) dt < \frac{L_S + L_R}{k_2} \int_{t^-}^{t^+} S_v(t) - S_u(t) dt$$

Then with $C = \frac{L_S + L_R}{k_2}$ we have :

$$\begin{aligned} W_v(T) &= \int_{[0, T] \setminus [t^-, t^+]} \dot{W}_v(t) dt + \int_{t^-}^{t^+} v(t) + C(S_{tol} - S_v(t)) dt \\ &< \int_{[0, T] \setminus [t^-, t^+]} \dot{W}_u(t) dt + \int_{t^-}^{t^+} u(t) + C(S_{tol} - S_u(t)) dt \end{aligned}$$

Furthermore the comparison between $S_v(\cdot)$ and $S_u(\cdot)$ leads to :

$$\begin{aligned} B_v(T) &= \int_{[0, T] \setminus [t^-, t^+]} \dot{B}_v(t) dt + \int_{t^-}^{t^+} \varphi(t) K_S(S_v(t)) dt \\ &> \int_{[0, T] \setminus [t^-, t^+]} \dot{B}_u(t) dt + \int_{t^-}^{t^+} \varphi(t) K_S(S_u(t)) dt \end{aligned}$$

To conclude, we have :

$$B_v(T) > B_u(T) \quad \text{and} \quad W_v(T) < W_u(T)$$

Thus the control $v(\cdot)$ is admissible for $(\widehat{P}j)$, and the control $u(\cdot)$ is therefore non optimal for $(\widehat{P}j)$. Hence one has :

$$\forall t \in [0, T], f(S_{\widehat{P}j}(t)) = 0$$

As a consequence, the trajectory $S_{\widehat{P}j}$ satisfies $S_{\widehat{P}j}(t) \geq S_{tol}$ for all $t \in [0, T]$, and the control $u_{\widehat{P}j}$ is thus admissible for the initial problem (Pj) . \square

3.2 Properties of optimal solutions

In this section, we aim at providing characteristic features of optimal solutions to the optimization problems $(\widetilde{P}j)$.

Proposition 3.2. *Let $u_{\widetilde{P}1}(\cdot), u_{\widetilde{P}2}(\cdot), u_{\widetilde{P}3}(\cdot)$ be optimal controls of their respective problems. Denote the corresponding solutions by $(S_{\widetilde{P}1}(\cdot), B_{\widetilde{P}1}(\cdot), W_{\widetilde{P}1}(\cdot)), (S_{\widetilde{P}2}(\cdot), B_{\widetilde{P}2}(\cdot), W_{\widetilde{P}2}(\cdot))$ and $(S_{\widetilde{P}3}(\cdot), B_{\widetilde{P}3}(\cdot), W_{\widetilde{P}3}(\cdot))$. Then one has :*

$$\begin{aligned} \text{(I)} : \quad & u_{\widetilde{P}j}(t) = 0 \quad , \text{ a.e. } t \in [0, t^*] \\ \text{(II)} : \quad & S_{\widetilde{P}j}(t) \leq S^* \quad , \forall t \in [t^*, T] \end{aligned}$$

Proof (I) For any feasible control $u(\cdot)$, consider $S_u(\cdot), B_u(\cdot), W_u(\cdot)$ the corresponding solutions. Suppose that u is optimal for the problem $(\widetilde{P}j)$, and denote $E = \{t \in [0, T], S_u(t) \leq S^*\}$ and $\hat{t} = \inf E \geq t^*$. One has :

$$\int_0^{\hat{t}} u^*(t) dt \leq \int_0^{\hat{t}} u(t) dt \quad \text{and} \quad K_S(S_0, S_0, u^*(\cdot))(t) = 1 \quad \text{for all } t \in [0, T]$$

where u^* is defined in Eq. (4). The inequality $\int_{\hat{t}}^T u(t) dt \geq \int_{\hat{t}}^T u^*(t) dt$ cannot be fulfilled, since it would imply :

$$W_u(T) \geq V^* \quad , \quad B_u(T) = B^* \quad \text{and} \quad g(B_u(T)) - c(W_u(T)) \leq g(B^*) - c(V^*)$$

that contradict the definition of the problems $(\widetilde{P}1)$ and $(\widetilde{P}2)$ and means that u would not be optimal for $(\widetilde{P}3)$.

Then the inequality $\int_{\hat{t}}^T u(t) dt < \int_{\hat{t}}^T u^*(t) dt$ holds, and one necessarily has $\hat{t} < T$. We set :

$$E_1 = \{t \in [\hat{t}, T], u(t) < 1\}$$

The set E_1 has non null measure, and the intersection $E \cap E_1$ has also non null measure (otherwise we would have $u = 1$ a.e. in E , which would imply S_u increasing over E , that prevents S_u to reach the domain below S^*).

Assume that $\hat{t} > t^*$. Then $\int_0^{\hat{t}} u^*(t) dt < \int_0^{\hat{t}} u(t) dt$. We consider the control $u_1^\dagger(\cdot)$, and its corresponding solutions $S_{u_1^\dagger}(\cdot), B_{u_1^\dagger}(\cdot), W_{u_1^\dagger}(\cdot)$, defined by :

$$\begin{cases} u_1^\dagger(t) = u^*(t) & , t \in [0, \hat{t}) \\ u_1^\dagger(t) = u(t) & , t \in [\hat{t}, T] \setminus (E \cap E_1) \text{ and } S_{u_1^\dagger}(t) < 1 \\ u_1^\dagger(t) = \min(u(t), 1/k_2) & , t \in [\hat{t}, T] \setminus (E \cap E_1) \text{ and } S_{u_1^\dagger}(t) = 1 \\ u_1^\dagger(t) \in [u(t), 1] & , t \in E \cap E_1 \end{cases} \quad (13)$$

with :

$$0 < \int_{E \cap E_1} u_1^\dagger(t) - u(t) dt < \int_0^{\hat{t}} u(t) - u^*(t) dt$$

Then we have :

$$\begin{aligned} W_{u_1^\dagger}(T) &= \int_0^{\hat{t}} u^*(t) dt + \int_{E \cap E_1} u_1^\dagger(t) dt + \int_{[\hat{t}, T] \setminus (E \cap E_1)} u_1^\dagger(t) dt \\ &< \int_0^{\hat{t}} u(t) dt + \int_{E \cap E_1} u(t) dt + \int_{[\hat{t}, T] \setminus (E \cap E_1)} u(t) dt \end{aligned}$$

The associated solution $S_{u_1^\dagger}(\cdot)$ satisfies :

$$\forall t \in [0, T], S_{u_1^\dagger}(t) \leq 1 \quad \text{and} \quad \forall t \in [\hat{t}, T], S_{u_1^\dagger}(t) \geq S_u(t)$$

with :

$$\int_{E \cap E_1} S_{u_1^\dagger}(t) dt > \int_{E \cap E_1} S_u(t) dt$$

Moreover one has $S_u(t) \leq S^*$ over $E \cap E_1$, and there exists $J \subset E \cap E_1$ of non null measure such that $S_u(t) < S^*$ over J . Then :

$$\int_{E \cap E_1} \varphi(t) K_S(S_{u_1^\dagger}(t)) dt > \int_{E \cap E_1} \varphi(t) K_S(S_u(t)) dt$$

This leads to :

$$B_{u_1^\dagger}(T) > \int_0^{\hat{t}} \varphi(t) K_S(S_u(t)) dt + \int_{E \cap E_1} \varphi(t) K_S(S_u(t)) dt \\ + \int_{[\hat{t}, T] \setminus (E \cap E_1)} \varphi(t) K_S(S_u(t)) dt$$

To conclude, we have $B_{u_1^\dagger}(T) > B_u(T)$ and $W_{u_1^\dagger}(T) < W_u(T)$.

Thus the control $u_1^\dagger(\cdot)$ is admissible for (\widetilde{P}^j) , and the control $u(\cdot)$ is therefore non optimal for (\widetilde{P}^j) . Hence one has :

$$\hat{t} = t^* \quad , \text{ i.e. } u_{\widetilde{P}^j}(t) = 0 \quad \text{a.e. } t \in [0, t^*]$$

(II) For any feasible control $u(\cdot)$ such that $u(t) = 0$ over $[0, t^*]$, we denote $S_u(\cdot)$, $B_u(\cdot)$, $W_u(\cdot)$ the corresponding trajectories. Suppose that u is optimal for the problem (\widetilde{P}^j) and consider the sets E and E_1 as above. We set :

$$G = \{t \in [t^*, T], S_u(t) > S^*\}$$

Assume that $G \neq \emptyset$. Then we have :

$$\int_G u_{sing}(t, S^*) dt < \int_G u(t) dt$$

Let us consider the following control $u_{11}^\dagger(\cdot)$, and its associated solution $S_{u_{11}^\dagger}(\cdot)$, $B_{u_{11}^\dagger}(\cdot)$, $W_{u_{11}^\dagger}(\cdot)$, defined by :

$$\begin{cases} u_{11}^\dagger(t) = 0 & , t \in [0, t^*] \\ u_{11}^\dagger(t) = u_{sing}(t, S^*) & , t \in G \\ u_{11}^\dagger(t) = u(t) & , t \in E \setminus E_1 \text{ and } S_{u_{11}^\dagger}(t) < 1 \\ u_{11}^\dagger(t) = \min(u(t), 1/k_2) & , t \in E \setminus E_1 \text{ and } S_{u_{11}^\dagger}(t) = 1 \\ u_{11}^\dagger(t) \in [u(t), 1] & , t \in E \cap E_1 \end{cases} \quad (14)$$

with :

$$0 < \int_{E \cap E_1} u_{11}^\dagger(t) - u(t) dt < \int_G u(t) - u_{sing}(t, S^*) dt$$

Then one has :

$$W_{u_{11}^\dagger}(T) = \int_G u_{sing}(t, S^*) dt + \int_{E \setminus E_1} u_{11}^\dagger(t) dt + \int_{E \cap E_1} u_{11}^\dagger(t) dt \\ < \int_G u(t) dt + \int_{E \setminus E_1} u(t) dt + \int_{E \cap E_1} u(t) dt$$

The associated solution $S_{u_{11}^\dagger}(\cdot)$ satisfies :

$$\forall t \in [0, T], S_{u_{11}^\dagger}(t) \leq 1, \forall t \in G, S_{u_{11}^\dagger}(t) \geq S^*, \forall t \in [0, T] \setminus G, S_{u_{11}^\dagger}(t) \geq S_u(t)$$

with :

$$\int_{E \cap E_1} S_{u_{11}^\dagger}(t) dt > \int_{E \cap E_1} S_u(t) dt$$

Moreover one has $S_u(t) \leq S^*$ over $E \cap E_1$, and there exists $J \subset E \cap E_1$ of non null measure such that $S_u(t) < S^*$ over J .

Then :

$$\int_{E \cap E_1} \varphi(t) K_S(S_{u_{11}^\dagger}(t)) dt > \int_{E \cap E_1} \varphi(t) K_S(S_u(t)) dt$$

This leads to :

$$B_{u_{11}^\dagger}(T) > \int_{[0, t^*] \cup G} \varphi(t) K_S(S_u(t)) dt + \int_{E \cap E_1} \varphi(t) K_S(S_u(t)) dt \\ + \int_{E \setminus E_1} \varphi(t) K_S(S_u(t)) dt$$

To conclude, we have $B_{u_{\dagger}}(T) > B_u(T)$ and $W_{u_{\dagger}}(T) < W_u(T)$.

Thus the control $u_{\dagger}(\cdot)$ is admissible for $(\widetilde{P}j)$, and the control $u(\cdot)$ is therefore non optimal for $(\widetilde{P}j)$. Hence one has :

$$G = \emptyset \quad , \text{ i.e. } \quad S_{\widetilde{P}j}(t) \leq S^* \quad , \forall t \in [t^*, T]$$

□

Remark 6. The upper state constraint $S \leq 1$ is then never saturated by an optimal trajectory $S_{\widetilde{P}j}(\cdot)$ under water scarcity assumption, except from initial time if the initial condition is $S(0) = 1$.

Proposition 3.3. Let $u_{\widetilde{P}1}(\cdot), u_{\widetilde{P}2}(\cdot)$ be the optimal controls of their respective problems and we denote their corresponding solutions by $(S_{\widetilde{P}1}(\cdot), B_{\widetilde{P}1}(\cdot), W_{\widetilde{P}1}(\cdot))$ and $(S_{\widetilde{P}2}(\cdot), B_{\widetilde{P}2}(\cdot), W_{\widetilde{P}2}(\cdot))$. Then :

$$(III) : \quad W_{\widetilde{P}1}(T) = Q$$

$$(IV) : \quad B_{\widetilde{P}2}(T) = Y$$

Proof (III) Consider a feasible control $u(\cdot)$ such that $u(t) = 0$ on $[0, t^*]$ and that the corresponding solution $S_u(\cdot), B_u(\cdot), W_u(\cdot)$ satisfies $S_u(t) \leq S^*$ for all $t \in [t^*, T]$. Suppose that u is optimal for the problem $(\widetilde{P}1)$, and consider the set E_1 as previously.

Assume that $W_u(T) < Q$. There exists $J \subset [t^*, T]$ of non null measure such that $S_u(t) < S^*$ over J , and $J \cap E_1$ has non null measure. Let us consider the control $u_{\dagger}(\cdot)$, and its associated solution $S_{u_{\dagger}}(\cdot), B_{u_{\dagger}}(\cdot), W_{u_{\dagger}}(\cdot)$, defined by :

$$\begin{cases} u_{\dagger}(t) = u(t) & , t \in [0, T] \setminus (J \cap E_1) \\ u_{\dagger}(t) = \min(u(t), 1/k_2) & , t \in J \cap E_1 \text{ and } S_{u_{\dagger}}(t) = 1 \\ u_{\dagger}(t) \in [u(t), 1] & , t \in J \cap E_1 \text{ and } S_{u_{\dagger}}(t) < 1 \end{cases} \quad (15)$$

with :

$$0 < \int_{(J \cap E_1) \cap \{S_{u_{\dagger}} < 1\}} u_{\dagger}(t) - u(t) dt < Q - W_u(T)$$

Then one has :

$$\begin{aligned} W_{u_{\dagger}}(T) &= \int_{(J \cap E_1) \cap \{S_{u_{\dagger}} < 1\}} u_{\dagger}(t) dt + \int_{(J \cap E_1) \cap \{S_{u_{\dagger}} = 1\}} u_{\dagger}(t) dt \\ &\quad + \int_{[0, T] \setminus (J \cap E_1)} u(t) dt \\ &< Q - W_u(T) + \int_{[0, T]} u(t) dt \end{aligned}$$

The associated solution $S_{u_{\dagger}}(\cdot)$ satisfies :

$$\forall t \in [0, T], \quad S_{u_{\dagger}}(t) \leq 1 \quad \text{and} \quad \forall t \in [0, T], \quad S_{u_{\dagger}}(t) \geq S_u(t)$$

with :

$$\int_{J \cap E_1} S_{u_{\dagger}}(t) dt > \int_{J \cap E_1} S_u(t) dt$$

Moreover one has $S_u(t) < S^*$ over $J \cap E_1$, then :

$$\int_{J \cap E_1} \varphi(t) K_S(S_{u_{\dagger}}(t)) dt > \int_{J \cap E_1} \varphi(t) K_S(S_u(t)) dt$$

This leads to :

$$B_{u_{\dagger}}(T) > \int_{J \cap E_1} \varphi(t) K_S(S_u(t)) dt + \int_{[0, T] \setminus (J \cap E_1)} \varphi(t) K_S(S_u(t)) dt$$

To conclude, we have $B_{u_{\dagger}}(T) > B_u(T)$ and $W_{u_{\dagger}}(T) < Q$.

Thus the control $u_{\text{III}}^\dagger(\cdot)$ is admissible for $(\widetilde{P1})$, and the control $u(\cdot)$ is therefore non optimal for $(\widetilde{P1})$. Hence one has :

$$W_{u_{\widetilde{P1}}} (T) = Q$$

(IV) Consider a feasible control $u(\cdot)$ such that $u(t) = 0$ on $[0, t^*]$ and that the corresponding solution $S_u(\cdot)$, $B_u(\cdot)$, $W_u(\cdot)$ satisfies $S_u(t) \leq S^*$ for all $t \in [t^*, T]$. Suppose that u is optimal for the problem $(\widetilde{P2})$.

Assume that $B_u(T) > Y$. We set :

$$E_2 = \{t \in [t^*, T], u(t) > 0\}$$

The set E_2 has non null measure, and let $t_2 = \text{ess sup } E_2$.

If $t_2 = T$, set $I_\varepsilon = [T - \varepsilon, T]$, for $\varepsilon > 0$ small enough, and consider the control $u_{\text{IV}}^\dagger(\cdot)$, with its associated solution $S_{u_{\text{IV}}^\dagger}(\cdot)$, $B_{u_{\text{IV}}^\dagger}(\cdot)$, $W_{u_{\text{IV}}^\dagger}(\cdot)$, defined by :

$$\begin{cases} u_{\text{IV}}^\dagger(t) = u(t) & , t \in [0, T] \setminus I_\varepsilon \\ u_{\text{IV}}^\dagger(t) = 0 & , t \in I_\varepsilon \end{cases}$$

If $t_2 < T$, set $I_\varepsilon = [t_2 - \frac{\varepsilon}{2}, t_2 + \frac{\varepsilon}{2}]$, for $\varepsilon > 0$ small enough, and consider the control $u_{\text{IV}}^\dagger(\cdot)$, with its associated solution $S_{u_{\text{IV}}^\dagger}(\cdot)$, $B_{u_{\text{IV}}^\dagger}(\cdot)$, $W_{u_{\text{IV}}^\dagger}(\cdot)$, defined by :

$$\begin{cases} u_{\text{IV}}^\dagger(t) = u(t) & , t \in [0, T] \setminus I_\varepsilon \\ u_{\text{IV}}^\dagger(t) = \tilde{u}_V^{\text{Sol}}(t) & , t \in I_\varepsilon \end{cases}$$

where \tilde{u}_V^{Sol} is the MRAP to the value S_{tol} from above, over the interval I_ε .

The following arguments hold in both cases. One has :

$$0 < \int_{I_\varepsilon} u(t) - u_{\text{IV}}^\dagger(t) dt \leq \varepsilon \quad (16)$$

The associated solution $S_{u_{\text{IV}}^\dagger}(\cdot)$ satisfies :

$$\forall t \in [0, T] \setminus I_\varepsilon, S_{u_{\text{IV}}^\dagger}(t) = S_u(t) \quad \text{and} \quad \forall t \in I_\varepsilon, S_{u_{\text{IV}}^\dagger}(t) \leq S_u(t)$$

with :

$$\int_{I_\varepsilon} S_u(t) - S_{u_{\text{IV}}^\dagger}(t) dt > 0$$

Moreover one has $S_u(t) \leq S^*$ over I_ε . Then :

$$\int_{I_\varepsilon} \varphi(t) K_S(S_{u_{\text{IV}}^\dagger}(t)) dt < \int_{I_\varepsilon} \varphi(t) K_S(S_u(t)) dt$$

And :

$$0 < \int_0^T \varphi(t) \left(K_S(S_u(t)) - K_S(S_{u_{\text{IV}}^\dagger}(t)) \right) dt$$

Let $t \in I_\varepsilon$. From the dynamics of S , one has :

$$\begin{aligned} S_u(t) - S_{u_{\text{IV}}^\dagger}(t) &= k_1 k_2 \int_{t^*}^t u(\tau) - u_{\text{IV}}^\dagger(\tau) d\tau \\ &+ k_1 \int_{t^*}^t \varphi(\tau) \left(K_S(S_{u_{\text{IV}}^\dagger}(\tau)) - K_S(S_u(\tau)) \right) d\tau \\ &+ k_1 \int_{t^*}^t (1 - \varphi(\tau)) \left(K_R(S_{u_{\text{IV}}^\dagger}(\tau)) - K_R(S_u(\tau)) \right) d\tau \end{aligned}$$

As a consequence, one has :

$$|S_u(t) - S_{u_{\text{IV}}^\dagger}(t)| \leq k_1 k_2 \varepsilon + k_1 \int_{t^*}^t \left(\varphi(\tau) L_S + (1 - \varphi(\tau)) L_R \right) |S_u(\tau) - S_{u_{\text{IV}}^\dagger}(\tau)| d\tau$$

By the Grönwall Lemma, we deduce that :

$$|S_u(t) - S_{u_{iv}^\dagger}(t)| \leq \varepsilon k_1 k_2 \exp\left(\int_0^T k_1 (\varphi(\tau)L_S + (1 - \varphi(\tau))L_R) d\tau\right) := \varepsilon K$$

Then :

$$0 < B_u(T) - B_{u_{iv}^\dagger}(T) \leq \varepsilon \int_{I_e} \varphi(t)L_S K dt \leq \varepsilon^2 L_S K$$

For $\varepsilon > 0$ small enough, we have :

$$0 < B_u(T) - B_{u_{iv}^\dagger}(T) \leq \varepsilon^2 L_S K < B_u(T) - Y$$

With Eq. (16), it leads to $Y < B_{u_{iv}^\dagger}(T)$ and $W_{u_{iv}^\dagger}(T) < W_u(T)$

To conclude, the control $u_{iv}^\dagger(\cdot)$ is admissible for $(\widetilde{P2})$, and the control $u(\cdot)$ is therefore non optimal for $(\widetilde{P2})$. Hence one has :

$$B_{\widetilde{P2}}(T) = Y$$

□

Remark 7. Consider an optimal solution $u_{\widetilde{P3}}$ to the problem $(\widetilde{P3})$, with its associated trajectory $(S_{\widetilde{P3}}(\cdot), B_{\widetilde{P3}}(\cdot), W_{\widetilde{P3}}(\cdot))$ and its associated optimal cost $J_{\widetilde{P3}} = g(B_{\widetilde{P3}}(T)) - c(W_{\widetilde{P3}}(T))$. From properties above and Lemma 2.2 we deduce that $W_{\widetilde{P3}}(T) \in [V_{tol}, V^*]$.

If $W_{\widetilde{P3}}(T) = V_{tol}$ then $u_{\widetilde{P3}} = u_{tol}$, and if $W_{\widetilde{P3}}(T) = V^*$ then $u_{\widetilde{P3}} = u^*$. They correspond to trivial trajectories and these situations will then not be considered in the rest of our analysis.

If $W_{\widetilde{P3}}(T) \in (V_{tol}, V^*)$, we can set $Q = W_{\widetilde{P3}}(T) \in (V_{tol}, V^*)$ and solve the problem $(\widetilde{P1})$ associated to the quota Q . The optimal solution $u_{\widetilde{P1}}$ and its corresponding trajectory $(S_{\widetilde{P1}}(\cdot), B_{\widetilde{P1}}(\cdot), W_{\widetilde{P1}}(\cdot))$ satisfy $W_{\widetilde{P1}}(T) = Q$ and $B_{\widetilde{P1}}(T) \geq B_{\widetilde{P3}}(T)$. To conclude, we have $J_{\widetilde{P3}} = g(B_{\widetilde{P1}}(T)) - c(W_{\widetilde{P1}}(T))$, i.e. there exists an optimal control of the problem $(\widetilde{P3})$ which is an optimal control of the problem $(\widetilde{P1})$.

4 Optimal synthesis

4.1 Application of the Pontryagin Maximum Principle

In this section, we apply the non-smooth Pontryagin Maximum Principle (see for instance [5, 11]) for the optimal control problems (\widetilde{Pj}) (without state constraint, due to Propositions 3.1 and 3.2). We write the optimality necessary conditions, and we show further properties about the signs of adjoint states.

4.1.1 Optimality necessary conditions

The Hamiltonian of the problem (\widetilde{Pj}) is written as follows :

$$\begin{aligned} H_{\widetilde{Pj}}(t, S, \lambda_S^{\widetilde{Pj}}, \lambda_B^{\widetilde{Pj}}, \lambda_W^{\widetilde{Pj}}, u) &= \lambda_S^{\widetilde{Pj}} k_1 \left(k_2 u - (\varphi(t)K_S(S) + (1 - \varphi(t))K_R(S)) \right) \\ &\quad + \lambda_B^{\widetilde{Pj}} \varphi(t)K_S(S) + \lambda_W^{\widetilde{Pj}} (u + Cf(S)) \end{aligned} \quad (17)$$

We also write the associated adjoint equations.

$$\begin{cases} \dot{\lambda}_S^{\widetilde{Pj}}(t) \in \varphi(t)(\lambda_S^{\widetilde{Pj}}(t)k_1 - \lambda_B^{\widetilde{Pj}}) \partial_C K_S(S_{\widetilde{Pj}}(t)) \\ \quad + \lambda_S^{\widetilde{Pj}}(t)(1 - \varphi(t))k_1 \partial_C K_R(S_{\widetilde{Pj}}(t)) - \lambda_W^{\widetilde{Pj}} C \partial_C f(S_{\widetilde{Pj}}(t)) \\ \dot{\lambda}_B^{\widetilde{Pj}} = 0 \\ \dot{\lambda}_W^{\widetilde{Pj}} = 0 \end{cases} \quad (18)$$

where ∂_C denotes the Clarke generalized gradient [5].

Denoting $X = (S, B, W)^\top$, for any optimal solution $u_{\tilde{P}^j}, X_{\tilde{P}^j}$ there exists an absolutely continuous solution of Eq. (18) denoted by $\Lambda^{\tilde{P}^j} = (\lambda_S^{\tilde{P}^j}, \lambda_B^{\tilde{P}^j}, \lambda_W^{\tilde{P}^j})$ that maximizes Eq. (17) and satisfies for all $t \in [0, T]$:

$$\left| \lambda_S^{\tilde{P}^j}(t) \right| + \left| \lambda_B^{\tilde{P}^j} \right| + \left| \lambda_W^{\tilde{P}^j} \right| \neq 0 \quad (19)$$

and the following necessary transversality condition, with $\lambda_0^{\tilde{P}^j} \geq 0$:

$$\Lambda^{\tilde{P}^j}(T) \in -N_{K^{\tilde{P}^j}}(X_{\tilde{P}^j}(T)) + \text{sgn}(\tilde{P}^j) \lambda_0^{\tilde{P}^j} \nabla_X L^{\tilde{P}^j}(X_{\tilde{P}^j}) \quad (20)$$

where $\text{sgn}(\tilde{P}^j) = 1$ if the cost is to be maximized, and $\text{sgn}(\tilde{P}^j) = -1$ if it is to be minimized. We used the notations N_K for the normal cone to the set K , $K^{\tilde{P}^j}$ for the target sets and $L^{\tilde{P}^j}$ for the cost functions.

More precisely, from the optimality necessary conditions, one has :

$$(\tilde{P}^j) : \lambda_S^{\tilde{P}^j}(T) \geq 0 \quad (21a)$$

$$(\tilde{P}^1) : \lambda_B^{\tilde{P}^1} = \lambda_0^{\tilde{P}^1} \quad , \quad \lambda_W^{\tilde{P}^1} \leq 0 \quad (21b)$$

$$(\tilde{P}^2) : \lambda_B^{\tilde{P}^2} \geq 0 \quad , \quad \lambda_W^{\tilde{P}^2} = -\lambda_0^{\tilde{P}^2} \quad (21c)$$

$$(\tilde{P}^3) : \lambda_B^{\tilde{P}^3} = \lambda_0^{\tilde{P}^3} g'(B_{\tilde{P}^3}(T)) \quad , \quad \lambda_W^{\tilde{P}^3} = -\lambda_0^{\tilde{P}^3} c'(W_{\tilde{P}^3}(T)) \quad (21d)$$

We define the switching function as follows :

$$\psi^{\tilde{P}^j}(t) = \lambda_S^{\tilde{P}^j}(t) k_1 k_2 + \lambda_W^{\tilde{P}^j} \quad (22)$$

Then maximizing the Hamiltonian Eq. (17) provides for an optimal solution and for a.e. $t \in [0, T]$:

$$\begin{cases} u_{\tilde{P}^j}(t) = 0 & , \text{ if } \psi^{\tilde{P}^j}(t) < 0 \\ u_{\tilde{P}^j}(t) \in [0, 1] & , \text{ if } \psi^{\tilde{P}^j}(t) = 0 \\ u_{\tilde{P}^j}(t) = 1 & , \text{ if } \psi^{\tilde{P}^j}(t) > 0 \end{cases} \quad (23)$$

Remark 8. From the dynamics of S and Hypothesis 1, we get that the function $S_{\tilde{P}^j}(\cdot)$ is decreasing when $\psi^{\tilde{P}^j} < 0$ and increasing when $\psi^{\tilde{P}^j} > 0$.

4.1.2 Properties of adjoints

Proposition 4.1. For any optimal solution to the problem (\tilde{P}^j) , one necessarily has $\lambda_S^{\tilde{P}^j}(t) \geq 0$ for all $t \in [0, T]$.

Proof Suppose that there exists $\hat{t} \in [0, T]$ such that $\lambda_S^{\tilde{P}^j}(\hat{t}) < 0$.

There exists $\varepsilon > 0$ such that $\lambda_S^{\tilde{P}^j}(t) < 0$ for all $t \in (\hat{t} - \varepsilon, \hat{t} + \varepsilon)$. Since $\lambda_W^{\tilde{P}^j} \leq 0$ we deduce that the switching function $\psi^{\tilde{P}^j}$ is negative over $(\hat{t} - \varepsilon, \hat{t} + \varepsilon)$. Therefore $S_{\tilde{P}^j}(\cdot)$ is decreasing on $(\hat{t} - \varepsilon, \hat{t} + \varepsilon)$, and then $S_{\tilde{P}^j}(t) \in (S_{tol}, 1)$ for all $t \in (\hat{t} - \varepsilon, \hat{t} + \varepsilon)$. Consequently one has for all $t \in (\hat{t} - \varepsilon, \hat{t} + \varepsilon)$:

$$\dot{\lambda}_S^{\tilde{P}^j}(t) \in \varphi(t) (\lambda_S^{\tilde{P}^j}(t) k_1 - \lambda_B^{\tilde{P}^j}) \partial_C K_S(S_{\tilde{P}^j}(t)) + \lambda_S^{\tilde{P}^j}(t) k_1 (1 - \varphi(t)) \partial_C K_R(S_{\tilde{P}^j}(t))$$

Since $0 \leq \varphi \leq 1$, $k_1 > 0$, $\lambda_B^{\tilde{P}^j} \geq 0$, $\partial_C K_S \subset \mathbb{R}^+$ and $\partial_C K_R \subset \mathbb{R}^+$, then one has :

$$\forall t \in (\hat{t} - \varepsilon, \hat{t} + \varepsilon), \dot{\lambda}_S^{\tilde{P}^j}(t) \in \mathbb{R}^-$$

Let $A = \{t \in [0, T], \lambda_S^{\tilde{P}^j}(t) < 0\}$. The function $\lambda_S^{\tilde{P}^j}(\cdot)$ is then non increasing over A , which leads to a contradiction regarding the necessary terminal condition $\lambda_S^{\tilde{P}^j}(T) \geq 0$. Hence one has :

$$\forall t \in [0, T], \lambda_S^{\tilde{P}^j}(t) \geq 0$$

□

Proposition 4.2. For any optimal solution to the problem $(\widetilde{P}j)$, one necessarily has $\lambda_W^{\widetilde{P}j} < 0$.

Proof Assume that $\lambda_W^{\widetilde{P}j} = 0$. Then, one has for all $t \in [0, T]$:

$$\dot{\lambda}_S^{\widetilde{P}j}(t) \in \varphi(t)(\lambda_S^{\widetilde{P}j}(t)k_1 - \lambda_B^{\widetilde{P}j})\partial_C K_S(S_{\widetilde{P}j}(t)) + \lambda_S^{\widetilde{P}j}(t)k_1(1 - \varphi(t))\partial_C K_R(S_{\widetilde{P}j}(t))$$

The function $\lambda_S^{\widetilde{P}j}(\cdot)$ is non negative, and can vanish on a time interval of non null measure only if $0 \in \partial_C K_S(S_{\widetilde{P}j}(t))$ on this interval. As a result, for any t such that $S_{\widetilde{P}j}(t) \in (S_{tol}, S^*)$ we have $\lambda_S^{\widetilde{P}j}(t) > 0$. Moreover one has $\psi^{\widetilde{P}j}(t) = \lambda_S^{\widetilde{P}j}(t)k_1k_2$, then for any t such that $S_{\widetilde{P}j}(t) \in (S_{tol}, S^*)$ one has $\psi^{\widetilde{P}j}(t) > 0$ thus $u_{\widetilde{P}j}(t) = 1$, i.e. $S_{\widetilde{P}j}(\cdot)$ increases when belonging to (S_{tol}, S^*) .

We deduce that the trajectory $S_{\widetilde{P}j}(\cdot)$ can not reach below S^* , thus from the comparison Lemma 2.2 we get $W_{\widetilde{P}j}(T) \geq V^*$, which contradicts the framework hypothesis. Hence one has :

$$\lambda_W^{\widetilde{P}j} < 0$$

□

Proposition 4.3. For any optimal solution to the problem $(\widetilde{P}j)$, one necessarily has $\lambda_B^{\widetilde{P}j} > 0$.

Proof Assume that $\lambda_B^{\widetilde{P}j} = 0$. Then, one has for all $t \in [0, T]$:

$$\begin{aligned} \dot{\lambda}_S^{\widetilde{P}j}(t) \in & \lambda_S^{\widetilde{P}j}(t)k_1 \left(\varphi(t)\partial_C K_S(S_{\widetilde{P}j}(t)) + (1 - \varphi(t))\partial_C K_R(S_{\widetilde{P}j}(t)) \right) \\ & - \lambda_W^{\widetilde{P}j} C \partial_C f(S_{\widetilde{P}j}(t)) \end{aligned}$$

We know from Proposition 3.2 that the trajectory of $S_{\widetilde{P}j}(\cdot)$ is decreasing over $[0, t^*)$ and $S_{tol} \leq S_{\widetilde{P}j}(t) \leq S^*$ for all $t \in [t^*, T]$. As a consequence, for any t , one has either $S_{\widetilde{P}j}(t) = S_{tol}$ or $S_{\widetilde{P}j}(t) > S_{tol}$.

Suppose that the set $N = \{t \in [0, T], S_{\widetilde{P}j}(t) = S_{tol}\}$ has non null measure. Define $t^- = \text{ess inf } N$ and $t^+ = \text{ess sup } N$. The function $S_{\widetilde{P}j}$ being neither decreasing nor increasing over N implies that $\psi^{\widetilde{P}j}(t) = 0$ for a.e. $t \in N$. Thus $\lambda_S^{\widetilde{P}j}$ is equal to $\gamma^{\widetilde{P}j} = -\frac{\lambda_W^{\widetilde{P}j}}{k_1k_2} > 0$ for a.e. $t \in N$.

First suppose that $t^+ < T$. Then one has $S_{\widetilde{P}j}(t) > S_{tol}$ for all $t \in (t^+, T]$, as a consequence the function $\lambda_S^{\widetilde{P}j}$ satisfies for all $t \in (t^+, T]$:

$$\dot{\lambda}_S^{\widetilde{P}j}(t) \in \lambda_S^{\widetilde{P}j}(t)k_1 \left(\varphi(t)\partial_C K_S(S_{\widetilde{P}j}(t)) + (1 - \varphi(t))\partial_C K_R(S_{\widetilde{P}j}(t)) \right) \quad (24)$$

Since $\lambda_S^{\widetilde{P}j}(t^+) = \gamma^{\widetilde{P}j} > 0$, then $\lambda_S^{\widetilde{P}j}$ is increasing over $(t^+, T]$, implying that $\psi^{\widetilde{P}j}(t) > 0$ for all $t \in (t^+, T]$, and then $S_{\widetilde{P}j}$ is increasing over $(t^+, T]$. As a result, $S_{\widetilde{P}j}(T) > S_{tol}$, then the transversality condition Eq. (20) brings $\lambda_S^{\widetilde{P}j}(T) = 0$, which is a contradiction.

We then have $t^+ = T$, and a similar argument as above brings that $\psi^{\widetilde{P}j}(t) < 0$ for all $t \in [0, t^-)$. We deduce that $S_{\widetilde{P}j}(t) = S_{0, S_{0, u_{tol}(\cdot)}}(t)$ for all $t \in [0, T]$. Then $B_{\widetilde{P}j}(T) = B_{tol}$ which contradicts the framework hypothesis.

Finally, the set N has null measure, and $S_{\widetilde{P}j}(t) > S_{tol}$ for a.e. $t \in [0, T]$. The equation Eq. (24) is thus satisfied for a.e. $t \in [0, T]$, and the function $\lambda_S^{\widetilde{P}j}$ is either increasing or constant equal to 0 on $[0, T]$. If there exists $\hat{t} \in (0, T)$ such that $\lambda_S^{\widetilde{P}j}(t) > \gamma^{\widetilde{P}j}$ for all $t \in (\hat{t}, T]$, then the trajectory $S_{\widetilde{P}j}$ is increasing on $(\hat{t}, T]$. As previously, this would imply $\lambda_S^{\widetilde{P}j}(T) = 0$ contradicting the fact that it is increasing. As a consequence, $\lambda_S^{\widetilde{P}j}(t) \in [0, \gamma^{\widetilde{P}j})$ for all $t \in [0, T]$, and the switching function thus verifies :

$$\forall t \in [0, T], \psi^{\widetilde{P}j}(t) < 0$$

We deduce that $u_{\widetilde{P}j}(t) = 0$ for a.e. $t \in [0, T]$, and then $B_{\widetilde{P}j}(T) \leq B_{tol}$ which contradicts the framework hypothesis. Hence one has :

$$\lambda_B^{\widetilde{P}j} > 0$$

□

Remark 9. Finally, the constants $\lambda_B^{\widetilde{P}j}$, $\lambda_W^{\widetilde{P}j}$ both have strict sign (either positive or negative). As a result, all $\lambda_0^{\widetilde{P}j} \neq 0$ and there is no abnormal extremal.

In addition we deduce from previous arguments that for an optimal control of a problem $(\widetilde{P}j)$, there necessarily exists $\alpha > 0$, such that $u_{\widetilde{P}j}(t) = 0$ for a.e. $t \in (T - \alpha, T]$.

4.2 Structures of optimal solutions

In this section, we aim at characterizing the structures of optimal solutions from the necessary optimality conditions. To this end, we first discuss about the possible existence of singular arcs and the loci where they can exist, and then we investigate the switching function in order to extract information about the number of switchings.

4.2.1 Singular arcs

Proposition 4.4. For any optimal solution to the problem $(\widetilde{P}j)$, a singular arc can only occur at $S_{\widetilde{P}j} = S_{tol}$ or $S_{\widetilde{P}j} = S^*$.

Proof The problems $(\widetilde{P}1)$ and $(\widetilde{P}2)$ are written with piecewise affine and continuous dynamics with respect to the state variables, and their cost can be rewritten as an integral cost with a piecewise affine continuous integrand function. Moreover, following Remark 7, either the problem $(\widetilde{P}3)$ is trivial (and presents singular arcs at S_{tol} or S^*), or it can be solved with an optimal control of the problem $(\widetilde{P}1)$.

As a consequence, the optimal control problems $(\widetilde{P}j)$ fulfill the framework of [3], and we deduce that there exists an optimal solution of $(\widetilde{P}j)$ composed of a succession of bang-bang arcs over time interval when the trajectory belongs to a region where the dynamics Eq. (11) is differentiable, and possible singular arcs located where the dynamics Eq. (11) is non-differentiable, or when a state constraint is active.

By definition of the functions K_S and K_R , and using optimality properties of Section 3, we deduce that a singular arc can only occur at the non-differentiable points S_{tol} and S^* . \square

Remark 10. We see that the loci of singular arcs are the non-differentiable points of the dynamics, therefore the use of more generic shapes for the functions K_S and K_R as discussed in Remark 1 is compatible with our methodological process. In particular, more non-differentiable points in K_S and K_R would simply generate a higher number of possible values of singular arcs.

4.2.2 Analysis of switching function

Proposition 4.5. For any optimal solution to the problem $(\widetilde{P}j)$, the set $I_{\widetilde{P}j}$ defined as follows is non-empty and connected.

$$I_{\widetilde{P}j} = \{t \in [t^*, T], \psi^{\widetilde{P}j}(t) \geq 0\}$$

Proof First notice that $I_{\widetilde{P}j}$ has non null measure, otherwise we would have $u_{\widetilde{P}j}(t) = 0$ for a.e. $t \in [0, T]$, which cannot be optimal.

Suppose that $I_{\widetilde{P}j}$ is not connected. In other words, assume that there exists an interval $(t_1, t_2) \subset [t^*, T]$ such that $\psi^{\widetilde{P}j}(t_1) = \psi^{\widetilde{P}j}(t_2) = 0$ and $\psi^{\widetilde{P}j}(t) < 0$ for all $t \in (t_1, t_2)$. This implies $u(t) = 0$ for a.e. $t \in (t_1, t_2)$ and thus $S_{\widetilde{P}j}(\cdot)$ is decreasing on (t_1, t_2) , which means that $S_{tol} < S_{\widetilde{P}j}(t) < S^*$ for all $t \in (t_1, t_2)$. Therefore the function $\lambda_S^{\widetilde{P}j}(\cdot)$ satisfies for all $t \in (t_1, t_2)$:

$$\lambda_S^{\widetilde{P}j}(t) \in \varphi(t)(\lambda_S^{\widetilde{P}j}(t)k_1 - \lambda_B^{\widetilde{P}j})\partial_C K_S(S_{\widetilde{P}j}(t)) + \lambda_S^{\widetilde{P}j}(t)k_1(1 - \varphi(t))\partial_C K_R(S_{\widetilde{P}j}(t))$$

Moreover, the function $\psi^{\widetilde{P}j}$ attains a local minimum over (t_1, t_2) , at a time denoted $\hat{t} \in (t_1, t_2)$. Since $\psi^{\widetilde{P}j}(t) = \lambda_S^{\widetilde{P}j}(t)k_1k_2 + \lambda_W^{\widetilde{P}j}$ then the function $\lambda_S^{\widetilde{P}j}$ also attains its local minimum over (t_1, t_2) at the time \hat{t} . At this local minimum, one has :

$$0 \in \varphi(\hat{t})(\lambda_S^{\widetilde{P}j}(\hat{t})k_1 - \lambda_B^{\widetilde{P}j})\partial_C K_S(S_{\widetilde{P}j}(\hat{t})) + \lambda_S^{\widetilde{P}j}(\hat{t})k_1(1 - \varphi(\hat{t}))\partial_C K_R(S_{\widetilde{P}j}(\hat{t})) \quad (25)$$

First note that Eq. (25) can be fulfilled only if $\lambda_S^{\widetilde{P}j}(\hat{t})k_1 - \lambda_B^{\widetilde{P}j} < 0$. Since we have $S_{tol} < S_{\widetilde{P}j}(t) < S^*$ for all $t \in (t_1, t_2)$, the quantities $K_S'(S_{\widetilde{P}j}(t))$ and $K_R'(S_{\widetilde{P}j}(t))$ are well defined and constant on (t_1, t_2) , respectively equal to $K_S'(S_{\widetilde{P}j}(\hat{t})) > 0$

and $K'_R(S_{\widetilde{P}_j}(\hat{t})) > 0$. Then one has for all $t \in (t_1, \hat{t})$:

$$\begin{aligned}
\dot{\lambda}_S^{\widetilde{P}_j}(t) &= \varphi(t)(\lambda_S^{\widetilde{P}_j}(t)k_1 - \lambda_B^{\widetilde{P}_j})K'_S(S_{\widetilde{P}_j}(t)) + \lambda_S^{\widetilde{P}_j}(t)k_1(1 - \varphi(t))K'_R(S_{\widetilde{P}_j}(t)) \\
&\geq \varphi(t)(\lambda_S^{\widetilde{P}_j}(\hat{t})k_1 - \lambda_B^{\widetilde{P}_j})K'_S(S_{\widetilde{P}_j}(\hat{t})) + \lambda_S^{\widetilde{P}_j}(\hat{t})k_1(1 - \varphi(t))K'_R(S_{\widetilde{P}_j}(\hat{t})) \\
&> \varphi(\hat{t})(\lambda_S^{\widetilde{P}_j}(\hat{t})k_1 - \lambda_B^{\widetilde{P}_j})K'_S(S_{\widetilde{P}_j}(\hat{t})) + \lambda_S^{\widetilde{P}_j}(\hat{t})k_1(1 - \varphi(\hat{t}))K'_R(S_{\widetilde{P}_j}(\hat{t}))
\end{aligned} \tag{26}$$

The last inequality holds since φ takes values in $[0, 1]$ and is increasing. We deduce from Eq. (26) that $\dot{\lambda}_S^{\widetilde{P}_j}(t) > 0$ for all $t \in (t_1, \hat{t})$, which contradicts the local minimality of $\lambda_S^{\widetilde{P}_j}(\hat{t})$ over (t_1, t_2) . Finally we conclude that the set $I_{\widetilde{P}_j}$ is connected. \square

4.3 Main result

We now define a class of time-varying feedback controls, and we prove that the optimal solutions to the problems (\widetilde{P}_j) belong to this class of strategies.

Definition 4.1. For $t_d \in [t^*, T]$, $W_c \in [V_{tol}, V^*]$, $t_c \in [t_d, T]$, we define the following time-varying feedback control :

$$U_{t_d, W_c, t_c}^{\widetilde{P}_j}(t, S, W) = \begin{cases} 0 & , (S \geq S_{tol} \wedge t \leq t_d) \vee W = W_c \vee t \geq t_c \\ u_{sing}(t, S_{tol}) & , S = S_{tol} \text{ and } t \leq t_d \\ u_{sing}(t, S^*) & , S = S^* \text{ and } (W < W_c \wedge t < t_c) \\ 1 & , \text{otherwise} \end{cases}$$

Theorem 4.6. *In the framework of Definition 3.1 for the optimal control problems given in Definition 3.2 with the dynamics Eq. (11), there exists $t_d \in [t^*, T]$, $W_c \in [V_{tol}, V^*]$, $t_c \in [t_d, T]$ such that the feedback control $U_{t_d, W_c, t_c}^{\widetilde{P}_j}$ is optimal for the problem (\widetilde{P}_j) .*

Proof Let $u_{\widetilde{P}_j}(\cdot)$ be an optimal control for the problem (\widetilde{P}_j) . As seen in Proposition 4.5, the set $I_{\widetilde{P}_j}$ is non-empty and connected. Since the function $\psi^{\widetilde{P}_j}(\cdot)$ is continuous, we deduce that $I_{\widetilde{P}_j}$ is an interval $[t_a, t_c]$ where $t^* \leq t_a < t_c \leq T$, and the control $u_{\widetilde{P}_j}(\cdot)$ is null outside this interval.

At any $t \in I_{\widetilde{P}_j}$ the switching function $\psi^{\widetilde{P}_j}(t)$ is non-negative, then for a.e. $t \in I_{\widetilde{P}_j}$ one has either $u_{\widetilde{P}_j}(t) = 1$ or $u_{\widetilde{P}_j}(t) = u_{sing}(t, \tilde{S})$ where $\tilde{S} \in \{S_{tol}, S^*\}$. As a consequence, the quantity $W_{\widetilde{P}_j}(\cdot)$ is increasing on $I_{\widetilde{P}_j}$ and $S_{\widetilde{P}_j}(\cdot)$ is non-decreasing on $I_{\widetilde{P}_j}$. More precisely the trajectory $S_{\widetilde{P}_j}(\cdot)$ is composed of one increasing part (with $u_{\widetilde{P}_j} = 1$) and possible constant parts (at $S_{\widetilde{P}_j} = S_{tol}$ before the increasing part, or at $S_{\widetilde{P}_j} = S^*$ after the increasing part). The increasing part is thus over an interval $[t_d, t_f]$ with $t_a \leq t_d < t_f \leq t_c$.

At $t = t_c$, we know that $S_{\widetilde{P}_j}(t_c) \leq S^*$ and since $W_{\widetilde{P}_j}$ is increasing on $I_{\widetilde{P}_j}$ we deduce that t_c and $W_{\widetilde{P}_j}(t_c)$ are equivalently defined, and we set $W_c = W_{\widetilde{P}_j}(t_c)$. We have either $t_a = t_d$ or t_a fully determined by the trajectory $S_{0, S_{0,0}}(\cdot)$ such that $S_{0, S_{0,0}}(t_a) = S_{tol}$. Moreover, the time t_f is either equal to t_c or fully determined by the trajectory $S_{t_d, S_{\widetilde{P}_j}(t_d), 1}(\cdot)$ such that $S_{t_d, S_{\widetilde{P}_j}(t_d), 1}(t_f) = S^*$.

Finally, the control $u_{\widetilde{P}_j}(\cdot)$ is fully determined by the two parameters t_d and t_c (or equivalently t_d and W_c). We check that it fulfills the feedback formulation $u_{\widetilde{P}_j}(t) = U_{t_d, W_c, t_c}^{\widetilde{P}_j}(t, S_{\widetilde{P}_j}(t), W_{\widetilde{P}_j}(t))$ for a.e. $t \in [0, T]$. \square

Remark 11. *It may seem that the optimal parameterized feedback in Definition 4.1 is defined with three parameters (t_d, W_c, t_c) . The proof of Theorem 4.6 hints that we only need two of these parameters. In fact we can be more precise for each problem (\widetilde{P}_j) .*

For the problem (\widetilde{P}_1) , we set $W_c = Q$, and we can posit $t_c = T$ as convention. This way, the control $U_{t_d, Q, T}^{\widetilde{P}_1}(t, S, W)$ belongs to the admissibility set $\mathcal{W}^{\widetilde{P}_1}$, and it suffices to perform a one-dimensional optimization over $t_d \in [t^, T]$ to find the optimal solution $u_{\widetilde{P}_1}$.*

For the problem $(\widetilde{P2})$, we can posit $W_c = V^*$ as convention, and for each t_d we can compute t_c such that the associated trajectory $(S(\cdot), B(\cdot), W(\cdot))$ reaches exactly $B(T) = Y$. This way, the control $U_{t_d, V^*, t_c}^{\widetilde{P2}}(t, S, W)$ belongs to the admissibility set $\mathcal{U}^{\widetilde{P2}}$, and it suffices to perform a one-dimensional optimization over $t_d \in [t^*, T]$ to find the optimal solution $u_{\widetilde{P2}}$.

For the problem $(\widetilde{P3})$, we don't have more information from its settings to help determining W_c or t_c , so we can just posit $W_c = V^*$ as convention. This way, the control $U_{t_d, V^*, t_c}^{\widetilde{P3}}(t, S, W)$ belongs to the admissibility set $\mathcal{U}^{\widetilde{P3}}$, and it will be needed perform a two-dimensional optimization over $t_d \in [t^*, T]$ and $t_c \in [t_d, T]$ to find the optimal solution $u_{\widetilde{P3}}$.

5 Conclusion

We dealt with a family of optimal control problems, and we showed, using dedicated tools (in particular an application of the extended bang-bang principle), that optimal irrigation strategies for these problems share the same structure of optimal feedback. These strategies can be interpreted as threshold-based strategies, where the values of thresholds correspond to non-differentiable points of the dynamics. This result allows to compute numerically the optimal solutions as a simple low dimensional optimization of the parameters of the structured optimal policies, hence enabling large numerical model exploration. Such threshold-based irrigation strategies can also be tested on more details models (for which the analytic study of the optimal feedback strategies would not be possible), and are suited to closed-loop experimental implementation, typically using soil moisture sensors.

Acknowledgements

This work is funded by INRAE-MathNum and Labex Numev (ANR-10-LABX-0020), and supported by the French National Research Agency under the Investments for the Future Program, referred as ANR-16-CONV-0004.

The authors are also thankful to the French Ministry for Europe and Foreign Affairs, and the French Ministry for Higher Education and Research for the "PHC PESSOA" programme (project number 52874SG : OITE (Optimizing Irrigation efficiency under Technical and Environmental constraints)).

References

- [1] Bertrand, N., Roux, S., Forey, O., Guinet, M., Wery, J. : Simulating plant water stress dynamics in a wide range of bi-specific agrosystems in a region using the BISWAT model. European Journal of Agronomy 99, 116-128 (2018)
- [2] Boumazza, K., Kalboussi, N., Rapaport, A., Roux, S., Sinfort, C. : Optimal control of a crop irrigation model under water scarcity. Optimal Control Applications and Methods 42, 1612-1631 (2021)
- [3] Chenevat, R., Cheviron, B., Roux, S., Rapaport, A. : About the Bang-Bang Principle for piecewise affine systems. IEEE 63rd Conference on Decision and Control (CDC), Milan, Italy, pp. 1004-1009 (2024)
- [4] Cheviron, B., Lo, M., Catel, J., Delmas, M., Elamri, Y., Akakpo, K., Ramos-Fuentes, I.-A., Dominguez-Bohorquez, J.-D., Cezar, M. G. de, Razavi-Ebrahimi, Y., Degenne, A., Chenevat, R., Mailhol, J.-C. : The Optirrig model for the generation, analysis and optimization of irrigation scenarios: rationale and scopes. ISHS Acta Horticulturae 1373 (2022)
- [5] Clarke, F. : Functional Analysis, Calculus of Variations and Optimal Control, Springer, Graduate Texts in Mathematics 264, London (2013)
- [6] Delgoda, D., Malano, H., Saleem, S. K., Halgamuge, M. N. : Irrigation control based on model predictive control (MPC): Formulation of theory and validation using weather forecast data and AQUACROP model. Environmental Modelling & Software 78, 40-53 (2016)
- [7] Ding, Y., Wang, L., Li, Y., Li, D. : Model predictive control and its application in agriculture: A review. Computers and Electronics in Agriculture 151, 104-117 (2018)

- [8] Geerts, S., Raes, D. : Deficit irrigation as an on-farm strategy to maximize crop water productivity in dry areas. *Agricultural Water Management* 96, 1275-1284 (2009)
- [9] Hartl, R. : Most rapid approach paths in dynamic economic problems. *Athenäum/Hain/Hanstein, Königstein, Methods Oper. Res.* 58, 397-410 (1989)
- [10] Hartl, R. F., Feichtinger, G. : A new sufficient condition for most rapid approach paths. *Journal of Optimization Theory and Applications* 54, 403-411 (1987)
- [11] Hartl, R. F., Sethi, S. P., Vickson, R. G. : A Survey of the Maximum Principles for Optimal Control Problems with State Constraints. *SIAM Review* 37, 181-218 (1995)
- [12] Hermes, H., LaSalle, J. P. : Functional analysis and time optimal control. *Mathematics in Science and Engineering* 56, Academic Press, New York-London (1969)
- [13] Jones, J. W., Hoogenboom, G., Porter, C. H., Boote, K. J., Batchelor, W. D., Hunt, L. A., Wilkens, P. W., Singh, U., Gijsman, A. J., Ritchie, J. T. : The DSSAT cropping system model. *European Journal of Agronomy* 18, 235-265 (2003)
- [14] Kalboussi, N., Roux, S., Boumaza, K., Sinfort, C., Rapaport, A. : About modeling and control strategies for scheduling crop irrigation. *IFAC Workshop on Control Methods for Water Resource Systems, IFAC-PapersOnLine* 52 (23), 43-48 (2019)
- [15] Lee, E.B., Markus, L. : Foundations of optimal control theory. *SIAM series in applied mathematics*. John Wiley & sons, inc., New York (1967)
- [16] Lemos-Paião, A. P., Lopes, S. O., Pinho, M. D. R. de : A parametric optimal control problem applied to daily irrigation. *Mathematical Modelling of Natural Phenomena* 20 (2025)
- [17] Linker, R., Ioslovich, I., Sylaios, G., Plauborg, F., Battilani, A. : Optimal model-based deficit irrigation scheduling using AquaCrop: A simulation study with cotton, potato and tomato. *Agricultural Water Management* 163, 236-243 (2016)
- [18] Lopes, S. O., Fontes, F. A. C. C., Pereira, R. M. S., Pinho, M. D. R. de, Gonçalves, A. M. : Optimal Control Applied to an Irrigation Planning Problem. *Mathematical Problems in Engineering* 2016 (2016)
- [19] Pelak, N., Revelli, R., Porporato, A. : A dynamical systems framework for crop models: Toward optimal fertilization and irrigation strategies under climatic variability. *Ecological Modelling* 365, 80-92 (2017)
- [20] Rao, N. H., Sarma, P. B. S., Chander, S. : Irrigation scheduling under a limited water supply. *Agricultural Water Management* 15, 165-175 (1988)
- [21] Romero, R., Muriel, J. L., García, I., Muñoz de la Peña, D. : Research on automatic irrigation control: State of the art and recent results. *Agricultural Water Management* 114, 59-66 (2012)
- [22] Saleem, S. K., Delgoda, D. K., Ooi, S. K., Dassanayake, K. B., Liu, L., Halgamuge, M. N., Malano, H. : Model Predictive Control for Real-Time Irrigation Scheduling. *IFAC Proceedings Volumes* 46, 299-304 (2013)
- [23] Schättler, H. : Local Fields of Extremals for Optimal Control Problems with State Constraints of Relative Degree 1. *J Dyn Control Syst* 12, 563-599 (2006)
- [24] Serra-Wittling, C., Molle, B., Cheviron, B. : Plot level assessment of irrigation water savings due to the shift from sprinkler to localized irrigation systems or to the use of soil hydric status probes. Application in the French context. *Agricultural Water Management* 223 (2019)
- [25] Shang, C., Chen, W.-H., Stroock, A. D., You, F. : Robust Model Predictive Control of Irrigation Systems With Active Uncertainty Learning and Data Analytics. *IEEE Transactions on Control Systems Technology* 28, 1493-1504 (2020)
- [26] Shani, U., Tsur, Y., Zemel, A. : Optimal dynamic irrigation schemes. *Optimal Control Applications and Methods* 25, 91-106 (2004)

- [27] Trélat, E. : Linear turnpike theorem. *Mathematics of Control, Signals, and Systems* 35, 685-739 (2023)
- [28] Trélat, E., Zuazua, E. : The turnpike property in finite-dimensional nonlinear optimal control. *Journal of Differential Equations* 258, 81-114 (2015)
- [29] Vinter, R. : *Optimal Control*. Birkhäuser, Boston (2010)
- [30] Wang, B., Dam, J. van, Yang, X., Ritsema, C., Du, T., Kang, S. : Reducing water productivity gap by optimizing irrigation regime for winter wheat-summer maize system in the North China Plain. *Agricultural Water Management* 280 (2023)