Structures of optimal solutions for a model of crop irrigation under biological and operating constraints

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Contents

1 Introduction

In the present context where the world population keeps growing and climate tensions intensify, one has to wonder about natural resources management, especially in agriculture. Rising temperatures, increasing soil evapotranspiration, decreasing groundwater levels, multiplication of extreme and non-easily predictable weather events are examples of phenomena that have huge impact on agricultural practices. In particular there is an increase of areas where irrigation has a more important part than rainfalls for crop production, while the amount of available water per capita is expected to decrease in the near future. In order to match with the upcoming need to step up the world food production while adapting to these new pressures, different approaches of improvement exist. It is possible to deal with saving water through agronomic or technological levers (choice of species, equipment, pilotage). The latter seems to have promising water saving potential in the framework of deficit irrigation (see [13], [18]).

Multiple mathematical models have been developed to study crop irrigation in various ways and various scales. A certain amount of them are complex models describing quite precisely the physiological processes of the plant evolution ([8], [6], [2], [10], [17]), where the numerical approach for irrigation optimization is privileged. On the other hand, some more parsimonious models with less parameters are adapted to a simpler dynamical system model and decision making through a limited number of decision variables. These models have inspired several mathematical formulations of optimal control problems representing crop growth and irrigation as the decision variable to optimize, in order to derive analytical results (see [1], [7], [11], [16], [19], [20]). They typically use a water balance with few hypotheses on the functions modeling the biological process, enabling a general study of their problem (maximizing a financial cost or minimizing the water used to keep the soil moisture above a threshold). Some of these models give a more detailed description of the evapotranspiration and maximizes the production of biomass under the context of water scarcity.

The interest of analytical results is to eventually provide strategies describing optimal irrigation through time-varying feedback control, and the agronomic point of view may have interest in many objectives : maximizing the biomass under water constraint, minimizing the water used for reaching a target of biomass, maximizing a financial cost, keeping the soil moisture above a stress threshold. In this work, we aim at studying these problems. Therefore we choose to use the formulation from [1] and to extend its work to the aforementioned objectives. Furthermore we will not only look for the best strategies, but we will also investigate the worst strategies since it seems crucial to be able to quantify the potential gain margin that we might expect from an appropriate water management. More precisely, we define six optimization problems (three objectives, best and worst cases) that we regroup in order to analyze them in a unified way. The chosen model and its conditionning bring a certain amount of mathematical difficulties in order to solve the optimization problem. Indeed we will have to deal with an optimal control problem with non-autonomous and non-smooth dynamics, with different types of terminal criterion and targets, and subject to some state inequality constraints. Through an analysis of the necessary conditions of optimality, we wonder if there exists a unified optimal structure for the optimal strategies (for the whole set of objectives). In other words, the spirit of our approach is to give preference to reasonably simple mathematical models for which an analytical characterization of the optimality is tractable, in particular in terms of state feedback. These feedback strategies are then expected to be implemented and tested in situations closer to real world.

In the section 2, we describe the model with its general hypotheses, and we present the different problems (depending on their different objectives) that we investigate altogether through a unified formulation. In particular we provide an insight of the unusual method we used to deal with the state constraints. After some general results on the system and on particular trajectories entailed by the so-called MRAP controls presented in the section 3, we treat in the section 4 some trivial cases and we restrict our study to the framework of deficit irrigation corresponding to water scarcity. Under this last hypothesis, we are able to show some necessary properties on optimal solutions at the end of the section 4. Then we apply the non-smooth Pontryagin's Maximum Principle and we derive and exploit the necessary conditions of optimality, to finally obtain in the section 5 the expressions of the optimal structures for the best and the worst strategies. We then illustrate the theoretical results with numerical simulations and we draw the crop production curbs and various comparison indicators to discuss about their application.

2 Model description and general hypothesis

2.1 Model description

The definition of the problems is based on a simplified crop model, only considering the soil moisture balance in terms of the soil evaporation, the plant transpiration, and the water contribution through irrigation (hence we do not consider the rainfall in this work), during the period from sowing to harvest. We use the following general water balance on any time interval : "$\Delta S = -Evaporation - Transpiration + Irrigation$".
We then consider a system of differential equations, the first equation being the direct translation of the formula above.

$$(\mathcal{S}_{basis}) : \begin{cases} \dot{S}(t) = k_1(-\varphi(t)K_S(S(t)) - (1-\varphi(t))K_R(S(t)) + k_2u(t)) \\ \dot{B}(t) = \varphi(t)K_S(S(t))f(B(t)) \\ \dot{V}(t) = u(t) \end{cases} ,\forall t \in [0,T]$$

with $B(0) = B_0 > 0$.

The system progresses from the sowing date $t = 0$ to the harvestiong date $t = T$. As we can see, the biomass variation is supposed to be proportional to the plant transpiration dynamics. The first two equations represent the dynamics of the soil moisture and the biomass, the third one enables us to follow the water consumption. The term $k_2u(t)$ describes the water contribution through irrigation, and we call $u(\cdot)$ the "control" of the system. This is the manipulating variable that we seek to optimise, i.e. we are looking for the "best" (or the "worst") $u(\cdot)$ amongst all the admissible controls, according to the criterium of the problem. We assume that irrigation occurs with a maximum flow of F_{max}, renormalising the control $u(\cdot)$ to take values between 0 and 1. Let us define the set of the admissible controls as follows :

$$\mathcal{U} = \{u : [0,T] \mapsto [0,1], \text{measurable}\}$$

Let us now deal with the class of functions representing the evaporation and the transpiration : we follow some well known and already studied models in previous works (cf .....), and we consider the functions K_S and K_R piecewise affine (note that this implies a non-smooth framework). Thus we have to set several critical points for some values of S referring to what we will then call "thresholds". These moisture thresholds are inner parts of the model hypothesis. The transpiration of the plant is maximum above a certain threshold S^* and it is minimal below another certain threshold S_w. Moreover the soil is not capable of evaporating water anymore under a third certain moisture threshold S_h.

Hypothesis 2.1.
These differents moisture thresholds are supposed to be satisfying $0 < S_h < S_w < S^* < 1$.

We are now able to give the expressions of the function K_S and K_R respectively modelising the plant transpiration and the soil evaporation.

Definition 2.1.
The functions K_S and K_R are piecewise affines, non-decreasing, and defined by the following expressions :

$$K_S(S) = \begin{cases} 0 & ,\text{if } S \in [0, S_w] \\ \frac{S-S_w}{S^*-S_w} & ,\text{if } S \in [S_w, S^*] \\ 1 & ,\text{if } S \geq 1 \end{cases}$$

$$K_R(S) = \begin{cases} 0 & ,\text{if } S \in [0, S_h] \\ \frac{S-S_h}{1-S_h} & ,\text{if } S \geq S_h \end{cases}$$

The function φ related to the cover rate by the plant naturally satisfies some regularity and monotony assumptions.

Hypothesis 2.2.
The function φ *is* $\mathcal{C}^1$*, increasing, with initial and final conditions* $\varphi(0) \geq 0$ *and* $\varphi(T) \leq 1$.

Finally, the constants k_1 and k_2 are normalising constants for the system and can be calibrated through real data. There is no additional requirement on k_1, but the constant k_2 has to be large enough to ensure that we are able to "refill" the soil through irrigation (otherwise the soil moisture would always be decreasing, which is not acceptable), which amounts to require the irrigation flow to be large enough. This can be seen as a controllability condition.

Hypothesis 2.3.
The parameters k_1 and k_2 are positive, with $k_2 > 1$.

The last term we didn't discuss in the system is the function f. It represents a growth coefficient of the plant depending on the quantity of biomass.

Hypothesis 2.4.
The function f is a non-negative Lipschitz continuous function with linear growth, and satisfies $f(B_0) > 0$.

Remark 2.1.
It is possible to drop that term from the equation on the biomass, thanks to the following change of variable for B :

$$B \mapsto \tilde{B} = g(B) = \int_{B_0}^{B} \frac{1}{f(b)} \mathrm{d}b \quad , \quad B \in [B_0, \bar{B})$$

This simplifies the corresponding equation in the dynamics :

$$\dot{\tilde{B}}(t) = \varphi(t) K_S(S(t)) \quad , \quad \tilde{B}_0 = 0$$

Thereby optimising $B(T)$ is equivalent to optimising $\tilde{B}(T)$, we will then write the equation on B without loss of generality :

$$\dot{B}(t) = \varphi(t) K_S(S(t)) \quad , \quad B_0 = 0$$

Moreover, in order to prepare the following work, we introduce the following system of equations, with an extended number of states which will be explained later. The function C_Y and C_Z are only supposed to be continuous for the moment. The final form of the system of differential equations that we are going to study thus rewrites :

$$(\mathcal{S}) \; : \quad \begin{cases} \dot{S}(t) = k_1(-\varphi(t)K_S(S(t)) - (1-\varphi(t))K_R(S(t)) + k_2 u(t)) \\ \dot{B}(t) = \varphi(t)K_S(S(t)) \\ \dot{V}(t) = u(t) \\ \dot{Y}(t) = C_Y(S(t)) \\ \dot{Z}(t) = C_Z(S(t)) \end{cases} \quad , \forall t \in [0,T]$$

with initial conditions :

$$(IC) \; : \quad S(0) = S_0 \in [0,1] \quad , \quad B(0) = 0 \quad , \quad V(0) = 0 \quad , \quad Y(0) = Y_0 \in \mathbb{R} \quad , \quad Z(0) = Z_0 \in \mathbb{R}$$

Remark 2.2. Some of these equations may be dropped later in accordance with the problems that we study.

In this section, we aim to formulate precisely the problems and framework.

2.2 Constraints and targets

(upcoming work) (+ physical definition of $(Bj), (Wj)$)

2.3 Mathematical correspondence

2.3.1 State constraints

The biological system induces a natural constraint for the soil moisture :

$$0 \leq S(t) \leq 1 \quad , \quad \forall t \in [0,T]$$

Moreover in this work we assume that the operator wants to prevent the plants to undergo too much of a water stress, therefore we add a supplementary constraint on the soil moisture as follows :

$$(\mathcal{C}): \quad S_{tol} \leq S(t) \leq 1 \quad , \quad \forall t \in [0,T]$$

with $S_w \leq S_{tol} \leq S^*$ representing the tolerance threshold.
In order to cope with this condition, the standard idea would be to add a state constraint to the system of equations $(\mathcal{S}_{basis})$, but as we seek to apply the Pontryagin Maximum Principle, we would like to avoid the complicated necessary conditions resulting of the statement of the PMP with running state constraints. There comes the explaination of the variables Y and Z. Replacing the constraint $(\mathcal{C})$ by the equations :

$$\dot{Y}(t) = C_Y(S(t)) \quad , \quad \dot{Z}(t) = C_Z(S(t))$$

where

$$C_Y(S) = \min(0, S - S_{tol}) \quad , \quad C_Z(S) = \max(0, S-1)$$

with the initial conditions $Y(0) = 0, Z(0) = 0$ and the final conditions $Y(T) \geq 0, Z(T) \leq 0$ gives an equivalent problem with two additional variables and final conditions, but without state constraint, as we show in the next lemma.

Lemma 2.5.
The initial system :

$$(\mathcal{S}_{basis}): \begin{cases} \dot{S}(t) = k_1(-\varphi(t)K_S(S(t)) - (1-\varphi(t))K_R(S(t)) + k_2 u(t)) \\ \dot{B}(t) = \varphi(t)K_S(S(t))f(B(t)) \\ \dot{V}(t) = u(t) \end{cases} , \forall t \in [0,T]$$

with initial conditions $S(0) = S_0 \in (S^*, 1]$, $B(0) = B_0 > 0$, $V(0) = 0$, *under the state constraints* $S_{tol} \leq S(t) \leq 1$, *and the system :*

$$(\mathcal{S}) \ : \ \begin{cases} \dot{S}(t) = k_1(-\varphi(t)K_S(S(t)) - (1-\varphi(t))K_R(S(t)) + k_2 u(t)) \\ \dot{B}(t) = \varphi(t)K_S(S(t)) \\ \dot{V}(t) = u(t) \\ \dot{Y}(t) = C_Y(S(t)) \\ \dot{Z}(t) = C_Z(S(t)) \end{cases} , \forall t \in [0,T]$$

with initial conditions $S(0) = S_0 \in (S^*, 1]$, $B(0) = 0$, $V(0) = 0$, $Y(0) = 0$, $Z(0) = 0$ *and final conditions* $Y(T) \geq 0$, $Z(T) \leq 0$, *are equivalent.*

2.3.2 Criteria and terminal conditions

(upcoming work)

2.3.3 General formulation

We then have the general formulation of our problems as follows :

$$(\mathcal{OC}) \ : \begin{cases} \underset{u(\cdot)\in\mathcal{F}}{opti} \ \Omega(X(T)) \\ X(T) \in \mathcal{K} \end{cases}$$

where '*opti*' might stand for 'sup' or 'inf', and $\Omega : \mathbb{R}^p \longrightarrow \mathbb{R}$ is a terminal cost function. This function, the set of feasible controls $\mathcal{F}$ and the non-empty closed convex target $\mathcal{K} \subset \mathbb{R}^p$ depend on the nature of the problem we want to study.
As said previously, we are eventually dropping some equations of the system for certain problems, thus we are not considering the corresponding coordinates, so that p is an integer in $\{4, 5\}$.

3 Preliminary results

In this section, we show some results which don't require the theory of optimal control, and which are inherent to the system of equations, no matter what the initial condition or the constraints are.

3.1 First lemmas

Let us make the following remark derived from standard considerations on ordinary differential equations.

Lemma 3.1.
Let $u \in \mathcal{U}$. Let $(S(\cdot), B(\cdot), V(\cdot), Y(\cdot), Z(\cdot))$ be a solution of $(\mathcal{S})$ with the initial conditions (IC).
If there exists $t_0 \in [0, T]$ such that $S(t_0) > S_h$, then :

$$\forall t \in [t_0, T],\ S(t) > S_h$$

Proof.
Denote $\tilde{S}(\cdot)$ the solution of $(\mathcal{S})_{|u=0}$ (the system $(\mathcal{S})$ with the constant control $u = 0$), such that $\tilde{S}(t_0) = S(t_0) > S_h$.
Since $u \geq 0$, we have $\dot{S}(t) \geq \dot{\tilde{S}}(t)$ for all $t \in [t_0, T]$, and then $S(t) \geq \tilde{S}(t)$ for all $t \in [t_0, T]$.
Moerover the trajectory $S_c(\cdot)$ such that $S_c(t) = S_h$ for all $t \in [t_0, T]$ is a solution of $(\mathcal{S})_{|u=0}$. By Cauchy-Lipschitz, we know that two trajectories of the same equation with different initial conditions don't intersect.
Thus for all $t \in [t_0, T]$ one has $\tilde{S}(t) > S_c(t)$.
As a consequence, one gets $S(t) > S_h$ for all $t \in [t_0, T]$. □

Lemma 3.2.
Let $u \in \mathcal{U}$. Let $(S(\cdot), B(\cdot), V(\cdot), Y(\cdot)), Z(\cdot))$ be a solution of $(\mathcal{S})$ with the initial conditions (IC).
Then $B(\cdot)$ and $V(\cdot)$ are uniformly bounded on $[0, T]$.

Proof.
Since $u \in [0, 1]$ and $K_S \in [0, 1]$, we obtain :

$$\forall t \in [0, T], \quad 0 \leq B(t) \leq \int_0^T \varphi(t)\mathrm{d}t \quad \text{and} \quad 0 \leq V(t) \leq T$$

□

Lemma 3.3.
Let $(S_1(\cdot), B_1(\cdot), V_1(\cdot), Y_1(\cdot)), Z_1(\cdot))$ and $(S_2(\cdot), B_2(\cdot), V_2(\cdot), Y_2(\cdot)), Z_2(\cdot))$ be two solutions of the system $(\mathcal{S})$ such that :

$$S_1(t_1) = S_2(t_1) \quad \textit{and} \quad S_1 \geq S_2 \ \textit{ on } \ [t_1, t_2] \quad , \quad \textit{for } \ 0 \leq t_1 < t_2 \leq T$$

Let $u_1, u_2 \in \mathcal{U}$ be their corresponding controls.
Then :

$$\forall t \in [t_1, t_2]\ , \quad \int_{t_1}^{t} u_1(\tau)\mathrm{d}\tau \geq \int_{t_1}^{t} u_2(\tau)\mathrm{d}\tau$$

Proof.
Let $t \in [t_1, t_2]$. One has :

$$\begin{aligned} S_1(t) - S_2(t) &= \int_{t_1}^{t} k_1(-\varphi(\tau)K_S(S_1(\tau)) - (1-\varphi(\tau))K_R(S_1(\tau)) + k_2 u_1(\tau))\mathrm{d}\tau \\ &\qquad - \int_{t_1}^{t} k_1(-\varphi(\tau)K_S(S_2(\tau)) - (1-\varphi(\tau))K_R(S_2(\tau)) + k_2 u_2(\tau))\mathrm{d}\tau \\ &= k_1 \int_{t_1}^{t} \varphi(\tau)(K_S(S_2(\tau)) - K_S(S_1(\tau))) + (1-\varphi(\tau))(K_R(S_2(\tau)) - K_R(S_1(\tau)))\mathrm{d}\tau \\ &\qquad + k_1 k_2 \int_{t_1}^{t} u_1(\tau) - u_2(\tau)\mathrm{d}\tau \end{aligned}$$

As the functions K_S and K_R are non-decreasing in S, we get :

$$K_S(S_2(\tau)) - K_S(S_1(\tau)) \leq 0 \quad \text{and} \quad K_R(S_2(\tau)) - K_R(S_1(\tau)) \leq 0$$

Then we have :

$$0 \leq S_1(t) - S_2(t) \leq k_1 k_2 \int_{t_1}^{t} u_1(\tau) - u_2(\tau)\mathrm{d}\tau$$

Hence :

$$\forall t \in [t_1, t_2]\ , \quad \int_{t_1}^{t} u_1(\tau)\mathrm{d}\tau \geq \int_{t_1}^{t} u_2(\tau)\mathrm{d}\tau$$

□

Remark 3.1.
Moreover if the trajectories of S_1 and S_2 belong to $[S_h, 1]$, since the function $\varphi K_S + (1-\varphi)K_R$ is increasing in S over $[S_h, 1]$, then the inequality above is strict unless the trajectories of S_1 and S_2 are identical.

3.2 Most Rapid Approach Path

Before analysing the specificity of each problem, we can already show some properties of a certain class of controls that will be usefull in the following. This class of controls, called MRAP (Most Rapid Approach Path) for a certain fixed moisture value, consists in bringing the soil moisture from the start to the fixed value as fast as possible, then keeping this constant value as long as possible, and finally meeting the end point as fast as possible.
This MRAP control can be explained qualitatively in two interesting cases : when the starting and end points are both above the certain fixed moisture value, and when these points are both below the certain fixed value. Indeed we will see later that in the first case the MRAP control is the least consuming way to reach the end point from the starting point, while remaining above the fixed value. Symmetrically, in the second case the MRAP control is the most consuming way to reach the end point from the strating point, while remaining below the fixed value.
Therefore we naturally introduce two definitions : the MRAP by upper values and the MRAP by lower values, respectively associated with the aforementionned cases. We then need to define the corresponding tools for each case.

In the following, we denote $S_{t_i,S_i,u(\cdot)}(\cdot)$ the trajectory of $S(\cdot)$ induced by the control $u(\cdot)$ with initial time t_i and initial condition S_i.

3.2.1 MRAP from above

Definition 3.1.
Let $0 < \bar{S} < 1$.

For any $(t_S, S_S) \in [0, T) \times [\bar{S}, 1]$, we define :

$$t^+_{up}(t_S, S_S) = \begin{cases} T & , \text{ if } S_{t_S,S_S,0}(t) > \bar{S} \text{ for } t \in [t_S, T] \\ \inf\{t > t_S \ , \ S_{t_S,S_S,0}(t) = \bar{S}\} & , \text{ otherwise} \end{cases}$$

For any $(t_S, S_S) \in (0, T] \times [\bar{S}, 1]$, we define :

$$t^-_{up}(t_S, S_S) = \begin{cases} 0 & , \text{ if } S_{t_S,S_S,1}(t) > \bar{S} \text{ for } t \in [0, t_S] \\ \sup\{t < t_S \ , \ S_{t_S,S_S,1}(t) = \bar{S}\} & , \text{ otherwise} \end{cases}$$

Definition 3.2.
Let $0 < \bar{S} < 1$.
For any $(t_1, S_1) \in [0, T) \times [\bar{S}, 1]$ and $(t_2, S_2) \in (0, T] \times [\bar{S}, 1]$ such that S_2 is attainable at the time t_2 from (t_1, S_1) with some admissible control $u(\cdot)$, we call MRAPup to the value $\bar{S}$ the control $\tilde{u}^{up}_{\bar{S}}(\cdot)$ on $[t_1, t_2]$ defined by the following :
i) If $t^+_{up}(t_1, S_1) \le t^-_{up}(t_2, S_2)$:

$$\tilde{u}^{up}_{\bar{S}}(t) = \begin{cases} 0 & , \text{ if } t \in [t_1, t^+_{up}(t_1, S_1)) \\ u_{sing}(t, \bar{S}) & , \text{ if } t \in [t^+_{up}(t_1, S_1), t^-_{up}(t_2, S_2)] \\ 1 & , \text{ if } t \in (t^-_{up}(t_2, S_2), t_2] \end{cases}$$

ii) If $t^-_{up}(t_2, S_2) < t^+_{up}(t_1, S_1)$:

$$\tilde{u}_{\ge \bar{S}}(t) = \begin{cases} 0 & , \text{ if } t \in [t_1, \bar{t}_{up}(t_1, S_1, t_2, S_2)) \\ 1 & , \text{ if } t \in [\bar{t}_{up}(t_1, S_1, t_2, S_2), t_2] \end{cases}$$

where $\bar{t}_{up}(t_1, S_1, t_2, S_2)$ is the unique $\bar{t} \in [t_1, t_2]$ such that $S_{t_1,S_1,0}(\bar{t}) = S_{t_2,S_2,1}(\bar{t}) > \bar{S}$.
(We indeed check that the function $I(t) = S_{t_1,S_1,0}(t) - S_{t_2,S_2,1}(t)$ is decreasing on $[t_1, t_2]$ and that $I(t_1) \ge 0$ and $I(t_2) \le 0$, hence the existence and uniqueness of $\bar{t}_{up}(t_1, S_1, t_2, S_2)$.)

3.2.2 MRAP from below

Definition 3.3.
Let $0 < \bar{S} < 1$.
For any $(t_S, S_S) \in [0, T) \times [0, \bar{S}]$, we define :

$$t^+_{low}(t_S, S_S) = \begin{cases} T & , \text{ if } S_{t_S,S_S,1}(t) < \bar{S} \text{ for } t \in [t_S, T] \\ \inf\{t > t_S \ , \ S_{t_S,S_S,1}(t) = \bar{S}\} & , \text{ otherwise} \end{cases}$$

For any $(t_S, S_S) \in (0, T] \times [0, \bar{S}]$, we define :

$$t^-_{low}(t_S, S_S) = \begin{cases} 0 & , \text{ if } S_{t_S,S_S,0}(t) < \bar{S} \text{ for } t \in [0, t_S] \\ \sup\{t < t_S \ , \ S_{t_S,S_S,0}(t) = \bar{S}\} & , \text{ otherwise} \end{cases}$$

Definition 3.4.
Let $0 < \bar{S} < 1$.
For any $(t_1, S_1) \in [0, T) \times [0, \bar{S}]$ and $(t_2, S_2) \in (0, T] \times [0, \bar{S}]$ such that S_2 is attainable at the time t_2 from (t_1, S_1) with some admissible control $u(\cdot)$, we call MRAPlow to the value $\bar{S}$ the control $\tilde{u}^{low}_{\bar{S}}(\cdot)$ on $[t_1, t_2]$ defined by the following :
i) If $t^+_{low}(t_1, S_1) \le t^-_{low}(t_2, S_2)$:

$$\tilde{u}^{low}_{\bar{S}}(t) = \begin{cases} 1 & , \text{ if } t \in [t_1, t^+_{low}(t_1, S_1)) \\ u_{sing}(t, \bar{S}) & , \text{ if } t \in [t^+_{low}(t_1, S_1), t^-_{low}(t_2, S_2)] \\ 0 & , \text{ if } t \in (t^-_{low}(t_2, S_2), t_2] \end{cases}$$

ii) If $t^-_{low}(t_2,S_2) < t^+_{low}(t_1,S_1)$:

$$\tilde{u}^{low}_{\bar{S}}(t) = \begin{cases} 1 & , \text{ if } t \in [t_1, \bar{t}_{low}(t_1,S_1,t_2,S_2)) \\ 0 & , \text{ if } t \in [\bar{t}_{low}(t_1,S_1,t_2,S_2), t_2] \end{cases}$$

where $\bar{t}_{low}(t_1,S_1,t_2,S_2)$ is the unique $\bar{t} \in [t_1,t_2]$ such that $S_{t_1,S_1,1}(\bar{t}) = S_{t_2,S_2,0}(\bar{t}) < \bar{S}$.
(We indeed check that the function $I(t) = S_{t_1,S_1,1}(t) - S_{t_2,S_2,0}(t)$ is increasing on $[t_1,t_2]$ and that $I(t_1) \leq 0$ and $I(t_2) \geq 0$, hence the existence and uniqueness of $\bar{t}_{low}(t_1,S_1,t_2,S_2)$.)

3.2.3 Properties of MRAP trajectories

We are now able to describe some properties of the solutions to the initial problem, by comparing them with the solutions induced by the MRAP controls (entailing sub-optimal solutions), taking full advantage of the previous definitions.

Proposition 3.4.
Let $S_h < \underline{S} < \bar{S} < 1$.
Let $S(\cdot)$ be a solution to the equation on $[t_1,t_2]$ (with $0 \leq t_1 < t_2 \leq T$) for an admissible control $u(\cdot)$ such that $\underline{S} \leq S(t) \leq \bar{S}$ for all $t \in [t_1,t_2]$. We denote $S_1 = S(t_1)$ and $S_2 = S(t_2)$.
Then, the respective solutions $\tilde{S}^{up}_{\underline{S}}(\cdot)$ and $\tilde{S}^{low}_{\bar{S}}(\cdot)$ on $[t_1,t_2]$ with $\tilde{S}^{up}_{\underline{S}}(t_1) = \tilde{S}^{low}_{\bar{S}}(t_1) = S_1$ and the respective MRAP controls $\tilde{u}^{up}_{\underline{S}}(\cdot)$ and $\tilde{u}^{low}_{\bar{S}}(\cdot)$, satisfy the following properties :

$$\tilde{S}^{up}_{\underline{S}}(t_2) = \tilde{S}^{low}_{\bar{S}}(t_2) = S_2$$

$$\tilde{S}^{up}_{\underline{S}}(t) \leq S(t) \leq \tilde{S}^{low}_{\bar{S}}(t) \quad , \quad \forall\, t \in [t_1,t_2]$$

$$\int_{t_1}^{t_2} \tilde{u}^{up}_{\underline{S}}(t)\mathrm{d}t \leq \int_{t_1}^{t_2} u(t)\mathrm{d}t \leq \int_{t_1}^{t_2} \tilde{u}^{low}_{\bar{S}}(t)\mathrm{d}t$$

Moreover the last inequalities are strict when $S(\cdot)$ is not identical to $\tilde{S}^{up}_{\underline{S}}(\cdot)$, respectively to $\tilde{S}^{low}_{\bar{S}}(\cdot)$.

4 The optimization problems

4.1 Notations and noteworthy trajectories

Let us first fix $S_0 \in (S^*, 1]$.
The first time when the trajectory of $S(\cdot)$ reaches S^* from the initial condition $(0, S_0)$ with a null control is written $t^* = \sup\{t \in [0,T],\ \text{st} : S_{0,S_0,0}(t) > S^*\}$.
The first time when the trajectory of $S(\cdot)$ reaches S_{tol} from the initial condition $(0, S_0)$ with a null control is written $t_{tol} = \sup\{t \in [0,T],\ \text{st} : S_{0,S_0,0}(t) > S_{tol}\}$.
The first time when the trajectory of $S(\cdot)$ reaches 1 from the initial condition $(0, S_0)$ with a control equal to 1 is written $t^1 = \sup\{t \in [0,T],\ \text{st} : S_{0,S_0,1}(t) < 1\}$.
With these special instants, let us define the following trajectories and their associated controls.

$$S^*_{traj}(t) = \tilde{S}^{up}_{S^*}(t) = \begin{cases} S_{0,S_0,0}(t) & , \ t \in [0,t^*) \\ S^* & , \ t \in [t^*,T] \end{cases} \quad ; \quad U^*(t) = \tilde{u}^{up}_{S^*}(t) = \begin{cases} 0 & , \ t \in [0,t^*) \\ u_{sing}(t,S^*) & , \ t \in [t^*,T] \end{cases}$$

$$S^{tol}_{traj}(t) = \tilde{S}^{up}_{S_{tol}}(t) = \begin{cases} S_{0,S_0,0}(t) & , \ t \in [0,t_{tol}) \\ S_{tol} & , \ t \in [t_{tol},T] \end{cases} \quad ; \quad U_{tol}(t) = \tilde{u}^{up}_{S_{tol}}(t) = \begin{cases} 0 & , \ t \in [0,t_{tol}) \\ u_{sing}(t,S_{tol}) & , \ t \in [t_{tol},T] \end{cases}$$

$$S^1_{traj}(t) = \tilde{S}^{low}_1(t) = \begin{cases} S_{0,S_0,1}(t) & , \ t \in [0,t^1) \\ 1 & , \ t \in [t^1,T] \end{cases} \quad ; \quad U^1(t) = \tilde{u}^{low}_1(t) = \begin{cases} 1 & , \ t \in [0,t^1) \\ u_{sing}(t,1) & , \ t \in [t^1,T] \end{cases}$$

where we used the expression of the control inducing a constant soil moisture S :

$$u_{sing}(t,S) = \frac{\varphi(t)K_S(S) + (1-\varphi(t))K_R(S)}{k_2} \qquad \in [0,1] \quad ,\forall t \in [0,T]$$

The corresponding quantity of water (renormalized by F_{max}) and biomass are :

$$V[U^*(\cdot)] = \int_{t^*}^{T} u_{sing}(t, S^*)\mathrm{d}t = V^* \quad , \quad V[U_{tol}(\cdot)] = \int_{t_{tol}}^{T} u_{sing}(t, S_{tol})\mathrm{d}t = V_{tol}$$

$$V[U^1(\cdot)] = t^1 + \int_{t^1}^{T} u_{sing}(t, 1)\mathrm{d}t = V^1$$

$$B[U^*(\cdot)] = B[U^1(\cdot)] = \int_0^T \varphi(t)\mathrm{d}t = B_{max} \quad , \quad B[U_{tol}(\cdot)] = B[0] = \int_0^T \varphi(t)K_S(S_{traj}^{tol}(t))\mathrm{d}t = B_{min}$$

We also denote V_{max} the following quantity :

$$V_{max} = \sup\{V \mid \exists u \in \mathcal{U},\ V[u(\cdot)] = V,\ \text{st} : \exists t \in [0, T],\ S_{0,S_0,u(\cdot)}(t) < S^*\}$$

which verifies $V^* < V_{max} < V^1$.

Remark 4.1. It is important to note that all the objects defined above depend on S_0.

4.2 Trivial cases for the terminal conditions

First we would like to maximize the production of biomass by using at most a certain fixed amount of water : problem $(B1)$.
Therefore we have for $(B1)$ an additional final condition which is $V(T) \leq V_{init}$ where V_{init} is the fixed amount. In our present work, we discard the following values of V_{init} that lead to a trivial problem :

$$V_{init} = 0 \quad \Longrightarrow \quad u_{opti}(t) = 0 \quad ; \quad V_{init} \geq V^* \quad \textit{(no water scarcity)} \quad \Longrightarrow \quad U^*(\cdot) \text{ is optimal}$$

Indeed the control $U^*(\cdot)$ generates the trajectory for $S(\cdot)$ which stays above S^* using the least amount of water (from proposition 3.4).

$$V_{init} < 0 \quad \Longrightarrow \quad \textit{no solution}$$

Also, we would like to solve its complementary problem, that is minimize the production of biomass by using at least a certain fixed amount of water : problem $(W1)$.
Therefore the additional final condition becomes $V(T) \geq V_{init}$. We then discard the following values of V_{init} for this problem :

$$V_{init} \leq V_{tol} \quad \Longrightarrow \quad U_{tol}(\cdot) \text{ is optimal} \quad ; \quad V_{init} \in [V_{max}, V^1] \quad \Longrightarrow \quad U^1(t) \text{ is optimal}$$

Indeed in the first case, among every admissible control generating a trajectory for $S(\cdot)$ that stays below S_{tol}, $U_{tol}(\cdot)$ is the one using the greatest amount of water ; in the second case, every admissible control generates a trajectory for $S(\cdot)$ that stays above S^*, and the control $U^1(\cdot)$ is the one using the greatest amount of water (from proposition 3.4).

$$V_{init} > V^1 \quad \Longrightarrow \quad \textit{no solution}$$

Second we would like to minimize the amount of water used in order to obtain a certain fixed target of biomass : problem $(B2)$.
Therefore we have for $(B2)$ an additional final condition which is $B(T) \geq B_{target}$. In our present work, we discard the following values of B_{target} that lead to a trivial problem :

$$B_{target} \leq B_{min} \quad \Longrightarrow \quad u_{opti}(t) = 0 \quad ; \quad B_{target} = B_{max} \quad \textit{(no water scarcity)} \quad \Longrightarrow \quad U^*(t) \text{ is optimal}$$

Indeed the control $U^*(\cdot)$ generates the trajectory for $S(\cdot)$ which stays above S^* using the least amount of water (from proposition 3.4).

$$B_{target} > B_{max} \quad \Longrightarrow \quad \textit{no solution}$$

Also, we would like to solve its complementary problem, that is maximize the quantity of water used that doesn't overpass a certain target of biomass : problem $(W2)$.

Therefore the additional final condition becomes $B(T) \leq B_{target}$. We then discard the following values of B_{target} for this problem :

$$B_{target} = B_{min} \quad \Longrightarrow \quad U_{tol}(\cdot) \text{ is optimal} \quad ; \quad B_{target} \geq B_{max} \quad \Longrightarrow \quad U^1(t) \text{ is optimal}$$

Indeed in the first case, among every admissible control generating a trajectory for $S(\cdot)$ that stays below S_{tol}, $U_{tol}(\cdot)$ is the one using the greatest amount of water ; in the second case, every admissible control generates a trajectory for $S(\cdot)$ that stays above S^*, and the control $U^1(\cdot)$ is the one using the greatest amount of water (from proposition 3.4).

$$B_{target} < B_{min} \quad \Longrightarrow \quad \textit{no solution}$$

Finally we would like to maximize a financial balance between the sellings due to the production of biomass and the cost due to the use of water : problem $(B3)$.
Therefore we suppose for $(B3)$ that we know a function describing the evolution of the total price of our sellings, and another function corresponding to the total price of the water used. We assume that these two functions are continuous and increasing, and since we look for a weighted cost there is no additional final condition.
Also, we would like to solve its complementary problem, that is minimize the same balance between these two financial cost functions : problem $(W3)$, and there is still no additional final condition for this problem.

4.3 Optimization framework

We can sum up all the considerations above by the following list of framework hypothesis.

Hypothesis 4.1.

$$t^* < T \quad \textit{and} \quad t_{tol} < T \quad \textit{and} \quad t^1 < T$$
$$(B1) \ : \quad \textit{final condition : } V(T) \leq V_{init} \quad , \quad 0 < V_{init} < V^*$$
$$(W1) \ : \quad \textit{final condition : } V(T) \geq V_{init} \quad , \quad V_{tol} < V_{init} < V_{max}$$
$$(B2) \ : \quad \textit{final condition : } B(T) \geq B_{target} \quad , \quad B_{min} < B_{target} < B_{max}$$
$$(W2) \ : \quad \textit{final condition : } B(T) \leq B_{target} \quad , \quad B_{min} < B_{target} < B_{max}$$
$$(B3),(W3) \ : \quad g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \textit{ and } c : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \textit{ such that : } \forall x, y \in \mathbb{R}^+, \ g'(x) > 0 \textit{ and } c'(y) > 0$$

Remark 4.2.
(Constraints qualifications to be precised)
Recalling the general formulation of our problems :

$$(\mathcal{OC}) \ : \begin{cases} \underset{u(\cdot)\in\mathcal{F}}{opti} \ \ \Omega(X(T)) \\ X(T) \in \mathcal{K} \end{cases}$$

we can now give more precisely the nature of Ω and $\mathcal{K}$ according to the optimization :

Problem	State vector	Optimization	$\Omega(X)$	$\mathcal{K}$
$(B1)$	$X = (S, B, V, Y)$	sup	B	$\mathbb{R}^2 \times]-\infty, V_{init}] \times \mathbb{R}^+$
$(B2)$	$X = (S, B, V, Y)$	inf	V	$\mathbb{R} \times [B_{target}, +\infty[\times \mathbb{R} \times \mathbb{R}^+$
$(B3)$	$X = (S, B, V, Y)$	sup	$g(B) - c(V)$	$\mathbb{R}^3 \times \mathbb{R}^+$
$(W1)$	$X = (S, B, V, Y, Z)$	inf	B	$\mathbb{R}^2 \times [V_{init}, +\infty[\times \mathbb{R}^+ \times \mathbb{R}^-$
$(W2)$	$X = (S, B, V, Y, Z)$	sup	V	$\mathbb{R} \times]-\infty, B_{target}] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^-$
$(W3)$	$X = (S, B, V, Y, Z)$	inf	$g(B) - c(V)$	$\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}^-$

Definition 4.1.
We say that an admissible control u is feasible for the problem (Bj) (resp. (Wj)) if $u \in \mathcal{U}$ and if the corresponding solution of $(\mathcal{P})$ with the conditions $(Init)$ satisfies the constraints $X(T) \in \mathcal{K}_{(Bj)}$ (resp. $X(T) \in \mathcal{K}_{(Wj)}$).
We denote $\mathcal{F}_{Bj}$ (resp. $\mathcal{F}_{Wj}$) the set of the feasible controls for the problem (Bj) (resp. (Wj)).

Hypothesis 4.2.
For each problem, the values of S_0, S_{tol}, V_{init}, B_{target}, the state constraints and the terminal conditions are compatible, ie. for each problem, the set of feasible controls is non-empty.

Remark 4.3.
This hyptothesis could be the subject of a viability analysis in further work.

4.4 Properties of optimal solutions

First, under assumption 4.2, we have the following lemma, which is the classical result for the existence of a solution to the optimal control problem.

Lemma 4.3.
For each problem $(Bj), (Wj)$ there exists u^* an optimal solution.

The following property gives preliminary results on necessary conditions for an optimal solution in the "Best" problems.

Proposition 4.4.
Let $u_{B1}(\cdot), u_{B2}(\cdot), u_{B3}(\cdot)$ the optimal controls of their respective problems and we denote the corresponding solutions by :

$$S_{B1}(\cdot), B_{B1}(\cdot), V_{B1}(\cdot), Y_{B1}(\cdot), Z_{B1}(\cdot) \quad ; \quad S_{B2}(\cdot), B_{B2}(\cdot), V_{B2}(\cdot), Y_{B2}(\cdot), Z_{B2}(\cdot)$$
$$S_{B3}(\cdot), B_{B3}(\cdot), V_{B3}(\cdot), Y_{B3}(\cdot), Z_{B3}(\cdot)$$

Then :

$$\text{(I)} \quad u_{Bj}(t) = 0 \quad , \quad a.e.\ t \in [0, t^*]$$
$$\text{(II)} \quad S_{Bj}(t) \leq S^* \quad , \forall t \in [t^*, T]$$

Proof.
(I) In this part, we drop the index Bj for simplicity.
Let $E = \{t \in [0,T],\ \text{st} : S(t) \leq S^*\}$. Let $\hat{t} = \inf E \geq t^*$.
For any feasible control $u(\cdot)$, consider $S(\cdot), B(\cdot), V(\cdot)$ the corresponding solutions.
We set :

$$v(t) = \begin{cases} \tilde{u}^{up}_{S^*}(t) & , \ t \in [0, \hat{t}) \\ u_{sing}(t, S^*) & , \ t \in [\hat{t}, T] \end{cases}$$

One has :

$$\int_0^{\hat{t}} v(t)\mathrm{d}t \leq \int_0^{\hat{t}} u(t)\mathrm{d}t \quad \text{and} \quad B[v(\cdot)] = B_{max}$$

If $\int_{\hat{t}}^T u(t)\mathrm{d}t \geq \int_{\hat{t}}^T v(t)\mathrm{d}t$, then one has :

$$V[u(\cdot)] \geq V[v(\cdot)] = V^* \quad , \quad B[u(\cdot)] \leq B[v(\cdot)] \quad , \quad g(B_u(T)) - c(V_u(T)) \leq g(B_v(T)) - c(V_v(T))$$

Thus, v is optimal for (Bj), and $v = 0$ on $[0, t^*]$.

If $\int_{\hat{t}}^T u(t)\mathrm{d}t < \int_{\hat{t}}^T v(t)\mathrm{d}t$, one has $\hat{t} < T$. We set :

$$E_1 = \{t \in [\hat{t}, T],\ \text{st} : u(t) < 1\}$$

The set E_1 has non null measure, because $v(t) < 1$ on $[\hat{t}, T]$, and the intersection $E \cap E_1$ has also non null measure (otherwise we would have $u = 1$ a.e. in E, which would imply S increasing over E, that prevents S to reach the domain below S^*).
Assume that $\hat{t} > t^*$.
Then $\int_0^{\hat{t}} v(t)\mathrm{d}t < \int_0^{\hat{t}} u(t)\mathrm{d}t$.
We consider the control $y_1(\cdot)$, and its corresonding solutions $S_{y_1}(\cdot), B_{y_1}(\cdot), V_{y_1}(\cdot)$, defined by :

$$\begin{cases} y_1(t) = v(t) & , \ t \in [0, \hat{t}) \\ y_1(t) = u(t) & , \ t \in [\hat{t}, T] \setminus (E \cap E_1) \text{ and } S_{y_1}(t) < 1 \\ y_1(t) = \min(u(t), 1/k_2) & , \ t \in [\hat{t}, T] \setminus (E \cap E_1) \text{ and } S_{y_1}(t) = 1 \\ y_1(t) \in [u(t), 1] & , \ t \in E \cap E_1 \end{cases}$$

with :

$$0 < \int_{E \cap E_1} y_1(t) - u(t)\mathrm{d}t < \int_0^{\hat{t}} u(t) - v(t)\mathrm{d}t$$

Then we have :

$$\begin{aligned} V[y_1(\cdot)] &= \int_0^{\hat{t}} v(t)\mathrm{d}t + \int_{E \cap E_1} y_1(t)\mathrm{d}t + \int_{[\hat{t},T]\setminus(E\cap E_1)} y_1(t)\mathrm{d}t \\ &< \int_0^{\hat{t}} u(t)\mathrm{d}t + \int_{E \cap E_1} u(t)\mathrm{d}t + \int_{[\hat{t},T]\setminus(E\cap E_1)} u(t)\mathrm{d}t = V[u(\cdot)] \end{aligned}$$

The associated solution $S_{y_1}(\cdot)$ satisfies :

$$\forall t \in [0, T],\ S_{y_1}(t) \leq 1 \quad ; \quad \forall t \in [\hat{t}, T],\ S_{y_1}(t) \geq S(t) \quad , \text{ with } \quad \int_{E \cap E_1} S_{y_1}(t)\mathrm{d}t > \int_{E \cap E_1} S(t)\mathrm{d}t$$

Moreover one has $S(t) \leq S^*$ over $E \cap E_1$, and there exists $J \subset E \cap E_1$ of non null measure such that $S(t) < S^*$ over J.
Then :

$$\int_{E \cap E_1} \varphi(t) K_S(S_{y_1}(t))\mathrm{d}t > \int_{E \cap E_1} \varphi(t) K_S(S(t))\mathrm{d}t$$

This leads to :

$$\begin{aligned} B[y_1(\cdot)] &= \int_0^{\hat{t}} \varphi(t) K_S(S_{y_1}(t))\mathrm{d}t + \int_{E \cap E_1} \varphi(t) K_S(S_{y_1}(t))\mathrm{d}t + \int_{[\hat{t},T]\setminus(E\cap E_1)} \varphi(t) K_S(S_{y_1}(t))\mathrm{d}t \\ &> \int_0^{\hat{t}} \varphi(t) K_S(S(t))\mathrm{d}t + \int_{E \cap E_1} \varphi(t) K_S(S(t))\mathrm{d}t + \int_{[\hat{t},T]\setminus(E\cap E_1)} \varphi(t) K_S(S(t))\mathrm{d}t = B[u(\cdot)] \end{aligned}$$

To conclude, we have :

$$B[y_1(\cdot)] > B[u(\cdot)] \quad \text{and} \quad V[y_1(\cdot)] < V[u(\cdot)]$$

Thus the control $y_1(\cdot)$ is feasible for (Bj), and the control $u(\cdot)$ is therefore non optimal for (Bj).
Hence :

$$\hat{t} = t^* \quad , \text{ ie : } \quad u_{Bj}(t) = 0 \quad \text{a.e. } t \in [0, t^*]$$

(II) In this part, we drop the index Bj for simplicity.
For any feasible control $u(\cdot)$ such that $u(t) = 0$ over $[0, t^*]$, we denote $S(\cdot), B(\cdot), V(\cdot)$ the corresponding trajectories.
We keep the same notations for the sets E and E_1, and we set :

$$F = \{t \in [t^*, T],\ \text{st} : S(t) > S^*\}$$

Assume that $F \neq \emptyset$. Then we have :

$$\int_F u_{sing}(t, S^*)\mathrm{d}t < \int_F u(t)\mathrm{d}t$$

Let us consider the following control $y_{\text{II}}(\cdot)$, and its associated solution $S_{y_{\text{II}}}(\cdot), B_{y_{\text{II}}}(\cdot), V_{y_{\text{II}}}(\cdot)$, defined by :

$$\begin{cases} y_{\text{II}}(t) = 0 & , t \in [0, t^*] \\ y_{\text{II}}(t) = u_{sing}(t, S^*) & , t \in F \\ y_{\text{II}}(t) = u(t) & , t \in E \setminus E_1 \text{ and } S_{y_{\text{II}}}(t) < 1 \\ y_{\text{II}}(t) = \min(u(t), 1/k_2) & , t \in E \setminus E_1 \text{ and } S_{y_{\text{II}}}(t) = 1 \\ y_{\text{II}}(t) \in [u(t), 1] & , t \in E \cap E_1 \end{cases}$$

with :

$$0 < \int_{E \cap E_1} y_{\text{II}}(t) - u(t)\mathrm{d}t < \int_F u(t) - u_{sing}(t, S^*)\mathrm{d}t$$

Then one has :

$$\begin{aligned} V[y_{\text{II}}(\cdot)] &= \int_F u_{sing}(t, S^*)\mathrm{d}t + \int_{E \setminus E_1} y_{\text{II}}(t)\mathrm{d}t + \int_{E \cap E_1} y_{\text{II}}(t)\mathrm{d}t \\ &< \int_F u(t)\mathrm{d}t + \int_{E \setminus E_1} u(t)\mathrm{d}t + \int_{E \cap E_1} u(t)\mathrm{d}t = V[u(\cdot)] \end{aligned}$$

The associated solution $S_{y_{\text{II}}}(\cdot)$ satisfies :

$$\forall t \in [0, T],\ S_{y_{\text{II}}}(t) \leq 1 \quad ; \quad \forall t \in F,\ S_{y_{\text{II}}}(t) \geq S^*$$

$$\forall t \in [0, T] \setminus F,\ S_{y_{\text{II}}}(t) \geq S(t) \quad , \text{ with } \quad \int_{E \cap E_1} S_{y_{\text{II}}}(t)\mathrm{d}t > \int_{E \cap E_1} \mathrm{d}t$$

Moreover one has $S(t) \leq S^*$ over $E \cap E_1$, and there exists $J \subset E \cap E_1$ of non null measure such that $S(t) < S^*$ over J.
Then :

$$\int_{E \cap E_1} \varphi(t) K_S(S_{y_{\text{II}}}(t))\mathrm{d}t > \int_{E \cap E_1} \varphi(t) K_S(S(t))\mathrm{d}t$$

This leads to :

$$\begin{aligned} B[y_{\text{II}}(\cdot)] &= \int_{[0,t^*] \cup F} \varphi(t) K_S(S_{y_{\text{II}}}(t))\mathrm{d}t + \int_{E \cap E_1} \varphi(t) K_S(S_{y_{\text{II}}}(t))\mathrm{d}t + \int_{E \setminus E_1} \varphi(t) K_S(S_{y_{\text{II}}}(t))\mathrm{d}t \\ &> \int_{[0,t^*] \cup F} \varphi(t) K_S(S(t))\mathrm{d}t + \int_{E \cap E_1} \varphi(t) K_S(S(t))\mathrm{d}t + \int_{E \setminus E_1} \varphi(t) K_S(S(t))\mathrm{d}t = B[u(\cdot)] \end{aligned}$$

To conclude, we have :

$$B[y_{\text{II}}(\cdot)] > B[u(\cdot)] \quad \text{and} \quad V[y_{\text{II}}(\cdot)] < V[u(\cdot)]$$

Thus the control $y_{\text{II}}(\cdot)$ is feasible for (Bj), and the control $u(\cdot)$ is therefore non optimal for (Bj).
Hence :

$$F = \emptyset \quad , \text{ ie : } \quad S_{Bj}(t) \leq S^* \quad , \forall t \in [t^*, T]$$

□

Remark 4.4.
This result implies that an optimal solution for the problems (Bj) doesn't saturate the state constraint $S(t) \leq 1$ over a non-null measured set. Therefore we can drop the equation over Z in the system $(\mathcal{S})$ for the optimal synthesis of the problems "Best".

Next we show that the optimal solutions of their repsective problems with additional terminal conditions necessarily reach their targets.

Proposition 4.5.
Let $u_{B1}(\cdot), u_{B2}(\cdot)$ and $u_{W1}(\cdot), u_{W2}(\cdot)$ the optimal controls of their respective problems and we denote the corresponding solutions by :

$$S_{B1}(\cdot), B_{B1}(\cdot), V_{B1}(\cdot), Y_{B1}(\cdot) \quad ; \quad S_{B2}(\cdot), B_{B2}(\cdot), V_{B2}(\cdot), Y_{B2}(\cdot)$$
$$S_{W1}(\cdot), B_{W1}(\cdot), V_{W1}(\cdot), Y_{W1}(\cdot), Z_{W1}(\cdot) \quad ; \quad S_{W2}(\cdot), B_{W2}(\cdot), V_{W2}(\cdot), Y_{W2}(\cdot), Z_{W2}(\cdot)$$

Then :

$$\text{(III)} \quad V[u_{B1}(\cdot)] = V[u_{W1}(\cdot)] = V_{init}$$
$$\text{(IV)} \quad B[u_{B2}(\cdot)] = B[u_{W2}(\cdot)] = B_{target}$$

Proof.
(III) In this first part, we drop the index $B1$ for simplicity.
Consider any feasible control $u(\cdot)$ such that $u(t) = 0$ on $[0, t^*]$ and that the corresponding solution $S(\cdot), B(\cdot), V(\cdot)$ satisfies $S(t) \leq S^*$ for all $t \in [t^*, T]$.
We keep the same notation for the set E_1. Assume that $V[u(\cdot)] < V_{init}$.
There exists $J \subset [t^*, T]$ of non null measure such that $S(t) < S^*$ over J, and $J \cap E_1$ has also non null measure.
Let us consider the following control $y_{\text{III}}(\cdot)$, and its associated solution $S_{y_{\text{III}}}(\cdot), B_{y_{\text{III}}}(\cdot), V_{y_{\text{III}}}(\cdot)$, defined by :

$$\begin{cases} y_{\text{III}}(t) = u(t) & , t \in [0, T] \setminus (J \cap E_1) \\ y_{\text{III}}(t) = \min(u(t), 1/k_2) & , t \in J \cap E_1 \text{ and } S_{y_{\text{III}}}(t) = 1 \\ y_{\text{III}}(t) \in [u(t), 1] & , t \in J \cap E_1 \text{ and } S_{y_{\text{III}}}(t) < 1 \end{cases}$$

with :

$$0 < \int_{(J \cap E_1) \cap \{S_{y_{\text{III}}} < 1\}} y_{\text{III}}(t) - u(t) \mathrm{d}t < V_{init} - V[u(\cdot)]$$

Then one has :

$$\begin{aligned} V[y_{\text{III}}(\cdot)] &= \int_{(J \cap E_1) \cap \{S_{y_{\text{III}}} < 1\}} y_{\text{III}}(t) \mathrm{d}t + \int_{(J \cap E_1) \cap \{S_{y_{\text{III}}} = 1\}} y_{\text{III}}(t) \mathrm{d}t + \int_{[0,T] \setminus (J \cap E_1)} u(t) \mathrm{d}t \\ &< V_{init} - V[u(\cdot)] + \int_{[0,T]} u(t) \mathrm{d}t = V_{init} \end{aligned}$$

The associated solution $S_{y_{\text{III}}}(\cdot)$ satisfies :

$$\forall t \in [0, T],\ S_{y_{\text{III}}}(t) \leq 1 \quad ; \quad \forall t \in [0, T],\ S_{y_{\text{III}}}(t) \geq S(t) \quad , \text{ with } \quad \int_{J \cap E_1} S_{y_{\text{III}}}(t) \mathrm{d}t > \int_{J \cap E_1} S(t) \mathrm{d}t$$

Moreover one has $S(t) < S^*$ over $J \cap E_1$, then :

$$\int_{J \cap E_1} \varphi(t) K_S(S_{y_{\text{III}}}(t)) \mathrm{d}t > \int_{J \cap E_1} \varphi(t) K_S(S(t)) \mathrm{d}t$$

This leads to :

$$\begin{aligned} B[y_{\text{III}}(\cdot)] &= \int_{J \cap E_1} \varphi(t) K_S(S_{y_{\text{III}}}(t)) \mathrm{d}t + \int_{[0,T] \setminus (J \cap E_1)} \varphi(t) K_S(S_{y_{\text{III}}}(t)) \mathrm{d}t \\ &> \int_{J \cap E_1} \varphi(t) K_S(S(t)) \mathrm{d}t + \int_{[0,T] \setminus (J \cap E_1)} \varphi(t) K_S(S(t)) \mathrm{d}t = B[u(\cdot)] \end{aligned}$$

To conclude, we have :

$$B[y_{\text{III}}(\cdot)] > B[u(\cdot)] \quad \text{and} \quad V[y_{\text{III}}(\cdot)] < V_{init}$$

Thus the control $y_{\text{III}}(\cdot)$ is feasible for $(B1)$, and the control $u(\cdot)$ is therefore non optimal for $(B1)$.
Hence :

$$V[u_{B1}(\cdot)] = V_{init}$$

In this second part, we drop the index $W1$ for simplicity.
Consider any feasible control $u(\cdot)$ such that the corresponding solution $S(\cdot), B(\cdot), V(\cdot)$ satisfies $S(t) \geq S_{tol}$ over $[0,T]$.
Let $\mathcal{E} = \{t \in [0,T],\ \text{st} : S(t) > S_{tol}\}$. We set :

$$\mathcal{E}_1 = \{t \in [\hat{t}, T],\ \text{st} : u(t) > 0\}$$

Assume that $V[u(\cdot)] > V_{init}$.
We consider the control $z_{\text{III}}(\cdot)$, and its corresponding solution $S_{z_{\text{III}}}(\cdot), B_{z_{\text{III}}}(\cdot), V_{z_{\text{III}}}(\cdot)$, defined by :

$$\begin{cases} z_{\text{III}}(t) = u(t) & , t \in [0,T] \setminus (\mathcal{E} \cap \mathcal{E}_1) \\ z_{\text{III}}(t) \in [0, u(t)] & , t \in \mathcal{E} \cap \mathcal{E}_1 \end{cases}$$

with :

$$0 < \int_{\mathcal{E}\cap\mathcal{E}_1} u(t) - z_{\text{III}}(t)\mathrm{d}t < V[u(\cdot)] - V_{init}$$

Then one has :

$$\begin{aligned} V[z_{\text{III}}(\cdot)] &= \int_{\mathcal{E}\cap\mathcal{E}_1} z_{\text{III}}(t)\mathrm{d}t + \int_{[0,T]\setminus(\mathcal{E}\cap\mathcal{E}_1)} u(t)\mathrm{d}t \\ &> V_{init} - V[u(\cdot)] + \int_{[0,T]} u(t)\mathrm{d}t = V_{init} \end{aligned}$$

The associated solution $S_{z_{\text{III}}}(\cdot)$ satisfies :

$$\forall t \in [0,T],\ S_{z_{\text{III}}}(t) \leq S(t) \quad , \text{ with } \int_{\mathcal{E}\cap\mathcal{E}_1} S_{z_{\text{III}}}(t)\mathrm{d}t < \int_{\mathcal{E}\cap\mathcal{E}_1} S(t)\mathrm{d}t$$

One has $S(t) > S_w$ over $\mathcal{E} \cap \mathcal{E}_1$, then :

$$\int_{\mathcal{E}\cap\mathcal{E}_1} \varphi(t)K_S(S_{z_{\text{III}}}(t))\mathrm{d}t < \int_{\mathcal{E}\cap\mathcal{E}_1} \varphi(t)K_S(S(t))\mathrm{d}t$$

This leads to :

$$\begin{aligned} B[z_{\text{III}}(\cdot)] &= \int_{\mathcal{E}\cap\mathcal{E}_1} \varphi(t)K_S(S_{z_{\text{III}}}(t))\mathrm{d}t + \int_{[0,T]\setminus(\mathcal{E}\cap\mathrm{E}_1)} \varphi(t)K_S(S_{z_{\text{III}}}(t))\mathrm{d}t \\ &< \int_{\mathcal{E}\cap\mathcal{E}_1} \varphi(t)K_S(S(t))\mathrm{d}t + \int_{[0,T]\setminus(\mathcal{E}\cap\mathrm{E}_1)} \varphi(t)K_S(S(t))\mathrm{d}t = B[u(\cdot)] \end{aligned}$$

To conclude, we have :

$$B[z_{\text{III}}(\cdot)] < B[u(\cdot)] \quad \text{and} \quad V[z_{\text{III}}(\cdot)] > V_{init}$$

Thus the control $z_{\text{III}}(\cdot)$ is feasible for $(W1)$, and the control $u(\cdot)$ is therefore non optimal for $(W1)$.
Hence :

$$V[u_{W1}(\cdot)] = V_{init}$$

(IV) In this first part, we drop the index $B2$ for simplicity.
Consider any feasible control $u(\cdot)$ such that $u(t) = 0$ on $[0, t^*]$ and that the corresponding solution $S(\cdot), B(\cdot), V(\cdot)$ satisfies $S(t) \leq S^*$ for all $t \in [t^*, T]$.
Assume that $B[u(\cdot)] > B_{target}$. We set :

$$E_2 = \{t \in [t^*, T],\ \text{st} : u(t) > 0\} \qquad \text{(has non null measure)}$$

Let $t_2 = \operatorname{ess\,sup} E_2$.
If $t_2 = T$, then we set $I_\varepsilon = [T-\varepsilon, T]$, for $\varepsilon > 0$ small enough, and we consider the following control $y_{\mathrm{IV}}(\cdot)$, with its associated solution $S_{y_{\mathrm{IV}}}(\cdot), B_{y_{\mathrm{IV}}}(\cdot), V_{y_{\mathrm{IV}}}(\cdot)$, defined by :

$$\begin{cases} y_{\mathrm{IV}}(t) = u(t) & , t \in [0,T] \setminus I_\varepsilon \\ y_{\mathrm{IV}}(t) = 0 & , t \in I_\varepsilon \end{cases}$$

If $t_2 < T$, then we set $I_\varepsilon = [t_2 - \varepsilon/2, t_2 + \varepsilon/2]$, for $\varepsilon > 0$ small enough, and we consider the following control $y_{\mathrm{IV}}(\cdot)$, with its associated solution $S_{y_{\mathrm{IV}}}(\cdot), B_{y_{\mathrm{IV}}}(\cdot), V_{y_{\mathrm{IV}}}(\cdot)$, defined by :

$$\begin{cases} y_{\mathrm{IV}}(t) = u(t) & , t \in [0,T] \setminus I_\varepsilon \\ y_{\mathrm{IV}}(t) = \tilde{u}^{up}_{S_{tol}}(t) & , t \in I_\varepsilon \end{cases}$$

Then one has :

$$0 < \int_{I_\varepsilon} u(t) - y_{\mathrm{IV}}(t)\mathrm{d}t \leq \varepsilon$$

The associated solution $S_{y_{\mathrm{IV}}}(\cdot)$ satisfies :

$$\forall t \in [0,T] \setminus I_\varepsilon,\ S_{y_{\mathrm{IV}}}(t) = S(t) \quad ; \quad \forall t \in I_\varepsilon,\ S_{y_{\mathrm{IV}}}(t) \leq S(t) \quad , \text{ with } \quad \int_{I_\varepsilon} S(t) - S_{y_{\mathrm{IV}}}(t)\mathrm{d}t > 0$$

Moreover one has $S(t) \leq S^*$ over I_ε.
Then :

$$\int_{I_\varepsilon} \varphi(t) K_S(S_{y_{\mathrm{IV}}}(t))\mathrm{d}t < \int_{I_\varepsilon} \varphi(t) K_S(S(t))\mathrm{d}t$$

And :

$$0 < \int_0^T \varphi(t)\big(K_S(S(t)) - K_S(S_{y_{\mathrm{IV}}}(t))\big)\mathrm{d}t = \int_{I_\varepsilon} \varphi(t)\big(K_S(S(t)) - K_S(S_{y_{\mathrm{IV}}}(t))\big)\mathrm{d}t$$

Let $t \in I_\varepsilon$. There exists $C_S > 0, C_R > 0$ such that :

$$|K_S(S(t)) - K_S(S_{y_{\mathrm{IV}}}(t))| \leq C_S |S(t) - S_{y_{\mathrm{IV}}}(t)| \quad , \quad |K_R(S(t)) - K_R(S_{y_{\mathrm{IV}}}(t))| \leq C_R |S(t) - S_{y_{\mathrm{IV}}}(t)|$$

From the equation on S, one has :

$$\begin{aligned} S(t) - S_{y_{\mathrm{IV}}}(t) = & k_1 k_2 \int_{t^*}^t u(\tau) - y_{\mathrm{IV}}(\tau)\mathrm{d}\tau \\ & + k_1 \int_{t^*}^t \varphi(\tau)\big(K_S(S_{y_{\mathrm{IV}}}(\tau)) - K_S(S(\tau))\big) + (1-\varphi(\tau))\big(K_R(S(\tau)) - K_R(S_{y_{\mathrm{IV}}}(\tau))\big)\mathrm{d}\tau \end{aligned}$$

As a consequence, one has :

$$|S(t) - S_{y_{\mathrm{IV}}}(t)| \leq k_1 k_2 \varepsilon + k_1 \int_{t^*}^t \big(\varphi(\tau) C_S + (1-\varphi(\tau)) C_R\big) |S(\tau) - S_{y_{\mathrm{IV}}}(\tau)|\mathrm{d}\tau$$

By the Grönwall Lemma, we deduce that :

$$\begin{aligned} |S(t) - S_{y_{\mathrm{IV}}}(t)| & \leq k_1 k_2 \varepsilon \exp\left(\int_{t^*}^t k_1 \big(\varphi(\tau) C_S + (1-\varphi(\tau)) C_R\big)\mathrm{d}\tau \right) \\ & \leq k_1 k_2 \varepsilon \exp\left(\int_0^T k_1 \big(\varphi(\tau) C_S + (1-\varphi(\tau)) C_R\big)\mathrm{d}\tau \right) = K\varepsilon \end{aligned}$$

Then :

$$0 < B[u(\cdot)] - B[y_{\mathrm{IV}}(\cdot)] \leq \int_{I_\varepsilon} \varphi(t) C_S K \varepsilon \mathrm{d}t \leq C_S K \varepsilon^2$$

For $\varepsilon > 0$ small enough, we have :

$$0 < B[u(\cdot)] - B[y_{\text{IV}}(\cdot)] \leq C_S K \varepsilon^2 < B[u(\cdot)] - B_{target}$$

Thus :

$$B_{target} < B[y_{\text{IV}}(\cdot)] \quad \text{, and} \quad 0 < \int_{I_\varepsilon} u(t) - y_{\text{IV}}(t)\mathrm{d}t \leq \varepsilon \quad \text{gives} \quad V[y_{\text{IV}}(\cdot)] < V[u(\cdot)]$$

To conclude, we have :

$$V[y_{\text{IV}}(\cdot)] < V[u(\cdot)] \quad \text{and} \quad B[y_{\text{IV}}(\cdot)] > B_{target}$$

Thus the control $y_{\text{IV}}(\cdot)$ is feasible for $(B2)$, and the control $u(\cdot)$ is therefore non optimal for $(B2)$.
Hence :

$$B[u_{B2}(\cdot)] = B_{target}$$

In this second part, we drop the index $W2$ for simplicity.
Consider any feasible control $u(\cdot)$ such that the corresponding solution $S(\cdot), B(\cdot), V(\cdot)$ satisfies $S(t) \geq S_w$ for all $t \in [0,T]$.
Assume that $B[u(\cdot)] < B_{target}$. We set :

$$\mathcal{E}_2 = \{t \in [0,T], \text{ st} : u(t) < 1\} \qquad \text{(has non null measure)}$$

Let $t_3 = \operatorname{ess\,sup} \mathcal{E}_2$.
If $t_3 = T$, then we set $\mathcal{I}_\varepsilon = [T-\varepsilon, T]$, for $\varepsilon > 0$ small enough, and we consider the following control $z_{\text{IV}}(\cdot)$, with its associated solution $S_{z_{\text{IV}}}(\cdot), B_{z_{\text{IV}}}(\cdot), V_{z_{\text{IV}}}(\cdot)$, defined by :

$$\begin{cases} z_{\text{IV}}(t) = u(t) & , \ t \in [0,T] \setminus \mathcal{I}_\varepsilon \\ z_{\text{IV}}(t) = 1 & , \ t \in \mathcal{I}_\varepsilon \end{cases}$$

If $t_3 < T$, then we set $\mathcal{I}_\varepsilon = [t_3 - \varepsilon/2, t_3 + \varepsilon/2]$, for $\varepsilon > 0$ small enough, and we consider the following control $z_{\text{IV}}(\cdot)$, with its associated solution $S_{z_{\text{IV}}}(\cdot), B_{z_{\text{IV}}}(\cdot), V_{z_{\text{IV}}}(\cdot)$, defined by :

$$\begin{cases} z_{\text{IV}}(t) = u(t) & , \ t \in [0,T] \setminus \mathcal{I}_\varepsilon \\ z_{\text{IV}}(t) = \tilde{u}_1^{low}(t) \quad , \ t \in \mathcal{I}_\varepsilon \end{cases}$$

Then one has :

$$0 < \int_{\mathcal{I}_\varepsilon} z_{\text{IV}}(t) - u(t)\mathrm{d}t \leq \varepsilon$$

The associated solution $S_{z_{\text{IV}}}(\cdot)$ satisfies :

$$\forall t \in [0,T] \setminus \mathcal{I}_\varepsilon, \ S_{z_{\text{IV}}}(t) = S(t) \quad ; \quad \forall t \in \mathcal{I}_\varepsilon, \ S_{z_{\text{IV}}}(t) \geq S(t) \quad \text{, with} \quad \int_{\mathcal{I}_\varepsilon} S_{z_{\text{IV}}}(t) - S(t)\mathrm{d}t > 0$$

Moreover one has $S(t) \geq S_w$ over $\mathcal{I}_\varepsilon$.
Then :

$$\int_{\mathcal{I}_\varepsilon} \varphi(t) K_S(S_{z_{\text{IV}}}(t))\mathrm{d}t > \int_{\mathcal{I}_\varepsilon} \varphi(t) K_S(S(t))\mathrm{d}t$$

And :

$$0 < \int_0^T \varphi(t)\big(K_S(S_{z_{\text{IV}}}(t)) - K_S(S(t))\big)\mathrm{d}t = \int_{\mathcal{I}_\varepsilon} \varphi(t)\big(K_S(S_{z_{\text{IV}}}(t)) - K_S(S(t))\big)\mathrm{d}t$$

As before, one has :

$$\forall t \in \mathcal{I}_\varepsilon, \ |K_S(S_{z_{\text{IV}}}(t)) - K_S(S(t))| \leq C_S |S_{z_{\text{IV}}}(t) - S(t)| \leq C_S K \varepsilon$$

Then :

$$0 < B[z_{\text{IV}}(\cdot)] - B[u(\cdot)] \leq \int_{\mathcal{I}_\varepsilon} \varphi(t) C_S K \varepsilon \mathrm{d}t \leq C_S K \varepsilon^2$$

For $\varepsilon > 0$ small enough, we have :

$$0 < B[z_{\text{IV}}(\cdot)] - B[u(\cdot)] \leq C_S K \varepsilon^2 < B_{target} - B[u(\cdot)]$$

Thus :

$$B_{target} > B[z_{\text{IV}}(\cdot)] \quad , \text{ and } \quad 0 < \int_{\mathcal{I}_\varepsilon} z_{\text{IV}}(t) - u(t) \mathrm{d}t \leq \varepsilon \quad \text{gives} \quad V[z_{\text{IV}}(\cdot)] > V[u(\cdot)]$$

To conclude, we have :

$$V[z_{\text{IV}}(\cdot)] > V[u(\cdot)] \quad \text{and} \quad B[z_{\text{IV}}(\cdot)] < B_{target}$$

Thus the control $z_{\text{IV}}(\cdot)$ is feasible for $(W2)$, and the control $u(\cdot)$ is therefore non optimal for $(W2)$.
Hence :

$$B[u_{W2}(\cdot)] = B_{target}$$

□

5 Optimal synthesis

In this section we apply the theory of optimal control.

5.1 Application of the Pontryagin Maximum Principle

Recall that we aim to study the general optimal control problem $(\mathcal{OC})$ derived with all the different conditions mentionned for the problems (Bj) and (Wj).

5.1.1 Optimality necessary conditions

We first write the general shape of the corresponding Hamiltonian in order to apply the Pontryagin's Maximum Principle.

$$(\mathcal{H}) : \quad H_{\mathcal{P}} = \lambda_S k_1 (k_2 u - (\varphi(t) K_S(S) + (1 - \varphi(t)) K_R(S))) + \lambda_B \varphi(t) K_S(S) + \lambda_V u + \lambda_Y C_Y(S) + \lambda_Z C_Z(S)$$

We also write the associated adjoint equations obtained from this Hamiltonian.

$$(\mathcal{L}) : \begin{cases} \dot{\lambda}_S \in \varphi(t)(\lambda_S k_1 - \lambda_B) \partial_C K_S(S(t)) + (1 - \varphi(t)) \lambda_S k_1 \partial_C K_R(S(t)) - \lambda_Y \partial_C C_Y(S(t)) - \lambda_Z \partial_C C_Z(S(t)) \\ \dot{\lambda}_B = 0 \\ \dot{\lambda}_V = 0 \\ \dot{\lambda}_Y = 0 \\ \dot{\lambda}_Z = 0 \end{cases}$$

Denoting $\bar{X}(\cdot)$ an optimal trajectory, and $\Lambda_{\bar{X}}$ the corresponding adjoints, we have the following necessary conditions from the PMP :

$$(\mathcal{N}) : \begin{cases} |\lambda_S(t)| + |\lambda_B(t)| + |\lambda_V(t)| + |\lambda_Y(t)| + |\lambda_Z(t)| \neq 0 \quad , \ t \in [0, T] \\ \Lambda_{\bar{X}(T)} \in -N_{\mathcal{K}}(\bar{X}(T)) - sgn(m) \nabla_{X(T)} \Omega(\bar{X}(T)) \end{cases}$$

where $sgn : \begin{cases} min & \mapsto 1 \\ max & \mapsto -1 \end{cases}$

More precisely, we have :

Problem	$\lambda_S(T)$	$\lambda_B(T)$	$\lambda_V(T)$	$\lambda_Y(T)$	$\lambda_Z(T)$
$(B1)$	0	1	≤ 0	≥ 0	
$(B2)$	0	≥ 0	-1	≥ 0	
$(B3)$	0	$\in \partial_C g(B(T)) \subset \mathbb{R}_+^*$	$\in -\partial_C c(V(T)) \subset \mathbb{R}_-^*$	≥ 0	
$(W1)$	0	-1	≥ 0	≥ 0	≤ 0
$(W2)$	0	≤ 0	1	≥ 0	≤ 0
$(W3)$	0	$\in -\partial_C g(B(T)) \subset \mathbb{R}_-^*$	$\in \partial_C c(V(T)) \subset \mathbb{R}_+^*$	≥ 0	≤ 0

We also define the commutation function :

$$\phi_c(t) = \lambda_S(t) k_1 k_2 + \lambda_V \qquad (1)$$

It gives the corresponding necessary optimal control :

$$\begin{cases} u(t) = 0 & \text{, if } \phi_c(t) < 0 \qquad \textit{(then } S \textit{ decreases)} \\ u(t) \in [0,1] & \text{, if } \phi_c(t) = 0 \\ u(t) = 1 & \text{, if } \phi_c(t) > 0 \qquad \textit{(then } S \textit{ increases)} \end{cases} \qquad (2)$$

Let us now indicate the outline of the following. We first show some properties over the adjoint coordinates to describe them more precisely, in particular their sign. Then we will investigate the case of singular arcs, and we are going to characterize the set of possible values of S where a singular arc can occur. Finally we will show a property of connexity which is a key ingredient to conclude over a structure of optimal strategy.

5.1.2 Properties of the adjoints

Proposition 5.1.
(I) *For any optimal solution to the problem (Bj), we have $\lambda_S^{(Bj)}(t) \geq 0$ for all $t \in [0,T]$.*
(II) *For any optimal solution to the problem (Wj), we have $\lambda_S^{(Wj)}(t) \leq 0$ for all $t \in [0,T]$.*

Proof.
(I) Suppose that there exists $t \in [0,T]$ such that $\lambda_S^{(Bj)}(t) < 0$. One has :

$$\dot{\lambda}_S^{(Bj)}(t) \in \varphi(t)(\lambda_S^{(Bj)}(t)k_1 - \lambda_B^{(Bj)})\partial_C K_S(S(t)) + (1-\varphi(t))\lambda_S^{(Bj)}(t)k_1\partial_C K_R(S(t)) - \lambda_Y \partial_C C_Y(S(t))$$

Since $0 \leq \varphi \leq 1$, $\lambda_B^{(Bj)} \geq 0$, $\partial_C K_S \subset \mathbb{R}^+$, $\partial_C K_R \subset \mathbb{R}^+$, $\lambda_Y^{(Bj)} \geq 0$ and $\partial_C C_Y \subset \mathbb{R}^+$, then one has :

$$\dot{\lambda}_S^{(Bj)}(t) \in \mathbb{R}^-$$

Let $G = \{t \in [0,T],\ \text{st} : \lambda_S^{(Bj)}(t) < 0\}$.
The function $\lambda_S^{(Bj)}(\cdot)$ is then non increasing over G, which contradicts the necessary terminal condition $\lambda_S^{(Bj)}(T) = 0$.
Hence :

$$\forall t \in [0,T],\ \lambda_S^{(Bj)}(t) \geq 0$$

(II) Suppose that there exists $\hat{t} \in [0,T]$ such that $\lambda_S^{(Wj)}(\hat{t}) > 0$.
Then there exists $\varepsilon > 0$ such that $\lambda_S^{(Wj)}(t) > 0$ for all $t \in (\hat{t}-\varepsilon, \hat{t}+\varepsilon)$. Therefore $\phi_c(t) > 0$ for all $t \in (\hat{t}-\varepsilon, \hat{t}+\varepsilon)$, and the function $S(\cdot)$ is increasing over this interval.
Since the trajectory of $S(\cdot)$ verifies the constraint $S_{tol} \leq S \leq 1$, then we deduce that $S(t) \in (S_{tol}, 1)$ for all $t \in (\hat{t}-\varepsilon, \hat{t}+\varepsilon)$. As a consequence, one has for $t \in (\hat{t}-\varepsilon, \hat{t}+\varepsilon)$:

$$\dot{\lambda}_S^{(Wj)}(t) \in \varphi(t)(\lambda_S^{(Wj)}(t)k_1 - \lambda_B^{(Wj)})\partial_C K_S(S(t)) + (1-\varphi(t))\lambda_S^{(Wj)}(t)k_1\partial_C K_R(S(t))$$

Since $0 \leq \varphi \leq 1$, $\lambda_B^{(Wj)} \leq 0$, $\lambda_Z^{(Wj)} \leq 0$, $\partial_C K_S : [0,1] \mapsto \mathbb{R}^+$, and $\partial_C K_R : [0,1] \mapsto \mathbb{R}^+$, then one has :

$$\dot{\lambda}_S^{(Wj)}(t) \in \mathbb{R}^+ \quad , \forall t \in (\hat{t} - \varepsilon, \hat{t} + \varepsilon)$$

Let $\mathcal{G} = \{t \in [0,T],\ \text{st} : \lambda_S^{(Wj)}(t) > 0\}$.
From above, we deduce that the function $\lambda_S^{(Wj)}(\cdot)$ is non decreasing over $\mathcal{G}$, which contradicts the necessary terminal condition $\lambda_S^{(Wj)}(T) = 0$.
Hence :

$$\forall t \in [0,T],\ \lambda_S^{(Wj)}(t) \leq 0$$

$\square$

Proposition 5.2.
(I) *We have* $\lambda_B^{(B2)} > 0$.
(II) *We have* $\lambda_V^{(B1)} < 0$.

Proof.
(I) Assume that $\lambda_B^{(B2)} = 0$.
Then, one has :

$$\dot{\lambda}_S^{(B2)}(t) \in \varphi(t)\lambda_S^{(B2)}(t)k_1\partial_C K_S(S(t)) + (1-\varphi(t))\lambda_S^{(B2)}(t)k_1\partial_C K_R(S(t)) - \lambda_Y \partial_C C_Y(S(t))$$

This rewrites as :

$$\dot{\lambda}_S^{(B2)}(t) \in \lambda_S^{(B2)}(t)k_1\big(\varphi(t)\partial_C K_S(S(t)) + (1-\varphi(t))\partial_C K_R(S(t))\big) - \lambda_Y \partial_C C_Y(S(t))$$

When S is strictly between S_{tol} and 1, the quantity $\partial_C C_Y(S)$ is null. Moreover we know from before that S is decreasing over $[0,t^*)$ and $S_{tol} \leq S \leq S^*$ over $[t^*,T]$. As a consequence, either $S = S_{tol}$ or $S > S_{tol}$, and in the latter case the function λ_S satisfies :

$$\dot{\lambda}_S^{(B2)}(t) \in \lambda_S^{(B2)}(t)k_1\big(\varphi(t)\partial_C K_S(S(t)) + (1-\varphi(t))\partial_C K_R(S(t))\big)$$

Suppose that the set $\{t \in [0,T],\ S(t) = S_{tol}\}$ has non null measure. The function S being neither decreasing nor increasing implies that $\phi_c(t) = 0$ over this set. Thus $\lambda_S^{(B2)}$ is contant equal to $-\frac{\lambda_V^{(B2)}}{k_1 k_2} > 0$. Since $\lambda_S^{(B2)}(T) = 0$ and $\lambda_S(\cdot)$ is continuous, there exists an interval $(T-\alpha, T]$ such that $\lambda_S^{(B2)} < \frac{\lambda_V^{(B2)}}{2k_1 k_2}$, then $S(t) > S_{tol}$ almost everywhere in $(T-\alpha, T]$. Consequently, one has :

$$\dot{\lambda}_S^{(B2)}(t) \in \lambda_S^{(B2)}(t)k_1\big(\varphi(t)\partial_C K_S(S(t)) + (1-\varphi(t))\partial_C K_R(S(t))\big) \qquad \text{ae. in } (T-\alpha, T]$$

Since $\lambda_S^{(B2)}(T) = 0$, we deduce that necessarily $\lambda_S^{(B2)}(t) = 0$ for all $t \in (T-\alpha, T]$.
Denoting $\hat{t} = \sup\{t \in [0,T],\ S(t) = S_{tol}\}$, we can use the same argument as above to show that $\lambda_S^{(B2)} = 0$ on $(\hat{t}, T]$, and thus $\lambda_S^{(B2)}(\hat{t}) = 0$, which leads by continuity again that $S(t) > S_{tol}$ for all $t \in (\hat{t} - \delta, \hat{t} + \delta)$ for $\delta > 0$, which contradicts the definition of $\hat{t}$. Finally, we deduce that $\lambda_S^{(B2)}(t) = 0$ for all $t \in [0,T]$.
Since $\lambda_V^{(B2)} = -1$, the switching function thus verifies :

$$\phi^{(B2)}(t) = \lambda_S^{(B2)}(t)k_1k_2 + \lambda_V^{(B2)} \quad < 0 \quad , \quad \forall t \in [0,T]$$

This would imply that $u_{B2}(t) = 0$ ae. $t \in [0,T]$, and that the optimal trajectory is $S_{B2}(t) = S_{0,S_0,0}(t)$.
Then $V[u_{B2}(\cdot)] = 0$ and $B[u_{B2}(\cdot)] = B_{min}$. Combined with the proposition 4.3, it contradicts the framework hypothesis for the problem $(B2)$.
Hence :

$$\lambda_B^{(B2)} > 0$$

(II) Assume that $\lambda_V^{(B1)} = 0$.
Then, one has :

$$\dot{\lambda}_S^{(B1)}(t) \in \varphi(t)(\lambda_S^{(B1)}(t)k_1 - \lambda_B^{(B1)})\partial_C K_S(S(t)) + (1-\varphi(t))\lambda_S^{(B1)}(t)k_1\partial_C K_R(S(t)) - \lambda_Y \partial_C C_Y(S(t))$$

The function $\lambda_S^{(B1)}(\cdot)$ is non negative, and can be null on a time interval of non null measure only if $0 \in \partial_C K_S(S(t))$ and $0 \in \lambda_Y C_Y(S(t))$ on this interval.
Moreover one has $\phi^{(B1)}(t) = \lambda_S^{(B1)}(t)k_1k_2$, then for any t such that $S(t) \in (S_{tol}, S^*)$ we necessarily have $\lambda_S^{(B1)}(t) > 0$ and $\phi^{(B1)}(t) > 0$ which implies $u_{B1}(t) = 1$.
We deduce that the trajectory $S(\cdot)$ can not go below S^*, thus $V[u_{B1}(\cdot)] = V^*$ and $B[u_{B1}(\cdot)] = B_{max}$.
Combined with the proposition 4.3, it contradicts the framework hypothesis for the problem $(B1)$.
Hence :

$$\lambda_V^{(B1)} < 0$$

□

Proposition 5.3.
(I) *We have :*

$$\lambda_B^{(W2)} < 0$$

(II) *We have :*

$$\lambda_V^{(W1)} > 0$$

Proof.
(I) We know that $\lambda_V^{(W2)} = 1$. Assume that $\lambda_B^{(W2)} = 0$.
Then one has :

$$\dot{\lambda}_S^{(W2)}(t) \in \lambda_S^{(W2)}(t)k_1\big(\varphi(t)\partial_C K_S(S(t)) + (1-\varphi(t))\partial_C K_R(S(t))\big) - \lambda_Y \partial_C C_Y(S(t)) - \lambda_Z \partial_C C_Z(S(t))$$

Moreover :

$$\phi^{(W2)}(t) = \lambda_S^{(W2)}(t)k_1k_2 + \lambda_V^{(W2)}$$

By the transversality condition, we thus have $\phi^{(W2)}(T) > 0$.
By continuity of the function $\phi^{(W2)}(\cdot)$ there exists $\varepsilon > 0$, such that $\phi^{(W2)}(t) > 0$ on $[T-\varepsilon, T]$, thus $u_{W2}(t) = 1$ ae. $t \in [T-\varepsilon, T]$.
The trajectory $S(\cdot)$ is then increasing over $[T-\varepsilon, T]$, and as it verifies $S_{tol} \leq S(\cdot) \leq 1$ we deduce that $S_{tol} < S(t) < 1$ for $t \in (T-\varepsilon, T)$.
As a consequence, over $(T-\varepsilon, T)$, the function $\lambda_S^{(W2)}(\cdot)$ satisfies :

$$\dot{\lambda}_S^{(W2)}(t) \in \lambda_S^{(W2)}(t)k_1\big(\varphi(t)\partial_C K_S(S(t)) + (1-\varphi(t))\partial_C K_R(S(t))\big)$$

By continuity, one has $\lambda_S^{(W2)}(t) \underset{t\to T}{\longrightarrow} \lambda_S^{(W2)}(T) = 0$. Then, as before, we necessarily have $\lambda_S^{(W2)}(t) = 0$ for all $t \in [T-\varepsilon, T]$.
We know that $\lambda_S^{(W2)} \leq 0$, and we consider the following set :

$$\mathcal{M} = \{t \in [0,T], \text{ st} : \lambda_S^{(W2)}(t) < 0\}$$

Suppose that this set has non null measure, and let $\hat{t} = \operatorname{ess\,sup} \mathcal{M}$.
From above, we know that $\hat{t} < T$, and we have the following :

$$\forall t \in [\hat{t}, T],\ \lambda_S^{(W2)}(t) = 0 \quad ; \quad \forall t \in (\hat{t}, T],\ \dot{\lambda}_S^{(W2)}(t) = 0$$

Then the function $\phi^{(W2)}(\cdot)$ is constant equal to $\phi^{(W2)}(T)$ over $[\hat{t}, T]$.
Since $\mathcal{M}$ has non null measure, $\hat{t} > 0$. By continuity of $\phi^{(W2)}$, there exists $\delta > 0$ such that $\phi^{(W2)}(t) > 0$ for $t \in [\hat{t}-\delta, \hat{t}]$.
As before, it follows that $\lambda_S^{(W2)}(t) = 0$ for $t \in [\hat{t}-\delta, \hat{t}]$, which contradicts the definition of $\hat{t}$.

Therefore the set $\mathcal{M}$ has null measure, and by continuity we get $\lambda_S^{(W2)} = 0$ on $[0, T]$. Thus the function $\phi(\cdot)$ is constant equal to $\phi(T) > 0$ on $[0, T]$, which means $u_{W2}(t) = 1$ ae. $t \in [0, T]$. This is not allowed by the framework hypothesis (the corresponding trajectory $S_{0,S_0,1}(\cdot)$ reaches 1 at time $t^1 < T$ and then has to verify the constraint $S \leq 1$), hence the contradiction.
To conclude, we have :

$$\lambda_B^{(W2)} < 0$$

(II) We know that $\lambda_B^{(W1)} = -1$. Assume that $\lambda_V^{(W1)} = 0$.
Then one has :

$$\dot{\lambda}_S^{(W1)}(t) \in \varphi(t)(\lambda_S^{(W1)}(t)k_1 - \lambda_B^{(W1)})\partial_C K_S(S(t)) + (1 - \varphi(t))\lambda_S^{(W1)}(t)k_1\partial_C K_R(S(t)) \\ - \lambda_Y \partial_C C_Y(S(t)) - \lambda_Z \partial_C C_Z(S(t))$$

Moreover :

$$\phi^{(W1)}(t) = \lambda_S^{(W1)}(t)k_1k_2$$

In particular, we have $\phi^{(W1)}(T) = \lambda_S^{(W1)}(T)k_1k_2 = 0$.
Consider the following sets :

$$D_1 = \{t \in [0, T], \text{ st} : S(t) = 1\} \quad , \quad D^* = \{t \in [0, T], \text{ st} : S(t) = S^*\} \quad , \quad D_{tol} = \{t \in [0, T], \text{ st} : S(t) = S_{tol}\}$$

$$\mathcal{D}_*^1 = \{t \in [0, T], \text{ st} : S^* \leq S(t) \leq 1\} \quad , \quad \mathcal{D}_{tol}^* = \{t \in [0, T], \text{ st} : S_{tol} \leq S(t) \leq S^*\}$$

We first note that $[0, T] \setminus D_1$ has non nul measure, since otherwise we would have $V[u_{W1}(\cdot)] = V_{max}$ and $B[u_{W1}(\cdot)] = B_{max}$, which, combined with the proposition 4.3, contradicts the framework hypothesis for the problem $(W1)$.

Suppose that D_1 has also non null measure. By continuity of the function $S(\cdot)$, there exists $t_a < t_b < t_c$ such that $[t_a, t_b] \subset D_1$ and $(t_b, t_c] \subset \mathcal{D}_*^1 \setminus D_1$ (or $[t_b, t_c] \subset D_1$ and $[t_a, t_b) \subset \mathcal{D}_*^1 \setminus D_1$, which can be treated the same way).
The function $\lambda_S^{(W1)}(\cdot)$ satisfies over $(t_b, t_c]$:

$$\dot{\lambda}_S^{(W1)}(t) \in (1 - \varphi(t))\lambda_S^{(W1)}(t)k_1\partial_C K_R(S(t))$$

Moreover, as $S(\cdot)$ is constant equal to 1 on $[t_a, t_b]$, we deduce that the control is equal to $1/k_2 \in (0, 1)$ on $[t_a, t_b]$, thus the function $\phi^{(W1)}(\cdot)$ verifies $\phi^{(W1)}(t) = 0$ for all $t \in [t_a, t_b]$. In particular, $\lambda_S^{(W1)}(t_b) = 0$.
By continuity, we have $\lambda_S^{(W1)}(t) \underset{t \to t_b^+}{\longrightarrow} \lambda_S^{(W1)}(t_b) = 0$, then necessarily we deduce that $\lambda_S^{(W1)}(t) = 0$ for all $t \in [t_b, t_c]$. It follows that $\lambda_S^{(W1)}(t) = 0$ for all $t \in \mathcal{D}_*^1$.
Note that, on $[0, T] \setminus D_1$, the function $\lambda_S^{(W1)}(\cdot)$ can be null on an unterval only when $0 \in \partial_C K_S(S(t))$ and $0 \in \lambda_Y \partial_C C_Y(S(t))$ over this interval. Therefore in $\mathcal{D}_{tol}^* \setminus (D^* \cup D_{tol})$ the function $\lambda_S^{(W1)}(\cdot)$ can not be null on an interval, thus $\lambda_S^{(W1)}(t) < 0$ ae. $t \in \mathcal{D}_{tol}^* \setminus (D^* \cup D_{tol})$, then $\phi^{(W1)}(t) < 0$ ae. $t \in \mathcal{D}_{tol}^* \setminus (D^* \cup D_{tol})$, then $u(t) = 0$ ae. $t \in \mathcal{D}_{tol}^* \setminus (D^* \cup D_{tol})$, and finally $S(\cdot)$ is decreasing on $\mathcal{D}_{tol}^* \setminus (D^* \cup D_{tol})$.
Let $t_s = \sup \mathcal{D}_*^1$, it verifies $t_s < T$.
There exists $\delta > 0$ such that $S_{tol} < S(t) < S^*$ for $t \in (t_s, t_s + \delta]$. Then on this interval we have :

$$\dot{\lambda}_S^{(W1)}(t) \in \varphi(t)\lambda_S^{(W1)}(t)k_1\big(\partial_C K_S(S(t)) - \partial_C K_R(S(t))\big) - \lambda_B^{(W1)}\varphi(t)\partial_C K_S(S(t)) + \lambda_S^{(W1)}(t)k_1\partial_C K_R(S(t))$$

On this interval, the quantities $\partial_C K_S(S(t))$ and $\partial_C K_R(S(t))$ are positive constants, we denote them by $C_1 = \partial_C K_S(S(t))$ and $C_2 = \partial_C K_R(S(t))$. One also has by hypothesis $C_1 - C_2 > 0$. Furthermore the function $\varphi(\cdot)$ satisfies $0 < \varphi(t_s) < \varphi(t) < 1$ for all $t \in (t_s, t_s + \delta]$.
We have the following inequalities for $t \in (t_s, t_s + \delta]$:

$$|\varphi(t)\lambda_S^{(W1)}(t)k_1(\partial_C K_S(S(t)) - \partial_C K_R(S(t))) + \lambda_S^{(W1)}(t)k_1\partial_C K_R(S(t))| \leq K|\lambda_S^{(W1)}(t)| \\ \text{with } K = k_1(\varphi(1)(C_1 - C_2) + C_2) > 0$$

and :

$$-\lambda_B^{(W1)}\varphi(t)\partial_C K_S(S(t)) \geq -\lambda_B^{(W1)}\varphi(t_s)C_1 > 0$$

As $\lambda_S^{(W1)}(\cdot)$ is continuous and $\lambda_S^{(W1)}(t_s) = 0$, there exists $0 < \delta' < \delta$ such that $K|\lambda_S^{(W1)}(t)| \leq \frac{-\lambda_B^{(W1)}\varphi(t_s)C_1}{2}$ for all $t \in (t_s, t_s + \delta']$.
As a result, we have :

$$\dot{\lambda}_S^{(W1)}(t) \geq \frac{-\lambda_B^{(W1)}\varphi(t_s)C_1}{2} > 0 \quad , \quad \forall t \in (t_s, t_s + \delta']$$

Then $\lambda_S^{(W1)}(\cdot)$ is increasing over $(t_s, t_s + \delta']$, which is impossible since we know that $\lambda_S^{(W1)} \leq 0$.
Therefore the set D_1 has null measure.

Moreover, if there exists an interval $I \subset \mathcal{D}_*^1 \setminus D_1$ such that $\lambda_S^{(W1)}(t) = 0$ for all $t \in I$, we can use again the reasoning above and we obtain the same contradiction. Then $\lambda_S^{(W1)}(t) < 0$ ae. $t \in \mathcal{D}_*^1$, and the function $S(\cdot)$ is decreasing over $\mathcal{D}_*^1$.
To conclude, we have shown that $u(t) = 0$ ae. $t \in \mathcal{D}_*^1 \cup \mathcal{D}_{tol}^* \setminus D_{tol}$, and $S(\cdot)$ is decreasing over $\mathcal{D}_*^1 \cup \mathcal{D}_{tol}^* \setminus D_{tol}$. Then the set $\mathcal{D}_*^1 \cup \mathcal{D}_{tol}^* \setminus D_{tol}$ is actually an interval, and is in fact denoted $[0, t_{tol})$, with $t_{tol} < T$ by hypothesis. Since we know that a trajectory has to satisfy $S \geq S_{tol}$, then we have $S(t) = S_{tol}$ for all $t \in [t_{tol}, T]$. Therefore we have :

$$V[u_{W1}(\cdot)] = V_{tol} \quad \text{and} \quad B[u_{W1}(\cdot)] = B_{min}$$

Combined with the proposition 4.3, it contradicts the framework hypothesis for the problem $(W1)$.
Hence :

$$\lambda_V^{(W1)} > 0$$

□

Finally, we have :

Problem	$\lambda_S(T)$	$\lambda_B(T)$	$\lambda_V(T)$	$\lambda_Z(T)$
$(B1)$	0	1	< 0	
$(B2)$	0	> 0	-1	
$(B3)$	0	$\in \partial_C g(B(T)) \subset \mathbb{R}_+^*$	$\in -\partial_C c(V(T)) \subset \mathbb{R}_-^*$	
$(W1)$	0	-1	> 0	≤ 0
$(W2)$	0	< 0	1	≤ 0
$(W3)$	0	$\in -\partial_C g(B(T)) \subset \mathbb{R}_-^*$	$\in \partial_C c(V(T)) \subset \mathbb{R}_+^*$	≤ 0

5.1.3 Singular arcs

Proposition 5.4.
A singular arc in the problems (Bj) or (Wj) can only occur at a corner point of one of the functions K_S, K_R, C_Y and C_Z.

Proof.
In this proof, we drop the indices Bj or Wj for simplicity.
Consider a closed interval $I = [t_1, t_2]$ of non-null measure, such that the switching function is equal to zero on I. This implies to have λ_S being constant equal to $\tilde{\lambda}_S = -\frac{\beta}{k_1 k_2} \neq 0$ on this interval (where $\beta \in \{\lambda_V^{(B1)}, \lambda_V^{(B2)}, \lambda_V^{(B3)}, \lambda_V^{(W1)}, \lambda_V^{(W2)}, \lambda_V^{(W3)}\}$). If the functions K_S, K_R, C_Y and C_Z are differentiable at $S(\hat{t})$ where $\hat{t}$ is an interior point of I, then by construction the quantities $K'_S(S(t))$, $K'_R(S(t))$, $C'_Y(S(t))$ and $C'_Z(S(t))$ are constant equal to $K'_S(S(\hat{t}))$, $K'_R(S(\hat{t}))$, $C'_Y(S(\hat{t}))$ and $C'_Z(S(\hat{t}))$ on a neighborhood $(\hat{t}-\varepsilon, \hat{t}+\varepsilon)$ of $\hat{t}$. More precisely, since we know that $S_{tol} \leq S(t) \leq 1$, one has $K'_R(S(\hat{t})) > 0$ and $C'_Y(S(\hat{t})) = C'_Z(S(\hat{t})) = 0$.
Thus, from the equation verified by λ_S, we get :

$$\varphi(t)\big((\tilde{\lambda}_S k_1 - \alpha)K'_S(S(\hat{t})) - \tilde{\lambda}_S k_1 K'_R(S(\hat{t}))\big) = -\tilde{\lambda}_S k_1 K'_R(S(\hat{t})) \neq 0 \qquad , \qquad \forall t \in I \cap (\hat{t} - \varepsilon, \hat{t} + \varepsilon)$$

with $\alpha \in \{\lambda_B^{(B1)}, \lambda_B^{(B2)}, \lambda_B^{(B3)}, \lambda_B^{(W1)}, \lambda_B^{(W2)}, \lambda_B^{(W3)}\}$.
As the function φ is strictly increasing, we deduce that this equality cannot be satisfied for all $t \in I \cap (\hat{t} - \varepsilon, \hat{t} + \varepsilon)$.
Consequently, a singular arc can only occur for some $S = \tilde{S}$ where one of the functions K_S, K_R, C_Y or C_Z is not differentiable. □

The set of corner points of the parameter functions is the following :

$$\mathcal{D} = \{S_{tol}, S^*, 1\}$$

Remark 5.1.
Furthermore, we notice now that the expression of the control during a singular arc has to enable its associated trajectory $S(\cdot)$ to be constant. Consequently, if a singular arc happens for the value $S = \tilde{S}$, then the corresponding control is explicitly known as $\tilde{u}(t) = u_{sing}(t, \tilde{S})$.

Let us now restrict even more the possibilities for the values of the singular arcs.

Proposition 5.5.
For any optimal solution of (Bj), a singular arc can only occur at $S = S_{tol}$ or $S = S^*$.

Proof.
We know from a previous property that an optimal solution of the problem (Bj) has to verify $u(t) = 0$ for a.e. $t \in [0, t^*]$. Thus implying that $S(\cdot)$ is decreasing over $[0, t^*]$. Moreover we also know that an optimal trajectory has to satisfy $S_{tol} \leq S(t) \leq S^*$ for all $t \in [t^*, T]$. Therefore it is not possible to observe a singular arc at $S = 1$. □

Proposition 5.6.
For any optimal solution of (Wj), a singular arc can only occur at $S = S_{tol}$ or $S = 1$.

Proof.
Assume that there exists $I = [t_1, t_2]$, with $0 \leq t_1 < t_2 \leq T$, such that $S(\cdot)$ describes a singular arc at S^* over I. Let $u(\cdot)$ be the associated control.
For $\gamma \in [t_1, t_2]$ to be determined later, we consider the control $z_\gamma(\cdot)$, and its corresponding solutions $S_{z_\gamma}(\cdot), B_{z_\gamma}(\cdot), V_{z_\gamma}(\cdot)$, defined by :

$$z_\gamma(t) = \begin{cases} u(t) & , t \in [0, t_1) \\ \tilde{u}_{S_{tol}}^{up}(t) & , t \in [t_1, \gamma) \\ \tilde{u}_1^{low}(t) & , t \in [\gamma, t_2) \\ u(t) & , t \in [t_2, T] \end{cases}$$

If $\gamma = t_1$, one has $\int_I z_{t_1}(t)\mathrm{d}t > \int_I u(t)\mathrm{d}t$.
If $\gamma = t_2$, one has $\int_I z_{t_2}(t)\mathrm{d}t < \int_I u(t)\mathrm{d}t$.
Let $C' = \frac{1}{2}\int_I z_{t_1}(t) - u(t)\mathrm{d}t > 0$. By the intermediate value theorem, there exists $\gamma \in (t_1, t_2)$ such that $\int_I z_\gamma(t) - u(t)\mathrm{d}t = C'$.
Then we have :

$$\begin{aligned} V[z_\gamma(\cdot)] &= \int_{[0,T]\setminus I} z_\gamma(t)\mathrm{d}t + \int_I z_\gamma(t)\mathrm{d}t \\ &= \int_{[0,T]\setminus I} u(t)\mathrm{d}t + \int_I u(t)\mathrm{d}t + C' > V[u(\cdot)] \end{aligned}$$

The associated solution $S_{z_\gamma}(\cdot)$ satisfies :

$$\forall t \in [0,T] \setminus I,\ S_{z_\gamma}(t) = S(t) \quad ; \quad \forall t \in (t_1, \gamma),\ S_{z_\gamma}(t) < S^* = S(t) \quad ; \quad \forall t \in (\gamma, t_2),\ S_{z_\gamma}(t) > S^* = S(t)$$

Then :

$$\forall t \in [0,T],\ K_S(S_{z_\gamma}(t)) \leq K_S(S(t)) \quad \text{with} \quad \int_I K_S(S_{z_\gamma}(t))\mathrm{d}t < \int_I K_S(S(t))\mathrm{d}t$$

This leads to :

$$\begin{aligned} B[z_\gamma(\cdot)] &= \int_{[0,T]\setminus I} \varphi(t)K_S(S_{z_\gamma}(t))\mathrm{d}t + \int_I \varphi(t)K_S(S_{z_\gamma}(t))\mathrm{d}t \\ &< \int_{[0,T]\setminus I} \varphi(t)K_S(S(t))\mathrm{d}t + \int_I \varphi(t)K_S(S(t))\mathrm{d}t = B[u(\cdot)] \end{aligned}$$

To conclude, we have :

$$B[z_\gamma(\cdot)] < B[u(\cdot)] \quad \text{and} \quad V[z_\gamma(\cdot)] > V[u(\cdot)]$$

Thus the control $z_\gamma(\cdot)$ is feasible for (Wj), and the control $u(\cdot)$ is therefore non optimal for (Wj). Hence there is no singular arc at the value S^*. □

5.2 Structures of optimal solutions

Proposition 5.7.
For any optimal solution to the problem (Bj), the set :

$$\mathcal{X}_B = \{t \in [t^*, T],\ \phi_c(t) \geq 0\}$$

is connected.

Proof.
Suppose that $\mathcal{X}_B$ is not connected. In other words, assume that there exists an interval $(t_1, t_2) \subset [t^*, T]$ such that $\phi_c(t_1) = \phi_c(t_2) = 0$ and $\phi_c(t) < 0$ for all $t \in (t_1, t_2)$. This implies $u(t) = 0$ ae. in (t_1, t_2) and thus $S(\cdot)$ decreasing on (t_1, t_2), which means that $S_{tol} < S(t) < S^*$ on (t_1, t_2).
Therefore the function $\lambda_S^{(Bj)}(\cdot)$ satisfies over (t_1, t_2) :

$$\dot{\lambda}_S^{(Bj)}(t) \in \varphi(t)(\lambda_S^{(Bj)}(t)k_1 - \lambda_B^{(Bj)})\partial_C K_S(S(t)) + (1-\varphi(t))\lambda_S^{(Bj)}(t)k_1\partial_C K_R(S(t))$$

Moreover, the function ϕ_c attains a local minimum over (t_1, t_2) at $\hat{t} \in (t_1, t_2)$, and since $\phi_c(t) = \lambda_S^{(Bj)}(t)k_1k_2 + \lambda_V^{(Bj)}$ the function $\lambda_S^{(Bj)}$ on (t_1, t_2) attains its minimum at $\hat{t}$. One then has :

$$0 \in \varphi(\hat{t})(\lambda_S^{(Bj)}(\hat{t})k_1 - \lambda_B^{(Bj)})\partial_C K_S(S(\hat{t})) + (1-\varphi(\hat{t}))\lambda_S^{(Bj)}(\hat{t})k_1\partial_C K_R(S(\hat{t})) \tag{3}$$

First note that this equation above can be fulfilled only if $\lambda_S^{(Bj)}(\hat{t})k_1 - \lambda_B^{(Bj)} < 0$.
As $S_{tol} < S(t) < S^*$ for all $t \in (t_1, t_2)$, the quantities $K'_S(S(t))$ and $K'_R(S(t))$ are defined and constant equal to $K'_S(S(\hat{t})) > 0$ and $K'_R(S(\hat{t})) > 0$ respectively on (t_1, t_2). Then, recalling that $\lambda_S^{(Bj)} \geq 0$, one has for all $t \in (t_1, \hat{t})$:

$$\begin{aligned} \dot{\lambda}_S^{(Bj)}(t) &= \varphi(t)(\lambda_S^{(Bj)}(t)k_1 - \lambda_B^{(Bj)})K'_S(S(t)) + (1-\varphi(t))\lambda_S^{(Bj)}(t)k_1K'_R(S(t)) \\ &\geq \varphi(t)(\lambda_S^{(Bj)}(\hat{t})k_1 - \lambda_B^{(Bj)})K'_S(S(\hat{t})) + (1-\varphi(t))\lambda_S^{(Bj)}(\hat{t})k_1K'_R(S(\hat{t})) \\ &> \varphi(\hat{t})(\lambda_S^{(Bj)}(\hat{t})k_1 - \lambda_B^{(Bj)})K'_S(S(\hat{t})) + (1-\varphi(\hat{t}))\lambda_S^{(Bj)}(\hat{t})k_1K'_R(S(\hat{t})) \end{aligned}$$

The last inequality comes from the fact that φ takes its values in $[0, 1]$ and is increasing.
We deduce from (3) that $\dot{\lambda}_S^{(Bj)}(t) > 0$ for all $t \in (t_1, \hat{t})$, which contradicts the minimality of $\lambda_S^{(Bj)}(\hat{t})$ on (t_1, t_2).
Finally, we conclude that the set $\mathcal{X}_B$ is connected. □

Proposition 5.8.
For any optimal solution to the problem (Wj), the set :

$$\mathcal{X}_W = \{t \in [0, T],\ \phi_c(t) \leq 0\}$$

is connected.

Proof.
Suppose that $\mathcal{X}_W$ is not connected. In other words, assume that there exists an interval $(t_1, t_2) \subset [0, T]$ such that $\phi_c(t_1) = \phi_c(t_2) = 0$ and $\phi_c(t) > 0$ for all $t \in (t_1, t_2)$. This implies $u(t) = 1$ ae. in (t_1, t_2) and thus $S(\cdot)$ increasing on (t_1, t_2), which means that $S_{tol} < S(t) < 1$ on (t_1, t_2).
Therefore the function $\lambda_S^{(Wj)}(\cdot)$ satisfies over (t_1, t_2) :

$$\dot{\lambda}_S^{(Wj)}(t) \in \varphi(t)(\lambda_S^{(Wj)}(t)k_1 - \lambda_B^{(Wj)})\partial_C K_S(S(t)) + (1 - \varphi(t))\lambda_S^{(Wj)}(t)k_1\partial_C K_R(S(t))$$

Moreover, the function ϕ_c attains a local maximum over (t_1, t_2) at $\check{t} \in (t_1, t_2)$, and since $\phi_c(t) = \lambda_S^{(Wj)}(t)k_1k_2 + \lambda_V^{(Wj)}$ the function $\lambda_S^{(Wj)}$ on (t_1, t_2) attains its maximum at $\check{t}$. One then has :

$$0 \in \varphi(\check{t})(\lambda_S^{(Wj)}(\check{t})k_1 - \lambda_B^{(Wj)})\partial_C K_S(S(\check{t})) + (1 - \varphi(\check{t}))\lambda_S^{(Wj)}(\check{t})k_1\partial_C K_R(S(\check{t})) \tag{4}$$

First note that this equation above can be fulfilled only if $\lambda_S^{(Wj)}(\check{t})k_1 - \lambda_B^{(Wj)} > 0$.
If $S_{tol} < S(\check{t}) < S^*$, then $S_{tol} < S(t) < S^*$ for all $t \in (t_1, \check{t}]$, and the quantities $K_S'(S(t))$ and $K_R'(S(t))$ are defined and constant equal to $K_S'(S(\check{t})) > 0$ and $K_R'(S(\check{t})) > 0$ respectively on $(t_1, \check{t}]$. Then, recalling that $\lambda_S^{(Wj)} \leq 0$, one has for all $t \in (t_1, \check{t})$:

$$\begin{aligned}
\dot{\lambda}_S^{(Wj)}(t) &= \varphi(t)(\lambda_S^{(Wj)}(t)k_1 - \lambda_B^{(Wj)})K_S'(S(t)) + (1 - \varphi(t))\lambda_S^{(Wj)}(t)k_1K_R'(S(t)) \\
&\leq \varphi(t)(\lambda_S^{(Wj)}(\check{t})k_1 - \lambda_B^{(Wj)})K_S'(S(\check{t})) + (1 - \varphi(t))\lambda_S^{(Wj)}(\check{t})k_1K_R'(S(\check{t})) \\
&< \varphi(\check{t})(\lambda_S^{(Wj)}(\check{t})k_1 - \lambda_B^{(Wj)})K_S'(S(\check{t})) + (1 - \varphi(\check{t}))\lambda_S^{(Wj)}(\check{t})k_1K_R'(S(\check{t}))
\end{aligned}$$

The last inequality comes from the fact that φ takes its values in $[0, 1]$ and is increasing.
We deduce from (4) that $\dot{\lambda}_S^{(Wj)}(t) < 0$ for all $t \in (t_1, \check{t})$, which contradicts the maximality of $\lambda_S^{(Wj)}(\check{t})$ on (t_1, t_2).
If $S(\check{t}) > S^*$, then $\partial_C K_S(S(\check{t})) = \{0\}$ and $\partial_C K_R(S(\check{t})) \subset \mathbb{R}_+^*$, therefore from (4) one has $\lambda_S^{(Wj)}(\check{t}) = 0$. Since $S(\cdot)$ is increasing on $[\check{t}, t_2)$, the function $\lambda_S^{(Wj)}(\cdot)$ satisfies for all $t \in [\check{t}, t_2)$:

$$\dot{\lambda}_S^{(Wj)}(t) \in \lambda_S^{(Wj)}(t)k_1(1 - \varphi(t))\partial_C K_R(S(t))$$

We then deduce as previously that $\lambda_S^{(Wj)}(t) = 0$ for all $t \in [\check{t}, t_2)$, which implies $\phi_c(t) = \lambda_V^{(Wj)}$ for all $t \in [\check{t}, t_2)$.
As $\lambda_V^{(Wj)} > 0$ and ϕ_c is continuous, it contradicts the fact that $\phi_c(t_2) = 0$.
Finally we necessarily have $S(\check{t}) = S^*$.
Since $S(\cdot)$ is increasing on (t_1, t_2), and $S(0) = S_0 > S^*$, we deduce that there exists $t_0 \in (0, t_1)$ such that $S(t_0) = S^*$. Moreover we know that there exists an interval $(T - \alpha, T]$ where the switching function ϕ_c is positive, then the trajectory $S(\cdot)$ is increasing over $(T - \alpha, T]$, and $S(T - \alpha) < 1$. We have $\phi_c(t_2) = 0$, and $S(t_2) \in (S^*, 1]$, then there can be a singular arc starting at t_2 only if $S(t_2) = 1$. Therefore, in any case the trajectory $S(\cdot)$ will have to be decreasing at some point and thus the switching function will be negative. As a consequence, there exists $\tilde{t} \in (t_2, T)$ that is a local minimum of ϕ_c, and thus a local minimum of $\lambda_S^{(Wj)}$. With a similar argument as above, we show that $S(\tilde{t})$ cannot be greater than S^*, then there exists $t_3 \in (t_2, \tilde{t}]$ such that $S(t_3) = S^*$.
Let us fix the quantity $\overline{U} = \int_{t_0}^{t_3} u(t)\mathrm{d}t$. This leads to consider an auxiliary problem defined on $[t_0, t_3]$ with the same dynamics :

$$\begin{cases}
\dot{s}(t) = k_1(-\varphi(t)K_S(s(t)) - (1 - \varphi(t))K_R(s(t)) + k_2u(t)) \\
\dot{b}(t) = \varphi(t)K_S(s(t)) \\
\dot{v}(t) = u(t) \\
\dot{y}(t) = C_Y(s(t)) \\
\dot{z}(t) = C_Z(s(t))
\end{cases}$$

with initial conditions $s(t_0) = S^*$, $b(0) = 0$, $v(0) = 0$, $y(0) = 0$, $z(0) = 0$ and final conditions $s(t_3) = S^*$, $v(t_3) = \overline{U}$, $y(t_3) \geq 0$, $z(t_3) \leq 0$. In these settings, we seek to minimize the criterium $\int_{t_0}^{t_3} \dot{b}(t)\mathrm{d}t$, and it

therefore respects the conditions of the "auxiliary periodic problem" that can be found in the Annex.
We deduce from this auxiliary problem that the optimal solution u^* has the structure "$1, 0, 1$" over $[t_0, t_3]$, thus the trajectory described above (especially with $S(\cdot)$ increasing on (t_1, t_2) and $S(\check{t}) = S^*$) is non optimal, which is a contradiction.
To conclude, the set $\mathcal{X}_W$ is connected. □

Definition 5.1.
For $t_{trig} \in [t^*, T]$, V_{cut}, t_{cut}, we define the time-varying feedback control :

$$\Psi^+_{t_{trig},V_{cut},t_{cut}}(t,S,B,V) = \begin{cases} 0 & \text{, if } \big(S \geq S_{tol} \ \wedge \ t \leq t_{trig}\big) \text{ or } V = V_{cut} \text{ or } t \geq t_{cut} \\ u_{sing}(t, S_{tol}) & \text{, if } S = S_{tol} \ \wedge \ t \leq t_{trig} \\ u_{sing}(t, S^*) & \text{, if } S = S^* \ \wedge \ \big(V < V_{cut} \text{ and } t < t_{cut}\big) \\ 1 & \text{, otherwise} \end{cases}$$

Definition 5.2.
For $V_{trig} \in [0, V^1)$, V_{cut}, t_{cut}, we define the time-varying feedback control :

$$\Psi^-_{V_{trig},V_{cut},t_{cut}}(t,S,B,V) = \begin{cases} 1 & \text{, if } \big(V < V_{trig} \wedge S < 1\big) \text{ or } V_{cut} - V = T - t \text{ or } t \geq t_{cut} \\ u_{sing}(t, 1) & \text{, if } \big(V < V_{trig} \wedge S = 1\big) \\ u_{sing}(t, S_{tol}) & \text{, if } S = S_{tol} \ \wedge \ \big(V_{cut} - V < T - t \text{ and } t < t_{cut}\big) \\ 0 & \text{, otherwise} \end{cases}$$

Theorem 5.9.
Under the framework for the problems (Bj), there exists $t_{trig} \in [t^*, T)$, $V_{cut} \in (0, V_{max}]$, $t_{cut} \in (t_{trig}, T)$ such that the feedback control $\Psi^+_{t_{trig},V_{cut},t_{cut}}$ is optimal for (Bj).

Proof.
Let $u(\cdot)$ be an optimal control for the problem (Bj).
As seen above, the set $\mathcal{X}_B$ is non-empty and connected, and since the function $\phi_c(\cdot)$ is continuous, we deduce that $\mathcal{X}_B$ is an interval $[t_a, t_b]$ where $0 \leq t_a < t_b \leq T$, and the control $u(\cdot)$ is null outside this interval.
From previous results, we know that $t_a \geq t^*$ and $t_b < T$. At any $t \in \mathcal{X}_B$ the switching function $\phi_c(t)$ is non-negative, then for ae. $t \in \mathcal{X}_B$ one has either $u(t) = 1$ or $u(t) = u_{sing}(t, \tilde{S})$ where $\tilde{S} \in \{S_{tol}, S^*\}$.
As a consequence, the quantities $V(\cdot)$ and t are increasing on $\mathcal{X}_B$ and $S(\cdot)$ is non-decreasing on $\mathcal{X}_B$. Furthermore, the trajectory $S(\cdot)$ is thus composed of an increasing part (with $u = 1$) and possible singular parts (with $u = u_{sing}$ and $S = S_{tol}$ or $S = S^*$). The increasing part is thus over an interval $[t_{trig}, t_f]$ with $t_a \leq t_{trig} < t_f \leq t_b$.
At $t = t_b$, we know that $S(t_b) \leq S^*$ and since V is increasing on $\mathcal{X}_B$ we deduce that t_b and $V(t_b)$ are equivalently defined. We denote the "cut-off time" by $t_{cut} = t_b$ and we set $V_{cut} = V(t_{cut})$.
According to what was said above, we then have either $t_a = t_{trig}$ or t_a fully determined by the trajectory $S_{0,S_0,0}(\cdot)$ such that $S_{0,S_0,0}(t_a) = S_{tol}$. Moreover, the time t_f is either equal to t_{cut} or fully determined by the trajectory $S_{t_{trig},S(t_{trig}),1}(\cdot)$ such that $S_{t_{trig},S(t_{trig}),1}(t_f) = S^*$.
Finally, the control $u(\cdot)$ is fully determined by the two parameters t_{trig} and t_{cut} (or equivalently t_{trig} and V_{cut}), and we easily check that it fulfills $u(t) = \Psi^+_{t_{trig},V_{cut},t_{cut}}(t, S(t), B(t), V(t))$ for ae. $t \in [0, T]$. □

Remark 5.2.
The structure is common for the three "Best" problems, and by the theorem, the optimal control for each case is fully determined by two parameters t_{trig} and t_{cut} (or equivalently t_{trig} and V_{cut}). However we can highlight a difference now.
Indeed for the problem $(B1)$ there is a terminal condition that has to be satisfied (proposition 4.5) known as $V(T) = V_{init}$. According to the structure for $u(\cdot)$, there is no irrigation on $[t_{cut}, T]$, thus $V(T) = V_{cut}$. As a consequence, V_{cut} (and equivalently t_{cut}) is fully determined by the problem data in the $(B1)$ case, and the optimal control $u(\cdot)$ is therefore only determined by one parameter $t_{trig} \in [t^*, T)$.
For the problem $(B2)$ the terminal condition that has to be satisfied (proposition 4.5) is $B(T) = B_{target}$. As above, we deduce that $B(T) = B(t_{cut}) + \int_{t_{cut}}^{T} \varphi(\tau) K_S(S_{t_{cut},S(t_{cut}),0}(\tau)) \mathrm{d}\tau$. We denote $S^{struc,t_{trig},t_{cut}}(\cdot)$ the

trajectory following the optimal structure and parametrized by t_{trig} and t_{cut}. For any fixed $t_{trig} \in [t^*, T)$, we verify that the function :

$$t_{cut} \in [t_{trig}, T] \;\mapsto\; B(t_{cut}) + \int_{t_{cut}}^{T} \varphi(\tau) K_S(S_{t_{cut}, S^{struc,t_{trig},t_{cut}}(t_{cut}),0}(\tau))\mathrm{d}\tau$$

is increasing. Moreover the target B_{target} is supposed to be attainable. As a consequence there exists a unique $t_{off} \in (t_{trig}, T)$ such that the corresponding quantity $B(T)$ is equal to B_{target}. As a consequence, t_{cut} (and equivalently V_{cut}) is fully determined by the problem data in the $(B2)$ case, and the optimal control $u(\cdot)$ is therefore only determined by one parameter $t_{trig} \in [t^*, T)$.
For the problem $(B3)$, there is no terminal condition. As a consequence, the optimal control $u(\cdot)$ is determined by two parameters t_{trig} and t_{cut} (or equivalently t_{trig} and V_{cut}).

Theorem 5.10.
Under the framework for the problems (Wj), there exists $V_{trig} \in [0, V_{max})$, $V_{cut} \in [V_{trig}, V_{max})$, $t_{cut} \in (0, T)$ such that the feedback control $\Psi^-_{V_{trig},V_{cut},t_{cut}}$ is optimal for (Wj).

Proof.
Let $u(\cdot)$ be an optimal control for the problem (Wj).
As seen above, the set $\mathcal{X}_W$ is non-empty and connected, and since the function $\phi_c(\cdot)$ is continuous, we deduce that $\mathcal{X}_W$ is an interval $[t_a, t_b]$ where $0 \le t_a < t_b \le T$, and the control $u(\cdot)$ is equal to 1 outside this interval.
From previous results, we know that $t_b < T$. At any $t \in \mathcal{X}_W$ the switching function $\phi_c(t)$ is non-positive, then for ae. $t \in \mathcal{X}_W$ one has either $u(t) = 0$ or $u(t) = u_{sing}(t, \tilde{S})$ where $\tilde{S} \in \{S_{tol}, 1\}$.
As a consequence, the quantity $T - t + V(\cdot)$ is decreasing on $\mathcal{X}_W$ and t increasing on $\mathcal{X}_W$, and $S(\cdot)$ is non-increasing on $\mathcal{X}_W$. Furthermore, the trajectory $S(\cdot)$ is thus composed of a decreasing part (with $u = 0$) and possible singular parts (with $u = u_{sing}$ and $S = S_{tol}$ or $S = 1$). The decreasing part is thus over an interval $[t_{trig}, t_f]$ with $t_a \le t_{trig} < t_f \le t_b$.
At $t = t_b$, we know that $S(t_b) \ge S_{tol}$ and since $T - t + V$ is decreasing on $\mathcal{X}_W$ we deduce that $T - t_b + V(t_b)$ and t_b are equivalently defined. We denote the "cut-off time" by $t_{cut} = t_b$ and we set $V_{cut} = V(t_{cut})$.
According to what was said above, we then have V increasing outside $\mathcal{X}_W$ and on singular parts of $\mathcal{X}_W$. Thus either $t_a = t_{trig}$ or t_a is fully determined by the trajectory $S_{0,S_0,1}(\cdot)$ such that $S_{0,S_0,1}(t_a) = 1$, and the quantities t and V are equivalently defined over $[0, t_{trig}]$, in particular we denote $V_{trig} = V(t_{trig})$. Moreover, the time t_f is either equal to t_{cut} or fully determined by the trajectory $S_{t_{trig},S(t_{trig}),0}(\cdot)$ such that $S_{t_{trig},S(t_{trig}),0}(t_f) = S_{tol}$.
Finally, the control $u(\cdot)$ is fully determined by the two parameters V_{trig} and t_{cut} (or equivalently V_{trig} and $T - t_{cut} + V_{cut}$), and we easily check that it fulfills $u(t) = \Psi^-_{V_{trig},V_{cut},t_{cut}}(t, S(t), B(t), V(t))$ for ae. $t \in [0, T]$. ☐

Remark 5.3.
The structure is common for the three "Worst" problems, and by the theorem, the optimal control for each case is fully determined by two parameters V_{trig} and t_{cut} (or equivalently t_{trig} and V_{cut}). However we can highlight a difference now.
Indeed for the problem $(W1)$ there is a terminal condition that has to be satisfied (proposition 4.5) known as $V(T) = V_{init}$. According to the structure for $u(\cdot)$, irrigation is equal to 1 on $[t_{cut}, T]$, thus $V(T) = V_{cut} + T - t_{cut}$. As a consequence, V_{cut} (and equivalently t_{cut}) is fully determined by the problem data in the $(W1)$ case, and the optimal control $u(\cdot)$ is therefore only determined by one parameter $V_{trig} \in [0, V_{max}]$.
For the problem $(W2)$ the terminal condition that has to be satisfied (proposition 4.5) is $B(T) = B_{target}$. As above, we deduce that $B(T) = B(t_{cut}) + \int_{t_{cut}}^{T} \varphi(\tau) K_S(S_{t_{cut},S(t_{cut}),1}(\tau))\mathrm{d}\tau$. We denote $S^{struc,V_{trig},t_{cut}}(\cdot)$ the trajectory following the optimal structure and parametrized by V_{trig} and t_{cut}. For any fixed $V_{trig} \in [0, V_{max}]$, we denote $t_{trig} \in [0, T)$ the corresponding time, and we verify that the function :

$$t_{cut} \in [t_{trig}, T] \;\mapsto\; B(t_{cut}) + \int_{t_{cut}}^{T} \varphi(\tau) K_S(S_{t_{cut}, S^{struc,V_{trig},t_{cut}}(t_{cut}),1}(\tau))\mathrm{d}\tau$$

is decreasing. Moreover the target B_{target} is supposed to be attainable. As a consequence there exists a unique $t_{off} \in (t_{trig}, T)$ such that the corresponding quantity $B(T)$ is equal to B_{target}. As a consequence, t_{cut} (and equivalently V_{cut}) is fully determined by the problem data in the $(W2)$ case, and the optimal control

$u(\cdot)$ is therefore only determined by one parameter $V_{trig} \in [0, V_{max}]$.
For the problem $(W3)$, there is no terminal condition. As a consequence, the optimal control $u(\cdot)$ is determined by two parameters V_{trig} and t_{cut} (or equivalently V_{trig} and V_{cut}).
Finally we have for the "Best" problems :

Problem	t_{trig}	V_{cut}	t_{cut}
$(B1)$	TBO in $[t^*, T)$	V_{init}	T
$(B2)$	TBO in $[t^*, T)$	V_{max}	t_{off}
$(B3)$	TBO in $[t^*, T)$	TBO in $[0, V_{max}]$	T

And we have for the "Worst" problems :

Problem	V_{trig}	V_{cut}	t_{cut}
$(W1)$	TBO in $[0, V_{max})$	V_{init}	T
$(W2)$	TBO in $[0, V_{max})$	V_{max}	t_{off}
$(W3)$	TBO in $[0, V_{max})$	TBO in $[V_{trig}, V_{max}]$	T

Remark 5.4.
Note that when we select one problem, some of the conditions in the definition 5.1 or 5.2 are always satisfied. The interest is only to have a unified structure for every type of problem, but since we optimize over different quantities (B, V) for the criteria and the targets it is not possible to keep the same parameters for every problem in the definition of the strategies.

6 Conclusion

Considering six optimization problems based on the same set of dynamics of a simplified crop model, we have highlighted two structures for optimal strategies depending on the criterion and target. These strategies can be implemented as parametrized controls in small dimension, hence the optimal solutions can be computed numerically and efficiently. The perspectives are first to perform a numerical exploration with a large number of parameter sets in order to determine the respective influence of the different variables on the problem, and then to assess the potential gain margin between the best and the worst theoretical practices in order to bring decision support with quantitative results in a concrete application.

Appendix

The auxiliary perioidic problem is defined as follows.

$$\dot{x}(t) = -\mu(t) f_1(x(t)) - (1 - \mu(t)) f_2(x(t)) + g u(t) \qquad , \forall t \in [a, b]$$

where $x \in \mathbb{R}$, $\mu : [a, b] \mapsto [0, 1]$ is C^1 and monotonous, $f_1, f_2 : \mathbb{R} \mapsto [0, 1]$ are continuous, Lipschitz and have the same monotony, $g > 1$ and $u : [a, b] \mapsto [0, 1]$ is measurable.
We add the periodic condition $x(a) = x(b) = x_0 \in \mathbb{R}$, and the constraint on the input $\int_a^b u(t)t = \overline{U}$ where $\overline{U}$ is reachable.
We aim to optimize the criterium $J = \int_a^b \mu(t) f_1(t) \mathrm{d}t$.

Proposition 6.1.
For any choice of monotony for μ and the functions f_1, f_2, and for any optimization of J, the optimal solution has at most two bang switches.
More precisely, the eight possibilities are regrouped in only two structures of solution, and are listed in the following table :

	μ	f_1, f_2	opti	Solution u^*
(1)	$\nearrow$	$\nearrow$	sup	"$0,1,0$"
(2)	$\nearrow$	$\nearrow$	inf	"$1,0,1$"
(3)	$\nearrow$	$\searrow$	sup	"$1,0,1$"
(4)	$\nearrow$	$\searrow$	inf	"$0,1,0$"
(5)	$\searrow$	$\nearrow$	sup	"$1,0,1$"
(6)	$\searrow$	$\nearrow$	inf	"$0,1,0$"
(7)	$\searrow$	$\searrow$	sup	"$0,1,0$"
(8)	$\searrow$	$\searrow$	inf	"$1,0,1$"

Proof.
It suffices to show the first two lines because the other settings can be reformulated into one of the situations (1) or (2).

Sketch of proof for (1) in the case f_1 concave and f_2 convex.
We write the following system over $[a, b]$:

$$\begin{cases} \dot{x}(t) = -\mu(t)f_1(x(t)) - (1-\mu(t))f_2(x(t)) + gu(t) \\ \dot{y}(t) = u(t) \end{cases}$$

with initial conditions $x(a) = x_0$ and $y(a) = 0$, and terminal conditions $x(b) = x_0$ and $y(b) = \overline{U}$.
We have the associated Hamiltonian :

$$H = p_x(t)\big(gu(t) - \big(\mu(t)f_1(x(t)) + (1-\mu(t))f_2(x(t))\big)\big) + p_y(t)u(t) + p_0\mu(t)f_1(x(t))$$

and the corresponding adjoint system :

$$\begin{cases} \dot{p}_x(t) \in \mu(t)(p_x(t) - p_0)\partial_C f_1(x(t)) + p_x(t)(1-\mu(t))\partial_C f_2(x(t)) \\ \dot{p}_y = 0 \end{cases}$$

We also define the switching function $\phi(t) = p_x(t)g + p_y$.
Consider the set :

$$\chi = \{t \in [a, b],\ \phi(t) \geq 0\}$$

If χ is not connected, then there exists $a \leq t_1 < t_2 \leq b$ such that $\phi(t_1) = \phi(t_2) = 0$ and $\phi(t) < 0$ for all $t \in (t_1, t_2)$ (then x is decreasing over (t_1, t_2).
Denote $\phi(\hat{t}) = \min\limits_{t\in(t_1,t_2)} \phi(t)$, thus $p_x(\hat{t}) = \min\limits_{t\in(t_1,t_2)} p_x(t)$.
At time $\hat{t}$ we have :

$$0 \in \mu(\hat{t})(p_x(\hat{t}) - p_0)\partial_C f_1(x(\hat{t})) + p_x(\hat{t})(1-\mu(\hat{t}))\partial_C f_2(x(\hat{t}))$$

We can show that $p_x \geq 0$ and $p_x(\hat{t}) - p_0 < 0$, and by assumptions on the monotony of μ, concavity of f_1 and convexity of f_2, we have for any $t \in (t_1, \hat{t})$:

$$\forall \xi \in \partial_C f_1(x(\hat{t}),\ \forall \zeta \in \partial_C f_1(x(t)),\ \mu(\hat{t})(p_x(\hat{t}) - p_0)\xi < \mu(t)(p_x(t) - p_0)\zeta$$

$$\forall \xi \in \partial_C f_2(x(\hat{t}),\ \forall \zeta \in \partial_C f_2(x(t)),\ (1-\mu(\hat{t}))p_x(\hat{t})\xi < (1-\mu(t))p_x(t)\zeta$$

Then for all $t \in (t_1, \hat{t})$:

$$\dot{p}_x(t) > \mu(\hat{t})(p_x(\hat{t}) - p_0)\xi + (1-\mu(\hat{t}))p_x(\hat{t})\zeta \quad , \forall \xi \in \partial_C f_1(x(\hat{t}),\ \forall \zeta \in \partial_C f_2(x(\hat{t})$$

Finally we get that p_x is increasing on $(t_1, \hat{t})$, which contradicts the minimality of $p_x(\hat{t})$.
Therefore the set χ is connected, and the optimal solution is of the form "$0, 1, 0$".

A similar proof can be written for the case (2) when f_1 is convex and f_2 is concave.

Numerical simulation with BOCOP seem to claim that the result is still true if f_1 and f_2 are only monotonous, not necessarily convex or concave. □

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